October, 2002

To: Zeev Rudnick

Dear Zeev,

Your talk in Park City led me to revisit the topic of the number variance for the eigenvalues of $SL(2,\mathbb{Z})\backslash\mathbb{H}$. In my opinion the critical problem, at least for smooth counting, is to analyze this quantity in intervals $\left[\lambda - \frac{H}{2}, \lambda + \frac{H}{2}\right]$ with H as small as possible. In particular, the Poissonian behavior probably kicks in for $H = o\left(\sqrt{\lambda}/\log\lambda\right)$. In any event the smaller that one can take H the more information can be gleaned about the multiplicities and spacings between the eigenvalues.

The problem appears to be a very difficult one. An examination of Chapter 2 of Hejhal [H], especially the discussion about $S(T)$, confirms this. Rather than the standard approach to these problems via the trace formula, I describe below how the Petersson-Kuznietzov formula allows one to see a little further. Roughly speaking, it allows one to halve the interval width that can be analyzed. This is very modest progress but it allows for a small window in which the number variance is shown to be Poissonian. Interestingly, in this form the large number variance is not a consequence of high multiplicities of lengths but is rather a consequence of the analytic aspects of the formula. This approach also makes transparent the Poissonian nature of the number variance.

Let, at first, X_Γ be any compact quotient $\Gamma\backslash\mathbb{H}$ of the upper half plane. Denote by $\lambda_\phi = \frac{1}{4} + t_\phi^2$ the eigenvalue of the eigenfunction ϕ. Fix h an even test function in $\mathcal{S}(\mathbb{R})$ with support $\hat{h} \subset (-1,1)$ and $\int_{-\infty}^{\infty} h(x)\,dx = 1$. Here

$$\hat{h}(\xi) \;=\; \int_{-\infty}^{\infty} h(x)\, e^{-2\pi i x\xi}\, dx\,. \tag{1}$$

For t large and $1 \ll L \ll t^{1-\epsilon}$ set

$$N_h(t,L) \;=\; \sum_{\phi} h((t_\phi - t)L)\,. \tag{2}$$

These are your "smooth" counting functions of eigenvalues t_ϕ near t in a window of size t/L [R]. The expected number $\widetilde{N}_h(t,L)$ is given by the identity term in the trace formula and it satisfies

$$\widetilde{N}_h(t,L) \;\sim\; \frac{Vol(X)}{4\pi}\,\frac{t}{L} \text{ as } t \longrightarrow \infty\,. \tag{3}$$

Applying the trace formula to the sum in (2) and estimating trivially the contribution of the conjugacy classes using the Prime Geodesic Theorem yields:

$$N_h(t, L) \sim \widetilde{N}_h(t, L) \tag{4}$$

as long as $L \leq \frac{\log t}{\pi}$.

While I expect that (4) holds with L as large as $t^{1/2-\epsilon}$, I don't see how to extend the range in (4). For the case of $\Gamma = SL(2, \mathbb{Z})$ the range can be doubled - see below.

If in (2) we choose h as above and also satisfying $h(x) \geq 0$, we obtain a bound for the multiplicity, $m_X(t)$, of the eigenvalue $t^2 + \frac{1}{4}$:

$$m_X(t) := \sum_{t_\phi = t} 1 \leq \frac{Vol(X)t}{h(0)\,4\log t}\,(1 + o(1))\,. \tag{5}$$

Optimizing the choice of such h, (see [I-L-S] page 115) yields

$$\overline{\lim_{t \longrightarrow \infty}} \frac{m_X(t)\,\log t}{t} \leq \frac{Vol(X)}{4}\,. \tag{6}$$

In my youth I would have been impressed with a bound of $m_X(t) = O(t^{1-\delta})$ for some $\delta > 0$. Today a bound $m_X(t) = o\left(\frac{t}{\log t}\right)$ looks pretty good. In any case (6) is the best I know for a general hyperbolic surface.

The number variance (smoothed) is defined to be

$$\sum(T, L) := \frac{1}{T}\int_T^{2T} \left(N_h - \tilde{N}_h\right)^2 dt\,. \tag{7}$$

The Poisson model for the t_ϕ's (that is to say that they are statistically independent) predicts that (we assume that $L(t) \longrightarrow \infty$ with t)

$$\sum(T, L) \sim \left(\int_{-\infty}^{\infty} h^2(y)\,dy\right)\frac{Vol(X)}{4\pi}\,\frac{1}{T}\int_T^{2T}\left(\frac{t}{L}\right)dt\,. \tag{8}$$

We turn to $SL(2, \mathbb{Z})$ in which case the ϕ's are taken to be Hecke-Maass cusp forms. In this case you show [R] that if $L = L(T) = o(\log T)$ then

$$\sum(T, L) \sim \frac{K}{2\pi L}\int_0^{\infty} |\hat{h}(u)|^2\,e^{\pi L u}\,du\,, \qquad \text{with} \qquad K = 1.328\ldots \tag{9}$$

While $\sum$ is growing with T it is still much smaller (in this range) than Poissonian. One can extend (9) to the range $L \leq \frac{\log T}{\pi}$. In fact, Luo and I [L-S] (see also [G-J-S] for a similar result for an arithmetic element in $\mathbb{C}[SU(2)]$) work a little more to show that $\sum$ (unsmoothed)* is bounded below by $\frac{T}{L^2}$ in this range. This is off by a factor of L from Poissonian.

To see more of the key features of the $SL_2(\mathbb{Z})$ spectrum we need to increase L beyond $\frac{\log t}{\pi}$. This can be done using the Kuznietzov Formula. To this end introduce the weights

$$|\nu_\phi|^2 \;=\; \frac{2\pi}{L(1,\,\mathrm{sym}^2\phi)}\,. \tag{10}$$

They vary mildly with ϕ;

$$t_\phi^{-\epsilon} \underset{\epsilon}{\ll} \;|\nu_\phi|^2\; \ll\; \log t_\phi\,. \tag{11}$$

Moreover

$$\mathrm{Average}_\phi\,\left(|\nu_\phi|^2\right) \;=\; \frac{4}{\mathrm{Vol}(X)} \;=\; \frac{12}{\pi} \tag{12}$$

One can compute $\mathrm{Average}_\phi(|\nu_\phi|^4)$ by the methods in [L1] and one finds that

$$\mathrm{Average}_\phi\,\left(|\nu_\phi|^4\right) \;=\; \frac{24}{\zeta(3)}\,. \tag{13}$$

The Kuznietzov Formula (see [I] page 140) reads (for $n, m \geq 1$).

$$\begin{aligned}&\sum_\phi h(t_\phi)\,|\nu_\phi|^2\,\lambda_\phi(n)\,\overline{\lambda_\phi(m)}\;+\;cts\\ &=\;\tfrac{\delta_{m,n}}{\pi}\int_{-\infty}^{\infty} t\;\tanh(\pi t)\,h(t)\,dt\\ &\qquad\qquad+\sum_{c=1}^{\infty}\frac{S(m,n,c)}{c}\,h^+\left(\frac{4\pi\sqrt{mn}}{c}\right)\,.\end{aligned} \tag{14}$$

Here $\quad h^+(x) \;=\; 2i\displaystyle\int_{-\infty}^{\infty} J_{2it}(x)\,\frac{h(t)t}{\cosh\pi t}\,dt$, $S(m,n,c)$ is the Kloosterman sum, $\lambda_\phi(n)$ are normalized Hecke eigenvalues of ϕ and cts is the continuous spectrum, which will be of no significance in what we do.

Consider the weighted number counting functions:

$$M_h(t,L) \;:=\; \sum_\phi h((t_\phi - t)\,L)\,|\nu_\phi|^2\,. \tag{15}$$

*unsmoothed counting functions carry information about eigenvalues in intervals of length smaller than $\frac{1}{L}$. I know of no nontrivial upper bounds for these.

According to (14) the expected value $\widetilde{M}$, of M satisfies

$$\widetilde{M}_h(t,L) \sim \frac{t}{\pi L} \tag{16}$$

(our assumptions on h are the same as those that we imposed on $N_h(t,L)$). Using asymptotic expansions of $J_{2it}(x)$ for t large and x small we obtain to leading order that for $h_t(x) = h(L(x-t))$,

$$h_t^+(x) \sim \frac{t^{1/2}}{L} \Im \left\{ (2i)^{1/2} \left(\frac{xe}{4\pi t}\right)^{2it} \hat{h}\left(\frac{-\log \frac{xe}{4t}}{\pi L}\right)\right\} . \tag{17}$$

Hence (essentially) we have

$$M_h(t,L) - \widetilde{M}_h(t,L) = \frac{t^{1/2}}{L} \Im \left\{ (2i)^{1/2} \sum_{c=1}^{\infty} \frac{S(1,1,c)}{c} \left(\frac{e}{ct}\right)^{2it} \hat{h}\left(\frac{\log \frac{ct}{\pi e}}{\pi L}\right)\right\} \tag{18}$$

From (18) we first note that if $L \leq \frac{\log t}{\pi}$ (ie the range (4)) then the right hand side of (18) is zero. So in this range M_h does not fluctuate. Moreover, as soon as $L > \frac{\log t}{\pi}$ there is immediately a $\frac{t^{1/2}}{L}$ fluctuation. Thus the large (Poisson like) fluctuations come from the $\frac{t^{1/2}}{L}$ factor rather than from multiplicities in the c-sum. Applying Weil's bound for $S(1,1,c)$ and estimating the RHS of (18) in absolute value yields:

For

$$L \leq \frac{2\log t}{\pi}, \ M_h - \widetilde{M}_h = o\left(\widetilde{M}_h\right) . \tag{19}$$

Thus in this range, which is double that of (4), we have

$$M_h \sim \widetilde{M}_h . \tag{20}$$

One can remove the weights $|\nu_\phi|^2$ using zero density theorems, thus establishing (4) in this extended range. Precisely;

$$N_h(t,L) \sim \widetilde{N}_h(t,L)$$

$$\text{for} \qquad L \leq \frac{2\log t}{\pi} . \tag{21}$$

To see this we proceed as follows:

Let $\delta > 0$ be small enough so that the density lemma 2.1 of [L1] ensures that at most $T^{1/10}$ of the $L(s, \mathrm{sym}^2\phi)$ with $|t_\phi - T| \le \frac{T}{2}$ have a zero in $1-\delta \le \sigma < 1$ and $|\Im(t)| \le \log^3 T$. Hence for all but at most $T^{1/10}$ of the ϕ's, $L(s, \mathrm{sym}^2\phi) \underset{\epsilon}{\ll} T^\epsilon$ for $1-\delta \le \sigma < 1$, $|t| \le (\log T)^2$. Denote by G the set of good ϕ's as above and by B the rest. We have $|B| \le T^{1/10}$. Fix $g \in C_0^\infty(0,\infty)$, $g \ge 0$ and $\int_0^\infty g(t)\,dt = 1$ for $\phi \in G$ we have and $M \ge 1$

$$\sum_n |\lambda_\phi(n)|^2\, g\left(\frac{n}{M}\right) = M\,\frac{L(1,\mathrm{sym}^2\phi)}{\zeta(2)} + O\left(M^{1-\delta}T^\epsilon\right). \tag{22}$$

Hence $\displaystyle\sum_n |\nu_\phi|^2\, |\lambda_\phi(n)|^2\, g\left(\frac{n}{M}\right)$

$$= \frac{2\pi 6}{\pi^2}\, M + O\left(M^{1-\delta}T^\epsilon\right)$$
$$= \frac{12}{\pi}\, M + O\left(M^{1-\delta}T^\epsilon\right). \tag{23}$$

Now summing over ϕ we have

$$\sum_\phi h((t_\phi - t)L) \sum_n |\nu_\phi|^2\, |\lambda_\phi(n)|^2\, g\left(\frac{n}{M}\right)$$

$$= \sum_{\phi\in G} h((t_\phi - t)L)\,\frac{12M}{\pi} + O\left(M^{1-\delta}\,T^\epsilon\,\frac{t}{L}\right)$$

$$+ O\left(t^{\frac{1}{10}}\,M^{1+\epsilon}\right)$$

$$= \frac{12}{\pi}\, M \sum_\phi h((t_\phi - t)L) + O\left(M^{1-\delta}\,\frac{t^{1+\epsilon}}{L}\right)$$

(we assume $M \ll \sqrt{t}$).

$$= \frac{12}{\pi}\, M\, N_h(t,L) + O\left(M^{1-\delta}\,t^{1+\epsilon}\right). \tag{24}$$

Switch orders of summation on the L.H.S. of (24) and apply (14). The $\delta_{n,n}$ terms yield

$$\frac{Mt}{\pi L}. \tag{25}$$

Let δ_1 be small enough so that $[-\delta_1,\delta_1]+$ (support $\hat{h}$) $\subset(-1,1)$. Then for $M=t^{\delta_1}$, the Kloosterman sum contributions to the R.H.S. of (24) is $o(Mt/L)$ as long as $L\le\frac{2\log t}{\pi}$. Thus under this assumption we have

$$\begin{aligned} N_h(t,L) &= \tfrac{t}{12L}+O\left(M^{-\delta}\,t^{1+\epsilon}\right)+o\left(\tfrac{t}{L}\right) \\ &= \tfrac{Vol\,(X_{SL(2,\mathbb{Z})})}{4\pi}\,\tfrac{t}{L}+o\left(\tfrac{t}{L}\right) \end{aligned} \tag{26}$$

$$\sim\ \widetilde{N_h}(t,L)\,. \tag{27}$$

This establishes the claim (21). As a consequence we have

$$\overline{\lim}_{t\longrightarrow\infty}\frac{m_{SL(2,\mathbb{Z})}(t)\log t}{t}\ \le\ \frac{Vol(X)}{8}\ =\ \frac{\pi}{24}\,. \tag{28}$$

It is rather embarrassing that this is the best multiplicity bound which I can give, given that in all likelihood $m_{SL(2,\mathbb{Z})}(t)\le 1$ for all t.

The extended range $L>\frac{\log t}{\pi}$ allows us to obtain a Poissonian number variance for M_h.

Set

$$\sum(T,L)\ :=\ \frac{1}{T}\int_0^\infty 4\left(\frac{t}{T}\right)\left(M_h(t,L)\,-\,\widetilde{M_h}(t,L)\right)^2\,dt\,. \tag{29}$$

Here $\psi\ge 0$ is a smooth function of compact support in $(0,\infty)$. If $L\le\frac{2\log T}{\pi}$ one can square out in (18) and as $T\longrightarrow\infty$ the diagonal is the dominant term. That is

$$\sum(T,L)\ \sim\ \frac{T}{L^2}\sum_{c=1}^{\infty}\frac{|S(1,1,c)^2}{c^2}\int_0^\infty t\psi(t)\left|\hat{h}\left(\frac{\log\left(\frac{ctT}{\pi c}\right)}{\pi L}\right)\right|^2 dt \tag{30}$$

$$\sim\ \frac{T}{L^2}\int_0^\infty t\psi(t)\,dt\left(\sum_{c=1}^{\infty}\frac{|S(1,1,c)|^2}{c^2}\left|\hat{h}\left(\frac{\log(Tc)}{\pi L}\right)\right|^2\right)\,. \tag{31}$$

Let

$$B(x)\ =\ \sum_{c\le x}\frac{|S(1,1,c)|}{c^2}\,. \tag{32}$$

Recent work [F-M] comes close to determining the asymptotic behavior of $B(x)$. They show that

$$\exp((\log\log x)^{5/17})\ \ll\ B(x)\ \ll\ (\log x)\,(\log\log x)^3\,. \tag{33}$$

One could use these bounds in (31) to give quite sharp bounds for $\sum(T,L)$. Rather than doing this let's postulate that

$$B(x) \sim A \log x \qquad \text{for some } A > 0 \tag{34}$$

(various considerations suggest this - numerics plus heuristics are needed here). Then it follows that

$$\sum(T,L) \sim A\pi \left(\int_0^\infty t\psi(t)\,dt\right)\left(\int_{\frac{\log t}{\pi L}}^\infty |\hat{h}(y)|^2\,dy\right)\frac{T}{L}$$

for

$$\frac{\log T}{\pi} < L \leq \frac{2\log T}{\pi}. \tag{35}$$

Except for the constant this yields exactly a Poissonian number variance. The Poisson model, that is to say that the t_ϕ's are independent as are the $|\nu_\phi|^2$ (in the last, of each other as well as of the t_ϕ's) leads to

$$\sum(T,L) \sim \frac{2}{\zeta(3)}\left(\int_0^\infty t\psi(t)\,dt\right)\left(\int_{-\infty}^\infty |h(y)|^2\,dy\right)\frac{T}{L}. \tag{36}$$

Thus the constant in (35) cannot be the Poissonian one since the functional dependence on h is not the same as long as $\frac{\log T}{\pi L}$ is bounded below. We expect that the diagonal approximation in (30) remains valid as even for $\frac{L}{\log T} \longrightarrow \infty$. If so (35) and (36) will agree as long as A is what it should be. One can speculate further along these lines. In view of the randomness that is present in the numbers $\frac{S(1,1,c)}{\sqrt{c}}$ one might believe that for $H \in C_0^\infty(0,\infty)$, we have

$$\frac{1}{T}\int_T^{2T}\left|\sum_c \frac{S(1,1,c)}{c}\,H\left(\frac{c}{X}\right)c^{2it}\right|^2 dt \underset{\epsilon}{\ll} T^\epsilon \sum_c \left|\frac{S(1,1,c)}{c}\right|^2 H\left(\frac{c}{X}\right)^2 \tag{37}$$

for X as large as e^T.

Assuming (37) and using the upper bound in (33) we would then have that for $\delta > 0$,

$$\sum(T,L) \underset{\epsilon}{\ll} \frac{T^{1+\epsilon}}{L} \tag{38}$$

for $L \leq T^{1-\delta}$. (38) together with (11) already lead to far-reaching statements about multiplicities. For example, (38) implies that

$$\sum_{t\leq T} m^2_{SL(2,\mathbb{Z})}(t) \underset{\epsilon}{\ll} T^{2+\epsilon} \tag{39}$$

and that

$$N_0(T) := \sum_{\substack{t \leq T \\ m_{SL(2,\mathbb{Z})}(t) \geq 1}} 1 \gg_{\epsilon} T^{2-\epsilon}\,. \tag{40}$$

My interest in the multiplicities for $SL_2(\mathbb{Z})$ lies in part in its connection to the deformation theory that Phillips and I began in the 80's and that was pushed significantly farther by Wolpert in the 90's. That theory is now essentially complete except for the issue of multiplicities for $SL_2(\mathbb{Z})$, $\Gamma_0(2)$ and $\Gamma_0(4)$. In fact, (38) together with the results of Luo [L2] on nonvanishing of special values of Rankin-Selberg L-functions, is sufficient to show that for the generic Γ in Teichmuller space, the scattering determinant $\phi_\Gamma(s)$ is a meromorphic function of order 2!

With best regards,

Peter Sarnak

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Addendum to letter to Rudnick

In connection with the issues of multiplicities of the eigenvalues, one can ask about joint multiplicities of the t_ϕ's and $\lambda_\phi(p)$'s. For S a subset of the primes P, let

$$Y^S = [0,\infty) \times \prod_{p \in S} J_p\,, \ J_p = \left[-p^{1/2} - p^{-1/2},\, p^{1/2} + p^{-1/2}\right].$$

Each ϕ gives a point $x_\phi \in Y^P$, $x_\phi = (t_\phi, \lambda_\phi(2), \lambda_\phi(3), \ldots)$, and by projection a point in Y^S for any S. When ordered by t_ϕ the points x_ϕ become equi-distributed in $\prod_p J_p$, w.r.t. $\prod_p d\mu_p$ where μ_p is the p-adic Plancherel measure [S]. In particular, the image $\phi \longrightarrow x_\phi$ is dense in $\prod_p [-2,2]$ since the support of each μ_p is $[-2,2]$. Strong multiplicity one asserts that $\phi \longrightarrow x_\phi \in Y^P$ is injective. For $y^S \in Y^S$ let

$$m(y^S) = \#\{\phi : x_\phi = y\}. \tag{1}$$

In what follows t_y or t will denote the archimedian component of y. For S fixed, the trace formula methods can be used to show that if $y^S \in Y^S$ then

$$m(y^S) \underset{S}{\ll} t/(\log t)^{|S|+1} \tag{2}$$

This suggests that if we let S grow slowly with t we should get sharp bounds. Indeed, assuming GRH for $L(s, \phi \times \phi')$ one can show that if S_ν is the set of the first ν primes and $\nu \gg (\log t)^2$, then for $y \in Y^{S_\nu}$

$$m(y) \le 1\,. \tag{3}$$

Assuming the Lindeloff Hypothesis for $L(s, \phi \times \phi')$ the same conclusion (3) can be drawn as long as $\nu \underset{\epsilon}{\gg} t^\epsilon$, for any $\epsilon > 0$.

One point of interest is that one can get rid of all hypotheses by using zero density theorems for $L(s, \phi \times \phi')$ (which can be developed as in [L1]) or one can proceed more directly using large sieve inequalities for Fourier coefficients, as is done in [D-K], to prove the following:

For any $\epsilon > 0$, there are $c_\epsilon > 0$, $c'_\epsilon < \infty$ s.t. if $\nu \ge c_\epsilon t^\epsilon$ and $y \in Y^{S_\nu}$ then

$$m(y) \le c'_\epsilon\, t^\epsilon\,. \tag{4}$$

It is perhaps worth noting that for these automorphic forms ϕ, t_ϕ is the "analytic conductor" introduced in [I-S] and that these last results which use L-functions can be obtained much more generally on GL_n with the role of t played by this conductor.

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