Thursday Morning Seminar

DIVISION ALGEBRAS I

R. Langlands

Introduction.

In the Friday afternoon seminar a method for comparing traces on different groups was described and applied to some low-dimensional groups of rank one. As a further test of its effectiveness we will consider here the comparison of the trace formula on $G = GL(n)$ and $G' = D^*$, D being a division algebra of degree n^2 over the global field F of characteristic zero.

It should be emphasized that the advantage of the method is that it does not require that the trace formula be made invariant, so that many problems in local harmonic analysis are avoided. On the other hand it cannot, so far as I can see, be applied when one is working, for whatever reason, with a single trace formula.

The procedure.

We suppose we are given a function $\phi' = \prod_v \phi'_v$ on G', smooth and of compact support, and a similar function $\phi = \prod \phi_v$ on G. We suppose that at each place v of F and for each regular γ in G_v the orbital integral $\Phi(\gamma, \phi_v)$ is equal to $\Phi(\gamma, \phi'_v)$ if γ occurs in G'_v and to 0 otherwise. We want to show that

$$\Theta_{G'}(\phi') = \Theta_G(\phi) \quad .$$

Observe that G and G' have no non-trivial cuspidal endoscopic groups, so that stabilization is superfluous.

The trace formula for G' reads simply

$$J_{G'}(\phi') = \Theta_{G'}(\phi') \quad .$$

The trace formula for G reads

$$\sum_M J_M^T(\phi) = \sum_M \Theta_M^T(\phi) \quad .$$

The sum over M runs over conjugacy classes of Levi subgroups of parabolic subgroups of G. They are indexed by unordered partitions of n. One lesson to be drawn from the following is that it is better to use a sum over $M \supseteq M_0$.

We take it for granted that $J_G^T(\phi) = J_G(\phi)$ is equal to $J_{G'}(\phi')$, obtaining an equality

$$\text{(1)} \qquad \Theta_G(\phi) - \Theta_{G'}(\phi') = \sum_{M \neq G} J_M^T(\phi) - \sum_{M \neq G} \Theta_M^T(\phi) \quad .$$

Before going on I underline a peculiar feature of the notation. Both J_M^T and Θ_M^T are distributions on $\mathbf{G}$ and thus depend on the pair M and G, the dependence on G being implicit in the function ϕ. If however we write $J_M^T(\psi_{M'})$ or $\Theta_M^T(\psi_{M'})$ where $M \subseteq M'$ and $\psi_{M'}$ is a function on $\mathbf{M}'$ then it is understood that the distributions involved are those for the pair M, M'. Since the T is of no concern to us we drop it from the notation.

What we want to do is show the existence of smooth, compactly supported functions ψ^M on M, $M \neq G$, such that

$$\sum_{M \neq G} J_M(\phi) = \sum_{M'} \sum_{M} J_M(\psi^{M'}) \quad , \tag{2}$$

the inner sum being taken over conjugacy classes of M in M'. The trace formula for M' yields

$$\sum_{M} J_M(\psi^{M'}) = \sum_{M} \Theta_M(\psi^{M'}) \quad ,$$

and the relation (1) becomes

$$\Theta_G(\phi) - \Theta_{G'}(\phi') = \sum_{M' \neq G} \sum_{M} \Theta_M(\psi^{M'}) - \sum_{M \neq G} \Theta_M(\phi) \quad .$$

This equality will allow us - provided the $\psi^{M'}$ satisfy a supplementary condition that is to be explained later - to proceed with the argument on decomposition of measures and to show not only that

$$\Theta_G(\phi) = \Theta_{G'}(\phi')$$

but also that for each $M \neq G$

$$\sum_{M'} \Theta_M(\psi^{M'}) = \Theta_M(\phi) \quad ,$$

appropriate care being taken with the range of summation on the left.

However, our concern at present is with (2) and indeed at first with a weaker statement. We anticipate that we will be provided with an expression for $J_M(\phi)$ of the following form,

$$J_M(\phi) = \frac{|\Omega^M|}{|\Omega^G|} \sum_{M'} \sum_{\sigma'} J_{M'}(\sigma', \phi) \quad ,$$

the outer sum running over all M' containing M_0 and conjugate to M and the inner sum over all elliptic conjugacy classes in M. Thus we may expect that (2) reduces to a collection of equalities, one for each $\sigma \subseteq G$,

$$(3) \quad \sum_{\substack{M \neq G \\ M_0 \subseteq M}} \frac{|\Omega^M|}{|\Omega^G|} \sum_{\sigma_M \subseteq \sigma} J_M(\sigma_M, \phi) = \sum_{M' \neq G} \sum_{M_0 \subseteq M \subseteq M'} \frac{|\Omega^M|}{|\Omega^{M'}|} \sum_{\sigma_M \subseteq \sigma} J_M(\sigma_M, \psi^{M'}) \quad .$$

It is thus convenient to fix a Levi subgroup M_0 of P_0 and to work only with M containing M_0. It will also be convenient to suppose that ψ^Q, a function on $\mathbf{M}_Q$, is defined for each Q containing M_0. So we are reformulating the problem, the functions we originally introduced being $m \longrightarrow \sum_{M'} \sum_{Q \in P(H')} \psi^Q(g^{-1}mg)$, with g in the normalizer of M_0, $g^{-1}Mg = M'$, and M' running over the conjugates of M.

In this lecture we are concerned with (3) only for regular σ, and it is clear it would follow from

$$(5) \qquad J_M(\sigma, \phi) = \sum_{\substack{M \subsetneq Q \\ Q \neq G}} J_M(\sigma, \psi^Q) \quad ,$$

σ now being a conjugacy class in M.

Let $\rho(M)$ be the dimension of the center of M minus the dimension of G and let $\rho(Q) = \rho(M_Q)$. The functions ψ^Q are to be defined inductively on $\rho(Q)$, starting with $\rho(Q) = 1$. The only condition is that (5) be satisfied at each stage.

Weighted orbital integrals.

Let S be a finite set of places containing all infinite places, all places at which D ramifies, and all places at which ϕ_v is not a spherical function. If $v \notin S$ and if M is the Levi factor of Q then

$$(6) \qquad \phi_{Q,v}(m) = \rho_Q(m) \int_{K_v} \int_{N_Q(F_v)} \phi_v(k^{-1}mnk)\,dn\,dk \quad ,$$

ρ_Q being the square root of the absolute value of the determinant of $\operatorname{adm} \backslash \mathfrak{n}_Q$, is a spherical function on M_v and is independent of Q. So we sometimes denote it $\phi_v^{M_Q}$. We demand, and this is the supplementary condition mentioned above, that

$$\psi^Q = \psi_S^Q \prod_{v \notin S} \phi_v^{M_Q} \quad .$$

Since the function ϕ has compact support there will be only finitely many conjugacy classes $\mathfrak{o}$ for which $J_M(\mathfrak{o}_M, \phi) \neq 0$ for some M and some $\mathfrak{o}_M \subseteq \mathfrak{o}$. It is easily seen that we can choose the finite set of places $S' = S(\phi)$ to be so large that for each such $\mathfrak{o}$ and $\mathfrak{o}_M \subseteq \mathfrak{o}$ the class $\mathfrak{o}_M$ has a representation γ which for $v \notin S'$ lies in K_v and is such that $g^{-1}\gamma g \in K_v$, $g \in G_v$ implies that $g \in G_\gamma(F_v)K_v$. The group K_v is of course the standard maximal compact subgroup of G_v.

As a consequence we may replace $J_M(\mathfrak{o}_M, \phi)$ by

$$c(\mathfrak{o}_M) \int_{G_\gamma(\mathbb{A}_{S'}) \backslash G(\mathbb{A}_S)} \phi(g^{-1}\gamma g) v_M^G(g)\,dg$$

or

$$c(\sigma_M)\int_{G_\gamma(\mathbf{A}_{S'})\backslash G(\mathbf{A}_S)} \phi_{S'}(g^{-1}\gamma g)V_M^G(\gamma, g)dg \quad .$$

Let

$$K^{S'} = \prod_{v \notin S} K_v$$

and let $\delta(\sigma_M)$ be 1 or 0 according as σ_M does or does not meet

$$\{g^{-1}K^{S'}g \mid g \in G(\mathbf{A}^{S'})\} \quad .$$

Then

$$c(\sigma_M) = \text{meas}(G_\gamma \backslash \mathbf{G}'_\gamma)\text{meas}(K^{S'} \cap \mathbf{G}_\gamma \backslash K^{S'})\delta(\sigma_M) \quad .$$

The weight factor $v_M^G(g)$ is that given by the trace formula and $V_M^G(\gamma, g)$ that given by Flicker's trick. For the γ being studied at the moment they are equal.

We may disregard the factor $c(\sigma_M)$, examining instead

$$J_M(\gamma, \phi) = |D^G(\gamma)|^{\frac{1}{2}}\int_{G_\gamma(\mathbf{A}_{S'})\backslash G(\mathbf{A}_{S'})} \phi_{S'}(g^{-1}\gamma g)V_M^G(\gamma, g)dg \quad .$$

The factor $|D^G(\gamma)|$ is that introduced on p. 30 of Arthur's Annals paper. It is 1 for regular semi-simple γ in G. We shall study $J_M^G(\gamma, \phi)$ for γ in $M(\mathbf{A}_{S'})$ and regular in G. We need to show, by the same inductive procedure, the existence of $\psi_{S'}^Q$ such that

$$(7) \qquad J_M(\gamma, \phi_{S'}) = \sum_{\substack{M \subset Q \\ Q \neq G}} J_M(\gamma, \psi^Q_{S'})$$

for every M. There is only one observation to make. We began by choosing $S' = S(\phi)$. It can always be made larger. As we construct the ψ^Q inductively we will introduce $S(\psi^Q)$. They may not be contained in S'. We simply enlarge S' at each stage to accommodate them.

This being understood we work entirely within the given set S', sometimes dropping it from the notation. All functions will be spherical outside of S.

Basic lemmas.

The distributions $\phi \longrightarrow J_M(\gamma, \phi) = J^G_M(\gamma, \phi)$ are very similar to the distributions $f \longrightarrow J^G_{M,\gamma}(f)$ studied in §8 of Arthur's Annals paper. Apart from the fact that our S' is his S the difference is that he works with v^G_M rather than V^G_M. The analogue of his Lemma 8.2 is valid and the proof is exactly the same.

LEMMA 1. Let $L \supseteq M$ be Levi factors of G over F. Let h lie in $L(\mathbf{A}_{S'})$ and let γ on $M(\mathbf{A}_{S'})$ be regular in $L(\mathbf{A}_{S'})$. If ψ is smooth and of compact support on $L(\mathbf{A}_{S'})$ then

$$J_M(\gamma, \psi^h) = \sum_{Q \in \mathcal{F}(M)} J_M(\gamma, \psi_{Q,h}) \quad .$$

The sum runs over the parabolic subgroups of L over F which contain M. Moreover $\psi_{Q,h}$ is a function on $M_Q(\mathbf{A}_{S'})$. So in agreement with our notational conventions $J_M(\gamma, \psi_{Q,h})$ is $J_M^{M_Q}(\gamma, \psi_{Q,h})$. The function

$\psi_{Q,h}$ is defined on p. 20 of Arthur's paper and is smooth and of compact support on $M_Q(\mathbf{A}_{S'})$.

It is important to observe that - this one sees immediately from the definition - if $h \in L(\mathbf{A}_S)$ then $f = \psi_{Q,h}$ is a product,

$$f = f_S \cdot \prod_{v \in S'-S} \phi_v^{M_Q} \quad .$$

The function $\phi_v^{M_Q}$ is defined by (6) and $f_S = \psi_{S,Q,h}$.

For technical reasons it is convenient to fuse all infinite places into a single place, denoted ∞. With this convention F_∞ denotes $\prod_{v \in S_\infty} F_v$ and $M_\infty = \prod_{v \in S_\infty} M_v$. Moreover $S = S_{fin} \cup \{\infty\}$, $S' = S'_{fin} \cup \{\infty\}$.

Suppose that L is a Levi factor of G containing M_0 and that f_S is a smooth, compactly supported function on $L(\mathbf{A}_S)$. A collection of functions F_S^Q, Q lying in L and F_S^Q being a smooth, compactly supported function $M_Q(\mathbf{A}_S)$, will be said to be adapted to f_S if for every $S' \supseteq S$ and every collection of spherical functions f_v, $v \in S'-S$, every M, $M_0 \subset M \subset L$, and every semi-simple γ in $M(\mathbf{A}_{S'})$ regular in G the equality

$$J_M^{L}(\gamma, f_S \cdot \prod_{v \in S'-S} f_v) = \sum_{\substack{Q \in \mathcal{F}^L(M) \\ Q \neq L}} J_M(\gamma, F_S^Q \cdot \prod_v f_{Q,v})$$

holds. The functions $f_{Q,v}$ are defined by the equation (6), f_v replacing ϕ_v.

LEMMA 2. <u>Suppose that</u> $f_S = \prod_{v \in S} f_v$ <u>and that for some given</u> v <u>the orbital integral</u> $\Phi(\gamma, f_v) = 0$ <u>for all semi-simple</u> γ <u>in</u> $\mathfrak{k}(F_v)$ <u>regular</u>

in L

in $L(F_v)$. Then a collection F_S^Q adapted to f_S exists.

For $v = \infty$ the proof of the lemma draws on various facts whose explanation it is convenient to postpone. For v finite it is however an immediate consequence of Lemma 1 and the following lemma of Vignéras (cf. §2 of Caractérisation des intégrales orbitales sur un groupe réductif p-adique and App 1.1 of Représentations des algèbras centrales simples p-adiques). Gelfand-Kazhdan?

LEMMA 3. If v is finite and $\Phi(\gamma, f_v) = 0$ for all regular semi-simple γ then f_v may be expressed as a sum $\sum_i f_{i,v}^{h_i} - f_{i,v}$, $h_i \in L(F_v)$.

We set $f_i = f_{i,v} \cdot \prod_{w \neq v} f_v$ and take

$$F_S^Q = \sum_i f_{i,Q,h_i} .$$

LEMMA 4. Suppose that f is a smooth function with compact support on $M(\mathbf{A}_S)$ and that for every regular semi-simple γ the orbital integral $\Phi(\gamma, f) = 0$. Then f is a sum $\sum_{v \in S} f^v$ where f^v is a sum of functions f_i^v such that each f_i^v is a product $\prod_{w \in S} F_w$ and $\Phi(\gamma, F_v) = 0$ for all regular semi-simple γ in $M(F_v)$.

Observe that the F_w depend on v and i. It is inconvenient and unnecessary to incorporate this in the notation.

The lemma is proved by induction on the cardinality of S. It is trivial if S_{fin} is empty. So choose $v \in S_{fin}$. If C is a compact subset of M_v and U an open compact subgroup of it, let $H(C//U)$

be the set of functions on M_v supported by C and bi-invariant under U. We write $m \in M(\mathbf{A}_S)$ as (m_1, m_2), $m_1 \in M(\mathbf{A}_{S_1})$, $m_2 \in M(F_v)$, $S_1 = S - \{v\}$. Choose C and U such that for each m_1 the function $m_2 \longrightarrow f(m_1, m_2)$ lies in $\mathcal{H}(C/\!/U)$.

Since $\mathcal{H}(C/\!/U)$ is finite dimensional we can find, for a suitable ℓ, functions $h_1, \ldots, h_\ell$ in it and ℓ regular semi-simple elements $\gamma_1, \ldots, \gamma_\ell$ in M_v such that $\Phi(\gamma_i, h_j) = \delta_{ij}$ and such that any other function in $\mathcal{H}(C/\!/U)$ is of the form $\sum_{j=1}^{\ell} a_j h_j + h$, where $h \in \mathcal{H}(C/\!/U)$ and all its orbital integrals are 0. In particular

$$f(m_1, m_2) = \sum a_j(m_1) h_j(m_2) + h(m_1, m_2)$$

with

$$a_j(m_1) = \int_{M_\gamma(F_v)\backslash M(F_v)} f(m_1, m_2^{-1}\gamma m_2)\, dm_2 \quad .$$

The lemma follows.

We shall be faced with the following problem. We shall be given a function $\gamma \longrightarrow \Psi(\gamma)$ on regular semi-simple classes in $M(\mathbf{A}_S)$ and we will want to show that there is a smooth, compactly supported function f such that $\Psi(\gamma) = \Phi(\gamma, f)$ for all such γ.

If v is a place in S we say that Ψ is satisfactory at v if for each regular semi-simple γ_1 in $M(\mathbf{A}_{S_1})$ there is a function $f^v_{\gamma_1}$ on $M(F_v)$ smooth and of compact support, such that:

(i) For all regular semi-simple γ_2 in $M(F_v)$ and all regular semi-simple γ_1 in $M(\mathbf{A}_{S_1})$

$$\Psi(\gamma_1, \gamma_2) = \Phi(\gamma_1, f^v_{\gamma_2}) \quad .$$

(ii) If v is finite then there is a C and U such that $f^v_{\gamma_2} \in H(C//U)$ for all γ_2.

LEMMA 5. <u>If</u> Ψ <u>is satisfactory at every</u> v <u>in</u> S <u>then there is a smooth, compactly supported function</u> f <u>on</u> $M(\mathbf{A}_S)$ <u>such that</u> $\Psi(\gamma) = \Phi(\gamma, f)$ <u>for all regular semi-simple</u> γ <u>in</u> $M(\mathbf{A}_S)$.

We argue by induction. If S contains only one element there is nothing to prove. So suppose S_{fin} contains v. Choosing $h_1, \ldots, h_\ell$ and $\gamma_1, \ldots, \gamma_\ell$ as above we may write

$$f^v_\gamma = \sum a_j(\gamma) h_j + h_\gamma \quad ,$$

with

$$a_j(\gamma) = \Phi(\gamma_j, \gamma) \quad .$$

By induction there are functions f'_{γ_j} on $M(\mathbf{A}_{S_1})$, smooth and of compact support, such that

$$\Phi(\gamma_j, \gamma) = \Phi(\gamma, f'_{\gamma_j}) \quad .$$

We may take

$$f(m_1, m_2) = \sum_j f'_{\gamma_j}(m_1) h_j(m_2) \quad .$$

LEMMA 6. *Suppose that* R *is a subset of* S *containing at least two elements and that for every* v *in* R *and every regular semi-simple* γ_2 *in* $M(F_v)$ *there is a smooth compactly supported function* f_{γ_2} *on* $M(\mathbf{A}_{S_1})$ *such that*

$$\Psi(\gamma_1, \gamma_2) = \Phi(\gamma_1, f_{\gamma_2})$$

for regular, semi-simple γ_1 *in* $M(\mathbf{A}_{S_1})$. *Suppose moreover that for each finite* $w \neq v$, *all* γ_2, *and all* $m' \in \prod_{x \neq w, v} M_x$ *the function* $m \longrightarrow f_{\gamma_2}(m', m)$ *lies in* $\mathcal{H}(C /\!/ U)$ *for some given* C *and* U. *Then* Ψ *is satisfactory at every place in* S.

This is clear.

A simple problem. The functions $\psi^{\mathbf{Q}}$ will be chosen inductively and the lack of unicity is somewhat disconcerting. To allay at least some of the unease we consider a Levi factor L of G and the trace formula for a function ψ on $\mathbb{L}$ so chosen that $\psi = \prod_v \psi_v$ and, for some v, all orbital integrals of ψ_v are zero. With this assumption the trace formula should be trivial and we should be able to take all terms from the left to the right without any difficulty.

This means that we should be able to find functions ψ on $\mathbf{M}^Q$, $Q \in \mathcal{F}(M_0)$. $Q \neq G$ such that the analogue of (5) is satisfied,

$$J_M^L(\sigma, \psi) = \sum_{M \subseteq Q \subsetneq L} J_M^L(\sigma, \psi^Q) \quad .$$

Lemma 2 guarantees the existence of ψ^Q satisfying this relation.

Existence.

We come now to the proof of the existence of the functions ψ^Q attached to $\phi = \prod_v \phi_v$. The critical property of the function ϕ is the following (cf. part C of Vignéras's notes).

Suppose $L \neq G$. Then there exist at least two places $v \in S$ such that for all $\gamma \in L(F_v)$ which are regular in G the orbital integrals $\Phi(\gamma, \phi_v)$ are zero.

Let $n_1, \ldots, n_r$ be the partition defining M and let d_v be the denominator of the invariant of D at v. In order that $\Phi(\gamma, \phi_v)$ is not zero for all γ in $L(F_v)$ all $n_1, \ldots, n_r$ must be divisible by d_v. If this were so at all but one v it would be so at all v and then r would be 1 and $L = G$.

For $\rho(M) = 1$ the factors $V_M^G(\gamma, g)$ are linear, and

$$J_M(\gamma, \phi) = |D^G(\gamma)|^{\frac{1}{2}} \sum_v \int_{G_\gamma(\mathbf{A}_{S'}) \backslash G(\mathbf{A}_{S'})} \phi(g^{-1}\gamma g) V_M^G(\gamma, g_v) = 0 .$$

Thus we may take $\psi^Q = 0$ if $\rho(Q) = 1$.

We now suppose that for $\rho(Q) < \rho$ we have so defined ψ^Q that

$$J_M(\gamma, \phi) = \sum_{\substack{M \subseteq Q \\ Q \neq G}} J_M(\gamma, \psi^Q) \tag{8}$$

for $\rho(M) < \rho$, and we prove the existence of ψ for $\rho(Q) = \rho$ satisfying the corresponding equation. We apply Lemma 6 to the difference

$$J_M(\gamma, \phi) - \sum_{\substack{M \subseteq Q \\ Q \neq G \\ Q \notin P(M)}} J_M(\gamma, \psi^Q) ,$$

proving that it is equal to $J_M(\gamma,\ f)$ for some smooth compactly supported function on $M(\mathbb{A}_{S'})$ and we set

$$\psi^Q = \frac{1}{|\mathcal{P}(M)|} f \quad ,$$

for $Q \in \mathcal{P}(M)$.

Let v be a place in S such that $\Phi(\gamma,\ \phi_v) = 0$ for all $\gamma \in M_v$ regular in G_v. We need only show that for such a v the condition of Lemma 6 is satisfied.

According to Lemma 6.3 of Arthur's Annals paper we have a decomposition of $V_M^G(g)$ as

$$\sum_{Q \in \mathcal{F}(M)} V_M^Q(g_1) U_Q(g_2)$$

where $g_1 \in G(\mathbb{A}_{S_1})$, $g_2 \in G(F_v)$. This leads to a decomposition

$$J_M(\gamma,\ \phi_S) = \sum_{Q \in \mathcal{F}(M)} J_M(\gamma_1,\ \phi_Q^1) L_Q(\gamma_2,\ \phi_2) \tag{9}$$

where $\phi(g) = \phi^1(g_1)\phi^2(g_2)$ and

$$\phi_Q^1(m) = \rho_Q(m) \int_{K_1} \int_{N_Q(\mathbb{A}_{S_1'})} \phi^1(k^{-1}mnk)\,dn\,dk \quad .$$

If $Q = G$ then $U_Q(g_2) \equiv 1$ and $L_Q(\gamma_2,\ \phi^2) = 0$, for it is an ordinary orbital integral and $\gamma_2 \in M_v$. So we drop the term corresponding to $Q = G$ from the sum (9).

For any other Q we apply Lemma 2 to the function ϕ_Q^1 and write

$$J_M(\gamma_1, \phi_Q^1) = \sum_{\substack{Q' \subsetneq Q \\ M \subseteq Q'}} J_M(\gamma_1, F_Q^{Q'}) \quad .$$

Since γ_2 and ϕ_2 play the role of fixed parameters in the discussion we remove them from the notation, setting

$$F^{Q'} = \sum_{Q' \subseteq Q} F_Q^{Q'} L_Q(\gamma_2, \phi_2) \quad .$$

We also have a decomposition

$$J_M(\gamma, \psi^Q) = \sum_{M \subseteq Q' \subseteq Q} J_M(\gamma_1, \psi_{Q'}^{Q,1}) L_{Q'}(\gamma_2, \psi^{Q,2})$$

We set

$$H^{Q'} = \sum_{\substack{Q' \subseteq Q \\ 2 \leq \rho(Q) < \rho}} \psi_{Q'}^{Q,1} L_{Q'}(\gamma_2, \psi^{Q,2}) \quad .$$

Both $H^{Q'}$ and $F^{Q'}$ are smooth, compactly supported functions on $M^{Q'}(\mathbb{A}_{S_1})$.

By assumption

$$(10) \qquad \sum_{Q' \in \mathcal{F}(M')} J_{M'}(\gamma_1, F^{Q'}) - \sum_{Q' \in \mathcal{F}(M')} J_{M'}(\gamma_1, H^{Q'}) = 0$$

if $\rho(M') < \rho$.

If $\rho = 2$ then

$$J_M(\gamma_1, \gamma_2, \phi) = \sum_{Q \in \mathcal{P}(M)} J_M(\gamma_1, F^Q) \quad .$$

Thus we may take

$$f_{\gamma_2} = \sum_{Q \in P(M)} F^Q \quad .$$

Now we suppose that $\rho > 2$ and we define inductively on $\rho(M')$, $2 \leq \rho(M') \leq \rho(M)$, functions $E^{M',Q}$ on $M_Q(\mathbf{A}_{S_1'})$, $Q \in \mathcal{F}^{M'}(M)$, $Q \neq M'$, which are smooth and of compact support.

They will be shown inductively to have the property that the orbital integrals of

$$f_{S_1}^{M''} = \sum_{Q'' \in P(M'')} F_{S_1}^{Q''} - \sum_{Q'' \in P(M'')} H_{S_1}^{Q''} + \sum_{\substack{M' \supsetneq M'' \\ \rho(M') \geq 2}} \sum_{Q'' \in P^{M'}(M'')} E_{S_1}^{H',Q''}$$

are zero on regular semi-simple elements if $\rho(M'') < \rho$.

This being so we will take the functions $E_{S_1}^{M'',Q}$ to be those attached to $f_{S_1}^{M''}$ by Lemma 2. The relation (10) assures us that for $\rho(M'') = 2$ the function

$$f_{S_1}^{M''} = \sum_{Q'' \in P(M'')} F_{S_1}^{Q''} - \sum_{Q'' \in P(M'')} H_{S_1}^{Q''}$$

has orbital integrals zero. So we can begin the process.

To verify that it continues we have to evaluate $J_{M''}(\gamma_1, f_{S_1}^{M''})$. Observe first of all that if $M''' \subseteq M''$ then

$$\sum_{Q'' \in P(M'')} J_{M'''}(\gamma_1, F_{S_1}^{Q''}) - \sum_{Q'' \in P(M'')} J_{M'''}(\gamma_1, H_{S_1}^{Q''}) \tag{11}$$

plus

$$\sum_{\substack{M' \supset M'' \\ \rho(\bar{M}') \geq 2}} \sum_{Q'' \in P^{M'}(M'')} J_{M'''}(\gamma_1, E_{S_1}^{M',Q''}) \tag{12}$$

is equal to

$$(13)\qquad \sum_{\substack{Q'''\in \mathcal{F}^{M''}(M''') \\ Q'''\neq M''}} J_{M'''}(\gamma_1, E_{S_1}^{M'',Q'''}) \quad .$$

We are interested in calculating $J_{M'''}(\gamma_1, f_{S_1}^{M'''})$. This will involve, for each $M'' \supseteq M'''$, the terms

$$(14)\qquad \sum_{Q'''\in \mathcal{P}^{M''}(M''')} J_{M'''}(\gamma_1, E_{S_1}^{M'',Q'''}) \quad .$$

They appear in (11) and thus can be calculated as the sum of (11) and (12) minus the difference of the remaining terms in (13). Moreover there will have to be a sum over $M'' \supsetneq M''$.

This will lead to an expression for $J_{M'''}(\gamma_1, f_{S_1}^{M'''})$ containing first of all

$$\sum_{Q\in \mathcal{F}(M''')} J_{M'''}(\gamma_1, F_{S_1}^{Q}) = J_{M'''}(\gamma, \phi_S)$$

and secondly

$$-\sum_{Q\in \mathcal{F}(M''')} J_{M'''}(\gamma_1, H_{S_1}^{Q}) = \sum_{Q\in \mathcal{F}(M''')} J_{M'''}(\gamma, \psi_S^{Q}) \quad .$$

By assumption these two terms cancel.

The sum of the contributions (12) gives for every triple $M' \supseteq M''$ (with $M'' \supseteq M'''$) and every $Q'' \in \mathcal{P}^{M'}(M'')$ a term $J_{M'''}(\gamma_1, E_{S_1}^{M',Q''})$. On the other hand if in the contributions from (13) we denote M'' by M' and Q''' by Q', denoting $M_{Q'}$ by M'' (which contains M''' but is different from it) we see that they cancel those of (12).

Thus the inductive definition is permitted. We cannot define $f^M_{S_1}$ for the given M with $\rho(M) = \rho$ because $H^Q_{S_1}$ is not defined for $Q \in \mathcal{P}(M)$. However we can introduce the function

$$f_{S_1} = \sum_{Q \in \mathcal{P}(M)} F^Q_{S_1} + \sum_{\substack{M' \supsetneq M \\ \rho(M') \geq 2}} \sum_{Q \in \mathcal{P}^{M'}(M)} E^{H',Q}_{S_1} - \sum_{Q \in \mathcal{P}(M)} \sum_{Q' \supsetneq Q} \psi_Q^{Q',1} L_Q(\gamma_2, \psi^{Q,2}) \quad .$$

The previous calculation now shows that

$$J_M(\gamma_1, f_{S_1}) = J_M(\gamma, \phi_S) - \sum_{\substack{Q \in \mathcal{F}(M) \\ Q \notin \mathcal{P}(M)}} J_M(\gamma, \psi^Q_S) \quad .$$

A similar calculation, using the properties of adapted collections, shows that the relation remains valid if S is replaced by S'. This completes the proof of the existence.