SPECIAL VALUES OF AUTOMORPHIC COHOMOLOGY CLASSES

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Contents

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Introduction

The objective of this work is to study aspects of the *automorphic cohomology groups* $H^q(X, L_\mu)$ on quotients $X = \Gamma\backslash D$ by an arithmetic group Γ acting on a class of homogeneous complex manifolds $D = G_\mathbb{R}/T$. Here $G_\mathbb{R}$ is the connected real Lie group associated to a reductive $\mathbb{Q}$-algebraic group G, $T \subset G_\mathbb{R}$ is a compact maximal torus, and μ the weight associated to a character of T that gives a homogeneous holomorphic line bundle $L_\mu \to D$. These D's may be realized as Mumford-Tate domains that arise in Hodge theory, and in general we shall follow the terminology and notations from the monograph [GGK1].[1] We shall say that D is *classical* if it equivariantly fibres holomorphically or anti-holomorphically over an Hermitian symmetric domain; otherwise it is *non-classical*, and this is the case of primary interest in this paper.

In the non-classical case it has been known for a long time that, at least when Γ is co-compact in $G_\mathbb{R}$,

- $H^0(X, L_\mu) = 0$ for any non-trivial μ;

[1] Cf. the Notations and Terminology section below.

- when μ is sufficiently non-singular,[2] then

$$\begin{cases} H^q(X, L_\mu) = 0, & q \neq q(\mu+\rho) \\ H^{q(\mu+\rho)}(X, L_\mu) \neq 0 \end{cases}$$

  where $q(\mu+\rho)$ will be defined in the text.

More precisely, for $k \geqq k_0$ and any non-singular μ

$$\dim H^{q(\mu+\rho)}(X, L_{k\mu}) = \mathrm{vol}(X) \cdot P_\mu(k)$$

where $P_\mu(k)$ is a Hilbert polynomial with leading term $C_\mu k^{\dim D}$ where $C_\mu > 0$ is independent of Γ. Thus, in the non-classical case there is a lot of automorphic cohomology and it does not occur in degree zero. In the classical case, the intensive study of the very rich geometric, Hodge theoretic, arithmetic and representation theoretic properties of automorphic forms has a long and venerable history and remains one of great current interest. In contrast, until recently in the non-classical case the geometric and arithmetic properties of automorphic cohomology have remained largely mysterious.[3]

For three reasons this situation has recently changed. One reason is the works [Gi], [EGW] that give a general method for interpreting analytic coherent cohomology on a complex manifold as holomorphic de Rham cohomology on an associated correspondence space $\mathcal{W}$.[4] In the two examples of this paper, of which a particular case of the first example is studied in [Gi] and [EGW], the associated space $\mathcal{W}$ will be seen to have a very rich geometric structure and the relevant holomorphic

[2] *Non-singular*, or *regular*, means that μ is not on the wall of a Weyl chamber; sufficiently non-singular means that μ is at a large enough distance $|\mu|$ from any wall.

[3] Important exceptions are the works Schmid [Schm1], Williams [Wi1], [Wi2], [Wi3], [Wi4], [Wi5], Wells and Wolf [WW1], [WW2], [WW3], and Wolf [Wo], some of which will be discussed below. These deal primarily with the representation-theoretic aspects of automorphic cohomology.

[4] For some time it has been known that in certain cases the cohomology group $H^{q_0}(D, L_\mu)$ may be realized as a subspace of the space of holomorphic sections of a holomorphic vector bundle over the cycle space $\mathcal{U}$ (cf. [Schm2], [BE] and [FHW]). Moreover this interpretation descends to quotients by Γ. The correspondence space will lie over the cycle space and in a number of ways appears to be a more fundamental object.

de Rham cohomology classes will turn out to have canonical representatives. The upshot is that in the situation of this work *automorphic cohomology classes can be "evaluated" at points of* $\mathcal{W}$.

A second reason is the very interesting work [C1], [C2], [C3] of Carayol.[5] In the case $G = \mathcal{U}(2,1)$, a case already considered in [EGW], Carayol uses the result in [EGW] applied to a diagram

$$\begin{tikzcd} & \mathcal{W} \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ D & & D', \end{tikzcd} \tag{1}$$

where D is non-classical and D' is classical, to construct a Penrose-type transform

$$\mathcal{P} : H^0(D', L'_{\mu'}) \to H^1(D, L_\mu) \tag{2}$$

that relates the classical object $H^0(D', L'_{\mu'})$ to the non-classical object $H^1(D, L_\mu)$. He also shows that (1) and (2) exist on the quotients by Γ. For special choices of μ' the group $H^0(X', L'_{\mu'})$ is interpreted as *Picard automorphic forms*. The construction of $\mathcal{P}$ is via the commutative diagram (the notations are explained below)

$$\begin{tikzcd} H^0_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^\bullet_{\pi'} \otimes \pi'^{-1} L'_{\mu'})\big) \arrow[r, dashed] & H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^\bullet_{\pi} \otimes \pi^{-1} L_{\mu})\big) \\ H^0(D', L'_{\mu'}) \arrow[u, phantom, "\wr\|"] \arrow[r, "\mathcal{P}"] & H^1(D, L_\mu) \arrow[u, phantom, "\wr\|"] \end{tikzcd} \tag{3}$$

where the vertical isomorphisms are the above mentioned result in [EGW].[6]

[5]The authors would like to thank Wushi Goldring for bringing this work to our attention and for some helpful discussions early in the preparation of this manuscript.

[6]Penrose transforms associated to the diagram

$$\begin{tikzcd} & \mathcal{I} \arrow[dl] \arrow[dr] & \\ D & & \mathcal{U}, \end{tikzcd} \tag{4}$$

where $\mathcal{I} \subset D \times \mathcal{U}$ is the incidence variety by means of "pull-back and push-down" are classical and over the years have been the subject of extensive work; cf. [Schm2], [BE], [EWZ], [FHW] and the references cited therein. The "Penrose-type" transforms we will be discussing in this paper are somewhat different and have more the

In [C1], [C2] for $\mathcal{U}(2,1)$ the dotted arrow above is constructed by explicit "coordinate calculations", and one of the main purposes of this paper is to give a general, intrinsic geometric construction of such maps. More specifically, the dotted arrow will be seen to be multiplication by the restriction to $\mathcal{W} \subset \check{\mathcal{W}}$ of a canonical form

$$\omega \in \Gamma\big(\check{\mathcal{W}}, \Omega^1_\pi \otimes L(\mu,\mu')\big)$$

where $L(\mu,\mu') \to \check{\mathcal{W}}$ is a homogeneous line bundle over $\check{\mathcal{W}} \cong \mathbb{G}_\mathbb{C}/T_\mathbb{C}$ associated to the characters μ, μ' and to the relative positions of the Borel subgroups B and B' associated to D and D'. For the analogous diagram to (3) for a general $H^{q'}(D', L'_\mu)$ and $H^q(D, L_\mu)$ one has

$$\omega = \prod_\alpha \omega^\alpha$$

where the product is over the positive roots associated to B' which change sign when they are considered as roots of B, and ω^α is the dual under the Cartan-Killing form to the root vector X_α. The form ω is invariant under the group action and thus the construction (3) descends to quotients by Γ.[7] As we shall see in section IV.B, the bottom row of (3) is (in this quotient) replaced by a map between two Lie algebra groups induced by "multiplication by X_α."

A third reason is the recent classification [GGK1] of the reductive, $\mathbb{Q}$-algebraic groups that can be realized as a Mumford-Tate group of a polarized Hodge structure and the related classification of the associated Mumford-Tate domains D. Although these domains and their quotients $X = \Gamma\backslash D$ by arithmetic groups arose as target spaces for period mappings $P : S \to X$ where S is a quasi-projective algebraic variety, it has since emerged that their geometry and the cohomology of homogeneous vector bundles over them is of interest in its own right. For the line bundles $L_\mu \to D$ for which the restriction $L_\mu|_S$ is ample,

flavor of the maps on cohomology induced by a correspondence in classical algebraic geometry induced by a cycle on the product of the two varieties.

[7]One will notice the similarity to the classical Borel-Weil-Bott theorem. This is of course not accidental and will be discussed below where the form ω will be seen to have a representation-theoretic interpretation; cf. the appendix to section III.D.

corresponding in the classical case to automorphic forms but for which in the non-classical case $H^0(X, L_\mu) = 0$, the automorphic cohomology $H^{q(\mu)}(X, L_\mu)$, $q(\mu) > 0$, seemed a curiosity of no particular relevance to variations of Hodge structure. It was through the interesting geometry of Mumford-Tate domains that from a Hodge-theoretic perspective automorphic cohomology has emerged as an object of interest. Just how interpreting automorphic cohomology as global holomorphic objects might be related to period mappings is a matter yet to be explored.

In this work, rather than attempt to formulate and establish results in generality, we shall focus on examples. One will be the *basic example*, essentially $\mathbb{P}^1$. Although the most elementary of cases, many of the main features of the general situation already arise here. The other two will be referred to as *example one* and *example two*. Example one will be the $\mathcal{U}(2,1)$ case; here we shall use the correspondence space from [EGW] and shall formulate intrinsically and reprove some of the results from [C1], [C2].[8] This approach suggests how the general case might go. In order to test the validity of this suggestion, we shall work out our second example of Sp(4). Here, a main step is to construct the correspondence space $\mathcal{W}$ for Sp(4), a construction that turns out to involve the concept of *Lagrange quadrilaterals*. In fact, from the two examples it is clear how the Penrose transform can be defined once one has the correspondence space $\mathcal{W}$ in hand. Although there is now a general construction of $\mathcal{W}$ and an analysis of its properties which will be given in a separate work, we have chosen to here focus on the two examples, in part because of the very beautiful geometry associated to each and in part because understanding them points to the way the general case should go.

The term *correspondence space* arises from the following consideration: The equivalence classes of homogeneous complex structures on

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[8]As will be explained below, for a given choice of positive Weyl chamber $S\mathcal{U}(2,1)/T_S$ and $\mathcal{U}(2,1)/T$ are the same as complex manifolds but are not the same as *homogeneous* complex manifolds. For Hodge-theoretic purposes the latter is more important.

$G_{\mathbb{R}}/T$ are indexed by the cosets in W/W_K where W is the Weyl group of $G_{\mathbb{C}}$ and W_K is the Weyl group of the maximal compact subgroup K of $G_{\mathbb{R}}$. We label these as D_w where $w \in W/W_K$. The correspondence space is then "universal" for maps $\mathcal{W} \to D_w$ and leads to diagrams

$$\begin{tikzcd} & \mathcal{W} \arrow[dl] \arrow[dr] & \\ D_w & & D_{w'} \end{tikzcd}$$

giving rise to Pensrose transforms between $H^q(D_w, L_\mu)$'s and $H^{q'}(D_{w'}, L_{\mu'})$'s. In particular, when one of the D_w is classical, which implies that $G_{\mathbb{R}}$ is of Hermitian type, this should lead to an identification of at least some non-classical automorphic cohomology with a classical object. This insight appears in [C1] and [C2] and is one hint that automorphic cohomology has a richer structure than previously thought.

We mention that as homogeneous complex manifolds for the complex Lie group $G_{\mathbb{C}}$ all of the domains D_w have a common *compact dual* $\check{D} \cong G_{\mathbb{C}}/B$ where B is a Borel subgroup. The D_w's are the $G_{\mathbb{R}}$-equivalence classes of the open $G_{\mathbb{R}}$-orbits in $\check{D}$. The correspondence space for the compact dual is $\check{\mathcal{W}} \cong G_{\mathbb{C}}/T_{\mathbb{C}}$, and $\mathcal{W} \subset \check{\mathcal{W}}$ turns out to be an open subset that is somewhat subtle to define.[9] In particular, it seems to be a somewhat new type of object; one that fibres over the cycle space $\mathcal{U}$, which has many of the characteristics of a bounded domain of holomorphy in $\mathbb{C}^N$, with affine algebraic varieties as fibres. It thus has a mixed complex function theoretic/algebro-geometric character. As mentioned above, this will be treated in a separate work.

In section III.C we shall discuss our *basic example*, the case of $G = \mathrm{SL}_2$. Although it is certainly "elementary", looking at it from the point of view of the correspondence space and Penrose transform gives new perspective on this most basic of cases and already suggests some aspects of what turns out to be the general mechanism. Of note is the

[9] $\check{\mathcal{W}}$ is sometimes referred to as the "enhanced flag variety" in the representation theory literature.

canonical identification of the group $H^1(\Omega^1_{\mathbb{P}^1}(k))$, $k \leqq 0$, with *global* holomorphic data; this is a harbinger of a fairly general situation.

The general mechanism was also suggested in part by formulating the calculations in the setting of *moving frames*. The elements of $G_{\mathbb{C}}$ may be identified as frames adopted to the geometry of the situation. The points of $G_{\mathbb{C}}/T_{\mathbb{C}}$ are the corresponding projective frames, and then the equations of the moving frame and their integrability conditions, the *Maurer-Cartan equations*, reveal the computational framework for the Penrose transform and suggest what the form ω above should be. Interpreting the formulas in terms of the roots of $G_{\mathbb{C}}$, $G_{\mathbb{R}}$ and the Borel subgroups B_w corresponding to D_w gives the suggested general prescription for ω that was mentioned above.

Associated to a domain is its *cycle space* $\mathcal{U}$, defined in this paper to be the set of $G_{\mathbb{C}}$-translates $Z = gZ_0$ of the maximal compact subvariety $Z_0 = K/T$. There is a comprehensive treatment of cycle spaces in [FHW], where they give a more general definition of the cycle space. It is known (loc. cit.) that in the non-classical case

$$\mathcal{U} \subset \check{\mathcal{U}} := G_{\mathbb{C}}/K_{\mathbb{C}}$$

where $\mathcal{U}$ is an open Stein domain in the affine algebraic variety $\check{\mathcal{U}}$.[10] There is the incidence diagram (4) but $\mathcal{I}$ is not Stein so the [EGW] method does not apply to this picture.[11] There is however a surjective map $\mathcal{W} \to \mathcal{U}$ where, in first approximation, the fibre lying over a point in $\mathcal{U}$ corresponding to $Z \cong K/T \subset D$ is the correspondence space $K_{\mathbb{C}}/T_{\mathbb{C}}$ for the homogeneous projective variety Z. For instance, in both the examples we shall consider we will have $Z \cong \mathbb{P}^1$ and the corresponding fibre will be $\mathbb{P}^1 \times \mathbb{P}^1 \backslash \{\text{diagonal}\}$. In some sense one may think of $\mathcal{W}$ as a common Stein refinement of the cycle spaces $\mathcal{U}_w$ for all

[10] The substantive statements here are (i) that $K_{\mathbb{C}} = \{g \in G_{\mathbb{C}} : gZ_0 = Z_0\}$, (ii) that $\mathcal{U}$ is Stein, and (iii) $\mathcal{U}$ is Kobayashi hyperbolic.

[11] As noted above, it is this picture to which much of the classical literature on Penrose transforms, given by "pull-back and push-down", pertains.

the domains D_w, a refinement to which the methods of [EGW] apply for all the D_w's.[12]

The cycle spaces will enter in an essential way in the proof of the injectivity of the Penrose transform for certain ranges of μ and μ'. Basically, the idea is that non-injectivity leads to an equation

$$F\omega = d_\pi G \tag{5}$$

where G is a holomorphic section of a line bundle $L(\mu, \mu') \to \mathcal{W}$. The equation (5) gives differential restrictions on G, and with these it is shown that G lives on a quotient variety $\mathcal{J}$ of $\mathcal{W}$ and that $\mathcal{J}$ is covered by the lifts $\widetilde{Z}$ of compact subvarieties $Z \subset D$. Then is is shown that for the range of weights μ of interest and for all such Z the restriction

$$G\big|_{\tilde{Z}} = 0\,.$$

Since $\mathcal{J}$ is covered by the $\widetilde{Z}$'s, this implies that $G = 0$.[13]

As mentioned above one primary objective of this work is to formulate and illustrate the general method of Penrose-type transforms. A second objective is to use this method to define and derive results about one *arithmetic aspect* of automorphic cohomology. Informally stated, the result is that, *up to a transcendental factor that depends only on the CM type, arithmetic automorphic cohomology classes assume arithmetic values at CM points in* $\mathcal{W}$.

To explain this, a first observation is that the compact dual $\check{D}$ is a homogeneous, rational projective variety defined over a number field k. We shall say that a complex vector space V has an arithmetic structure in case there is a number field L with an embedding $L \hookrightarrow \mathbb{C}$ together with an L-vector space $V_L \subset V$ such that $V \cong \mathbb{C} \otimes_L V_L$. In all cases considered below the arithmetic structures will be natural in

[12]It is a non-trivial result [FHW] that the $\mathcal{U}_w$ are all the *same* open set in $\check{\mathcal{U}} = G_{\mathbb{C}}/K_{\mathbb{C}}$, but the compact subvarieties of D_w parametrized by $\mathcal{U}_w$ are different. This *universality property*, which is closely related to Matsuki duality, will play an important role in the definition and analysis of the properties of $\mathcal{W}$.

[13]This method will not apply to $\mathcal{W}$ itself since, being Stein, it contains no compact subvarieties. In order for it to apply we must quotient $\mathcal{W}$ on the right by parabolic subgroups $P_{\mathbb{C}}$ with $T_{\mathbb{C}} \subset P_{\mathbb{C}} \subset K_{\mathbb{C}}$. The roots of $P_{\mathbb{C}}$ are the ones that appear in the definition of the form ω mentioned above.

a sense that we hope will be clear from the context. For example, for any number field $L \supset k$, at an L-rational point of $\check{D}$ the fibres of $G_{\mathbb{C}}$-homogeneous vector bundles that are defined over k will have a natural arithmetic structure.

A second type of arithmetic structure arises when we realize D as a Mumford-Tate domain. There are then defined the set of complex multiplication, or CM, points $\varphi \in D$. The action of the CM field L_φ on the fibres at φ of the Hodge bundles then gives an arithmetic structure to these vector spaces. A basic result [GGK1] is that this arithmetic structure is comparable with the previously mentioned one for $\varphi \in \check{D}$, in the sense that there is a number field L' with $k \subset L'$, $L_\varphi \subset L'$ and such that when tensored with L' these two arithmetic structures coincide.

In our two examples, via the Penrose transform with the resulting natural isomorphism of the images of cuspidal automorphic forms[14]

$$H^1_o(X, L_\mu) \cong H^0_o(X', L'_{\mu'}) \tag{6}$$

an arithmetic structure on the RHS will induce one on the LHS. For the $\mathcal{U}(2,1)$ and $\mathrm{Sp}(4)$ examples and for a special choice of μ', the RHS consists of cuspidal *Picard*, respectively *Siegel modular forms*. If $H \xrightarrow{\pi} \Gamma\backslash H =: Y$ is the quotient of the Hermitian symmetric domain H to which X' maps, then the RHS is the cuspidal subspace of $H^0(Y, \omega_Y^{\otimes l/3})$.

It is known that Y has a *canonical model*, which is a projective variety defined over a number field $\mathbf{k}$ with homogeneous coordinate ring $\bigoplus_{l \geqq 0} H^0(Y, \omega_Y^{\otimes l/3})$ that is defined over $\mathbf{k}$.[15] The vector space $H^0(Y(\mathbf{k}), \omega_{Y(\mathbf{k})}^{\otimes l/3}) := H^0(Y, \omega_Y^{\otimes l/3})_{\mathbf{k}}$ are the *modular forms of weight l defined over* $\mathbf{k}$. For $y \in Y(\mathbf{k})$ a k-rational point, the fibre $\omega_{Y,y}$ of $\mathbb{C} \otimes_k \omega_{Y(\mathbf{k}),y}$ at y is defined

[14]Cf. section IV.A for the notation and terminology. For $\Gamma \subset G$ co-compact, the subscript "o" may be dropped on both sides.

[15]For the examples considered in this work, the boundary components of Y in the Baily-Borel compactification will have codimension at least two, so finiteness conditions at the cusps are not necessary.

over $\mathbf{k}$, and if $\psi \in H^0(Y, \omega_Y^{\otimes l/3})_{\mathbf{k}}$ then the value

$$\psi(y) \in \omega_{Y(\mathbf{k}),y}^{\otimes l/3}\,.$$

In our two examples, H will be realized as a Mumford-Tate domain and the notion of a CM point $h \in H$ is well-defined. As noted above, the fibres $\mathbb{F}_h^p$ of the Hodge bundles $\mathbb{F}^p \to H$ then have an arithmetic structure, so that there is a number field L and an L-vector space $\mathbb{F}_{h,L}^p \subset \mathbb{F}_h^p$ with $\mathbb{F}_h^p = \mathbb{C} \otimes_L \mathbb{F}_{h,L}^p$. The canonical bundle ω_H is constructed from the Hodge bundles, and therefore at a CM point h we have $\omega_{H,h,L}^{\otimes l/3} \subset \omega_{H,h}^{\otimes l/3}$. A classical result [Shi] is that *there is a fixed transcendental factor $\Delta \in \mathbb{C}^*/\overline{\mathbb{Q}}^*$ that depends only on the CM field associated to h, together with a choice of positive embeddings of the field, and a finite extension $L' \supset L$ such that for $\psi \in H^0(Y, \omega_H^{\otimes l/3})_{\mathbf{k}}$*

$$\Delta^{-l}(\pi^*\psi)(h) \in \omega_{H,h;L'}^{\otimes l/3}\,.$$

In other words, in the sense just explained up to the factor Δ arithmetic automorphic forms assume arithmetic values at CM points.

Using the isomorphism (6), for suitable characters μ we may define an arithmetic structure on the cuspidal automorphic cohomology group $H_o^1(X, L_\mu)$. Using the [EGW] method we may then evaluate an automorphic cohomology class $\alpha \in H_o^1(X, L_\mu)$ in the fibres of bundles at $\mathbf{w} \in \mathcal{W}$ constructed from the Hodge bundles. At a CM point of $\mathcal{W}$ these vector spaces have arithmetic structures, and our result (IV.D.3) is that *at such a point* $\mathbf{w}$ *whose CM structure is compatible with that on its image $\psi \in H$, up to a fixed transcendental factor as above the value $\alpha(\mathbf{w})$ is arithmetic.* Moreover, these points are dense in the analytic topology.[16]

[16]We remark that a geometrically more natural "evaluation" of automorphic cohomology classes in $H_0^1(X, L_\mu)$ would be to classes in $H^1\big(S, \mathcal{O}_S(L_\mu)\big)$

$$S = \Gamma_S \cap \mathcal{H}$$

where $\mathcal{H} \subset D$ is an equivalently embedded copy of the upper half plane $\mathcal{H} = \mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}(2)$ arising from an inclusion $\mathrm{SL}_2(\mathbb{Q}) \hookrightarrow G$ and $\Gamma_S = \Gamma \cap \mathrm{SL}_2(\mathbb{Q})$ is an arithmetic group. The notation "S" stands for *Shimura curve*. In general, equivariantly embedded Hermitian symmetric domains in non-classical Mumford-Tate domains have independently arisen from a number of perspectives ([FL], [R] and

We remark that this work is one dealing primarily with the complex geometry and coherent cohomology of Mumford-Tate domains and their quotients by discrete groups. It is written from the perspective of the geometry of a class of interesting locally homogeneous complex manifolds that independently arise from Hodge theory and from representation theory. The deeper geometric and cohomological aspects of representation theory are treated here only superficially. We refer to the paper [Schm3] for an exposition of some of these aspects that will be used in the sequel to this paper [GG] where the general properties of correspondence spaces will be discussed. We also refer to the introduction to [CK] for a lucid overview of some related aspects of arithmetic automorphic representation theory and the role of TDLDS's in this theory. One of our main points is that different coherent cohomology groups may be associated to the same representation, either finite dimensional of $G_{\mathbb{C}}$ in the compact case or infinite dimensional of $G_{\mathbb{R}}$ in the non-compact case (including both $G_{\mathbb{R}}/T$'s and $\Gamma\backslash G_{\mathbb{R}}/T$'s), and that in some generality the connection between these different manifestations may be realized geometrically. Although we here have informally mentioned some of these general results, for the reasons stated above we have in this work focused on our examples.

It is the authors' pleasure to thank Sarah Warren for a marvelous job of converting an at best barely legible handwritten manuscript into mathematical text.

Outline

The following is an outline of the contents of the various sections of this paper.

We begin in section I.A with a general discussion of the homogeneous complex manifolds that will be considered in this work. Here, and later, we emphasize the distinction between equivalence of *homogenous* complex manifolds and *homogeneous* vector bundles over them, rather

the above) and would seem to be objects worthy of further study. Some comments about this issue will be given in the forthcoming CBLS volume [GGK2].

than just equivalence as complex manifolds and holomorphic vector bundles.

In section I.B we discuss our first example, which is the non-classical complex structure on $\mathcal{U}(2,1)/T := D$, realized as one of the three open orbits of $\mathcal{U}(2,1)$ acting on the homogeneous projective variety of flags $(0) \subset F_1 \subset F_2 \subset F_3 = \mathbb{C}^3$ where $\dim F_i = i$. Here $\mathbb{C}^3$ has the important additional structure of being the complexification of $\mathbb{F}^3$ where $\mathbb{F} = \mathbb{Q}(\sqrt{-d})$ is a quadratic imaginary number field. It is this additional structure that leads to the realization of D as a Mumford-Tate domain, thereby bringing Hodge theory into the story. The other two open $G_{\mathbb{R}}$-orbits D' and D'' are classical and may also be realized as Mumford-Tate domains, or what is more relevant to this work, the set of Hodge flags asssociated to Mumford-Tate domains consisting of polarized Hodge structures of weight one with additional structure.

All three of the above domains have three descriptions: geometric, group-theoretic and Hodge-theoretic. The interplay between these different perspectives is an important part of the exposition. Especially important is the book-keeping between the tautological, root and weight, and Hodge theoretic descriptions of the $\mathcal{U}(2,1)$-homogeneous line bundles over the domains, which is given in section II.B.

In our first example the three domains may be pictured as

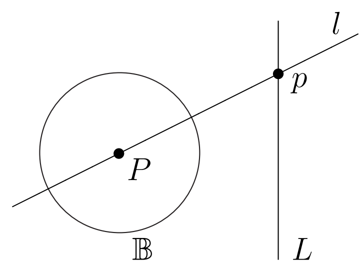


Figure 1

where $\mathbb{B}$ is the unit ball in $\mathbb{C}^2 \subset \mathbb{P}^2$ defined by the Hermitian form with matrix $\mathrm{diag}(1, 1-1)$ and

$$\begin{cases} D &= \{(p,l)\} \\ D' &= \{(P,l)\} \\ D'' &= \{(p,L)\}\ . \end{cases}$$

All of thse domains are quotients of the correspondence variety $\mathcal{W}$ given by the set of configurations

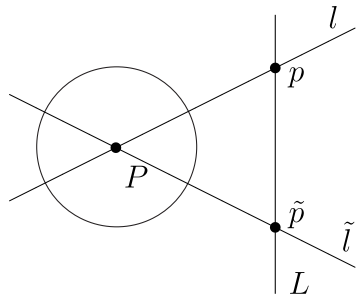


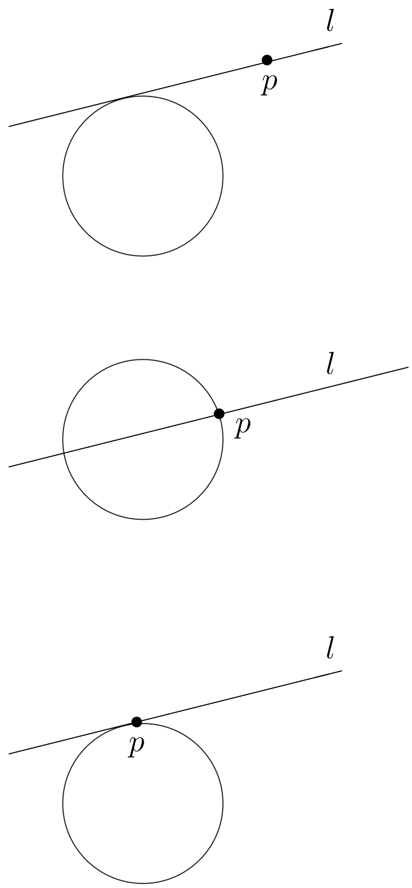
FIGURE 1a

Correspondence varieties play a central role in the theory.[17]

Although it will not be needed for the present work, we mention that three other orbits of the action of $G_{\mathbb{R}}$ on the flag manifold may be

[17]We observe that $\mathcal{W}$ may be described as the set of projective frames $(p, \tilde{p}, P)$ in the above figure; this is a general phenomenon.

pictured as



where the third is the unique closed orbit. These and their Mutsuki dual $K_{\mathbb{C}}$-orbits will play a central role in the sequel.[18]

The second example is the non-classical complex structure on $\mathrm{Sp}(4,\mathbb{R})/T$ $:= D$ where D is realized as the period domain of polarized Hodge structures of weight $n = 3$ and with all Hodge numbers $h^{p,q} = 1$, an example that arises in the study of the mirror-quintic Calabi-Yau varieties. In this case there are four inequivalent complex structures, of which two, the D mentioned above and one classical one D', will play important

[18]In the above example the pictured orbits in ∂D will be seen to have Hodge-theoretic significance in terms of the Kato-Usui theory [KU] of limiting mixed Hodge structures and in representation theory where the TDLDS is constricted by parabolic inductions from the unique closed $G_{\mathbb{R}}$-orbit given by the third figure above (cf. [KP] and [GGK2]).

roles in this work. Again, the three descriptions — geometric, group-theoretic and Hodge-theoretic — and their interplay are important in this work.

The geometric description of the domains D and D' will be given by configurations

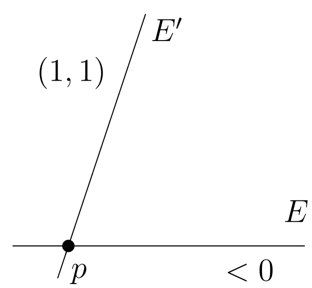


FIGURE 2

where

$$D = \{(p, E')\}$$
$$D' = \{(p, E)\}.$$

Here we are given a non-degenerate alternating form Q and conjugation σ on a four dimensional complex vector space V with $\mathbb{P}^3 = \mathbb{P}V$. The form Q defines the Hermitian form $H(u, v) = iQ(u, \sigma v)$. In the above figure, E and E' are Lagrange lines and the $(1, 1)$ and < 0 denote the signature of H restricted to them. The correspondence space $\mathcal{W}$ consists of all configurations

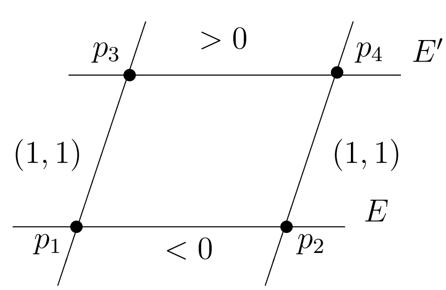


FIGURE 2a

which we will refer to as Lagrange quadrilaterals.[19]

In section II we discuss the equivalence classes of homogeneous line bundles over the two examples. Of particular importance is the "dictionary" between the three descriptions — geometric, representation-theoretic, and Hodge-theoretic — of these line bundles. We note that when D is considered as a homogeneous space for $\mathcal{U}(2,1)$, all the homogeneous line bundles are Hodge bundles but this is not when the complex manifold D is considered as a homogeneous space for $S\mathcal{U}(2,1)$ or the adjoint group $S\mathcal{U}(2,1)_{\mathrm{ad}}$. As there are several sign and notational conventions in the literature, and since the signs are critical for the considerations below, in a brief appendix to section II.B we have given our sign conventions and the rationale for them.

In each of our two examples, the classical homogeneous complex structures D' will fibre holomorphically with $\mathbb{P}^1$ as fibre over an Hermitian symmetric domain H. In the first example, H is the unit ball $\mathbb{B}$ in $\mathbb{C}^2$ and in the second H is Siegel's generalized upper-half-plane $\mathcal{H}_2$. Of particular importance are the homogeneous line bundles $\omega_{\mathbb{B}}^{\prime\otimes k/3} \to D'$ that are pullbacks of $\omega_{\mathbb{B}}^{\otimes k/3} \to H$ where $\omega_{\mathbb{B}}$ is the canonical line bundle, as the Γ-invariant sections over H, or equivalently over D', of these bundles give Picard, respectively Siegel modular forms of weight k. As will be explained below, these are relevant to automorphic cohomology in the non-classical case via the Penrose-type transforms.

In section III we discuss correspondence and cycle spaces and the Penrose transforms associated to the former. Our definition of a correspondence space $\mathcal{W}$ is motivated by the example in [EGW] and the uses of this example in [C1], [C2],[C3].[20] In this paper we will use diagrams

[19]Again we note that the above figure is specified by the projective Lagrange frame p_1, p_2, p_3, p_4.

[20]The general definition and properties of $\mathcal{W}$ will be given in [GG]; it was motivated by careful analysis of the two examples studied here.

like

$$\begin{tikzcd} & \mathcal{W} \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ D & & D' \end{tikzcd}$$

where the correspondence space $\mathcal{W}$ is Stein and the fibres of π, π' are contractible. By the results in [EGW] the coherent cohomology of homogeneous line bundles over each of D, D' is represented by global, holomorphic relative de Rham cohomology on $\mathcal{W}$. In the examples in this work there will be canonical "harmonic" representations of the de Rham classes, so that coherent cohomology on D, D' is canonically represented by global, holomorphic objects on the correspondence space. As explained below, in the cases of particular interest here the cohomology groups arising from D, D' and represented holomorphically on $\mathcal{W}$ will then be related by multiplication by a canonical invariant differential form ω.

In addition to the correspondence spaces we shall make use of the cycle space $\mathcal{U}$ associated to D. For the definition of $\mathcal{U}$ used here we first note that $D = G_{\mathbb{R}}/T$ contains maximal compact subvarieties, one of which is $Z_0 := K/T$. Then $\mathcal{U}$ is defined to be the set of translates $Z = gZ_0$, $g \in G_{\mathbb{C}}$, such that $Z \subset D$. Just as with Hilbert schemes in algebraic geometry, cycle spaces lead to diagrams

$$\begin{tikzcd} & \mathcal{I} \arrow[dl] \arrow[dr] & \\ D & & \mathcal{U}\,. \end{tikzcd}$$

Using the basic result that $\mathcal{U}$ is Stein, for certain homogeneous line bundles $L \to D$ the standard "pull-back and push-down" method enables the groups $H^s(D, L)$, $s = \dim Z_0$, to be represented by global holomorphic objects over $\mathcal{U}$. As noted in the introduction above, cycle spaces and their use as just described have been the subject of considerable work over the years, but for our purposes the correspondence spaces, which lie over the cycle spaces and contain more information, will play a more fundamental role.

Following brief introductory comments in section III.A, in section III.B the correspondence and cycles spaces for our two examples will be constructed and their properties described. For example one, $\mathcal{W}$ is given by the set of configurations $(p, P, \tilde{p})$ in Figure 1a above, and $\mathcal{U}$ is given by the configurations (P, L) there. For example two, $\mathcal{W}$ is given by the set of configurations (p_1, p_2, p_3, p_4) in Figure 2a, and $\mathcal{U}$ is given by configurations (E, E') there.

In section III.C we study in some detain the basic example of $\check{D} = \mathbb{P}^1$ and $D = \mathcal{H} \subset \mathbb{P}^1$. Although not essential for what follows, many of the main features of correspondence spaces and their uses already appear here. Moreover, the explicit formulas — especially their arithmetic interpretations — indicate the flavor of our two examples discussed in section IV.

In section III.D, and especially in the appendix to that section, we discuss the relation in the compact case between the Penrose transform and the Borel-Weil-Bott (BWB) theorem. As is well known the BWB theorem shows that the same irreducible $G_{\mathbb{C}}$-module may appear in multiple ways as cohomology groups $H^q(M, L_\mu)$ where $L_\mu \to M = G_{\mathbb{C}}/B$ is a homogeneous line bundle. The EGW method gives a way of geometrically realizing these identifications. For use later for our two examples we give the explicit identification of the Maurer-Cartan forms ω on $G_{\mathbb{C}}$ such that, suitably interpreted and for particular q' and q, multiplication by ω realizes the isomorphisms $H^{q'}(M, L_{\mu'}) \xrightarrow{\omega} H^q(M, L_\mu)$ of $G_{\mathbb{C}}$-modules as given by the BWB theorem. In the appendix to this section the general expression for ω when $q' = 0$ is given and is related to the classical paper [Ko] where ω appears in the setting of $\mathfrak{n}$-cohomology.[21] Also, in the appendix as a harbinger of the results in section IV we discuss the arithmetic aspects of the groups $H^q(M, L_\mu)$, the form ω and the resulting Penrose transforms, the result being that "the Penrose transform is arithmetic."

[21]Geometrically, ω corresponds to the fundamental class of the Schubert cycle associated to the set of positive roots that "change sign" when one passes from the curvature form of $L_{\mu'}$ to that of L_μ.

In section III.E we discuss the Penrose transform in the first example. The main result is that, in certain ranges of indices and for the line bundles in question, the Penrose transform is injective. For a different range of indices a similar result is given in [C1] and [C2]. The reason for the difference is that in those works the proof of injectivity is computational whereas our argument is geometric, making use of the maximal compact subvarieties lying in a quotient $\mathcal{J}$ of $\mathcal{W}$ over those parametrized by the cycle space for D. Both ranges of indices include the case of particular interest corresponding to Picard modular forms that is used in section IV below.

In section III.F we discuss the Penrose transform in the second example. Again the main result is its injectivity in certain ranges, which is established by an argument using the cycle space similar to that used in the first example.[22]

Section IV is devoted to one of the new results in this work. Section IV.A is devoted first to a discussion of Williams' lemma, which is the main technical tool we shall use in the computations of the relevant $\mathfrak{n}$-cohomology groups in our two examples. Then following [C1] we give the definition of cuspidal automorphic cohomology, and the somewhat delicate proof of the injectivity, which by dimension count implies isomorphism, of the Penrose transform in our two examples.

In section IV.B Picard and Siegel cuspidal automorphic forms are discussed; again the proof of injectivity of the Penrose transform, using Williams' lemma which turns out to just work, is somewhat subtle.

Section IV.C is devoted to a discussion of the arithmetic structures on vector spaces, including such structures in special fibres of the homogeneous vector bundles of interest. There are three types of such structures: (i) Hodge-theoretic or HT, defined in the fibres of the Hodge bundles at complex multiplication (CM) points; (ii) projective, defined using the embedding $D \subset \check{D}$ where $\check{D}$ is a projective algebraic variety

[22]The general question of ranges of injectivity of Penrose transforms is an interesting one from both a geometric and a representation theoretic perspective; it will be discussed in [GGK2].

defined over a number field; and (iii) algebro-geometric or AG, defined only when D is classical and using the canonical model of $\Gamma\backslash H$ where H is the Hermitian symmetric domain over which D fibres. In (ii) and (iii) the arithmetic structures are defined in the fibres over points which are defined over the number fields, this set of points being quite different in the two cases. The comparison between the arithmetic structures in cases (i) and (ii) from [GGK1], chapter V; the result is that these structures are "comparable" at common points. The deep arithmetic result we shall use is the comparison between the HT and AG arithmetic structures in the fibres of the Hodge bundles at CM points. In the appendix to section IV.C we discuss the explicit canonical models in our two examples.

In section IV.D we state and begin the proof of the result, Theorem (IV.D.3), concerning the field of definition of cuspidal arithmetic automorphic classes evaluated in the fibres of Hodge bundles at CM points. The two main points in the argument are: (i) that using the [EGW] formalism automorphic cohomology classes of higher degree may be evaluated at points of the correspondence space, and (ii) the comparison between the HT and AG arithmetic structures in the classical case. In the appendix to this section we discuss the Penrose transform and arithmeticity of automorphic cohomology classes in $H^1(X, L_\mu)$ where $X = \Gamma\backslash D$ and $\mu + \rho$ is in the anti-dominant Weyl chamber. In some sense the anti-dominant one is the most important Weyl chamber for representation theory but it is not the Weyl chamber where the Penrose transform of Picard and Siegel modular forms ends up.[23] One point of note here is that there are many Penrose transforms, a topic that will be discussed elsewhere when the general definitions and properties of correspondence spaces will be treated.

[23] In the appendix to section IV.D we give an alternate method for evaluating cohomology classes in the case $\mu + \rho$ is anti-dominant and pose an interesting question that arises from this construction.

In section IV.E we complete the proof of theorem (IV.D.3). The argument involves a somewhat intricate analysis of compatible pairs of CM points in correspondence spaces.

Finally, in section IV.F we will present an exposition of the main result in [C1] and [C2] concerning the relation of the cup-products of the Penrose transform of Picard automorphic forms and their conjugates to the automorphic cohomology group $H^2(X, L_{-\rho})$. It is this group that appears in the automorphic representation of the adele group $\mathcal{U}(2,1;\mathbb{A})$ whose infinite component is a totally degenerate limit of discrete series (TDLDS); i.e., one whose Harish-Chandra infinitesimal character is zero (cf. [CK]). It is known that such a representation cannot arise from the cohomology, either l-adic or coherent, associated to a Shimura variety of Hodge type. To be able to define an arithmetic structure on $H^2(X, L_{-\rho})$ was a motivating question for Carayol, and his result is given in theorem (IV.F.1) with his proof presented in the context of this work.[24] In Carayol's work he used the explicit construction of the TDLDS represented by functions on the closed $S\mathcal{U}(2,1)$-orbit, which is the 3-sphere as depicted by the third of the above pictures of non-open orbits acted on by $S\mathcal{U}(2,1)$ through linear functional transformations. In the appendix to section IV.F we shall discuss a different geometric realization of a part of the TDLDS and relate this to $\mathfrak{n}$-cohomology considerations. This construction, interpreted in the context of Beilinson-Bernstein localization [BB] and the duality theorem in [HMSW] and coupled with the general construction of correspondence spaces should allow the methods of this work to be extended to further interesting geometric examples.

Some notations and assumptions.

- G will be a reductive $\mathbb{Q}$-algebraic group; $G_{\mathbb{R}}, G_{\mathbb{C}}$ will denote the corresponding real and complex Lie groups. We always

[24]Although Carayol's result does not yet give the sought for arithmetic structure on $H^2(X, L_{-\rho})$, and by duality one on $H^1(X, L_{-\rho})$, we feel that his work is extremely interesting and the arguments bring new and deep insight into automorphic cohomology.

assume that $G_{\mathbb{R}}$ is connected and that $G_{\mathbb{R}}$ contains a compact maximal torus;

- $\Gamma \subset G$ will denote an arithmetic subgroup;
- $T \subset G_{\mathbb{R}}$ will be a compact maximal torus with Lie algebra $\mathfrak{t}$; $\mathfrak{t}_{\mathbb{C}}$ will denote the corresponding Cartan sub-algebra of $\mathfrak{g}_{\mathbb{C}}$ (the more customary notation is $\mathfrak{h}$ or $\mathfrak{h}_{\mathbb{C}}$, but these will be used elsewhere);
- $\Phi \subset i\check{\mathfrak{t}}$ will be the roots, and Φ_c, Φ_{nc} will denote the compact, respectively non-compact roots. Upon the choice of a set of positive roots Φ^+, we have $\Phi = \Phi^+ \cup \Phi^-$ where $\Phi^- = \overline{\Phi^+}$, the conjugation being relative to the real form $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}_{\mathbb{C}}$;
- $X_\alpha, \alpha \in \Phi$, will denote the root vectors, normalized as in [K1];
- We set $\mathfrak{n}^{\pm} = \bigoplus_{\alpha \in \Phi^{\pm}} \mathfrak{g}^{\alpha}$ where $\mathfrak{g}^{\alpha} = \mathbb{C}X_\alpha$ is the root space. In section IV we will simply set $\mathfrak{n} = \mathfrak{n}^+$;
- $K \subset G_{\mathbb{R}}$ will denote a maximal compact subgroup with $T \subset K$;
- W and W_K will denote the Weyl groups of $G_{\mathbb{C}}, K$ respectively;
- A homogeneous complex manifold $D \cong G_{\mathbb{R}}/T$ will be *classical* if it fibres holomorphically or anti-holomorphically over an Hermitian symmetric domain;
- $\overline{D}$ will denote the homogeneous complex manifold with the conjugate complex structure; it may or may not happen that $\overline{D} \cong D$ as homogeneous complex manifolds (cf. the beginning of section I for a discussion of this point);
- $\mathbb{B}$ will denote the unit ball in $\mathbb{C}^2 \subset \mathbb{P}^2$, and $\mathbb{B}^c$ will denote the complement in $\mathbb{P}^2$ of the closure of $\mathbb{B}$ (we do not use $\mathbb{P}^2\backslash\overline{\mathbb{B}}$ because $\overline{\mathbb{B}}$ refers to $\mathbb{B}$ with the conjugate complex structure);
- A polarized Hodge structure (V, Q, φ) is given by a $\mathbb{Q}$-vector space V, a non-degenerate bilinear form $Q : V \otimes V \to \mathbb{Q}$ with $Q(u, v) = (-1)^n Q(v, u)$ where n is weight, and a circle $\varphi : S^1 \to \mathrm{Aut}(V_{\mathbb{R}}, Q)$ such that the eigenspace decomposition $V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$ where $\varphi(z) = z^p\overline{z}^q$ on $V^{p,q}$ gives a Hodge

structure that is polarized by Q. We refer to [GGK1] whose notations and terminology we shall generally follow;

- *Mumford-Tate domains* in this paper will be homogeneous complex manifolds that parametrize a connected component of the set of polarized Hodge structures on (V, Q) where V is a $\mathbb{Q}$-vector space, $Q : V \otimes V \to \mathbb{Q}$ is a non-degenerate form with $Q(u, v) = (-1)^n Q(v, u)$ where n is the weight, and where the Hodge numbers $h^{p,q} = \dim V^{p,q}$ are given, and where the polarized Hodge structures have the additional structure of containing a given set of Hodge tensors. We will denote by G the *Mumford-Tate group* of a generic $\varphi \in D$. For much of this work it is only the complex structure that is relevant, but the arithmetic part requires the additional structure as a Mumford-Tate domain. Moreover, the Hodge-theoretic interpretation of our examples enriches the geometry. In general, as noted above we shall follow the terminology and notations of [GGK1], one exception being that these Mumford-Tate groups were denoted there by "M" whereas here they are "G";
- The homogeneous complex manifolds $D = G_{\mathbb{R}}/T$ that we shall consider have *compact duals* $\check{D} = G_{\mathbb{C}}/B$ where $B \subset G_{\mathbb{C}}$ is a Borel subgroup with $B \cap G_{\mathbb{R}} = T$;
- In some subsections, such as I.A and III.D, we will consider these manifolds not as compact duals of D's but as homogeneous complex manifolds in their own right, and there they will be denoted by M;
- We shall consider three types of *correspondence spaces:* $\check{\mathcal{W}}$ for $\check{D}$, $\mathcal{W}$ for D and $\mathfrak{X}$ for M (cf. section III.B for the definitions);
- $\mathcal{U}$ will denote the cycle space for D (also defined in section III.B);
- There are notations for each of our examples, introduced individually at the beginning of sections I.B and I.C. These notations are consistent with two exceptions: (i) Because of

the necessity of keeping careful track of the notation for the three different realizations of the one complex manifold D as a *homogeneous* complex manifold, we shall use a slightly non-standard but hopefully clear notation for the roots of $\mathcal{U}(2,1)$. (ii) In the first example V will be a $\mathbb{Q}$-vector space, whereas in the second example it will be a complex vector space.

- $S^q(X, L_\mu)$ will denote the space of *cuspidal automorphic cohomology* as defined in section IV.A. It maps to a subspace $H^q_o(X, L_\mu) \subset H^q(X, L_\mu)$ of the usual automorphic cohomology.

I. Geometry of the Mumford-Tate domains

I.A. **Generalities on homogeneous complex manifolds.** Before turning to our two examples we give some general remarks on homogeneous complex manifolds $M = G/H$. Here, G and H are Lie groups with H a closed subgroup such that G/H has a G-invariant complex structure. A *morphism of homogeneous complex manifolds*

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ \| & & \| \\ G/H & & G'/H' \end{array}$$

is given by a homomorphism $G \to G'$ taking H to H' and such that the induced map f is holomorphic. We shall consider two types of examples.

Compact case. Here $M = G_{\mathbb{C}}/B$ where $G_{\mathbb{C}}$ is a complex semi-simple Lie group and B is a Borel subgroup. The choice of B is equivalent to the choice of a set of positive roots where the Lie algebra $\mathbf{u}$ of the unipotent radical U is the span of the *negative root vectors*. Then the holomorphic tangent space to $M = G_{\mathbb{C}}/B$ at the identity coset is identified as

$$TM = \operatorname{span}\{X_\alpha : \alpha \in \Phi^+\}.$$

A different choice of a set of positive roots, or equivalently of a Weyl chamber, is given by an element $w \in W$ and the corresponding homogeneous complex manifold is

$$M_w = G_{\mathbb{C}}/B_w$$

where

$$B_w = wBw^{-1}\,.$$

M and M_w are equivalent as homogeneous complex manifolds under the automorphism $G_{\mathbb{C}} \to G_{\mathbb{C}}$ given by $g \to wgw^{-1}$ which takes B to B_w.

Non-compact case. Here we use the notation D instead of M, and then $D = G_{\mathbb{R}}/T$ where $G_{\mathbb{R}}$ is a real, semi-simple Lie group and $T \subset G_{\mathbb{R}}$ is a compact maximal torus. There are two ways of describing the homogeneous complex structure.

(i) By a choice Φ^+ of positive roots. Then with the identifications at the identity coset

$$T_{\mathbb{R}}G_{\mathbb{R}}/T \cong \mathfrak{g}_{\mathbb{R}}/\mathfrak{t}$$
$$T_{\mathbb{R}}G_{\mathbb{R}}/T \otimes \mathbb{C} := T_{\mathbb{C}}G_{\mathbb{R}}/T \cong \mathfrak{n}^+ \oplus \mathfrak{n}^-$$

where $\mathfrak{n}^+ = \operatorname{span}\{X_\alpha, \alpha \in \Phi^+\}$ and $\mathfrak{n}^- = \overline{\mathfrak{n}^+}$, setting

$$T^{1,0}G_{\mathbb{R}}/T = \mathfrak{n}^+$$

gives an integrable almost complex structure making $D = G_{\mathbb{R}}/T$ a homogeneous complex manifold. A different choice of positive roots is equivalent to the choice of an element $w \in W$ leading to a possibly different homogeneous complex structure D_w on $G_{\mathbb{R}}/T$. An important observation is

> *$D \cong D_w$ as homogeneous complex manifolds if, and only if, $w \in W_K$.*

Thus there are $|W/W_K|$ inequivalent homogeneous complex structures on $G_{\mathbb{R}}/T$.

Example. The unit ball $\mathbb{B} \subset \mathbb{C}^2$ is *not* equivalent as a homogeneous complex manifold for $G_\mathbb{R} = S\mathcal{U}(2,1;\mathbb{R})$ to $\overline{\mathbb{B}}$, the ball with the conjugate complex structure.

(ii) The second method is by the choice of an open orbit of $G_\mathbb{R}$ acting on $M = G_\mathbb{C}/B$. These two descriptions are related as follows: Let $\varphi_0 =: eB \in G_\mathbb{C}/B$ be the identity coset and $D =: G_\mathbb{R}(\varphi_0) \cong G_\mathbb{R}/T$ with $T = G_\mathbb{R} \cap B$ the corresponding $G_\mathbb{R}$-orbit. For $w \in W$, we choose $\mathbf{w} \in N_{G_\mathbb{C}}(T_\mathbb{C})$ inducing w and set $D_\mathbf{w} = G_\mathbb{R} \cdot (\mathbf{w}^{-1}(\varphi_0)) \subset M$. Then the isotropy group in $G_\mathbb{C}$ of $\mathbf{w}^{-1}(\varphi_0)$ is $B_\mathbf{w} = \mathbf{w}B\mathbf{w}^{-1}$, so that $D_\mathbf{w}$ is the $G_\mathbb{R}$-orbit of the identity coset in $G_\mathbb{C}/B_w$ corresponding to the choice $\mathbf{w}(\Phi^+) = \Phi^+_\mathbf{w}$ of positive roots. The above identification of $G_\mathbb{C}/B$ with $G_\mathbb{C}/B_w$ as $G_\mathbb{C}$-homogeneous complex manifolds induces an identification $D \cong D_w$ as $G_\mathbb{R}$-homogeneous complex manifolds if, and only if, $\mathbf{w} \in G_\mathbb{R} \cap N_{G_\mathbb{C}}(T_\mathbb{C}) = W_K$.

I.B. The first example ($\mathcal{U}(2,1)$).

Notations. We assume given

- a $\mathbb{Q}$-vector space V of dimension six;
- a non-degenerate alternating form $Q : V \otimes V \to \mathbb{Q}$; and
- an action of $\mathbb{F} = \mathbb{Q}(\sqrt{-d})$ on V, i.e.

$$\mathbb{F} \hookrightarrow \operatorname{End}_Q(V)\,.$$

Here, d is a square-free positive integer. Setting $V_\mathbb{F} = \mathbb{F} \otimes_\mathbb{Q} V$ we have a decomposition

$$V_\mathbb{F} = V_+ \oplus V_-$$

into the $\pm$ eigenspaces of the action of $\mathbb{F}$. If $\rho \in \operatorname{Gal}(\mathbb{F}/\mathbb{Q})$ is the generator, then ${}^\rho V_+ = V_-$. For later reference we note that

$$V_\mathbb{F} \cap V_\mathbb{R} = V\,.$$

We assume that

- $Q(F_+, F_+) = 0$ and $Q(F_-, F_-) = 0$, and
- the Hermitian form

$$H(u,v) = \pm i Q(u, \bar{v}) \qquad\qquad u, v \in V_{+,\mathbb{C}}$$

has signature $(2,1)$.

The $\pm$ sign will be chosen later depending on the weight of the Hodge structure we are considering.

We assume that we may choose a basis $e_1, e_2, e_3 \in V_+$ so that $H(e_i, \bar{e}_j)$ is the matrix H given below. Then

- $V_{+,\mathbb{C}} \cong \mathbb{C}^3$, written as column vectors with $e_1 = \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right)$, $e_3 = \left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)$;
- in this basis

$$H = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}.$$

We observe that if we take any basis e_1, e_2, e_3 for the $\mathbb{F}$-vector space V_+ and define the Hermitian form H by the above matrix, then the alternating form Q on $V_{\mathbb{F}}$ defined by

$$Q(V_+, V_+) = 0 = Q(V_-, V_-)$$

and for $u, v \in V_+$, identifying $\overline{V}_+ = V_-$, requiring Q to be alternating and setting

$$Q(u, \bar{v}) = \sqrt{-d}H(u, v),$$

is actually defined over $\mathbb{Q}$. This is because the vectors

$$\begin{cases} \frac{1}{2}(e_i + \bar{e}_i) \\ \frac{\sqrt{-d}}{2}(e_i - \bar{e}_i) \end{cases}$$

give a basis for $V_{\mathbb{R}} \cap V_{\mathbb{F}} = V$, and on these vectors Q takes values in $\mathbb{R} \cap \mathbb{F} = \mathbb{Q}$.

- The group $\mathrm{Aut}_{\mathbb{F}}(V, Q)$ of automorphisms in $\mathrm{Aut}(V, Q)$ that commute with the $\mathbb{F}$-action is an algebraic group defined over $\mathbb{F}$, and we set

  $$\mathcal{U}(2,1) = \mathrm{Aut}_{\mathbb{Q}}(V, Q) \cap \mathrm{Res}_{\mathbb{F}/\mathbb{Q}} \mathrm{GL}_{\mathbb{F}}(V)\,.$$

  This is a $\mathbb{Q}$-algebraic group, and is a $\mathbb{Q}$-form of the real Lie group $\mathcal{U}(2,1)_{\mathbb{R}}$ of automorphisms of $\mathbb{C}^3$ preserving the Hermitian form H.

If $A \in \mathrm{Aut}_{\mathbb{F}}(V, Q)$, then $A : V_{\mathbb{F}} \to V_{\mathbb{F}}$ preserves the decomposition into the $V_{\pm}$ eigenspaces, and the induced map $A_+ : V_+ \to V_+$ determines A. Thus, we shall think of $\mathcal{U}(2,1)$ and $\mathcal{U}(2,1)_{\mathbb{R}}$ as subgroups of $\mathrm{SL}(3, \mathbb{F})$ and $\mathrm{SL}(3, \mathbb{C})$. We shall also consider the subgroup $S\mathcal{U}(2,1)$ that corresponds to a subgroup of $\mathrm{Res}_{\mathbb{F}/\mathbb{Q}}(\mathrm{SL}_{3,\mathbb{F}})$ and the quotient group $S\mathcal{U}(2,1)_{\mathrm{ad}}$ of $S\mathcal{U}(2,1)$.

- We denote by e_1^*, e_2^*, e_3^* the dual basis to e_1, e_2, e_3, considered as row vectors.
- The maximal torus T of $\mathcal{U}(2,1)_{\mathbb{R}}$ is
$$\left\{ g = \begin{pmatrix} e^{2\pi i\theta_1} & & \\ & e^{2\pi i\theta_2} & \\ & & e^{2\pi i\theta_3} \end{pmatrix} \right\}.$$
Then the isomorphism between T and $\mathfrak{t}/\Lambda$ is given explicitly by
$$g \to \boldsymbol{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} =: \theta_1 e_1 + \theta_2 e_2 + \theta_3 e_3 \,.$$
Here, $\mathfrak{t} \cong \mathbb{R}^3$ and $\Lambda \cong \mathbb{Z}^3$.
- The maximal tori T_S and T_{ad} of $S\mathcal{U}(2,1)_{\mathbb{R}}$ and $S\mathcal{U}(2,1)_{\mathrm{ad},\mathbb{R}}$ are given by
$$\begin{cases} T_S &= \mathfrak{t}_S/\Lambda_S \\ T_{\mathrm{ad}} &= \mathfrak{t}_{\mathrm{ad}}/\Lambda_{\mathrm{ad}} \,. \end{cases}$$
The inclusion $S\mathcal{U}(2,1) \hookrightarrow \mathcal{U}(2,1)$ induces
$$\mathfrak{t}_S \hookrightarrow \mathfrak{t}$$
where $\mathfrak{t}_S \cong \mathbb{R}^2 \cong \mathrm{span}_{\mathbb{R}}\{u_1, u_2\}$ where
$$\begin{aligned} u_1 &\to \left(\begin{smallmatrix} 1 \\ -1 \\ 0 \end{smallmatrix} \right) = e_1 - e_2 \\ u_2 &\to \left(\begin{smallmatrix} 0 \\ 1 \\ -1 \end{smallmatrix} \right) = e_2 - e_3 \,. \end{aligned}$$
We have $\mathfrak{t}_S \cong \mathfrak{t}_{\mathrm{ad}}$ and the inclusion
$$\Lambda_{\mathrm{ad}} \hookrightarrow \Lambda_S$$

is given by letting $\Lambda_{\rm ad} = \operatorname{span}_{\mathbb{Z}}\{v_1, v_2\}$ and setting

$$\begin{cases} u_1 = 2v_1 - v_2 \\ u_2 = 2v_2 - v_1 \,. \end{cases}$$

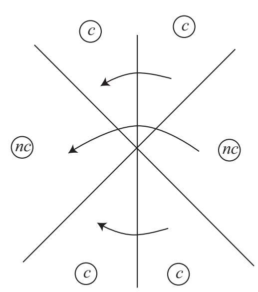
Then $\Lambda_{\rm ad}/\Lambda_S \cong \mathbb{Z}/3\mathbb{Z}$, and $\Lambda_{\rm ad}$ in $\mathbb{R}^3$ is spanned by

$$\begin{cases} v_1 & = \begin{pmatrix} 2/3 \\ -1/3 \\ -1/3 \end{pmatrix} \\ v_2 & = \begin{pmatrix} 1/3 \\ 1/3 \\ -2/3 \end{pmatrix} \,. \end{cases}$$

- We will view $\mathfrak{t}_S \subset \mathfrak{t}$ as the linear subspace defined by

$$e_1^* + e_2^* + e_3^* = 0\,.$$

Then the roots of $\mathcal{U}(2,1)$ are the roots of $S\mathcal{U}(2,1)$ and are generated by the restrictions to $\mathfrak{t}_S$ of $e_1^* - e_2^*$ (compact root), $e_2^* - e_3^*$, $e_3^* - e_1^*$.[25] We also have the picture



where the arrows $\curvearrowleft$ and $\leftarrow$ denote the action of the compact Weyl group and where the Weyl chambers corresponding to classical complex structures are labelled ⓒ and those corresponding to the non-classical complex structures are labelled (nc).

- For the usual root picture of su$(2,1)$ we have

[25]In the literature (cf. [K1]) the roots of $S\mathcal{U}(2,1)$ are frequently denoted by $e_i - e_j$, i.e. omitting the *'s. For the calculations below it will be better to keep separate the vector space $\mathfrak{t}$ and its dual.

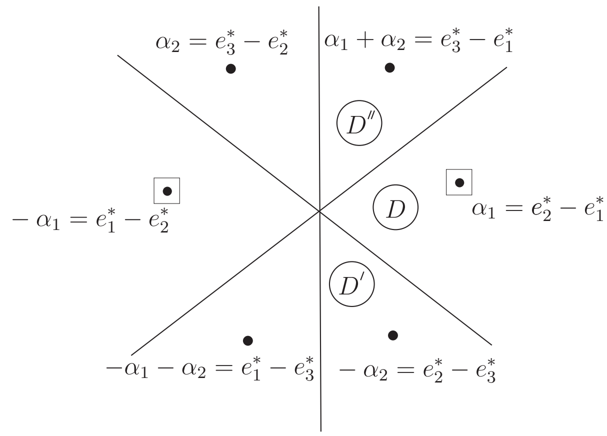


FIGURE 1.

The Weyl chambers are the indicated sectors, and the compact roots have a box around them. We will explain the notations D'', D and D' below.

Description of D**.** We will give three descriptions of D:

(i) *geometric*, when D is realized as the homogeneous complex manifold $D_S = S\mathcal{U}(2,1)_{\mathbb{R}}/T_S$;

(ii) *group theoretic*, when the complex structure on the homogeneous manifold $S\mathcal{U}(2,1)_{\mathbb{R}}/T_S$ is specified by a choice of Weyl chamber;

(iii) *Hodge theoretic*, when D is realized as a Mumford-Tate domain $D = \mathcal{U}(2,1)_{\mathbb{R}}/T$.

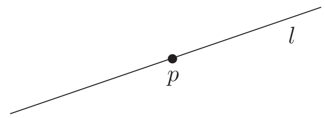
We begin with the description of the compact dual $\check{D} = \mathrm{GL}(3,\mathbb{C})/B$ where $\mathrm{GL}(3,\mathbb{C})$ is the complex Lie group $\mathrm{GL}(V_{+,\mathbb{C}})$ of which $\mathcal{U}(2,1)_{\mathbb{R}}$ is a particular real form. In fact, $\check{D}$ is just the flag manifold $\mathbb{F}(1,2)$ of flags $F_1 \subset F_2 \subset F_3 = V_{+,\mathbb{C}}$ where $\dim F_i = i$. We may realize $\check{D}$ geometrically as the incidence variety

$$\check{D} \subset \mathbb{P}^2 \times \check{\mathbb{P}}^2$$

where $\mathbb{P}^2 = \mathbb{P}V_{+,\mathbb{C}}$ and $\check{\mathbb{P}}^2 = \mathbb{P}\check{V}_{+,\mathbb{C}} = \mathbb{P}V_{-,\mathbb{C}}$, the latter identification coming from

$$\check{V}_+ \cong V_-$$

using the form Q. The points of D are



where $p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$ and $l = [l_1, l_2, l_3]$.

We set $G_{\mathbb{C}} = \mathrm{GL}(V_{+,\mathbb{C}})$ and identify $G_{\mathbb{C}}$ with the set of frames $f_1, f_2, f_3 \in V_{+,\mathbb{C}} \cong \mathbb{C}^3$, the element of $g \in G_{\mathbb{C}}$ corresponding to the frame

$$f_i = ge_i\,.$$

We have the equations of a moving frame and Maurer-Cartan equation

$$\begin{cases} df_i = \omega_i^j \wedge f_j \\ d\omega_i^k = \omega_i^j \wedge \omega_j^k\,. \end{cases}$$

Here we are using the summation convention and thinking of f_i as a vector-valued map $f_i : G_{\mathbb{C}} \to \mathbb{C}^3$ with differential df_i. The *Maurer-Cartan matrix* $\omega = \|\omega_j^i\|$ is given by $\omega = g^{-1}dg$.

The domain D will be an open $\mathcal{U}(2,1)_{\mathbb{R}}$-orbit under the action of $\mathcal{U}(2,1)_{\mathbb{R}} \subset \mathrm{GL}(3,\mathbb{C})$ on $\check{D}$. To describe it, we have the unit ball

$$\mathbb{B} = \left\{ p \in \mathbb{P}^2 : H(p) < 0 \right\}\,.$$

The notation means that we choose a vector $v \in \mathbb{C}^3 \backslash \{0\}$ lying over p and require $H(v, \bar{v}) < 0$. In homogeneous coordinates this is

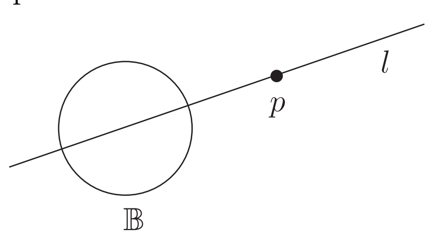
$$|p_1|^2 + |p_2|^2 < |p_3|^2\,.$$

We then have the

Geometric definition of D.

$$D = \{(p,l) : p \in \mathbb{B}^c,\ \ l \cap \mathbb{B} \neq \phi\}$$

where $\mathbb{B}^c$ is the complement in $\mathbb{P}^2$ of the closure of $\mathbb{B}$.



Using the notations from above, D is the $\mathcal{U}(2,1)_{\mathbb{R}}$-orbit of the reference flag

$$[e_1] \subset [e_1, e_3] \subset [e_1, e_3, e_2]$$

where $[*]$ denotes the span over $\mathbb{C}$ of the indicated vectors. We note that the point $[e_1] \in \mathbb{P}^2$ lies outside $\mathbb{B}$, and since $H(e_3, \bar{e}_3) < 0$ the line $[e_1, e_3] \in \check{\mathbb{P}}^2$ meets $\mathbb{B}$. The Borel subgroup that stabilizes the above flag is

$$B = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix} \right\}.$$

Group theoretic definition of D. As noted above, this is obtained by identifying the tangent space to $S\mathcal{U}(2,1)_{\mathbb{R}}/T_S$ at the identity coset e with $\mathrm{su}(2,1)_{\mathbb{R}}/\mathfrak{t}_S$ and choosing a set Φ^+ of positive roots whose root vectors X_α, $\alpha \in \Phi^+$, span the $(1,0)$ part of the tangent space

$$\begin{cases} T_e(S\mathcal{U}(2,1)_{\mathbb{R}}/T_S) \otimes \mathbb{C} \cong T^{1,0} \oplus \overline{T}^{1,0} \\ T^{1,0} = \mathrm{span}_{\mathbb{C}}\{X_\alpha : \alpha \in \Phi^+\}\,. \end{cases}$$

This defines an invariant almost-complex structure, which is integrable because the sum of positive roots is either zero or a positive root. From

$$T^{1,0} \cong \mathrm{gl}(3,\mathbb{C})/\mathfrak{b}$$

and the above picture of B, we see that the positive roots are

$$e_3^* - e_1^*,\ e_2^* - e_1^*,\ e_2^* - e_3^*\,.$$

The root picture of the positive roots is

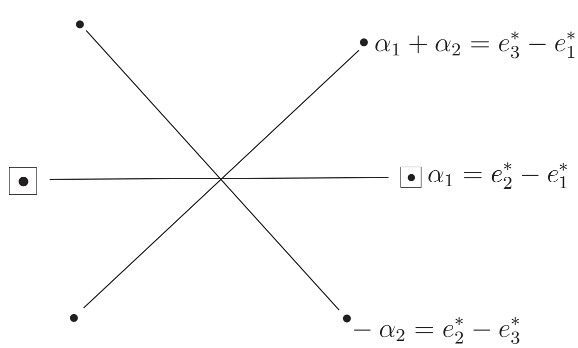


FIGURE 2

Using the correspondence between choices of positive roots and Weyl chambers, the Weyl chamber for this complex structure is the one labelled D in Figure 1 above.

Hodge-theoretic definition of D**.** We define a polarized Hodge structure of weight three on (V, Q) by taking as Hodge basis

$$\begin{cases} e_1 \text{ for } V^{3,0} & \bar{e}_1 \text{ for } V^{0,3} \\ e_3, \bar{e}_2 \text{ for } V^{2,1} & \bar{e}_3, e_2 \text{ for } V^{1,2}\,. \end{cases}$$

Since we want

$$i^3 Q(e_1, \bar{e}_1) > 0,\ iQ(e_3, \bar{e}_3) > 0 \text{ and } iQ(\bar{e}_2, e_2) > 0$$

we choose the minus sign in the above definition of H. The "picture" of the polarized Hodge structure is

$*$	$*$	$*$		V_+
	$*$	$*$	$*$	V_-
$V^{3,0}$	$V^{2,1}$	$V^{1,2}$	$V^{0,3}$	

where the number of $*$'s in a box is the dimension of the corresponding vector space $V_{\pm}^{p,q}$. Clearly this polarized Hodge structure has an $\mathbb{F}$-action. In fact, it is a complex multiplication (CM) polarized Hodge structure (cf. section IV.D).

We now let D be the set of polarized Hodge structures on (V, Q) with the above Hodge numbers and which admit an $\mathbb{F}$-action. It is easy to see that

- $\mathcal{U}(2,1)_{\mathbb{R}}$ acts transitively on this set with isotropy group T of the reference Hodge structure given above, so that
$$D = \mathcal{U}(2,1)_{\mathbb{R}}/T;$$
- the generic point of D has Mumford-Tate group $\mathcal{U}(2,1)$.

We observe that the circle

$$\varphi : S^1 \to \mathrm{Aut}_{\mathbb{F}}(V_{\mathbb{R}}, Q)$$

that gives polarized Hodge structure of the above type with

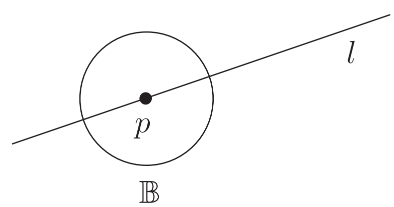
$$\varphi(z) = z^{p-q} \text{ on } V^{p,q}$$

has image $\varphi(S^1) \subset \mathcal{U}(2,1)_{\mathbb{R}}$ and not in $S\mathcal{U}(2,1)_{\mathbb{R}}$; in fact, $S\mathcal{U}(2,1)$ cannot be the Mumford-Tate group of an odd weight polarized Hodge structure (cf. [GGK1]).

Description of D', D'' and $\mathbb{B}$. There are two other $G_{\mathbb{R}}$-equivalence classes of open orbits of $\mathcal{U}(2,1)_{\mathbb{R}}$ acting on $\check{D}$, which we shall call D' and D''. We begin with a description of D'.

D'-geometric: It is given by

$$D' = \{(p,l) : p \in \mathbb{B},\ p \in l\}$$



We take as reference flag

$$[e_3] \subset [e_3, e_1] \subset [e_3, e_1, e_2]$$

with Borel group stabilizing this flag given by

$$B' = \left\{ \begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ * & * & * \end{pmatrix} \right\}$$

D'-group-theoretic: The corresponding positive roots are

$$e_2^* - e_1^*,\ e_2^* - e_3^*,\ e_1^* - e_3^*$$

with the picture having the positive roots labelled

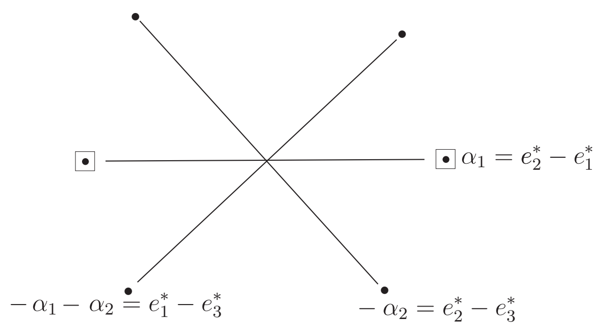


FIGURE 2′

The corresponding Weyl chamber is D' in Figure 1.

D'-Hodge theoretic: Here the minimal weight is five with all Hodge numbers $h^{p,q} = 1$, and a reference Hodge structure has the picture

*	*		*			V_+
		*		*	*	V_-
$V^{5,0}$	$V^{4,1}$	$V^{3,2}$	$V^{2,3}$	$V^{1,4}$	$V^{0,5}$	

Here

$$\begin{cases} e_3 \in V_+^{5,0} \\ e_1 \in V_+^{4,1} \\ e_2 \in V_+^{2,3} \end{cases}$$

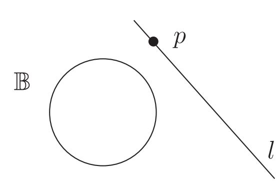
and we take the + sign for H to have for $i^5 Q(V^{5,0}, \overline{V}^{5,0}) > 0$

$$H(e_3, e_3) = iQ(e_3, \bar{e}_3) > 0$$

as well as $i^3 Q(e_1, \bar{e}_1) > 0$, $i^{-1} Q(e_2, \bar{e}_2) > 0$.

Description of D''.

Geometric: For the geometric description of D'' we have the picture



and $D'' = \{(p,l) \in \check{D} : l \subset \mathbb{B}^c\}$. Here we take as reference flag

$$[e_1] \subset [e_1, e_2] \subset [e_1, e_2, e_3]$$

with corresponding Borel subgroup

$$B'' = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\}.$$

Group-theoretic: The corresponding positive roots are

$$e_3^* - e_1^*,\ e_2^* - e_1^*,\ e_3^* - e_2^*$$

with the picture

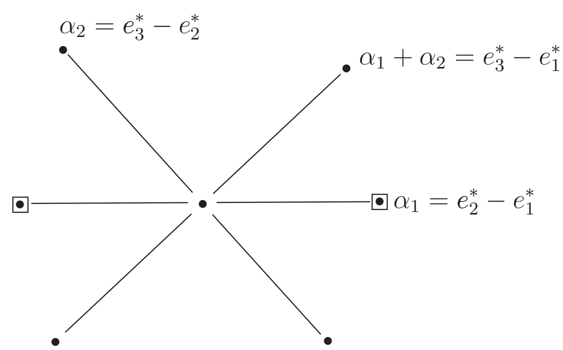


FIGURE $2''$

and Weyl chamber D'' in Figure 1.

Hodge-theoretic: Here the minimal weight is five with all Hodge numbers $h^{p,q} = 1$ and picture

		*		*	*	V_+
*	*		*			V_-

where

$$\begin{cases} e_1 \in V_+^{3,0} \\ e_2 \in V_+^{1,2} \\ e_3 \in V_+^{0,3}. \end{cases}$$

Note. We note that for D the positive compact is *not* simple, whereas for D' and D'' it is simple. It is for this reason that D will be related to the TDLDS for $S\mathcal{U}(2,1)_{\mathbb{R}}$, whereas D' and D'' will not (cf. [CK]).

The ball $\mathbb{B}$*:* As a homogeneous space

$$\mathbb{B} = \mathcal{U}(2,1)_{\mathbb{R}}/K$$

where K is the maximal compact subgroup. Taking a reference point $[e_3] \in \mathbb{P}^2$, we have

$$\mathbb{P}^2 = \mathrm{GL}(3,\mathbb{C})/P$$

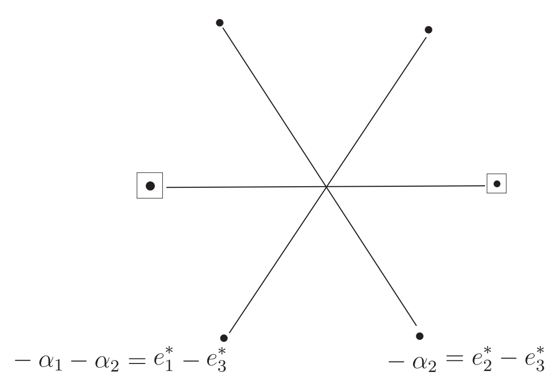
where P is the parabolic subgroup

$$\left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \right\},$$

and $\mathbb{B}$ is the open $\mathcal{U}(2,1)_{\mathbb{R}}$-orbit of $[e_3]$ where

$$K = \left\{ \begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix} : A \in \mathcal{U}(2),\ a \in \mathcal{U}(1) \right\}.$$

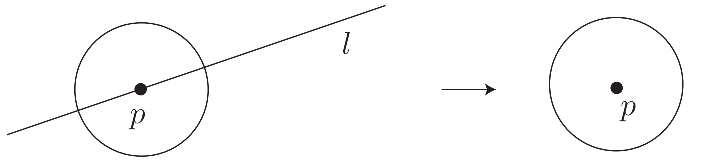
Identifying $T_{[e_3]}\mathbb{B}$ with $u(2,1)_{\mathbb{R}}/\mathfrak{k}$ with $u(2,1)$ the Lie algebra of $\mathcal{U}(2,1)$, after complexifying the $(1,0)$ tangent space $T^{1,0}_{[e_1]}\mathbb{B}$ of $\mathbb{B}$, the root vectors spanning $T^{1,0}_{[e_1]}\mathbb{B}$ correspond to the positive roots $e_2^* - e_3^*$ and $e_1^* - e_3^*$ with the picture



Comparing with Figure 2′ we see that there is an equivariant holomorphic fibration

$$D' \to \mathbb{B}$$

given geometrically by



From a Hodge-theoretic perspective, we may describe $\mathbb{B}$ as the Mumford-Tate domain for polarized Hodge structures of weight one on (V, Q) and which admit an $\mathbb{F}$-action. The picture is

$$\begin{array}{|c|c|l} \hline * & ** & V_+ \\ \hline ** & * & V_- \\ \hline \multicolumn{1}{c}{V^{1,0}} & \multicolumn{1}{c}{V^{0,1}} & \end{array}$$

where

$$\begin{cases} e_3 \in V_+^{1,0} \\ e_1, e_2 \in V_+^{0,1} \,. \end{cases}$$

As maps of Hodge structures, $D' \to \mathbb{B}$ is given by

$$\begin{cases} V_+^{1,0} = V^{5,0} \\ V_+^{0,1} = V^{4,1} \oplus V^{2,3} \end{cases}$$

and the conjugate of that.

In fact, it will be more convenient to interpret D' not as a Mumford-Tate domain as above, but rather as the $\mathbb{F}$-*Hodge flags* for the Mumford-Tate domain $\mathbb{B}$ as just described. Here an $\mathbb{F}$-Hodge flag for $V_{\mathbb{C}} = V^{1,0} \oplus \overline{V^{1,0}}$ is a partial flag

$$\{0\} \subset F_1 \subset V^{1,0}$$

where $\dim F_1 = 1$ and the partial flag is invariant under the action of $\mathbb{F}$. In fact,

$$F_1 = V_+ \cap V^{1,0} \,.$$

All of this will become particularly relevant when we relate CM points on D, D' and $\mathbb{B}$ in section IV.D.

I.C. **The second example** (Sp(4)). In this section the non-classical Mumford-Tate domain will be the period domain for polarized Hodge structures of weight $n = 3$ and all Hodge numbers $h^{p,q} = 1$, an example that has been much studied from the point of view of mirror symmetry.

Notations:

- V is a *complex* vector space of dimension four;
- $Q : V \otimes V \to \mathbb{C}$ is a non-degenerate alternating form;
- there is a basis $v_{-e_1}, v_{-e_2}, v_{e_2}, v_{e_1}$ for V such that $Q = \begin{pmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$. (The reason for this notation will appear below.)
- there is a complex conjugation $\boldsymbol{\sigma} : V \to V$ where $\boldsymbol{\sigma} v_{-e_1} = iv_{e_1}, \boldsymbol{\sigma} v_{-e_2} = iv_{e_2}$ (and then $\boldsymbol{\sigma} v_{e_2} = iv_{-e_2}, \sigma v_{e_1} = iv_{-e_1}$);
- There is a $\mathbb{Q}$-form $V_{\mathbb{Q}} \subset V$ given by $V_{\mathbb{Q}} = \mathrm{span}_{\mathbb{Q}}\{w_1, w_2, w_3, w_4\}$ where
$$\begin{cases} w_1 = \frac{1}{\sqrt{2}i}(v_{-e_1} - iv_{e_1}) \\ w_2 = \frac{1}{\sqrt{2}}(v_{-e_1} + iv_{e_1}) \\ w_3 = \frac{1}{\sqrt{2}i}(v_{-e_2} - iv_{e_2}) \\ w_4 = \frac{1}{\sqrt{2}}(v_{-e_2} + iv_{e_2}) \,. \end{cases}$$
The matrix $Q_{\mathbf{w}}$ of Q in this basis is
$$\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix};$$
- $H : V \otimes V \to \mathbb{C}$ is the Hermitian form $H(u, v) = iQ(u, \boldsymbol{\sigma} v)$. It has signature $(2, 2)$;
- $H(v, \boldsymbol{\sigma} v) = 0$ defines a real quadratic hypersurface Q_H in $\mathbb{P}V \cong \mathbb{P}^3$, which we picture as

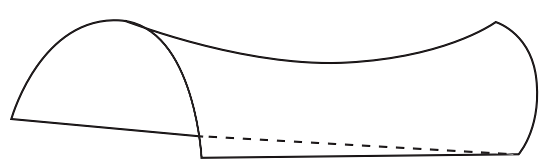


FIGURE 3

- $G_{\mathbb{C}} = \mathrm{Aut}(V, Q)$;
- $G_{\mathbb{R}} = \mathrm{Aut}_{\boldsymbol{\sigma}}(V, Q)$. Then $G_{\mathbb{R}}$ is a real form of $G_{\mathbb{C}}$ containing a compact maximal torus T;[26]
- $G_{\mathbb{R}}$ is also the subgroup of $\mathrm{GL}(V_{\mathbb{C}})$ that preserves both Q and H.

  Proof. For $g \in G_{\mathbb{C}} = \mathrm{Aut}(V_{\mathbb{C}}, Q)$ we have

$$\begin{aligned} H\big(g(v), g(w)\big) &= iQ\big(g(v), \boldsymbol{\sigma} g(w)\big) \\ &= iQ\big(g(v), {}^{\boldsymbol{\sigma}}g(\boldsymbol{\sigma}(w))\big) \end{aligned}$$

  where $g \in \mathrm{GL}(V_{\mathbb{C}}) \subset \check{V}_{\mathbb{C}} \otimes V_{\mathbb{C}}$ and ${}^{\boldsymbol{\sigma}}g$ is the induced conjugation

$$\begin{aligned} &= iQ\big(v, g^{-1}\, {}^{\boldsymbol{\sigma}}g(\boldsymbol{\sigma}(w))\big) \\ &= H\big(w, \boldsymbol{\sigma} g^{-1}\, {}^{\boldsymbol{\sigma}}g(\boldsymbol{\sigma}(w))\big)\,. \end{aligned}$$

  Since v is arbitrary this gives $g^{-1}\, {}^{\boldsymbol{\sigma}}g =$ identity or $g = {}^{\boldsymbol{\sigma}}g$, which was to be proved.
- $T_{\mathbb{C}}$ is given by

$$\begin{pmatrix} \lambda_1^{-1} & & & \\ & \lambda_2^{-1} & & \\ & & \lambda_2 & \\ & & & \lambda_1 \end{pmatrix};$$

- $v_{-e_1}, v_{-e_2}, v_{e_2}, v_{e_1}$ are the eigenvectors for the action of T;
- The weight lattice $\Lambda \cong \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, and the root lattice $R \cong \mathbb{Z}(e_1+e_2) \oplus \mathbb{Z}(e_1-e_2)$ (equivalently, it is $\{n_1e_1+n_2e_2 : n_1+n_2 \equiv 0(2)\}$;[27]
- With the usual ordering, $e_1 > e_2 > -e_2 > -e_1$, $v_{-e_1}, v_{-e_2}, v_{e_2}, v_{e_1}$ are the weight vectors for the indicated weights with the ordering inverse to that of the weights;

[26]In fact, $G_{\mathbb{R}} = \mathrm{Aut}(V_{\mathbb{R}}, Q_{\mathbf{w}}) \cong \mathrm{Sp}(4, \mathbb{R})$.
[27]In this example we use the customary notation [K1] for the roots and weights.

- The picture of the root diagram is

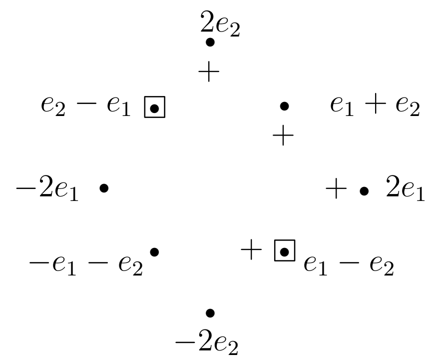


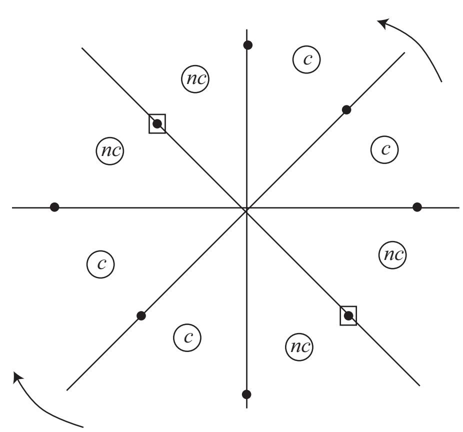
FIGURE 4

where the compact roots have a box around them.

- We denote by B_0 the Borel subgroup that stabilizes the *weight vector flag*

$$[v_{-e_1}] \subset [v_{-e_1}, v_{-e_2}] \subset [v_{-e_1}, v_{-e_2}, v_{e_2}] \subset [v_{-e_1}, v_{-e_2}, v_{e_2}, v_{e_1}]\,,$$

where [ ] denotes the span of the indicated vectors. With the choice of positive roots in Figure 4, $\mathfrak{b}_0$ is spanned by $\mathfrak{t}_{\mathbb{C}}$ and the negative root vectors $X_{-2e_2}, X_{-e_1-e_2}, X_{-2e_1}, X_{e_2-e_1}$.
- In terms of Weyl chambers one has the picture, where c means a classical complex structure and nc a non-classical one



The action of the compact Weyl group is pictured by  and ; thus there are four equivalence classes of homogeneous complex structures on $G_{\mathbb{R}}/T$.

- $G_L(V) \subset \mathrm{Grass}(2, V)$ is the set of Lagrangian planes in V. We picture points E of $G_L(V)$ as lines in $\mathbb{P}^3$ lying in a quadric. As a variety it is the transverse intersection of two non-singular quadrics in $\mathbb{P}(\Lambda^2 V)$.

Description of $\check{D}$.

Note. The two non-classical chambers are those for which the compact root is *not* simple. These correspond to the two TDLDS's for Sp(4).

Definition. A *Lagrangian flag* F is given by a flag

$$(0) \subset F_1 \subset F_2 \subset F_3 \subset F_4 = V, \qquad \dim F_j = j$$

where $F_2 = F_2^{\perp}$, $F_3 = F_1^{\perp}$, the $^{\perp}$ being with respect to Q.

The Lagrangian flag is determined by $F_1 \subset F_2$, and we denote by $\check{D}$ the set of Lagrangian flags. Then $\check{D} \to G_L(V)$ given by $(F_1, F_2) \to F_2$ is a $\mathbb{P}^1$-bundle. Upon choice of a reference flag we have an identification

$$\check{D} = G_{\mathbb{C}}/B$$

where B is a Borel subgroup of $G_{\mathbb{C}}$. The weight vector flag is a Lagrangian flag.

We may think of $G_{\mathbb{C}}$ as the set of *Lagrange frames* $\mathbf{f}_\bullet = (f_1, f_2, f_3, f_4)$, meaning those where $Q(f_i, f_j)$ is the matrix $\begin{pmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$ and where $[f_1] \subset [f_1, f_2] \subset [f_1, f_2, f_3] \subset [f_1, f_2, f_3, f_4]$ is a Lagrange flag. The principal bundle $G_{\mathbb{C}} \to \check{D}$ is given by $\mathbf{f}_\bullet \to ([f_1], [f_1, f_2])$, where $[\quad]$ denotes the span.[28]

We have the equations of a moving frame (using summation convention)

$$\begin{cases} df_i = \omega_i^j f_j \\ d\omega_i^k = \omega_i^j \wedge \omega_j^k \end{cases}$$

[28]In particular, for $0 \neq f \in V$, $[f] \in \mathbb{P}V$, and for linearly independent $f_1, f_2 \in V$, $[f_1 \wedge f_2] \in \mathbb{P}\Lambda^2 V$ gives the Plücker coordinates of $[f_1, f_2]$.

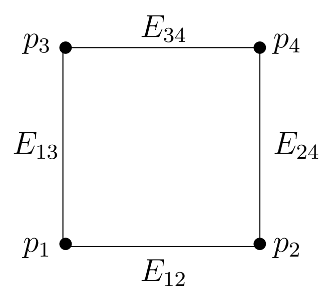
where the *Maurer-Cartan forms* ω_i^j on $G_{\mathbb{C}}$ satisfy

$$\text{(I.C.1)} \qquad \begin{cases} \omega_1^3 = \omega_2^4, & \omega_3^1 = \omega_4^2 \\ \omega_1^1 + \omega_4^4 = 0, & \omega_2^2 + \omega_3^3 = 0 \\ \omega_1^2 + \omega_3^4 = 0, & \omega_2^1 + \omega_4^3 = 0\,. \end{cases}$$

By definition, upon choice of a *projective Lagrange frame* $[\mathbf{f}_\bullet] =: ([f_1],[f_2],[f_3],[f_4])$ where $\mathbf{f}_\bullet = (f_1, f_2, f_3, f_4)$ is a Lagrange frame, we have an identification of sets

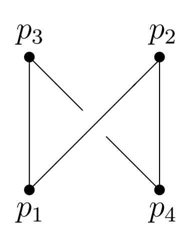
$$\{\text{projective Lagrange frames}\} \longleftrightarrow G_{\mathbb{C}}/T_{\mathbb{C}}\,.$$

Definition. A *Lagrange quadrilateral* (LQ) is given by the following picture associated to a projective Lagrange frame $\mathfrak{p} = (p_1, p_2, p_3, p_4)$ where $p_j = [f_j]$



The lines $E_{ij} = \overline{p_i p_j}$ drawn in are Lagrangian, and the omitted lines E_{ij} are not Lagrangian.

An alternative, perhaps more suggestive but for our purposes less convenient, picture is



The Weyl group W is of order 8 and the Weyl group W_K is of order 2 in W. We may realize W as the group of symmetries of a square given by the above Lagrange quadrilateral.

Description of D. As in the case of the first example we will give three descriptions of D

(i) geometric;
(ii) group theoretic;
(iii) Hodge theoretic.

Referring to figures 3 and 4 the position of a Lagrange quadrilateral relative to the real hyperquadric Q_H may be pictured as

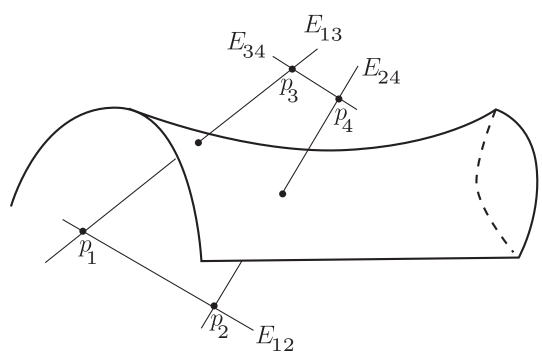


FIGURE 5

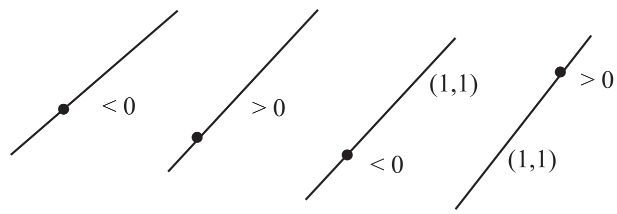
that is, the pictured Lagrangian lines E_{ij} are of three types

- E_{12} lies "inside" Q_H, meaning that $H < 0$ on the corresponding Lagrangian 2-plane $\tilde{E}_{12}$ in V;
- E_{13} meets Q_H in a real circle; as a consequence H has signature $(1,1)$ on $\tilde{E}_{13}$; E_{24} has a similar property;[29]
- E_{34} lies "outside" Q_H, meaning that $H > 0$ on $\tilde{E}_{34}$.

There are eight orbits of the four Lagrange flags in the above picture; thus we have (p_1, E_{12}) and (p_2, E_{12}) associated to E_{12}. These orbits give eight complex structures on $G_{\mathbb{R}}/T$, of which four pairs are equivalent under the action of W_K. The four types may be pictured as the orbits

[29]We do not include the case where the circle has shrunk to a point, because as will be seen below any Lagrangian line meets Q_H transversely in a proper circle.

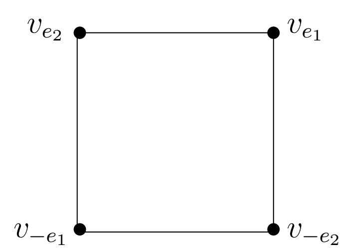
of



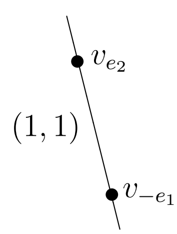
The notations mean that $H < 0$, $H > 0$ on the first two, H has signature $(1,1)$ on the second two, and the two where H has signature $(1,1)$ we have indicated the sign of H on the marked points.

From the previously noted fact that $G_{\mathbb{R}}$ is the subgroup of $\mathrm{GL}(V_{\mathbb{C}})$ that preserves both Q and H, we infer that $G_{\mathbb{R}}$ *acts transitively on pointed Lagrange lines of each of the above four types.* These are the four inequivalent complex structures on $G_{\mathbb{R}}/T$.

We now choose as reference Lagrange quadrilateral



Definition. The domain D is the $G_{\mathbb{R}}$-orbit of the Lagrange flag $([v_{-e_1}], [v_{-e_1}, v_{e_2}])$



From a group-theoretic view, D is the homogeneous complex manifold given by the set of positive roots as indicated in the following figure:
As will be explained below, the shaded Weyl chamber is one of those in which we "expect" to have non-zero L^2-cohomology group $H^1_{(2)}(D, L_\mu)$

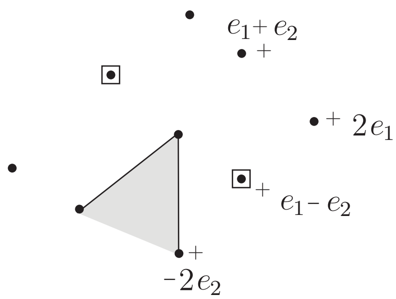


FIGURE 6

where μ is a weight such that $\mu+\rho$ is a non-singular element of that Weyl chamber.

The Borel subgroup of $G_{\mathbb{C}}$ that stabilizes the reference frame is given by matrices in $\mathfrak{g}_{\mathbb{C}}$ of the form

$$\begin{pmatrix} * & * & * & * \\ 0 & * & 0 & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}. \tag{I.C.2}$$

Remark that $\mathfrak{g}_{\mathbb{C}}$ consists of matrices

$$\begin{pmatrix} a_{11} & a_{21} & c_{11} & c_{21} \\ a_{12} & a_{22} & c_{12} & c_{22} \\ b_{11} & b_{21} & -a_{22} & -a_{21} \\ b_{12} & b_{11} & -a_{12} & -a_{11} \end{pmatrix}, \tag{I.C.3}$$

so that (I.C.2) is given by the four conditions

$$a_{12}=0,\ \ b_{11}=0,\ \ b_{12}=0,\ \ c_{12}=0$$

which is the right count.

From a Hodge theoretic perspective D *is the period domain for polarized Hodge structures of weight* $n=3$ *and with all Hodge numbers* $h^{p,q}=1$. For the reference polarized Hodge structure we have

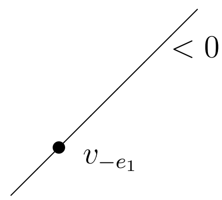
$$\begin{cases} V^{3,0}=[v_{-e_1}], & V^{0,3}=[v_{e_1}] \\ V^{2,1}=[v_{e_2}], & V^{1,2}=[v_{-e_2}]\,. \end{cases}$$

The conjugation $\overline{V}^{p,q} = V^{q,p}$ is understood as

$$\boldsymbol{\sigma} V^{p,q} = V^{q,p}\,.$$

Description of D'.

Geometric: D' is the $G_{\mathbb{R}}$-orbit of



Group theoretic: The analogue of Figure 6 is

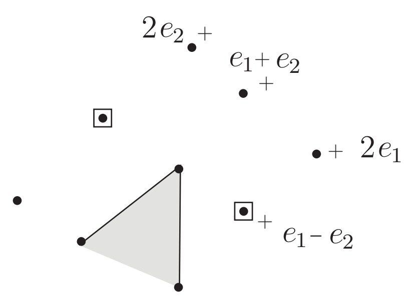


FIGURE 6′

The shaded Weyl chamber is the one where we expect to have non-zero $H^0_{(2)}(D', L'_{\mu'})$.

The Borel subgroup at the reference flag $([v_{-e_1}], [v_{-e_1}, v_{-e_2}])$, denoted by B_0 above, is given by matrices (cf. (I.C.3) above)

$$\begin{pmatrix} a_{11} & a_{21} & c_{11} & c_{21} \\ 0 & a_{22} & c_{12} & c_{22} \\ 0 & 0 & -a_{22} & -a_{21} \\ 0 & 0 & 0 & -a_{11} \end{pmatrix}. \tag{I.C.4}$$

Referring to (I.C.3) it is defined by the four equations

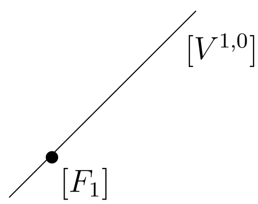
$$a_{12} = 0, \;\; b_{11} = 0, \;\; b_{12} = 0, \;\; b_{21} = 0\,.$$

From a Hodge theoretic perspective one possibility is to realize D' as a sub-domain for polarized Hodge structures of weight $n = 5$ and with Hodge numbers $h^{5,0} = h^{0,5} = 1$, $h^{4,1} = h^{1,4} = 0$, $h^{3,2} = h^{2,3} = 1$. In fact, it will be more convenient to interpret D' not as this Mumford-Tate domain, but rather as the Hodge flags for the Mumford-Tate domain to be described now.

The Siegel space $\mathcal{H}$.

Definition. We define $\mathcal{H}$ to the open set in $G_L(2, V)$ consisting of Lagrange lines $E \subset \mathbb{P}^3$ with $H < 0$ on E.

By reversing the sign of H we see that $\mathcal{H}$ is biholomorphic to the Siegel generalized upper plane $\mathcal{H}_g$ for $g = 2$, which is the Mumford-Tate domain for polarized Hodge structures of weight $n = 1$ and with Hodge number $h^{1,0} = 2$. Given $V^{1,0} \in \mathcal{H}$, a *Hodge flag* is a line $F_1 \subset V^{1,0}$; projectively we have the picture



where the brackets represent the projectivezation of $V^{1,0}$ and F_1. There is an obvious map

$$D' \to \mathcal{H}$$

that represents D' as the $\mathbb{P}^1$-bundle $\mathbb{P}\mathbb{V}^{1,0}$ over $\mathcal{H}$.

CR geometry of Q_H. The CR geometry of the real hyperquadric Q_H in $\mathbb{P}^3$ given by

$$|z_1|^2 + |z_2|^2 = |z_3|^2 + |z_4|^2$$

is standard. The *Levi form* $\mathcal{L}$, which is a $(1,1)$ form defined at each $p \in Q_H$ on

$$T_p^{\mathrm{CR}} Q_H =: T_{\mathbb{R},p} Q_H \cap J T_{\mathbb{R},p} Q_H \cong \mathbb{C}^2$$

where $T_{\mathbb{R}} Q_H$ is the real tangent space and $J T_{\mathbb{R},p} Q_H$ is its image under the almost-complex structure J in the real tangent bundle to D. $\mathcal{L}$ is is an Hermitian form of signature $(1,1)$. The regions $\Omega^{\pm}$ in $\mathbb{P}^3$ where $H > 0$, $H < 0$ "look the same"; at the boundary each is "1/2 pseudo-convex and 1/2 pseudoconcave". However, when one adds the additional condition that $H(u,v) = iQ(u, \boldsymbol{\sigma} v)$ where Q is an alternating form and $\boldsymbol{\sigma}$ is the conjugation described above ($\boldsymbol{\sigma}$ is *not* the standard conjugation on $\mathbb{C}^4$), then an additional feature arises in the CR geometry.

(I.C.5) **Proposition:** *At each point of Q_H, the null-direction of $\mathcal{L}$ on $T_p^{\mathrm{CR}} Q_H$ is a Lagrangian line contained in Q_H.*

Proof. By the homogeneity of Q_H under the group $G_{\mathbb{R}}$, it will suffice to verify the result at one point p. From the beginning of this section we have, up to a factor of i,

$$Q(z, \tilde{z}) = (z_1 \tilde{z}_4 - z_4 \tilde{z}_1) + (z_2 \tilde{z}_3 - z_3 \tilde{z}_2)\,.$$

Then

$$[t_0, t_1] \to [t_0, t_1, -t_1, t_0]$$

gives a Lagrangian line contained in Q_H.

To explain where this comes from, let $p = [1,0,0,1]$ and choose the affine coordinates $(u,v,w) \to [u,v,w,1]$ in a neighborhood of p. The form H is

$$|u|^2 + |v|^2 - |w|^2 - 1$$

and at the point p given by $u = 1, v = w = 0$

$$T_p^{\mathrm{CR}} Q_H = \mathrm{span}\{\partial/\partial v, \partial/\partial w\}$$

and the Levi form is

$$\mathcal{L} = i(dv \wedge d\overline{v} - dw \wedge d\overline{w})\,.$$

Then at p and for the tangent vector $X = \partial/\partial v - \partial/\partial w$ we have

$$\mathcal{L}(X, \overline{X}) = 0\,.$$

The above Lagrangian line is given in affine coordinates $t_0 = 1$, $t_1 = t$ by

$$t \to (1, t, -t)$$

with tangent X at p. $\square$

(I.C.6) **Corollary:** *Let E be a Lagrangian line in $\mathbb{P}^3$ that meets, but does not lie in, Q_H. Then E meets Q_H transversely in a circle.*

In other words, Lagrangian lines not in Q_H meet Q_H transversely. This will be important when we prove below that the correspondence space $\mathcal{W}$ for D is Stein.

II. Homogeneous line bundles over the Mumford-Tate domains

II.A. **Generalities on homogeneous vector bundles.** We will be dealing with homogeneous vector bundles over a homogeneous manifold $M = G/H$ where G is a Lie group and H is a closed subgroup. A *homogeneous vector bundle* is a vector bundle $\mathbb{E} \to M$ together with a G-action such that for $g \in G$

$$\begin{tikzcd} \mathbb{E} \arrow[r, "g"] \arrow[d] & \mathbb{E} \arrow[d] \\ M \arrow[r, "g"] & M \end{tikzcd}$$

commutes. There is an evident notion of a morphism between homogeneous manifolds and of a morphism between homogeneous vector bundles. We shall be working in the setting of holomorphic homogeneous vector bundles over homogeneous complex manifolds. The following are three observations.

(i) *Given a representation $r : H \to \mathrm{GL}(E)$ there is an associated homogeneous vector bundle*

$$\mathbb{E} = G \times_H E$$

where $(g, e) \sim (gh^{-1}, r(h)e)$. Any homogeneous vector bundle is trivial as a homogeneous vector bundle if, and only if, r is the restriction to H of a representation $r : G \to \mathrm{GL}(E)$.

Remark. We shall be working in the holomorphic setting, and as noted at the beginning of section I, there will be two cases for G/H. One is when G is a real, reductive Lie group and H is a compact subgroup that contains a compact maximal torus. The other is when G is a complex reductive Lie group and H is a parabolic subgroup.

(ii) *Two holomorphic homogeneous vector bundles may be equivalent as holomorphic vector bundles but not as homogeneous vector bundles.*

(iii) *The set of equivalence classes of holomorphic homogeneous vector bundles depends on the particular representation of $M = G/H$ as a homogeneous complex manifold.*

We will illustrate this below.

II.B. **Homogeneous line bundles over the first example.** In this section we will use the notations

$$D = \mathcal{U}(2,1)_{\mathbb{R}}/T,\ D_S = S\mathcal{U}(2,1)_{\mathbb{R}}/T_S,\ D_{\mathrm{ad}} = S\mathcal{U}(2,1)_{\mathrm{ad},\mathbb{R}}/T_{\mathrm{ad}}\,.$$

As complex manifolds these are all the same, but as *homogeneous* complex manifolds they are different. The maps $S\mathcal{U}(2,1) \hookrightarrow \mathcal{U}(2,1)$ and $S\mathcal{U}(2,1) \twoheadrightarrow S\mathcal{U}(2,1)_{\mathrm{ad}}$ induce maps of homogeneous complex manifolds

$$\begin{array}{c} D_S \hookrightarrow D \\ \downarrow\!\!\downarrow \\ D_{\mathrm{ad}}\ . \end{array}$$

We shall denote by $\mathrm{Pic}_h(*)$ the equivalence classes of holomorphic homogeneous line bundles in each case. We then have the

(II.B.7) **Proposition:** *The above maps induce*

(i) $\mathrm{Pic}_h(D) \xrightarrow{\sim} \mathrm{Pic}_h(D_S)$

(ii) $0 \to \mathrm{Pic}_h(D_{\mathrm{ad}}) \to \mathrm{Pic}_h(D_S) \to \mathbb{Z}/3\mathbb{Z} \to 0.$

Proof. For a torus $T^\# = \mathfrak{t}^\#/\Lambda^\#$ one has the usual description

$$X(T^\#) \cong \mathrm{Hom}(\Lambda^\#, \mathbb{Z})$$

of the characters. In each of the above three cases, $\mathrm{Pic}_h(*)$ is isomorphic as an abelian group to the corresponding character group, modulo those characters that come by restriction from the whole group.

For (i) we have

$$0 \to \Lambda_S \to \Lambda \xrightarrow{\pi} \mathbb{Z} \to 0$$

where $\pi\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = n_1 + n_2 + n_3$. This gives

$$0 \to \mathbb{Z} \xrightarrow{\check{\pi}} X(T) \to X(T_S) \to 0$$

where $\check{\pi}(1) = \det$. By (i) in the previous section we obtain (i) in the proposition.

For (ii) we have

$$0 \to \Lambda_S \to \Lambda_{\mathrm{ad}} \to \mathbb{Z}/3\mathbb{Z} \to 0$$

which gives

$$\begin{array}{ccccccc} 0 \to & \operatorname{Hom}(\Lambda_{\mathrm{ad}}, \mathbb{Z}) & \to & \operatorname{Hom}(\Lambda, \mathbb{Z}) & \to & \mathbb{Z}/3\mathbb{Z} \to 0 \\ & \wr\| & & \wr\| & & \\ & X(T_{\mathrm{ad}}) & & X(T_S)\,. & & \end{array}$$ □

There are three types of homogeneous line bundles over D that we shall consider.

(a) the line bundles $F_{(a,b)}$ obtained by restricting $\mathcal{O}_{\mathbb{P}^2}(a) \boxtimes \mathcal{O}_{\check{\mathbb{P}}^2}(b)$;
(b) the line bundles $L_{\mathfrak{k}}$ obtained from the character

$$\chi_{\mathfrak{k}}(g) = e^{2\pi i \langle \mathfrak{k}, \boldsymbol{\theta} \rangle} = e^{2\pi i (k_1\theta_1 + k_2\theta_2 + k_2\theta_3)}$$

where $\mathfrak{k} = (k_1, k_2, k_3) \in \check{\mathbb{Z}}^3 = \operatorname{Hom}(\Lambda, \mathbb{Z})$;
(c) the Hodge bundles $\mathbb{V}^{p,q}$.

Remark. Because of (ii) in the proposition not all of these homogeneous line bundles will exist over D_{ad}. For example, the condition that $L_{\mathfrak{k}}$ exist on D_{ad} is

$$k_1 + k_2 + k_3 = 3m$$

for some integer m.

We will now give the relations among the line bundles of types a), b) and c) above. First, we note that as homogeneous $\mathcal{U}(2,1)_{\mathbb{R}}$ line bundles

$$L_{\mathfrak{k}} \cong L_{\mathfrak{k}'} \Leftrightarrow \mathfrak{k} = \mathfrak{k}' + m(1,1,1)\,.$$

Taking $m \in \left(\frac{1}{3}\right)\mathbb{Z}$, we can normalize so that $k_1 + k_2 + k_3 = 0$, in which we can write $\mathfrak{k}$ as a linear combination of the roots

$$\begin{cases} e_2^* - e_1^* = \alpha_1 \\ e_3^* - e_2^* = \alpha_2 \, . \end{cases}$$

(II.B.8) **Proposition:** *We have*

$$F_{(a,b)} = L_{\left(\frac{2a+b}{3}\right)(e_2^*-e_1^*)+\left(\frac{a-b}{3}\right)(e_3^*-e_2^*)} = L_{\left(\frac{2a+b}{3}\right)\alpha_1+\left(\frac{a-b}{3}\right)\alpha_2}.$$

Proof. The fibre $F_{(-1,0)}$ at the reference flag is the line $[e_1]$. The character of T acting on the line is $e^{2\pi i\theta_1}$ whose differential, using our conventions, is e_1^*. Thus $F_{(-1,0)} = L_{e_1^*}$. Similarly, the fibre of $F_{(0,-1)}$ is the line $[e_2]^\perp \subset \check{\mathbb{C}}^3$, which corresponds to $-e_2^*$. Thus

$$F_{(a,b)} = L_{-ae_1^*+be_2^*} \, .$$

To normalize as above we take $m = \frac{1}{3}(b-a)$, so that the normalized

$$\mathfrak{k} = \frac{1}{3}(-2a-b, 2b+a, a-b) \, .$$

Note that $\mathfrak{k} + m(1,1,1) = (-a, b, 0)$. Then

$$\begin{aligned} \mathfrak{k} &= \frac{1}{3}(-2a-b, 2a+b, 0) + \frac{1}{3}(0, -a+b, a-b) \\ &= \left(\frac{2a+b}{3}\right)(e_2^* - e_1^*) + \left(\frac{a-b}{3}\right)(e_3^* - e_2^*) \, . \end{aligned}$$

□

Note: The normalized $\mathfrak{k}$ above is in the weight lattice for $S\mathcal{U}(2,1)_{\mathbb{R}}$. The necessary and sufficient condition that it be in the root lattice, so that $F_{(a,b)}$ is a homogeneous line bundle in D_{ad}, is $a \equiv b \pmod 3$.

Remark. In [C2], pages 309ff., we observe that in the notation there where $e_1(\mathrm{diag}(x,y,z)) = x - y$ corresponds to our $e_1^* - e_2^*$, one finds $F_{e_1} = F_{(-1,-1)}$, which is in agreement with the above.

Of special interest are the $F_{(a,b)}$ which are *anti-dominant* in the sense that the corresponding character is in the anti-dominant Weyl chamber.

Proposition. *$F_{(a,b)}$ is anti-dominant for D if, and only if,*

$$\begin{cases} 2a + b < 0 \\ a + 2b < 0 \, . \end{cases}$$

Proof. We write

$$x\alpha_1 + y\alpha_2 = \mu(\alpha_1 + \alpha_2) + \lambda(-\alpha_2)\,.$$

The anti-dominant Weyl chamber is given by

$$\mu < 0,\ \lambda < 0\,.$$

In terms of x, y these conditions are

$$x < 0,\ x < y\,.$$

Taking $x = \frac{2a+b}{3}$, $y = \frac{a-b}{3}$ gives the result. □

From the theory of the discrete series one is particularly interested in the $F_{(a,b)}$ such that $F_{(a,b)} \otimes L_\rho$ is anti-dominant, where $\rho = \frac{1}{2}$ (sum of the positive roots) $= \alpha_1$.

Proposition. *$F_{(a,b)} \otimes L_\rho$ is anti-dominant if, and only if,*

$$\begin{cases} 2a + b + 3 < 0 \\ a + 2b + 3 < 0\,. \end{cases}$$

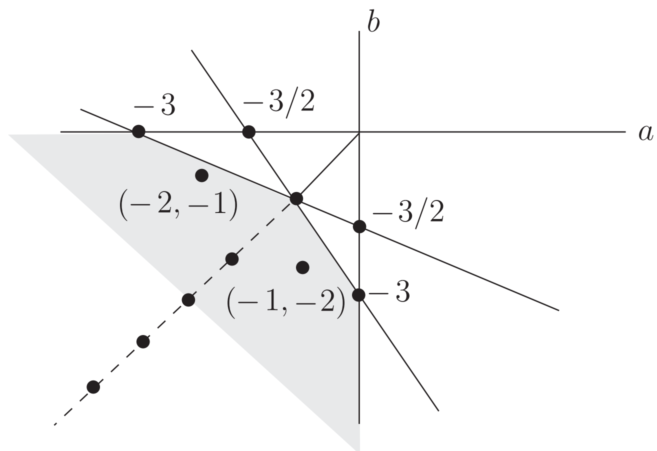
Proof. This follows from the previous proposition and $L_\rho = F_{(1,1)}$, where this equality follows from the proposition above where $\rho = \alpha_1$. □

We note that the *canonical line bundle*

$$\omega_D = L_{-2\rho} = F_{(-2,-2)}\,.$$

Corollary. *$\omega_D^{\otimes k} \otimes L_\rho$ is the anti-dominant Weyl chamber for $k \geqq 2$.*

The picture in the (a,b) plane of the $F_{(a,b)}$ such that $F_{(a,b)} \otimes L_\rho$ is anti-dominant is



The dots on the dotted line are $(-1,-1),(-3,-3),(-5,-5),\dots$. The minimal integral (a,b) in the shaded region are the points

$$\begin{cases} a=-2, & b=-1 \\ a=-1, & b=-2\,. \end{cases}$$

The dots on the dashed line are the powers $\omega_D^{\otimes k}$, $k \geqq 1$. This diagram will appear when we discuss the Penrose transforms of Picard modular forms.

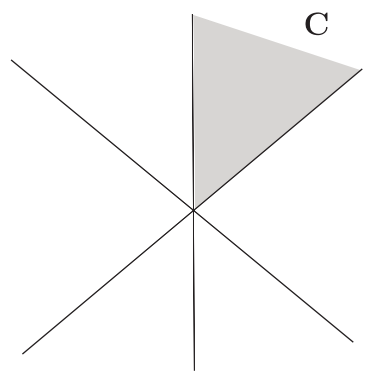
Turning to the Hodge bundles $\mathbb{V}_{\pm}^{p,q} \to D$, we have from the choice $[e_1] \subset [e_1,e_3] \subset [e_1,e_3,e_2]$ of a reference flag and definition of the fibres of the Hodge bundles at that point that

$$\begin{cases} \mathbb{V}_+^{3,0} = L_{(1,0,0)} \\ \mathbb{V}_+^{2,1} = L_{(0,0,1)} \\ \mathbb{V}_+^{1,2} = L_{(0,1,0)}\,. \end{cases}$$

Note that this is consistent with $\overset{p,q}{\otimes}\, \mathbb{V}_+^{p,q} \cong \det(\mathbb{V}_{+,\mathbb{C}})$, with $\det(V_{+,\mathbb{C}}) = V_{+,\mathbb{C}} \times D$ being trivial as a homogeneous line bundle. We note also that

$$\mathbb{V}_+^{3,0} = F_{(-1,0)}\,.$$

For later use we want to focus on another Weyl chamber, namely, the shaded one in



The reason is that this is the Weyl chamber where both $H^1_{(2)}(D, L_\mu)$ and $H^0_{(2)}(\mathbb{B}, \omega_{\mathbb{B}}^{\otimes k})$, $k \geqq 2$, will be non-zero. We recall from [Schm1] that the sufficient condition to have $H^1_{(2)}(D, L_\mu)$ non-zero is that $\rho + \mu$ should be non-singular, and that

$$q(\mu+\rho) := \# \left\{\alpha \in \Phi_c^+ : (\mu+\rho, \alpha) < 0\right\} + \# \left\{\beta \in \Phi_{nc}^+ : (\mu+\rho, \beta) > 0\right\}$$

should be equal to one.

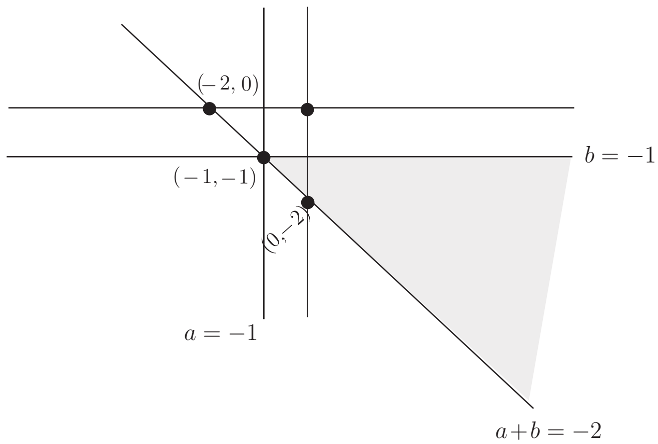
(II.B.9) **Proposition:** *The above condition for* $\mu = x\alpha_1 + y\alpha_2$ *on* $L_{x\alpha_1+y\alpha_2}$ *is*

$$\begin{cases} x+1+y > 0 \\ x+1 < 2y \\ 2x+2 > y\,. \end{cases}$$

In terms of $F_{(a,b)}$ *the conditions are*

$$\begin{cases} a+1 > 0 \\ b+1 < 0 \\ a+b+2 > 0\,. \end{cases}$$

The picture in (a,b) space is



Conclusion: *The positive root is* $\alpha =: e_2^* - e_1^*$ *and* $E, F = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, *and* $H = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ *is a basis for* sl_2.

- the line bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$ has fibre at the reference point the line $[e_1]$. Then

$$h_x \cdot e_1 = xe_1\,,$$

so that identifying characters with weights

$$\mathcal{O}_{\mathbb{P}^1}(-1) \longleftrightarrow \text{weight } e_1^*\,.$$

Here, we are identifying $\check{\mathfrak{h}}$ with $\check{\mathbb{C}}^2/\{e_1^* + e_2^* = 0\}$.

As a check on the signs, we have for the canonical bundle the two descriptions

$$\begin{cases} \omega_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2) \\ \omega_{\mathbb{P}^1} \longleftrightarrow \text{homogeneous line bundle with weight } -\alpha\,. \end{cases}$$

The second is because $E \in T^{1,0}\mathbb{P}^1$ has weight α. Since $-\alpha = e_1^* - e_2^* = 2e_1^*$, the two descriptions of $\omega_{\mathbb{P}^1}$ agree.

As a final check, when we restrict homogeneous line bundles to $\mathcal{H}$, the curvature reverses sign. Thus, $\omega_{\mathcal{H}} \cong \mathcal{O}_{\mathbb{P}^1}(-2)\big|_{\mathcal{H}}$ has positive curvature (= negative Gaussian curvature).

Homogeneous line bundles over D' and $\mathbb{B}$. We shall denote the homogeneous line bundles over D' by $F'_{(a,b)}, L'_{\mathfrak{k}}$ etc. We recall our reference flag $[e_3] \subset [e_3, e_1] \subset [e_3, e_1, e_2]$ for which D' is a $\mathcal{U}(2,1)$-orbit in $\check{D}$.

(II.B.10) **Proposition:** *We have*

$$F'_{(a,b)} = L'_{\frac{1}{3}(a-b,2b+a,-2a-b)} = L'_{\frac{1}{3}(b-a)\alpha_1+\frac{1}{3}(-2a-b)\alpha_2}\,.$$

Proof. Using the reference flag gives

$$F'_{(1,0)} = L'_{-e_3^*}, F'_{(0,1)} = L'_{e_2^*}$$

so that

$$F'_{(a,b)} = L'_{-ae_3^*+be_2^*}\,.$$

As in the case of D above, $m = \frac{1}{3}(b-a)$ and the normalized $\mathfrak{k}' = (0, b, -a) \oplus (m, m, m)$ is

$$\begin{aligned} \mathfrak{k}' &= \frac{1}{3}(a-b, 2b+a, -2a-b) \\ &= \frac{(b-a)}{3}(e_2^* - e_1^*) + \left(\frac{-2a-b}{3}\right)(e_3^* - e_2^*)\,. \end{aligned}$$ □

The condition that $F'_{(a,b)}$ be anti-dominant is

$$F'_{(a,b)} = L_{k\alpha_1+l(-\alpha_1-\alpha_2)}, \qquad k, l < 0\,.$$

This gives the same conditions

$$a + 2b < 0,\ 2a + b < 0$$

as for D.

For D', we have

$$\rho' = \frac{1}{2}(\alpha_1 - \alpha_2 - \alpha_1 - \alpha_2) = -\alpha_2$$

which gives for the canonical bundle

$$\omega_{D'} = L'_{\alpha_2} = F'_{(-1,-1)}\,.$$

The conditions in terms of a and b that $F'_{(a,b)} \otimes L'_{\rho'}$ be anti-dominant are again the same as for D.

We next ask the question:

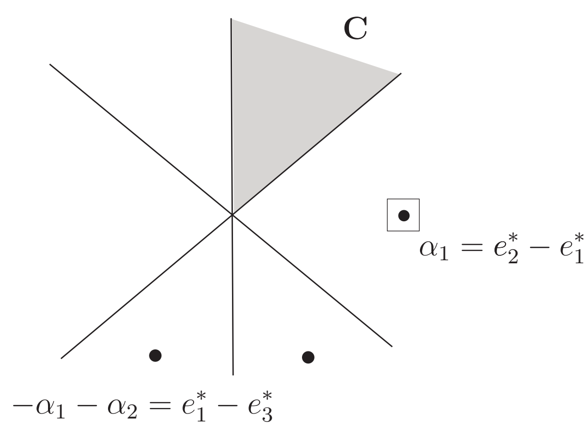
Which Weyl chamber $\mathbf{C}$ has the property that for a normalized $\mathfrak{k}' \in \mathbf{C}$

$$\begin{cases} (\mathfrak{k}' + \rho', \alpha_1) > 0 \\ (\mathfrak{k}' + \rho', -\alpha_2) < 0, \quad (\mathfrak{k}' + \rho', -\alpha_1 - \alpha_2) < 0\,? \end{cases}$$

From [Schm1] these are the conditions that give $H^0_{(2)}(D', L'_{\mathfrak{k}'}) \neq 0$. The first and second imply the third. We write

$$\mathfrak{k}' = k(e_2^* - e_1^*) + l(e_3^* - e_2^*) = k\alpha_1 + l\alpha_2\,.$$

From the picture



and setting $\mu = \mathfrak{k}' + \rho' = k\alpha_1 + (l-1)\alpha_2$ we want the conditions that give

$$(\mu, \alpha_1) > 0,\ (\mu, -\alpha_2) < 0\,.$$

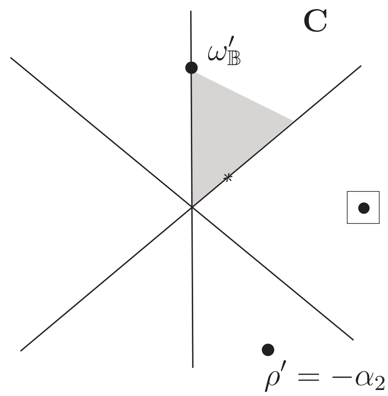
These work out to be

$$\begin{cases} 2k + 1 > l \\ k + 2 < 2l\,. \end{cases}$$

We are interested in the sections in $H^0_{(2)}(\mathbb{B}, \omega_{\mathbb{B}}^{\otimes k})$. From the Leray spectral sequence, and denoting by $\omega'_{\mathbb{B}}$ the pullback of $\omega_{\mathbb{B}}$ to D', we want $H^0_{(2)}(D', \omega_{\mathbb{B}}^{\prime\otimes k}) \neq 0$. Now on D'

$$\omega'_{\mathbb{B}} = L'_{2\alpha_2 + \alpha_1} = L'_{2e_3^* - e_1^* - e_2^*}\,.$$

The picture for $\omega'_{\mathbb{B}}$ is given by the dot



We observe that $\omega'_{\mathbb{B}}$ is orthogonal to the compact root, which must be the case. Then $\omega'_{\mathbb{B}} \otimes L'_{\rho'} = L'_{\alpha_1+\alpha_2}$, which is the $*$ in the picture.

Conclusion: *We have* $\omega_{\mathbb{B}}'^{\otimes k} \otimes L'_{\rho'} \in \mathbf{C}$ *for* $k > 1$.

Next, we claim that

$$\omega'_{\mathbb{B}} \otimes L'_{\rho'} = F'_{(-2,1)}\,.$$

The LHS is

$$L'_{2e_3^*-e_1^*-e_2^*} \otimes L'_{e_2^*-e_3^*} = L'_{e_3^*-e_1^*} = L'_{\alpha_1+\alpha_2}\,.$$

From the proposition, for $F'_{(a,b)} = L'_{\alpha_1+\alpha_2}$ we have

$$b-a=3,\ -2a-b=3. \qquad \square$$

Finally we note that

$$\omega'_{\mathbb{B}} = F'_{(-3,0)}\,,$$

which is consistent with $\mathbb{B} \subset \mathbb{P}^2$ and $\omega_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$.

For the Hodge bundle, from the picture

$$\begin{array}{|c|c|l} \hline * & ** & V_{+,\mathbb{C}} \\ \hline ** & * & V_{-,\mathbb{C}} \\ \hline \multicolumn{1}{c}{V^{1,0}} & \multicolumn{1}{c}{V^{0,1}} & \end{array}$$

where the fibre of $\mathbb{V}_+^{1,0}$ at the reference point is $[e_3]$,

$$V_+^{1,0} = \mathcal{O}_{\mathbb{B}}(-1)$$

and

$$\omega_{\mathbb{B}} = \otimes^3 \mathbb{V}_+^{1,0} .$$

We shall abuse notation and write

$$\omega_{\mathbb{B}}^{\prime \otimes k/3} = F'_{(-k,0)} = \mathbb{V}_+^{1,0} .$$

Appendix to section II.B: Sign conventions. Since the signs and Weyl chambers are critical, as a check we will here do the simple case of $\mathcal{H} \subset \mathbb{P}^1$. We set

- $\mathbb{C}^2$ = column vectors with basis $e_1 = \binom{1}{0}$, $e_2 = \binom{0}{1}$;
- $\check{\mathbb{C}}^2$ = dual space of row vectors with dual basis $e_1^* = (1,0)$, $e_2^* = (0,1)$;
- with the reference point $[e_1]$ = line in $\mathbb{C}^2$ spanned by e_1, we have the identification

$$\mathbb{P}^1 = \mathrm{SL}_2(\mathbb{C})/B$$

  where $B = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix} \right) \right\}$;
- for the Cartan subalgebra $\mathfrak{h} = \{ h_x = \left(\begin{smallmatrix} x & 0 \\ 0 & -x \end{smallmatrix} \right) \}$ we identify $\mathfrak{h}$ with $\mathbb{C}$ and h_x with $x(e_1 - e_2)$;
- For $E = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} \right)$, we may identify the holomorphic tangent space to $\mathbb{P}^1$ at $[e_1]$ with $\mathrm{sl}_2(\mathbb{C})/\mathfrak{b} \cong \mathbb{C}E$. Then

$$[h_x, E] = -2xE = \langle e_2^* - e_1^*, h_x \rangle \ .$$

II.C. **Homogeneous line bundles in the second example.** In this section we shall use the following notations.

- $G = \mathrm{Sp}(4)$;
- D and D' are the homogenous complex structures on $G_{\mathbb{R}}/T$ given by the respective root diagram
- $\check{D}$ will be the compact dual of all Lagrangian flags $F_\bullet = \{F_1 \subset F_2 \subset F_3 \subset F_4 = V\}$ where $F_2^{\perp} = F_2$, $F_1^{\perp} = F_3$;
- $F_{(a,b)} \to \check{D}$ will be the homogeneous line bundle defined to be the homogeneous line bundle associated to the character whose corresponding weight is $ae_1 + be_2$;

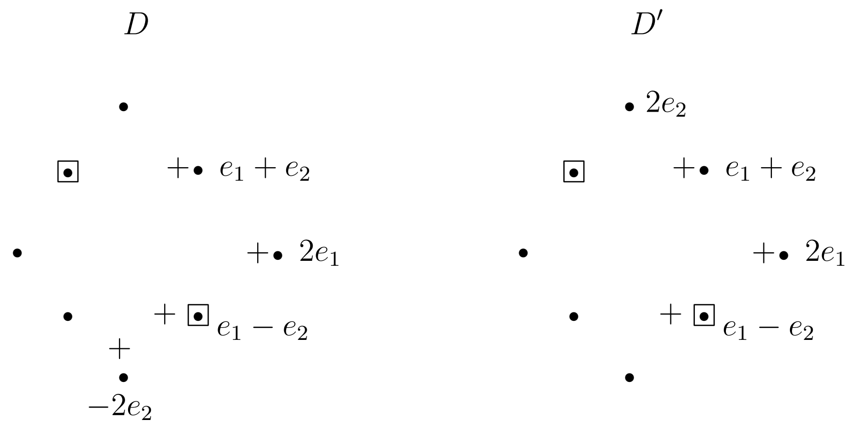


FIGURE 7

- the restrictions of $F_{(a,b)} \to \check{D}$ to D and D' will be denoted respectively by

$$\begin{cases} F_{(a,b)} \to D \\ F'_{(a,b)} \to D' . \end{cases}$$

When we construct the correspondence space $\mathcal{W} \subset G_{\mathbb{C}}/T_{\mathbb{C}}$ with maps

$$\begin{tikzcd} & \mathcal{W} \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ D & & D', \end{tikzcd}$$

we will have the isomorphism of homogeneous line bundles

$$\pi^{-1} F_{(a,b)} \cong \pi'^{-1} F'_{(a,b)} .$$

Homogeneous line bundles over D. Recall that in $\check{D}$ we have the reference flag

$$[v_{-e_1}] \subset [v_{-e_1}, v_{-e_2}] \subset [v_{-e_1}, v_{-e_2}, v_{e_2}] \subset [v_{-e_1}, v_{-e_2}, v_{e_2}, v_{e_1}]$$

with corresponding Borel subgroup denoted by B_0 above. Recalling that a Lagrange flag is determined by the first two subspaces in the reference flag over the point $F_\bullet \in \check{D}$ whose $G_{\mathbb{R}}$-orbit is D, we take for the first two subspaces

$$[v_{-e_1}], [v_{-e_1}, v_{e_2}] .$$

It follows that the fibres of $F_{(a,b)} \to D$ at the reference flag are determined by

$$\begin{cases} F_{(1,0),F_\bullet} = [\check{v}_{-e_1}] \longleftrightarrow e_1 \\ F_{(0,1),F_\bullet} = [v_{e_2}] \longleftrightarrow e_2\,. \end{cases} \tag{II.C.1}$$

Here the notation "$\longleftrightarrow$" means that homogeneous line bundles are given by the characters of T corresponding to the indicated weight.

When we realize D as the Mumford-Tate domain for polarized Hodge structures of weight $n = 3$ and all $h^{p,q} = 1$, we then have for the Hodge bundles

$$\begin{cases} \mathbb{V}^{3,0} = F_{(-1,0)} \\ \mathbb{V}^{2,1} = F_{(0,1)}\,. \end{cases} \tag{II.C.2}$$

Homogeneous line bundles over D'. Here the reference flag $F'_\bullet$ is the same as that for $\check{D}$ given above. It follows that

$$\begin{cases} F'_{(1,0),F'_\bullet} = [\check{v}_{-e_1}] \longleftrightarrow e_1 \\ F'_{(0,1),F'_\bullet} = [\check{v}_{-e_2}] \longleftrightarrow e_2\,. \end{cases} \tag{II.C.3}$$

As previously noted, the Hodge-theoretic interpretation of D' that we shall use is the following.

- $\mathcal{H}$ is the space of polarized Hodge structures of weight one on V given by Lagrangian 2-planes $F_2 \subset V$ such that $H < 0$ on F_2, where H is the Hermitian form given above;
- $D' \to \mathcal{H}$ is the space of *Hodge flags*

$$F_1 \subset F_2$$

where $F_2 \in \mathcal{H}$.

Thus, D' is a $\mathbb{P}^1$-bundle over an Hermitian symmetric domain. If we denote by

$$\mathbb{V}'^{1,0} \to D'$$

the pullback to D' of the Hodge bundle over $\mathcal{H}$, then

$$\det \mathbb{V}^{1,0} = F'_{(-1,-1)}\,. \tag{II.C.4}$$

Of importance for later use will be the pullback $\omega'_{\mathcal{H}}$ to D' of the canonical line bundle $\omega_{\mathcal{H}} \to \mathcal{H}$. Referring to figure 7 above, the $(1,0)$ tangent space to $\mathcal{H}$ at the image of the reference flag for D' is

$$T^{1,0}\mathcal{H} \cong \operatorname{span}\{X_{2e_2}, X_{e_1+e_2}, X_{2e_1}\} \ .$$

It follows that

$$\omega'_{\mathcal{H}} = F_{(-3,-3)} \longleftrightarrow -3(e_1+e_2)\,. \tag{II.C.5}$$

An important Weyl chamber. Denoting by Φ^+ the set of positive roots corresponding to the complex structure on D, we have

$$\Phi^+ = \Phi^+_c \cup \Phi^+_{nc}$$

where Φ^+_c, Φ^+_{nc} are respectively the compact, non-compact positive roots. For a weight μ recall the definition

$$q(\lambda) = \#\left\{\alpha \in \Phi^+_c : (\lambda,\alpha) < 0\right\} + \#\left\{\beta \in \Phi^+_{nc} : (\lambda,\beta) > 0\right\} \ .$$

For the complex structure on D', we have similarly

$$\Phi'^+ = \Phi'^+_c \cup \Phi'^+_{nc}\,,$$

and for a weight λ we set

$$q'(\lambda) = \#\left\{\alpha \in \Phi'^+_c : (\lambda,\alpha) < 0\right\} + \#\left\{\beta \in \Phi^+_{nc} : (\lambda,\beta) > 0\right\}.$$

Referring to Figure 7 above, we denote by $\mathbf{C}$ the following Weyl chamber

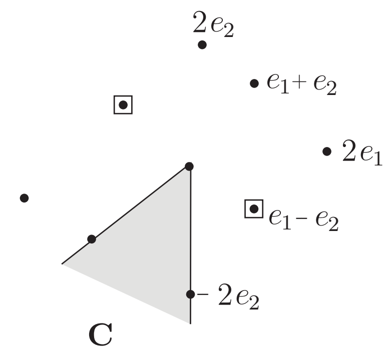


FIGURE 8

From inspection of this picture we infer the

(II.C.6) **Proposition:** *For* $\lambda \in \mathbf{C}$ *we have*

$$\begin{cases} q(\lambda) = 1 \\ q'(\lambda) = 0 \,. \end{cases}$$

The importance of the Weyl chamber **C** is the following: With the usual notations

$$\begin{cases} \rho \ = \frac{1}{2}\left(\sum_{\alpha\in\Phi^+} \alpha\right) = 2e_1 - e_2 \\ \rho' = \frac{1}{2}\left(\sum_{\alpha\in\Phi'^+} \alpha\right) = 2e_1 + e_2 \,, \end{cases} \tag{II.C.7}$$

from the work of Schmid [Schm1] we have

$$\begin{cases} \bullet \quad \text{If } \mu+\rho \in \mathbf{C}, \text{ then } H^1_{(2)}(D, L_\mu) \neq 0 \\ \bullet \quad \text{If } \mu'+\rho' \in \mathbf{C}, \text{ then } H^0_{(2)}(D', L'_{\mu'}) \neq 0 \,. \end{cases} \tag{II.C.8}$$

The same result will hold for quotients $X = \Gamma\backslash D$ and $X' = \Gamma\backslash D'$ where Γ is a co-compact discrete subgroup of G acting freely on D and D', provided that $\mu+\rho$ and $\mu'+\rho'$ are "sufficiently far" from the walls of **C**.

For a line bundle $L'_{\mu'} \to D'$, we shall say that

$$L'_{\mu'} \in \mathbf{C}$$

if the weight of $\mu'+\rho' \in \mathbf{C}$. Then we have the

(II.C.9) **Proposition:** *For* $m \geqq 1$ *we have*

$$\omega'^{\otimes m}_{\mathcal{H}} \otimes L'_{\rho'} \in \mathbf{C} \,.$$

The sections in $H^0(X', \omega_{\mathcal{H}}'^{\otimes k/3})$ will correspond to Siegel modular forms of weight k. The Penrose transform will give a natural map

$$\mathcal{P} : H^0(X', \omega_{\mathcal{H}}'^{\otimes k/3}) \to H^1(X, L_{\mu_k})$$

for a weight μ_k to be determined below. In this way, Siegel modular forms are mapped to the non-classical automorphic cohomology $H^1(X, L_{\mu_k})$.

III. Correspondence and cycle spaces; Penrose transforms

III.A. **Introduction.** For this paper correspondence and cycle spaces will provide a Penrose-transfer type mechanism for transforming higher cohomology of homogeneous line bundles over Mumford-Tate domains, and the quotients of such by discrete groups, into global holomorphic objects. In particular, for the cases we shall consider the correspondence spaces will provide a way to "evaluate" automorphic cohomology classes α at CM points and then to be able to say that α has an "arithmetic value" there. The correspondence spaces will have three descriptions

- geometric
- group theoretic
- Hodge theoretic

and it will be the interplay between these, especially the explicit formulas, that will be of use in this work.

The cycle spaces, which first appeared from Hodge-theoretic considerations in the 1960's and have been the subject of substantial recent work (cf. [FHW] and the references cited therein), will for us provide an intermediary between the correspondence spaces and the Mumford-Tate domains.

III.B. **Basic definitions and examples.**

Definition. Let M be a complex manifold. A *correspondence space* for M is a complex manifold $\mathcal{X}$ together with a holomorphic submersion

$$\pi : \mathcal{X} \to M$$

such that (i) $\mathcal{X}$ is a Stein manifold, and (ii) the fibres of π are contractible.

Being closed complex submanifolds of a Stein manifold, the fibres $\pi^{-1}(m) = \mathcal{X}_m$, $m \in M$ of π are Stein manifolds. The term *correspondence space* is used because in the examples we shall consider there will be a diagram

$$\begin{tikzcd} & \mathcal{X} \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ M & & M' \end{tikzcd}$$

where each projection satisfies (i), (ii) above, and where there will be maps arising from the EGW theorem below and a canonical class on $\mathcal{X}$ that combine to give a "Penrose transform" map

$$\mathcal{P} : H^{q'}(M', L') \to H^q(M, L)$$

relating cohomology on M' to that on M.

Our use of correspondence spaces arises from the fundamental works [Gi] and [EGW]. The basic result we shall use is this: Let $F \to M$ be a holomorphic vector bundle and $\pi^{-1}F \to \mathcal{X}$ the pullback of F to $\mathcal{X}$. The sheaves

$$\Omega^q_\pi =: \Lambda^q \Omega^1_\pi \text{ where } \Omega^1_\pi = \Omega^1_{\mathcal{X}} / \pi^* \Omega^1_M$$

of relative differentials are defined and form a complex $(\Omega^\bullet_\pi, d_\pi)$. Thinking of $\pi^{-1}F \to \mathcal{X}$ as having transition functions that are constant along the fibres of $\mathcal{X} \to M$, we may define $d_\pi : \Omega^q_\pi(\pi^{-1}F) \to \Omega^{q+1}_\pi(\pi^{-1}F)$, and then from this define the global, relative de Rham cohomology groups

$$H^*_{\mathrm{DR}}\big(\Gamma(\mathcal{X}, \Omega^\bullet_\pi(\pi^{-1}F))\big) .$$

Here, and below, we shall omit reference to the differential d_π.[30]

Theorem ([Gi], [EGW]). *There is a natural isomorphism*

$$\boxed{H^*(M,F)\cong H^*_{\mathrm{DR}}\big(\Gamma(\mathcal{X},\Omega^\bullet_\pi(\pi^{-1}F))\big)\,.} \tag{III.B.1}$$

In the examples we shall consider, there will be natural "harmonic" representatives of the cohomology classes on the RHS of (III.B.1). In this way the higher cohomology classes on M on the LHS will be represented on $\mathcal{X}$ by global holomorphic objects.

The examples we shall consider will be when M is a homogeneous complex manifold. They will be of two types.

Compact case: $M=K/T=K_{\mathbb{C}}/B_K$ where K is a compact, reductive real Lie group, $T\subset K$ is a compact maximal torus and $B_K\subset K_{\mathbb{C}}$ is a Borel subgroup. We think of M as "flags," and then we will have

$$\mathcal{X}=\{\text{pairs of flags in general position}\}\cong K_{\mathbb{C}}/T_{\mathbb{C}}\,.$$

More precisely, if B_K stabilizes a reference flag and if B_K^* is the opposite Borel subgroup, then $T_{\mathbb{C}}=B_K\cap B_K^*$. Here one may think of $B_K=T_{\mathbb{C}}U$ where $T_{\mathbb{C}}$ is a Cartan subgroup and U, the unipotent radical of B_K, is the exponential of the negative root spaces. Then $B_K^*=T_{\mathbb{C}}U^*$ where U^* is the exponential of the positive root spaces.

Basic example in the compact case: $M=\mathbb{P}^1$ and $\mathcal{X}=\mathbb{P}^1\times\mathbb{P}^1\backslash\{\text{diagonal}\}$ consists of pairs of distinct points in $\mathbb{P}^1$. The cohomology groups are the standard $H^q(\mathbb{R}^1,\mathcal{O}_{\mathbb{P}^1}(k))$, with the groups for $q=1$ and $k\leqq 2$ being interpreted by global holomorphic data via the RHS of (III.B.1). We will analyze this example in detail in section III.C below.

Mumford-Tate domain case: In this case we will have $M=D=G_{\mathbb{R}}/T$ where T is a compact maximal torus. We then have the compact dual

[30] These groups are commonly written as $H^q_{\mathrm{DR}}(\mathcal{X}/M,\pi^{-1}F) = H^q\big(\Gamma(\mathcal{X},\Omega^\bullet_\pi(\pi^{-1}F));d_\pi\big)$, the relative de Rham cohomology groups with coefficients in the local system $\pi^{-1}F$ over $\mathcal{X}$.

$\check{D}$, and denoting by $\mathcal{W}, \check{\mathcal{W}}$ the respective correspondence spaces we will have

$$\begin{array}{ccc} \mathcal{W} & \subset & \check{\mathcal{W}} \\ \downarrow & & \downarrow \\ D & \subset & \check{D} \end{array}$$

where $\mathcal{W}$ is an open set in $\check{\mathcal{W}}$. Although $\check{\mathcal{W}} \to \check{D}$ is an example of the compact case, we shall reserve the notation $\mathcal{X}$ for the correspondence space $\mathcal{X} \to K/T$ where $K \subset G_{\mathbb{R}}$ is a maximal compact subgroup.

In our examples we will have for $\check{D}$ an identification

$$\check{\mathcal{W}} = G_{\mathbb{C}}/T_{\mathbb{C}}\,,$$

so that again we may think of $\check{\mathcal{W}}$ as pairs of flags in general position. Much more subtle is the definition of the correspondence space $\mathcal{W} \to D$ for a homogeneous complex manifold $D = G_{\mathbb{R}}/T$ which is the $G_{\mathbb{R}}$-orbit of a point φ in $\check{D}$ (these include the Mumford-Tate domains we shall consider). Letting W denote the Weyl group of $G_{\mathbb{C}}$ when $G_{\mathbb{R}}$ is of Hermitian type, there is the description

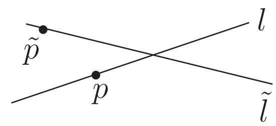
$$\mathcal{W} = \left(\bigcap_{w \in W} G_{\mathbb{R}} \cdot B_w \right) / T_{\mathbb{C}} \tag{III.B.2}$$

where $B \subset G_{\mathbb{C}}$ is the stabilizer of $\varphi \in \check{D}$ and $B_w = wBw^{-1}$.[31] We note here that $T_{\mathbb{C}} = \bigcap_{w \in W} B_w$, so that (III.B.2) is defined. We will work out the description of (III.B.2) explicitly in our two examples. The geometry of the correspondence spaces is a very rich story and is the subject of [GG].

First example: $D = \mathcal{U}(2,1)_{\mathbb{R}}/T$ as in section I. Then $\check{D} \subset \mathbb{P}^2 \times \check{\mathbb{P}}^2$ is the incidence or flag variety, and $\check{\mathcal{W}}$ is the set of configurations of pair

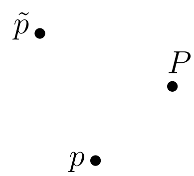
[31] When $G_{\mathbb{R}}$ is not of Hermitian type this description needs to be refined; this happens already for $\mathrm{SO}(4,1)$. The general definition of $\mathcal{W}$ is somewhat subtle and requires use of the *universality property* of the correspondence space, related to that property of the cycle space.

$(p, l; \check{p}, \tilde{l}) \in \check{D} \times \check{D}$



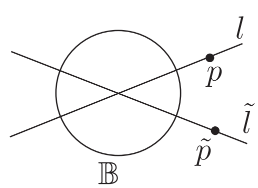
in general position.

We may obviously identify $\check{\mathcal{W}}$ with the set of *projective frames* $(p, P, \tilde{p})$

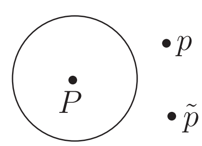


by drawing in the lines $\overline{pP}$ and $\overline{\tilde{p}P}$ to obtain the first picture and by omitting them in the first to obtain the second. We have chosen to use the first, and also the notations $p, P, \tilde{p}$ so as to be consistent with the rest of this work. Remark that the second picture is the more fundamental and is the one that will generalize.

For $\mathcal{W}$ we have the picture



of configurations above where $l \cap \tilde{l} \in \mathbb{B}$ and the line $\overline{p\tilde{p}} \subset \mathbb{B}^c$. We may identify this picture with sets of triples of points



where $P \in \mathbb{B}$ and where the line $\overline{p\tilde{p}}$ lies in $\mathbb{B}^c$. This description may then be seen to agree with (III.B.2) in this special case.

Below we will describe $\check{\mathcal{W}}$ and $\mathcal{W}$ for the second example.

Next, we turn to the

Definition. Let $D = G_{\mathbb{R}}/T$ be a non-classical Mumford-Tate domain. Let $Z = K/T$ be a maximal compact complex analytic subvariety of

D. Then the *cycle space*

$$\mathcal{U} = \{gZ : g \in G_{\mathbb{C}} \text{ and } gZ \subset D\}$$

is the set of translates of Z by elements g of the complex group such that gZ_0 remains in D. If $g_1 Z_0 = g_2 Z_0$ they represent the same point of $\mathcal{U}$. For $u \in \mathcal{U}$ we denote by $Z_u \subset D$ the corresponding subvariety.

Because D is non-classical, i.e., it does not fibre holomorphically or antiholomorphically over an Hermitian symmetric domain, it is a non-trivial result (cf. [FHW] and the references cited there) that

$$\{g \in G_{\mathbb{C}} : gZ_0 = Z_0\} = K_{\mathbb{C}}\,.$$

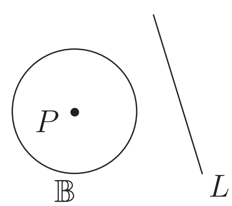
It follows that

$$\mathcal{U} \subset G_{\mathbb{C}}/K_{\mathbb{C}}$$

is an open set, and this gives $\mathcal{U}$ as an open set in the affine variety $G_{\mathbb{C}}/K_{\mathbb{C}}$. Another basic result (loc. cit.) is that

$\mathcal{U}$ *is a Stein manifold.*

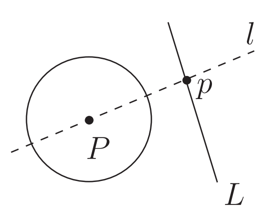
First example (continued): In this case $\mathcal{U}$ has the picture



That is

$$\mathcal{U} = \{(P, L) : P \in \mathbb{B},\ L \subset \mathbb{B}^c\} \cong \mathbb{B} \times \overline{\mathbb{B}}\,.$$

The corresponding maximal compact subvariety $Z(P, L)$ is given by the picture



Thus $Z(P, L) = \{(p, l) \in D\} \cong \mathbb{P}^1$.

In general, denoting by Z_u the maximal compact subvariety corresponding to $u \in \mathcal{U}$, the correspondence space and cycle space are related through the

Definition. The *incidence variety* $\mathcal{I} \subset D \times \mathcal{U}$ is defined by $\mathcal{I} = \{(\varphi, u) : \varphi \in Z_u\}$.

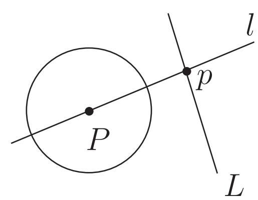
We then have the projections

$$\begin{tikzcd} & \mathcal{I} \arrow[dl, "\tau"'] \arrow[dr, "\eta"] & \\ D & & \mathcal{U} \end{tikzcd}$$

where

$$\begin{cases} \tau^{-1}(\varphi) & = \{(\varphi, u) : \varphi \in Z_u\} = \text{all } Z_u \text{ passing through } \varphi \\ \eta^{-1}(u) & = \{(\varphi, u) : \varphi \in Z_u\} \cong Z_u\,. \end{cases}$$

First example (continued): The picture of $\mathcal{I}$ is



In this example we have the following

Observation: There is a map $\mathcal{W} \xrightarrow{\sigma} \mathcal{I}$ that gives a factorization

$$\begin{tikzcd} \mathcal{W} \arrow[rr, "\sigma"] \arrow[dr, "\pi"'] & & \mathcal{I} \arrow[dl, "\tau"] \\ & D & \end{tikzcd}$$

defined by the conditions that if $(\varphi, \widetilde{\varphi}) = (p, l; \tilde{p}, \tilde{l}) \in \mathcal{W}$, then

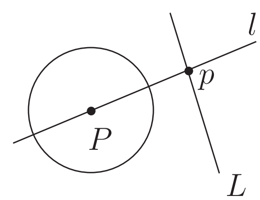
$$\sigma(\varphi, \widetilde{\varphi}) = (p, l; l \cap \tilde{l}, \overline{p\tilde{p}}) \in \mathcal{I} \subset D \times \mathcal{U}\,;$$

i.e., the conditions give that $P = l \cap \tilde{l} \in \mathbb{B}$ and $L = \overline{p\tilde{p}}$ lies in $\mathbb{B}^c$, and consequently $(P, L) \in \mathcal{U}$.

This observation has the following

Consequences. The above diagram factors the map π into two simpler maps. Namely

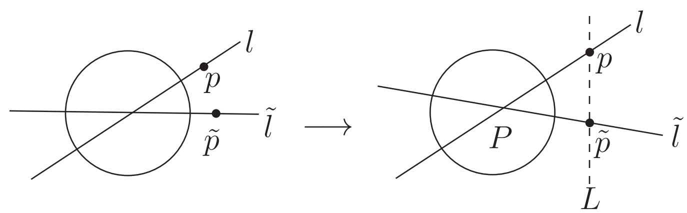
(i) In our example



$$\tau^{-1}(p,l) = \left\{ \begin{array}{l} P \in l \cap \mathbb{B} \cong \Delta \\ L \text{ through } p, L \subset \mathbb{B}^c \cong \Delta \end{array} \right\} \cong \Delta \times \Delta$$

where Δ is the unit disc in $\mathbb{C}$. Thus, even though $\mathcal{I}$ is not Stein the fibres over D are Stein and contractible.

(ii) In this example, the map $\mathcal{W} \xrightarrow{\sigma} \mathcal{I}$ is



$$(p,l;\tilde{p},\tilde{l}) \qquad \longrightarrow \qquad (p,l;P,L)$$

The fibre

$$\begin{aligned} \sigma^{-1}(p,l;P,L) &\cong \{\tilde{p} \in L \backslash \{p\}\} \\ &\cong \{\tilde{l} \text{ through } P, l \neq \tilde{l}\} \cong \mathbb{C}\,. \end{aligned}$$

Thus, the EGW theorem applies to $\mathcal{W} \xrightarrow{\sigma} \mathcal{I}$.

(iii) In this example

$$\mathcal{W} = \bigcup_{u \in \mathcal{U}} \mathcal{X}_u\,,$$

i.e., *the correspondence space* $\mathcal{W}$ *is fibered over* $\mathcal{U}$ *by the simpler correspondence spaces given in the basic example.*

In particular, for each $u \in \mathcal{U}$ by restriction there is a commutative diagram

$$\begin{tikzcd} H^*_{\mathrm{DR}}\left(\Gamma(\mathcal{W}, \Omega^\bullet_\pi(L_\mu))\right) \arrow[r] \arrow[d, phantom, "\wr\Vert"] & H^*_{\mathrm{DR}}\big(\Gamma(\mathcal{X}_u, \Omega^\bullet_\gamma(L_\mu))\big) \arrow[d, phantom, "\wr\Vert"] \\ H^*(D, L_\mu) \arrow[r] & H^*(Z_u, L_\mu)\,. \end{tikzcd} \tag{III.B.3}$$

It is known that for a weight μ such that $\mu+\rho$ is anti-dominant, the cohomology $H^d(D, L_\mu)$ is essentially captured by its restrictions to the maximal compact subvarieties (Schmid's identity theorem [Schm1]), and as will be seen below the RHS of this diagram can be very explicitly analyzed. This will be carried out in the appendix to section IV.D.

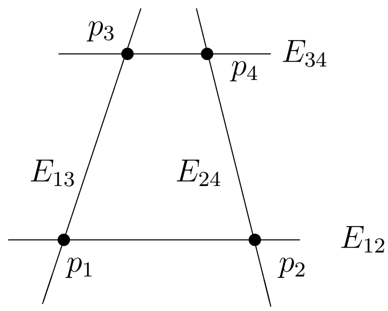
The second example. Our purpose here is to construct and give properties of the correspondence space in the second example. Specifically, we will show that

(III.B.4) **Proposition:** *The construction* (III.B.2) *gives a correspondence space.*

We will also (i) give the geometric picture of $\mathcal{W}$, (ii) give the geometric pictures of the cycle space $\mathcal{U}$ and the analogue in this example of $\mathcal{I}$ above, and (iii) give the geometric description of the maps between these various spaces. The discussion will be given in a sequence of steps.

Step one. We recall from section I.C the description of the correspondence space $\check{\mathcal{W}} = G_{\mathbb{C}}/T_{\mathbb{C}}$ for the compact dual as

$$\check{\mathcal{W}} = \left\{ \begin{array}{c} \text{set of Lagrange} \\ \text{quadrilaterals} \end{array} \right\}$$



Such a Lagrange quadrilateral is given by the projective frame (p_1, p_2, p_3, p_4) associated to a frame $(f_1, f_2, f_3, f_4) \in G_{\mathbb{C}}$ where $p_i = [f_i]$. The correspondence space $\mathcal{W}$ will be the open set in $\check{\mathcal{W}}$ described geometrically as the points of $\check{\mathcal{W}}$ given by

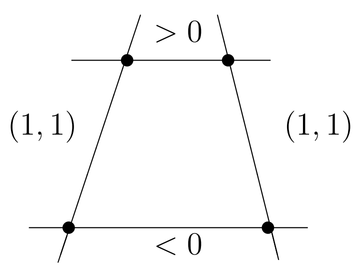
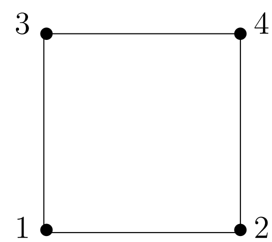


FIGURE 9

This means that the restrictions of the Hermitian form H to the Lagrange lines E_{ij} have the indicated signature.

Step two. We recall that the Weyl group W is the subgroup of $\mathcal{S}_4 =$ permutations of $\{1, 2, 3, 4\}$ realized geometrically as the symmetries of the square



For $w \in W$ we denote by $\sigma_w \in \mathcal{S}_4$ the corresponding permutation, and we denote by

$$f_1 = v_{-e_1}, f_2 = v_{-e_2}, f_3 = v_{e_2}, f_4 = v_{e_1}\ ,$$

the reference frame. Setting $B_w = wB_0w^{-1}$ we have the

(III.B.5) **Lemma:** (i) $B_w = \{g \in G_{\mathbb{C}} : g$ *preserves the reference flag* $([f_{\sigma_w(1)}],\ [f_{\sigma_w(1)}, f_{\sigma_w(2)}])\}$; (ii) $G_{\mathbb{R}}B_w = \{g \in G_{\mathbb{C}} : H$ *has the correct signature on the flag* $([g(f_{\sigma_w(1)})], [g(f_{\sigma_w(1)}), g(f_{\sigma_w(2)})])\}$.

Proof. We recall that any Lagrange flag $F_1 \subset F_2 \subset F_3 \subset V$ is determined by $F_1 \subset F_2$. Then (i) follows from the description of B_w as the subgroup of $G_{\mathbb{C}}$ preserving the reference flag given in (i).

As for (ii), it is convenient to think in Hodge-theoretic terms. It will suffice to prove the result when w is the identity. Then we may think of $D_w = D$ as the set of filtrations $F_1 \subset F_2 \subset F_3 \subset F_4 = V$ satisfying the second Hodge-Riemann bilinear relation (the first one is satisfied in $\check{D}$). This is the same as the set of partial flags $F_1 \subset F_2$ satisfying the second bilinear relation, which is just the condition in (ii). Letting $\varphi_0 \in D$ be the identity coset, we need then to show that $G_{\mathbb{R}} = \{g \in G_{\mathbb{C}} : g \cdot \varphi_0 \in D\}$. But $D = G_{\mathbb{R}} \cdot \varphi$, so that we have $g\varphi_0 = g_{\mathbb{R}} \cdot \varphi_0$ and then $g = g_{\mathbb{R}}b$ where $b \in B_w$. $\square$

Step three. We define

$$D_w = G_{\mathbb{R}}B_w/B_w \subset G_{\mathbb{C}}/B_w\,.$$

Then from the discussion at the beginning of section I we have

$$\text{(III.B.6)} \qquad \begin{cases} D_w = G_{\mathbb{R}}\text{-orbit of } wB \text{ in } G_{\mathbb{C}}/B \\ D_w \cong D_{w'} \text{ if, and only if, } wW_K = w'W_K\,. \end{cases}$$

In the second statement the isomorphism is as homogeneous complex manifolds given by a complex structure on $G_{\mathbb{R}}/T$.

Definition. *We set*

$$\text{(III.B.7)} \qquad \mathcal{W} = \left(\bigcap_{w \in W} G_{\mathbb{R}} B_w \right) / T_{\mathbb{C}}\,.$$

We will show that $\mathcal{W}$ is an open set in $\check{\mathcal{W}}$ and that *it gives a correspondence space for all the domains D_w.*[32]

We note that $\mathcal{W}$ *is universal* in the sense that we have surjective holomorphic submersions

$$\mathcal{W} \xrightarrow{\pi_w} D_w$$

for all $w \in W$. This mapping is given by

$$\mathcal{W} \to G_{\mathbb{R}} B_w / B_w \cong G_{\mathbb{R}}/T$$

where $G_{\mathbb{R}}/T$ has the homogeneous complex structure corresponding to $w \in W$.

From (ii) in the Lemma in step two we have the identification

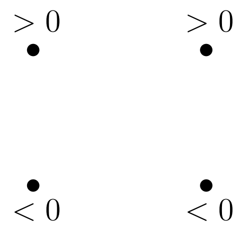
$$G_{\mathbb{R}} B_w / T_{\mathbb{C}} \longleftrightarrow \left\{ \begin{array}{c} \text{flags } ([f_{\sigma_w(1)}], [f_{\sigma_w(1)}, f_{\sigma_w(2)}]) \text{ for} \\ \text{which } H \text{ has the same signature as} \\ \text{for the reference flag } ([f_1], [f_1, f_2]) \end{array} \right\}.$$

It follows that we have the identification of sets

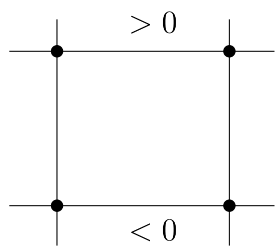
$$\mathcal{W} = \text{all pictures given by Figure 9.}$$

[32]As previously noted, the definition (III.B.7) will work in general whenever $G_{\mathbb{R}}$ is of Hermitian type, but in general must be modified.

This gives the inclusion $\mathcal{W} \subset \check{\mathcal{W}}$, where $\mathcal{W}$ is the open set given by the inequalities



Here one notes that this configuration can be completed to a Lagrange quadrilateral

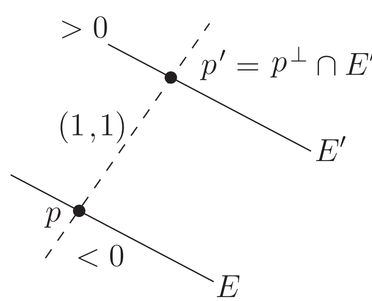


Step four. We next describe $\mathcal{U}$ = set of maximal compact subvarieties $Z \subset D$.

Denote by $\mathcal{H}$ the Siegel upper space parametrizing polarized Hodge structures of weight one on V.

(III.B.8) **Proposition:** $\mathcal{U} \cong \mathcal{H} \times \overline{\mathcal{H}}$.

Proof. We think of $\mathcal{H}$ as the set of Lagrange lines E in $\mathbb{P}^3$ on which $H < 0$, and $\overline{\mathcal{H}}$ as the set of Lagrange lines E' where $H > 0$. Given the picture

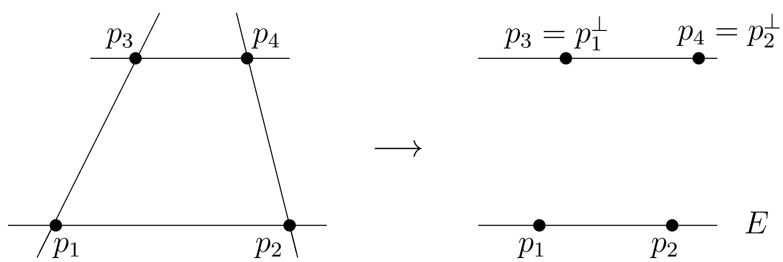


to each $p \in E$ we have the unique point $p' = p^{\perp} \cap E' \in E'$, where $^{\perp}$ means orthogonal with respect to Q. Then $\overline{pp'}$ is the Lagrange line giving a partial filtration $F_1 \subset F_2$ satisfying the condition that $H < 0$ on F_1 and H has a signature $(1,1)$ on F_2. As p varies over $E \cong \mathbb{P}^1$ we obtain a maximal compact subvariety $Z(E, E') \subset D$. It then follows either directly, or from the general results in [FHW], that the above description gives $\mathcal{U}$. □

Step five. We willl show that

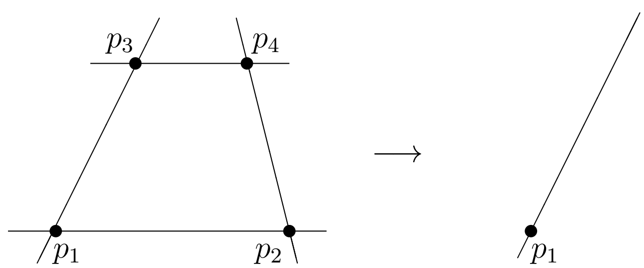
(III.B.9) **Proposition:** (i) *$\mathcal{W}$ is a Stein manifold, and* (ii) *the fibres of any projection $\mathcal{W} \to D_w$ are contractible.*

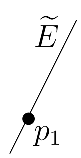
Proof. We map $\mathcal{W} \to \mathcal{U}$ by



where $p_1^\perp$, $p_2^\perp$ are the points on the Lagrangian line that are orthogonal under Q to p_1 and p_2. The fibre $E \times E \backslash \{\text{diagonal}\}$ is Stein as is $\mathcal{U}$, from which it may be seen that $\mathcal{W}$ is a Stein fibration over $\mathcal{U}$. □ for (i)

For the proof of (ii), the map $\mathcal{W} \to D$ is



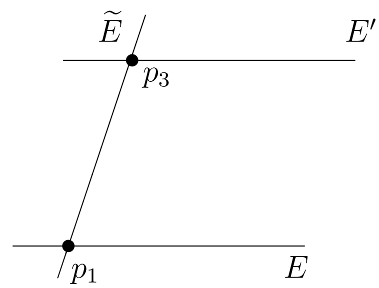
then the fibre over 

, where $H(p_1) < 0$ and H has signature $(1,1)$ on $\widetilde{E}$, is described as follows:

- pick a point $p_3 \in \widetilde{E}$ where $H(p_3) > 0$; this is a disc in $\widetilde{E} \cong \mathbb{P}^1$;

- pick a Lagrangian line E through p_1 and where $H < 0$ on E; this is a disc in the set of Lagrangian lines through p_1, which is a $\mathbb{P}^1$;
- pick a Lagrangian line E' through p_3 and where $H > 0$ on E'; another disc;

At this stage we have

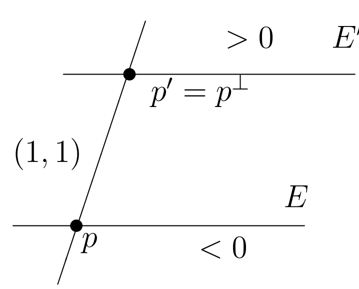


- finally, pick $p_2 \neq p_1$ on E, which is a $\mathbb{C}$, and set $p_4 = p_2^\perp \cap E'$.

This completes the figure just above to a Lagrange quadrilateral, and the fibres are successively $\Delta, \Delta, \Delta, \mathbb{C}$.

This completes the proof of (ii) for D. The arguments for the other D_w's are similar. $\square$

Step six. We have $\mathcal{I} \subset D \times \mathcal{U}$ given by the set of configurations



That is

$$\mathcal{I} = \{(p, \overline{pp'}, (E, E')\} \in D \times \mathcal{U}$$

as in the above picture. The fibres of the maps in

$$\begin{tikzcd} & \mathcal{J} \arrow[dl, "\tau"'] \arrow[dr, "\eta"] & \\ D & & \mathcal{U} \end{tikzcd}$$

may be analyzed as above. We note that $\tau^{-1}(p, \overline{pp'})$ is contractible Stein.

III.C. **The basic example.** This section is not essential for the main results in the paper. It is in part based on [Gi] and [EGW], and its purpose is to describe in detail the isomorphism in Theorem III.B.1 in the case of the compact form of the basic example. The description will be given both in coordinates and group-theoretically ($\mathfrak{n}$-cohomology) and in both the compact and non-compact cases. These descriptions will be shown to agree (up to constants). The point is that both descriptions will be used in the two examples in this work, where although the formulas are of course more complicated the conceptual framework and method of calculation are essentially the same. Additionally we can use the maps in the diagram (III.B.3) to analyze cohomology in the example in terms of the simpler cohomology in the basic example. The basic conclusion of this discussion of the compact case is summarized in step six below. Following that we will give a discussion of the non-compact case. The conclusion of that is (III.C.13) below.

We will proceed in several steps. In this section V will be a 2-dimensional *complex* vector space.

Step one. We observe that *a non-degenerate alternating form*

$$Q : V \otimes V \to \mathbb{C}$$

gives an $\mathrm{SL}_2(\mathbb{C})$*-equivariant isomorphism*

$$\Omega^1_{\mathbb{P}V} \cong \mathcal{O}_{\mathbb{P}V}(-2)\,.$$

Proof. We think of Q as giving an isomorphism

$$\Lambda^2 V \cong \mathbb{C}\,,$$

and then $\mathrm{Aut}(V, Q) = \mathrm{SL}_2(\mathbb{C})$. Let $[z] = \left[\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right]$ be homogeneous coordinates and $[\xi] = [\xi^0, \xi^1]$ the dual homogeneous coordinates in $\mathbb{P}\check{V}$. It is convenient to use the physicists notation $\epsilon^{ij} = -\epsilon^{ji}$ for a skew-symmetric matrix with dual ϵ_{ij} and where using summation convention $\epsilon_{ij}\epsilon^{jk} = \delta_i^k$. Then

$$Q(z, \tilde{z}) = \epsilon^{ij} z_i \tilde{z}_j \,.$$

We claim that the 1-form

$$\omega(z) =: \epsilon^{ij} z_i dz_j$$

gives a nowhere-zero section of $\Omega^1_{\mathbb{P}V}(2)$, and thus an $\mathrm{SL}_2(\mathbb{C})$ equivariant isomorphism $\Omega^1_{\mathbb{P}V} \cong \mathcal{O}_{\mathbb{P}V}(-2)$. To see this we note that $\omega(z)$ as given by the formula is a 1-form on $V\backslash\{0\}$, homogeneous of degree 2 and which satisfies

$$\langle \omega(z), Z \rangle = 0$$

where $Z = z_k \partial/\partial z_k$ is the Euler vector field that is tangent to the fibres of $V\backslash\{0\} \to \mathbb{P}V$. Thus $\omega(z)$ is *semi-basic* or *horizontal*; i.e., at each point of $V\backslash\{0\}$ it is in the image of the pullback on cotangent spaces of the map $\check{T}\mathbb{P}V \to \check{T}(V\backslash\{0\})$, and it is homogeneous of degree 2 under the $\mathbb{C}^*$-action. The conclusion follows from this. $\square$

Remark. If $f(z)$ is a holomorphic function defined locally over an open set in $\mathbb{P}V$ and pulled back to the inverse image in $V\backslash\{0\}$, then for $\lambda \neq 0$ we have $f(\lambda z) = f(z)$ and Euler's relation gives

$$\left(\frac{1}{z_1}\right) \frac{\partial f}{\partial z_0}(z) = -\left(\frac{1}{z_0}\right) \frac{\partial f}{\partial z_1}(z) \,.$$

Using the form Q to identify V with $\check{V}$, for the pullback of the differential of f there is the formula

$$df = \left(\frac{1}{2}\right) \left(\frac{\nabla f}{z_0 z_1}\right) \omega(z) \tag{III.C.1}$$

where $\omega(z) = z_1 dz_0 - z_0 dz_1$ and

$$\nabla f = z_0 \frac{\partial f}{\partial z_0} - z_1 \frac{\partial f}{\partial z_1} \,.$$

We will encounter a similar expression for the formula for d_π given below.

Step two. We set

$$\mathcal{X} = \mathbb{P}V \times \mathbb{P}\check{V} \backslash \{\langle \xi, z \rangle = 0\}\,.$$

Using the isomorphism $V \cong \check{V}$ given by Q, we have an identification

$$\mathcal{X} \cong \mathbb{P}V \times \mathbb{P}V \backslash \{\text{diagonal}\}$$

where the diagonal is $\{([z], [\tilde{z}]) : Q(z, \tilde{z}) = 0\}$.[33] For notational convenience we will use $([z], [\xi])$ for points of $\mathcal{X}$ in the second description, so that $[\xi^0, \xi^1] \in \mathbb{P}\check{V}$ corresponds to the point $\begin{pmatrix} \xi^1 \\ -\xi^0 \end{pmatrix} \in$ PV. We also assume that $Q = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Let $\mathcal{O}(k, l)$ be the sheaf $\mathcal{O}_{\mathbb{P}V}(k) \boxtimes \mathcal{O}_{\mathbb{P}\check{V}}(l)$ on $\mathbb{P}V \times \mathbb{P}\check{V}$ restricted to $\mathcal{X}$. Denoting by

$$\pi : \mathcal{X} \to \mathbb{P}V$$

the projection on the first factor, from step one we have the $\operatorname{Aut}(V, Q)$-invariant identification

$$\Omega^1_\pi \cong \mathcal{O}(0, -2)\,.$$

We set $\mathcal{O}_{\mathbb{P}V}(k)_\pi = \pi^{-1}\mathcal{O}_{\mathbb{P}V}(k)$, $\Omega^1_\pi(k) = \Omega^1_\pi \otimes \mathcal{O}_{\mathbb{P}V}(k)_\pi$, and using this identification we have a diagram

$$\begin{array}{ccc} \mathcal{O}_\pi(k, 0) & \xrightarrow{d_\pi} & \Omega^1_\pi(k) \\ \wr\| & & \wr\| \\ \mathcal{O}_\pi(k) & \longrightarrow & \mathcal{O}_\pi(k, -2)\,. \end{array}$$

For $f(z, \xi)$ a holomorphic function defined in an open set in $V \backslash \{0\} \times \check{V} \backslash \{0\}$ lying over an open set in $\mathcal{X}$ and which is homogeneous of degree k in z and degree zero in ξ, we claim that the top map is given by

$$d_\pi f(z, \xi) = \frac{\epsilon^{ij} z_i}{\langle \xi, z \rangle} \frac{\partial f(z, \xi)}{\partial \xi^j} \omega(\xi) \tag{III.C.2}$$

where $\omega(\xi) = \epsilon_{ij} \xi^i d\xi^j$.

For the proof of this formula, taking again for Q the standard form and using Euler's relation, we have

$$z_0 \frac{\partial f}{\partial \xi^0} - z_1 \frac{\partial f}{\partial \xi^1} = \frac{(z_0 \xi^0 + z_1 \xi^1) \nabla_\xi f}{2\xi^0 \xi^1} = \frac{\langle \xi, z \rangle\, \nabla_\xi f}{2\xi^0 \xi^1}$$

[33]If $Q(z, \tilde{z}) = z \wedge \tilde{z}$, then we are taking out from $\mathbb{P}^1 \times \mathbb{P}^1$ the usual diagonal $[z] = [\tilde{z}]$.

where $\nabla_\xi f = \xi^1 \frac{\partial f}{\partial \xi^1} + \xi^0 \frac{\partial f}{\partial \xi^0}$. This establishes the above claim and explains the factor $\langle \xi, z\rangle$ in the denominator.

The bottom map is given by the same formula (III.C.2) omitting the $\omega(\xi)$. From this we see that $d_\pi f = 0$ implies that $f = f(z)$ depends only on z, and this gives

$$H^0_{\mathrm{DR}}\big(\Gamma(\mathcal{X}, \Omega^\bullet_\pi(k))\big) \cong H^0\big(\mathbb{P}V, \mathcal{O}_{\mathbb{P}V}(k)\big)\,.$$

Step three. More interestingly, for $k \geq 2$ we will now make explicit the isomorphism

$$H^1\big(\mathbb{P}V, \Omega^1_{\mathbb{P}V}(-k)\big) \cong H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{X}, \Omega^\bullet_\pi(-k-2))\big)\,,$$

and then will show that each class on the RHS has a canonical representative. For this we use the isomorphism $V \cong \check{V}$ given by Q to have

$$\begin{aligned} H^1\big(\Omega^1_{\mathbb{P}V}(-k)\big) &\cong H^1\big(\mathcal{O}_{\mathbb{P}V}(-k-2)\big) && \text{(using } Q\text{)} \\ &\cong H^0\big(\mathcal{O}_{\mathbb{P}V}(k)\big)^\vee && \text{(Kodaira-Serre duality)} \\ &\cong (\operatorname{Sym}^k \check{V})^\vee \\ &\cong \operatorname{Sym}^k \check{V} && \text{(using } Q\text{).} \end{aligned}$$

Given $g(z) \in \operatorname{Sym}^k \check{V}$ a homogeneous form of degree k, we define $\check{g}(\xi) \in \operatorname{Sym}^k V$ by setting $z_i = \epsilon_{ij}\xi^j$ in $g(z)$. We then define the map

$$\operatorname{Sym}^k \check{V} \to \Gamma(\mathcal{X}, \mathcal{O}(-k-2, -2))$$

by

$$g(z) \to \frac{\check{g}(\xi)}{\langle \xi, z\rangle^{k+2}}\,.$$

(III.C.3) **Proposition III:** *This map induces a canonical isomorphism*

$$H^1\big(\Omega^1_{\mathbb{P}V}(-k)\big) \cong H^1_{\mathrm{DR}}\left(\Gamma\big(\mathcal{X}, \Omega^\bullet_\pi(-k-2)\big)\right)\,.$$

The element on the RHS corresponding to $g(z)$ is $\frac{\check{g}(\xi)\omega(\xi)}{\langle \xi, z\rangle^{k+2}}$.

Here, canonical means with respect to the form Q and group $\mathrm{Aut}(V,Q)$.

A direct proof of the proposition would be to show that the equation

$$\frac{\check{g}(\xi)}{\langle \xi, z\rangle^{k+2}} = \frac{\epsilon^{ij} z_j}{\langle \xi, z\rangle} \frac{\partial f(z,\xi)}{\partial \xi^i}$$

has no non-trivial solution for $f(z,\xi) \in \Gamma(\mathcal{X}, \mathcal{O}(-k-2,0))$. The more conceptual argument is to define an "adjoint"

$$d_\pi^* : \mathcal{O}(-k-2,-2) \to \mathcal{O}(-k-2,0)$$

and to show that for $k \geqq 0$ there is a unique representative ψ of a class in $H^1_{\mathrm{DR}}\big(\Gamma\big(\mathcal{X}, \Omega_\pi^\bullet \otimes \mathcal{O}(-k-2,0)\big)\big)$ that satisfies $d_\pi^* \psi = 0$. The coordinate formula for d_π^* is given by

$$f \to \langle \xi, z\rangle\, \epsilon_{ij} \xi^j \frac{\partial f}{\partial z_i}\,. \tag{III.C.4}$$

Using $\epsilon_{ij}\xi^i\xi^j = 0$ we see that

$$\epsilon_{ij}\xi^j \frac{\partial}{\partial z_i}\left(\frac{\check{g}(\xi)}{\langle \xi, z\rangle^{k+2}}\right) = 0\,,$$

so that the form given by $\frac{\check{g}(\xi)\omega(\xi)}{\langle \xi, z\rangle^{k+2}}$ is in fact the unique "harmonic" representative in its cohomology class.

Because similar arguments will be used in more elaborate situations below, we will here give the

Direct proof: We take for ϵ^{ij} the standard skew-symmetric form and must show that the equation

$$z_0 \frac{\partial f(z,\xi)}{\partial \xi^1} - z_1 \frac{\partial f(z,\xi)}{\partial \xi^0} = \frac{\check{g}(\xi)}{\langle \xi, z\rangle^{k+1}} \tag{III.C.5}$$

implies that $g(z) = 0$. Here, $f(z,\xi)$ is homogeneous by degree $-k-2$ in z and 0 in ξ. We expand

$$f(z,\xi) = \sum_n \frac{f_n(z,\xi)}{\langle \xi, z\rangle^n}$$

where $f_n(z,\xi)$ is holomorphic in z and ξ and has degree $n-k-2$ in z and degree n in ξ. It follows that we have

$$\begin{cases} k \geqq 0 & \text{(by assumption)} \\ n \geqq k+2 & \Rightarrow n \geqq 2 \text{ in the sum on the right}\,. \end{cases}$$

On the other hand the equation (III.C.5) gives

$$z_0 \frac{\partial f_1}{\partial \xi^1} - z_1 \frac{\partial f_1}{\partial \xi^0} = \check{g}\,,$$

but since $k \geqq 0$ implies $n \geqq 2$, and because the RHS of the above equation corresponds to $n = 1$ we see that $\check{g} = 0$. □

Step four. Here we will

- describe $\mathcal{X}$ group-theoretically
- describe $H^*_{\mathrm{DR}}\big(\Gamma(\mathcal{X}, \Omega^\bullet_\pi(k))\big)$ in terms of $\mathfrak{n}$-cohomology.

Step five will relate these descriptions to the coordinate description.

In this step we will assume given (V, Q) as above. Thus our symmetry group is

$$G_{\mathbb{C}} =: \mathrm{Aut}(V, Q) \cong \mathrm{SL}_2(\mathbb{C})\,.$$

Using the identification $V \cong \check{V}$ we have

$$\mathcal{X} = \mathbb{P}V \times \mathbb{P}V \backslash \{\text{diagonal}\}.$$

We choose an isomorphism $V \cong \mathbb{C}^2$ and the reference frame $e_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ with $Q(e_1, e_2) = 1$ in V, and then we denote by $T_{\mathbb{C}}$ the subgroup of $G_{\mathbb{C}}$ stabilizing the reference frame. We then claim that *we have an identification of homogeneous spaces*

$$\mathcal{X} \cong G_{\mathbb{C}}/T_{\mathbb{C}}\,. \tag{III.C.6}$$

Proof. We have the $G_{\mathbb{C}}$-equivariant description

$$\mathcal{X} = \{\text{paris of points } [z], [\tilde{z}] \in \mathbb{P}V \text{ in general position}\}\,.$$

Here, $G_{\mathbb{C}}$ acts by

$$g([z], [\tilde{z}]) = ([gz], [g\tilde{z}])\,,$$

and the stability groups of $[e_1], [e_2]$ are respectively

$$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \quad B^* = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}.$$

Thus the stability group of $([e_1], [e_2])$ is

$$B \cap B^* = T_{\mathbb{C}}\,.$$

It is clear that $G_{\mathbb{C}}$ acts transitively on pairs of distinct points in $\mathbb{P}V$. □

We have $\mathbb{P}V \cong G_{\mathbb{C}}/B$, and at the reference point $[e_1]$ there is the identification of the $(1,0)$ tangent space

$$\begin{aligned} T^{1,0}_{[e_1]}\mathbb{P}V &\cong \mathfrak{g}_{\mathbb{C}}/\mathfrak{b} \\ &\cong \mathbb{C}E \end{aligned}$$

where $E = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Following the usual convention we choose as positive root

$$\alpha = e_2^* - e_1^*$$

so that $E = X_\alpha$. Then $F =: \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = X_{-\alpha}$, and with $H = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ we have a basis $\{H, E, F\}$ of sl_2. Using the Cartan-Killing form the dual of X_α is $X^*_{-\alpha} = \left(-\frac{1}{4}\right) X_\alpha$ and we set

$$\mathfrak{n} = \mathbb{C}X^*_{-\alpha} = \mathbb{C}E\ .$$

Denoting by $\mathcal{O}(G_{\mathbb{C}})$ the algebra of global holomorphic functions on $G_{\mathbb{C}}$ and by $\mathbb{C}_k$ the $T_{\mathbb{C}}$-module given by the character that gives the $G_{\mathbb{C}}$-homogeneous line bundle $\mathcal{O}_{\mathbb{P}V}(k)$, we claim that we have the canonical identification of vector spaces

$$\boxed{\Gamma\big(\mathcal{X}, \Omega^\bullet_\pi(k)\big) \cong \big(\mathcal{O}(G_{\mathbb{C}}) \otimes \mathbb{C}_k \otimes \Lambda^\bullet \mathfrak{n}\big)^{T_{\mathbb{C}}}\,.} \tag{III.C.7}$$

The RHS is also a complex, and below we will give the group-theoretic formula for the differential d_π.

Proof. We denote by $X_{\pm\alpha}$ the corresponding left-invariant holomorphic vector field on $G_{\mathbb{C}}$, acting as usal on the right on $\mathcal{O}(G_{\mathbb{C}})$, and by $\omega^{\pm\alpha}$ the corresponding left-invariant holomorphic 1-forms. The vertical tangent space to $G_{\mathbb{C}} \to G_{\mathbb{C}}/T_{\mathbb{C}}$ at the reference point is $\mathbb{C}X_\alpha \oplus \mathbb{C}X_{-\alpha}$. In this the vertical tangent space to $G_{\mathbb{C}}/T_{\mathbb{C}} \to G_{\mathbb{C}}/B = \mathbb{P}V$ is $\mathbb{C}X_{-\alpha}$. It follows that at the reference point the space of relative differentials is $\mathbb{C}\omega^{-\alpha} \cong \mathfrak{n}$. Thus the pullback to $G_{\mathbb{C}}$ of $\Gamma(\mathcal{X}, \Omega^\bullet_\pi(k))$ is identified with a subspace of $\mathcal{O}(G_{\mathbb{C}}) \otimes \mathbb{C}_k \otimes \Lambda^\bullet\mathfrak{n}$, and then invariance under $T_{\mathbb{C}}$ is exactly the condition to be such a pullback. $\square$

Next we have

$$\mathfrak{b} = \mathfrak{t}_{\mathbb{C}} \oplus \check{\mathfrak{n}}\,,$$

and thus $\mathbb{C}_k \otimes \Lambda^\bullet \mathfrak{n}$ is a $\mathfrak{b}$-module. For $f \in \mathcal{O}(G_\mathbb{C}) \otimes \mathbb{C}_k$ we will see that

$$d_\pi f = (X_{-\alpha} f)\omega^{-\alpha}\,.$$

The "adjoint" d_π^* is defined by

$$d_\pi^*(g\omega^{-\alpha}) = -(X_\alpha \cdot g)\,.$$

A form $\psi = g\omega^{-\alpha}$ is "harmonic" if

$$\begin{cases} d_\pi \psi = 0 \\ d_\pi^* \psi = 0\,. \end{cases}$$

In our case the first equation is automatic. In [EGW] it is proved that:

> *For $k \geqq 0$, there is at most one harmonic form in each class in $H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{X}, \Omega^\bullet_\pi(-k))\big)$.*

In the next step we will show that the form $\frac{\check{g}(\xi)}{\langle \xi, z\rangle^{k+2}}\omega(\xi)$ is harmonic in the group theoretic way just defined.

Step five. For $f(z)$ a holomorphic function defined in an open set in $\mathbb{P}V$ and considered as a homogeneous function of degree zero in the inverse image of that open set, in the remark at the end of step one we have given the formula

$$df = \left(\frac{1}{2}\right) \frac{\langle \check{z}, \nabla f\rangle\, \omega(z)}{\langle \check{z}, z\rangle} \tag{III.C.8}$$

for the pullback of df. Here we are using Q to give an isomorphism $V \cong \check{V}$ taking $z \in V$ to $\check{z} \in \check{V}$ and where $\nabla f = \left(\frac{\partial f}{\partial z_0}, \frac{\partial f}{\partial z_1}\right)$. In coordinates, $\check{z}^i = \epsilon^{ij} z_j$ so that $\check{z} = (-z_1, z_0)$ and $\langle \check{z}, z\rangle = 2z_0 z_1$. We want to express the pullback to $G_\mathbb{C}$ of the RHS of this formula.

We begin by using the reference point to identify $G_\mathbb{C} = \mathrm{Aut}(V, Q)$ with $\mathrm{SL}_2(\mathbb{C})$ and $\mathbb{P}V$ with $\mathbb{P}^1$, the action of $\mathrm{SL}_2(\mathbb{C})$ on $[e_1]$ being given by

$$\begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & \delta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}.$$

Here, we write δ for lower right entry because d will be the exterior derivative. The pullback ω to $\mathrm{SL}_2(\mathbb{C})$ of $\omega(z)$ is then

$$\omega = a\,dc - c\,da\,.$$

The Maurer-Cartan matrix $\Omega = g^{-1}dg$ on $\mathrm{SL}_2(\mathbb{C})$ is given by

$$\Omega = \begin{pmatrix} \delta da - bdc & \delta db - bd\delta \\ -cda + adc & -cdb + ad\delta \end{pmatrix} := \begin{pmatrix} -\theta & \omega^{-\alpha} \\ \omega^{\alpha} & \theta \end{pmatrix}.$$

It follows that (see below for a proof)

$$\omega = \omega^{\alpha}$$

so that

$$df = (X_\alpha \cdot f)\omega^\alpha = X_\alpha \rfloor df$$

where

$$X_\alpha \cdot f = \left(-\frac{1}{2}\right)\frac{\langle \check{z}, \nabla f\rangle}{z_0 z_1}.$$

The denominator and numerator reflect a particular choice of a pair of distinct points in $\mathbb{P}V$. Below we will vary over all pairs of distinct points and this will be reflected in a more symmetric formula.

For $f(g)$ a holomorphic function defined in an open set in $\mathrm{SL}_2(\mathbb{C})$ we write

$$df = f_\alpha \omega^\alpha + f_{-\alpha}\omega^{-\alpha} + f_\theta \theta\,.$$

The condition that $f(g)$ depend only on the first column of $g = \left(\begin{smallmatrix} a & e \\ b & \delta \end{smallmatrix}\right)$ is $f_{-\alpha} = 0$, and the condition that $f(z)$ be homogeneous of degree k in $z = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ is $f_\theta = kf$. When $k = 0$, we simply have $df = f_\alpha \omega^\alpha$.

This being so, we may ask why is $X_\alpha \cdot f$ a section of $\mathcal{O}_{\mathbb{P}V}(-2)$? Denoting by X_θ the vector field given by $f \to f_\theta$ above, we have $X_\theta \cdot f = 0$, and then

$$\begin{aligned} X_\theta \cdot (X_\alpha f) &= [X_\theta, X_\alpha] f + X_\alpha(X_\theta \cdot f) \\ &= 2X_\alpha f\,. \end{aligned}$$

We now apply these considerations to

$$\begin{array}{c} \mathcal{X} \subset \mathbb{P}V \times \mathbb{P}\check{V} \\ \Big\downarrow{\scriptstyle \pi} \\ \mathbb{P}V. \end{array}$$

Since we are interested in

$$d_\pi : \mathcal{O}(k,0) \to \mathcal{O}(k,-2)$$

the above considerations will be applied to the second factor. Thus, with the above notation we replace z by ξ, $\check{z}$ by z, X_α, ω^α by $X_{-\alpha}, \omega^{-\alpha}$ and $\omega(z)$ by $\omega(\xi)$. This gives

$$d_\pi f = (X_{-\alpha} f)\omega^{-\alpha}$$

or in coordinates

$$d_\pi f(z,\xi) = \left(-\frac{1}{4}\right) \frac{\langle z, \nabla_\xi f(z,\xi)\rangle}{\langle \xi, z\rangle}\,.$$

Here the numerator is

$$\epsilon^{ij} z_i \frac{\partial f}{\partial \xi^j}(z,\xi)\,.$$

As above we may ask why is $X_{-\alpha} f$ a section of $\mathcal{O}(k,-2)$? A subtlety is that *as an* $\mathrm{SL}_2(\mathbb{C})$*-homogeneous line bundle over* $\mathcal{X}$,

$$\mathcal{O}(l,l) \cong \mathcal{O}(0,0)\,. \tag{III.C.9}$$

It is of course not the case that $\mathcal{O}_{\mathbb{P}V}(l) \boxtimes \mathcal{O}_{\mathbb{P}\check{V}}(l) \cong \mathcal{O}_{\mathbb{P}V} \boxtimes \mathcal{O}_{\mathbb{P}\check{V}}$ over $\mathbb{P}V \times \mathbb{P}\check{V}$. The reason for (III.C.9) is that $\langle \xi, z\rangle$ gives an $\mathrm{SL}_2(\mathbb{C})$-invariant, non-vanishing section of $\mathcal{O}(1,1)$. In fact, in terms of the above coordinates on $\mathrm{SL}_2(\mathbb{C})$ we have $z = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$, $\xi = (\delta, -b)$ so that $\langle \xi, z\rangle = 1$ on $\mathrm{SL}_2(\mathbb{C})$. This reflects the fact that $\mathrm{SL}_2(\mathbb{C})$ only acts transitively on the "slice" $\langle \xi, z\rangle = 1$ of $(V\backslash\{0\}) \times (\check{V}\backslash\{0\}) \to \mathbb{P}V \times \mathbb{P}\check{V}$. We note $\omega(\xi) = -\omega^{-\alpha}$ in the Maurer-Cartan matrix Ω.

The proof that, for f a section of $\mathcal{O}(k,-2)$, $(X_{-\alpha} f)\cdot \omega^{-\alpha}$ is a section of $\Omega^1_\pi(k,0)$ is now the same as the argument given above, taking into account the double sign change in scaling factors passing from V to $\check{V}$ and X_α to $X_{-\alpha}$.

Step six: Summary. For $\mathbb{P}^1 = \mathbb{P}V$ with the alternating form Q giving an isomorphism $V \cong \check{V}$, we have a commutative diagram of isomorphisms

$$\begin{array}{ccc} H^0_{\mathrm{DR}}\big(\Gamma(\mathcal{X}, \Omega^\bullet_\pi(k)\big) & \longrightarrow & H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{X}, \Omega^\bullet_\pi(-k-2)\big) \\ \wr\Vert & & \wr\Vert \\ H^0\big(\mathcal{O}_{\mathbb{P}^1}(k)\big) & \xrightarrow{\ \mathcal{P}\ } & H^1\big(\mathcal{O}_{\mathbb{P}^1}(-k-2)\big) \end{array} \tag{III.C.10}$$

where the top row is multiplication by $\omega(\xi)/\langle\xi,z\rangle^{k+2}$ and $g(z)\in\operatorname{Sym}^k\check{V}$ is identified with $\tilde{g}(\xi)\in\operatorname{Sym}^k V$. *The diagram* (III.C.10) *is a commutative diagram of* $G_{\mathbb{C}}=\mathrm{SL}_2(\mathbb{C})$*-modules.* The two ways of realizing the $G_{\mathbb{C}}$-module $\operatorname{Sym}^k\check{V}$ as a cohomology group are identified via the map $\mathcal{P}$, which is an example of a *Penrose transform.* As will be explained and illustrated in the next section and in the appendix to that section, in general the Borel-Weil-Bott theorem gives multiple ways of realizing the same $G_{\mathbb{C}}$-module as cohomology groups $H^{q(\mu)}(G_{\mathbb{C}}/B,L_\mu)$. These different realizations are identified geometrically via Penrose transforms.

The non-compact case of the basic example. We shall use the real form $\mathrm{SL}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{C})$. In this case there are two open orbits of $\mathrm{SL}_2(\mathbb{R})$ acting on $\mathbb{P}^1=\mathbb{P}V$, the upper-half-plane $\mathcal{H}$ and the lower-half-plane $\overline{\mathcal{H}}$, which we denote by $\mathcal{H}'$ to conform to the notation in the rest of this paper. As coordinate on $\mathcal{H}$ we take $z=\left[\begin{smallmatrix}z\\1\end{smallmatrix}\right]$, $\operatorname{Im}z>0$, and on $\mathcal{H}'$ we use $z'=\left[\begin{smallmatrix}z'\\1\end{smallmatrix}\right]$, $\operatorname{Im}z'<0$. We let $\Gamma\subset\mathrm{SL}_2(\mathbb{R})$ be a co-compact discrete group acting freely on $\mathcal{H}$ and denote by

$$X=\Gamma\backslash\mathcal{H},\quad X'=\Gamma\backslash\mathcal{H}'$$

the corresponding compact Riemann surfaces. Then $X'=\overline{X}$ is the Riemann surface with the conjugate complex structure to that on X.

For the correspondence space $\mathcal{W}\subset\mathfrak{X}$, recalling that we have $\mathfrak{X}\subset\mathbb{P}^1\times\mathbb{P}^1$ and projections

$$\begin{tikzcd} & \mathfrak{X} \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ \mathbb{P}^1 & & \mathbb{P}^1, \end{tikzcd}$$

we take

$$\mathcal{W}=\pi^{-1}(\mathcal{H})\cap\pi'^{-1}(\mathcal{H}')$$

so that we have a diagram

$$\text{(III.C.11)}\qquad \begin{tikzcd} & \mathcal{W} \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ \mathcal{H} & & \mathcal{H}', \end{tikzcd}$$

where $\mathcal{W} \cong \mathcal{H} \times \mathcal{H}'$ is Stein and the fibres of π, π' are contractible. Thus the [EGW] formalism applies to (III.C.11) to give a diagram

$$\begin{array}{ccc} H^0_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^\bullet_{\pi'}(k')\big) & \xrightarrow{\ \omega\ } & H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^\bullet_{\pi}(k)\big) \\ \wr\| & & \wr\| \\ H^0\big(X', \mathcal{O}_{X'}(k')\big) & \xrightarrow{\ \mathcal{P}\ } & H^1\big(X, \mathcal{O}_X(k)\big)\,. \end{array} \tag{III.C.12}$$

Then we have

(III.C.13) *For $k \geqq -1$ and $k' = k+2$, multiplication by the form $\omega = \omega(z')/Q(z,z')^{k+2}$ induces an isomorphism in the top row of* (III.C.12).

Below we shall interpret the resulting isomorphism

$$H^0\big(X', \mathcal{O}_{X'}(k')\big) \overset{\mathcal{P}}{\xrightarrow{\sim}} H^1\big(X, \mathcal{O}_X(k)\big)$$

given by the Penrose transform defined by the bottom row in the commutative diagram (III.C.12).

The indices in the above may be understood as follows: The root diagram for SL_2 is

$$\begin{array}{ccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & -2 & -1 & 0 & 1 & 2=\alpha & \end{array}$$

and identifying the roots with $\mathbb{Z}$, with our convention $\alpha = 2$. The positive Weyl chamber for the complex structure on $\mathcal{H}$ is given by the $k \in Z$ with $k \geqq 0$, and that for the complex structure on $\mathcal{H}'$ is given by the $k' \in \mathbb{Z}$ with $k' \leqq 0$. We thus have

$$\rho = 1, \quad \rho' = -1$$

and the condition on k and k' in (III.C.13) is

$$k + \rho = k' + \rho'\,. \tag{III.C.14}$$

The analogue of this condition will appear in both our examples discussed in sections III.E, III.F below. It is of course a familiar one from the perspective of the representation theory of semi-simple, real Lie groups.

We shall not give the formal proof of (III.C.14), which may be done by showing that multiplication by ω takes harmonic forms to harmonic forms, where "harmonic" is in the sense of [EGW]. It follows that $\mathcal{P}$ is injective, and then the isomorphism statement follows from the equality of dimensions. For $k \geqq 0$ this equality of dimensions is just the Riemann-Roch theorem for the compact Riemann surfaces X and X'. For $k = -1$ a special argument is required.

The sections in $H^0\big(X', \mathcal{O}_{X'}(k')\big)$ are automorphic forms $\psi_f = f(z')(dz')^{\otimes k'/2}$ of weight k'. Here, $f(z')$ is holomorphic for $z' \in \mathcal{H}'$, i.e. $\operatorname{Im} z' < 0$, and ψ_f is invariant under the action of Γ on $\mathcal{H}'$. The mapping $\mathcal{P}$ is given by

$$f \to \big[f(\bar z)(d\bar z)^{\otimes k/2}\big]\, d\bar z\,. \tag{III.C.15}$$

Here the expression in brackets is an *anti*-holomorphic section of the line bundle $\mathcal{O}_X(k)$, and the RHS is interpreted as a $(0,1)$ form on X with coefficients in this line bundle.

When $k = 0$, (III.C.15) gives the standard identification

$$H^1(X, \mathcal{O}_X) \cong H^0(X, \Omega^1_X)\,.$$

When $k = 1$ the line bundle $\mathcal{O}_X(-1) = \omega_X^{1/2}$ is a square root of the canonical bundle[34] so that by Kodaira-Serre duality

$$H^1\big(X, \mathcal{O}_X(-1)\big) \cong H^0\big(X, \mathcal{O}_X(-1)\big)^\vee\,.$$

Here $H^0(X, \mathcal{O}_X(-1))$ is the space of automorphic forms on X of weight one. The Penrose transform gives

$$H^0\big(X', \mathcal{O}_{X'}(1)\big) \xrightarrow{\sim} H^1\big(X, \mathcal{O}_X(-1)\big)\,,$$

and using this identification the pairing between the space $H^0(X', \mathcal{O}_{X'}(1))$ of weight one automorphic forms on X' and $H^0(X, \mathcal{O}_X(1))$ is, up to constant, given by

$$(f'(z')dz'^{1/2}) \otimes (f(z)dz^{1/2}) \to \int_X f'(\bar z) f(z)\, dz \wedge d\,\bar z\,.$$

[34]In classical terms, $\mathcal{O}_X(-1)$ is a *theta-characteristic*. Among the set of 2^{2g} such that it is special as it arises from the uniformization of X by $\mathcal{H}$. It is also the Hodge bundle for the standard realization of $\mathcal{H}$ as the period domain for elliptic curves.

III.D. **The Penrose transform in the compact case.** We begin by recalling the statement of the Borel-Weil-Bott (BWB) theorem. Let $G_{\mathbb{C}}$ be a complex simple Lie group and $B \subset G_{\mathbb{C}}$ a Borel subgroup. We assume that B contains a Cartan subgroup $T_{\mathbb{C}}$ that is the complexification of a compact maximal torus T in a compact form G_c of $G_{\mathbb{C}}$.[35] We set

$$M = G_{\mathbb{C}}/B$$

and denote by

$$L_\mu \to M$$

the homogeneous holomorphic line bundle corresponding to a weight μ. The Borel subgroup B singles out a set Φ^+ of positive roots and as usual we set

$$\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha\,.$$

(III.D.1) **Theorem (BWB):** *If $\mu+\rho$ is singular, then $H^q(M, L_\mu) = 0$ for all q. If $\mu+\rho$ is non-singular we let $q(\mu+\rho) = \#\{\alpha \in \Phi^+ : (\mu+\rho, \alpha) < 0\}$. Then $H^q(M, L_\mu) = 0$ for $q \neq q(\mu+\rho)$, and $H^{q(\mu+\rho)}(M, L_\mu)$ is the irreducible $G_{\mathbb{C}}$-module W_μ with highest weight $w(\mu+\rho)-\rho$, where w is the element of the Weyl group that moves $\mu+\rho$ into the positive Weyl chamber.*

We next set[36]

$$\check{\mathcal{W}} = G_{\mathbb{C}}/T_{\mathbb{C}}\,.$$

Then $\check{W}$ is an affine algebraic variety, and since $B = T_{\mathbb{C}}U$, where the unipotent radical U of B is affine and contractible, the EGW theorem applies to

$$\check{\mathcal{W}} \xrightarrow{\pi} M$$

to give a $G_{\mathbb{C}}$-equivariant isomorphism

$$H^q(M, L_\mu) \cong H^q_{\mathrm{DR}}\big(\Gamma(\check{W}, \Omega^\bullet_\pi \otimes \pi^{-1}L_\mu)\big). \tag{III.D.2}$$

[35]The reason for this notation will appear below.
[36]Again, the reason for this notation will appear below.

let $B' \subset G_{\mathbb{C}}$ be another Borel subgroup with $T_{\mathbb{C}} \subset B'$, $M' = G_{\mathbb{C}}/B'$ and μ' a weight giving $L'_{\mu'} \to M'$. Using the evident notations, we assume that $\mu' + \rho'$ is non-singular and set $q(\mu' + \rho') = \#\{\alpha \in \Phi'^{+} : (\mu' + \rho', \alpha) < 0\}$. We also assume that

$$w(\rho + \mu) - \rho = w'(\rho' + \mu') - \rho' ,$$

where w' for $(G_{\mathbb{C}}, B', T_{\mathbb{C}})$ is defined in the same way as for $(G_{\mathbb{C}}, B, T_{\mathbb{C}})$, so that the $G_{\mathbb{C}}$-modules W_μ and $W_{\mu'}$ are isomorphic. There is a diagram of $G_{\mathbb{C}}$-modules

(III.D.3)

$$\begin{array}{ccc} H^{q(\mu+\rho)}_{\mathrm{DR}}\big(\Gamma(\check{\mathcal{W}}, \Omega^{\bullet}_{\pi} \otimes \pi^{-1} L_\mu\big) & \dashleftarrow & H^{q(\mu'+\rho')}_{\mathrm{DR}}\big(\Gamma(\check{\mathcal{W}}, \Omega^{\bullet}_{\pi'} \otimes \pi'^{-1} L'_\mu\big) \\ \wr\Vert & & \wr\Vert \\ H^{q(\mu+\rho)}(M, L_\mu) & \cong & H^{q(\mu'+\rho')}(M', L'_\mu) . \end{array}$$

(III.D.4) **Theorem:** *Assume that $q(\mu' + \rho') = 0$ and $q(\mu + \rho) \neq 0$. Then there is a unique $G_{\mathbb{C}}$-invariant form*

$$\omega \in \Gamma(\check{\mathcal{W}}, \Omega^{q(\mu+\rho)}_{\pi} \otimes \pi^{-1} L_\mu \otimes \pi'^{-1} \check{L}'_{\mu'}) ,$$

such that multiplication by ω gives a map given by the dotted line in (III.D.3) *for which* (III.D.3) *is a commutative diagram of $G_{\mathbb{C}}$-modules.*

We will first give the proof in the special case of the first example and in section III.F we will give it for the second example. Then in the appendix to this section we will show how the general argument goes.[37] Thus we assume

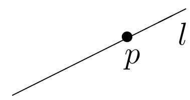
$$G_{\mathbb{C}} = \mathrm{SL}(3, \mathbb{C})$$

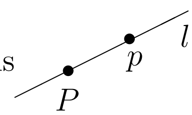
$$B = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix} \right\}$$

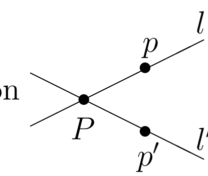
$$B' = \left\{ \begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ * & * & * \end{pmatrix} \right\} .$$

[37]The reason for doing things this way is that the computations for this result in the two examples will be the essential ones to be used in the proof of the main results in this paper.

The intersection $A = B \cap B'$ is a subgroup of $G_{\mathbb{C}}$ that contains $T_{\mathbb{C}}$, and we have the geometric interpretations

- $G_{\mathbb{C}}/B \longleftrightarrow$ flags 

- $G_{\mathbb{C}}/A \longleftrightarrow$ configurations 

- $G_{\mathbb{C}}/T_{\mathbb{C}} \longleftrightarrow$ pairs of flags in general position 

We note that $G_{\mathbb{C}}/A =: \check{\mathfrak{J}}$ is the set of pairs p, P of distinct points in $\mathbb{P}^2$, the line l being $\overline{pP}$. We have a diagram

(III.D.5)
$$\begin{tikzcd}
 & & G_{\mathbb{C}} \arrow[d] & & \\
 & & \check{\mathcal{W}} \arrow[d] \arrow[ddll, "\pi"'] \arrow[ddrr, "\pi'"] & & = G_{\mathbb{C}}/T_{\mathbb{C}} \\
 & & \check{\mathfrak{J}} \arrow[dll] \arrow[drr] & & = G_{\mathbb{C}}/A \\
G_{\mathbb{C}}/B = M & & & & M' = G_{\mathbb{C}}/B'
\end{tikzcd}$$

where the maps are

$$\begin{tikzcd}
 & (p, l; p', l') \arrow[d] & \\
 & (p, l, P = l \cap l') \arrow[dl] \arrow[dr] & \\
(p, l) & & (P, l)
\end{tikzcd}$$

Next, we denote by

$$g^{-1}dg = \begin{pmatrix} \omega_1^1 & \omega_2^1 & \omega_3^1 \\ \omega_1^2 & \omega_2^2 & \omega_3^2 \\ \omega_1^3 & \omega_2^3 & \omega_3^3 \end{pmatrix}, \qquad g \in G_{\mathbb{C}}$$

the Maurer-Cartan matrix of left-invariant forms on $G_{\mathbb{C}}$. We will show that:

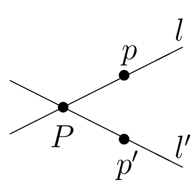
(III.D.6) *ω_3^1 gives the pullback to $G_{\mathbb{C}}$ of the form ω in Theorem III.D.4.*

From the pictures of B and B', we observe that ω_3^1 corresponds to a non-compact root that changes sign when we pass from B to B'.

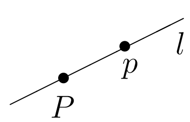
Step one: We will identify $\mathrm{SL}(3,\mathbb{C})$ with the set of frames f_1, f_2, f_3, given by a basis for $\mathbb{C}^3$ with $f_1 \wedge f_2 \wedge f_3 = 1$, and thought of as mappings $f_i : G_{\mathbb{C}} \to \mathbb{C}^3$. The exterior derivative of these vector-valued functions and the integrability condition $d^2 = 0$ give the equations for a moving frame (summation convention)

$$\begin{cases} df_i &= \omega_i^j f_j \\ d\omega_i^k &= \omega_i^j \wedge \omega_j^k \qquad \text{(Maurer-Cartan equation).} \end{cases}$$

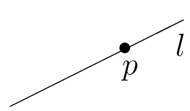
The mappings $G_{\mathbb{C}} \to \check{\mathcal{W}}, G_{\mathbb{C}} \to \check{\mathcal{J}}$ and $G_{\mathbb{C}} \to M$ are respectively

$\{f_i\} \to$ 

$p = [f_1], p' = [f_2], P = [f_3]$

$\{f_i\} \to$ 

$p = [f_1], P = [f_3]$

$\{f_i\} \to$ 

$p = [f_1], l = [f_1, f_3]\,.$

We will think of the mapping $G_{\mathbb{C}} \to M$ as given by

$$\text{(III.D.7)} \qquad \begin{cases} \omega_1^2 = 0,\ \omega_1^3 = 0 \\ \omega_3^2 = 0\,. \end{cases}$$

This means: The Pfaffian equations (III.D.7) give a Frobenius system on $G_{\mathbb{C}}$, and the leaves of the resulting foliation are the fibres of $G_{\mathbb{C}} \to M$. Geometrically, the first two equations mean that "p doesn't move," and then the third means that "in addition, l doesn't move."

The fibres of $G_{\mathbb{C}} \to \check{\mathfrak{J}}$ are given by (III.D.7) plus

$$\omega^1_3 = 0\,, \tag{III.D.8}$$

meaning that "P doesn't move."

We note that (III.D.7) and (III.D.8) correspond to the zeroes in the matrices in $\mathfrak{b}$, $\mathfrak{a}$ respectively. They have an evident root-theoretic meaning.

Step two: We observe that

$$\omega^1_3 \textit{ spans a line sub-bundle } J \subset \Omega^1_\pi\,.$$

This means: ω^1_3 is semi-basic for $G_{\mathbb{C}} \to \check{W}$, and it spans the pullback to $G_{\mathbb{C}}$ of a line sub-bundle $J \subset \Omega^1_\pi$. Geometrically, ω^1_3 measures how dP moves along the line l.

Step three: We next observe that

$$\textit{The line bundle } J \subset \Omega^1_\pi \textit{ is integrable.}$$

Geometrically this means that on the fibres of $\check{\mathcal{W}} \to M$, J is integrable. Analytically it means that for a section ψ of J

$$\begin{cases} d\psi \equiv 0 \text{ in } \Omega^2_\pi/(\operatorname{Im} J \otimes \Omega^1_\pi), \\ d_\pi\psi \equiv \psi \wedge \eta \text{ is zero in } \Omega^2_\pi\,. \end{cases}$$

Indeed, the Maurer-Cartan equation

$$d\omega^1_3 = \omega^1_3 \wedge \omega^1_1 + \omega^2_3 \wedge \omega^3_2 + \omega^3_3 \wedge \omega^1_3$$

gives the result, noting that the middle term is zero in Ω^2_π.

Step four: In our situation

$$\begin{tikzcd} & G_{\mathbb{C}} \arrow[d] & \\ & G_{\mathbb{C}}/T_{\mathbb{C}} \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ G_{\mathbb{C}}/B & & G_{\mathbb{C}}/B' \end{tikzcd}$$

the line bundles will be trivialized when pulled back up to $G_{\mathbb{C}}$. Sections of a line bundle $L'_{\mu'} \to G_{\mathbb{C}}/B'$ will be given by holomorphic functions F on $G_{\mathbb{C}}$ that transform by the character μ' when $T_{\mathbb{C}}$ acts on the right on $G_{\mathbb{C}}$. We recall that sections s of $\pi'^{-1}L'_{\mu'} \to G_{\mathbb{C}}/T_{\mathbb{C}}$ that satisfy $d_{\pi'}s = 0$ are constant along the fibres of $G_{\mathbb{C}}/T_{\mathbb{C}} \to G_{\mathbb{C}}/B'$.

Observation. *A holomorphic function F on $G_{\mathbb{C}}$ is the pullback to $G_{\mathbb{C}}$ of a section s of $\pi'^{-1}L'_{\mu'} \to G_{\mathbb{C}}/T_{\mathbb{C}}$ with $d_{\pi'}s = 0$ satisfies*

$$dF \equiv 0 \operatorname{mod}\{\omega_1^1, \omega_2^2, \omega_3^3, \omega_1^2, \omega_3^1, \omega_3^2\}\,. \tag{III.D.9}$$

Put pictorially, dF does not involve the Maurer-Cartan forms ω_j^i corresponding to the off-diagonal terms in the Lie algebra $\mathfrak{b}'$. Geometrically, F is invariant under the right action of the unipotent radical U' of B'.

A consequence of the above is

$$d_\pi F \equiv 0 \operatorname{mod}\{\omega_1^1, \omega_2^2, \omega_3^3, \omega_3^1\}\,. \tag{III.D.10}$$

Step five: Let F be the pullback to $G_{\mathbb{C}}$ of a section of $\pi'^{-1}L'_{\mu'} \to G_{\mathbb{C}}/B'$. Then

$$d_\pi(F\omega_3^1) \equiv 0\{\omega_1^1, \omega_2^2, \omega_3^3\}\,. \tag{III.D.11}$$

Proof. This follows from (III.D.9) and the Maurer-Cartan equation

$$d\omega_3^1 \equiv 0 \operatorname{mod}\{\omega_1^1 \wedge \omega_3^1, \omega_3^3 \wedge \omega_3^1, \omega_3^2 \wedge \omega_2^1\}$$

which gives

$$d_\pi\omega_3^1 \equiv 0 \operatorname{mod}\{\omega_1^1 \wedge \omega_3^1, \omega_3^3 \wedge \omega_3^1\}\,.$$

Completion of the proof of Theorem III.D.4. From the above calculations (using $\omega_1^1+\omega_2^2+\omega_3^3=0$) we have

$$\begin{cases} d_\pi F & \equiv a\omega_1^1 + b\omega_2^2 \operatorname{mod}\{\omega_3^1\} \\ d_\pi\omega_3^1 & \equiv (\tilde{a}\omega_1^1+\tilde{b}\omega_2^2)\wedge\omega_3^1 \,. \end{cases}$$

Choosing $a=-\tilde{a}$, $b=-\tilde{b}$ we obtain

$$d_\pi(F\omega_3^1)=0\,. \tag{III.D.12}$$

The choices of a,b correspond to the weight $\mu-\mu'$. Now

$$H^0_{\mathrm{DR}}\big(\Gamma(\check{W},\Omega^\bullet_{\pi'}\otimes\pi'^{-1}L'_{\mu'})\big)\cong\left\{\begin{array}{c}\text{holomorphic functions } F\\ \text{on } G_{\mathbb{C}} \text{ satisfying (III.D.9)}\end{array}\right\}.$$

For μ,μ' chosen as above, the map

$$F\to F\omega_3^1$$

induces a map

$$H^0_{\mathrm{DR}}\big(\Gamma(\check{W},\Omega^\bullet_{\pi'}\otimes\pi'^{-1}L'_{\mu'})\big)\to H^1_{\mathrm{DR}}\big(\Gamma(\check{W},\Omega^\bullet_{\pi}\otimes\pi^{-1}L_{\mu})\big)\,.$$

This map is injective on the form level. To show that it induces an isomorphism on cohomology we need to show that

- the map is injective on cohomology
- both groups have the same dimension.

The second follows from the BWB theorem (III.D.1), and the first follows from noting that, by the same argument as in steps four and five in section III.C, *the form $F\omega_3^1$ is harmonic in the sense of* [EGW].[38] The results in that work then imply that if $F\neq 0$, then $F\omega_3^1$ is non-zero in $H^1_{\mathrm{DR}}\big(\Gamma(\check{W},\Omega^\bullet_{\pi}\otimes\pi^{-1}L_{\mu})\big)$.[39] □

[38]In the appendix to this section we will give the general formula for the harmonic form that gives the Penrose transform.

[39]We do not give the details of this argument as the statement of the result will not be needed in the proof of the main results in this paper. Moreover, a proof can be given along the lines of the analogous result (III.C.3) for the basic example. We will give the proof of the injectivity of the Penrose transform for the relevant μ and μ' in the more subtle non-compact cases of our two examples and their quotients by an arithmetic group.

Appendix to section III.D: Arithmetic aspects of the Penrose transform in the compact case. Anticipating the discussion in section IV, and using the terminology and notations from that section, we want to observe that

the Penrose transform is arithmetic.

We begin by noting that M, M' and $\check{\mathcal{W}}$ are algebraic varieties defined over a number field k, and $\check{\mathcal{W}} \xrightarrow{\pi} M$, $\check{\mathcal{W}} \xrightarrow{\pi'} M'$ as well as the line bundles $L_\mu \to M$ and $L'_{\mu'} \to M'$ are also defined over k. In our two examples we have $k = \mathbb{F}$, $\mathbb{Q}$ respectively. In general, an algebraic variety X that is the set of complex points of a variety G/P where G and P are algebraic groups defined over k is itself defined over k. It follows that each of the groups $H^{q(\mu+\rho)}(M, L_\mu)$ and $H^{q(\mu'+\rho')}(M', L'_{\mu'})$ in (III.D.3) are vector spaces defined over k; in the terminology of section IV, these vector spaces have an *arithmetic structure*.

The key observations are:

(i) the vector spaces in the top row of the diagram (III.D.3) have arithmetic structures; and

(ii) the form ω will be a relative differential form on $G_{\mathbb{C}}$ that is defined over k.

In the basic example discussed in section III.C, both of these points are clear from the explicit formulas: The vector space V and bilinear form Q are defined over k, as is then $\operatorname{Sym}^k \check{V}$. Similarly, in our two examples both (i) and (ii) will be clear from the explicit formulas given in the next two sections.

In general, because it is of some interest in its own right we will give the formula for the harmonic form ω that, via the diagram (III.D.3) when $\mu'+\rho'$ is dominant, i.e. $q(\mu'+\rho') = 0$, gives the Penrose transform. For this we identify M with M' and for simplicity of notation set $\mu' = \lambda$. Then there is an element w in the Weyl group such that

$$w(\mu + \rho) = \lambda + \rho\,.$$

For any $w \in W$ we set

$$\Psi_w = w\Phi^- \cap \Phi^+ ,$$

and for any subset $\Psi = \{\psi_1, \dots, \psi_q\} \subset \Phi^+$ we set

$$\begin{cases} \langle \Psi \rangle & = \psi_1 + \cdots + \psi_q \\ \omega^{-\Psi} & = \omega^{-\psi_1} \wedge \cdots \wedge \omega^{-\psi_q} . \end{cases}$$

If $\alpha_1, \dots \alpha_r$ are the positive roots, from [Ko], pages 356 ff. we have

(i) $\rho - \langle \Psi \rangle = \frac{1}{2}(\pm\alpha_1 \pm \alpha_2 + \cdots \pm \alpha_r)$ for some choice of the signs, and as Ψ runs through all subsets of Φ^+ all choices of signs are possible;
(ii) Ψ_w and $\Psi_w^c := \Phi^+ \backslash \Psi_w$ are both closed under addition;
(iii) if $\Psi \subset \Phi^+$ has this property, then $\Psi = \Psi_\omega$ for a unique $w \in W$;
(iv) $w(\rho) = \rho - \langle \Psi_w \rangle$; and
(v) $\langle \Psi \rangle = \langle \Psi_w \rangle \Rightarrow \Psi = \Psi_w$.

From (iii) we have

$$\mu = w(\lambda) - \langle \Psi_w \rangle \ . \tag{III.D.13}$$

Let V^λ be the irreducible $G_{\mathbb{C}}$-module with highest weight λ. The dual $\check{V}^\lambda$ is the irreducible $G_{\mathbb{C}}$-module with the lowest weight $-\lambda$, and we let $\check{v}_{-\lambda}$ be a lowest weight vector. Then $w(-\lambda)$ is an extremal weight[40] for $\check{V}^\lambda$, and we let $\check{v}_{w(-\lambda)}$ be a corresponding weight vector, which is well-defined up to scaling. With this notation we set

$$\omega = \omega^{-\langle \Psi_w \rangle} \otimes \check{v}_{w(-\lambda)} \, . \tag{III.D.14}$$

(III.D.15) **Theorem:** *In the diagram* (III.D.3), *multiplication by* ω *gives the Penrose transform*

$$\mathcal{P} : H^0(M, L_\lambda) \xrightarrow{\sim} H^{q(\mu+\rho)}(M, L_\mu) \, .$$

Proof. We will use the proof to explain the way in which ω can be considered as a section in $\Gamma(\check{\mathcal{W}}, \Omega_\pi^q \otimes \pi^{-1} L_\lambda)$ when $q = q(\mu + \rho)$. Denoting

[40] That means that $|w(-\lambda)| \geqq |\xi|$ for any weight ξ. If equality holds then $\xi = w'(-\lambda)$ for some $w' \in W$ and the corresponding weight space is 1-dimensional.

by $\mathcal{O}(G_{\mathbb{C}})$ the global holomorphic functions on $G_{\mathbb{C}}$ and by $\mathcal{O}(G_{\mathbb{C}})^{\mathrm{alg}}$ the coordinate ring of $G_{\mathbb{C}}$ as an affine algebraic variety, the inclusion

$$\mathcal{O}(G_{\mathbb{C}})^{\mathrm{alg}} \hookrightarrow \mathcal{O}(G_{\mathbb{C}})$$

induces a morphism of complexes

$$\Gamma^{\mathrm{alg}}(\check{\mathcal{W}}, \Omega^{\bullet}_{\pi} \otimes \pi^{-1} L_{\lambda}) \to \Gamma(\check{\mathcal{W}}, \Omega^{\bullet}_{\pi} \otimes \pi^{-1} L_{\lambda})\,.$$ [41]

On the other hand, we have the isomorphism of $G_{\mathbb{C}}$-modules

$$\mathcal{O}(G_{\mathbb{C}})^{\mathrm{alg}} \cong \bigoplus_{\lambda} V^{\lambda} \otimes \check{V}^{\lambda} \tag{III.D.16}$$

where λ runs over the positive Weyl chamber and the matrix entries of the $G_{\mathbb{C}}$-module $V^{\lambda} \otimes \check{V}^{\lambda}$ give functions in the coordinate ring $\mathcal{O}(G_{\mathbb{C}})^{\mathrm{alg}}$ of $G_{\mathbb{C}}$.

Next we claim that, using (III.D.16), we have the isomorphism of $G_{\mathbb{C}}$-modules

$$H^{*}_{\mathrm{DR}}\big(\Gamma^{\mathrm{alg}}(\check{\mathcal{W}}, \Omega^{\bullet}_{\pi} \otimes \pi^{-1} L_{\mu})\big) \cong \bigoplus_{\lambda} V^{\lambda} \otimes H^{*}(\mathfrak{n}^{-}, \check{V}^{\lambda} \otimes \mathbb{C}_{\mu})^{T_{\mathbb{C}}} \tag{III.D.17}$$

where $\mathbb{C}_{\mu}$ is the $T_{\mathbb{C}}$-module corresponding to the weight μ. Here,

$$\mathfrak{n}^{-} = \bigoplus_{\alpha \in \Phi^{+}} \mathfrak{g}^{-\alpha}$$

and $G_{\mathbb{C}}$ acts on the summands on the RHS via its left action on V^{λ}. As a check on signs we observe that when pulled back to $G_{\mathbb{C}}$ in the diagram

$$\begin{array}{l} G_{\mathbb{C}} \\ \big\downarrow \\ \check{\mathcal{W}} = G_{\mathbb{C}}/T_{\mathbb{C}} \\ \big\downarrow{\scriptstyle \pi} \\ M \ = G_{\mathbb{C}}/B \end{array}$$

the 1-forms ω^{α}, $\alpha \in \Phi^{+}$, are semi-basic for $\check{\mathcal{W}} \to M$. This is because the holomorphic tangent space to M at the identity coset is identified

[41] A variant of Grothendieck's algebraic de Rham theorem gives that this map induces an isomorphism on cohomology, but we do not need this result.

with $\mathfrak{n}^+ = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}^\alpha$. Thus the pullback to $G_{\mathbb{C}}$ of the relative differentials Ω^1_π are spanned by the $\omega^{-\alpha}$ where $\alpha \in \Phi^+$.

With this understood we have the isomorphism of complexes

$$\Gamma^{\mathrm{alg}}(\check{\mathcal{W}}, \Omega^\bullet_\pi \otimes \pi^{-1}L_\mu) \cong \bigoplus_\lambda V^\lambda \otimes C^*(\mathfrak{n}^-, \check{V}^\lambda \otimes \mathbb{C}_\mu)^{T_{\mathbb{C}}},$$

which gives (III.D.17).

We now write

$$H^*(\mathfrak{n}^-, \check{V}^\lambda \otimes \mathbb{C}_\mu)^{T_{\mathbb{C}}} = H^*(\mathfrak{n}^-, \check{V}^\lambda)_{-\mu},$$

where the RHS denotes that part of the $T_{\mathbb{C}}$-module $H^*(\mathfrak{n}^-, \check{V}^\lambda)$ that transforms under $T_{\mathbb{C}}$ by $-\mu$. With these conventions, for the form ω in (III.D.14) we have that $V^\lambda \otimes \omega \in \Gamma(\check{\mathcal{W}}, \Omega^q_\pi \otimes \pi^{-1}L_\mu)$. The assertion in the theorem may then be inferred by the classic paper [Ko]; cf. section 5, especially lemma 5.12, there. $\square$

Remarks. There are sign and duality issues in which the above differs from [Ko], and we now comment on some of these.

First we note that when $\mu = \lambda$ belongs to the positive Weyl chambers,

$$H^0(\mathfrak{n}^-, \check{V}^\lambda) = (\check{V}^\lambda)^{\mathfrak{n}^-} = \text{lowest weight space},$$

which then transforms under $T_{\mathbb{C}}$ by $-\mu$. We also note that using (III.D.13) the form ω transforms under the action of $T_{\mathbb{C}}$ by

$$\langle \Psi_w \rangle + w(-\lambda) = -\mu.$$

Next we want to check that

$$d_\pi \omega = 0. \tag{III.D.18}$$

For this we begin with the

(III.D.19) **Lemma:** $d_\pi \omega^{-\langle \Psi_w \rangle} = 0$.

Proof. Here we are working up on $G_{\mathbb{C}}$. For any root α the Maurer-Cartan equation is

$$d\omega^{-\alpha} \equiv \sum_{\beta,\gamma} c^\alpha_{\beta\gamma} \omega^{-\beta} \wedge \omega^{-\gamma} \operatorname{mod} \check{\mathfrak{t}}_{\mathbb{C}} \wedge \check{\mathfrak{g}}_{\mathbb{C}},$$

where the summation is over the set Φ of roots and where

$$c^{\alpha}_{\beta\gamma} \neq 0 \implies \alpha = \beta + \gamma\,.$$

It follows that if $\alpha \in \Phi^+$,

$$d_\pi \omega^{-\alpha} \equiv \sum_{\beta,\gamma\in\Phi^+} c^{\alpha}_{\beta\gamma}\omega^{-\beta} \wedge \omega^{-\gamma} \bmod \check{\mathfrak{t}}_{\mathbb{C}} \wedge \check{\mathfrak{g}}_{\mathbb{C}}\,.$$

The terms in $\check{\mathfrak{t}}_{\mathbb{C}} \wedge \check{\mathfrak{g}}_{\mathbb{C}}$ represent the action of $T_{\mathbb{C}}$ on the right and will disapper at the end; we will denote modding them out simply by $\equiv$.[42] It follows that if $\Psi_w = \{\psi_1, \dots, \psi_q\} \subset \Phi^+$,

$$d_\pi \omega^{-\langle\Psi_w\rangle} \equiv \sum_j (-1)^j c^{\psi_j}_{\beta\gamma}\omega^{-\beta} \wedge \omega^{-\gamma} \wedge \omega^{-\psi_1} \wedge \cdots \wedge \widehat{\omega^{-\psi_j}} \wedge \cdots \wedge \omega^{-\psi_q}$$

where for each ψ_j the sum is over $\beta, \gamma \in \Psi^c_w$. Non-zero terms could occur only when $\psi_j = \beta + \gamma$, and by property (ii) above this does not happen. $\square$

By the lemma we have

$$d_\pi \omega = (-1)^q \omega^{-\langle\Psi_w\rangle} \wedge d_\pi \check{v}_{w(-\lambda)}\,.$$

Now

$$d_\pi \check{v}_{w(-\lambda)} = \sum_{\beta\in\Phi^+} X_{-\beta} \cdot \check{v}_{w(-\lambda)} \otimes \omega^\beta\,.$$

Only the terms where $\beta \in \Psi^c_w$ will contribute non-trivially to $d_\pi\omega_i$ and for these we have the

(III.D.20) **Lemma:** *For $\beta \in \Psi^c_w$, $X_{-\beta} \cdot \check{v}_{w(-\lambda)} = 0$.*

Proof. We have $\Phi^+ = (-\Psi_w) \cup (\Psi^c_w)$ (disjoint union). Now for every $\alpha \in \Phi^+$, $X_{-\alpha} \cdot \check{v}_{-\lambda} = 0$. It follows that for every $\beta \in w\Phi^+$,

$$X_{-\beta} \cdot \check{v}_{w(-\lambda)} = 0\,.$$

But $\beta \in \Psi^c_w \implies \beta \in w\Phi^+$, which gives the lemma. $\square$

[42]Equivalently, the symbol $\equiv$ means that up on $G_{\mathbb{C}}$ we only retain forms that are semi-basic for the fibering $G_{\mathbb{C}} \to G_{\mathbb{C}}/T_{\mathbb{C}}$.

We conclude with a remark about the form $\omega^{-\alpha}$ where α is a simple root. This corresponds to the form $\omega^{-\Psi_w}$ when $w = w_\alpha$ is the reflection in the root plane corresponding to α. In order for a root α to have the property that for $\Psi = \{\alpha\}$ both Ψ and Ψ^c are closed under addition, it is necessary that α be simple. Note that $\omega^{-\alpha}$ gives a class in $H^1(\mathfrak{n}^-, \mathbb{C})_\alpha$ where $\mathbb{C}$ is the trivial $\mathfrak{n}^-$-module. For the line bundle $L_\alpha \to M$, if as above w_α is the reflection in the α root plane, then using $w_\alpha(\rho) = \rho - \alpha$ we have

$$w_\alpha(-\alpha + \rho) - \rho = \alpha + \rho - \alpha - \rho = 0\,.$$

By the BWB theorem, $H^1(M, L_{-\alpha})$ is the trivial 1-dimensional $G_\mathbb{C}$-module, and then $\omega^{-\alpha}$ represents the generating class in $H^1(M, L_{-\alpha})$ via the EGW formalism.

This class has the following geometric interpretation: Since $G_\mathbb{C}$ acts trivially on $H^1(M, \Omega^1_M) \cong H^2(M, \mathbb{C})$, using the EGW formalism we have

$$H^1(M, \Omega^1_M) \cong H^1(\mathfrak{n}^-, \mathfrak{n}^-)\,.$$

We note that

$$\mathfrak{n}^-/[\mathfrak{n}^-, \mathfrak{n}^-] \cong \bigoplus_{\left\{\substack{\alpha\in\Phi^+ \\ \alpha \text{ simple}}\right.} \mathfrak{g}^{-\alpha}$$

so that

$$\begin{aligned} H^1(\mathfrak{n}^-, \mathfrak{n}^-/[\mathfrak{n}^-, \mathfrak{n}^-])^{T_\mathbb{C}} &\cong \bigoplus_{\left\{\substack{\alpha\in\Phi^+ \\ \alpha \text{ simple}}\right.} H^1(\mathfrak{n}^-, \mathfrak{g}^{-\alpha})^{T_\mathbb{C}} \\ &\cong \bigoplus_{\left\{\substack{\alpha\in\Phi^+ \\ \alpha \text{ simple}}\right.} H^1(\mathfrak{n}^-, \mathbb{C})_\alpha\,. \end{aligned}$$

The mapping

$$H^1(\mathfrak{n}^-, \mathfrak{n}^-)^{T_\mathbb{C}} \to H^1(\mathfrak{n}^-, \mathfrak{n}^-/[\mathfrak{n}^-, \mathfrak{n}^-])^{T_\mathbb{C}}$$

may be proved to be an isomorphism, so that combining the above we have

$$H^1(M, \Omega^1_M) \cong \operatorname{span}_\mathbb{C}\{\omega^{-\alpha} : \alpha \in \Phi^+ \text{ and } \alpha \text{ simple}\}\,.$$

With this interpretation, up to a non-zero constant $\omega^{-\alpha}$ represents $c_1(L_\alpha)$, which is consistent with the classical result that

$$H^1(M,\mathbb{Z}) \cong \operatorname{span}_{\mathbb{Z}}\{c_1(L_\alpha) : \alpha \in \Phi^+ \text{ and } \alpha \text{ simple}\}\,.$$

III.E. **The Penrose transform in the first example.** In the previous section, including the appendix to that section, we have discussed (a special case of) the following general principle:

> *Given two realizations, via the BWB theorem, $H^q(G_{\mathbb{C}}/B, L_\mu)$ and $H^{q'}(G_{\mathbb{C}}/B', L'_{\mu'})$ of an irreducible $G_{\mathbb{C}}$-module, the EGW method gives canonical holomorphic realizations of these groups, and then for certain values of q' these holomorphic realizations are canonically isomorphic via multiplication by a canonical form ω.*

The resulting identification

$$\mathcal{P} : H^{q'}(G_{\mathbb{C}}/B', L'_{\mu'}) \xrightarrow{\sim} H^q(G_{\mathbb{C}}/B, L_\mu)$$

is termed a *Penrose transform.*

In Carayol [C2] these methods are used to relate $H^1(D, L_\mu)$ and $H^0(D', L'_{\mu'})$ where $D = S\mathcal{U}(2,1)_{\mathbb{R}}/T$ with the non-classical complex structure and $D' = S\mathcal{U}(2,1)_{\mathbb{R}}/T$ with a classical complex structure. Moreover, for $\Gamma \subset S\mathcal{U}(2,1)$ a co-compact discrete group, for $X = \Gamma\backslash D$ and $X' = \Gamma\backslash D'$, Carayol uses the same methods to establish an injection

$$\mathcal{P} : H^0(X', L'_{\mu'}) \xrightarrow{\sim} H^1(X, L_\mu) \tag{III.E.1}$$

for certain μ's and μ''s satisfying $\mu + \rho = \mu' + \rho'$. The LHS of (III.E.1) has arithmetic-algebro-geometric significance, in particular a Galois action, whereas the RHS does not have such, at least in any direct fashion. Taking two different choices of $\mu, \tilde{\mu}$, Carayol shows that for $\alpha \in H^1(X, L_\mu)$ and $\widetilde{\alpha} \in H^1(X, L_{\tilde{\mu}})$ the cup-product

$$\alpha\widetilde{\alpha} \in H^2(X, L_{\mu+\tilde{\mu}})$$

is non-zero. The character $\mu + \tilde{\mu}$ corresponds to a degenerate limit of discrete series, and it is known that such can never be the infinite component of an automorphic representation arising from the cohomology-coherent or l-adic — of a Shimura variety. The calculations in [C1] and [C2] are explicit "in coordinates," and one of the purposes of this work is to present proofs of the results of Carayol in a conceptual, geometric framework, one which shows what form extensions of the method might take. The essential idea appears in the proof of (III.D.6) above, and we will now show how this applies to the non-compact case, and in the next section to quotients of such by a co-compact discrete group. The results in this section are (III.E.13) and (III.F.14) below.

Step one: With the notations from section II.B, we consider the restriction of the diagram (III.D.5) to the correspondence space $\mathcal{W}$

(III.E.2)
$$\begin{tikzcd} & \mathcal{W} \arrow[ddl, "\pi"'] \arrow[d, "\tau"] \arrow[ddr, "\pi'"] & \\ & \mathcal{J} \arrow[dl, "\sigma"] \arrow[dr, "\sigma'"'] & \\ D & & D' \end{tikzcd}$$

We will now denote by ω_i^j the restriction to the open subset lying over $\mathcal{W}$ in $G_{\mathbb{C}} = \mathrm{SL}(3, \mathbb{C})$ of the Maurer-Cartan forms and we set

$$\omega = \omega_3^1 .$$

(III.E.3) **Proposition:** *ω is a holomorphic section of*

$$\Omega_\pi^1 \otimes \pi^{-1} F_{(-2,1)} .$$

Proof. Denoting congruence modulo $\Omega_\pi^\bullet$ by $\equiv_\pi$, by the Maurer-Cartan equation we have

$$d\omega_3^1 \equiv_\pi (\omega_3^3 - \omega_1^1) \wedge \omega_3^1 .$$

From $\omega_1^1 + \omega_2^2 + \omega_3^3 = 0$ we obtain

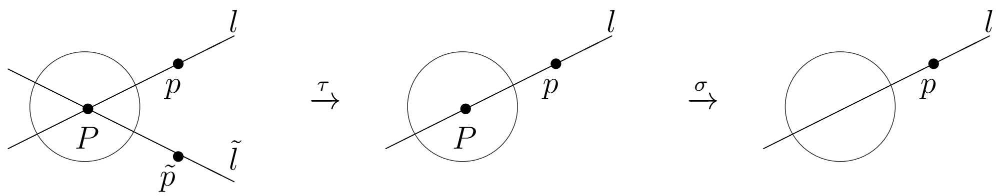
$$d\omega_3^1 \equiv_\pi (-2\omega_1^1 - \omega_2^2) \wedge \omega_3^1 .$$

From Proposition (III.B.3), we obtain that over D

$$F_{(a,b)} = L_{-ae_1^* + be_2^*}\,,$$

from which the result follows. $\square$

Remark. The maps in (III.E.2) are



The fibres are

- $$\tau^{-1}(p,l,P) = \left\{ \begin{array}{l} \text{set of lines } \tilde{l} \text{ through } P,\ \tilde{l} \neq l, \\ \text{and points } \tilde{p} \in \tilde{l} \text{ such that } \overline{p\tilde{p}} \subset \mathbb{B}^c \end{array} \right\}$$
  $$= \text{disc bundle over } \mathbb{C}$$
- $$\sigma^{-1}(p,l) = \{P \in l \cap \mathbb{B}\} \cong \Delta\,.$$

These are contractible Stein manifolds, so that at least one half of the proof of the EGW theorem applies to each map. However,

$$\mathcal{J} \cong \{(p,P) : P \in \mathbb{B} \text{ and } p \in \mathbb{B}^c\}$$

is not Stein. Thus even though the diagram

(III.E.4)
$$\begin{tikzcd} & \mathcal{J} \arrow[dl] \arrow[dr] & \\ D & & D' \end{tikzcd}$$

is the most natural one to interpolate between D and D', we need to go up to the correspondence space $\mathcal{W}$ to be able to apply the EGW theorem to holomorphically realize the cohomologies of D and D' and then to relate them via the Penrose transform. This situation is the general one when B and B' are not "opposite" Borel subgroups. In the non-opposite case for the group $A = B \cap B'$ we may expect to have

$$\begin{tikzcd} & \mathcal{J} \subset G_{\mathbb{C}}/A \arrow[dl] \arrow[dr] & \\ D & & D' \end{tikzcd}$$

as the natural space to connect D and D'.

Referring to (III.E.4), even though $\mathcal{J}$ is not Stein, the geometry is reflected in the exact sequence

$$0 \to \tau^*\Omega^1_\sigma \to \Omega^1_\pi \to \Omega^1_\tau \to 0\,, \tag{III.E.5}$$

where the geometric meanings are

- $\tau^*\Omega^1_\sigma$ means dP moves along l,
- Ω^1_π means $d\tilde{p}, d\tilde{l}$ move subject to $d\left\langle \tilde{l}, \tilde{p}\right\rangle = 0$,
- Ω^1_τ means $d\tilde{p}$ moves ($\tilde{l} = \overline{P\tilde{p}}$).

The exact sequence (III.E.5) gives a filtration of $\Omega^\bullet_\pi$. For any line bundle $L \to D$ we may tensor (III.E.5) with

$$\pi^{-1}L \cong \tau^{-1}(\sigma^{-1}L)$$

to obtain a spectral sequence

$$E_0^{p,q} = \Gamma\big(\mathcal{W}, \Omega^q_\tau \otimes \tau^*\Omega^p_\sigma(\pi^{-1}L)\big) \Rightarrow H^{p+q}_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi(\pi^{-1}L))\big)\,.$$

For fixed p, the relative differentials are d_τ; since the fibres of τ are contractible and Stein we may apply the proof of EGW to infer that

$$E_1^{p,q} = H^1\big(\mathcal{J}, \Omega^p_\sigma(\sigma^{-1}L)\big)\,.$$

One may then identify the canonical form ω as representing a class in the image of the natural mapping

(III.E.6)

$$\begin{array}{ccc} H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{J}, \Omega^\bullet_\sigma(\sigma^{-1}\mathcal{O}_D(-2,1)))\big) & \xrightarrow{\tau^*} & H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi(\pi^{-1}\mathcal{O}_D(-2,1)))\big) \\ \wr\| & & \\ E_2^{1,0}\,. & & \end{array}$$

Step two: We want to relate the following

- over D we have the line bundles $F_{(a,b)}$;
- over D' we have the line bundles $F'_{(a',b')}$;
- over $\mathcal{W}$ we have the homogeneous line bundles
$$\mathcal{L}_{e_i^*} \to \mathcal{W}$$
given by the identification $\check{\mathcal{W}} \cong G_{\mathbb{C}}/T_{\mathbb{C}}$ and the characters of $T_{\mathbb{C}}$ corresponding to the e_i^*.

(III.E.7) **Proposition:** *Over $\mathcal{W}$ we have*

$$\begin{cases} {\pi'}^{-1}F'_{(-1,0)} \cong \pi^{-1}F_{(1,-1)} \\ {\pi'}^{-1}F'_{(0,-1)} \cong \pi^{-1}F_{(0,-1)} \,. \end{cases}$$

Corollary. *Over $\mathcal{W}$ we have*

$$\begin{cases} {\pi'}^{-1}F'_{(a',b')} \cong \pi^{-1}F_{(-a',a'+b')} \\ {\pi'}^{-1}F'_{(a',b')} \otimes \pi^{-1}F_{(-2,1)} \cong \pi^{-1}F_{(-a'-2,a'+b'+1)} \,. \end{cases}$$

Proof. The result follows from

$$\begin{cases} \pi^{-1}F_{(-1,0)} & \cong \mathcal{L}_{e_1^*}, \pi^{-1}F_{(0,-1)} \cong \mathcal{L}_{-e_2^*} \\ {\pi'}^{-1}F'_{(-1,0)} & \cong \mathcal{L}_{e_3^*}, {\pi'}^{-1}F'_{(0,-1)} \cong \mathcal{L}_{-e_2^*} \,. \end{cases}$$ □

Definition. *The Penrose-transform*

(III.E.8) $$\mathcal{P} : H^0(D', L'_{(a',b')}) \to H^1(D, L_{(a,b)})\,,$$

where $a = -a' - 2$ and $b = a' + b' + 1$, is defined by the commutative diagram

$$\begin{array}{ccc} H^0_{DR}\big(\Gamma(\mathcal{W}, \Omega^\bullet_{\pi'} \otimes {\pi'}^{-1}L'_{(a',b')})\big) & \xrightarrow{\omega} & H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^\bullet_{\pi} \otimes \pi^{-1}L_{(a,b)})\big) \\ \wr\!\| & & \wr\!\| \\ H^1(D', L'_{(a',b')}) & \xrightarrow{\quad\mathcal{P}\quad} & H^1(D, L_{(a,b)})\,. \end{array}$$

Step three: We begin with the

(III.E.9) **Observation:** *For $F \in H^0(D', L'_{(a',b')}) \cong H^0_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^\bullet_{\pi'} \otimes {\pi'}^{-1}L'_{(a',b')}\big)$,*

$$F\omega \in \Gamma(\mathcal{W}, \Omega^1_\pi \otimes \pi^{-1}L_{(a,b)})$$

is harmonic.[43]

Proof. Lifted up to the open set in $G_\mathbb{C}$ lying over $\mathcal{W}$, F is a function of f_1, f_2, f_3 of the form

$$F = F(f_3, f_1 \wedge f_3)\,.$$

If $\alpha = e_3^* - e_1^*$ is the root with

$$\omega^1_3 = \omega^{-\alpha}\,,$$

[43]This observation is not needed logically for the sequel. We have put it in to connect to the methods used in [EGW].

then the harmonic condition from [EGW] is

$$X_\alpha \cdot (X_{-\alpha} \rfloor F\omega) = X_\alpha \cdot F = 0\,.$$

This is equivalent to

$$F_3^1 = 0 \iff \text{the coefficient of } \omega_1^3 \text{ in } dF \text{ is zero.}$$

By the chain rule, dF will be a linear combination of the forms in df_3 and in $d(f_1 \wedge f_3)$. The former are the ω_3^j, and for the latter we have

$$\begin{aligned} d(f_1 \wedge f_3) &\equiv (df_1) \wedge f_3 \operatorname{mod}\{\omega_3^3, \omega_3^2, \omega_3^1\} \\ &\equiv 0 \operatorname{mod}\{\omega_1^1, \omega_1^2, \omega_3^3, \omega_3^2 \wedge \omega_3^1\} \end{aligned}$$

since $f_3 \wedge f_3 = 0$.

Since ω_1^3 does not appear in the bracket term we have $F_3^1 = 0$. $\square$

Theorem 2.13 in [EGW] gives conditions on (a,b) such that a de Rham class in $H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^1_\pi \otimes \pi^{-1}L_{(a,b)})\big)$ has a unique harmonic representative. Unfortunately, this result is not sharp enough for our purposes. Geometrically, one may say that the reason for this is that the [EGW] proof uses the diagram

$$\begin{tikzcd} \mathcal{W} \arrow[rr] \arrow[dr] & & \mathcal{I} \arrow[dl] \\ & D & \end{tikzcd}$$

rather than the diagram (III.E.4) which more clearly captures the geometric relationship between D and D'. This brings us to the

(III.E.10) **Proposition:** *(i) If $H^0(D', L'_{(a',b')}) \neq 0$, then $b' \geqq 0$.*

(ii) The Penrose transform (III.E.8) *is injective if $b' \leqq 0$. The common solutions to (i) and (ii) are $b' = 0$.*

Remarks: (i) In terms of (a,b) these conditions are

$$\begin{cases} a+b+1 \geqq 0 \\ a+b+1 \leqq 0\,. \end{cases}$$

(ii) The Weyl chamber where $H^0_{(2)}(D', L'_{(a',b')})$ is non-zero is given by

$$\begin{cases} b'+1 > 0 \\ a'+b'+2 < 0\,. \end{cases} \tag{III.E.11}$$

If $b' = 0$ these reduce to

(III.E.12) $$a' \leqq -3\,.$$

As we have seen, the pullback $\omega'_{\mathbb{B}}$ to D' of the canonical bundle on $\mathbb{B}$ is given by

$$\omega'_{\mathbb{B}} = F'_{(-3,0)}\,.$$

Also, we have noted that the pullback $\mathbb{V}'^{1,0}_{+}$ to D' of the Hodge bundle $\mathbb{V}^{1,0}_{+}$ over $\mathbb{B}$ is given by

$$\mathbb{V}'^{1,0}_{+} = F'_{(-1,0)}\,.$$

Thus

$$\omega'_{\mathbb{B}} = F'_{(-3,0)}\,.$$

We set $\omega'^{\otimes k/3}_{\mathbb{B}} = F'_{(-k,0)} = \otimes^k V'^{1,0}_{+}$ and in section IV will define Picard modular forms of weight k to be Γ-invariant sections of $\omega'^{\otimes k/3}_{\mathbb{B}}$. Picard modular forms of weight $k \geqq 1$ then give sections of

$$F'_{(-k,0)} \to D'\,.$$

Comparing with (III.E.12) we have the

(III.E.13) **Corollary:** *The Penrose transform*

$$\mathcal{P} : H^0(D', F'_{(-3-l,0)}) \to H^1(D, F_{(l_1,-2-1)})$$

is injective for $l \geqq 0$.

In particular, $\mathcal{P}$ will be seen to be injective on Picard modular forms of weight $k \geqq 3$.

Remark. In Carayol (cf. Proposition (3.1) in [C2]) it is proved that

(III.E.14) $\mathcal{P}$ *is injective for* $b' \geqq 0, a' + b' + 2 \leqq 0\,.$

The common solutions to (III.E.10) and (III.E.14) are

$$b' = 0, \qquad a' \leqq -2\,.$$

The solutions to (III.E.11) when $b' = 0$ are

$$a' < -2$$

which is exactly the range in (III.E.13).

Proof of (ii). The first step is to use the diagram (III.E.4) and the spectral sequence arising from (III.E.5) to reduce the question to one on $\mathcal{J}$. The spectral sequence leads to the maps

$$\begin{aligned} H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{J},\Omega^\bullet_\sigma\otimes\sigma^{-1}L_{(a,b)})\big) &\xrightarrow{\tau^*} H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{W},\tau^*\Omega^\bullet_\sigma\otimes\pi^{-1}L_{(a,b)})\big) \\ &\longrightarrow H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{W},\Omega^\bullet_\pi\otimes\pi^{-1}L_{(a,b)})\big). \end{aligned}$$

We will show that

(a) $F\omega\in\Gamma(\mathcal{J},\Omega^1_\sigma\otimes\sigma^{-1}L_{(a,b)})$;

(b) the image of $F\omega$ under the natural map

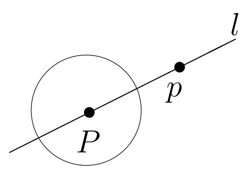
$$H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{J},\Omega^\bullet_\sigma\otimes\sigma^{-1}F_{(a,b)}\big)\to H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{W},\Omega^\bullet_\pi\otimes\pi^{-1}F_{(a,b)})\big)$$

is non-zero in $H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{W},\Omega^\bullet_\pi\otimes\pi^{-1}L_{(a,b)})\big)$ for (a,b) in the range stated in the Proposition.

Proof of (b): We let $G_{\mathbb{C}}(\mathcal{J})$ be the inverse image of $\mathcal{J}$ under the mapping

$$(f_1,f_2,f_3)\to([f_1]\in\mathbb{B}^c,[f_3]\in\mathbb{B},[f_1\wedge f_3])$$

where the RHS is the point



of $\mathcal{J}$ given by $p=[f_1]$, $P=[f_3]$ and $l=\overline{Pp}=f_1\wedge f_3$. The 1-forms $\omega^2_1,\omega^3_1,\omega^2_3,\omega^1_3$ are semi-basic for $G_{\mathbb{C}}(\mathcal{J})\to\mathcal{J}$, and $\omega^2_1,\omega^3_1,\omega^2_3$ are semi-basic for $G_{\mathbb{C}}(\mathcal{J})\to D$. Then $\omega=\omega^1_3$, $F=F(f_3,f_1\wedge f_3)$ and

$$d(F\omega)\equiv 0\bmod\{\omega^1_1,\omega^2_2,\omega^3_3,\omega^2_1,\omega^3_1,\omega^2_3\}$$

implies that $F\omega\in\Gamma(\mathcal{J},\Omega^1_\sigma\otimes\sigma^{-1}L_{(a,b)})$.[44]

Suppose now that

$$F\omega=d_\pi G$$

where $G\in\Gamma(\mathcal{W},\pi^{-1}L_{(a,b)})$. Pulling G back to the open subset $G_{\mathbb{C}}(\mathcal{J})$ of $G_{\mathbb{C}}$ we have that $G=G(f_1,f_2,f_3,f_1\wedge f_3,f_2\wedge f_3)$. Then $F\omega=$

[44]Here, the ω^i_i's give the scaling, as we have seen above.

$d_\pi G$ implies that dG has no ω_2^1, ω_2^3 term, which then gives that $G = G(f_1, f_3, f_1 \wedge f_3)$, and when the scaling is taken into account

$$G \in \Gamma(\mathcal{J}, \sigma^{-1} L_{(a,b)})\,.$$

This reduces the question to one on $\mathcal{J}$; we have to show that the equation on $\mathcal{J}$

$$F\omega = d_\sigma G \tag{III.E.15}$$

implies that $F = 0$. We will prove the stronger result

> *For* (a', b') *in the range stated in Proposition* (III.E.10)*, equation* (III.E.15) *implies that* $G = 0$.

The idea is to show that (i) the maximal compact subvarieties $Z \subset D$ have natural lifts to compact subvarieties $\widetilde{Z} \subset \mathcal{J}$, and the $\widetilde{Z}$ cover $\mathcal{J}$; (ii) the restrictions $G\big|_{\widetilde{Z}}$ are zero. In fact, we have that $Z \cong \mathbb{P}^1$ and under the projection $\sigma : \widetilde{Z} \to Z$ we will show that

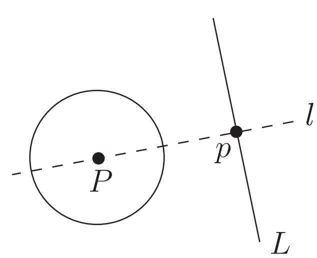
$$\sigma^{-1} F_{(a,b)}\big|_{\widetilde{Z}} \cong \mathcal{O}_{\mathbb{P}^1}(a+b)\,. \tag{III.E.16}$$

Thus

$$G\big|_{\widetilde{Z}} \in H^0(\mathcal{O}_{\mathbb{P}^1}(a+b))\,,$$

and from (III.E.8) we see that the range of (a', b') in (III.E.10) is exactly $a + b < 0$.

For the details, we identify $\mathcal{J}$ with pairs $(P, p) \in \mathbb{B} \times \mathbb{B}^c$ and $\overline{\mathbb{B}}$ with lines $L \subset \mathbb{B}^c$. Then $\mathbb{B} \times \overline{\mathbb{B}} = \mathcal{U}$ is the cycle space, and we have seen that each point $(P, L) \in \mathcal{U}$ gives a maximal compact subvariety $Z(P, L) \subset D$ as in the picture



where

$$Z(P, L) = \{(p, l) \in D\} \cong \mathbb{P}^1\,.$$

The lift $\widetilde{Z}(P,L)\subset \mathcal{J}$ of $Z(P,L)$ is then given by the picture

$$\widetilde{Z}(P,L)=\{(p,l,P)\in\mathcal{J}\}$$

where P is constant. We have

$$\begin{tikzcd} & \widetilde{Z}(P,L) \arrow[dl, "f"'] \arrow[dr, "\check{f}"] & \\ \mathbb{P}^2 & & \check{\mathbb{P}}^2 \end{tikzcd}$$

where

$$\begin{cases} f(p,l,P)=p \\ \check{f}(p,l,P)=l\,. \end{cases}$$

From this we may infer (III.E.16) where $\widetilde{Z}=\widetilde{Z}(P,L)$. This completes the proof of (ii) in Proposition (III.E.10).

The proof of (i) is similar. Given $(P,L)\in\mathcal{U}$ we define $Z'(P,L)\subset D'$ by

$$Z'(P,L)=\{(P,L)\in D'\}$$

in the above figure. Then we have

$$\begin{tikzcd} & Z'(P,L) \arrow[dl, "f'"'] \arrow[dr, "\check{f}'"] & \\ \mathbb{P}^2 & & \check{\mathbb{P}}^2 \end{tikzcd}$$

where

$$\begin{cases} f'(P,l)=P \\ \check{f}'(P,l)=l\,. \end{cases}$$

Since P is fixed we have that

$$F'_{(a',b')}\big|_{Z'(P,L)}\cong \mathcal{O}_{\mathbb{P}^1}(b')\,.$$

If follows that

$$b'<0\Rightarrow \Gamma(\mathcal{J},\sigma'^{-1}F'_{(a',b')})=0\Rightarrow \Gamma(D',F'_{(a',b')})=0$$

where $\sigma'(P,p)=P$. $\square$

Discussion: The argument in [C2] is rather different in that Carayol uses the pseudo-concavity of $\mathcal{J}$ rather than the compact subvarieties. In outline his proof goes as follows.

Since f_1 and f_3 obviously determine $f_1 \wedge f_3$, we may write

$$G(f_1, f_3, f_1 \wedge f_3) = H(f_1, f_3)\,.$$

Then for each fixed f_3 the LHS is bi-homogeneous of degree (a, b) in f_1 and $f_1 \wedge f_3$. The RHS is then bi-homogeneous of degree $a + b = b' - 1$ in f_1 and $b = a' + b' + 1$ in f_3. Now as noted above

$$\mathcal{J} \cong \mathbb{B} \times \mathbb{B}^c$$

where $[f_3] \in \mathbb{B}$ and $[f_1] \in \mathbb{B}^c$. For fixed f_3, $H(f_1, f_3)$ is a holomorphic function defined for $f_1 \in (\mathbb{C}^3 \backslash \{0\}) \backslash \widetilde{\mathbb{B}}^c$, where $\widetilde{\ }$ denotes the inverse image in $\mathbb{C}^3 \backslash \{0\}$ of $\mathbb{B}^c \subset \mathbb{P}^2$. By Hartogs' theorem, $H(f_1, f_3)$ extends to a holomorphic function of f_1 to all of $\mathbb{C}^3$ where it is homogeneous of degree $b' - 1$. Then if $b' \leqq 1$, the case we shall be primarily interested in, it follows that $G = 0$.

Remark. The above raises the following interesting point. The form $F\omega$, and in particular the Maurer-Cartan form $\omega = \omega_3^1$, are defined on the open set $G_{\mathbb{C}}(\mathcal{J}) \subset G_{\mathbb{C}}$. Now $\mathrm{GL}(3, \mathbb{C})$ is given by non-singular matrices

$$g = (f_1, f_2, f_3)\,,$$

and the Maurer-Cartan form ω_3^1 is a holomorphic, rational function in the matrix entries of g. In fact, aside from scaling ω_3^1 depends only on f_1, f_3 and one may ask for the behavior of ω_3^1 along the locus $f_1 \wedge f_3 = 0$. The answer is that in the open set $f_1 \wedge f_2 \neq 0$, $f_3 \wedge f_2 \neq 0$ *along the divisor* $f_1 \wedge f_2 \wedge f_3 = 0$ *the form* ω_3^1 *has a pole of order two.* In fact, we have already seen this illustrated in the basic example.

As noted in [C2], the above argument gives the following

(III.E.17) **Observation:** *Every section* $s \in \Gamma(D, F_{(a,b)})$ *is the restriction to* D *of a section* $\hat{s} \in \Gamma(\check{D}, \check{F}_{(a,b)})$.

Proof. The section s lifts to a function $(f_1, f_1 \wedge f_3)$ defined on an open set of $G_{\mathbb{C}}$ and homogeneous of degree (a, b) in $(f_1, f_1 \wedge f_3)$. We then define

$$S(f_1, f_3) = s(f_1, f_1 \wedge f_3)$$

and apply Hartogs' theorem to S to give (III.E.17) (cf. [C2] for the details). □

(III.E.18) **Corollary:**

$$H^0(D, F_{(k-2,1-k)}) = \{0\} \text{ for all } k \in \mathbb{Z}\,.$$

Proof. By (III.E.17), we must show $H^0(\check{D}, F_{(k-2,1-k)}) = \{0\}$. Referring to (III.D.1), for $k \in \mathbb{Z}$ (and $\mu_k = \frac{k-3}{3}\alpha_1 + \frac{2k-3}{3}\alpha_2$) we have

$$\begin{cases} \mu_k + \rho \text{ singular } (k = 1, 2) \\ \text{ or} \\ q(\mu_k + \rho) = 1 \end{cases}$$

which gives the result. □

III.F. **The Penrose transform in the second example.** The objectives of this section are

(i) to define the Penrose transform

(III.F.1) $$\mathcal{P} : H^0(D', L'_{\mu'}) \to H^1(D, L_\mu)$$

in the second example, where D and D' are $\mathrm{Sp}(4, \mathbb{R})/T$ with the non-classical and classical complex structures described in section II.B above;

(ii) to show that $\mathcal{P}$ is injective for certain μ and μ'.

The discussion will be carried out in several steps, the main intermediate result being Proposition (III.F.9) and end result being Theorem (III.F.14) below.

Step one: We first will carry out for Sp(4) the calculations that were given for $S\mathcal{U}(2, 1)$ just below the statement of Theorem (III.D.4). As was done there, we first discuss the compact case where we have

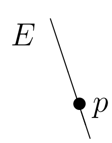
$$\begin{cases} M = G_{\mathbb{C}}/B \\ M' = G_{\mathbb{C}}/B' \end{cases}$$

where B, B' are the Borel subgroups given by (I.C.2) and (I.C.4). Of course, $M = \check{D}$ and $M' = \check{D}'$ are isomorphic as homogeneous complex manifolds, but after making this identification D and D' will be different $G_{\mathbb{R}}$ orbits.

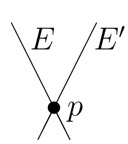
The first step is to describe in the compact case the diagram

(III.F.2)
$$\begin{tikzcd}
 & & G_{\mathbb{C}} \arrow[d] & & \\
 & & \check{\mathcal{W}} \arrow[d] \arrow[ddll, "\pi"'] \arrow[ddrr, "\pi'"] & & = G_{\mathbb{C}}/T_{\mathbb{C}} \\
 & & \check{\mathcal{J}} \arrow[dll] \arrow[drr] & & = G_{\mathbb{C}}/A \\
G_{\mathbb{C}}/B = M & & & & M' = G_{\mathbb{C}}/B'
\end{tikzcd}$$

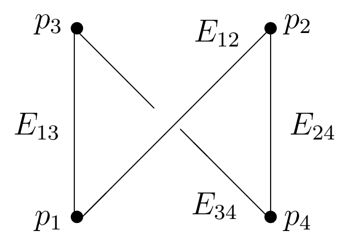
which is the analogue of (III.D.5). Here the pictures are

- $G_{\mathbb{C}}/B \longleftrightarrow$ 

$=$ Lagrange flag

- $G_{\mathbb{C}}/A \longleftrightarrow$ 

$= \left\{ \begin{array}{c} \text{pairs of} \\ \text{Lagrange flags} \\ \text{meeting in} \\ \text{a point} \end{array} \right\}$

- $G_{\mathbb{C}}/T_{\mathbb{C}} \longleftrightarrow$  $=$ Lagrange quadrilaterals

- $G_{\mathbb{C}} =$ frames (f_1, f_2, f_3, f_4).

The maps in (III.F.2) are

$$\begin{cases} (f_1, f_2, f_3, f_4) \longrightarrow (p_1, p_2, p_3, p_4), & p_i = [f_i] \\ (p_1, p_2, p_3, p_4) \longrightarrow (p_1, E_{13}, E_{12}), & \\ \quad (p, E, E') \longrightarrow (p, E) \text{ and } (p, E, E') \to (p, E'), & p = p_1 \text{ and} \\ & E = E_{13}, E' = E_{12}\,. \end{cases}$$

Step two: From (III.B.1) we have

$$\text{(III.F.3)} \qquad \begin{array}{ccc} H^1_{\mathrm{DR}}\big(\Gamma(\check{\mathcal{W}}, \Omega^\bullet_\pi \otimes \pi^{-1} L_\mu\big) & \overset{\omega}{\dashleftarrow} & H^0_{\mathrm{DR}}\big(\Gamma(\check{\mathcal{W}}, \Omega^\bullet_{\pi'} \otimes \pi'^{-1} L'_{\mu'}\big) \\ \wr\Vert & & \wr\Vert \\ H^1(M, L_\mu) & & H^0(M', L'_{\mu'}). \end{array}$$

We shall show that (cf. Proposition (III.F.9) below)

(III.F.4) *The form ω^3_2 gives the pullback to $G_{\mathbb{C}}$ of a canonical form*

$$\omega \in \Gamma\big(\mathcal{W}, \Omega^1_\pi \otimes \pi^{-1} L_\mu \otimes \pi'^{-1} \check{L}'_{\mu'}\big)$$

that gives the map indicated by the dotted line in (III.F.3)*.*

Here, μ and μ' are characters of $T_{\mathbb{C}}$ that give homogeneous line bundles $L_\mu, L'_{\mu'}$ over M, M', where $\mu + \rho = \mu' + \rho'$ (see below). The calculations are parallel to those given below (III.D.5).

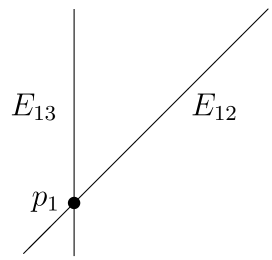
Proof of (III.F.4)*.* The method is similar to that used below (III.D.6). The fibres of the map $G_{\mathbb{C}} \to M$ are given by

$$\text{(III.F.5)} \qquad \begin{cases} \omega^2_1 = 0,\ \omega^3_1 = 0,\ \omega^4_1 = 0 \\ \omega^2_3 = 0 \end{cases}$$

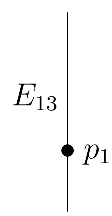
where we have used $\omega_1^2 + \omega_3^4 = 0$ and $\omega_1^3 + \omega_2^4 = 0$ from (I.C.1). The fibres of $G_{\mathbb{C}} \to \check{\mathcal{J}}$ are given by (III.F.5) together with

$$\omega_2^3 = 0\,. \tag{III.F.6}$$

Geometrically the above means that along the fibres of $G_{\mathbb{C}} \to \check{\mathcal{J}}$ the configuration



is constant, while along the fibres of $G_{\mathbb{C}} \to M$ the configuration



is constant.

We next observe that

ω_2^3 *spans an integrable sub-bundle* $J \subset \Omega_\pi^1$.

Indeed, using $\omega_2^4 + \omega_1^3 = 0$, the Maurer-Cartan equation

$$\begin{aligned} d\omega_2^3 &= \omega_2^1 \wedge \omega_1^3 + \omega_2^2 \wedge \omega_2^3 + \omega_2^3 \wedge \omega_3^3 + \omega_2^4 \wedge \omega_4^3 \\ &= \omega_2^1 \wedge \omega_1^3 + (\omega_2^2 - \omega_3^3) \wedge \omega_2^3 + \omega_4^3 \wedge \omega_1^3 \end{aligned}$$

gives

$$d\omega_2^3 \equiv_\pi (\omega_2^2 - \omega_3^3) \wedge \omega_2^3\,. \tag{III.F.7}$$

This implies first that J is a sub-bundle and secondly that it is integrable.

Step three: We next have the observation

> *Let F be a holomorphic function, defined in an open set in $G_{\mathbb{C}}$ that is the pullback of a holomorphic section of $L'_{\mu'} \to M'$. Then*

$$dF \equiv 0 \operatorname{mod} \left\{\omega_j^j, \omega_1^2, \omega_1^3, \omega_1^4, \omega_2^3\right\}. \tag{III.F.8}$$

Here, $1 \leqq j \leqq 4$. It follows that, where again $1 \leqq j \leqq 4$,

$$d_\pi F \equiv 0 \operatorname{mod} \left\{\omega_j^j, \omega_2^3\right\}.$$

From (III.F.7) and (III.F.8) we conclude that

$$d_\pi(F\omega_2^3) \equiv 0\,.$$

We now let ω be the form on $\check{\mathcal{J}}$ that pulls back to ω_2^3 on $G_{\mathbb{C}}$. More precisely, there is a line bundle $L \to \check{\mathcal{J}}$ that will be identified below and then

$$\omega \;\; \textit{is a section of} \;\;\; J \otimes L \subset \Omega_\pi^1 \otimes L\,.$$

The above calculations then give the

(III.F.9) **Proposition:** *For $F \in H^0(M, L'_{\mu'})$, the map*

$$F \to F\omega$$

induces a map given by the dotted arrow in (III.F.3).

Finally it remains to identify the relation among the line bundles $L'_{\mu'}, L_\mu$ and L. Let

$$\begin{cases} L'_{\mu'} &= F'_{(a',b')} \\ L_\mu &= F_{(a,b)}\,. \end{cases}$$

Then from (III.F.7) it follows that

$$L = \pi^{-1}F_{(0,2)}\,.$$

Using this and $\pi^{-1}F_{(a,b)} = \pi'^{-1}F'_{(a,b)}$ on $\mathcal{J}$, the vertical identifications in (III.F.3) give for the Penrose transformation

$$\mathcal{P} : H^0(D', L'_{(a',b')}) \to H^1(D, L_{(a,b)})$$

$$\begin{cases} a = a' \\ b = b' + 2\,. \end{cases} \tag{III.F.10}$$

From (II.C.7) this is the same as

$$\mu + \rho = \mu' + \rho' .$$

Step four: For $F \in H^0(D', F'_{(a',b')})$, we may pull F back to an open set in $G_{\mathbb{C}}$ where it is a holomorphic function

$$F(f_1, f_1 \wedge f_2) .$$

It follows that

$$F\omega \in \operatorname{Image}\left\{ H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{J}, \Omega^\bullet_\sigma \otimes \sigma^{-1}F_{(a,b)})\big) \to H^1\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi \otimes \pi^{-1}F_{(a,b)})\big)\right\} .$$

Suppose that

$$F\omega = d_\pi G \tag{III.F.11}$$

where $G \in H^0_{\mathrm{DR}}\big(\Gamma(\mathcal{W}, \Omega^\bullet_\sigma \otimes \pi^{-1}F_{(a,b)})\big)$. We will show that

(III.F.12) *The pullback of G to an open set in $G_{\mathbb{C}}$ is a function of the form $G(f_1, f_1 \wedge f_2, f_1 \wedge f_3)$.*

Proof. As in the first example, we shall work modulo the differential scaling coefficients ω^j_j, which will take care of themselves at the end. We recall that

$$\Omega^1_\pi = \operatorname{span}\left\{\omega^2_1 = -\omega^4_3,\ \omega^3_1 = -\omega^4_2, \omega^4_2, \omega^2_3\right\} .$$

Then (III.F.11) implies that

$$dG \ \textit{ does not involve } \omega^1_2 = -\omega^3_4,\ \omega^1_3 = -\omega^2_4, \omega^1_4 .$$

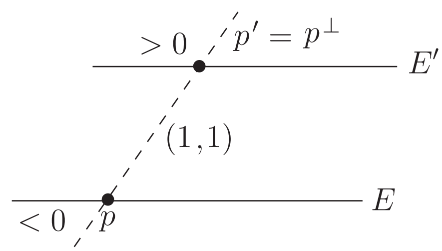
It follows first that $G = G(f_1, f_2, f_3)$. Next, since ω^1_2 and ω^1_3 do not appear in dG, we infer that

$$G = G(f_1, f_1 \wedge f_2, f_1 \wedge f_3) .$$

This gives the

Conclusion: *If* (III.F.11) *holds, then* $G \in \Gamma(\mathcal{J}, \sigma^{-1}F_{(a,b)})$.

Step five: The space $\mathcal{J}$ has maximal compact subvarieties $Z = Z(E, E')$ that have the picture



That is, the locus

$$\{p, \overline{pp'}, E\}, \qquad E \text{ fixed}$$

gives a $\mathbb{P}^1$ in $\mathcal{J}$. The line $\overline{pp'}$ is Lagrangian since $p' = p^{\perp}$, and H has signature $(1,1)$ on $\overline{pp'}$ since $H(p) < 0$ and $H(p') > 0$. Since $\mathcal{J}$ is covered by such $Z(E, E')$, to show that the equation

$$d_{\pi} G \equiv F\omega, \qquad G \in \Gamma(\mathcal{J}, \sigma^{-1} F_{(a,b)})$$

cannot hold non-trivially it will suffice to establish the stronger result that all

$$G\big|_{Z(E,E')} \equiv 0\,.$$

But we have seen that

$$F_{(a,b)}\big|_{Z(E,E')} = \mathcal{O}_{\mathbb{P}^1}(a-b)\,.$$

This gives the

(III.F.13) **Theorem:** *The Penrose transform*

$$\mathcal{P} : H^0(D', L'_{(a',b')}) \to H^1(D, L_{(a,b)})$$

is injective for $a < b$, or equivalently for

$$a' + b' + 1 < 0\,.$$

(III.F.14) **Corollary:** *The Penrose transform*

$$\mathcal{P} : H^0(D, \omega_{\mathcal{H}}^{'\otimes k/3}) \to H^1(D, F_{(-k,-k+2)})$$

is injective for $k \geqq 1$.

Remark. As a check on the signs we recall that the distinguished Weyl chamber $\mathbf{C}$ is the unique one where

$$\begin{cases} \mu' + \rho' \in \mathbf{C} \Rightarrow H^0_{(2)}(D', L'_{\mu'}) \neq 0 \\ \mu + \rho \in \mathbf{C} \Rightarrow H^1_{(2)}(D, L_\mu) \neq 0\,. \end{cases}$$

Then for $\mu' = a'e_1 + b'e_2$

$$\text{(III.F.15)}' \qquad \mu' + \rho' \in \mathbf{C} \Longleftrightarrow \begin{cases} a' < -2 \\ b' < a' + 1\,, \end{cases}$$

and for $\mu = ae_1 + be_2$

$$\text{(III.F.15)} \qquad \mu + \rho \in \mathbf{C} \Longleftrightarrow \begin{cases} a < -2 \\ b < a + 3\,. \end{cases}$$

The Penrose transform

$$\begin{cases} \mathcal{P} : H^0(D', L'_{(a',b')}) \to H^1(D, L_{(a,b)}) \\ \qquad\qquad a = a', b = b' + 2 \end{cases}$$

exactly takes the μ' satisfying (III.F.15)$'$ to the μ satisfying (III.F.15).

IV. The Penrose transform in the automorphic case and the main result

IV.A. **Cuspidal automorphic cohomology.** The objectives of this and the next section are

- to define *cuspidal automorphic cohomology* and to express it in terms of Lie algebra cohomology (in this case, $\mathfrak{n}$-cohomology);
- to recall a lemma of Williams ([Wi1]) that will provide the crucial ingredient in the computation of cuspidal automorphic cohomology;
- in the next section, in case of our two examples to define the Penrose transform on cuspidal automorphic cohomology and, using Williams' lemma, show that it is an isomorphism for cuspidal Picard and Siegel modular forms of weights $k \geqq 4$.

Generalities. We consider a non-compact homogeneous complex manifold embedded in its compact dual

$$D = G_{\mathbb{R}}/T \subset G_{\mathbb{C}}/B = \check{D}\,,$$

where B is the subgroup of $G_{\mathbb{C}}$ stabilizing a point $\varphi \in \check{D}$ and where D is the $G_{\mathbb{R}}$-orbit of φ. We write

$$\mathfrak{b} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}$$

where

$$\mathfrak{n} = \bigoplus_{\alpha\in\Phi^-} \mathfrak{g}^{\alpha}$$

is the direct sum of the *negative* root spaces. We may think of the dual $\check{\mathfrak{n}} := \mathfrak{n}^+$ in two ways:

(i) as the $(1,0)$ tangent space to D at φ;
(ii) as the $(0,1)$ cotangent space to D at φ.

For (i) we use the Cartan-Killing form to identify $\check{\mathfrak{n}}$ with $\bigoplus_{\alpha\in\Phi^+} \mathfrak{g}^{\alpha}$; this is the interpretation of $\check{\mathfrak{n}}$ that we shall use when we stay completely in a holomorphic setting as in the [EGW] formalism in section III. For (ii) we use the conjugation on $\mathfrak{g}_{\mathbb{C}}$ relative to a compact real form $\mathfrak{g}_c$ of $\mathfrak{g}_{\mathbb{C}}$ to identify $\mathfrak{n} = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}^{-\alpha}$ with the $(0,1)$ tangent space to D at φ; this is the interpretation we shall use when we discuss $\overline{\partial}$-cohomology below.

We let $\Gamma \subset G$ be an arithmetic group and set $X = \Gamma\backslash D$. For simplicity we make the inessential assumption that Γ acts freely on D. For a weight μ the corresponding homogeneous line bundle $L_\mu \to D$ descends to one on X that we also denote by $L_\mu \to X$. We will be interested in a subgroup $H^q_o(X, L_\mu)$ of $H^q(X, L_\mu)$ that we will call cuspidal automorphic cohomology. In order to define and discuss some of its properties we need to recall some aspects of the representation theory of $G_{\mathbb{R}}$. A general reference for this discussion is [K2].

We let $\widetilde{V}$ be an irreducible representation of $G_{\mathbb{R}}$ and $V = \widetilde{V}_{K\text{-finite}}$ the underlying Harish-Chandra module. Denoting by $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$, the *infinitesimal character*

$$\chi_V : \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$$

is defined by

$$z \cdot v = \chi_V(z)v \qquad\qquad z \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}), v \in V\,.$$

Recalling that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n} \oplus \mathfrak{n}^+$ by Poincaré-Birkhoff-Witt we have the additive isomorphism

$$\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) = \mathcal{U}(\mathfrak{t}_{\mathbb{C}}) \oplus \big(\mathcal{U}(\mathfrak{g}_{\mathbb{C}})\mathfrak{n}^+ \oplus \mathfrak{n}\mathcal{U}(\mathfrak{g}_{\mathbb{C}})\big).$$

The projection of $z \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ onto the second factor lies in $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})\mathfrak{n}^+$, and therefore it annihilates the highest weight vector $v \in V$ if such exists. It follows that the projection of z to the first factor above

$$\gamma' : \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \to \mathcal{Z}(\mathfrak{t}_{\mathbb{C}}) \cong \mathrm{Sym}(\mathfrak{t}_{\mathbb{C}}) \cong \mathbb{C}[\check{\mathfrak{t}}_{\mathbb{C}}]$$

has the same action on v as does z.[45] Composing γ' with the translation map

$$\begin{cases} \tau : \mathbb{C}[\check{\mathfrak{t}}_{\mathbb{C}}] \to \mathbb{C}[\check{\mathfrak{t}}_{\mathbb{C}}] \\ \tau(\phi(\cdot)) = \phi(\cdot - \rho) \end{cases}$$

yields the *Harish-Chandra homomorphism*

$$\gamma := \tau \circ \gamma' : \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}[\check{\mathfrak{t}}_{\mathbb{C}}]^W \tag{IV.A.1}$$

where W is the complex Weyl group.

For a weight λ we may use (IV.A.1) to define a homomorphism

$$\chi_\lambda : \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$$

by

$$\chi_\lambda(z) = \gamma(z)(\lambda)\,.$$

[45] $\mathbb{C}[\check{\mathfrak{t}}_{\mathbb{C}}]$ means the coordinate ring of $\check{\mathfrak{t}}_{\mathbb{C}}$; i.e., the polynomial functions $\phi(\cdot)$ on $\check{\mathfrak{t}}_{\mathbb{C}}$.

For $v \in V^{\mathfrak{n}^+}$ a highest weight vector of weight μ, from

$$\begin{aligned}
\chi_V(z)v &= z \cdot v \\
&= \gamma'(z)v && \text{(since } v \text{ is a highest weight vector)} \\
&= \big(\gamma'(z)(\mu)\big)v && \text{(since } v \text{ has weight } \mu\text{)} \\
&= \big(\gamma(z)(\mu+\rho)\big)v && \text{(definition of } \gamma\text{)} \\
&= \chi_{\mu+\rho}(z)v && \text{(definition of } \chi_\lambda\text{)}
\end{aligned}$$

we see that *if V has highest weight vector with weight μ, then V has infinitesimal character $\chi_{\mu+\rho}$.*

We will think of the highest weight vector of weight μ as an element

$$v \in H^0(\mathfrak{n}^+, V)_\mu$$

where, as usual, $H^0(\mathfrak{n}^+, V) = V^{\mathfrak{n}^+}$ and the subscript μ means that as a T-module $H^0(\mathfrak{n}^+, V)_\mu$ is that part of the Lie algebra cohomology that transforms by the weight μ. The *Casselman-Osborne lemma* [CO] says that the above result extends to $H^k(\mathfrak{n}^+, V)$:

(IV.A.2) *If $H^k(\mathfrak{n}^+, V)_\mu \neq \{0\}$, then $\chi_V = \chi_{\mu+\rho}$.*

Note that by the W-invariance of γ,

$$\chi_\lambda = \chi_{\mu+\rho} \text{ if } \lambda = w(\mu+\rho) \text{ for some } w \in W\,.$$

The converse also holds, from which one concludes that *if $H^k(\mathfrak{n}^+, V)_\mu \neq \{0\}$, then the other weights of the T-module $H^k(\mathfrak{n}^+, V)$ are $w(\mu+\rho)-\rho$ for $w \in W$.*

We shall use (IV.A.2) in the form obtained by replacing $\mathfrak{n}^+$ by $\mathfrak{n}$, μ by $-\mu$, and ρ by $-\rho$ to have

(IV.A.3) *If $\widetilde{V}$ is a unitary representation of $G_\mathbb{R}$ and $H^k(\mathfrak{n}, V)_{-\mu} \neq \{0\}$, then V has infinitesimal character $\chi_{-(\mu+\rho)}$.*

A consequence is that there are only finitely many possibilities for V, and hence up to infinitesimal equivalence for $\widetilde{V}$. If $\mu+\rho$ is regular, then $|W/W_K|$ of these are equivalence classes of discrete series representations. In order to ensure that $\widetilde{V}$ itself is in the discrete series we need an extra hypothesis, which is given by

(IV.A.4) **Williams Lemma:** *Given an irreducible unitary representation of $G_{\mathbb{R}}$ and a weight μ satisfying*

(i) *$\mu+\rho$ is regular;*

(ii) **Property P:** *For each $\alpha \in \Phi_{nc}$ with $(\mu+\rho, \alpha) > 0$,*

$$\Bigg(\mu+\rho-\tfrac{1}{2}\sum_{\left\{\substack{\beta\in\Phi \\ (\mu+\rho,\beta)>0}\right.}\beta, \alpha\Bigg) > 0\,.$$

Then

$$H^k(\mathfrak{n}, V)_{-\mu} \neq \{0\} \Longrightarrow \begin{cases} \bullet \quad k = q(\mu+\rho) \\ \bullet \quad \dim H^k(\mathfrak{n}, V)_{-\mu} = 1 \\ \bullet \quad \widetilde{V} \;\; \textit{is} \;\; V_{-(\mu+\rho)} \end{cases}$$

where $V_{-(\mu+\rho)}$ is the discrete series representation with Harish-Chandra character $\Theta_{-(\mu+\rho)}$.

Here we continue to use the notation

$$q(\mu+\rho) = \#\{\alpha \in \Phi_c^+ : (\mu+\rho, \alpha) < 0\} + \#\{\beta \in \Phi_{nc}^+ : (\mu+\rho, \beta) > 0\}\,.$$

Cuspidal automorphic cohomology. For Γ an arithmetic but not necessarily co-compact subgroup of G, in the non-classical case and for $X = \Gamma\backslash D$ very little seems to be known (e.g. finite dimensionality (Köecher principle) for a range of k's depending on Γ) about the groups $H^k(X, L_\mu)$. The general behavior of these groups at the Kato-Usui boundary components (cf. [KU]) is a subject yet to be explored. Here we shall define the subgroup $H^k_o(X, L_\mu)$ to be the image of cuspidal automorphic cohomology as defined below.

In order to study the groups $H^q(X, L_\mu)$ by standard methods, say using $\overline{\partial}$-cohomology, by lifting everything up to $\Gamma\backslash G_{\mathbb{R}}$ and trivializing the homogeneous bundles over $G_{\mathbb{R}}$ one encounters $C^\infty(\Gamma\backslash G_{\mathbb{R}})$. Replacing this by a sub-object that decomposes into an algebraic direct sum of Harish-Chandra modules is necessary to proceed. Two such objects are

(i) the L^2 *automorphic forms*

$$\mathcal{A}^2(G_{\mathbb{R}}, \Gamma) := \mathcal{A}(G_{\mathbb{R}}, \Gamma) \cap L^2(\Gamma\backslash G_{\mathbb{R}})$$

where $\mathcal{A}(G_{\mathbb{R}},\Gamma)$ are the usual automorphic forms as defined, e.g., in [Bor]. From their properties one has

$$\mathcal{A}^2(G_{\mathbb{R}},\Gamma) \subset L^2_{\text{disc}}(\Gamma\backslash G_{\mathbb{R}})$$

where the RHS is the discrete spectrum of the unitary $G_{\mathbb{R}}$-module $L^2(\Gamma\backslash G_{\mathbb{R}})$. We note that the irreducible $G_{\mathbb{R}}$-factors in $\mathcal{A}^2(G_{\mathbb{R}},\Gamma)$ will in general not all be discrete series representations of $G_{\mathbb{R}}$.

(ii) the *cuspidal automorphic forms* ([Bor], loc. cit.)

$$\text{(IV.A.5)} \qquad \begin{array}{ll} \mathcal{A}_o(G_{\mathbb{R}},\Gamma) \cong \underset{\pi\in\hat{G}_{\mathbb{R}}}{\oplus} V_\pi^{\oplus m_\pi(\Gamma)} & \qquad \text{(algebraic } \oplus) \\ \cap & \\ L^2_o(\Gamma\backslash G_{\mathbb{R}}) \cong \underset{\pi\in\hat{G}_{\mathbb{R}}}{\hat{\oplus}} \widetilde{V}_\pi^{\oplus m_\pi(\Gamma)} & \qquad \text{(Hilbert space } \oplus) \end{array}$$

where

$$m_\pi(\Gamma) = \left\{ \begin{array}{c} \text{multiplicity of } V_\pi \text{ in } \mathcal{A}_o(G_{\mathbb{R}},\Gamma), \\ \text{which is equal to the} \\ \text{multiplicity of } \widetilde{V}_\pi \text{ in } L^2_o(\Gamma\backslash G_{\mathbb{R}}) \end{array} \right\}.$$

With assumptions on μ (see below), is seems reasonably clear that replacing $C^\infty(\Gamma\backslash G_{\mathbb{R}})$ by $\mathcal{A}^2(G_{\mathbb{R}},\Gamma)$ or $\mathcal{A}_o(G_{\mathbb{R}},\Gamma)$ should lead to a subgroup of $H^q(X,L_\mu)$. For our purposes we will use (ii) and reasoning essentially as in Carayol [C1] and denoting by $A^{0,q}(X,L_\mu)$ the smooth L_μ-valued $(0,q)$-forms on X, we have

$$\begin{aligned} H^q(X,L_\mu) &= H^q(A^{0,\bullet}(X,L_\mu),\overline{\partial}) \\ &= H^q\left\{\left(A^{0,\bullet}(\Gamma\backslash G_{\mathbb{R}},L_\mu)\right)^T\right\} \\ &= H^q\left\{\operatorname{Hom}_T\left(\Lambda^\bullet\mathfrak{n}, C^\infty(\Gamma\backslash G_{\mathbb{R}})\otimes\mathbb{C}_\mu\right)\right\} \\ &= H^q\left(\mathfrak{n}, C^\infty(\Gamma\backslash G_{\mathbb{R}})\right)_{-\mu} \end{aligned}$$

where we have omitted the operator corresponding to $\overline{\partial}$ in the second and third steps. In the last step this operator becomes the usual coboundary operator in Lie algebra cohomology due to the second identification above of $\check{\mathfrak{n}}$ as the $(0,1)$ cotangent space to D at the identity. Also, $\mathbb{C}_\mu$ is the T-module corresponding to the character whose weight is μ, and the steps in the chain of equalities are the standard ones used to replace $\overline{\partial}$-cohomology by Lie algebra cohomology. With this as motivation we give the

(IV.A.6) **Definition:** The *cuspidal automorphic cohomology* is defined by

$$S^q(X, L_\mu) := H^q\big(\mathfrak{n}, \mathcal{A}_o(\Gamma\backslash G_{\mathbb{R}})\big)_{-\mu}\,.$$

From (IV.A.5) we have

$$\text{(IV.A.7)} \qquad \begin{aligned} S^q(X, L_\mu) &\cong \bigoplus_{\pi\in\hat{G}_{\mathbb{R}}} H^q(\mathfrak{n}, V_\pi)_{-\mu}^{\oplus m_\pi(\Gamma)} \\ &\cong \bigoplus_{\left\{\substack{\pi\in\hat{G}_{\mathbb{R}} \\ \chi_\pi=\chi_{-(\mu+\rho)}}\right.} H^q(\mathfrak{n}, V_\pi)_{-\mu}^{\oplus m_\pi(\Gamma)} \end{aligned}$$

which is a *finite* direct sum.

There is an obvious natural mapping

$$\text{(IV.A.8)} \qquad S^q(X, L_\mu) \to H^q(X, L_\mu)$$

and we shall denote by $H^q_o(X, L_\mu)$ the image of this mapping. Although very little seems to be known about this mapping, in the cases of the Penrose transform of Picard and Siegel modular forms we will see that it is injective.

(IV.A.9) **Proposition:** *Assume that $\mu+\rho$ is regular and satisfies property* **P**. *Then*

$$S^q(X, L_\mu) \cong \begin{cases} 0, & q \neq q(\mu) \\ \mathbb{C}^{m_\pi(\Gamma)}, & q = q(\mu+\rho) \end{cases}$$

where $\pi = \pi_{-(\mu+\rho)}$.

Proof. This follows immediately from the definition of $S^q(X, L_\mu)$ and Williams' lemma □

Remark. Using the extension of the Hirzebruch proportionality principle made possible by Atiyah-Singer to the case of $X = \Gamma\backslash D$ when D is non-classical and Γ is co-compact, it may be shown that

$$m_{\pi_{-(\mu+\rho)}}(\Gamma) = \mathrm{vol}(X) P_\mu$$

where P_μ is a Hilbert-type polynomial in μ whose coefficients are independent of Γ. In fact, for μ non-singular and setting $N = \dim D$ and $P_{k\mu} = P_\mu(k)$, we have

$$P_\mu(k) = C_\mu k^N + (\text{lower order terms in } k)$$

for a constant $C_\mu > 0$ that is independent of Γ (cf. [Wi5]).

As noted above, one may ask about the relation between the cuspidal automorphic cohomology $S^q(X, L_\mu)$ and the ordinary automorphic cohomology $H^q(X, L_\mu)$. A first result along these lines is the

(IV.A.10) **Proposition:** *The map* (IV.A.8) *is injective in each of the following cases:*

(i) *$\mu + \rho$ is regular antidominant and satisfies property* $\mathbf{P}$, *and* $q = q(\mu + \rho)$;
(ii) $q = 0$; *or*
(iii) Γ *cocompact (in which case it is an isomorphism).*

Unfortunately, for the case of the Penrose transform of Picard modular forms (example one) or Siegel modular forms (example two), $\mu+\rho$ is not anti-dominant, or even K-anti-dominant and the issue of injectivity is somewhat delicate.

Proof of (i). Using (IV.A.7) and (IV.A.9) we have

$$\begin{aligned} S^{q(\mu+\rho)}(X, L_\mu) &\cong \bigoplus_{\pi \in \hat{G}_{\mathbb{R}}} \operatorname{Hom}_{(\mathfrak{g}_{\mathbb{C}}, K)}(V_{-(\mu+\rho)}, V_\pi)^{\oplus m_\pi(\Gamma)} \\ &\cong \operatorname{Hom}_{(\mathfrak{g}_{\mathbb{C}}, K)}\big(V_{-(\mu+\rho)}, \mathcal{A}_o(G_{\mathbb{R}}, \Gamma)\big) \\ &\subset \operatorname{Hom}_{(\mathfrak{g}_{\mathbb{C}}, K)}\big(V_{-(\mu+\rho)}, C^\infty(\Gamma \backslash G_{\mathbb{R}})_{K\text{-finite}}\big) \\ &\cong H^0\big(\Gamma, \mathrm{MG}(V_{\mu+\rho})\big) \end{aligned}$$

where the last step uses [Schm1], theorem 7 on page 109, and MG means "maximal globalization". Using [Schm1], the corollary on page 109, this last term is

$$\cong H^0\big(\Gamma, H^{q(\mu+\rho)}(D, L_\mu)\big).$$

Next, using [Schm1], part (a) of the theorem on page 98 and theorem 3 on page 106, we have $H^q(D, L_\mu) = \{0\}$ for $q \neq q(\mu + \rho)$, and from the Cartan-Serre spectral sequence

$$H^i(\Gamma, H^j(D, L_\mu)) \Rightarrow H^{i+j}(\Gamma\backslash D, L_\mu),$$

the last term above is

$$\cong H^{q(\mu+\rho)}(X, L_\mu)\,.$$

Next, (ii) is essentially trivial. We write simply

$$\begin{aligned} S^0(\Gamma\backslash D, L_\mu) &= \mathcal{A}_o(\Gamma\backslash G_{\mathbb{R}})^{\mathfrak{n}}_{-\mu} \\ &\subseteq C^\infty(\Gamma\backslash G_{\mathbb{R}})^{\mathfrak{n}}_{-\mu} = H^0(\Gamma\backslash D, L_\mu)\,. \end{aligned}$$

Turning to (iii), in the co-compact case we clearly have

$$\mathcal{A}_o(G_{\mathbb{R}}, \Gamma) \subset C^\infty(\Gamma\backslash G_{\mathbb{R}}) \subset L^2_o(\Gamma\backslash G_{\mathbb{R}})$$

where the terms on either end decompose respectively into an algebraic direct sum of Harish-Chandra modules and a Hilbert space direct sum of globalizations of these modules with the same multiplicities. By reasoning as above we have

$$\begin{aligned} H^{q(\mu+\rho)}(X, L_\mu) &\cong H^{q(\mu+\rho)}\big(\mathfrak{n}, C^\infty(\Gamma\backslash G_{\mathbb{R}})\big)_{-\mu} \\ &\cong \bigoplus_{\pi\in\hat{G}_{\mathbb{R}}} H^{q(\mu+\rho)}(\mathfrak{n}, \widehat{V}^\infty_\pi)^{\oplus m_\pi(\Gamma)}_{-\mu} \end{aligned}$$

and from lemma 1 on page 98 of [Schm1] this is

$$\begin{aligned} &\cong \bigoplus_{\pi\in\hat{G}_{\mathbb{R}}} H^{q(\mu+\rho)}(\mathfrak{n}, V_\pi)^{\oplus m_\pi(\Gamma)}_{-\mu} \\ &\cong S^{q(\mu+\rho)}(X, L_\mu)\,. \end{aligned}$$

□

IV.B. **Picard and Siegel cuspidal automorphic forms.**

The purposes of this section are (see the conclusion (IV.B.10) at the end):

(IV.B.1) *to show that the μ' corresponding to Picard and Siegel automorphic forms satisfy Williams' property* $\mathbf{P}$*, together with and the accompanying regularity condition for $q = q(\mu' + \rho') = 1$ and for $k \geqq 4$, for both Picard and Siegel,* and

(IV.B.2) *to show that for the* μ' *giving weight* k *Picard automorphic forms for* $S\mathcal{U}(2,1)$*, respectively Siegel automorphic forms for* $\mathrm{Sp}(4)$*, the Penrose transform*

$$\mathcal{P}_\Gamma : S^0(X', L'_{\mu'}) \to S^1(X, L_\mu), \tag{IV.B.3}$$

where $\rho' + \mu' = \rho + \mu$*, is defined and in both cases is an isomorphism for* $k \geqq 4$*.*

We observe that in case Γ is co-compact, injectivity of $\mathcal{P}$ follows from (III.E.13) and (III.F.14); and then surjectivity will follow from the equality of dimensions, which is a consequence of (IV.B.1).[46]

Proof of (IV.B.1) *for* $S\mathcal{U}(2,1)$*.* We recall that the Penrose transform is a map

$$\mathcal{P} : H^0(D', F'_{(a,b)}) \to H^1(D, F_{(-a-2,a+b+1)})\,.$$

In terms of the roots α_1, α_2 we have

$$F'_{(a,b)} = L_{\mu'} \ \text{ where } \ \mu' = \left(\frac{b-a}{3}\right)\alpha_1 + \left(\frac{-2a-b}{3}\right)\alpha_2$$

and

$F_{(-a-2,a+b+1)} = L_\mu$ where

$$\mu = \left(\frac{2(-a-2)+(a+b+1)}{3}\right)\alpha_1 + \left(\frac{(-a-2)-(a+b+1)}{3}\right)\alpha_2\,.$$

We note that

$$\mu' + \rho' = \left(\frac{b-a}{3}\right)\alpha_1 + \left(\frac{-2a-b-3}{3}\right)\alpha_2 = \mu + \rho\,.$$

Denoting by $\omega'_{\mathbb{B}}$ the pullback under the map $D' \to \mathbb{B}$ of the canonical bundle $\omega_{\mathbb{B}} \to \mathbb{B}$, we have seen that $\omega_{\mathbb{B}}$ (and hence $\omega'_{\mathbb{B}}$) has a canonical

[46] In the co-compact case and for μ sufficiently far from the walls of the Weyl chamber, as previously noted $H^q(X', L'_{\mu'}) = 0$ for $q \neq 0$ and $H^q(X, L_\mu) = 0$ for $q \neq 1$. The equality of dimensions then follows from the Atiyah-Singer Hirzebruch Riemann-Roch theorem, as both sides may be computed to be the same multiple of $\mathrm{vol}(\Gamma\backslash G_{\mathbb{R}})$.

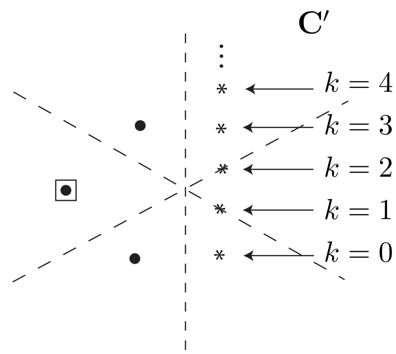
cube root. The Penrose transform for Picard modular forms of weight k then corresponds to

$$\omega_{\mathbb{B}}^{\prime\otimes k/3} = F'_{(-k,0)} \to F_{(k-2,1-k)}$$

where

$$\mu' + \rho' = \left(\frac{k}{3}\right)\alpha_1 + \left(\frac{2k-3}{3}\right)\alpha_2 = \mu + \rho\,.$$

The picture is



so that for $k \geqq 3$, $\mu' + \rho' = \mu + \rho$ lies in the Weyl chamber $\mathbf{C}'$. Recalling Williams' property $\mathbf{P}$ from (IV.A.3), the noncompact roots α with $(\mu+\rho,\alpha) > 0$ are $\alpha_1 + \alpha_2$ and α_2, where we have used $(\alpha_1,\alpha_1) = (\alpha_2,\alpha_2) = 2$ and $(\alpha_1,\alpha_2) = -1$ and assumed $k \geqq 3$. Moreover,

$$\left(\frac{1}{2}\right) \sum_{\left\{\substack{\beta\in\Phi \\ (\mu+\rho,\beta)>0}\right.} \beta = \alpha_1 + \alpha_2$$

and

$$\begin{cases} (\mu+\rho-\alpha_1-\alpha_2, \alpha_1+\alpha_2) = \frac{2}{3}(k-3) \\ (\mu+\rho-\alpha_1-\alpha_2, \alpha_2) = k-3, \end{cases}$$

so condition $\mathbf{P}$ holds for $k > 3$ (i.e. $k \geqq 4$). We emphasize that $k = 1$ or 2 makes $\mu + \rho$ irregular, and that $\mathbf{P}$ fails for $k = 3$.

Proof of (IV.B.2) *for* $S\mathcal{U}(2,1)$. We have
(IV.B.4)

$$S^0(X', \omega_{\mathbb{B}}^{\prime\otimes k/3}) \cong H^0\left(\mathfrak{n}', V_{-(\mu'+\rho')}\right)_{-\mu'}^{\oplus m_{-(\mu'+\rho')}(\Gamma)} \cong \mathcal{S}_k(S\mathcal{U}(2,1),\Gamma)$$

where the last term denotes the *cuspidal Picard modular forms of weight k in the classical sense*; i.e., this space is the space of cuspidal forms in $H^0(\Gamma\backslash\mathbb{B}, \omega_{\Gamma\backslash\mathbb{B}}^{\otimes k/3})$. The sections on the LHS of (IV.B.4) are constant along the $\mathbb{P}^1$ fibres of $X' \to \Gamma\backslash\mathbb{B}$. Moreover, that the cuspidal condition given by $\mathcal{A}_o(G_{\mathbb{R}}, \Gamma) \subset L_o^2(\Gamma\backslash G_{\mathbb{R}})$ specializes to the usual cuspidal condition for automorphic forms on quotients of bounded symmetric domains by arithmetic groups is standard ([Bor], loc. cit.).

The issue of defining the Penrose transform (IV.B.3) and showing that it is injective is an interesting one. The most natural way to proceed would be to have a diagram

(IV.B.5)
$$\begin{tikzcd} S^0(X', L'_{\mu'}) \arrow[r, "j_0"] \arrow[d, dashed, "\mathcal{P}_\Gamma"'] & H^0(X', L'_{\mu'}) \arrow[d, "\mathcal{P}"] \\ S^1(X, L_\mu) \arrow[r, "j_1"] & H^1(X, L_\mu) \end{tikzcd}$$

where

(i) the horizontal maps are injective, and
(ii) $\mathcal{P}_\Gamma\big(j_0(S^0(X', L'_{\mu'}))\big) \subseteq j_1\big(S^1(X, L_\mu)\big)$;

i.e. we can fill in the dotted arrow in the diagram to define $\mathcal{P}_\Gamma$. It turns out that (i) and (ii) are true, but the proof will be indirect. In preparation for that we have the

(IV.B.6) **Lemma:** *For μ' corresponding to Picard or to Siegel automorphic forms, and $\mu = \mu' - \rho + \rho'$ to their Penrose transforms*

$$\begin{cases} H^0(X', L'_{\mu'}) = H^0(D', L'_{\mu'})^\Gamma \\ H^1(X, L_\mu) \quad = H^1(D, L_\mu)^\Gamma \,. \end{cases}$$

Proof. The result is obvious for $q = 0$. In general the Cartan-Leray spectral sequence

$$H^p\big(\Gamma, H^q(D, F_{(a,b)})\big) \Rightarrow H^{p+q}(X, F_{(a,b)})\,,$$

leads to an exact sequence

$$0 \to H^1\big(\Gamma, H^0(D, F_{(a,b)})\big) \to H^1(X, F_{(a,b)}) \to H^1(D, F_{(a,b)})^\Gamma \\ \to H^2\big(\Gamma, H^0(D, F_{(a,b)})\big)\,.$$

The lemma then follows since we have seen in (III.E.18) and section III.F that $H^0(D, F_{(a,b)}) = 0$ for the case of Picard and Siegel automorphic forms. $\square$

As a consequence we may replace the diagram (IV.B.5) by

$$\text{(IV.B.7)} \qquad \begin{array}{ccc} S^0(X', L'_{\mu'}) & \xrightarrow{j'_0} & H^0(D', L'_{\mu'})^\Gamma \\ {\scriptstyle \mathcal{P}_\Gamma} \downarrow & & \downarrow {\scriptstyle \overline{\mathcal{P}}} \\ S^1(X, L_\mu) & \xrightarrow{j_1} & H^1(D, L_\mu)^\Gamma \end{array}$$

where j'_0 and $\overline{\mathcal{P}}$ are injective by (IV.A.10)(iii) and (III.E.13).

To *define* $\mathcal{P}_\Gamma$ we recall that

$$S^1(X, F_{(k-2,1-k)}) \cong H^1(\mathfrak{n}, V_{-(\mu+\rho)})_{-\mu}^{m_{\pi_{-(\mu+\rho)}}(\Gamma)}$$

where $\mu+\rho = \left(\frac{k}{3}\right)\alpha_1 + \left(\frac{2k}{3}-1\right)\alpha_2$ and where the $\mathfrak{n}$-cohomology group is 1-dimensional. Since

$$\begin{cases} \mathfrak{n} = \operatorname{span}\{X_{-(\alpha_1+\alpha_2)}, X_{\alpha_2}, X_{-\alpha_1}\} \\ \mathfrak{n}' = \operatorname{span}\{X_{\alpha_1+\alpha_2}, X_{\alpha_2}, X_{-\alpha_1}\} \end{cases}$$

we will define $\mathcal{P}_\Gamma$ as a map

$$H^0\left\{\operatorname{Hom}_T\left(\Lambda^\bullet \mathfrak{n}', \mathcal{A}_o(G_\mathbb{R}, \Gamma)\otimes \mathbb{C}_{\mu'}\right)\right\} \to H^1\left\{\operatorname{Hom}_T\left(\Lambda^\bullet \mathfrak{n}, \mathcal{A}_o(G_\mathbb{R}, \Gamma)\otimes \mathbb{C}_{\mu}\right)\right\}.$$

For this we need a T-invariant homomorphism

$$\text{(IV.B.8)} \qquad \mathfrak{n} \to \mathbb{C}_{\mu-\mu'}.$$

But $\mu - \mu' = -\alpha_1 - \alpha_2$, and so we may define the map (IV.B.8) by

$$\text{(IV.B.9)} \qquad X_{-\alpha_1-\alpha_2} \to 1 \in \mathbb{C}_{\mu-\mu'}$$

and $X_{\alpha_2}, X_{-\alpha_1}$ map to zero. Now, and this is the key point,

> *The map* (IV.B.8) *is just that given by the form* $\omega = \omega_3^1$ *in* (III.E.8).

The link between the two Penrose transforms is given by the isomorphism

$$H^q_{\mathrm{DR}}\left\{\Gamma\big(\Gamma\backslash\mathcal{W}, \Omega^\bullet_\pi \otimes \pi^{-1}Z_\mu\big)\right\}$$
$$\cong H^q\left\{\operatorname{Hom}_{T_\mathbb{C}}\big(\Lambda^\bullet\mathfrak{n}, \mathcal{O}(\Gamma\backslash \cap_{w\in W} G_\mathbb{R}B_w)\otimes\mathbb{C}_\mu\big)\right\}$$

(cf. (III.B.7)) together with the identification $\omega_1^3|_{\mathrm{id}_G} = \omega^{-(\alpha_1+\alpha_2)}$.

A first consequence is that, for $\mathcal{P}_\Gamma$ defined in this way and noting that $\mathcal{P}$ is $G_\mathbb{R}$-invariant, the *the diagram* (IV.B.7) *is commutative.* Consequently, $\mathcal{P}_\Gamma$ is injective. Moreover, by Williams' formula and using $\mu'+\rho'=\mu+\rho$, the two spaces have the same dimension; it follows that $\mathcal{P}_\Gamma$ is an isomorphism and that j_1 is also injective. This completes the proof of (IV.B.2) for $S\mathcal{U}(2,1)$. $\square$

Proof of (IV.B.1) *for* Sp(4). In this case we have

$$\begin{cases} F_{(a,b)} = F_{(-k,-k+2)} = L_{-ke_1^*+(-k+2)e_2^*} \\ \rho = 2e_1^* - e_2^* \end{cases}$$

which gives

$$\mu+\rho = (-k+2)e_1^* + (-k+1)e_2^*\,.$$

The non-compact roots are

$$\pm(e_1^*+e_2^*), \pm 2e_1^*, \pm 2e_2^*$$

and for α denoting these roots the values of $(\mu+\rho,\alpha)$ are successively

$$\pm(-2k+3),\ \pm(-2k+4),\ \pm(-2k+2)\,.$$

Thus, for $k \geqq 3$ the roots entering into Williams' condition $\mathbf{P}$ are $\alpha = -2e_2^*, -2e_1^*, -(e_1^*+e_2^*)$. (For $k=1,2$ the weight $\mu+\rho$ is non-regular). The roots β satisfying $(\mu+\rho,\beta)>0$ are

$$-2e_2^*, e_1^*-e_2^*, -e_1^*-e_2^*, -2e_1^*$$

so that

$$\frac{1}{2}\sum_{\begin{cases}\scriptstyle\beta\in\Phi\\ \scriptstyle(\mu+\rho,\beta)>0\end{cases}} \beta = -e_1^* - 2e_2^*\,.$$

Then

$$\mu+\rho-\frac{1}{2}\sum_{\begin{cases}\scriptstyle\beta\in\Phi\\ \scriptstyle(\mu+\rho,\beta)>0\end{cases}} \beta = (-k+3)e_1^* + (-k+3)e_2^*$$

and for α each of the above three roots

$$\left(\mu+\rho-\frac{1}{2}\sum_{\begin{cases}\scriptstyle\beta\in\Phi\\ \scriptstyle(\mu+\rho,\beta)>0\end{cases}} \beta, \alpha\right) = 2k-6,$$

which is positive for $k \geqq 4$. $\square$

The proof of (IV.B.2) for Siegel modular forms is the same as that for Picard modular forms.

(IV.B.10) **Conclusion:** *For Picard modular forms of weight $k \geqq 4$ and Siegel modular forms of weight $k \geqq 4$, the Penrose transform $\mathcal{P}_\Gamma$ may be defined and is an isomorphism leading to a commutative diagram*

(IV.B.11)
$$\begin{array}{ccccc} H^0_o(X', L'_{\mu'}) & \hookrightarrow & H^0(X', L'_{\mu'}) & \cong & H^0(D', L'_{\mu'})^\Gamma \\ {\scriptstyle \mathcal{P}_\Gamma}\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathcal{P}} \\ H^1_o(X, L_\mu) & \hookrightarrow & H^1(X, L_\mu) & \cong & H^1(D, L_\mu)^\Gamma . \end{array}$$

(IV.B.12) **Remark:** It is interesting, and will be of use in IV.F, to compare the inequality for the Williams Lemma with the one for integrability of $V_{-(\mu_k+\rho)}$:

$$|(\mu_k+\rho,\alpha)| > \frac{1}{2}\sum_{\beta\in\Phi}|(\beta,\alpha)| \text{ for all } \alpha\in\Phi_{nc} \text{ with } (\mu_k+\rho,\alpha)>0 .$$

For both Picard and Siegel, this is equivalent to $k \geq 5$.

IV.C. **Arithmetic structures on vector spaces.**

Notations: In this section $k, k_0, k', \ldots$ will be number fields with given embeddings $k \hookrightarrow \mathbb{C}, k_0 \hookrightarrow \mathbb{C}, k' \hookrightarrow \mathbb{C}$, etc.

Definition: Let E be a complex vector space. An *arithmetic structure* on E is given by a k-vector subspace $E_k \subset E$ such that $E = \mathbb{C}\otimes_k E_k$.

This means that $E_k \subset E$ is a subgroup together with an action $k\otimes_{\mathbb{Z}} E_k \to E_k$ that makes E_k into a vector space defined over k.

Of course any vector space has many arithmetic structures. The ones we shall use will be "natural" in the sense that should be clear from the context.

Definitions: (i) Two arithmetic structures given by $E_k \subset E$ and $E_{k'} \subset E$ are *comparable* if there is a number field k_0 such that $k \subset k_0, k' \subset k_0$ and $k_0 \otimes_k E_k = k_0 \otimes_{k'} \cdot E_{k'}$. (ii) Two arithmetic structures as above are *proportional* if there is a k_0 also as above and a complex number $\Delta \in \mathbb{C}^*$ such that $\Delta(k_0 \otimes_k E_k) = k_0 \otimes_{k'} E_{k'}$.

In other words, the two arithmetic structures E_k and $E_{k'}$ are proportional if, after finite algebraic extensions, the vector spaces are the same up to scaling by a generally transcendental number Δ.

The standard operations of linear algebra make sense when arithmetic structures are included. Thus, if we have $E = \mathbb{C} \otimes_k E_k$ and $F = \mathbb{C} \otimes_k F_k$ then $E \otimes F = \mathbb{C} \otimes_k (E_k \otimes_k F_k)$. More useful will be extending the operations of linear algebra to *equivalence classes* of *comparable arithmetic structures*; this is done in the evident way.

Definitions: (i) Let E have an arithmetic structure and $e \in E$. Then e is *arithmetic* if $e \in E_{k'}$ for some field $k' \supset k$ with $E_{k'} = k' \otimes_k E_k$. (ii) Keeping the same notation, e is *proportionally arithmetic* if there is a proportional arithmetic structure E_{k_0} to the given E_k such that $e \in E_{k_0}$.

Concretely, for a fixed transcendental number Δ, we will have $\Delta e \in E_{k_0}$ for all of the e's under consideration.

The motivation for these terms will hopefully become clear through the examples given below. One could presumably simplify by replacing all number fields with $\overline{\mathbb{Q}}$, but for reasons to be explained below we prefer not to do this.

Example (i): Let X be a smooth algebraic variety defined over k and $\mathbb{E} \to X$ an algebraic vector bundle also defined over k. Then if $\varphi \in X(k)$, the fibre $\mathbb{E}_\varphi$ has an arithmetic structure.

Example (ii): With X as above, and working in both the Zariski and analytic topologies and using the comparison theorem,

$$H^*_{\mathrm{DR}}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{C}) \cong \mathbb{H}^*(\Omega^\bullet_{X_{\mathbb{C}}/\mathbb{C}}) \cong \mathbb{H}^*(\Omega^\bullet_{X_k/k}) \otimes_k \mathbb{C}$$

has an arithmetic structure. Also,

$$H^{p,q}(X) \cong H^q(X^{\mathrm{an}}, \Omega^p_{X_{\mathbb{C}}}) \cong H^q(X^{\mathrm{Zar}}, \Omega^{\bullet}_{X_k}) \otimes_k \mathbb{C}$$

has an arithmetic structure, compatible with that on H^*_{DR} by the (algebraic) Hodge to de Rham spectral sequence.

Example (iii): Still keeping X as above, the Betti cohomology $H^*_B(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{C}) \cong H^*_B(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$ has an arithmetic structure.

The matter of the *non-proportionality* between the arithmetic structure in Examples (ii) and (iii), identifying $H^*_B(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{C})$ and $H^*_{\mathrm{DR}}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{C})$, is the subject of extensive and deep work and conjectures. In particular, one should mention Deligne's theory of absolute Hodge cycles; cf. [D] and the discussion in chapter VIII of [GGK1].

A simple example that does however illustrate the notion of proportionality is that for $X = \mathbb{C}^*$ viewed as the complex points of an algebraic variety defined over $\mathbb{Q}$,

$$\int_{|z|=1} \frac{dz}{z} = 2\pi i$$

gives that $H^1_B(X(\mathbb{C}), \mathbb{C})$ and $H^1(X(\mathbb{C}), \mathbb{C})$ have proportional arithmetic structures with $\Delta = 2\pi i$.

One can also consider the case of $X = E$, an elliptic curve defined over a number field. The arithmetic structures on H^1_B resp. H^1_{DR} are those arising from "classes with algebraic periods" and "classes represented by algebraic differential forms"; they are not in general proportional. We can also look at the arithmetic structures on F^1H^1 induced from algebraic 1-forms or "from periods"; but the latter structure may depend on the choice of 1-cycle. In fact, it is well-defined precisely when E has complex multiplication. This observation generalizes nicely, as we shall see in (IV.C.7) below.

Example (iv): Let $\check{D} = G_{\mathbb{C}}/B$ be a rational homogeneous variety where $G_{\mathbb{C}}$ is the complex Lie group associated to a reductive, $\mathbb{Q}$-algebraic group G. Then $\check{D}$ is defined over a number field k, as are the homogeneous vector bundles $\mathbb{F} \to \check{D}$ associated to representations of B defined over k. In particular, if D is a Mumford-Tate domain with compact dual $\check{D}$, then for $\varphi \in D \cap \check{D}(k)$ the fibres $\mathbb{F}^p_\varphi$ of Hodge filtration bundles $\mathbb{F}^p \to D$ have arithmetic structures which are preserved under the action by an arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$.

Remark: If X as in example (i) is complete and projective, and if $\varphi(X) \in D$ is the period point associated to the polarized Hodge structure on $H^n(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q})$, then to say that $\varphi(X) \in \check{D}(k)$ imposes very strong arithmetic restrictions on the polarized Hodge structure $\varphi(X)$.

Example (v): Let D be a Mumford-Tate domain and $\varphi \in D$ a complex-multiplication (CM) polarized Hodge structure (cf. chapter V of [GGK1]).

(IV.C.1) **Theorem:** *$\varphi \in \check{D}$ is defined over a number field constructed from L.*

Example (vi): The fibres $\mathbb{F}^p_\varphi$ of the Hodge filtration bundles, as well as the quotients $\mathbb{V}^{p,q}_\varphi = \mathbb{F}^p_\varphi / \mathbb{F}^{p+1}_\varphi$, have arithmetic structures arising from a CM field L.

Example (vii): Let $H = G_{\mathbb{R}}/K$ be a Mumford-Tate domain parametrizing polarized Hodge structures of weight $n = 1$ and whose generic member has Mumford-Tate group G. Then H is an Hermitian symmetric domain that may be equivariantly embedded in Siegel's generalized upper half space $\mathcal{H}_g \cong \mathrm{Sp}(2g)_{\mathbb{R}}/\mathcal{U}(g)$ where g is the dimension of the abelian variety A_φ corresponding to $\varphi \in H$. In example one, $H = \mathbb{B}$ is the ball in $\mathbb{C}^2$ and $g = 3$; in example two $H = \mathcal{H}_2$.

Let Γ be an arithmetic group and set $Y = \Gamma \backslash H$. Then we have the

(IV.C.2) **Theorem:** *Y is a quasi-projective algebraic variety defined over a number field.*

More specifically, replacing Γ by a subgroup of finite index if necessary, a step that will not affect our main result, we may assume that Y is smooth. Let ω_Y be the canonical bundle and assume that the boundary components in the Baily-Borel-Satake compactificaiton $Y \hookrightarrow \overline{Y}$ have codimension at least two. Then

$$R_Y := \bigoplus_{l \geqq 0} H^0(Y, \omega_Y^{\otimes l}) \tag{IV.C.3}$$

gives the coordinate ring for a projective variety $\overline{Y}$ in which Y is a Zariski open set. Moreover, R_Y has an arithmetic structure, meaning that

- the individual spaces $H^0(Y, \omega_Y^{\otimes l})$ have arithmetic structures; and
- these arithmetic structures are compatible with the homogeneous coordinate ring structure on R_Y.

These observations are corollaries to the existence of canonical models for Shimura varieties of Hodge type, due to Shimura and Deligne. (We refer the reader to [Mi] for further discussion and references.) For examples of the algebro-geometric information carried by these models in the Picard and Siegel cases, see the appendix below.

A consequence is

(IV.C.4) *For k a number field and $y \in Y(k)$, the fibres $\omega_{Y,y}^{\otimes l}$ have arithmetic structures that are compatible with the evident multiplication mappings.*

We will call these algebro-geometric or AG- *arithmetic structures.*

For certain points of Y there are other arithmetic structures in the fibres $\omega_{Y,y}^{\otimes l}$ that arise from the Hodge-theoretic interpretation of H as a Mumford-Tate domain, and therefore of Y as the moduli space of Γ-equivalence classes of polarized Hodge structures of weight one having additional structure. That is, they are inherited from the natural $\overline{\mathbb{Q}}$-algebraic structure on $\omega_H^{\otimes l}$. In example one we may think of the additional structure as the action of $\mathbb{F} = \mathbb{Q}(\sqrt{-d})$ on the polarized Hodge structure plus an additional level structure. In example two, the additional structure may be thought of as a level structure. In both cases G is a linear $\mathbb{Q}$-algebraic group and Γ is a linear group whose matrix entries are integers in a number field.

Besides ω_Y there are other natural line bundles to consider. Sitting over Y is a canonical family $\mathcal{A}$ of abelian varieties defined over $\overline{\mathbb{Q}}$, the relative algebraic 1-forms on which define a Hodge bundle $\mathbb{V}_Y^{1,0}$ (which is also $\overline{\mathbb{Q}}$-algebraic). Its pullback $\mathbb{V}_H^{1,0}$ to H gives the tautological weight-one VHS, and extends to a bundle over $\check{H}$, from which it inherits a different sort of $\overline{\mathbb{Q}}$-algebraic structure as follows. The pullback family $\widetilde{\mathcal{A}} \twoheadrightarrow H$ is a quotient of $\mathbb{C}^g \times H$; and writing $z_1, \ldots, z_g$ for the coordinates on $\mathbb{C}^g$, $dz_1, \ldots, dz_g$ provide a basis for the sections of $\mathbb{V}_H^{1,0}$. Passing to the g^{th} exterior power, for $H = \mathbb{B}$ or $\mathcal{H}_g$ we have isomorphisms

$$\begin{cases} \omega_Y \equiv (\det \mathbb{V}_Y^{1,0})^{\otimes m} \\ \omega_H \equiv (\det \mathbb{V}_H^{1,0})^{\otimes m} \end{cases}$$

replacing the $\overline{\mathbb{Q}}$-algebraic structures, for some $m \in \mathbb{Q}$ which depends on H.[47]

When $H = \mathcal{H}_g$ (instead of some proper subdomain), we have $m = g+1$, and sections $\psi \in H^0(Y, (\det \mathbb{V}_Y^{1,0})^{\otimes l})$ identify with *Siegel modular forms of weight l*. More precisely, writing $\widetilde{\psi} = f_\psi \cdot (dz_1 \wedge \cdots \wedge dz_g)^{\otimes l}$ for the pullback to H, f_ψ is the modular form. For more general H, we will still call such $\{f_\psi\}$ *restricted-Siegel modular forms.*

On the other hand, for $H = \mathbb{B}$ the 2-ball and $G = S\mathcal{U}(2,1)$, the $\mathbb{F}$-action produces a line sub-bundle $\mathbb{V}_+^{1,0} \subset \mathbb{V}^{1,0}$. Over H, we can take this to be generated by dz_1. One again has $\overline{\mathbb{Q}}$-algebraic isomorphisms

$$\begin{cases} \omega_Y^{\otimes 1/3} \cong \mathbb{V}_{Y,+}^{1,0} \cong (\det \mathbb{V}_Y^{1,0})^{\otimes 1/2} \\ \omega_H^{\otimes 1/3} \cong \mathbb{V}_{H,+}^{1,0} \cong (\det \mathbb{V}_Y^{1,0})^{\otimes 1/2}, \end{cases}$$

where the powers are consequences of our weight computations in earlier sections. Given $\psi \in H^0(Y, (\mathbb{V}_{Y,+}^{1,0})^{\otimes k})$, $f_\psi \in \mathcal{O}(\mathbb{B})$ (defined by $\widetilde{\psi} = f_\psi \cdot dz_1$) is known as a *Picard modular form of weight k*.

We will now write the points of Y as $y = [\varphi]$ where $[\varphi]$ is the Γ-equivalence class of a point $\varphi \in H \subset \check{H}$. Since $\check{H}$ is a projective variety defined over a number field k_0 the notion of an arithmetic point $\varphi \in \check{H}$ is defined; it means that $\varphi \in \check{H}(k_0')$ where $k_0' \supset k_0$ is a finite

[47]The meaning of $\mathcal{L}_1^{\otimes a/b} \cong \mathcal{L}_2$ being $\mathcal{L}_1^{\otimes a} \cong \mathcal{L}_2^{\otimes b}$.

extension. These arithmetic points are invariant under the action of Γ, and we denote by $Y_{\mathrm{HT}} \subset Y$ the set of such points.

The natural lift of this action to $\widetilde{\mathcal{A}}$ also respects the arithmetic structure derived from $dz_1, \dots, dz_g$ on $\mathbb{V}_H^{1,0}$. So for $y = [\varphi] \in Y_{\mathrm{HT}}$, the fibre $\mathbb{V}_{Y,y}^{1,0}$ of the Hodge bundle has a well-defined arithmetic structure induced from the complex-analytic isomorphism with $\mathbb{V}_{H,\varphi}^{1,0}$; and there are clearly compatible structures on $\det \mathbb{V}_{Y,y}^{1,0}$, $\omega_{Y,y}$, etc. We shall call these *Hodge-theoretic* or *HT-arithmetic structures.*

The AG-arithmetic structures and HT-arithmetic structures in $\omega_{Y,y}^{\otimes l}$, $\det \mathbb{V}^{1,0}$ etc. are defined for different sets of points $y \in Y$. We will denote these sets by $Y(\overline{\mathbb{Q}})$ and Y_{HT} respectively. *For common points $y \in Y(\overline{\mathbb{Q}}) \cap Y_{\mathrm{HT}}$ these arithmetic structures are in general not comparable.* An example will illustrate this.

Example. Let $\mathcal{H}$ be the upper-half-plane with standard coordinate τ and $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ a congruence subgroup. We think of $\mathcal{H}$ as the SL_2-orbit of i (and $\check{\mathcal{H}} \cong \mathbb{P}^1$), parametrizing Hodge structures of type $h^{1,0} = h^{0,1} = 1$ on a 2-dimensional vector space V. Then there is the standard trivialization

$$\omega_{\mathcal{H}} \cong \mathcal{O}_{\mathcal{H}}$$

of the canonical line bundle given by the section $d\tau$ of $\omega_{\mathcal{H}} \to \mathcal{H}$. Using this trivialization, sections $\psi \in H^0(Y, \omega_Y^{\otimes l})$ that are finite at the cusps are represented by automorphic functions $f_\psi(\tau)$ satisfying the usual condition

$$f_\psi\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^l f_\psi(\tau)$$

and where f_ψ is finite at the cusps. Denote this space of functions by $M_l(\mathcal{H}, \Gamma)$. It is well-known that $M_l(\mathcal{H}, \Gamma)$ has an arithmetic structure, and that for $l \gg 0$ the functions in $M_l(\mathcal{H}, \Gamma)_k$ projectively embed Y as a curve defined over a number field.[48] The arithmetic points $Y(\overline{\mathbb{Q}})$ may be defined by the condition that for all pairs $f, g \in M_l(\mathcal{H}, \Gamma)_k$

[48]Of course, much more precise results are known about this.

with $g(\tau) \neq 0$, we have

$$(f/g)(\tau) \in \overline{\mathbb{Q}}\,.$$

The Hodge bundle $\mathbb{V}^{1,0} \to \mathcal{H}$ has an arithmetic structure as described above, and therefore

$$\omega_{\mathcal{H}} = \otimes^2 \mathbb{V}^{1,0} \tag{IV.C.5}$$

has an induced arithmetic structure. We observe that

(IV.C.6) *$d\tau$ is an arithmetic section of $\omega_{\mathcal{H}} \to \mathcal{H}$.*

This means the following: For $\tau \in \mathbb{P}^1(k) \cap \mathcal{H}$ the filtration $\mathbb{V}^{1,0}_\tau \subset V_{\mathbb{C}}$ is defined over k; thus

$$\mathbb{V}^{1,0}_\tau = \mathbb{C} \otimes (\mathbb{V}^{1,0}_\tau)_k\,.$$

With the identification (IV.C.5)

$$d\tau \in \check{T}_\tau \mathcal{H} \cong \left(\mathbb{C} \otimes (\mathbb{V}^{1,0}_\tau)_k\right)^{\otimes 2}$$

and in fact

$$d\tau \in \left((\mathbb{V}^{1,0}_\tau)_k\right)^{\otimes 2}.$$

The proof consists in writing out the identification (IV.C.5) in coordinates.

Returning to the discussion of this example, it is well known [Wu] that for $\tau \in \mathbb{P}^1(\overline{\mathbb{Q}}) \cap \mathcal{H}$ and f, g as above, $(f/g)(\tau)$ is in general transcendental. It is only for those τ giving a CM Hodge structure, i.e. for τ a quadratic imaginary complex number, that we have $(f/g)(\tau) \in \overline{\mathbb{Q}}$. Moreover, even in this case, for ψ (AG-) arithmetic we have that $f_\psi(\tau)$ is transcendental — although its value in $\mathbb{C}^*/\overline{\mathbb{Q}}^*$ depends only on l and the imaginary quadratic field $\mathbb{Q}(\tau)$.

This is a a special case of the following deep result, in which (b) is adapted from (11.1) on page 77 of [Shi].

(IV.C.7) **Theorem:**

(a) [C], [SW] *Suppose given $\varphi \in H$ such that $[\varphi] \in Y(\overline{\mathbb{Q}}) \cap Y_{\mathrm{HT}}$; in particular, $\varphi \in H \cap \check{H}(\overline{\mathbb{Q}})$. Then φ is a CM point. There is a finite extension $k_0 \supset k$ depending only on Γ and the CM type associated to φ, such that $[\varphi] \in Y(k_0)$.*

(b) [Shi] *Let $\varphi \in H$ be a CM point (and k_0 as above). Then there is a transcendental number Δ_φ, which modulo $\overline{\mathbb{Q}}^*$ depends only on the CM type associated to φ, such that for every $l \in \mathbb{N}$ and $\psi \in H^0(Y, (\det \mathbb{V}_Y^{1,0})^{\otimes l})_k$, we have*

$$\Delta_\varphi^{-l} \cdot \widetilde{\psi}(\varphi) \in (\det \mathbb{V}_H^{1,0})_{k_0}^{\otimes l}.$$

Alternately, writing $\widetilde{\psi} = f_\psi \cdot (dz_1 \wedge \cdots \wedge dz_g)^{\otimes l}$, we have

$$f_\psi(\varphi) \in \Delta_\varphi^l \overline{\mathbb{Q}}. \qquad \square$$

(IV.C.8) **Remark:** Informally, this says (a) that the points where one can compare the AG- and HT-arithmetic structures on fibres of the canonical Hodge line bundle $\det \mathbb{V}^{1,0}$ are precisely the CM points; and (b) the proportionality factor between these structures[49] has good invariance properties. For example, given $\gamma \in G(\mathbb{Q})$ we have $\Delta_\varphi = \Delta_{\gamma \cdot \varphi}$.

(IV.C.9) **Example:** For a CM point $P \in \mathbb{B}$, there is a CM field L_P with $[L_P : \mathbb{F}] =: e = 1, 2,$ or 3 and P/L_P. Let $\psi^+, \psi^- = \overline{\psi^+}$ denote the embeddings of $\mathbb{F}$ in $\mathbb{C}$, and

$$\begin{cases} \theta_1^+, \dots, \theta_e^+ \\ \theta_1^-, \dots, \theta_e^- \end{cases} \quad \text{where} \quad \begin{cases} \theta_i^- = \overline{\theta_i^+} \text{ and} \\ \theta_i^+ |_{\mathbb{F}} = \psi^+ \text{ for each } i \end{cases}$$

denote the embeddings of L_P in $\mathbb{C}$. Then (up to reordering) $P = [1 : \theta_1^+(x) : \theta_1^+(y)]$ for some $x, y \in L_P$,[50] and we set

$$\Theta_P := \begin{cases} \{\theta_1^+\} = \{\psi^+\}, & e = 1 \\ \{\theta_1^+, \theta_2^-\}, & e = 2 \\ \{\theta_1^+, \theta_2^-, \theta_3^-\}, & e = 3. \end{cases}$$

Associated to the CM type (L_P, Θ_P) is an abelian 3-fold A_P, analytically isogenous to

$$\mathbb{C}^e / \Theta_P(\mathcal{O}_{L_P}) \times \left(\mathbb{C} / \psi^+(\mathcal{O}_{\mathbb{F}}) \right)^{3-e},$$

[49] Trivially, any two rank-1 arithmetic structures *are* proportional.

[50] So P is really defined over $\theta_1^+(L_P)$, which we informally identify with L_P.

which is defined over a finite abelian extension $\widetilde{L}_P$ of L_P^c. Taking any nonzero $\eta \in \Gamma(\Omega^3_{A_P/\overline{\mathbb{Q}}})$ and[51] $[\delta] \in H_3(A_P, \mathbb{Q})$, define

$$\Delta_P := \int_\delta \eta \in \mathbb{C}^* .$$

Theorem (IV.C.7) says that the class of Δ_P in $\mathbb{C}^*/\overline{\mathbb{Q}}^*$ depends only on (L_P, Θ_P) and is invariant under the section of $G(\mathbb{Q})$ on $\mathbb{B}$. However, in this special case it turns out that for each given L_P, the possible $\mathbb{F}$-compatible types are permuted transitively by the Galois group $\mathrm{Gal}(L_P^c/\mathbb{F})$, acting on the right. Consequently Δ_P depends only on L_P. □

In general, any *irreducible, polarizable* CM Hodge structure of type $(1,0)+(0,1)$ is obtained by taking

- a CM field L (say, of degree $2g$ over $\mathbb{Q}$), considered as a $\mathbb{Q}$-vector space; and
- a set $\Theta = \{\theta_1, \ldots, \theta_g\}$ of embeddings satisfying $\Theta \coprod \overline{\Theta} = \mathrm{Hom}(L, \mathbb{C})$.

Noting that $L \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{\theta \in \mathrm{Hom}(L,\mathbb{C})} \mathbb{C}_\theta$, we then set

$$F^1(L \otimes_{\mathbb{Q}} \mathbb{C}) := \bigoplus_{\theta \in \Theta} \mathbb{C}_\theta .$$

The resulting Hodge structure identifies with H^1 of a $\overline{\mathbb{Q}}$-abelian variety $A_{(L,\Theta)}$ analytically isogenous to $\mathbb{C}^g/\Theta(\mathcal{O}_L)$, where $\Theta(\beta) := (\theta_1(\beta), \ldots, \theta_g(\beta))$. The corresponding Δ-factor is any period of a $\overline{\mathbb{Q}}$-holomorphic form of top degree.

For $\varphi \in H$, Δ_φ is the product of Δ-factors of φ's irreducible summands. This is nothing but a special value of a period of an algebraic section of $\det \mathbb{V}_Y^{1,0}$. In (III.C.7)(b) it was therefore most natural to consider restricted-Siegel modular forms, but we could also have replaced $(\det \mathbb{V}^{1,0})^{\otimes l}$ by $\omega^{\otimes m}$ and pluricanonical forms.

(IV.C.10) **Example:**

[51]More precisely, we must require $[\delta]$ to be in the image of $H_2(\mathbb{C}^2/\Theta_P(L_P)) \otimes H_1(\mathbb{C}/\psi^+(\mathcal{O}_F))^{\otimes 3-e}$.

(a) In the first, respectively second examples ($H = \mathbb{B}$ resp. $\mathcal{H}_2$), this means replacing Δ_φ^l by $\Delta_\varphi^{3m/2}$, respectively Δ_φ^{3m}.
(b) For the first example, we could instead use $f \in H^0(Y, (\mathbb{V}_+^{1,0})^{\otimes k})$, which makes f a Picard modular form of weight k, replacing Δ_φ^l by $\Delta_\varphi^{k/2}$.

For the two examples, we shall set henceforth

$$\widetilde{\Delta}_\varphi := \begin{cases} \Delta_\varphi^{1/2} & \text{(Picard)} \\ \Delta_\varphi & \text{(Siegel).} \end{cases}$$

The choice of square root does not matter.

Appendix to section IV.C: Explicit canonical models for the two examples. The generators of the coordinate ring R_Y parametrize algebro-geometric structures lying over $Y = \Gamma\backslash H$. These algebro-geometric structures are not only families of abelian varieties, but also families of curves and K3 surfaces. In this appendix we summarize, for specific choices of Γ, how this looks in our two examples.

We first recall the classical definition of Siegel modular forms of degree $g \geqq 2$ and weight l with respect to $\Gamma \subseteq \mathrm{Sp}(2g, \mathbb{Z})$. A holomorphic function $F \in \mathcal{O}(\mathcal{H}_g)$ belongs to the space $M_l(\mathcal{H}_g, \Gamma)$ of such forms if for each $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$ we have

$$F(\gamma[\tau]) = \{\det(C[\tau] + D)\}^l f([\tau]) \,.$$

Here, $A, B, C, D, [\tau]$ are $g \times g$ matrices with ${}^t[\tau] = [\tau]$, $\mathrm{Im}[\tau] > 0$ and

$$\gamma[\tau] = (A[\tau] + B)(C[\tau] + D)^{-1} \,.$$

By the Köcher principle we need not impose any growth conditions.

In the special case $g = 2$ and $\Gamma = \mathrm{Sp}(4, \mathbb{Z})$ Clingher and Doran [CD] have constructed a family of K3 surfaces of generic Picard number 17 and given by the minimal resolution $\widetilde{X}_{\boldsymbol{\alpha}}$ of the hypersurface

$$X_{\boldsymbol{\alpha}} = \{Y^2 ZW - 4X^3 Z + 3\alpha_2 XZW^2 + \alpha_3 ZW^3 \\ + \alpha_5 XZ^2 W - \tfrac{1}{2}(\alpha_6 Z^2 W^2 + W^4) = 0\}$$

in $\mathbb{P}^3$. Here

$$\boldsymbol{\alpha} = [\alpha_2, \alpha_3, \alpha_5, \alpha_6] \in \mathcal{M} := \mathbb{WP}(2,3,5,6)\backslash\{\alpha_5 - \alpha_6 = 0\}$$

where $\mathbb{WP}$ denotes the weighted projective space with the indicated weights. The isomorphism class of $\widetilde{X}_{\boldsymbol{\alpha}}$ as an $H \oplus E_8 \oplus E_7$-polarized K3 surface is recorded exactly by $\mathcal{M}$ ([CD]), where here $H = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The period map gives an isomorphism

$$\mathcal{M} \xrightarrow{\sim} \Gamma\backslash\mathcal{H}_2$$

whose inverse is given by explicit modular forms: in fact,

$$\alpha_j \in M_{2j}(\mathcal{H}_2, \Gamma) \text{ for } j = 2,3,5,6$$

where two of the α_j are Eisenstein series and two are cusp forms. These generate an index-two subring of $\underset{l \geqq 0}{\oplus} M_l(\mathcal{H}_2, \Gamma)$. The details are in [CD].

For the next example we take $g = 3$ and $\mathbb{F} = \mathbb{Q}(\sqrt{-3})$, and assume $M \in \mathrm{Sp}_6(\mathbb{Z})$ satisfies $M^2 + M + 1 = 0$ and $\mathrm{Sp}_6(\mathbb{Z})^M = \mathcal{U}((2,1), \mathcal{O}_{\mathbb{F}})$. The fixed point set of M acting on $\mathcal{H}_3$ is a two-ball $\mathbb{B}$, and we define Picard modular forms with respect to $\Gamma' \subset \mathcal{U}((2,1), \mathcal{O}_{\mathbb{F}})$ as follows: Writing $\gamma = [\gamma_{ij}]_{i,j=0}^2 \in \Gamma'$ and representing points of $\mathbb{B}$ by $z = (z_1, z_2)$, we set

(IV.C.11) $$\begin{cases} j_\gamma(z_1, z_2) := \gamma_{00} + \gamma_{01}z_1 + \gamma_{02}z_2 \\ \gamma(z_1, z_2) := \left(\frac{\gamma_{10}+\gamma_{11}z_1+\gamma_{12}z_2}{j_\gamma(z_1,z_2)}, \frac{\gamma_{20}+\gamma_{21}z_1+\gamma_{22}z_2}{j_\gamma(z_1,z_2)}\right). \end{cases}$$

For $g(z) \in \mathcal{O}(\mathbb{B})$ we have $G \in M_{l'}(\mathbb{B}, \Gamma')$ if

(IV.C.12) $$G(\gamma(z)) = (j_\gamma(z_1, z_2))^{l'} G(z)$$

for all $\gamma \in \Gamma'$. The natural restriction map $F \to F|\mathbb{B}$ gives

$$M_l(\mathcal{H}_3, \Gamma) \to M_{2l}(\mathbb{B}, \Gamma^M)\,.$$

Taking $\Gamma' = \mathcal{U}((2,1), \mathcal{O}_{\mathbb{F}})$ and $S\Gamma' = S\mathcal{U}((2,1), \mathcal{O}_{\mathbb{F}})$ we consider the family of quartic curves C_β given by

$$\{Y^3Z = X^4 + \beta_2X^2Z^2 + \beta_3XZ^3 + \beta_4Z^4\} \subset \mathbb{P}^2\,.$$

Write the RHS of this equation as $\prod_{i=1}^{4}(X - t_i Z)$ with $\sum_i t_i = 0$ and take

$$\begin{aligned} &\mathbb{P}^2_{\mathfrak{t}} \subset \mathbb{P}^3 \text{ to be defined by } \sum_i t_i = 0 \\ &\cup \\ &\Delta := \bigcup_{i<j} \{t_i = t_j\} \\ &\cup \\ &\delta := \bigcup_{i<j<k} \{t_i = t_j = t_k\}\,. \end{aligned}$$

The above family of curves lives naturally over

$$\mathcal{M}' := (\mathbb{P}^2_{\mathfrak{t}} \backslash \Delta)/\mathcal{S}_4$$

where $\mathcal{S}_4$ is the permutation group of the t_i. The period map extends to an isomorphism

$$(\mathbb{P}^2_{\mathfrak{t}} \backslash \Delta)/\mathcal{S}_4 \xrightarrow{\sim} \Gamma' \backslash \mathbb{B}$$

and composing its inverse with the quotient $S\Gamma' \backslash \mathbb{B} \twoheadrightarrow \Gamma' \backslash \mathbb{B}$ exhibits

$$\beta_j \in M_{3j}(\mathbb{B}, S\Gamma'), \qquad j = 2, 3, 4\,.$$

The monograph [Ho] is the standard reference.

Remark. One reason for putting in the discussion is this: As mentioned in the introduction, historically higher degree automorphic cohomology in the non-classical case was a somewhat mysterious object; one had not actually explicitly "seen" such a class. Via the [EGW] formalism this is now possible. For example, if we let $G = \beta_j \in M_{3j}(\mathbb{B}, S\Gamma')$ and pick μ' so that $L'_{\mu'} = (\omega'_{\mathbb{B}})^{\otimes j}$ then the $S\Gamma'$-invariant pluricanonical form $G(z)(dz_1 \wedge dz_2)^{\otimes j}$ lifts to define

$$\sigma_G \in H^0_{\mathrm{DR}} \left\{ \Gamma\big(S\Gamma' \backslash \mathcal{W}, \Omega^{\bullet}_{\pi'} \otimes (\pi')^{-1} L'_{\mu'}\big) \right\}$$

with Penrose transform

$$\sigma_G \omega_1^3 \in H^1 \left\{ \Gamma\big(S\Gamma' \backslash \mathcal{W}, \Omega^{\bullet}_{\pi} \otimes \pi^{-1} L_{\mu}\big) \right\} \cong H^1\big(S\Gamma' \backslash D, L_{\mu}\big).$$

IV.D. **Special values of cuspidal automorphic cohomology classes.** Given a class $\alpha \in H^q_o(X, L_\mu)$, its pullback[52] $\widetilde{\alpha}$ to D has a canonical representative in the guise of a relative differential form on $\mathcal{W}$. In order to study the "special values" of these representatives in parallel with (IV.C.7), we need a notion of special points for $\mathcal{W}$. To this end, recall from section III.D the general notion of correspondence spaces $\mathcal{W} \subset \check{\mathcal{W}}$ with compatible holomorphic projections $\pi_w : \mathcal{W} \twoheadrightarrow D_w$ and $\check{\pi}_w : \check{\mathcal{W}} \twoheadrightarrow \check{D}_w$, where the $\check{\pi}_w$ are clearly defined over a number field.

(IV.D.1) **Definition:** *A point $\mathbf{\Phi} \in \mathcal{W}$ is* **CM** *if all $\varphi_w := \pi_w(\mathbf{\Phi}) \in D_w$ are CM Hodge structures.*

This notion does not depend on the choices of Hodge numbers one makes for the $\{D_w\}_{w\in W}$. It also forces $\mathbf{\Phi} \in \mathcal{W} \cap \check{\mathcal{W}}(\overline{\mathbb{Q}})$.

If D is nonclassical and D' fibres holomorphically over an Hermitian symmetric domain H (both in $\{D_w\}_{w\in W}$), then one has a diagram

(IV.D.2)
$$\begin{tikzcd} & \mathcal{W} \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] \arrow[dd, dashed, "\pi_H"'] & \\ D & & D' \arrow[dl] \\ & H & \end{tikzcd}$$

which is what we shall mainly work with. Note that π_H of a CM point is also CM.

Now let $X = \Gamma\backslash D$ be one of our two examples, with $G = S\mathcal{U}(2,1)$, respectively $\mathrm{Sp}(4)$. Assume $k \geq 4$. We shall write throughout this section
$$\mu'_k := \begin{cases} ke_3^*, & 1^{\text{st}} \text{ example} \\ -k(e_1^* + e_2^*), & 2^{\text{nd}} \text{ example} \end{cases}$$
and $\mu_k := \mu'_k - \rho + \rho'$, for the weights corresponding to Picard resp. Siegel modular forms and their Penrose transforms. Let $\mathfrak{F}_k \to \mathcal{W}$ denote the vector bundles $\Omega^1_\pi \otimes \pi^{-1} L_{\mu_k}$; at any CM point, or even any

[52]Recall that in the cases of interest in this work, the pullback map $H^q_o(X, L_\mu) \to H^q(D, L_\mu)$ is injective.

point in $\mathcal{W} \cap \check{\mathcal{W}}(\overline{\mathbb{Q}})$, it has a natural HT-arithmetic structure. In this section we shall prove the

(IV.D.3) **Theorem:**

(i) *The cuspidal automorphic cohomology $H^1_o(X, L_{\mu_k})$ has a natural arithmetic structure; cf.* (IV.D.5) *below.*

(ii) *Let $\mathbf{\Phi} \in \mathcal{W}$ be CM, with image $\varphi_H := \pi_H(\mathbf{\Phi})$ in H, and let $\alpha \in H^1_o(X, L_\mu)$ be an arithmetic class. Then*[53]

(IV.D.4) $$\widetilde{\Delta}^{-k}_{\pi_H(\Phi)}\tilde{\alpha}(\mathbf{\Phi}) \in (\mathfrak{F}_k \,|_{\Phi})_{\overline{\mathbb{Q}}}\,.$$

(iii) *CM points are dense in $\mathcal{W}$.*

This result (ii) is an analogue of (IV.C.7)(b), again providing a link between HT- and AG- notions of arithmeticity.

Classically there are at least four ways of describing arithmeticity of automorphic forms:

(A) in terms of the canonical model, as was done above;
(B) by the values of automorphic forms at CM points, as in Theorem (IV.C.7);
(C) in terms of the coefficients of Fourier expansions around rational boundary components;
(D) using a basis consisting of normalized Hecke eigenfunctions.[54]

For automorphic cohomology, (A)–(C) are available[55] in modified form: (A) via the Penrose transform; (B) in the sense described above; and (C) with Fourier coefficients as defined in [C3]. In fact, (C) turns out to be quite interesting as it gets into the arithmetic structure of *extensions*, or *partial compactifications*, of Mumford-Tate domains (cf. [KU] and the recent work [KP]). It is planned for this to be the subject of a future work, one in which, for example, we hope to discuss the arithmeticity of cuspidal automorphic representations of the exceptional group G_2.

[53] The precise meaning of (IV.D.4) will be defined in proof.
[54] Assuming a multiplicity one result for the Hecke eigenvalues, normalizing the leading Fourier coefficients to 1 forces the remaining ones into $\overline{\mathbb{Q}}$. Hence, $\overline{\mathbb{Q}}$-linear combinations will also have this property.
[55] At least, for (A) and (C), in special cases such as those considered here.

As for (D), Hecke operators (and hence eigenfunctions) are available directly, leaving the issue of normalization. The latter may in fact be accessible directly (without the aid of Penrose transforms) when the boundary component quotient admits a canonical model.

The proof of theorem (IV.D.1) will be given in several steps.

Step one: The definition of the arithmetic structure is given by the image

$$\mathcal{P}_\Gamma\big(H^0_o(X', L'_{\mu'})_k\big) \subset H^1_o(X, L_\mu) \tag{IV.D.5}$$

of the Penrose transform in (IV.B.11). This definition is, at least in principal, available whenever G is of Hermitian type: in this case, the quotient of the corresponding Hermitian symmetric domain by an arithmetic group has a canonical model defined over a number field.

Step two: The Penrose transform (IV.D.5) is induced from the diagram

$$\begin{array}{ccc} H^0\big(\Gamma(\mathcal{W}, \Omega^\bullet_{\pi'} \otimes {\pi'}^{-1} L'_{\mu'})\big) & \xrightarrow{\ \omega\ } & H^1\big(\Gamma(\mathcal{W}, \Omega^\bullet_{\pi} \otimes \pi^{-1} L_\mu)\big) \\ \wr\| & & \wr\| \\ H^0(D', L^1_{\mu'}) & \xrightarrow{\quad\mathcal{P}\quad} & H^1(D, L_\mu)\,, \end{array} \tag{IV.D.6}$$

where $\omega \in \Gamma(\check{\mathcal{W}}, \Omega^1_{\pi} \otimes \check{\pi}^{-1} L_{\rho'-\rho})$, and we are making use of the $\overline{\mathbb{Q}}$-isomorphism

$$\check{\pi}^{-1} L_{\rho'-\rho} \cong \check{\pi}^{-1} L_{\mu_k} \otimes \left({\check{\pi}}'^{-1} L'_{\mu'_k}\right)^{\vee}$$

of line bundles over the quasi-projective $\overline{\mathbb{Q}}$-variety $\check{\mathcal{W}}$. Locally at any point $\mathbf{\Phi} \in \check{\mathcal{W}}$, ω descends from a matrix entry of $g^{-1}dg$, where the matrix is computed with respect to the frame recorded projectively by $\mathbf{\Phi}$. Consequently, ω is $\overline{\mathbb{Q}}$-rational at $\overline{\mathbb{Q}}$-points (and is thus itself defined over $\overline{\mathbb{Q}}$).

Step three: Given an (AG-) arithmetic class ψ as in (IV.C.7) with pullbacks $\Psi \in H^0(X', L'_{\mu'_k})$ and $\widetilde{\Psi} \in H^0(D', L'_{\mu'_k})$, we wish to show that $\alpha := \mathcal{P}_\Gamma(\psi)$ satisfies (IV.D.4). By (IV.D.6), $\tilde{\alpha}$ has the canonical representative

$${\pi'}^*(\widetilde{\Psi}) \cdot \omega \in \Gamma(\mathcal{W}, \mathfrak{F}_k);$$

and (IV.D.4) now clearly follows from step 2 and Theorem (IV.C.7)(b), since the CM point $\mathbf{\Phi}$ is $\overline{\mathbb{Q}}$-rational and $(\pi'^*\widetilde{\Psi})(\mathbf{\Phi}) = \widetilde{\psi}(\pi_H(\mathbf{\Phi}))$. Having thus disposed of (IV.D.3)(i) and (ii), we take up (iii) in the next section.

Remark: The arithmetic structure on $H^1_o(X, L_\mu)$ is "classical" in the sense that it is derived from the arithmetic structure on the space of Picard or Siegel automorphic forms. From the point of view of the representation of $G_{\mathbb{R}}$ on $L^2(\Gamma\backslash G_{\mathbb{R}})$, or rather the adelic version of this ([C1]), since $\mu + \rho$ is regular and satisfies property $\mathbf{P}$ the only irreducible unitary $G_{\mathbb{R}}$-modules that make a nonzero contribution to H^1_o are discrete series, and these representations all occur elsewhere as the infinite component of automorphic representations arising from the coherent cohomology of Shimura varieties.

For Γ co-compact, much deeper is the sought for arithmetic structure on $H^2(\Gamma\backslash D, L_{-\rho})$ given by the results in [C1] and [C2]. To explain these, we recall that to a weight μ and Weyl chamber C such that $\mu+\rho$ is on a wall of $\mathbf{C}$ (and is therefore singular), there is associated a unitary $G_{\mathbb{R}}$-module $V_{(\mu+\rho,C)}$ called a *limit of discrete series* (cf. [K2]). The limit of discrete series is *totally degenerate* if $\mu+\rho = 0$ and it is associated to a Weyl chamber $\mathbf{C}$ for which no $\mathbf{C}$-simple root is compact. It is known that such a representation cannot occur as the infinite component of an automorphic representation arising from the coherent cohomology of Shimura varieties.

For the case of $S\mathcal{U}(2,1)$, $\mu = -\rho$ and $\mathbf{C}$ the dominant Weyl chamber, Carayol shows that

- $V_{(\mu+\rho,C)}$ occurs as the infinite component of the adelic version of an automorphic representation of $G_{\mathbb{R}}$ corresponding to $H^2(X, L_{-\rho})$;
- in the cup product mapping given below to the vector space $H^2(X, L_{-\rho})$, the two factors have an arithmetic structure, and therefore if the kernel has one then so does the image;

- If one takes the limit over Γ's then the image of the cup product spans.

We will give an exposition, in the setting of this work, of Carayol's argument in section IV.F below.

Appendix to section IV.D: An alternate method for evaluating cohomology classes and a question. In several respects the "preferred" Weyl chamber for D is the anti-dominant one.[56] Among the reasons for this are

(i) This is the Weyl chamber where the L^2-cohomology $H^1_{(2)}(D, L_\mu)$ and ordinary cohomology $H^1(D, L_\mu)$ best "line up;" more precisely, in a natural way $H^1(D, L_\mu)$ is the Harish-Chandra module associated to the discrete series representation $H^1_{(2)}(D, L_\mu)$;

(ii) Relatedly, the K-type of the Harish-Chandra module is given by "expanding" $H^1(D, L_\mu)$ about a maximal compact subvariety Z, via the sequence of maps

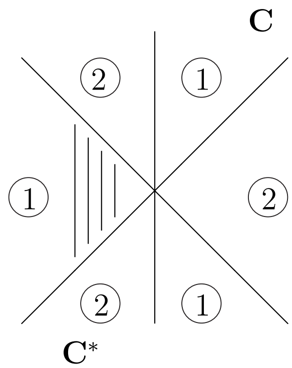
$$\begin{gathered} H^1(D, \mathcal{I}_Z \otimes L_\mu) \longrightarrow H^1(D, L_\mu) \longrightarrow H^1(Z, L_\mu) \longrightarrow 0 \\ H^1(D, \mathcal{I}_Z^2 \otimes L_\mu) \to H^1(D, \mathcal{I}_Z \otimes L_\mu) \longrightarrow H^1(Z, L_\mu \otimes \check{N}) \longrightarrow 0 \\ H^1(D, \mathcal{I}_Z^3 \otimes L_\mu) \to H^1(D, \mathcal{I}_Z^2 \otimes L_\mu) \to H^1(Z, L_\mu \otimes \operatorname{Sym}^2 \check{N}) \to 0 \\ \vdots \end{gathered}$$

where $\mathcal{I}_Z \subset \mathcal{O}_D$ is the ideal sheaf of Z and $N = N_{Z/D} \to Z$ is the normal bundle of Z in D (cf. [Schm2] for the origin of this method). This method will be used in the appendix to section IV.F to give a geometric derivation of the K-type of the totally degenerate limit of discrete series that was used in the proof of the main result in [C1].

(iii) We recall the picture of the Weyl chambers for D with the numbers giving the degree in which $H^q_{(2)}(D, L_\mu) \neq 0$ for $\mu + \rho$

[56] Suitably interpreted the following remarks are valid for a general $D = G_{\mathbb{R}}/T$. We shall illustrate the main point in the case $G_{\mathbb{R}} = S\mathcal{U}(2,1)$ of the first example.

non-singular, the shaded one being the anti-dominant one:

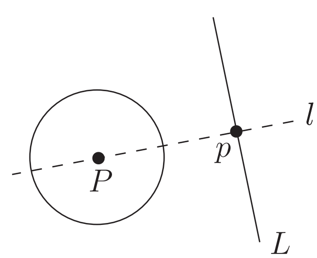


For Γ co-compact, using the Penrose transform we may define an arithmetic structure on the group $H^1(X, L_\mu)$ when $\mu + \rho \in \mathbf{C}$, and by Kodaira-Serre duality an arithmetic structure on $H^2(X, L_{\mu^*})$ when $\mu^* + \rho \in \mathbf{C}^*$.

Remark that for the Weyl chamber $\mathbf{C}$ such that for $\mu + \rho \in \mathbf{C}$ the group $H^1(D, L_\mu)$ contains the image of the Penrose transform, and since $(\mu, \alpha) > 0$ for α the positive compact root, the group $H^1(Z, L_\mu)$ is zero. Thus the expansion about Z method does not work for $\mathbf{C}$. We will explain below a method for evaluating classes in $H^1(D, L_\mu)$ at points of the correspondence space $\mathcal{W}$ when $\mu + \rho$ is anti-dominant.

The question remains of how to define an arithmetic structure on $H^q(X, L_\mu)$ for the remaining Weyl chambers, in particular when $\mu + \rho$ is anti-dominant.

Evaluation of classes in $H^1(X, L_\mu)$ when $\mu + \rho$ is anti-dominant. For this we recall the cycle space $\mathcal{U}$ whose points are $u = (P, L)$ in the figure



and when $u = (P, Z)$ corresponds to the maximal compact subvariety $Z_u = Z(P, L) = \{(p, l) \in D\}$ in the above figure. For each $Z_u \cong \mathbb{P}^1$ we have its cycle space $\mathcal{X}_u$ as in the basic example in section III.C. We set

$$\mathcal{X} = \bigcup_{u\in\mathcal{U}} \mathcal{X}_u\,.$$

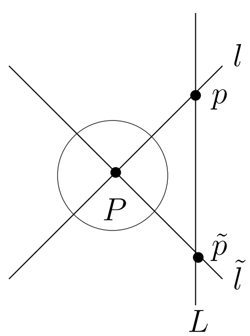
We also recall our notation for the incidence correspondence

$$\begin{tikzcd}[column sep=small] & \mathcal{I} \arrow[dl, "\pi_u"'] \arrow[dr] & = \{(\varphi, u) : \varphi \in Z_u\} \\ D & & \mathcal{U}\,. \end{tikzcd}$$

Proposition. *There is a commutative diagram of holomorphic mappings where F is biholomorphic:*

$$\begin{tikzcd} \mathcal{W} \arrow[r, "F"] \arrow[d, "\pi"'] & \mathcal{X} \arrow[d, "\tilde{\pi}"] & \supset & \mathcal{X}_u \arrow[d, "\tilde{\pi}_u"] \\ D & \mathcal{I} \arrow[l, "\pi_D"] & \ni & (\varphi, u)\,. \end{tikzcd}$$

Proof. We have our usual picture



$$\begin{cases} \varphi = (p, l) \in D \\ u = (P, L) \\ \mathcal{X}_u = \{(p, \tilde{p})\} \in Z_u \times Z_u \backslash \{\text{diagonal}\} \end{cases}$$

and if we define

$$\begin{cases} F(p, P, \tilde{p}) = (u = (P, L = \overline{p\tilde{p}}); p, \tilde{p}) \in \mathcal{X} \\ \tilde{\pi}(u = (P, L); p, \tilde{p}) = (\varphi = (p, \overline{Pp} = l), u = (P, L)) \\ \pi_D(\varphi, u) = \varphi \end{cases}$$

then it is straightforward to check commutativity and that F is biholomorphic. $\square$

From the diagram and the observation that the EGW theorem applies to both π and to $\tilde{\pi}_u$, fixing $u = u_0 \in \mathcal{U}$ and setting

$$Z = Z_{u_0}, \mathcal{X}_Z = \mathcal{X}_{u_0}, \pi_Z = \tilde{\pi}_{u_0}$$

we have a commutative diagram

$$\begin{tikzcd} H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{W},\Omega^\bullet_\pi\otimes\pi^{-1}L_\mu)\big) \arrow[r] & H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{X}_Z,\Omega^\bullet_{\pi_Z}\otimes\pi_Z^{-1}(L_\mu))\big) \\ H^1(D,L_\mu) \arrow[u, "\wr\|"] \arrow[r] & H^1(Z,L_\mu) \arrow[u, "\wr\|"] \end{tikzcd}$$

that reflects the restriction map given by the bottom arrow interpreted via the EGW method in the top arrow. We have seen in section III.C that each class in $H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{X}_Z,\Omega^\bullet_{\pi_Z}\otimes\pi^{-1}F_\mu)\big)$ has a canonical harmonic representative, and it is using this that we may evaluate a class in $H^1(D,L_\mu)$ at a point $\mathfrak{w}\in\mathcal{W}$.

Arithmetic structure on $H^1(X,L_\mu)$ when $\mu+\rho$ is anti-dominant. Above we have used the Penrose transform to give a map

$$H^0(X',L'_{\mu'})\xrightarrow{\mathcal{P}} H^1(X,L_\mu)$$

when $\mu'+\rho'=\mu+\rho\in\mathbf{C}$, and this led to an arithmetic structure on $H^1(X,L_\mu)$. The question arises if a similar method may be used in other Weyl chambers, e.g. the anti-dominant one. The answer is yes, and to explain it we note that

There are many Penrose transforms.

This is not surprising, because different cohomology groups $H^q(D,L_\mu)$ for D, and even $H^q(D,L_\mu)$ and $H^{q'}(D',L'_{\mu'})$ for different D and D', may represent the same $G_{\mathbb{R}}$-module. Rather than list the possibilities we shall illustrate them with the

Proposition. *Using the* [EGW] *formalism we have*

$$\begin{tikzcd} H^1_{\mathrm{DR}}\big(\Gamma(\mathcal{W},\Omega^\bullet_\pi\otimes\pi^{-1}L_\mu)\big) \arrow[r, "\omega_1^3"] & H^2_{\mathrm{DR}}\big(\Gamma(\mathcal{W},\Omega^\bullet_{\pi'}\otimes\pi'^{-1}L^1_{\mu'})\big) \\ H^1(D,L_\mu) \arrow[u, "\wr\|"] \arrow[r, "\mathcal{P}'"] & H^2(D',L'_{\mu'}) \arrow[u, "\wr\|"] \end{tikzcd}$$

where $\mu+\rho=\mu'+\rho'$.

Remark that when $\mu+\rho$ is anti-dominant for D, with a suitable topology $H^1(D,L_\mu)$ is the Harish-Chandra module associated to the discrete series $V_{\mu+\rho}$. Also, for $\mu'+\rho'$ in the same Weyl chamber but with the complex structure D', $q(\mu'+\rho')=2$ so that $H^2_{(2)}(D',L'_{\mu'})\neq 0$ and has the same Harish-Chandra module as $H^1_{(2)}(D,L_\mu)$.

Proof. The proposition will follow by showing that multiplication by ω_1^3 induces a morphism of complexes

$$(\Omega^\bullet_\pi\otimes\pi^{-1}L_\mu,d_\pi)\xrightarrow{\omega_1^3}(\Omega^{\bullet+1}_{\pi'}\otimes{\pi'}^{-1}L'_{\mu'},d_{\pi'})\,.$$

By lifting up to $G_{\mathbb{C}}$ and using the Maurer-Cartan forms that are semi-basic for $G_{\mathbb{C}}\to\mathcal{W}$, we have

$$\begin{cases}\Omega^1_\pi &=\Omega^1_{\mathcal{W}}/\{\omega_1^2,\omega_1^3,\omega_3^2\}\\ \Omega^1_{\pi'} &=\Omega^1_{\mathcal{W}}/\{\omega_3^1,\omega_3^2,\omega_1^2\}\end{cases}$$

where the brackets denote the span over $\mathcal{O}_{\mathcal{W}}$. It follows that there is an induced map

$$\Omega^\bullet_\pi\xrightarrow{\omega_1^3}\Omega^{\bullet+1}_{\pi'}\otimes{\pi'}^{-1}F_{(2,-1)}\,.$$

From the Maurer-Cartan equation

$$d\omega_1^3\equiv\omega_1^2\wedge\omega_2^3\operatorname{mod}\{\omega_1^1,\omega_2^2,\omega_3^3\}$$

we have

$$\begin{cases}d\omega_1^3\equiv_\pi 0\\ d\omega_1^3\equiv_{\pi'} 0\end{cases}$$

which implies that ω_1^3 induces a map of complexes. $\square$

Passing to the quotient by Γ there is an induced map

$$H^1(X,L_\mu)\xrightarrow{\mathcal{P}'}H^2(X',L'_{\mu'})\,.$$

Because of the representation-theoretic interpretations, it is feasible that $\mathcal{P}'$ is an isomorphism. If so, then we may use $\mathcal{P}'$ to define an arithmetic structure on $H^1(X,L_\mu)$ from that on the classical object $H^2(X',L'_{\mu'})$. Above we have shown how to evaluate classes $\alpha\in H^1(X,L_\mu)$ at points of $\Gamma\backslash\mathcal{W}$. This raises the

Question: *If $(\varphi_1, \varphi_2) \in \mathcal{W}$ is a compatible pair of CM points, then is there a number $\Delta \in \mathbb{C}^*$ such that for $\alpha \in H^1(X, L_\mu)$ the value $\Delta\alpha(\varphi_1, \varphi_2)$ is arithmetic?*[57]

To explain why this question is feasible, we note there is also a well-defined map of complexes

$$\Omega^{\bullet}_{\pi'} \otimes L'_{\mu'} \xrightarrow{\omega_2^1} \Omega^{\bullet+1}_{\pi'} \otimes L'_{\lambda'}$$

inducing a map

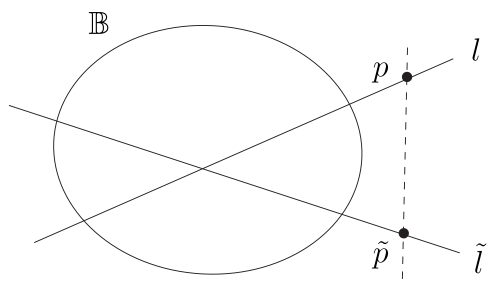
$$H^q(X', L'_{\mu'}) \xrightarrow{\omega_2^1} H^{q+1}(X', L'_{\lambda'})\,.$$

Now ω_2^1 is an arithmetic section of $\Omega^1_{\pi'}$ relative to the HT-arithmetic structure. Since the HT and AG-arithmetic structures are proportional at CM points, it is plausible, but we don't have a proof, that ω_2^1 induces a map of the AG-arithmetic structures on the cohomology groups.

IV.E. **CM points on correspondence spaces.** In this section we compute the proof of (IV.D.3) by proving existence and density of such points in the two running examples. For the first of these, we have

$$\mathcal{W} \subset D \times D$$

visualized by the picture $\mathbb{P}V_+ (\cong \mathbb{P}^2)$



with maps to the 6 D_w's[58] given by

$$(p, l),\ \ (\tilde{p}, \tilde{l}),\ \ (P, l),\ \ (P, \tilde{l}),\ \ (p, L),\ \ (\tilde{p}, L)$$

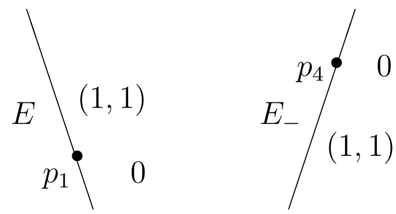
where $L := \overline{p\tilde{p}}$ and $P := l \cap \tilde{l}$.

[57]We do not expect that Δ is independent of the choice of (φ_1, φ_2).
[58]These come in 3 isomorphism classes, but that is not needed here.

In the second example,

$$\mathcal{W} \subset D \times D_{-}$$

where D_{-} is the "conjugate" domain, with points corresponding to configurations[59] in $\mathbb{P}V (\cong \mathbb{P}^3)$



with $H(p_1) < 0 < H(p_4)$ and E, E_{-} and Lagrangian with the restriction of H having signature $(1,1)$. Writing $p_2 = p_1^{\perp} \cap E_{-}$, $p_3 := p_4^{\perp} \cap E$, $E' := \overline{p_3 p_4}$, $E'' := \overline{p_1 p_2}$, the maps to the $\{D_w\}$ are given by the 8 evident "point $\in$ line" pairs.

In either case, we can recast CM points on $\mathcal{W}$ as pairs (φ_1, φ_2) (in $D \times D$, respectively $D \times D_{-}$) of CM Hodge structures[60] satisfying a compatibility condition. *A priori* this condition is unchanged from (IV.D.1), but at least in the first example there is a substantial simplification: in addition to (p, l) and $(\tilde{p}, \tilde{l})$ being CM points of D, *it is enough to assume that P and $L^{\perp}$ (see below) are points of $\mathbb{B}$.* Before proceeding to the comparatively simple proofs of density, we shall look in depth at some consequences of this observation — specifically, an approach to the classification of compatible CM pairs (in the first example) via their behavior under Galois conjugation.

We begin by recalling the description of the polarized Hodge structures involved. We are given the data $(V, Q, \mathbb{F})$ where:

- V is a 6-dimensional $\mathbb{Q}$-vector space;
- $Q : V \otimes V \to \mathbb{Q}$ is a non-degenerate alternating form;
- $\mathbb{F} = \mathbb{Q}(\sqrt{-d})$ acts faithfully on V;

[59] We need NOT have p_1 and p_4 (or E and E_{-}) conjugate.

[60] That is, their Mumford-Tate groups are $\mathbb{Q}$-rational tori. For our purposes here, however, we need the more explicit characterization of polarizable CM Hodge structures from [GGK1, §V.B].

- if $V_{\mathbb{F}} = V_+ \oplus V_-$ is the eigenspace decomposition for the $\mathbb{F}$-action, then $V_\pm$ are $\mathbb{Q}$-isotropic; and
- the Hermitian form

$$\langle v, w\rangle := \sqrt{-d}Q(v, \overline{w})$$

has signature $(2, 1)$.

The reason for the $\sqrt{-d}$ is so that, writing $v = \sum v_i\xi_i$ and $w = \sum w_j\xi_j$ for arbitrary vectors in $V_{+,\mathbb{C}}$, the matrix $k_{ij} = \overline{k_{ji}}$ defined by $\langle v, w\rangle =: \sum k_{ij}v_i\overline{w_j}$ has entries in $\mathbb{F}$. Now set $\mathcal{G}_{\mathbb{Q}} = \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$, $\mathcal{G}_{\mathbb{F}} = \mathrm{Gal}(\mathbb{C}/\mathbb{F})$ and let ρ = complex conjugation. Then any element of $\mathcal{G}_{\mathbb{Q}}$ is the product of ρ with a $\sigma \in \mathcal{G}_{\mathbb{F}}$, and any $\sigma \in \mathcal{G}_{\mathbb{F}}$ preserves $V_\pm$. The behavior of $\langle\ ,\ \rangle$ under Galois conjugation is thus described by

$$\begin{cases} \sigma\langle v, w\rangle = \langle \sigma v, \psi_\rho(\sigma)w\rangle\,, & \sigma \in \mathcal{G}_{\mathbb{F}} \\ \rho\langle v, w\rangle = \langle w, v\rangle \end{cases}$$

where $\psi_\rho(\sigma) = \rho\sigma\rho$.

Let now L be a CM field containing $\mathbb{F}$ and with normal closure L^c, which is also a CM field. One characterization of CM fields ([GGK1], section V.A) says that the restriction of ρ to L^c belongs to the center of $\mathrm{Gal}(L^c/\mathbb{Q})$. Since any element of $\mathcal{G}_{\mathbb{Q}}$ or $\mathcal{G}_{\mathbb{F}}$ acts on $v \in V_{+,L}$ through $\mathrm{Gal}(L^c/\mathbb{Q})$, we have

$$\text{(IV.E.1)} \qquad {}^{\sigma}\overline{v} = \sigma(\rho(v)) = \rho(\sigma(v)) = \overline{{}^{\sigma}v}\,.$$

We shall be working with

$$\begin{cases} \mathbb{P}^2 = \mathbb{P}V_{+,\mathbb{C}} \\ \check{\mathbb{P}}^2 = \mathbb{P}\check{V}_{+,\mathbb{C}} \end{cases}$$

and we shall use the notational correspondence

$$p \in \mathbb{P}V_{+,\mathbb{C}} = [v] \ \text{ where } \ 0 \neq v \in V_{+,\mathbb{C}}\,.$$

Then from section I.B we have the following descriptions of the Mumford-Tate domains $\mathbb{B}$ and D

$$\begin{aligned}
\mathbb{B} &= \{[v] \in \mathbb{P}V_{+,\mathbb{C}} : \langle v, v\rangle < 0\} \\
&= \left\{ \begin{array}{c} \text{Mumford-Tate domain for weight one polarized} \\ \text{Hodge structures for } (V,Q) \text{ which have an} \\ \mathbb{F}\text{-action and where } \dim(V_{+,\mathbb{C}} \cap V^{1,0}) = 1 \end{array} \right\} \\
D &= \{([v],[w]) \in \mathbb{P}^2 \times \mathbb{P}^2 : \langle v, v\rangle \ \text{ and } \ \langle w, w\rangle > 0, \langle v, w\rangle = 0\} \\
&= \{(p,l) \in \mathbb{P}^2 \times \check{\mathbb{P}}^2 : p \in l, p \in \mathbb{B}^c, l \cap \mathbb{B} \neq \phi\} \\
&= \left\{ \begin{array}{c} \text{Mumford-Tate domain for weight three polarized Hodge} \\ \text{structures for } (V,Q) \text{ which have an } \mathbb{F}\text{-action} \\ \text{and where } h^{3,0} = 1, h^{2,1} = 2 \text{ and } \dim V_{+,\mathbb{C}} \cap V^{3,0} = 1 \end{array} \right\}.
\end{aligned}$$

The correspondence between the first two equalities for D is

$$[w] = l^{\perp}$$

where $^{\perp}$ is the bijection $\mathbb{P}^2 \overset{\cong}{\longleftrightarrow} \check{\mathbb{P}}^2$ induced by $\langle\ ,\ \rangle$. We also recall the Hodge-theoretic meaning of the map $\pi_{\mathbb{B}} : \mathcal{W} \to \mathbb{B}$: Writing $\varphi_{\mathbb{B}} := \pi_{\mathbb{B}}(\varphi_1, \varphi_2)$, it is given by $F^1_{\varphi_{\mathbb{B}}} V_{+,\mathbb{C}} := F^2_{\varphi_1} V_{+,\mathbb{C}} \cap F^2_{\varphi_2} V_{+,\mathbb{C}}$.

From [GGK1] we recall the notion of a *Hodge-Galois basis* ω consisting of vectors $\omega_i \in V_{\mathbb{C}}$ which are of pure Hodge type, are defined over L^c, and have

$$^{\sigma}\omega_i = \omega_{\sigma(i)} \qquad\qquad \text{(up to constants)}$$

for some permutation of the indices. A polarized Hodge structure is CM if and only if it has a Hodge-Galois basis over a CM field which satisfies $Q(\omega_i, \overline{\omega_j}) = 0 \Leftrightarrow i \neq j$. In the present setting, this means that for $\omega_i \in V_{+,\mathbb{C}}$ and $\sigma \in \mathcal{G}_{\mathbb{F}}$, ω_i and $^{\sigma}\omega_i$ are either equal or orthogonal under $\langle\ ,\ \rangle$. Together with the above characterization of CM fields, this leads to:

(IV.E.2) *$p \in \mathbb{B}$ is a CM point if, and only if, for all $\sigma \in \mathcal{G}_{\mathbb{F}}$*

$$\begin{cases} \text{(a)} \quad {}^{\sigma}\overline{p} = \overline{{}^{\sigma}p}, \qquad \textit{and} \\ \text{(b)} \quad {}^{\sigma}p = p \ \textit{ or } \ {}^{\sigma}p \perp p; \end{cases}$$

(IV.E.3) *$(p,l) \in D$ is a CM point if, and only if, for all $\sigma \in \mathcal{G}_{\mathbb{F}}$*

$$\begin{cases} \text{(a)} & \text{(i) } {}^\sigma\overline{p} = \overline{{}^\sigma p} \textit{ and } \text{(ii) } {}^\sigma\overline{l} = \overline{{}^\sigma l}, \textit{ and} \\ \text{(b)} & \text{(i) } {}^\sigma p = p \textit{ or } {}^\sigma p \perp p \textit{ and } \text{(ii) } {}^\sigma l = l \textit{ or } {}^\sigma l \perp l\,. \end{cases}$$

This last statement means that ${}^\sigma l^\perp \perp l^\perp$ where $l^\perp$ is the unique point in $\mathbb{P}^2$ perpendicular under $\langle\ ,\ \rangle$ to all points of l.

To see the "if" part of (IV.E.3) one has to show that (a) and (b) imply that σ permutes the three points $p, p^\perp \cap l$, and $l^\perp$; that is, this set of points is closed under σ. This is similar to the proofs below and is left to the reader.

Thus the statement that

$$(p,l;\tilde{p},\tilde{l}) \textit{ is a compatible CM pair},$$

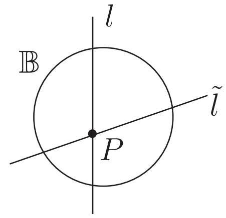
i.e. a CM point of $\mathcal{W}$, is equivalent to

$$\text{(IV.E.4)} \qquad \begin{cases} \text{(i)} & (p,l) \text{ and } (\tilde{p},\tilde{l}) \text{ belong to } D \\ & \text{and satisfy (IV.E.3)} \\ \text{(ii)} & P \text{ belongs to } \mathbb{B} \text{ and satisfies (IV.E.2)} \\ \text{(iii)} & L^\perp \text{ belongs to } \mathbb{B} \text{ and satisfies (IV.E.2).} \end{cases}$$

One can also look at this as saying that the configuration $(p,l;\tilde{p},\tilde{l})$ in $\mathbb{P}^2$ *and its dual configuration in* $\check{\mathbb{P}}^2$ *both* satisfy (i)–(ii). Consequently, one can obtain significant insight into the situation by understanding the possibilities for *semi-compatible CM pairs* — those configurations satisfying (IV.E.4)(i–ii) — and this is something we can understand well:

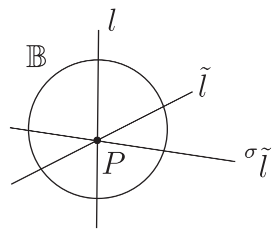
(IV.E.5) *Let $(p,l;\tilde{p},\tilde{l})$ be a semi-compatible CM pair, and $\sigma \in \mathcal{G}_{\mathbb{F}}$. Then the union of images of the configuration $l \cup \tilde{l} \subset \mathbb{P}V_{+,\mathbb{C}}$ under successive powers of σ takes one of the following forms (up to swapping tildes):*

(A)



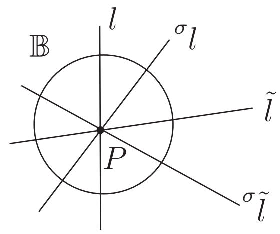
(i) ${}^{\sigma}l = l$ and ${}^{\sigma}\tilde{l} = \tilde{l}$
(ii) ${}^{\sigma}\tilde{l} = \tilde{l}$ and ${}^{\sigma}\tilde{l} = l$

(B)



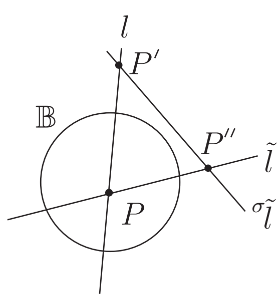
${}^{\sigma}l = l$ and ${}^{\sigma^2}\tilde{l} = \tilde{l}$

(C) *



${}^{\sigma^2}l = l$ and ${}^{\sigma^2}\tilde{l} = \tilde{l}$

(D)



(i) ${}^{\sigma}l = l$ and ${}^{\sigma^2}\tilde{l} = \tilde{l}$
(ii) ${}^{\sigma}l = \tilde{l}$ and ${}^{\sigma^2}\tilde{l} = l\,.$

In case (D) we also have that P, P', and P'' (hence also $l, \tilde{l}$, and ${}^{\sigma}\tilde{l}$) are all perpendicular, and that $\tilde{\mathrm{p}} = P''$. In subcase (ii) we have additionally that $\mathrm{p} = P'$.

Sketch of proof: Assuming (IV.E.3) for (p, l) and $(\tilde{p}, \tilde{l})$, it is clear that (IV.E.2)(a) will hold for P; it is the separate imposition of $P := l \cap \tilde{l} \in \mathbb{B}$ and (IV.E.2)(b) that creates significant restrictions. The key step is to show that the situation

(IV.E.6) $$ {}^{\sigma}l \neq l, \;\; {}^{\sigma}\tilde{l} \neq \tilde{l}, \;\; {}^{\sigma}P \neq P $$

can only lead to case (D)(ii). We shall prove this, as well as the assertions about case (D)(i), and leave the rest to the reader.

Assuming (IV.E.7), we have $P \perp {}^{\sigma}P$, $l^{\perp} \in {}^{\sigma}l$, and ${}^{\sigma}l^{\perp} \in l$. If ${}^{\sigma}P \neq l^{\perp}$ then since $P \perp {}^{\sigma}P, l^{\perp}$,

$$P \perp \mathrm{span}({}^{\sigma}P, l^{\perp}) = {}^{\sigma}l \implies P = {}^{\sigma}l^{\perp} .$$

Conversely, if $P \neq {}^{\sigma}l^{\perp}$, one finds ${}^{\sigma}P = l^{\perp}$. Now since P, ${}^{\sigma}P$, ${}^{\sigma^2}P$ etc. are all mutually perpendicular, we must have either ${}^{\sigma^2}P = P$ or ${}^{\sigma^3}P = P$.

Case ${}^{\sigma^2}P = P$*:* Since ${}^{\sigma}P = l^{\perp} \implies P = {}^{\sigma^2}P = {}^{\sigma}l^{\perp}$ and $P = {}^{\sigma}l^{\perp} \implies {}^{\sigma}P = {}^{\sigma^{-1}}P = l^{\perp}$, both ${}^{\sigma}P = l^{\perp}$ and $P = {}^{\sigma}l^{\perp}$ hold. The arguments thus far apply equally to $\tilde{l}$, and so

$$\tilde{l}^{\perp} = {}^{\sigma}P = l^{\perp} \implies \tilde{l} = l ,$$

a contradiction.

Case ${}^{\sigma^3}P = P$*:* Clearly then ${}^{\sigma^3}l = l$, otherwise ${}^{\sigma^2}l = l$ and so $P = l \cap {}^{\sigma}l = {}^{\sigma}P$ (contradiction); likewise for $\tilde{l}$. The only possibilities which do not lead to a contradiction (namely, $l = \tilde{l}$) are then

$${}^{\sigma}P = l^{\perp} \text{ and } P = {}^{\sigma}\tilde{l}^{\perp}$$

and the same thing with tildes swapped. This gives $l = {}^{\sigma^2}\tilde{l}$ hence ${}^{\sigma}l = \tilde{l}$, which is case (D)(ii).

To deal with case (D)(i), we first check the perpendicularity (i.e. $P = {}^{\sigma}\tilde{l}^{\perp}$, $P' = l^{\perp}$, and ${}^{\sigma}P = \tilde{l}^{\perp}$). Since ${}^{\sigma}\tilde{l} \neq \tilde{l}$ and ${}^{\sigma}P = P' \neq P$, we have

$${}^{\sigma}\tilde{l}^{\perp} \perp \tilde{l}^{\perp} \implies \begin{cases} \tilde{l}^{\perp} \in {}^{\sigma}\tilde{l} \\ {}^{\sigma}\tilde{l}^{\perp} \in \tilde{l} \end{cases} \implies {}^{\sigma}\tilde{l}^{\perp} \in {}^{\sigma^2}\tilde{l}$$

while

$$\begin{aligned} P \perp {}^{\sigma}P \text{ and } {}^{\sigma}l = l &\implies {}^{\sigma}P \perp {}^{\sigma^2}P \text{ (and both in } l) \\ &\implies {}^{\sigma^2}P = P \implies P \in {}^{\sigma^2}\tilde{l} . \end{aligned}$$

Suppose ${}^{\sigma^2}\tilde{l} \neq \tilde{l}$. Then ${}^{\sigma}\tilde{l}^{\perp}$, $P \in {}^{\sigma^2}\tilde{l} \cap \tilde{l} \implies {}^{\sigma}\tilde{l}^{\perp} = P \implies {}^{\sigma}P = {}^{\sigma^{-1}}P = \tilde{l}^{\perp}$. It follows that ${}^{\sigma}\tilde{l} = P^{\perp}$ and ${}^{\sigma^2}\tilde{l} = {}^{\sigma}P^{\perp} = \tilde{l}$, a contradiction.

So we must have ${}^{\sigma^2}\tilde{l} = \tilde{l}$, and ${}^{\sigma}P' = {}^{\sigma}({}^{\sigma}\tilde{l} \cap \tilde{l}) = {}^{\sigma^2}\tilde{l} \cap {}^{\sigma}\tilde{l} = \tilde{l} \cap {}^{\sigma}\tilde{l} = P'$. If $P \neq {}^{\sigma}\tilde{l}^{\perp}$, then P can be perpendicular to only one point of ${}^{\sigma}\tilde{l}$, and since it is perpendicular to ${}^{\sigma}P$ and $\tilde{l}^{\perp}$, we have ${}^{\sigma}P = \tilde{l}^{\perp} \implies P = {}^{\sigma}\tilde{l}^{\perp}$, a contradiction.

So (again) we must have $P = {}^{\sigma}\tilde{l}^{\perp}$, and thus ${}^{\sigma}P = \tilde{l}^{\perp}$. But then $P' = \tilde{l} \cap {}^{\sigma}\tilde{l}$ is perpendicular to both P and ${}^{\sigma}P$, hence to l, so that $P' = l^{\perp}$.

Finally we check that $\tilde{p} = P'$; otherwise, ${}^{\sigma}\tilde{p} \neq \tilde{p}$, and so ${}^{\sigma}\tilde{p} \perp \tilde{p}$. Now, $\tilde{p} \notin \mathbb{B}$ while $P \in \mathbb{B} \implies \tilde{p} \neq P$. Since $P' \in \operatorname{span}\{\tilde{p}, P\}$ and $P' \in \operatorname{span}\{{}^{\sigma}\tilde{p}, {}^{\sigma}P\}$, where $\tilde{p}, P$ are both perpendicular to ${}^{\sigma}\tilde{p}, {}^{\sigma}P$, we get $P' \perp P'$, a contradiction. $\square$

We turn to existence and density, starting with the first example. In the choice of basis above, we may assume that (say) $\langle \xi_2, \xi_2 \rangle < 0$. The standard orthogonalization procedure then shows that we may choose ξ_1, ξ_3 so that $k_{ij} = 0 \Leftrightarrow i \neq j$, while keeping the basis over $\mathbb{F}$. If we define (φ_1, φ_2) by

$$\begin{cases} p = [\xi_1] \\ l = [\langle \xi_1, \xi_2 \rangle] \end{cases} \quad \text{and} \quad \begin{cases} \tilde{p} = [\xi_3] \\ \tilde{l} = [\langle \xi_2, \xi_3 \rangle]\,, \end{cases}$$

then $P = [\xi_2] \in \mathbb{B}$ and $\overline{p\tilde{p}} \cap \mathbb{B} = \emptyset$ as desired, while φ_1 and φ_2 each split into 3 rank-two sub-Hodge structures with CM by $\mathbb{F}$. Clearly (IV.E.4) is satisfied trivially, and we are in case (A) of (IV.E.5) for every $\sigma \in \mathcal{G}_{\mathbb{F}}$.

Now D is a homogeneous space for the real points $G(\mathbb{R})$ of the $\mathbb{Q}$-algebraic group G, and $G(\mathbb{Q}) \subset G(\mathbb{R})$ is dense. The $G(\mathbb{Q})$ translates of either φ_1 or φ_2 in D are therefore analytically dense, and the same may be said for

$$\mathcal{W} \cap \{(G(\mathbb{Q}) \times G(\mathbb{Q})) \cdot (\varphi_1, \varphi_2)\}$$

in $\mathcal{W}$. Since this set consists of pairs trivially satisfying (IV.E.3)–(IV.E.4), we are done.

For the second example, one easily sees that any point $(p_1, E; p_4, E_-) \in \mathcal{W} \subset D \times D_-$ defined over an imaginary quadratic field[61] $\mathbb{F}$, is CM. These are dense by the same argument as above, finishing the proof of (IV.D.3)(iii).

[61]equivalently, p_1, p_2, p_3, and $p_4 \in \mathbb{P}^3(\mathbb{F})$.

(IV.E.7) **Remark:** Another way of constructing CM points on $\mathcal{W}$ (say, in the second example), would be to identify V with a degree 4 CM field L, and take the projective frame (for p_1, p_2, p_3, p_4) corresponding to the four eigenvectors for the multiplicative action. But just as general Lagrange frames (those (p_1, p_2, p_3, p_4) giving points of $\mathcal{W}$) need *not* be projectivized Hodge bases, this will give only a small slice of the CM points of $\mathcal{W}$. Moreover, it is only clear that "diagonal" $G(\mathbb{Q})$ — translates of such CM points (remaining in $\mathcal{W}$) would remain CM — not enough to establish density.

IV.F. **On a result of Carayol.** In this concluding section we shall given an expository account of the main theorem of [C1], in which the notion of *totally degenerate limits of discrete series* (TDLDS) plays a crucial role. To state a form of this result, fix

$$\begin{aligned}&G = \text{a } \mathbb{Q}\text{-anisotropic form of } S\mathcal{U}(2,1)\\ &\Gamma \subseteq G(\mathbb{Z}) \text{ a congruence subgroup}\end{aligned}$$

so that $\Gamma\backslash G_{\mathbb{R}}$ and $X(\Gamma) = \Gamma\backslash D = \Gamma\backslash G_{\mathbb{R}}/T$ are compact.[62] For each congruence subgroup $\Gamma_0 \subseteq \Gamma$ we denote by p_0 the covering map $X(\Gamma_0) \twoheadrightarrow X(\Gamma)$. Picard modular forms $M_k(\mathbb{B}, \Gamma)$ are defined as in (IV.C.10); their Penrose transforms and those of their conjugates correspond to the weights

$$\begin{aligned}\mu_k^{(1)} &:= \tfrac{k}{3}(\alpha_1 + 2\alpha_2) - \alpha_1 - \alpha_2\\ \overline{\mu}_k^{(1)} &:= \tfrac{-k}{3}(\alpha_1 + 2\alpha_2) + \alpha_2\,;\end{aligned}$$

cf. section IV.B.

[62] D, D', and D'' will be as above.

Now consider the composition

$$\begin{tikzcd}
M_k(\mathbb{B},\Gamma)\otimes\overline{M_k(\mathbb{B},\Gamma)} \arrow[d, "\mathcal{P}'_\Gamma\otimes\mathcal{P}''_\Gamma=:\mathcal{P}^{(k)}_\Gamma"] \arrow[dd, bend left=60, "\Theta^{(k)}_\Gamma"] \\
H^1(X(\Gamma),L_{\mu^{(1)}_k})\otimes H^1(X(\Gamma),L_{\overline{\mu}^{(1)}_k}) \arrow[d, "\widetilde{\Theta}^{(k)}_\Gamma"] \\
H^2(X(\Gamma),L_{-\rho})
\end{tikzcd}$$

where $\widetilde{\Theta}^{(k)}_\Gamma$ is the cup product and the Penrose transforms $\mathcal{P}^{(k)}_\Gamma$ are known to be surjective for $k\geq 4$.

(IV.F.1) **Theorem ([C1]):** *For $k\geq 5$, we have the "virtual surjectivity"*

(IV.F.2)
$$\begin{aligned} H^2(X(\Gamma),L_{-\rho}) &= \varinjlim_{\Gamma_0\subseteq\Gamma}(p_0)_*\left\{\mathrm{Im}\left(\widetilde{\Theta}^{(k)}_{\Gamma_0}\right)\right\} \\ &= \varinjlim_{\Gamma_0\subseteq\Gamma}(p_0)_*\left\{\mathrm{Im}\left(\Theta^{(k)}_{\Gamma_0}\right)\right\}. \end{aligned}$$

An immediate corollary of this result is that

(IV.F.3) *Assuming that the kernel* $\mathrm{Ker}\,\Theta^{(k)}_\Gamma$ *is defined over* $\overline{\mathbb{Q}}$*,* $H^2(\Gamma\backslash D,L_{-\rho})$ *inherits a dense* $\overline{\mathbb{Q}}$*-submodule*[63] $H^2(\Gamma\backslash D,L_{-\rho})_{\overline{\mathbb{Q}}}\subset H^2(\Gamma\backslash D,L_{-\rho})$

from the $\overline{\mathbb{Q}}$-arithmetic structures on the spaces of modular forms. This is of particular interest considering that this group corresponds to a TDLDS. Of course, the assumption that $\ker\Theta^{(k)}_\Gamma$ is defined over $\overline{\mathbb{Q}}$ means that the result is just a suggestion of where the hoped for $\overline{\mathbb{Q}}$-structure on the up until now very elusive group $H^2(\Gamma\backslash D,L_{-\rho})$ might come from. In our view, the depth of Carayol's argument suggest that there is more here than has been established thus far.

The proof, which occupies the remainder of this section, will consist of several steps, during which we will briefly elaborate on limits of discrete series for $S\mathcal{U}(2,1)$. There is further discussion of this in the

[63] not necessarily a $\overline{\mathbb{Q}}$-arithmetic structure, because we do not know that the $\ker(\Theta^{(k)}_\Gamma)$ are defined over $\overline{\mathbb{Q}}$.

appendix to this section. Obviously we have only to prove the first line of (IV.F.2). Before commencing, we have the following

(IV.F.4) **Remark:** (i) While not necessary for the statement, for the last step of the proof, the adelic point of view will be required. Given a compact open subgroup $K_f \leq G(\mathbb{A}_f)$, we set

$$X_{K_f} := G(\mathbb{Q})\backslash(G(\mathbb{R}) \times G(\mathbb{A}_f))/(T_{\mathbb{R}} \times K_f) \cong \coprod_{g \in C(K_f)} X(\Gamma_g)$$

with components indexed by the finite set

$$C(K_f) := G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f$$

and where for given $g \in G(\mathbb{A}_f)$ the

$$\Gamma_g := gK_f g^{-1} \cap G(\mathbb{Q})$$

are always congruence subgroups. (We shall also write $\Gamma := K_f \cap G(\mathbb{Q})$.) Associated to any[64] $\alpha \in G(\mathbb{A}_f)$ is an analytic Hecke correspondence in $X_{K_f} \times X_{K_f}$; cf. [Ke].

(ii) The spaces of automorphic forms $\mathcal{A}(G,\Gamma) \subset C^{\infty}(\Gamma\backslash G(\mathbb{R}))$ are the $\mathcal{Z}(\mathfrak{g})$-finite and right $K_{(\mathbb{R})}(= U(2))$-finite vectors. The Hecke correspondences lift to the self-product of

$$G(\mathbb{Q})\backslash\big(G(\mathbb{R}) \times G(\mathbb{A}_f)\big)/K_f \cong \coprod_{g \in C(K_f)} \Gamma_g\backslash G(\mathbb{R})\,,$$

and so operate on the spaces of functions

$$\mathcal{A}^G_{K_F} := \mathop{\times}_{g \in \mathcal{C}(K_F)} \mathcal{A}(G,\Gamma_g)\,.$$

Step 1*: Classifying the unitary irreducible representations of* G *with integral infinitesimal character.* These are

(a) discrete series and limits thereof: $\begin{cases} \text{(i)} & \text{holomorphic} \\ \text{(ii)} & \text{antiholomorphic} \\ \text{(iii)} & \text{nonholomorphic;} \end{cases}$

(b) characters (which have no cohomology in degrees 1 and 2);

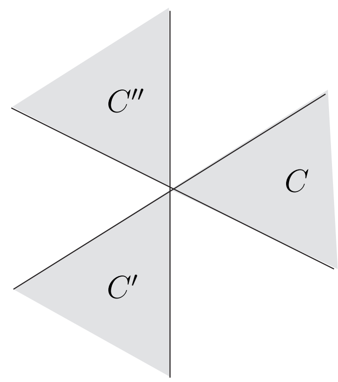
(c) certain non-tempered representations (which do not occur in $\mathcal{A}(G,\Gamma)$ under the anisotropy assumption).

[64]more precisely, to the decomposition $K_f \alpha K_f = \coprod_q K_f a_i$

We shall need no further information about (b) or (c). Representations (a) are in one-to-one correspondence with the set

$$\left\{ \begin{array}{l} (\lambda, \mathbf{C}) \mid \mathbf{C} \text{ (open) Weyl chamber,} \quad \lambda \in X^*(T) \cap \overline{\mathbf{C}}, \\ \lambda \text{ is not orthogonal to any } \mathbf{C}\text{-simple compact root} \end{array} \right\}$$

modulo the action of the Weyl group W_K. Referring to the picture



they therefore break into the classes $V_{(\lambda,\mathbf{C})}$ with

(i) [holo] $\mathbf{C} = C'$, $\lambda \in \overline{C'}$ but not on the vertical line;
(ii) [anti] $\mathbf{C} = C''$, $\lambda \in \overline{C''}$ but not on the vertical line;
(iii) [non] $\mathbf{C} = C$, $\lambda \in \overline{C}$.

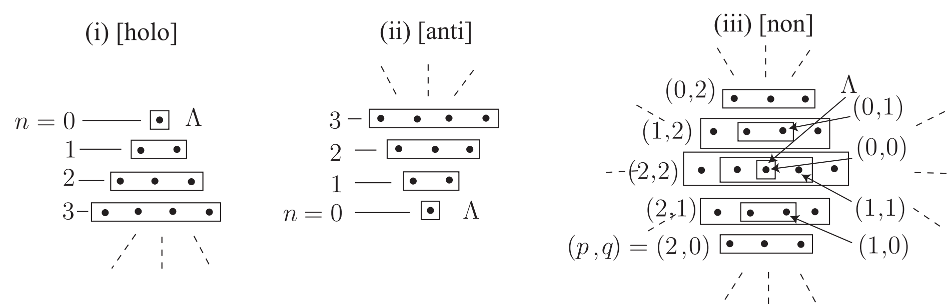
A given $V_{(\lambda,\mathbf{C})}$ with λ regular is the unique discrete series representation with Harish-Chandra parameter λ. If λ is instead on a wall, $V_{(\lambda,\mathbf{C})}$ is a limit of discrete series representation, the special case $V_{(0,C)}$ being the TDLDS. Abstractly, this can be constructed from discrete series by choosing $\nu \in X^*(T)$ such that $\lambda+\nu$ belongs to $\mathbf{C}$, taking $U_{-\nu}$ the finite-dimensional irreducible representation of G with lowest weight $-\nu$, then taking (roughly) the sub-irreducible representation of $V_{(\lambda+\nu,\mathbf{C})} \otimes U_{-\nu}$ with infinitesimal character χ_λ; cf. [K2, pp. 460ff.].

Step 2: Explicit description of discrete series and their limits. Carayol [C1] completely describes the decomposition of the Harish-Chandra module $V_{(\lambda,\mathbf{C})}$ into K-types, as well as the action of the generators

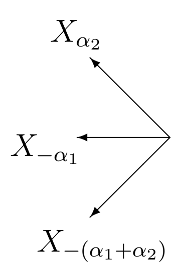
$\{X_{-\alpha_1}, X_{\alpha_2}, X_{-(\alpha_1+\alpha_2)}\}$ of $\mathfrak{n}$.[65] We restrict ourselves to the decomposition into $(\mathfrak{k}, T)$-modules. Let $\mathcal{S}^n(m)$ be the lift of the n^{th} standard representation from $(\mathrm{su}(2), T)$ to $(u(2), T)$ for which the $T = \{\left(\begin{smallmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{smallmatrix}\right) \mid \gamma_1, \gamma_2 \in S^1\}$-type ranges from $(\det)^{-3k/2} \otimes (\gamma_2/\gamma_1)^{-n/2}$ to $(\det)^{-3k/2} \otimes (\gamma_2/\gamma_1)^{n/2}$; and let $\mathbb{C}_\Lambda$ be the "trivial" lift[66] of $\Lambda \in X^*(T)$. Then for the three choices of $\mathbf{C}$ we have isomorphisms of $(u(2), T)$-modules

(i) $V_{(\lambda, \mathbf{C}')} \cong \bigoplus_{n \geq 0} \mathcal{S}^n(-n) \otimes \mathbb{C}_\Lambda, \quad \Lambda = \lambda - \alpha_2;$
(ii) $V_{(\lambda, \mathbf{C}'')} \cong \bigoplus_{n \geq 0} \mathcal{S}^n(n) \otimes \mathbb{C}_\Lambda, \quad \Lambda = \lambda + \alpha_1 + \alpha_2;$
(iii) $V_{(\lambda, \mathbf{C})} \cong \bigoplus_{p,q \geq 0} \mathcal{S}^{p+q}(q-p) \otimes \mathbb{C}_\Lambda, \quad \Lambda = \lambda;$

in each case the weight Λ, called the *Blattner parameter*, corresponds to the minimal K-type of V. In pictures, we have[67]



The action of $\mathfrak{n}$ is given by



[65]Carayol's description is based on [JW] in which a space of functions on the 3-sphere giving the corresponding $G_\mathbb{R}$-module are explicitly written out. In the appendix to this section we shall give a more conceptual and geometric method for obtaining the K-type.
[66]that is, X_{α_1} acts trivially
[67]These pictures will be "explained" in the appendix to this section.

Writing $\varepsilon_k := \frac{k}{3}(\alpha_1 + 2\alpha_2)$ ($= \det^{-k}$) for the weights along the vertical axis, the special cases we shall care about are

(i) $V_k := V_{(\lambda'_k, C')}$ where $\lambda'_k = \varepsilon_{-k} + \alpha_2$, $\Lambda'_k = \varepsilon_{-k}$;
(ii) $\overline{V}_k := V_{(\lambda''_k, C'')}$ where $\lambda''_k = \varepsilon_k - \alpha_1 - \alpha_2$, $\Lambda'' = \varepsilon_k$;
(iii) $V_0 := V_{(0,C)}$ with $\lambda_0 = 0 = \Lambda_0$.

(IV.F.5) **Remark:** These have analytic realizations: using the notation from (IV.C.9), and writing

$$f\Big|_{\gamma}^{(p,q)}(\underline{z}) := \frac{f(\gamma(\underline{z}))}{(j_\gamma(\underline{z}))^p\overline{(j_\gamma(\underline{z}))}^q} \qquad \left(f \in \begin{cases} C^\infty(\mathbb{B}) \\ C^\infty(\partial\mathbb{B}) \end{cases} \text{ and } \gamma \in G(\mathbb{R}) \right)$$

(i) $\widetilde{V}_k \subset C^\infty(\mathbb{B})$ with $(\gamma, f) \mapsto f\big|_{\gamma^{-1}}^{(k,0)}$;
(ii) $\widetilde{\overline{V}}_k \subset C^\infty(\mathbb{B})$ with $(\gamma, f) \mapsto f\big|_{\gamma^{-1}}^{(0,k)}$;
(iii) $\widetilde{V}_0 \subset C^\infty(\partial\mathbb{B})$ with $(\gamma, f) \mapsto f\big|_{\gamma^{-1}}^{(1,1)}$.

Step 3: Compute the automorphic cohomology of X_{K_f} in degrees $q = 1$ or 2. We have by section IV.A, Remark (IV.F.4) and Steps 1 and 2

$$\begin{aligned} H^q(X_{K_f}, L_\mu) &= \bigoplus_{g \in C(K_f)} H^q(X(\Gamma_g), L_\mu) \\ &= \bigoplus_g H^q(\mathfrak{n}, \mathcal{A}(G, \Gamma_g))_{-\mu} \\ &= H^q(\mathfrak{n}, V_{-(\mu+\rho)})_{-\mu}^{\oplus\{\sum_g m_{-(\mu+\rho)}(\Gamma_g)\}} \end{aligned}$$

where the group $H^q(\mathfrak{n}, V_{-(\mu+\rho)})_{-\mu}$ is always of dimension 0 or 1. For example, writing $\xi_{p,q} \in V_0$ for highest-weight vectors of $\mathcal{S}^{p+q}(q-p)$ Carayol computes (loc. cit.)

(IV.F.6)

$$H^q(\mathfrak{n}, V_0) \text{ is generated by } \begin{cases} \xi_{0,0}\omega^{-\alpha_1} - \xi_{0,1}\omega^{\alpha_2} - \xi_{1,0}\omega^{-(\alpha_1+\alpha_2)}, & q = 1 \\ \xi_{0,0}\omega^{-(\alpha_1+\alpha_2)} \wedge \omega^{\alpha_2}, & q = 2; \end{cases}$$

while one easily checks, writing $v_k \in V_k$ for the vector of minimal weight Λ'_k, that

$$\text{(IV.F.7)} \quad H^q(\mathfrak{n}, V_k) \text{ is generated by } \begin{cases} v_k\omega^{-(\alpha_1+\alpha_2)}, & q = 1 \\ v_k\omega^{-\alpha_1} \wedge \omega^{-(\alpha_1+\alpha_2)}, & q = 2\,. \end{cases}$$

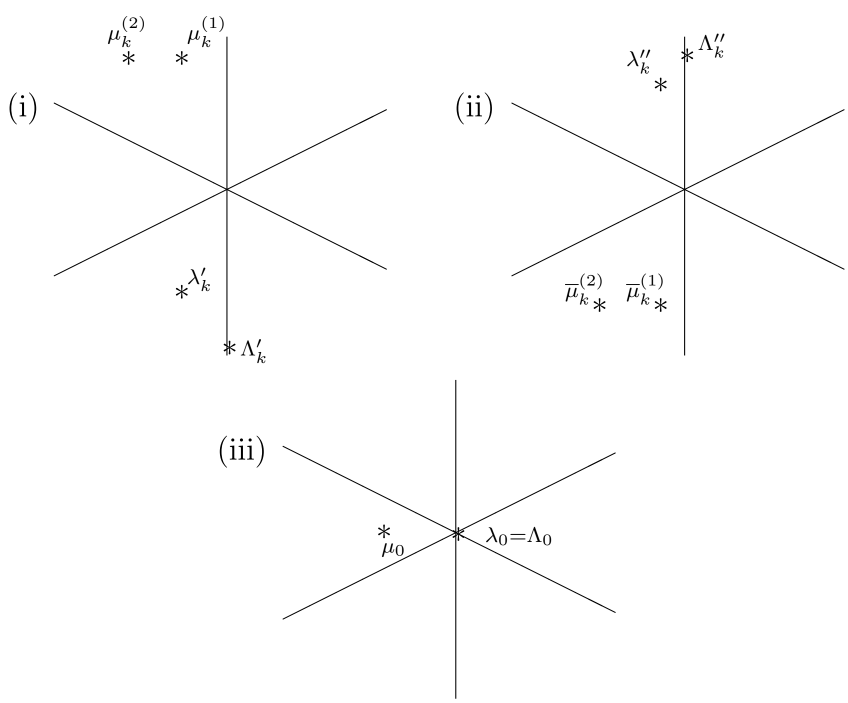
The point is that a more precise description of the representations, which Carayol extracts from [JW], can get us further than the Williams lemma. In general, if $w \in W_K \cong \mathbb{Z}/2$ is the generator which flips about the vertical axis, then for $H^q(\mathfrak{n}, V_\lambda)_{-\mu}$ to be nonzero we must have λ *or* $w(\lambda) = -(\mu + \rho)$. The three cases we need are taken care of by the following:

$$\text{(i)}\ \ H^q(\mathfrak{n}, V_k)_{-\mu} \neq \{0\} \iff \begin{cases} \mu = \mu_k^{(1)} := \varepsilon_k - \alpha_1 - \alpha_2 & \text{and } q = 1 \\ \text{or} \\ \mu = \mu_k^{(2)} = \varepsilon_k - 2\alpha_1 - \alpha_2 & \text{and } q = 2; \end{cases}$$

$$\text{(ii)}\ \ H^q(\mathfrak{n}, \overline{V}_k)_{-\mu} \neq \{0\} \iff \begin{cases} \mu = \overline{\mu}_k^{(1)} := \varepsilon_{-k} + \alpha_2 & \text{and } q = 1 \\ \text{or} \\ \mu = \overline{\mu}_k^{(2)} := \varepsilon_{-k} - \alpha_1 + \alpha_2 & \text{and } q = 2; \end{cases}$$

$$\text{(iii)}\ \ H^q(\mathfrak{n}, V_0)_\mu \neq \{0\} \iff \begin{cases} \mu = \mu_0 := -\rho \\ \text{and } q = 1 \text{ or } 2, \end{cases}$$

with pictures



The first two groups, as we know, are in the image of $\mathcal{P}$ for $k \geq 4$.

Step 4: Nonvanishing of the cup product in Lie algebra cohomology. Carayol applies a criterion of [HL] for deciding when a given discrete series representation of G occurs in the restriction to G of a unitary irreducible representation of a larger reductive group with "compatible" minimal types. At issue here is the diagonal restriction to G of the representation $\widetilde{V}_k \boxtimes \widetilde{V}_0$ of $G \times G$. The crucial observation is that for $\widetilde{V}_k$ integrable, which is equivalent to $k \geq 5$, by (IV.B.12), the matrix coefficients of $\widetilde{V}_k \otimes \widetilde{V}_0 = (\widetilde{V}_k \boxtimes \widetilde{V}_0)\big|_G$ are L^2, so that the criterion applies to yield:

(IV.F.8) *for $k \geq 5$, $\widetilde{V}_k$ is a subrepresentation of $\widetilde{V}_k \otimes \widetilde{V}_0$.*

He then uses the explicit description of V_k and V_0 as Lie algebra representations to deduce that $V_k \otimes V_0$ contains a unique 1-dimensional subspace killed by X_{α_2} and on which K acts via $\det^k$. Consequently, $\widetilde{V}_k$ is of multiplicity one in $\widetilde{V}_k \otimes \widetilde{V}_0$, and the corresponding projection restricted to Harish-Chandra modules

$$\pi^{(k)} : V_0 \otimes V_k \twoheadrightarrow V_k$$

induces an isomorphism between tensors of minimal K-types, sending

$$\xi_{0,0} \otimes v_k \mapsto cv_k \ \text{ for some } \ c \neq 0\,.$$

Together with the cup product, $\pi^{(k)}$ induces a map

$$\pi_*^{(k)} : H^1(\mathfrak{n}, V_0)_{-\mu_k^{(1)}} \otimes H^1(\mathfrak{n}, V_k)_\rho \to H^2(\mathfrak{n}, V_k)_{-\mu_k^{(2)}}\,. \tag{IV.F.9}$$

Wedging together generators of the LHS factors (cf. (IV.F.6)–(IV.F.7)) gives

$$\begin{aligned}(\xi_{0,0}\omega^{-\alpha_1} - \xi_{0,1}\omega^{\alpha_2} - \xi_{1,0}\omega^{-(\alpha_1+\alpha_2)}) \wedge (v_k\omega^{-(\alpha_1+\alpha_2)}) \\ = (\xi_{0,0} \otimes v_k)\omega^{-\alpha_1} \wedge \omega^{-(\alpha_1+\alpha_2)} - (\xi_{0,1} \otimes v_k)\omega^{\alpha_2} \wedge \omega^{-(\alpha_1+\alpha_2)}\,,\end{aligned}$$

whereupon applying $\pi^{(k)}$ evidently yields

$$cv_k\omega^{-\alpha_1} \wedge \omega^{-(\alpha_1+\alpha_2)}$$

which is the generator of the RHS. Since all three groups are 1-dimensional, (IV.F.9) is therefore an isomorphism.

Step 5: Nonvanishing of cup products in automorphic cohomology. By Step 3, any nonzero $\phi_k^{(1)} \in H^1(X_{K_f}, L_{\mu_k^{(1)}})$ belongs to the image of the map

$$\left(j_k^{(1)}\right)_* : H^1(\mathfrak{n}, V_k)_{-\mu_k^{(1)}} \hookrightarrow H^1\left(X_{K_f}, L_{\mu_k^{(1)}}\right)$$

induced by some $j_k^{(1)} \in \mathrm{Hom}_{(\mathfrak{g},K)}(V_k, \mathcal{A}^G_{K_f})$. Similarly, for $\phi_0 \in H^1(X_{K_f}, L_{-\rho})$ there is some $j_0 \in \mathrm{Hom}_{(\mathfrak{g},K)}(V_0, \mathcal{A}^G_{K_f})$. Since V_k is integrable, results of [Ha, §7] imply that there exist a compact open $K_f^0 \subset K_f$, a Hecke correspondence $\widetilde{\mathcal{C}} = \mathcal{C} \otimes \mathrm{id}$ on $X_{K_f^0} \times X_{K_f^0}$, and a *nonzero* $\tilde{j}_k^{(2)} \in \mathrm{Hom}_{(\mathfrak{g},K)}(V_k, \mathcal{A}^G_{K_f^0})$ such that

$$\begin{tikzcd}
V_k \otimes V_0 \arrow[r, hook, "j_k^{(1)} \otimes j_0"] \arrow[d, "\pi^{(k)}"] & \mathcal{A}^{G\times G}_{K_f \times K_f} \arrow[r, "\widetilde{\mathcal{C}}_* \circ p_0^*"] & \mathcal{A}^{G\times G}_{K_f^0 \times K_f^0} \arrow[d, "|_\Delta"] \\
V_k \arrow[rr, hook, "\tilde{j}_k^{(2)}"] & & \mathcal{A}^G_{K_f^0}
\end{tikzcd}$$

commutes. It follows easily from this that

$$\begin{tikzcd}
H^1(\mathfrak{n}, V_k)_{-\mu_k^{(1)}} \otimes H^1(\mathfrak{n}, V_0)_\rho \arrow[r, "(\mathcal{C}_* \circ p_0^* \circ (j_k^{(1)})_*) \otimes", "(p_0^* \circ (j_0)_*)"'] \arrow[d, "\pi_*^{(k)}"', "\cong"] & H^1(X_{K_f^0}, L_{\mu_k^{(1)}}) \otimes H^1(X_{K_f^0}, L_{-\rho}) \arrow[d, "\cup"] \\
H^2(\mathfrak{n}, V_k)_{-\mu_k^{(2)}} \arrow[r, hook, "(\tilde{j}_k^{(2)})_*"] & H^2(X_{K_f^0}, L_{\mu_k^{(2)}})
\end{tikzcd}$$

commutes as well. So for any $k \geq 5$ and any element $\phi_0 \in H^1(X(\Gamma), L_{-\rho}) \subset H^1(X_{K_f}, L_{-\rho})$ there exists K_f^0 such that

$$\cup p_0^* \phi_0 \in \mathrm{Hom}\left(H^1\big(X(\Gamma_0), L_{\mu_k^{(1)}}\big), H^2\big(X(\Gamma_0), L_{\mu_k^{(2)}}\big)\right)$$

is nonzero.[68] Dually, this says that in the limit over Γ_0,

$$H^1\big(X(\Gamma_0), L_{\mu_k^{(1)}}\big) \otimes H^2\big(X(\Gamma_0), L_{\mu_k^{(2)}}\big)^\vee \underset{\text{Serre}}{\cong} H^1\big(X(\Gamma_0), L_{\mu_k^{(1)}}\big) \otimes H^1\big(X(\Gamma_0), L_{-2\rho-\mu_k^{(2)}=\overline{\mu}_k^{(1)}}\big)$$

[68]Here $\Gamma_0 = K_f^0 \cap G(\mathbb{Q})$ after possibly conjugating K_f^0; the point is that this map is nonzero for *some* component of $X_{K_f^0}$ over $X(\Gamma) \subset X_{K_f}$, and we may assume it is $X(\Gamma_0)$.

surjects onto

$$H^1\big(X(\Gamma), L_{-\rho}\big)^\vee \underset{\text{Serre}}{\cong} H^2\big(X(\Gamma), L_{-\rho}\big)$$

via $(p_0)_* \circ \cup$. This completes the proof of (IV.F.1). $\square$

There remains, however, a loose end: we do not know for which (if any) Γ one has $H^1(X(\Gamma), L_{-\rho}) \neq \{0\}$, without which (IV.F.1) has no content. Since $L_{-\rho} = F_{(-1,-1)}$ on D, and $F_{(a,b)}|_Z \cong \mathcal{O}_{\mathbb{P}^1}(a+b)$ on the compact $K_{\mathbb{R}}$-orbit $Z \cong \mathbb{P}^1 \subset D$, we have $L_{-\rho}|_Z \cong \mathcal{O}(-2)$ which implies $H^0(D, L_{-\rho}) = \{0\}$. By the same argument as in the proof of (IV.B.6), we therefore have

$$H^1(X(\Gamma), L_{-\rho}) \cong \left(H^1(D, L_{-\rho})\right)^\Gamma.$$

It also seems plausible, but it has not yet been verified, that for sufficiently small Γ this has nontrivial Γ-invariants.

Appendix to section IV.F: Geometric construction of K-types and discussion of totally degenerate limits of discrete series

In the proof of theorem (IV.F.1) Carayol used the explicit "formula constructions" [JW] of certain limits of discrete series, including the totally degenerate limit of discrete series (TDLDS) for $S\mathcal{U}(2,1)$. In this appendix we shall give a geometric description of the K-types which, in particular, will "explain" the patterns in the diagrams given in step 2 in the proof of Carayol's result. We note that the general construction of TDLDS's via induced representations is given in [CK]. In a future work we shall discuss how the geometric construction below may be extended to give a $(\mathfrak{g}_{\mathbb{C}}, K)$-module and how this in turn relates to that in [CK] through the intermediate steps of Beilinson-Bernstein localization [BB] and the duality theorem of [HMSW].[69]

We shall first do the construction of the dual Harish-Chandra module to the TDLDS $V_0 = (0, \mathbf{C})$. For this we shall use the Harish-Chandra module associated to the group $H^1(D, L_{-\rho})$ is the Harish-Chandra

[69]In this regard we especially recommend the lecture notes [Schm3] as well as [Schm4] for heuristic discussion of the duality theorem.

module associated to V_0. For μ such that $\mu+\rho$ is anti-dominant, this is due to Schmid [Schm1]. In a conversation with the second author he has explained that this result also holds for the TDLDS.[70]

We consider the $G_{\mathbb{R}}$-module $H^1(D,L_{-\rho})$ and shall use the method explained in the appendix to section (IV.D) to expand this group about the maximal compact subvariety $Z\cong\mathbb{P}^1$ in D. Specifically, we consider the K-invariant filtration

$$F^kH^1(D,L_{-\rho})=\mathrm{image}\{H^1(D,\mathcal{I}_Z^k\otimes L_{-\rho})\to H^1(D,L_\rho)\}$$

with associated graded K-module

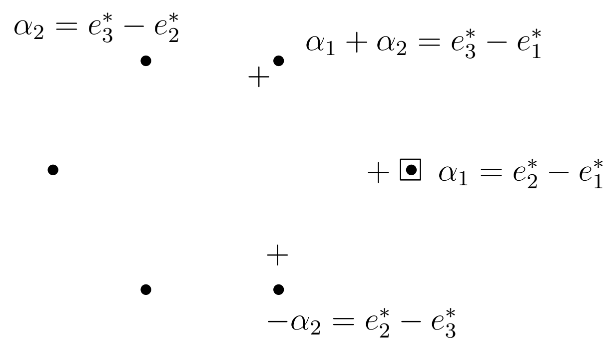
$$\bigoplus_{n\geqq 0} H^1(Z,L_{-\rho}\otimes \mathrm{Sym}^n\check{N}) \tag{A.IV.F.1}$$

where $N=N_{Z/D}$ is the normal bundle of Z in D. Here $K=\mathcal{U}(2)\subset S\mathcal{U}(2,1)$,

$$Z=K/T=K_{\mathbb{C}}/B_K$$

where $B_K=B\cap K_{\mathbb{C}}$, and in Proposition (A.IV.F.6) below we shall determine N as a $K_{\mathbb{C}}$-homogeneous vector bundle.

For reference we recall the root diagram of $S\mathcal{U}(2,1)$



where the positive roots giving the Weyl chamber corresponding to the non-classical complex structure on D are labelled with a $+$. The maximal torus of $S\mathcal{U}(2,1)$ is

$$T_S=\left\{\begin{pmatrix} e^{2\pi i\theta_1} & & \\ & e^{2\pi i\theta_2} & \\ & & e^{2\pi i\theta_3}\end{pmatrix} : \theta_1+\theta_2+\theta_3\equiv 0 \bmod \mathbb{Z}\right\}$$

[70]We will see that these Harish-Chandra modules have the same K-type.

and the maximal torus of

$$K = \begin{pmatrix} A & 0 \\ 0 & \det A^{-1} \end{pmatrix} : A \in \mathcal{U}(2)$$

is

$$T_K = \begin{pmatrix} e^{2\pi i\theta_1} & \\ & e^{2\pi i\theta_2} \end{pmatrix}.$$

The lattice of T and T_K corresponds to the weight lattice spanned by

$$\begin{cases} e_3^* - e_1^* = -2e_1^* - e_2^* \\ e_3^* - e_2^* = -2e_2^* - e_1^* \, . \end{cases}$$

From $\alpha_1 = e_2^* - e_1^* =$ compact root and

$$\begin{cases} (e_2^* - e_1^*, -2e_1^* - e_2^*) = 1 \\ (e_2^* - e_1^*, -2e_2^* - e_1^*) = -1 \end{cases}$$

we infer that

$$\begin{cases} L_{-2e_1^*-e_2^*}\big|_Z = \mathcal{O}(1) \\ L_{-2e_2^*-e_1^*}\big|_Z = \mathcal{O}(-1) \, . \end{cases} \tag{A.IV.F.2}$$

If we think of $T_K = \mathbb{R}^2/\mathbb{Z}^2$ where $\mathbb{R}^2$ has coordinates (θ_1, θ_2), then the weights e_1^*, e_2^* restrict to $\mathfrak{t}_K$ to give

$$\begin{cases} \langle e_1^*, (\theta_1, \theta_2) \rangle = \theta_1 \\ \langle e_2^*, (\theta_1, \theta_2) \rangle = \theta_2 \, . \end{cases}$$

We write this in shorthand as

$$e_1^* \leftrightarrow \theta_1, \quad e_2^* \leftrightarrow \theta_2 \, .$$

Then

$$\begin{cases} e_3^* - e_2^* \leftrightarrow -2\theta_2 - \theta_1 \\ e_3^* - e_1^* \leftrightarrow -2\theta_1 - \theta_2 \end{cases} \tag{A.IV.F.3}$$

which gives

$$\tfrac{1}{2}(e_2^* - e_1^*) \leftrightarrow \tfrac{1}{2}(\theta_2 - \theta_1) \, .$$

Thus, over $Z = \mathcal{U}(2)/T_K$ we have

$$\mathcal{O}_Z(1) = L_{\frac{1}{2}(\theta_2 - \theta_1)} \, . \tag{A.IV.F.4}$$

For the $\mathcal{U}(2)$-homogeneous bundle δ given by the character det, we have[71]

(A.IV.F.5) $$\delta = L_{\theta_1+\theta_2}\,.$$

Denoting by

$$\delta(n) = \delta \otimes \mathcal{O}_Z(n)$$

the indicated $K_{\mathbb{C}}$-homogenous line bundle over Z we then have the

(A.IV.F.6) **Proposition:** *For the normal bundle N to Z in D we have the isomorphism of $K_{\mathbb{C}}$-homogeneous vector bundles*

$$N \cong \delta^{-3/2}(1) \oplus \delta^{3/2}(1)\,.$$

Proof. Setting $\mathfrak{n}_K = \mathfrak{n} \cap \mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{n}_K^+ = \mathfrak{n}^+ \cap \mathfrak{k}_{\mathbb{C}}$, we have

$$\begin{cases} \mathfrak{n}_K = \mathbb{C}X_{e_1^*-e_2^*} \\ \mathfrak{n}^+/\mathfrak{n}_K^+ \cong \mathbb{C}X_{e_3^*-e_1^*} \oplus \mathbb{C}X_{e_1^*-e_3^*}\,, \end{cases}$$

so that $\mathfrak{n}_K$ acts trivially on $\mathfrak{n}^+/\mathfrak{n}_K^+$. Thus, as a $K_{\mathbb{C}}$-homogeneous bundle

$$\begin{aligned} N &\cong L_{e_3^*-e_1^*}\big|_Z \oplus L_{e_2^*-e_3^*}\big|_Z \\ &\cong L_{-2\theta_1-\theta_2} \oplus L_{2\theta_2+\theta_1}\,. \end{aligned}$$

But

$$\begin{cases} -2\theta_1-\theta_2 = \frac{-3}{2}(\theta_1+\theta_2) + \frac{1}{2}(\theta_2-\theta_1) \\ \quad 2\theta_2+\theta_1 = \frac{3}{2}(\theta_1+\theta_2) + \frac{1}{2}(\theta_2-\theta_1) \end{cases}$$

and the result follows from (A.IV.F.4) and (A.IV.F.5). □

Since $L_{-\rho}\big|_Z \cong \mathcal{O}_Z(-2)$ we have the

(A.IV.F.7) **Corollary:** *As $\mathcal{U}(2)$-homogeneous bundles over Z we have*

$$L_{-\rho}\big|_Z \otimes \operatorname{Sym}^n \check{N} \cong \bigoplus_{0\leqq p\leqq n} \delta^{3/2(n-2p)}(-n-2)\,.$$

The pattern is

$$\begin{array}{ll} n=0 & \mathcal{O}(-2) \\ n=1 & \delta^{3/2}(-3) \oplus \delta^{-3/2}(-3) \\ n=2 & \delta^{6/2}(-4) \oplus \mathcal{O}(-4) \oplus \delta^{-6/2}(-4) \\ n=3 & \delta^{9/2}(-5) \oplus \delta^{1/2}(-5) \oplus \delta^{-1/2}(-5) \oplus \delta^{-4/2}(-5)\,. \end{array}$$

[71] δ is trivial as a holomorphic, but not $\mathcal{U}(2)$-homogeneous, line bundle.

In the notation of step (iii), the dual of the $H^1(Z, \delta^{3/2(n-2p)}(-n-2))$'s, $0 \leqq p \leqq n$, is $\bigoplus_{p+q=n} \mathcal{S}^{p+q}(q-p)$.

Turning to the classical case we let $B_K = B \cap K_{\mathbb{C}}$ and we have the

(A.IV.F.8) **Proposition:** *As a $K_{\mathbb{C}}$-homogeneous bundle the normal bundle N' of $Z' \subset D'$ is given by*

$$N = \delta^{-1} \otimes \check{E}$$

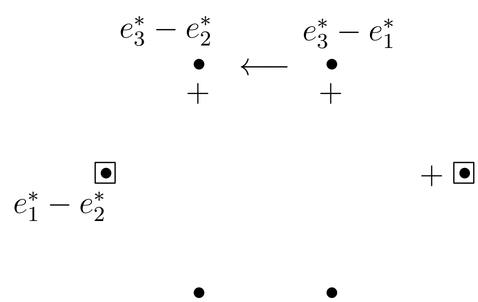
where E is the $K_{\mathbb{C}}$-homogeneous vector bundle given by the restriction to B_K of the standard representation on $W \cong \mathbb{C}^2$ of $\mathcal{U}(2)_{\mathbb{C}} \cong \mathrm{GL}(2, \mathbb{C})$.

Thus, as a holomorphic vector bundle

$$E \cong \mathcal{O}_Z \oplus \mathcal{O}_Z$$

is holomorphically trivial, as is N.[72]

Proof. From the diagram



we see that $X_{e_1^*-e_2^*} \in \mathfrak{n}'_K$ acts on $\mathfrak{n}'^{+}/\mathfrak{n}'^{+}_K$ by the arrow. Thus using (A.IV.F.3)

$$\begin{aligned} N &= L_{-2\theta_1-2\theta_2} \otimes E \\ &\cong L_{-\theta_2-\theta_2} \otimes \check{E} \\ &= \delta^{-1} \otimes \check{E}\,. \end{aligned}$$ □

[72]Geometrically this is clear, since in the holomorphic fibration $D' \to H$ of D' over an Hermitian symmetric domain, Z is the inverse image of a point $x \in H$ and $N \to Z$ is holomorphically the trivial bundle $Z \times T_x H$.

(A.IV.F.9) **Corollary:** *As $\mathcal{U}(2)$-homogeneous bundles over Z',*

$$L'_{-2\rho'}\big|_{Z'} \otimes \mathrm{Sym}^n \check{N}' \cong \mathcal{O}_{Z'}(-2) \otimes \delta^n \otimes \mathrm{Sym}^n E\,.$$

The picture is

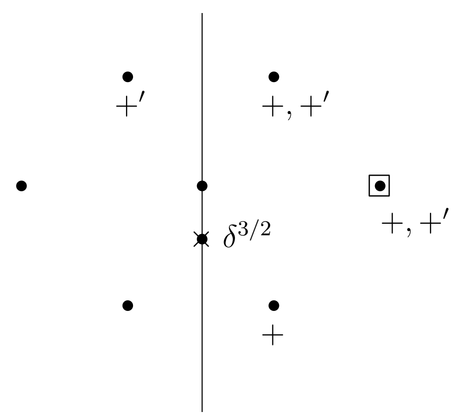
$$\begin{array}{ll} n=0 & \mathcal{O}(-2) \\ n=1 & \delta \otimes \check{W}(-2) \\ n=2 & \delta^2 \otimes \mathrm{Sym}^2 \check{W}(-2) \end{array}$$

which gives the pattern (i) in step 2 above.

Remark. It is interesting to note that the difference in patterns in the K-types as pictured in step 2 is explained geometrically by the difference in the normal bundles to the maximal compact subvariety

- $N = \delta^{-3/2}(1) \oplus \delta^{3/2}(1)$ non-classical case
- $N' = \delta^{-1} \otimes \check{E}$ classical case

where $\delta =$ "det" and E is the trivial bundle whose global sections are the standard $\mathcal{U}(2)$-module. The following picture explains the crucial sign change in the power of $\delta^{3/2}$



Here the positive roots for D are labelled $+$ and those for D' are labelled $+'$ and the determinant bundle is on the negative vertical axis. For D' both non-compact roots have a negative inner product with $\delta^{3/2}$, while for D there is one positive and one negative inner product.

Discussion of totally degenerate limits of discrete series (TDLDS's). The purpose of this concluding sub-section is to comment on several aspects of TDLDS's. We begin by discussing the

(A.IV.F.10) *Analogy between TDLDS's and special divisors of degree* $g-1$ *on an algebraic curve of genus* g.

In this discussion we will let $X = \Gamma\backslash D$ where D is a homogeneous complex manifold $G_{\mathbb{R}}/T$ and Γ is a co-compact discrete group acting freely on D; we will eventually specialize to the cases $G_{\mathbb{R}} = \mathrm{SL}_2(\mathbb{R})$, $S\mathfrak{U}(2,1)$ and $\mathrm{Sp}(4)$. We will also denote by Y a compact Riemann surface of genus g.

> *There is an analogy between limits of discrete series (LDS's) whose Harish-Chandra modules are represented by an* $H^q(D, L_\mu)$ *where* $\mu+\rho$ *is singular and by special divisors represented by classes on* $H^p(Y, L)$ *where* $\deg L = g-1$.

An elementary observation is that

(A.IV.F.11) *In both cases the sheaf Euler characteristic is zero.*

In the algebraic curve case this follows from the Riemann-Roch theorem. For X as above we first note that trivially from the Borel-Weil-Bott theorem the sheaf Euler characteristic of $L_\mu \to G_{\mathbb{C}}/B$ is zero. By the Hirzebruch-Riemann-Roch theorem this Euler characteristic is given by an integral over $G_{\mathbb{C}}/B$ of an invariant polynomial in the Chern forms, an expression that when pulled back to $G_{\mathbb{C}}$ is a polynomial in the Maurer-Cartan forms. If $\mu+\rho$ is singular this polynomial vanishes. By the Atiyah-Singer extension of the Hirzebruch-Riemann-Roch theorem and the Hirzebruch proportionality principle, the sheaf Euler characteristic of $L_\mu \to X$ is given by integrating over X a polynomial in the Chern classes whose pullback to $G_{\mathbb{R}}$ is, up to a sign, given by the same polynomial as that for the Maurer-Cartan forms on $G_{\mathbb{C}}$.[73] $\square$

Corollary. *We cannot have that just one* $H^q(X, L_\mu) \neq 0$.

A further observation is

[73]The sign is $(-1)^{nc}$ where nc is the number of positive, non-compact roots.

TDLDS correspond to special divisors given by theta characteristics.

This is because for a TDLDS one has $\mu = -\rho$ and $L_{-\rho} = \omega_X^{1/2}$ is a square root of the canonical bundle, which for curves is the definition of a theta characteristic. For SL_2 this analogy is exact:

Example. Let V_n^-, $n = 0, -1, -2, \dots$ be the Harish-Chandra module corresponding to the discrete series ($n = -1, -2, \dots$) or TDLDS ($n = 0$) with infinitesimal character n and standard positive Weyl chamber $\mathbb{Z}^{\geqq 0}$ for SL_2, and denote by $v_n^- \in V_n$ a generator. We also denote by V_n^+, $n = 0, 1, 2, \dots$, the similar ones for the opposite Weyl chamber. In this case $X = \Gamma\backslash\mathcal{H}$ is a compact Riemann surface and, denoting by $L_n \to X$ the line bundle associated to the weight n, we are interested in the groups

$$H^q(X, L_{-1}), \qquad q = 0, 1$$

where $L_{-1} = \omega_X^{1/2}$. Then Kodaira-Serre duality is a non-degenerate pairing

$$H^0(X, \omega_X^{1/2}) \otimes H^1(X, \omega_X^{1/2}) \to \mathbb{C}\,. \tag{A.IV.F.12}$$

From (IV.A.7) we have

$$H^q(X, L_{-1}) \cong \oplus\, H^q(\mathfrak{n}, V_n^+)_1^{\oplus\, m_\pi(\Gamma)},$$

while $H^0(\mathfrak{n}, V_n^-)_1 = (0)$ unless $n = 0$ and then

$$H^0(\mathfrak{n}, V_0^-)_1 = \mathbb{C}v_0\,.$$

Similarly, $H^1(\mathfrak{n}, V_n^-)_1 = 0$ unless $n = 0$ and then

$$H^1(\mathfrak{n}, V_0^-) \cong \mathbb{C}v_0^-\omega^{-\alpha}$$

where $\alpha =$ "2" is the positive root. The pairing (A.IV.F.12) is induced by

$$v_0 \otimes (v_0^-\omega^{-\alpha}) \to c\int_X \omega^\alpha \wedge \omega^{-\alpha}$$

where $\left(\frac{\sqrt{-1}}{2}\right)\omega^\alpha \wedge \omega^{-\alpha}$ is the volume form on X and the non-zero constant is computed from $\langle h_\alpha, v_0^\pm\rangle$.

Example. The case of $\mathrm{Sp}(4)$ has a similarity to $\mathrm{SL}_2 = \mathrm{Sp}(2)$ in that there are two TDLDS's V_k corresponding to $H^1(D_k, L_{-\rho_k})$ where $k = 1, 2$ and D_1, D_2 are representatives of the two non-classical complex structures on $\mathrm{Sp}(4)/T$. The pictures are

The K-type is obtained as in the $S\mathcal{U}(2,1)$ case by expanding cohomology about the maximal compact subvariety. Denoting by D one of the D_k's and by $Z = \mathcal{U}(2)/T \subset D$ the maximal compact subvariety, for W the standard $\mathcal{U}(2)$-module and $\mathbf{W} = \mathcal{U}(2) \times_T W$ the corresponding $\mathcal{U}(2)$-homogeneous vector bundle over $Z \cong \mathbb{P}^1$, one has for the normal bundle N to $Z \subset D$ that

$$N \cong \mathcal{O}(2) \oplus \mathbf{W}(1) \,.$$

Then $h^1(Z, N) = 0$ and

$$\dim H^0(Z, N) = 7 \,.$$

Thus there are deformations of Z in D other than those arising from the cycle space, a significant difference with the $S\mathcal{U}(2,1)$ case (cf. [FHW], chapters 17 and 20). The geometry and TDLDS in this case will be taken up in the sequel to this work.

We note that since $L_{-\rho}\big|_Z \cong \mathcal{O}(-3)$ and X is covered by images of deformations of Z we have

$$h^0(X, L_{-\rho}) = h^4(X, L_{-\rho}) = 0 \,,$$

and then $\sum_q (-1)^q h^q(X, L_{-\rho}) = 0$ gives $h^2(X, L_{-\rho}) = 0$. Thus

$$h^1(X, L_{-\rho}) = h^3(X, L_{-\rho})$$

are the only possible non-zero dimensions, from which we infer that

$$h^1(\mathfrak{n}, V^\vee)_\rho = h^3(\mathfrak{n}, V^\vee)_\rho\,,$$

so again we have a situation reminiscent of special divisors on algebraic curves.

$S\mathcal{U}(2,1)$. Here it will be convenient to use $\alpha_1+\alpha_2$, α_1, $-\alpha_2$ as positive roots, and then $\rho=\alpha_1$ and $\delta=-\frac{1}{3}(\alpha_1+2\alpha_2)$. For the normal bundle

$$\begin{tikzcd}[column sep=small] N^\vee \cong \mathcal{O}_Z(L_{-(\alpha_1+\alpha_2)}) & \oplus & \mathcal{O}_Z(L_{\alpha_2}) \arrow[ll, bend left, "3\delta"] \end{tikzcd}$$

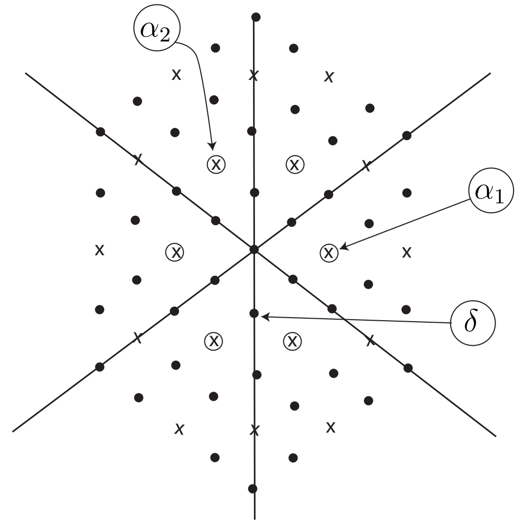
$$\begin{tikzcd}[column sep=small] \operatorname{Sym}^2 N^\vee \cong \mathcal{O}_Z(L_{-2(\alpha_1+\alpha_2)}) & \oplus & \mathcal{O}_Z(L_{-\alpha_1}) \arrow[ll, bend left, "3\delta"] & \oplus & \mathcal{O}_Z(L_{2\alpha_2}) \arrow[ll, bend left, "3\delta"] \end{tikzcd}$$

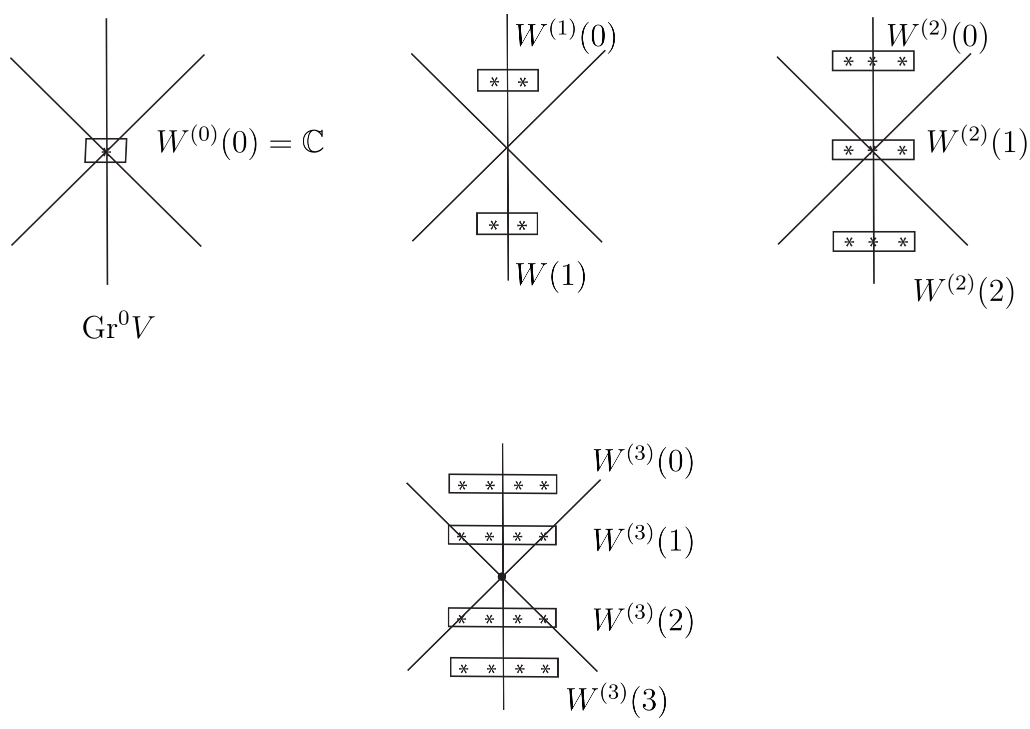
$$\vdots$$

The k^{th} right end term is $\mathcal{O}_Z(L_{k\alpha_2})$, and the terms to the left are successively obtained by "$\otimes L_{3\delta}$" for $k+1$ steps. The K-type then has as its k^{th} term the $\mathcal{U}(2)$-module

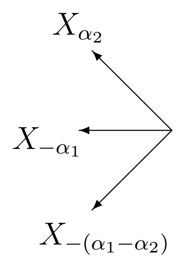
$$\text{(IV.F.13)}_k \qquad \bigoplus_{j=0}^{k} H^1(Z, \mathcal{O}_Z(L_{k\alpha_2+3j\delta-\alpha_1})) =: \bigoplus_{j=0}^{k} W^{(k)}(j) =: \operatorname{Gr}^k V_0$$

where W is the standard representation of $S\mathcal{U}(2)$, $W^{(k)}$ is the k^{th} symmetric product, and $W^{(k)}(j) = W^{(k)} \otimes (\det j)$. This notation is different from that in step 2 above, but it is more convenient for displaying the picture of V_0 in terms of its K-type and for conceptualizing the $\mathfrak{n}$-cohomology calculations given below. The picture of the weight diagram and the first few pictures of the $\operatorname{Gr}^k V_0$ are





Each $*$ represents a one-dimensional weight space for V_0. The diagram (iii) in Step 2 is the overlay of these pictures, and the action of the negative root vectors is pictured by[74]



In (IV.F.6) in Step 3 above, Carayol has used the detailed Johnson-Wallach description of V_0 to produce explicit generators of the groups $H^q(\mathfrak{n}, V_0)_\rho$. However, as we have seen in examples using geometric considerations it may be relatively easy to determine the K-type, and one may ask *to what extent can the* $\mathfrak{n}$*-cohomology be determined by the* K*-type*? Interestingly, as we shall illustrate the better question may be: *does knowing the* K*-type and* $\mathfrak{n}$*-cohomology determine the Harish-Chandra module*?[75] Specifically we shall show:

(A.IV.F.14) *There is one free parameter in determining an action of* $\mathfrak{g}_\mathbb{C}$ *on* $\bigoplus_{k=0}^{3}$ *(IV.F.13)*$_k$ $=: V_0^{[3]}$*, and this parameter is uniquely specified*

[74]There are two issues here: (i) From the K-type, how can one conclude that the action of $\mathfrak{n}$ is given by this diagram? (ii) Given the diagram, this only determines the action of $\mathfrak{n}$ up to scaling in the 1-dimensional weight spaces, and how can one determine these scalings? To address these questions there is an additional piece of geometric information for the TDLDS that is not present for other Harish-Chandra modules with the same K-type. Namely, for $S\mathcal{U}(2,1)$ and setting $\mathfrak{p} = \mathfrak{n}^+ \oplus \mathfrak{n}$, we have that the inclusion $\mathfrak{p} \subset \mathfrak{g}_\mathbb{C}$ induces an isomorphism

$$\mathfrak{p} \cong H^0(Z, N).$$

Then the action of $\mathfrak{p}$ *on the* K*-type* (IV.F.13)$_\infty$ *is given by dualizing the cup-products*

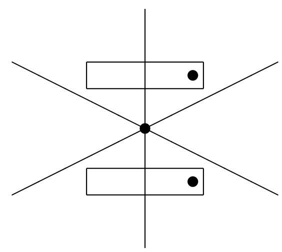
$$H^0(Z, N) \otimes H^0(Z, \operatorname{Sym}^k N \otimes L_\rho) \to H^0(Z, \operatorname{Sym}^{k+1} N \otimes L_\rho).$$

This statement requires explanation which will be given in a later work. The starting point is the observation that the action of normal vector fields along Z shifts the filtration given by $\mathcal{I}_Z^k \otimes_{\mathcal{O}_Z} L_{-\rho} \subset \mathcal{O}_Z(L_{-\rho})$ by at most one, so that the action of $\mathfrak{n}$ on $\mathrm{Gr}^k V_0$ goes to $\mathrm{Gr}^{k-1} V_0 \oplus \mathrm{Gr}^{k+1} V_0$. Weight considerations then give the above picture, and then a more detailed analysis is required to determine the scalings.

[75]In this regard we call particular attention to the paper [Schm5].

by knowing that there is a non-zero class in $H^1(\mathfrak{n}, V_0^{[3]})_\rho$ *that restricts to a generator of* $H^1(\mathfrak{n}_K, \mathbb{C})_\rho$.[76]

For $V_0 = H^1(D, L_{-\rho})$ there is such a class; viz., the class in $H^1(\mathfrak{n}, V_0)_\rho$ corresponding to the class in $H^1(D, L_{-\rho})$ that restricts to a generator of $H^1(Z, \omega_Z)$. In the second diagram the picture of this class is



Manipulating these and similar diagrams pictorially is what lies behind the calculations we are about to give.

For the calculations it is notationally more convenient to set $X_{ij} = X_{e_i^* - e_j^*}$ with dual ω^{ij}.[77] The bracket relations are

$$\begin{cases} [X_{ij}, X_{jk}] = X_{ik} & i \neq k \\ [X_{ij}, X_{ji}] = H_{ij} & \\ [X_{ij}, X_{kl}] = 0 \text{ if } j \neq k \text{ and } i \neq l\,. & \end{cases}$$

Here H_{ij} acts by $(i-j)\mathrm{Id}$ on the (i,j) weight space (i,j). We are indexing the weights of the maximal torus $\left\{ g = \begin{pmatrix} e^{2\pi\sqrt{-1}\theta_1} & \\ & e^{2\pi\sqrt{-1}\theta_2} \end{pmatrix} \right\}$ of $\mathcal{U}(2)$ by (i,j) where $g(i,j) = e^{2\pi\sqrt{-1}(i\theta_1 + j\theta_2)}(i,j)$. With this notation

$$X_{21}^i X_{13}^j X_{32}^k = k(-1,-2) + j(2,1) + i(-1,1)\,, \tag{A.IV.F.15}$$

meaning that the LHS acts on the weight space (p,q) by taking it to the weight space obtained by adding to (p,q) the RHS. We let v_0 be a

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[76] In the remark at the end of this section we shall explain what is meant by restricting a class in $H^1(\mathfrak{n}, V_0)_\rho$ to $H^1(\mathfrak{n}_K, V_0)_\rho$.

[77] The correspondence with the previous notation is

$$\begin{cases} X_{\alpha_2} & = X_{32} \\ X_{-\alpha_1} & = X_{12} \\ X_{-\alpha_1-\alpha_2} & = X_{13}\,. \end{cases}$$

generator of $W^{(0)}(0)$. As an application of the Poincaré-Birkhoff-Witt theorem it may be shown that

(A.IV.F.16) *The vectors* $X_{21}^i X_{13}^j X_{32}^k v_0 : j,k \geqq 0,\ j+k \geqq i \geqq 0$ *is a basis for the Harish-Chandra module* V_0*. Each spans a weight space given by* (A.IV.F.15)*.*

On may also prove that

(A.IV.F.17) (i) *For any constant* c*, setting*

$$(X_{13}X_{31} + X_{23}X_{32})v_0 = cv_0$$

the relations in $\mathrm{su}(2,1)$ *acting on* $V_0^{[3]}$ *are satisfied.*

(ii) *If*

$$\Phi = v_0\omega^{12} + (X_{23}v_0)\omega^{13} - (X_{31}v_0)\omega^{32}$$

then

$$d\Phi = 0 \iff c = 1\,.$$

This is the case for Harish-Chandra module V_0 *as described by* [JW]*.*

(iii) *The form* Φ *is the unique harmonic form in its cohomology class.*

The proof of (A.IV.F.17) is a calculation using (A.IV.F.15), (A.IV.F.16) and is illustrated by the pictures

$$\begin{array}{c} v_0 \\ (0,0) \end{array}$$

$$\begin{array}{c} (1,2) \xrightarrow{X_{12}} (2,1) \\ X_{23}v_0 \to X_{13}v_0 \end{array} \qquad \begin{array}{c} (-2,-1) \xrightarrow{X_{12}} (-1,-2) \\ X_{31}v_0 \to X_{32}v_0 \end{array}$$

$$\begin{array}{ccccc} (2,4) & \xrightarrow{X_{12}} & (3,3) & \xrightarrow{X_{12}} & (4,2) \\ X_{23}^2v_0 & \to & 2X_{23}X_{13}v_0 & \to & 2X_{13}^2v_0 \\ & & \| & & \\ & & 2X_{13}X_{23}v_0 & & \end{array}$$

$$\begin{array}{ccccc} & & (-1,1) \to (0,0) \to (-1,-1) & & \\ X_{31}X_{23}v & \to & (X_{13}X_{31} - X_{23}X_{32})v_0 & \to & -2X_{13}X_{32}v_0 \\ \| & & & & \| \\ X_{23}X_{31}v_0 & & & & -2X_{32}X_{13}v_0 \end{array}$$

$$\begin{array}{c} (-4,-2) \xrightarrow{X_{12}} (-3,3) \xrightarrow{X_{12}} (-2,-4) \\ \text{(similar to the left side)} \end{array}$$

It follows that

$$X_{13}X_{31}v_0 = X_{31}X_{13}v_0, X_{32}X_{23}v_0 = X_{23}X_{32}v_0$$

which leads to a relation

$$(X_{13}X_{31} + X_{23}X_{32})v_0 = cv_0\,.$$

At this point the only unknown is the action of $\mathrm{su}(2,1)$ on $V_0^{[3]}$ is the constant c.

Using the above one computes that

$$d\Phi = (v_0 - (X_{13}X_{31} + X_{23}X_{32})v_0)\omega^{13} \wedge \omega^{32}$$

which implies the assertion about when $d\Phi = 0$.

The general condition that $\Phi = v_0\omega^{12} + A\omega^{13} + B\omega^{32}$ be harmonic [EGW] is the divergence-type relation

$$d^*\Phi =: X_{21}v_0 + X_{31}A + X_{23}B = 0\,.$$

Taking $A = X_{23}v_0$, $B = -X_{31}v_0$ and using $X_{21}v_0 = 0$ this condition is

$$\begin{aligned} 0 &= X_{31}X_{23}v_0 - X_{23}X_{31}v_0 \\ &= -X_{21}v_0 \end{aligned}$$

which is satisfied. □

Remark. Setting $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$ where $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ with $\mathfrak{p}^+ = \bigoplus\limits_{\alpha \in \Phi^+_{nc}} \mathfrak{g}^\alpha$ the direct sum of the positive, non-compact root spaces, and using the duality $\mathfrak{n}^\vee \cong \mathfrak{p}^+$ given by the Cartan-Killing form, the Hochschild-Serre spectral sequence has the picture

$$\begin{tikzcd} H^1(\mathfrak{n}_K, V_0)_\rho \arrow[r, "d_1"] \arrow[drr, dashed, "d_2"] & H^1(\mathfrak{n}_K, V_0 \otimes \mathfrak{p}^+)_\rho \arrow[r, "d_1"] & H^1(\mathfrak{n}_K, V_0 \otimes \Lambda^2\mathfrak{p}^+)_\rho \\ H^0(\mathfrak{n}_K, V_0)_\rho \arrow[r] & H^0(\mathfrak{n}_K, V_0 \otimes \mathfrak{p}^+)_\rho \arrow[r] & H^0(\mathfrak{n}_K, V_0 \otimes \Lambda^2\mathfrak{p}^2)_\rho. \end{tikzcd}$$

There is a map

$$H^1(\mathfrak{n}, V_0)_\rho \to (\ker d_1 \cap \ker d_2) \subset H^1(\mathfrak{n}_K, V_0)_\rho\,,$$

and this is the meaning of the restriction map.

We note that $v_0\omega^{12}$ gives a class in $H^1(\mathfrak{n}_K, V_0)_\rho$, and $d_1[v_0\omega^{12}] = 0$ gives the coefficients $X_{23}v_0$ and $-X_{31}v_0$ in the above expression for Φ. Then $d_2[\Phi] = 0$ is equivalent to $c = 1$.

The details of this and similar calculations will be given in a future work.

Summary. For the TDLDS for $S\mathfrak{U}(2,1)$

- From geometric considerations we know the K-type $\bigoplus_k (\text{IV.F.13})_k$;
- Also from geometric considerations we know that there is a form Φ representing a class in $H^1(\mathfrak{n}, V_0)_\rho$ whose "restriction" to $H^1(\mathfrak{n}_K, \mathbb{C})_\rho$ is represented by ω^{12};
- There is a non-zero class in $H^2(\mathfrak{n}, V_0)_\rho$ which under duality pairs non-trivially with Ω; this class is represented by $\omega^{13} \wedge \omega^{32}$.

Note added in proof. Schmid has given a proof that the Hochschild-Serre spectral sequence degenerates at E_1 in the cases

- su$(2,1)$ and $E_1^{p,q} = H^q(\mathfrak{n}_K, \wedge^p \mathfrak{n}^+ \otimes V_0)_\rho \Rightarrow H^{p+q}(\mathfrak{n}, V_0)_\rho$ for V the TDLDS; and
- sp(4) and $E_1^{p,q} = H^1(\mathfrak{n}_K, \wedge^p \mathfrak{n}^+ \otimes V_k)_\rho \Rightarrow H^{p+q}(\mathfrak{n}, V_k)_\rho$ for V_k, $k = 1, 2$, the two TDLDS's.

His argument, together with a discussion of further related results, will be given in a future work.

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