THE QUANTUM VARIANCE OF THE MODULAR SURFACE

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Abstract. The variance of observables of quantum states of the Laplacian on the modular surface is calculated in the semiclassical limit. It is shown that this hermitian form is diagonalized by the irreducible representations of the modular quotient and on each of these it is equal to the classical variance of the geodesic flow after the insertion of a subtle arithmetical special value of the corresponding L-function.

1. Introduction

Let $G = PSL(2,\mathbb{R})$, $\Gamma = PSL(2,\mathbb{Z})$ and $X = \Gamma\backslash\mathbb{H}$ be the modular surface. X is a hyperbolic surface of finite area and it has a large discrete spectrum for the Laplacian (see [16] and [43]). The corresponding eigenfunctions can be diagonalized and we denote these Hecke-Maass forms by ϕ_j, $j = 1, 2, \cdots$. They are real valued and satisfy

$$\Delta\phi_j + \lambda_j\phi_j = 0, \quad T_n\phi_j = \lambda_j(n)\phi_j \tag{1}$$

and we normalize them by

$$\int_X \phi_j(z)^2 dA(z) = 1. \tag{2}$$

Here dA is the hyperbolic area form and write $\lambda_j = \frac{1}{4} + t_j^2$. If $\lambda > 0$ then it is known that such a ϕ is a cusp form [16]. ϕ_j has a Fourier expansion,

$$\phi_j(z) = \sum_{n\neq 0} \frac{c_j(|n|)}{\sqrt{|n|}} W_{0,it_j}(4\pi|n|y)e(nx), \tag{3}$$

where W_{0,it_j} is the Whittaker function. X carries a further symmetry induced by the orientation reversing isometry $z \to -\overline{z}$ of $\mathbb{H}$ and our ϕ's are either even or odd with respect to this symmetry r

$$\phi_j(rz) = \epsilon_j\phi_j(z), \qquad \epsilon_j = \pm 1. \tag{4}$$

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Correspondingly

$$c_j(n) = \epsilon_j c_j(-n). \tag{5}$$

The Iwasawa decomposition of $g \in G$ takes the form

$$g = n(x)a(y)k(\theta) \tag{6}$$

where

$$n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, a(y) = \begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix}, k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

$\mathbb{H}$ may be identified with G/K where $K = SO(2)/(\pm I)$ and then $\Gamma\backslash G$ is identified with the unit tangent space or phase space for the geodesic flow on X. The objects whose fluctuations we study in this paper are the Wigner distributions ω_j on $\Gamma\backslash G$. These are quadratic functionals of the ϕ_j's and are given by (see the recent paper [1] for a detailed description of these distributions as well as their basic invariance properties),

$$\omega_j = \phi_j(z) \sum_{k\in\mathbb{Z}} \phi_{j,k}(z) e^{2ik\theta} \tag{7}$$

Here the $\phi_{j,k}$ are the shifted (by raising and lowering operators) Maass cusp forms of weight k. They are eigenfunctions of the Casimir operator Ω, which acts on $C^\infty(\Gamma\backslash G)$.

The basic question concerning the ω_j's is their behavior in the semiclassical limit $t_j \to \infty$. Lindenstrauss [30] and Soundararajan [46] have shown that for an "observable" $\psi \in C(\Gamma\backslash G)$

$$\omega_j(\psi) \to \int_{\Gamma\backslash G} \psi(g)dg, \quad \text{as} \quad j \to \infty \tag{8}$$

where dg is normalized Haar measure (i.e. a probability measure), this is the so called "QUE" property.

It is known after Watson [49] and Jakobson [24] that the generalized Lindelof Hypothesis implies that if

$$\int_{\Gamma\backslash G} \psi(g)dg = 0 \tag{9}$$

then, for $\epsilon > 0$

$$\omega_j(\psi) \ll_\epsilon t_j^{-\frac{1}{2}+\epsilon} \tag{10}$$

For the rest of the paper we will assume that the mean value of ψ is 0, i.e. (9) holds. The main result below is the determination of the quantum variance, namely the mean-square of the $\omega_j(\psi)$'s. These are computed for special observables (ones depending only on $z \in X$) in

[37] where the ϕ_j's are replaced by holomorphic forms, and in [54] for the ω_j's at hand. The extension to the general observable that is carried out here is substantially more complicated and intricate. It comes with a reward in that the answer on the phase space is conceptually much more transparent and elegant.

The variance sums

$$S_\psi(T) := \sum_{t_j \le T} |\omega_j(\psi)|^2 \tag{11}$$

were introduced by Zelditch who showed (in much greater generality) that $S_\psi(T) = O(\frac{T^2}{\log T})$ [53]. Corresponding to (10) we expect that in our setting $S_\psi(T)$ will be at most $T^{1+\epsilon}$, since by Weyl's law [45], $\sum_{t_j \le T} 1 \sim \frac{T^2}{12}$.

Theorem 1. *Denote by $A_0(\Gamma\backslash G)$ the space of smooth right K-finite functions on $\Gamma\backslash G$ which are of mean 0 and of rapid decay. There is a sesquilinear form Q on $A_0(\Gamma\backslash G) \times A_0(\Gamma\backslash G)$ such that*

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t_j \le T} \omega_j(\psi_1)\overline{\omega}_j(\psi_2) = Q(\psi_1, \psi_2). \tag{12}$$

We call Q the quantum variance. The proof of Theorem 1 proceeds by proving the existence of the limit which comes with an explicit but formidable expression for Q see (34) of section 2. It involves infinite sums over arithmetic-geometric terms (twisted Kloosterman sums) and it appears very difficult to read any properties of Q directly from (34). For example even that Q is not identically zero (which is the case so that the exponent of T in the theorem is the correct one) is not clear. Using some apriori invariance properties of Q as well as some others that are derived from special cases of general versions of the daunting expression (34) allows us to eventually diagonalize Q.

In order to describe the result we need some more notation. The fluctuations of an observable $\psi \in C_0(\Gamma\backslash G)$ under the classical motion $\mathcal{G}_t$ by geodesics was determined in [40] and it asserts that for almost all g,

$$\frac{1}{\sqrt{T}} \int_0^T \psi(\mathcal{G}_t(g))dt \tag{13}$$

is Gaussian with mean zero and variance V given by

$$V(\psi_1, \psi_2) = \int_{-\infty}^{\infty} \int_{\Gamma\backslash SL(2,\mathbf{R})} \psi_1 \left(g \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix} \right) \overline{\psi_2(g)} dg dt. \tag{14}$$

Note that (14) converges due to the rapid decay of correlations for the geodesic flow. The correspondence principle suggests, and it has been conjectured in [7], that for chaotic systems such as the one at hand, the quantum fluctuations are also Gaussian with a variance which agrees with the classical one in (14).

The distributions ω_j enjoy some invariance properties that are inherited by Q and which are critical for its determination. The first is that ω_j is invariant under time reversal which is a reflection of ϕ being real. That is $w\omega_j = \omega_j$, where w is the involution of $\Gamma\backslash G$ given by

$$\Gamma g \to \Gamma g \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{15}$$

Thus

$$Q(w\psi_1, \psi_2) = Q(\psi_1, w\psi_2) = Q(\psi_1, \psi_2) \tag{16}$$

The second symmetry is special to X and follows from (4);

$$r\omega_j = \omega_j, \quad Q(r\psi_1, \psi_2) = Q(\psi_1, r\psi_2) = Q(\psi_1, \psi_2) \tag{17}$$

So if the quantum variance is to be compared with the classical variance then it should be to the symmetrized form

$$V^{\mathrm{sym}}(\psi_1, \psi_2) := V(\psi_1^{\mathrm{sym}}, \psi_2^{\mathrm{sym}}) \tag{18}$$

where

$$\psi^{\mathrm{sym}} := \frac{1}{4} \sum_{h\in H} h\psi \tag{19}$$

for $H = \{1, w, r, wr\}$.

These same symmetries arose in connection with the arithmetic measures on $\Gamma\backslash G$ studied in [32]. In fact the arithmetic variance B introduced in that paper turns out as we will show, to be very close to our quantum variance Q. We employ freely some of the techniques and notations in [32].

The classical variance V is diagonalized by the decomposition of $L^2_{\mathrm{cusp}}(\Gamma\backslash G)$ into irreducible representations under right translations by G. For simplicity we will restrict ourselves to examining Q on $L^2_{\mathrm{cusp}}(\Gamma\backslash G)$, the continuous spectrum can be investigated similarly. We have

$$L^2_{\mathrm{cusp}}(\Gamma\backslash G) = \bigoplus_{j=1}^{\infty} W_{\pi_j}, \tag{20}$$

where W_{π_j}'s are irreducible cuspidal automorphic representations, each also invariant under the Hecke algebra. The π_j's come in two types, the discrete series $W_{\pi_j^k}$, k even, $j = 1, 2, \cdots, d_k$, d_k being the dimension of the space of holomorphic and antiholomorphic forms of weight k, and the spherical representations π_k^0 (see [32]). Thus

$$\begin{aligned} L^2_{cusp}(\Gamma\backslash G) &= \sum_{j=1}^{\infty} W_{\pi_j} \oplus \sum_{k\geq 12}\sum_{j=1}^{d_k}\Big(W_{\pi_j^k}\oplus W_{\pi_j^{-k}}\Big) \\ &:= \sum_{j=1}^{\infty} U_{\pi_j^0} \oplus \sum_{k\geq 12}\sum_{j=1}^{d_k} U_{\pi_j^k} \qquad (21) \end{aligned}$$

where d_k is either $[k/12]$ or $[k/12]+1$ depending if $k/2 = 1 \bmod 6$ or not.

To each π_j is associated its standard L-function $L(s,\pi_j)$[1] which has an analytic continuation and functional equation relating its value at s to $1-s$. In particular, the number $L(\frac{1}{2},\pi_j)$ is real and it is a very subtle and much studied arithmetical invariant of π_j.

We can finally state our main result,

Theorem 2. *Both V^{sym} and Q are diagonalized by the orthogonal decomposition (21) and on each summand $U_{\pi_j^k}$, we have*

$$Q|_{U_{\pi_j^k}} = L(\frac{1}{2},\pi_j^k)V^{\mathrm{sym}}|_{U_{\pi_j^k}}. \qquad (22)$$

Remark 1. *The precise meaning in Theorem 2 is that it holds when evaluated on any ψ_1,ψ_2 in $L^2_{cusp}(\Gamma\backslash G)\cap A_0(\Gamma\backslash G)$.*

Remark 2. *The theorem asserts that the quantum variance is equal to the classical variance after inserting the "correction factor" of $L(\frac{1}{2},\pi)$ on each irreducible subspace. As we have noted Q is very close to the arithmetic variance B in* [32]*. Comment (1.4.6) of that paper indicates heuristically why one might expect this to be so. However our proof that these Hermitian forms are essentially the same goes through a very different route.*

We outline briefly the proofs of Theorem 1 and 2 and the contents of the paper. Section 2 is devoted to the proof of Theorem 1. The variance sums are studied for functions in $A_0(\Gamma\backslash G)$, all of which are realized by Poincare series. For technical reasons we weight the sums in (12) by an analytic function $u(\frac{t}{T})$ and also by mild arithmetical weights $L(1,\mathrm{sym}^2\varphi_j)$. This facilitates the use of the Petersson-Kuznetzov formula and the weights are only removed at the end. This technique was introduced in [36] and used in subsequent investigations [25], [37] and

[1]Our notation throughout is that $L(s,\pi)$ denotes the finite part of the L-function and $\Lambda(s,\pi)$ the completed L-function with its archimedean factors.

[54] with progressively more complicated answers. The present case is given in Section 2 equation (13) and is (as we have noted) very complicated. We have to pass through versions of it as it is the only way that we know of proving the existence of the limit at this scale and we also need to use these formulae later to prove (23) below.

The rest of the paper, Sections 3 and 4 are concerned with diagonalizing Q. A key role is played by the asymptotic invariance of ω_j under the geodesic flow $\mathcal{G}_t$ on $\Gamma\backslash G$. This alone does not suffice to get the corresponding invariance property for Q, since we are working at the level slightly sharper than the bounds (10). To this end the recent results of Anantharaman and Zelditch [1] clarify the exact error terms in the invariance properties of ω_j under $\mathcal{G}_t$. This together with well known multiplicity one results for linear functionals on irreducible representations of G, which are $\mathcal{G}_t$, w and r invariant, reduce the determination of Q to $Q(\xi,\eta)$, where ξ and η are vectors which generate the irreducible π_j^k and $\pi_{j'}^{k'}$ respectively (see [32]). If $\pi_j^k \neq \pi_{j'}^{k'}$, we need to show that $Q(\xi,\eta)=0$. This is done by establishing a self-adjointness property of Q with respect to the finite Hecke operators T_p. Namely that for such ξ and η,

$$Q(T_p\xi,\eta) = Q(\xi,T_p\eta) \tag{23}$$

The proof of this is given in Propositions 4 and 5 and requires one to prove several of identities for the corresponding twisted Kloosterman sums. This is similar to the analysis in applications of the trace formula to prove spectral identities, after comparisons of orbital integrals (the fundamental lemma as it is known in general). With (23) the vanishing of $Q(\xi,\eta)$, when $\pi_j^k \neq \pi_{j'}^{k'}$ follows from the multiplicity one theorem for automorphic cusp forms on GL_2. Finally when $\pi_j^k = \pi_{j'}^{k'}$ the sum (12) may be analyzed using Watson's triple product formula [49] and its generalization by Ichino [18] together with techniques from averaging special values of L-functions over families. One needs an explicit form of these triple product identities for forms which are ramified at infinity. This is provided in Appendix A. This leads to the explicit evaluation of $Q(\xi,\eta)$, and in particular it introduces the magic factor of $L(\frac{1}{2},\pi)$. Finally in Section 5, we remove the mild weights and derive Theorem 2.

2. Poincaré Series

In this section we calculate the quantum variance sum of the weight $2k$ incomplete Poincaré series against $d\omega_j$ on $\Gamma\backslash G$.

Let $h(t)$ be a smooth function on $(0,\infty)$ with compact support. On $C^\infty(0,\infty)$, define $\|\cdot\|_A$ by

$$\|h\|_A = \max_{0\le i,j\le A, t\in(0,\infty)} \left|\frac{h^i(t)}{t^j}\right|$$

For $m \in \mathbb{Z}$, define the incomplete Poincaré series of weight $-2k$:

$$P_{h,m,2k}(z,\theta) = e^{2ik\theta} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} h(y(\gamma z))(\epsilon_\gamma(z))^{2k} e(mx(\gamma z)).$$

For $m = 0$, it becomes the incomplete Eisenstein series of the same weight.

On $\Gamma\backslash G$, define the Wigner distributioon

$$d\omega_j = \varphi_j(z) \sum_{k\in\mathbb{Z}} \bar{\varphi}_{j,k}(z) e^{-2ik\theta} d\omega$$

where

$$d\omega = \frac{dxdy}{y^2}\frac{d\theta}{2\pi}.$$

φ_j is the j-th Hecke-Maass eigenform with the corresponding Laplacian eigenvalue $\lambda_j = \dfrac{1}{4} + t_j^2$, Hecke eigenvalues $\lambda_j(n)$ and we normalize $\|\varphi_j\|_2 = 1$. $\varphi_{j,k}(z)$ are shifted Maass cusp forms of weight $2k$, $\varphi_{j,k}(z)e^{-2ik\theta}$ is an eigenfunction of Casimir operator

$$\Omega = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + y\frac{\partial^2}{\partial x\partial\theta} = \Delta + y\frac{\partial^2}{\partial x\partial\theta}$$

with the same eigenvalue $\dfrac{1}{4}+t_j^2$ for every k. (Ω acts as $\Delta_{2k} = \Delta - 2iky\frac{\partial}{\partial x}$ on weight $2k$ forms.)

We fix an even function $u(t)$ be analytic in the strip $|\mathrm{Im} t| < \frac{1}{2}$ and real analytic on $\mathbb{R}$ satisfying $u^{(n)}(t) \ll (1+|t|)^{-N}$ for any $n > 0$ and large N, and $u(t) \ll t^{10}$ when $t \to 0$.

We have the following

Proposition 1. *For $h_1,\ h_2 \in C_c^\infty(0,\infty)$, $m_1, m_2, k_1, k_2 \in \mathbb{Z}$, and $P_{h_1,m_1,2k_1}$, $P_{h_2,m_2,2k_2}$ satisfying (9), there is a sesquilinear form Q, such that*

$$\begin{aligned}&\lim_{T\to\infty} \frac{1}{T}\sum_{j\ge1} u\left(\frac{t_j}{T}\right) L(1,\mathrm{sym}^2\varphi_j)\omega_j(P_{h_1,m_1,2k_1})\overline{\omega}_j(P_{h_2,m_2,2k_2})\\ = \ & Q(P_{h_1,m_1,2k_1}, P_{h_2,m_2,2k_2}).\end{aligned}$$

Moreover, there is a constant A and C (depending on k_1, k_2) such that the sesquilinear form Q satisfies

$$|Q(P_{h_1,m_1,2k_1}, P_{h_2,m_2,2k_2})| \le C((|m_1|+1)(|m_1|+1))^A \|h_1\|_A \|h_2\|_A.$$

Proof. We prove the proposition for weight $-2k$, $k>0$ and it is analogous for functions of weight $2k$ (the case of $k_1 = k_2 = 0$ being dealt with in [54]). Let $m_1 m_2 \neq 0$, without loss of generality, we assume $m_1, m_2 \in \mathbb{N}$. By the Iwasawa decomposition and unfolding we have

$$\begin{aligned} \omega_j(P_{h,m,2k}) &= \int_{\Gamma\backslash G} (e^{2ik\theta} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} h(y(\gamma z))(\epsilon_\gamma(z))^{2k} e(mx(\gamma z))) d\omega_j \\ &= \int_{\Gamma_\infty\backslash\mathbb{H}} h(y)e(mx)\varphi_j(z)\varphi_{j,k}(z)d\mu(z) \qquad (24) \end{aligned}$$

Apply the Fourier expansion of $\varphi_{j,k}(z)$ [24],

$$\varphi_{j,k}(z) = (-1)^k \Gamma(1/2+it_j) \sum_{n\neq 0} \frac{c_j(|n|) W_{\mathrm{sgn}(n)k,it_j}(4\pi|n|y)e(nx)}{\sqrt{|n|}\Gamma(\frac{1}{2}+\mathrm{sgn}(n)k+it_j)},$$

and

$$\varphi_j(z) = \sum_{n\neq 0} \frac{c_j(|n|)}{\sqrt{|n|}} W_{0,it_j}(4\pi|n|y)e(nx).$$

From the relation $c_j(n) = c_j(1)\lambda_j(n)$ and the well-known multiplicativity of Hecke eigenvalues

$$\lambda_j(n)\lambda_j(m) = \sum_{d|(n,m)} \lambda_j\left(\frac{mn}{d^2}\right),$$

we have

$$\begin{aligned} \omega_j(P_{h,m,2k}) &= 4\pi(-1)^k \Gamma(\frac{1}{2}+it_j)c_j(1) \sum_{d|m} \sum_{q\neq 0,-\frac{m}{d}} \frac{c_j(q^2+\frac{qm}{d})}{\sqrt{|1+\frac{m}{qd}|}} \\ & \int_0^\infty \frac{W_{\mathrm{sgn}(q)k,it_j}(y)}{\Gamma(\frac{1}{2}+\mathrm{sgn}(q)k+it_j)} W_{0,it_j}\left(y\left|1+\frac{m}{qd}\right|\right) h\left(\frac{y}{4\pi|qd|}\right)\frac{dy}{y^2}. \qquad (25) \end{aligned}$$

Let $H(s)$ be the Mellin transform of $h(y)$,

$$H(s) = \int_0^\infty h(y)y^{-s}\frac{dy}{y}.$$

By the Mellin inversion,

$$h(y) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} H(s)y^s ds,$$

the inner integral (4) can be written as

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{H(s)}{|4\pi qd|^s}\int_0^\infty y^{s-2}\frac{W_{\mathrm{sgn}(q)k,it_j}(y)}{\Gamma(\frac{1}{2}+\mathrm{sgn}(q)k+it_j)}W_{0,it_j}\left(y\Big|1+\frac{m}{qd}\Big|\right)dyds$$

Since $W_{0,\mu}(y)=\sqrt{y/\pi}K_\mu(y/2)$, we can denote the inner integral as

$$A_k(s)=\int_0^\infty y^{s-\frac{3}{2}}W_{\mathrm{sgn}(q)k,it_j}(2y)K_{it_j}\left(y\Big|1+\frac{m}{qd}\Big|\right)dy$$

When $k=0$, the integral involves a product of two K-Bessel functions, which was evaluated by Luo-Sarnak [36]. Jakobson [25] evaluated $A_1(s)$ using the standard properties of K-Bessel and Whittaker functions,

$$W_{1,it_j}=\sqrt{\frac{2}{\pi}}(y^{\frac{3}{2}}K_{it_j}(y)-y^{\frac{1}{2}}(\frac{1}{2}+it_j)K_{it_j}(y)+y^{\frac{3}{2}}K_{it_j+1}(y))$$

in which one gets

$$A_1(s)=A_0(s+1)-(\frac{1}{2}+it_j)A_0(s)+\sqrt{\frac{2}{\pi}}B(s)$$

where

$$B(s)=\int_0^\infty y^sK_{it_j+1}(y)K_{it_j}\left(y\Big|1+\frac{m}{qd}\Big|\right)dy$$

Hence,

$$\begin{aligned}\sqrt{\frac{\pi}{2}}A_1(s) &= 2^{s-2}\Gamma\left(\frac{s+1+2it_j}{2}\right)\Gamma\left(\frac{s+1-2it_j}{2}\right)(1+\frac{m}{qd})^{it_j}\\ &\quad\int_0^1\tau^{\frac{s-1}{2}}(1-\tau)^{\frac{s-1}{2}}(1+\frac{2\tau m}{qd}+\tau(\frac{m}{qd})^2)^{-\frac{s+1}{2}-it_j}d\tau\\ &\quad-(\frac{1}{2}+it_j)2^{s-3}\Gamma\left(\frac{s+2it_j}{2}\right)\Gamma\left(\frac{s-2it_j}{2}\right)(1+\frac{m}{qd})^{it_j}\\ &\quad\int_0^1\tau^{\frac{s-2}{2}}(1-\tau)^{\frac{s-2}{2}}(1+\frac{2\tau m}{qd}+\tau(\frac{m}{qd})^2)^{-\frac{s}{2}-it_j}d\tau\\ &\quad+2^{s-2}\Gamma\left(\frac{s+2+2it_j}{2}\right)\Gamma\left(\frac{s-2it_j}{2}\right)(1+\frac{m}{qd})^{it_j}\\ &\quad\int_0^1\tau^{\frac{s-2}{2}}(1-\tau)^{\frac{s}{2}}(1+\frac{2\tau m}{qd}+\tau(\frac{m}{qd})^2)^{-\frac{s}{2}-1-it_j}d\tau\end{aligned}\tag{26}$$

Similarly, we can obtain $A_{-1}(s)$ by the formula

$$A_{-1}(s)=\frac{A_0(s+1)}{\frac{1}{4}+t_j^2}+\frac{A_0(s)}{\frac{1}{2}-it_j}-\sqrt{\frac{2}{\pi}}\frac{B(s)}{\frac{1}{4}+t_j^2}.$$

Then plug $A_1(s)$ and $A_{-1}(s)$ into (42) and by Stirling formula, Mellin inversion and the fact that [34]

$$|c_j(1)|^2 = \frac{2}{L(1, \mathrm{sym}^2\varphi_j)},$$

we have

$$\omega_j(P_{h,m,2}) = \frac{1}{L(1, \mathrm{sym}^2\varphi_j)} \sum_{d|m} \sum_{q>0} \lambda_j(q^2 + \frac{qm}{d}) \tilde{H}(t_j, d, q, m)$$

where

$$\tilde{H}(t_j, d, q, m) = \tilde{H}_1(t_j, d, q, m) + \tilde{H}_2(t_j, d, q, m) + \tilde{H}_3(t_j, d, q, m))$$

and

$$\tilde{H}_1(t_j, d, q, m) = \int_0^1 \left(\frac{(1 + \frac{m}{qd})}{1 + \frac{2\tau m}{qd} + \tau(\frac{m}{qd})^2} \right)^{it_j} (\tau(1-\tau)(1 + \frac{2\tau m}{qd} + \tau(\frac{m}{qd})^2))^{-\frac{1}{2}}$$
$$h\left(\frac{t_j \sqrt{\tau(1-\tau)}}{\pi dq \sqrt{1 + \frac{2\tau m}{qd} + \frac{\tau m^2}{(qd)^2}}} \right) d\tau, \quad (27)$$

$$\tilde{H}_2(t_j, d, q, m) = -2 \int_0^1 \left(\frac{(1 + \frac{m}{qd})}{1 + \frac{2\tau m}{qd} + \tau(\frac{m}{qd})^2} \right)^{it_j} (\tau(1-\tau))^{-1}$$
$$h\left(\frac{t_j \sqrt{\tau(1-\tau)}}{\pi dq \sqrt{1 + \frac{2\tau m}{qd} + \frac{\tau m^2}{(qd)^2}}} \right) d\tau, \quad (28)$$

and

$$\tilde{H}_3(t_j, d, q, m) = \int_0^1 \left(\frac{(1 + \frac{m}{qd})}{1 + \frac{2\tau m}{qd} + \tau(\frac{m}{qd})^2} \right)^{it_j} (\tau(1 + \frac{2\tau m}{qd} + \tau(\frac{m}{qd})^2))^{-1}$$
$$h\left(\frac{t_j \sqrt{\tau(1-\tau)}}{\pi dq \sqrt{1 + \frac{2\tau m}{qd} + \frac{\tau m^2}{(qd)^2}}} \right) d\tau, \quad (29)$$

For $i = 1, 2$, we denote

$$\omega_j(P_{h,m,2}) = \frac{1}{L(1, \mathrm{sym}^2\varphi_j)} \sum_{d_i|m_i} \sum_{q_i>0} \lambda_j(q_i^2 + \frac{q_i m_i}{d_i})(\tilde{H}_{i1}(t_j, d_i, q_i, m_i)$$
$$+\tilde{H}_{i2}(t_j, d_i, q_i, m_i) + \tilde{H}_{i3}(t_j, d_i, q_i, m_i)). \quad (30)$$

Now, plug into

$$\sum_{j\geq 1} u\left(\frac{t_j}{T}\right) L(1,\mathrm{sym}^2\varphi_j)\omega_j(P_{h_1,m_1,2})\overline{\omega}_j(P_{h_2,m_2,2})$$

and apply Kuznetsov's formula [29] to the inner sum, we obtain

$$\sum_{j\geq 1}\lambda_j(q_1(q_1+\frac{m_1}{d_1}))\overline{\lambda_j(q_2(q_2+\frac{m_2}{d_2}))}\frac{1}{L(1,\mathrm{sym}^2\varphi_j)}\widetilde{h}(t_j)$$

$$= \frac{\delta_{q_1(q_1+\frac{m_1}{d_1}),q_2(q_2+\frac{m_2}{d_2})}}{\pi^2}\int_{-\infty}^{\infty} t\tanh(\pi t)\widetilde{h}(t)dt - \frac{2}{\pi}\int_0^{\infty}\frac{\widetilde{h}(t)d_{it}(q_1^2+q_1m_1/d_1)}{|\zeta(1+2it)|^2}$$

$$d_{it}(q_2^2+q_2m_2/d_2)dt + \frac{2i}{\pi}\sum_c c^{-1}S(q_1^2+q_1m_1/d_1, q_2^2+q_2m_2/d_2;c)$$

$$\int_{-\infty}^{\infty} J_{2it}(\frac{4\pi\sqrt{(q_1^2+q_1m_1/d_1)(q_2^2+q_2m_2/d_2)}}{c})t\frac{\widetilde{h}(t)}{\cosh(\pi t)}dt.$$

Here

$$S(m,n;c) = \sum_{ad\equiv 1 \mod c} e(\frac{dm+an}{c})$$

is the Kloosterman sum and

$$d_{it}(n) = \sum_{d_1d_2=n}\left(\frac{d_1}{d_2}\right)^{it}.$$

and

$$\tilde{h}(t) = \frac{1}{t^2}H_1(t,d_1q_1,m_1)\overline{H_2(t,d_2q_2,m_2)}u\left(\frac{t}{T}\right).$$

Thus, we have

$$\sum_{j\geq 1} u\left(\frac{t_j}{T}\right) L(1,\mathrm{sym}^2\varphi_j)\omega_j(P_{h_1,m_1,2})\overline{\omega}_j(P_{h_2,m_2,2})$$

$$= \frac{\pi^2}{32}\sum_{d_1,d_2,q_1,q_2}(\frac{\delta_{q_1(q_1+\frac{m_1}{d_1}),q_2(q_2+\frac{m_2}{d_2})}}{\pi^2}\int_{-\infty}^{\infty} t\tanh(\pi t)\widetilde{h}(t)dt$$

$$-\frac{2}{\pi}\int_0^{\infty}\frac{\widetilde{h}(t)}{|\zeta(1+2it)|^2}d_{it}(q_1^2+q_1m_1/d_1)d_{it}(q_2^2+q_2m_2/d_2)dt \tag{31}$$

$$+\frac{2i}{\pi}\sum_c c^{-1}S(q_1^2+q_1m_1/d_1, q_2^2+q_2m_2/d_2;c)$$

$$\int_{-\infty}^{\infty} J_{2it}(\frac{4\pi\sqrt{(q_1^2+q_1m_1/d_1)(q_2^2+q_2m_2/d_2)}}{c})t\frac{\widetilde{h}(t)}{\cosh(\pi t)}dt), \tag{32}$$

Next, we will estimate each of these terms respectively.

First, we treat the diagonal terms. Since $q_1(q_1+\frac{m_1}{d_1}) = q_2(q_2+\frac{m_2}{d_2})$ has at most finitely many solutions if $m_1/d_1 \neq m_2/d_2$, and the integer solutions to $q_1(q_1+\frac{m_1}{d_1}) = q_2(q_2+\frac{m_2}{d_2})$ are only $q_1 = q_2$ if $\frac{m_1}{d_1} = \frac{m_2}{d_2}$. Thus, the diagonal terms are

$$\int_{-\infty}^{\infty} u\left(\frac{t}{T}\right) \sum_{m_1/d_1=m_2/d_2} \sum_{q\geq 1} H_1(t,d_1q,m_1)\overline{H}_2(t,d_2q,m_2)dt$$

where

$$\begin{aligned}&H_1(t,d_1q,m_1)\overline{H}_2(t,d_2q,m_2)\\ =&\sum_{i,j=1}^{3}\tilde{H}_{1i}(t,d_1q,m_1)\tilde{H}_{2j}(t,d_2q,m_2)\end{aligned}$$

Here, we treat the following one of the nine terms

$$\begin{aligned}&\tilde{H}_{11}(t,d_1q,m_1)\tilde{H}_{21}(t,d_2q,m_2)\\ =&\int_0^1\int_0^1 \frac{1}{\tau\eta(1-\tau)(1-\eta)}\cos(\frac{m_1}{d_1q}t(2\tau-1))\cos(\frac{m_2}{d_2q}t(2\eta-1))\\ &h_1\left(\frac{t\sqrt{\tau(1-\tau)}}{\pi d_1q\sqrt{1+\frac{2\tau m_1}{d_1q}+\frac{\tau m_1^2}{d_1^2q^2}}}\right) h_2\left(\frac{t\sqrt{\eta(1-\eta)}}{\pi d_2q\sqrt{1+\frac{2\eta m_2}{d_2q}+\frac{\eta m_2^2}{d_2^2q^2}}}\right)d\tau d\eta\end{aligned}$$

For $i=1,2$; restricting h_i on $\mathbb{R}$ and h_i satisfy $h_i^{(n)}(t) \ll (1+|t|)^{-N}$ for any $n>0$ sufficiently large N and $h_i(t) \ll t^{10}$ when $t\to 0$. Thus, h_i are continuous uniformly on $\mathbb{R}$. For the sum over q, we estimate it as

$$\begin{aligned}&\sum_{q\geq 1} H_1(t,d_1q,m_1)\overline{H}_2(t,d_2q,m_2)\\ =&\int_0^1\int_0^1\int_0^\infty \cos(\frac{m_1}{d_1q}t(2\tau-1))\cos(\frac{m_2}{d_2q}t(2\eta-1))h_1\left(\frac{t\sqrt{\tau(1-\tau)}}{\pi d_1q\sqrt{1+\frac{2\tau m_1}{d_1q}+\frac{\tau m_1^2}{d_1^2q^2}}}\right)\\ &h_2\left(\frac{t\sqrt{\eta(1-\eta)}}{\pi d_2q\sqrt{1+\frac{2\eta m_2}{d_2q}+\frac{\eta m_2^2}{d_2^2q^2}}}\right)dq\frac{1}{\tau\eta(1-\tau)(1-\eta)}d\tau d\eta + O(1)\end{aligned}$$

$$= \int_0^1 \int_0^1 \int_0^\infty \cos(\frac{m_1}{d_1 q} t(2\tau - 1)) \cos(\frac{m_2}{d_2 q} t(2\eta - 1)) h_1 \left(\frac{t\sqrt{\tau(1-\tau)}}{\pi d_1 q} \right)$$

$$h_2 \left(\frac{t\sqrt{\eta(1-\eta)}}{\pi d_2 q} \right) dq \frac{1}{\tau\eta(1-\tau)(1-\eta)} d\tau d\eta + O(1)$$

$$= \frac{t}{\pi} \int_0^1 \int_0^1 \int_0^\infty \frac{\cos(\frac{\pi m_1}{d_1}\xi(2\tau - 1)) \cos(\frac{\pi m_2}{d_2}\xi(2\eta - 1))}{\tau\eta(1-\tau)(1-\eta)} h_1 \left(\frac{\xi\sqrt{\tau(1-\tau)}}{d_1} \right)$$

$$h_2 \left(\frac{\xi\sqrt{\eta(1-\eta)}}{d_2} \right) \frac{d\xi}{\xi^2} d\tau d\eta + O(1)$$

Similarly, we can evaluate the other 8 terms and we obtain the main term of the diagonal term is

$$\sum_{\frac{m_1}{d_1} = \frac{m_2}{d_2}} \int_0^\infty \int_0^1 \int_0^1 \sum_{i,j=1}^3 \tilde{h}_{1i}(\xi, m_1, d_1, \tau_1) \tilde{h}_{2j}(\xi, m_2, d_2, \tau_2)$$

where

$$\tilde{h}_{i1}(\xi, m_i, d_i, \tau_i) = \frac{\cos(\frac{\pi m_i}{d_i}\xi(2\tau_i - 1)}{\sqrt{\tau_i(1-\tau_i)}} h_i(\frac{\xi\sqrt{\tau_i(1-\tau_i)}}{d_i}),$$

$$\tilde{h}_{i2}(\xi, m_i, d_i, \tau_i) = \frac{\cos(\frac{\pi m_i}{d_i}\xi(2\tau_i - 1)}{\tau_i(1-\tau_i)} h_i(\frac{\xi\sqrt{\tau_i(1-\tau_i)}}{d_i})$$

$$\tilde{h}_{i3}(\xi, m_i, d_i, \tau_i) = \frac{\cos(\frac{\pi m_i}{d_i}\xi(2\tau_i - 1)}{\tau_i} h_i(\frac{\xi\sqrt{\tau_i(1-\tau_i)}}{d_i})$$

for $i = 1, 2$.

For the non-diagonal terms which is the following

$$\sum_{\substack{d_1|m_1 \\ d_2|m_2}} \sum_{q_1,q_2} \sum_{c \geq 1} \frac{S(q_1(q_1 + \frac{m_1}{d_1}), q_2(q_2 + \frac{m_2}{d_2}); c)}{c}$$

$$\times \int_{\mathbb{R}} J_{2it} \left(\frac{4\pi\sqrt{q_1 q_2 (q_1 + \frac{m_1}{d_1})(q_2 + \frac{m_2}{d_2})}}{c} \right) \frac{\tilde{h}(t)t}{\cosh(\pi t)} dt$$

where

$$\tilde{h}(t) = \frac{1}{t^2} H_1(t, d_1 q_1, m_1) \overline{H_2(t, d_2 q_2, m_2)} u\left(\frac{t}{T}\right),$$

$$H_j(t,k,m)=\int_0^1\left(\frac{1+\frac{m}{k}}{1+\frac{2\tau m}{k}+\frac{\tau m^2}{k^2}}\right)^{it}\frac{1}{\tau(1-\tau)}h_j\left(\frac{t\sqrt{\tau(1-\tau)}}{\pi k\sqrt{1+\frac{2\tau m}{k}+\frac{\tau m^2}{k^2}}}\right)d\tau;$$

for $j=1,2$.

Let $x=\frac{4\pi\sqrt{q_1q_2(q_1+\frac{m_1}{d_1})(q_2+\frac{m_2}{d_2})}}{c}$, the inner integral in the non-diagonal terms is

$$I_T(x)=\frac{1}{2}\int_{\mathbb{R}}\frac{J_{2it}(x)-J_{-2it}(x)}{\sinh(\pi t)}\tilde{h}(t)\tanh\pi t dt$$

Since $\tanh(\pi t)=\mathrm{sgn}(t)+O(e^{-\pi|t|})$ for large $|t|$, we can remove $\tanh(\pi t)$ by getting a negligible term $O(T^{-N})$ for any $N>0$.

Next we apply the Parseval identity and the Fourier transform in [3]

$$\left(\frac{\widehat{J_{2it}(x)-J_{-2it}(x)}}{\sinh(\pi t)}\right)(y)=-i\cos(x\cosh(\pi y)).$$

By the evaluation of the Fresnel integrals, we have

$$\begin{aligned} I_T(x) &= \frac{-i}{2}\int_0^\infty u\left(\frac{t}{T}\right)\sqrt{\frac{2}{xy}}\int_0^1\int_0^1\frac{\cos(\frac{m_1}{d_1k}\sqrt{\frac{xy}{2}}(2\tau-1))\cos(\frac{m_2}{d_2k}\sqrt{\frac{xy}{2}}(2\eta-1))}{\tau\eta(1-\tau)(1-\eta)}\\ &\quad h_1\left(\frac{\sqrt{\frac{xy}{2}}\sqrt{\tau(1-\tau)}}{\pi d_1k\sqrt{1+\frac{2\tau m_1}{d_1k}+\frac{\tau m_1^2}{d_1^2k^2}}}\right)h_2\left(\frac{\sqrt{\frac{xy}{2}}\sqrt{\eta(1-\eta)}}{\pi d_2k\sqrt{1+\frac{2\eta m_2}{d_2k}+\frac{\eta m_2^2}{d_2^2k^2}}}\right)\\ &\quad d\tau d\eta\cos(x-y+\frac{\pi}{4})\frac{dy}{\sqrt{\pi y}} \end{aligned}$$

Thus, the non-diagonal terms are equal to

$$\begin{aligned} &\frac{-i}{2}\sum_{\substack{d_1|m_1\\ d_2|m_2}}\sum_{q_1,q_2}\sum_{c\geq1}\frac{S(q_1(q_1+\frac{m_1}{d_1}),q_2(q_2+\frac{m_2}{d_2});c)}{c}\int_0^\infty u\left(\frac{\sqrt{\frac{xy}{2}}}{T}\right)\sqrt{\frac{2}{xy}}\int_0^1\int_0^1\\ &\frac{\cos(\frac{m_1}{d_1k}\sqrt{\frac{xy}{2}}(2\tau-1))\cos(\frac{m_2}{d_2k}\sqrt{\frac{xy}{2}}(2\eta-1))}{\tau\eta(1-\tau)(1-\eta)}h_1\left(\frac{\sqrt{\frac{xy}{2}}\sqrt{\tau(1-\tau)}}{\pi d_1q_1}\right)\\ &h_2\left(\frac{\sqrt{\frac{xy}{2}}\sqrt{\eta(1-\eta)}}{\pi d_2q_2}\right)d\tau d\eta\cos(x-y+\frac{\pi}{4})\frac{dy}{\sqrt{\pi y}} \end{aligned}$$

Since both $h_1(t)$ and $h_2(t)$ satisfy $h_i^{(n)}\ll(1+|t|)^{-N}$ for any $n>0$ and sufficiently large N, and $h_i(t)\ll t^{10}$ when $t\to0$, the above sum is concentrated on

$$\left|\frac{\sqrt{\frac{xy}{2}}}{T}\right| \ll 1$$

$$T^{-\frac{1}{10}} \ll \frac{xy\tau(1-\tau)}{q_1^2} \ll 1$$

$$T^{-\frac{1}{10}} \ll \frac{xy\eta(1-\eta)}{q_2^2} \ll 1$$

Thus we can get the following range

$$\sqrt{\frac{xy}{2}} \sim T.$$

Note that here $x \sim q_1 q_2 c^{-1}$, the ranges for q_1, q_2, c are as follows

$$T\sqrt{\tau(1-\tau)} \ll q_1 \ll T^{\frac{21}{20}}\sqrt{\tau(1-\tau)},$$
$$T\sqrt{\eta(1-\eta)} \ll q_2 \ll T^{\frac{21}{20}}\sqrt{\eta(1-\eta)},$$
$$c \ll yT^{\frac{1}{10}}$$

Here by the above relations and partial integration sufficiently many times, we will get sufficiently large power of y, q_1 and q_2 occurring in the denominator, so we get the terms with $c \gg T^{\frac{1}{10}}$ contribute $O(1)$.

Denote the above sum as

$$\sum_{\substack{d_1|m_1\\ d_2|m_2}} \sum_{q_1,q_2} \sum_{c\geq 1} \frac{S(q_1(q_1+\frac{m_1}{d_1}), q_2(q_2+\frac{m_2}{d_2}); c)}{c} J_{q_1,q_2,c} + O(1).$$

Making the change of variable $t = \frac{\sqrt{\frac{xy}{2}}}{T}$, we get $J_{q_1,q_2,c}$ is

$$\frac{2^{\frac{3}{2}}}{\sqrt{\pi x}} \int_0^\infty u\left(\frac{t}{T}\right)\frac{1}{t}\sin(-x+\frac{2(tT)^2}{x}-\frac{\pi}{4}) \int_0^1\int_0^1 \frac{\cos(\frac{m_1}{d_1 k}tT(2\tau-1))}{\tau(1-\tau)}$$
$$\frac{\cos(\frac{m_2}{d_2 k}tT(2\eta-1))}{\eta(1-\eta)} h_1\left(\frac{tT\sqrt{\tau(1-\tau)}}{\pi d_1 q_1}\right) h_2\left(\frac{tT\sqrt{\eta(1-\eta)}}{\pi d_2 q_2}\right) d\tau d\eta dt$$

By Taylor expansion,

$$\begin{aligned} xi &= \frac{4\pi i}{c}\sqrt{q_1 q_2 (q_1+\frac{m_1}{d_1})(q_2+\frac{m_2}{d_2})} \\ &= \frac{2\pi i}{c}(2q_1q_2 + \frac{m_2 q_1}{d_2} + \frac{m_1 q_2}{d_1} + \cdots) \end{aligned}$$

So we can write

$$J_{q_1,q_2,c} = \Im(e_c(-(2q_1q_2 + \frac{m_2q_1}{d_2} + \frac{m_1q_2}{d_1}))f_c(q_1,q_2)),$$

where

$$f_c(q_1,q_2) = e_c(\frac{m_1m_2}{2d_1d_2}-\frac{m_1^2q_2}{4d_1^2q_1}-\frac{m_2^2q_1}{4d_2^2q_2}+\cdots)\frac{2^{\frac{3}{2}}}{\sqrt{\pi x}}\int_0^\infty u\left(\frac{t}{T}\right)\frac{1}{t}$$
$$e^{i(\frac{2(tT)^2}{x}-\frac{\pi}{4})}\int_0^1\int_0^1\frac{\cos(\frac{m_1}{d_1k}tT(2\tau-1))\cos(\frac{m_2}{d_2k}tT(2\eta-1))}{\tau\eta(1-\tau)(1-\eta)}$$
$$h_1\left(\frac{tT\sqrt{\tau(1-\tau)}}{\pi d_1q_1}\right)h_2\left(\frac{tT\sqrt{\eta(1-\eta)}}{\pi d_2q_2}\right)d\tau d\eta dt$$

and we use the notation $e_c(z)=e^{\frac{2\pi iz}{c}}$.

Reducing the summation over q_1,q_2 into congruence classes mod c, we have,

$$\sum_{q_1,q_2\geq 1}S(q_1(q_1+\frac{m_1}{d_1}),q_2(q_2+\frac{m_2}{d_2});c)e_c(-(2q_1q_2+\frac{m_2q_1}{d_2}+\frac{m_1q_2}{d_1}))f_c(q_1,q_2)$$

$$= \sum_{a,b \mod c}S(a(a+\frac{m_1}{d_1}),b(b+\frac{m_2}{d_2});c)e_c(-(2ab+\frac{m_2a}{d_2}+\frac{m_1b}{d_1}))$$
$$\sum_{q_1\equiv a,q_2\equiv b \mod c}f_c(q_1,q_2)$$
$$= \frac{1}{c^2}\sum_{u,v \mod c}\sum_{a,b \mod c}S(a(a+\frac{m_1}{d_1}),b(b+\frac{m_2}{d_2});c)$$
$$e_c(-(2ab+(\frac{m_2}{d_2}+u)a+(\frac{m_1}{d_1}+v)b))(\sum_{q_1,q_2}f_c(q_1,q_2)e_c(-uq_1-vq_2)).$$

Apply the Poisson summation for the sum in q_1, q_2 and obtain,

$$\sum_{q_1,q_2}f_c(q_1,q_2)e_c(-uq_1-vq_2)=\sum_{l_1,l_2}\int\int_{\mathbb{R}^2}f_c(q_1,q_2)e((l_1-\frac{u}{c})q_1+(l_2-\frac{v}{c})q_2)dq_1dq_2.$$

We can assume $|u|\leq\frac{c}{2}$, $|v|\leq\frac{c}{2}$, by partial integration sufficiently many times, we get

$$\sum_{q_1,q_2}f_c(q_1,q_2)e_c(-uq_1-vq_2)=\int\int_{\mathbb{R}^2}f_c(q_1,q_2)e(-\frac{u}{c}q_1-\frac{v}{c}q_2)dq_1dq_2+O(T^{-A})$$

for any $A>1$.

For $(u,v)\neq(0,0)$, by partial integration sufficiently many times, we also obtain

$$\int\int_{\mathbb{R}^2}f_c(q_1,q_2)e(-\frac{u}{c}q_1-\frac{v}{c}q_2)dq_1dq_2\ll T^{-A},$$

for any $A > 0$. Thus only $(u, v) = (0, 0)$ contributes. We can allow $c \gg T^{\frac{1}{10}}$ in the c-summation, notice that we have the term $\frac{T^2 c}{q_1 q_2}$ in $f_c(q_1, q_2)$, so by partial integration sufficiently many times,

$$\int\int_{\mathrm{R}^2} f_c(q_1, q_2) dq_1 dq_2 \ll c^{-A} T^2,$$

for any $A > 0$.

For fixed d_i, m_i $(i = 1, 2)$, denote

$$S_c = \sum_{a,b \mod c} S(a(a + \frac{m_1}{d_1}), b(b + \frac{m_2}{d_2}); c) e_c(-(2ab + \frac{m_2 a}{d_2} + \frac{m_1 b}{d_1}))$$

Thus, the non-diagonal contribution is

$$\begin{aligned}
&\sum_{\substack{d_1|m_1\\ d_2|m_2}} \sum_{c\geq 1} \Im(\frac{S_c}{c^2} \int\int_{\mathrm{R}^2} f_c(q_1, q_2) dq_1 dq_2) + O(1)\\
= &\sum_{\substack{d_1|m_1\\ d_2|m_2}} \sum_{c\geq 1} \Im(\frac{S_c}{c^2} \int\int_{\mathrm{R}^2} e_c(\frac{m_1 m_2}{2d_1 d_2} - \frac{m_1^2 q_2}{4d_1^2 q_1} - \frac{m_2^2 q_1}{4d_2^2 q_2}) \frac{2^{\frac{3}{2}}}{\sqrt{\pi x}} \int_0^\infty u\left(\frac{t}{T}\right)\frac{1}{t}\\
&e^{i(\frac{2(tT)^2}{x} - \frac{\pi}{4})} \int_0^1 \int_0^1 \frac{\cos(\frac{m_1}{d_1 q_1} tT(2\tau - 1)) \cos(\frac{m_2}{d_2 q_2} tT(2\eta - 1))}{\tau\eta(1-\tau)(1-\eta)} h_1\left(\frac{tT\sqrt{\tau(1-\tau)}}{\pi d_1 q_1}\right)\\
&h_2\left(\frac{tT\sqrt{\eta(1-\eta)}}{\pi d_2 q_2}\right) d\tau d\eta dt dq_1 dq_2) + O(1)\\
= &\,T \sum_{\substack{d_1|m_1\\ d_2|m_2}} \sum_{c\geq 1} \Im(\frac{S_c \zeta_8}{c^{\frac{3}{2}}} \int\int_{\mathrm{R}^2} e_c(\frac{m_1 m_2}{2d_1 d_2} - \frac{m_1^2 \phi}{4d_1^2 \xi} - \frac{m_2^2 \xi}{4d_2^2 \phi}) \frac{2^{\frac{3}{2}}}{(\xi\phi)^{\frac{3}{2}}}\\
&e(\xi\phi c) \int_0^1 \int_0^1 \frac{\cos(\frac{m_1 \xi}{d_1}(2\tau - 1)) \cos(\frac{m_2 \phi}{d_2}(2\eta - 1))}{\tau\eta(1-\tau)(1-\eta)} h_1\left(\frac{\xi\sqrt{\tau(1-\tau)}}{\pi d_1}\right)\\
&h_2\left(\frac{\phi\sqrt{\eta(1-\eta)}}{\pi d_2}\right) d\tau d\eta d\xi d\phi) + O(1).
\end{aligned}$$

Thus, we obtain the following asymptotic formula including the diagonal and non-diagonal terms:

$$\lim_{T\to\infty}\frac{1}{T}\sum_{j\geq 1}u\left(\frac{t_j}{T}\right)L(1,\mathrm{sym}^2\varphi_j)\omega_j(P_{h_1,m_1,2})\overline{\omega}_j(P_{h_2,m_2,2})$$

$$= \int_0^\infty u(t)dt(\sum_{\frac{m_1}{d_1}=\frac{m_2}{d_2}}\int_0^\infty\int_0^1\int_0^1\sum_{i,j=1}^{3}\tilde{h}_{1i}(\xi,m_1,d_1,\tau_1)\tilde{h}_{2j}(\xi,m_2,d_2,\tau_2) \tag{33}$$

$$d\tau_1 d\tau_2\frac{d\xi}{\xi^2}+\sum_{d_1|m_1,d_2|m_2}\sum_{c\geq 1}\int_0^\infty\int_0^\infty\int_0^1\int_0^1\Im\{\frac{S_c\zeta_8}{c^{\frac{5}{2}}}e_c(\frac{m_1m_2}{2d_1d_2}-\frac{m_1^2\xi_1}{4d_1^2\xi_2}-\frac{m_2^2\xi_2}{4d_2^2\xi_1})$$

$$e((d_1d_2)^2\xi_1\xi_2c)\}\sum_{i,j=1}^{3}\tilde{h}_{1i}(\xi_1,m_1,d_1,\tau_1)\tilde{h}_{2j}(\xi_2,m_2,d_2,\tau_2)d\tau_1d\tau_2\frac{d\xi_1d\xi_2}{(\xi_1\xi_2)^{3/2}}) \tag{34}$$

where

$$\tilde{h}_{i1}(\xi,m_i,d_i,\tau_i)=\frac{\cos(\frac{\pi m_i}{d_i}\xi(2\tau_i-1)}{\sqrt{\tau_i(1-\tau_i)}}h_i(\frac{\xi\sqrt{\tau_i(1-\tau_i)}}{d_i}),$$

$$\tilde{h}_{i2}(\xi,m_i,d_i,\tau_i)=\frac{\cos(\frac{\pi m_i}{d_i}\xi(2\tau_i-1)}{\tau_i(1-\tau_i)}h_i(\frac{\xi\sqrt{\tau_i(1-\tau_i)}}{d_i})$$

$$\tilde{h}_{i3}(\xi,m_i,d_i,\tau_i)=\frac{\cos(\frac{\pi m_i}{d_i}\xi(2\tau_i-1)}{\tau_i}h_i(\frac{\xi\sqrt{\tau_i(1-\tau_i)}}{d_i})$$

for $i=1,2$.

In the non-diagonal terms (34), S_c is a sum involving Kloosterman sums which is explicitly

$$S_c=\sum_{a,b \mod c}S(a(a+\frac{m_1}{d_1}),b(b+\frac{m_2}{d_2});c)e_c(-(2ab+\frac{m_2a}{d_2}+\frac{m_1b}{d_1}))$$

This gives the existence of the limiting variance for the case $k_1=k_2=1$.

Now, by the induction and the recurrence formula

$$A_{k+1}(s)=-2kA_k(s)+2A_k(s+1)-[(k-\frac{1}{2})^2+t_j^2]A_{k-1}(s)$$

we can obtain the existence of $B(P_{h_1,m_1,k_1},P_{h_2,m_2,k_2})$ for any $k_1,k_2\in\mathbb{Z}$. Precisely, for the term $[(k-\frac{1}{2})^2+t_j^2]A_{k-1}(s)$, the involving Gamma factors are,

$$\begin{aligned}&\frac{[(k-\frac{1}{2})^2+t_j^2]\Gamma(\frac{1}{2}+it_j)A_{k-1}(s)}{\Gamma(k+\frac{3}{2}+it_j)}\\ =\ &\frac{\Gamma(\frac{1}{2}+it_j)A_{k-1}(s)}{\Gamma(k-\frac{1}{2}+it_j)}\cdot\frac{[(k-\frac{1}{2})^2+t_j^2]\Gamma(k-\frac{1}{2}+it_j)}{\Gamma(k+\frac{3}{2}+it_j)}\end{aligned}$$

Thus, we can evaluate using the induction assumption for the first factor and Stirling formula for the second factor.

For the term $kA_k(s)$, we can use the similar argument to evaluate. While for the terms involving $A_0(s+k)$ and $B(s+k)$, the Gamma factors are easy to handle since they are simply

$$\frac{\Gamma(\frac{s+k}{2})^2}{\Gamma(s+k)},\quad \frac{\Gamma(\frac{s+k}{2})\Gamma(\frac{s+k}{2}+1)}{\Gamma(s+k+1)}$$

Moreover, by keeping track of the independence on h_1 and h_2 and integration by parts in the double integrals of (33) and (34), we obtain that there is a constant A (depending on k_1, k_2), such that the sesquilinear form Q satisfies

$$|Q(P_{h_1,m_1,k_1},P_{h_2,m_2,k_2})|\ll_{k_1,k_2}((|m_1|+1)(|m_1|+1))^A\|h_1\|_A\|h_2\|_A. \tag{35}$$

If any incomplete Poincaré series in this proposition is replaced by incomplete Eisenstein series, i.e. $m_i=0$ with mean zero satisfying (9), the proposition is still valid. For the case $m_1=m_2=0$, there is a slight change for Q:

$$\begin{aligned}&\sum_{d_1,d_2\geq 1}\int_0^\infty\int_0^1 h_1\left(\frac{\xi\sqrt{\tau(1-\tau)}}{d_1}\right)d\tau\int_0^1 h_2\left(\frac{\xi\sqrt{\eta(1-\eta)}}{d_2}\right)d\eta\frac{d\xi}{\xi^2}\\ =\ &\int_0^\infty\int_0^1\sum_{d_1\geq 1}h_1\left(\frac{\xi\sqrt{\tau(1-\tau)}}{d_1}\right)d\tau\int_0^1\sum_{d_2\geq 1}h_2\left(\frac{\xi\sqrt{\eta(1-\eta)}}{d_2}\right)d\eta\frac{d\xi}{\xi^2}\end{aligned}$$

By Euler-MacLaurin summation formula, we have

$$\sum_{d_1\geq 1}h_1\left(\frac{\xi\sqrt{\tau(1-\tau)}}{d_1}\right)=-\int_0^\infty b_2(\alpha)H_1\left(\frac{\xi\sqrt{\tau(1-\tau)}}{\alpha}\right)\frac{d\alpha}{\alpha^2},$$

where $b_2(\alpha)$ is the Bernoulli polynomial of degree 2, $H_1(x)=(h_1'(x)x^2)'$. For the sum over d_2, we have the similar expression.

□

This completes the proof of the existence of the quantum variance for vectors $\psi_1 = P_{h_1,m_1,2k_1}$ and $\psi_2 = P_{h_2,m_2,2k_2}$ in Theorem 1. To obtain the result for the general ψ_1, ψ_2 asserted in the Theorem one proceeds by the approximation arguments in Section 4 of [37], which requires keeping track of the dependence of the remainders in the analysis leading to (34) and (35) above. This is a straightforward generalization and we omit the details. In the next section we derive an explicit version of (34) for special Poincaré series of various weights.

3. Symmetry Properties of Q

We begin by showing that the sesquilinear form Q is invariant under the geodesic flow as well as under time reversal. This is true much more generally as can be seen from the recent work of Anatharaman and Zelditch [1] in the context of $\Gamma\backslash\mathbb{H}$ where Γ is any lattice (not just $SL_2(\mathbb{Z})$, in fact they deal with cocompact lattices but their results are easily extended to finite volume as in [51]). In this generality, they relate the Wigner distributions to what they call Patterson-Sullivan distributions. Since the latter are geodesic flow as well time reversal invariant, this yields a complete asymptotic expansion measuring this invariance. This is given in their Theorem 1.1 and the expansion on page 386. Taken to second order this reads:

If f is smooth and $\tau \in \mathbb{R}$ are fixed and $f_\tau(x) = f(x\mathcal{G}_\tau)$, where $\mathcal{G}_\tau$ is the geodesic flow, then

$$\begin{aligned} &< Op(f_\tau)\phi_j, \phi_j > \\ (36) \qquad = \; &< Op(f)\phi_j, \phi_j > + \frac{< Op(L_2(f_\tau - f))\phi_j, \phi_j >}{t_j} + O(\frac{1}{t_j^2}) \end{aligned}$$

where L_2 is a second order differential operator generated by the vector field $X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

First we apply (36) with the first term only, that is

$$(37) \qquad < Op(f_\tau)\phi_j, \phi_j > = < Op(f)\phi_j, \phi_j > + O(\frac{1}{t_j})$$

to the variance sums.

$$\begin{aligned} &\sum_{t_j \le T} < Op(f_\tau)\phi_j, \phi_j > \overline{< Op(g)\phi_j, \phi_j >} \\ = &\sum_{t_j \le T} < Op(f)\phi_j, \phi_j > \overline{< Op(g)\phi_j, \phi_j >} + O(\sum_{t_j \le T} \frac{1}{t_j} | < Op(g)\phi_j, \phi_j > |) \end{aligned}$$
(38)

Now the general quantum ergodicity theorem in this context [52] asserts that as $y \to \infty$,

$$\sum_{t_j \le y} | < Op(g)\phi_j, \phi_j > | = o(y^2) \tag{39}$$

Hence by partial summation in the second sum in (38), we get that

$$\begin{aligned} & \sum_{t_j \le T} < Op(f_\tau)\phi_j, \phi_j > \overline{< Op(g)\phi_j, \phi_j >} \\ = & \sum_{t_j \le T} < Op(f)\phi_j, \phi_j > \overline{< Op(g)\phi_j, \phi_j >} + o(T) \end{aligned} \tag{40}$$

A similar statement is true if f_τ is replaced by time reversal applied to f. Hence in this generality (and with no arithmetic assumptions) the quantum variance sums are geodesic flow and time reversal invariant to the order required in our Theorem 1.

In our arithmetic setting of $\Gamma = SL_2(\mathbb{Z})$ we can use Theorem 1 together with the relation (36) (to second order) to deduce (with or without the arithmetic weights) that as $T \to \infty$,

$$\begin{aligned} & \sum_{t_j \le T} < Op(f_\tau)\phi_j, \phi_j > \overline{< Op(g)\phi_j, \phi_j >} - \sum_{t_j \le T} < Op(f)\phi_j, \phi_j > \overline{< Op(g)\phi_j, \phi_j >} \\ = & \ Q(Op(L_2(f_\tau - f)), g) \log T + o(\log T) \end{aligned}$$

In any case we deduce from the above that Q is bilinearly invariant under both the geodesic flow and time reversal.

Therefore, from the symmetry consideration as in Luo-Rudnick-Sarnak [32], we know that the space of such Hermitian forms $B(f, h)$ restricted to subspaces associated to each irreducible representation is at most one dimension.

To use this further, we need show the orthogonality that $Q(\phi_j, \phi_k) = 0$ if ϕ_j, ϕ_k are in the different irreducible representations π_j, π_k. It suffices to show for the generator vectors of the representation, i.e. $Q(\phi_j, \phi_k) = 0$ if ϕ_j, ϕ_k is either holomorphic form or Maass form. To show this, we need first evaluate $Q(\phi_j, \phi_k)$ and then use the explicit Hermitian form B to deduce the self-adjointness with respect to Hecke operators. We consider the following three cases:

(a) Both ϕ_j and ϕ_k are holomorphic;
(b) ϕ_j is holomorphic and ϕ_k is Maass form;
(c) Both ϕ_j and ϕ_k are Maass forms, while this case was dealt in [54].

In case (a), we first use holomorphic Poincaré series to find an explicit form of $Q(P_{m_1,k_1}, P_{m_2,k_2})$. For holomorphic Poincaré series

$$P_{m,k}(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma, z)^{-k} e(m(\gamma z)).$$

By unfolding, we have

$$< P_{m,k}, d\omega_j > \; = \int_{\Gamma_\infty\backslash\mathbb{H}} e^{-2\pi my} e(mx)\varphi_j(z)\varphi_{j,k}(z)d\mu(z) \tag{41}$$

Apply the Fourier expansion of $\varphi_{j,k}(z)$ [24],

$$\varphi_{j,k}(z) = (-1)^k\Gamma(1/2 + it_j) \sum_{n\neq 0} \frac{c_j(|n|)W_{\mathrm{sgn}(n)k,it_j}(4\pi|n|y)e(nx)}{\sqrt{|n|}\Gamma(\frac{1}{2} + \mathrm{sgn}(n)k + it_j)},$$

and

$$\varphi_j(z) = \sum_{n\neq 0} \frac{c_j(|n|)}{\sqrt{|n|}} W_{0,it_j}(4\pi|n|y)e(nx).$$

From the relation $c_j(n) = c_j(1)\lambda_j(n)$ and the well-known multiplicativity of Hecke eigenvalues

$$\lambda_j(n)\lambda_j(m) = \sum_{d|(n,m)} \lambda_j\left(\frac{mn}{d^2}\right),$$

we have

$$\begin{aligned} < P_{m,k}, d\omega_j > \; &= 4\pi(-1)^k\Gamma(\frac{1}{2} + it_j)c_j(1) \sum_{d|m} \sum_{q\neq 0,-\frac{m}{d}} \frac{c_j(q^2 + \frac{qm}{d})}{\sqrt{|1 + \frac{m}{qd}|}} \\ & \int_0^\infty \frac{W_{\mathrm{sgn}(q)k,it_j}(y)}{\Gamma(\frac{1}{2} + \mathrm{sgn}(q)k + it_j)} W_{0,it_j}\left(y(1 + \frac{m}{qd})\right)\left(\frac{y}{qd}\right)^k e^{(\frac{-my}{2qd})} \frac{dy}{y^2}. \end{aligned} \tag{42}$$

For the inner integral, we apply the formula 7.671 in [13]

$$\begin{aligned} & \int_0^\infty x^{-k-\frac{3}{2}} e^{-\frac{1}{2}(a-1)x} K_\mu(\frac{1}{2}ax)W_{k,\mu}(x)dx \\ = \; & \frac{\pi\Gamma(-k)\Gamma(2\mu - k)\Gamma(-2\mu - k)}{\Gamma(\frac{1}{2} - k)\Gamma(\frac{1}{2} + \mu - k)\Gamma(\frac{1}{2} - \mu - k)} 2^{2k+1} a^{k-\mu} F(-k, 2\mu - k; -2k; 1 - \frac{1}{a}) \end{aligned}$$

by letting $a = 1 + m/d$, $\mu = it_j$ and for the hypergeometric series $F(-k, 2\mu - k; -2k; 1 - \frac{1}{a})$, we use 9.111 in [13]

$$F(\alpha, \beta; \gamma; z) = \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 t^{\beta-1}(1 - t)^{\gamma-\beta-1}(1 - tz)^{-\alpha} dt$$

By Stirling formula and similar method of calculating $< P_{h,m,k}, d\omega_j >$ in Section 2, we have

$$
\begin{aligned}
& < P_{m,k}, d\omega_j > \\
= \;& \frac{1}{L(1,\mathrm{sym}^2\varphi_j)} \sum_{d|m} \sum_{q>0} \lambda_j(q^2 + \frac{qm}{d}) \int_0^1 \left(\frac{(1+\frac{m}{qd})}{1+\frac{2\tau m}{qd} + \tau(\frac{m}{qd})^2} \right)^{it_j} \\
& (\tau(1-\tau)(1+\frac{2\tau m}{qd} + \tau(\frac{m}{qd})^2))^{k-\frac{1}{2}} \exp\left(\frac{-mt_j\sqrt{\tau(1-\tau)}}{2dq\sqrt{1+\frac{2\tau m}{qd}+\frac{\tau m^2}{(qd)^2}}} \right) d\tau
\end{aligned}
$$

By the similar treatment on Kuznetsov formula as we did in [54], we obtain

$$
\begin{aligned}
& \frac{1}{T} \sum_{j\geq 1} u\left(\frac{t_j}{T}\right) L(1,\mathrm{sym}^2\varphi_j)\omega_j(P_{m_1,k_1})\overline{\omega}_j(P_{m_2,k_2}) \\
= & \int_0^\infty u(t)dt \sum_{\frac{m_1}{d_1}=\frac{m_2}{d_2}} \int_0^\infty \int_0^1 \cos(\frac{\pi m_1}{d_1}\xi(2\tau-1)) \exp\left(\frac{-m_1\xi\sqrt{\tau(1-\tau)}}{d_1} \right) \\
& (\tau(1-\tau))^{k_1} d\tau \int_0^1 \cos(\frac{\pi m_2}{d_2}\xi(2\eta-1)) \exp\left(\frac{-m_2\xi\sqrt{\eta(1-\eta)}}{d_2} \right) (\eta(1-\eta))^{k_2} \\
& \cdot d\eta \xi^{k_1+k_2} \frac{d\xi}{\xi^2} + \int_0^\infty u(t)dt \sum_{\substack{d_1|m_1 \\ d_2|m_2}} \sum_{c\geq 1} \Im(\frac{S_c\zeta_8}{c^{\frac{3}{2}}} \int\int_{\mathrm{R}^2} e_c(\frac{m_1m_2}{2d_1d_2} - \frac{m_1^2\xi}{4d_1^2\phi} - \frac{m_2^2\phi}{4d_2^2\xi}) \\
& \frac{2^{\frac{3}{2}}\xi^{k_1}\phi^{k_2}}{(\xi\phi)^{\frac{3}{2}}} e((d_1d_2)^2\xi\phi c) \int_0^1 \int_0^1 \cos(\pi m_1 d_2 \xi(2\tau-1)) \cos(\pi m_2 d_1 \phi(2\eta-1)) \\
& \tau^{k_1}\eta^{k_2}(1-\tau)^{k_1}(1-\eta)^{k_2} \exp(-m_1\xi d_2\sqrt{\tau(1-\tau)}) \exp(-m_2\phi d_1\sqrt{\eta(1-\eta)}) \\
& d\tau d\eta d\xi d\phi
\end{aligned}
$$

Now, we can use this explicit form to show the self-adjointness of $B(\phi_j, \phi_k)$ with respect to Hecke operators for holomorphic ϕ_j, ϕ_k, in fact we can check it for each Hecke operator T_p, where p is a prime, i.e.

Proposition 2.

$$Q(T_pP_{m_1,k_1}, P_{m_2,k_2}) = Q(P_{m_1,k_1}, T_pP_{m_2,k_2}).$$

Proof. This is a direct generalization of Appendix A.3 in [37], which deals with the Maass case with $k = 0$. We use the fact (Theorem 6.9

in [20])

$$T_n P_{m,k}(z) = \sum_{d|(m,n)} \left(\frac{n}{d}\right)^{k-1} P_{\frac{mn}{d^2},k}(z), \tag{43}$$

and the explicit evaluation of $S_{c,\frac{m_1}{m_2},\frac{m_2}{d_2}}(\gamma)$ (Appendix A.2 in [37])to verify it.

We denote

$$Q(P_{m_1,k_1}, P_{m_2,k_2}) = Q_D(P_{m_1,k_1}, P_{m_2,k_2}) + Q_{ND}(P_{m_1,k_1}, P_{m_2,k_2})$$

as the diagonal and non-diagonal terms, and we consider the following 4 cases:

(i) If $p \nmid m_1 m_2$, $Q_D(T_p P_{m_1,k_1}, P_{m_2,k_2}) = Q_D(P_{m_1,k_1}, T_p P_{m_2,k_2})$;
(ii) If $p \nmid m_1 m_2$, $Q_{ND}(T_p P_{m_1,k_1}, P_{m_2,k_2}) = Q_{ND}(P_{m_1,k_1}, T_p P_{m_2,k_2})$;
(iii) If $p^a \parallel (m_1, m_2)$, $Q_D(T_p P_{m_1,k_1}, P_{m_2,k_2}) = Q_D(P_{m_1,k_1}, T_p P_{m_2,k_2})$;
(iv) If $p^a \parallel (m_1, m_2)$, $Q_{ND}(T_p P_{m_1,k_1}, P_{m_2,k_2}) = Q_{ND}(P_{m_1,k_1}, T_p P_{m_2,k_2})$.

To prove (i), we use the fact

$$T_p P_{m,k}(z) = p^{k-1} P_{pm,k}(z)$$

from (43). Also, from the conditions $d_1 | pm_1$, $d_2 | m_2$ and $\frac{pm_1}{d_1} = \frac{m_2}{d_2}$ we have $p | d_1$. For our convenience, we denote

$$\tilde{h}(\xi, m_i, d_i, k_i, \tau_i) = \cos\left(\frac{\pi m_i}{d_i}\xi(2\tau_i - 1)\right) \exp\left(\frac{-m_i \xi \sqrt{\tau_i(1-\tau_i)}}{d_i}\right) (\tau_i(1-\tau_i))^{k_i}$$

Thus, by making the change of variables $d_1 \to pd_1$, $\frac{\xi}{p} \to \xi$ and $d_2 \to pd_2$, $\frac{\xi}{p} \to \xi$ for $Q_D(P_{pm_1,k}, P_{m_2,k})$ and $Q_D(P_{m_1,k}, P_{pm_2,k})$ respectively, we have

$$\begin{aligned}
& Q_D(T_p P_{m_1,k_1}, P_{m_2,k_2}) \\
= \ & p^{k_1-1} Q_D(P_{pm_1,k_1}, P_{m_2,k_2}) \\
= \ & p^{-1} \sum_{\frac{m_1}{d_1} = \frac{m_2}{d_2}} \int_0^\infty \int_0^1 \int_0^1 \prod_{i=1}^2 \tilde{h}_i(\xi, m_i, d_i, l_i, \tau_i) d\tau_i \frac{\xi^{k_1+k_2} d\xi}{\xi^2} \\
= \ & p^{k_2-1} Q_D(P_{m_1,k_1}, P_{pm_2,k_2}) \\
= \ & Q_D(P_{m_1,k_1}, T_p P_{m_2,k_2}).
\end{aligned}$$

For (ii), we have

$$\begin{aligned}
& Q_{ND}(T_pP_{m_1,k_1},P_{m_2,k_2}) \\
= \;& p^{k_1-1}Q_{ND}(P_{pm_1,k_1},P_{m_2,k_2}) \\
= \;& p^{k_1-1}\sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2}\sum_{\substack{d_1|pm_1\\ d_2|m_2}}\sum_{c\geq 1}\int_0^\infty\int_0^\infty\int_0^1\int_0^1 \Im\{\frac{S_c\zeta_8}{c^{\frac{5}{2}}}e_c(\frac{pm_1m_2}{2d_1d_2}-\frac{p^2m_1^2\xi}{4d_1^2\phi}-\frac{m_2^2\phi}{4d_2^2\xi}) \\
& e((d_1d_2)^2\xi\phi c)\}\tilde{h}_1(p\xi_1,m_1,d_1,l_1,\tau_1)\tilde{h}_2(\xi_2,m_2,d_2,l_2,\tau_2)d\tau_i\frac{d\xi_1d\xi_2}{\xi_1^{3/2-k_1}\xi_2^{3/2-k_2}} \\
= \;& p^{-1}\sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2}\sum_{\substack{d_1|m_1\\ d_2|m_2}}\sum_{c\geq 1}\int_0^\infty\int_0^\infty\int_0^1\int_0^1 \Im\{\frac{\tilde{S}_c\zeta_8}{c^{\frac{5}{2}}}e_c(\frac{pm_1m_2}{2d_1d_2}-\frac{p^2m_1^2\xi}{4d_1^2\phi}-\frac{m_2^2\phi}{4d_2^2\xi}) \\
& e((d_1d_2)^2\xi\phi c)\}\tilde{h}_1(p\xi_1,m_1,d_1,l_1,\tau_1)\tilde{h}_2(\xi_2,m_2,d_2,l_2,\tau_2)d\tau_i\frac{d\xi_1d\xi_2}{\xi_1^{3/2-k_1}\xi_2^{3/2-k_2}} \\
& +p^{-1}\sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2}\sum_{\substack{d_1|m_1\\ d_2|m_2}}\sum_{c\geq 1}\int_0^\infty\int_0^\infty\int_0^1\int_0^1 \Im\{\frac{S_c\zeta_8}{c^{\frac{5}{2}}}e_c(\frac{m_1m_2}{2d_1d_2}-\frac{m_1^2\xi}{4d_1^2\phi}-\frac{m_2^2\phi}{4d_2^2\xi}) \\
& e((d_1d_2)^2\xi\phi c)\}\tilde{h}_i(\xi_i,m_i,d_i,l_i,\tau_i)d\tau_i\frac{d\xi_1d\xi_2}{\xi_1^{3/2-k_1}\xi_2^{3/2-k_2}}
\end{aligned}$$

The above two sums correspond to the conditions $p\nmid d_1$, and $p|d_1$ respectively.

Similarly, we have

$$\begin{aligned}
& Q_{ND}(P_{m_1,k_1},T_pP_{m_2,k_2}) \\
= \;& p^{-1}\sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2}\sum_{\substack{d_1|m_1\\ d_2|m_2}}\sum_{c\geq 1}\int_0^\infty\int_0^\infty\int_0^1\int_0^1 \Im\{\frac{\tilde{S}'_c\zeta_8}{c^{\frac{5}{2}}}e_c(\frac{pm_1m_2}{2d_1d_2}-\frac{m_1^2\xi}{4d_1^2\phi}-\frac{p^2m_2^2\phi}{4d_2^2\xi}) \\
& e((d_1d_2)^2\xi\phi c)\}\tilde{h}_1(\xi_1,m_1,d_1,l_1,\tau_1)\tilde{h}_2(p\xi_2,m_2,d_2,l_2,\tau_2)d\tau_i\frac{d\xi_1d\xi_2}{\xi_1^{3/2-k_1}\xi_2^{3/2-k_2}} \\
& +p^{-1}\sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2}\sum_{\substack{d_1|m_1\\ d_2|m_2}}\sum_{c\geq 1}\int_0^\infty\int_0^\infty\int_0^1\int_0^1 \Im\{\frac{S_c\zeta_8}{c^{\frac{5}{2}}}e_c(\frac{m_1m_2}{2d_1d_2}-\frac{m_1^2\xi}{4d_1^2\phi}-\frac{m_2^2\phi}{4d_2^2\xi}) \\
& e((d_1d_2)^2\xi\phi c)\}\tilde{h}_i(\xi_i,m_i,d_i,l_i,\tau_i)d\tau_i\frac{d\xi_1d\xi_2}{\xi_1^{3/2-k_1}\xi_2^{3/2-k_2}}
\end{aligned}$$

Make the change of variables $\xi \to \frac{\xi}{p}$, $\phi \to p\phi$. Moreover, by the evaluation of the sum S_c which involving the Salie sum, precisely

$$S_{c,pm_1/d_1,m_2/d_2} = S_{c,m_1/d_1,pm_2/d_2}.$$

We can see $Q_{ND}(T_pP_{m_1,k_1}, P_{m_2,k_2}) = Q_{ND}(P_{m_1,k_1}, T_pP_{m_2,k_2})$.

For the cases (iii) and (iv), we use the fact

$$T_pP_{m,k}(z) = p^{k-1}P_{pm,k}(z) + P_{\frac{m}{p},k}(z).$$

where if $p \nmid m$, we understand that $P_{h(\frac{\cdot}{p}),\frac{m}{p}}(z) = 0$.

Thus, for the case (iii), we have

$$\begin{aligned} & Q_\infty(T_pP_{h_1,m_1,k_1}, P_{h_2,m_2,k_2}) \\ = \;& p^{k_1-1}Q_\infty(P_{h_1(p\cdot),pm_1,k_1}, P_{h_2,m_2,k_2}) + Q_\infty(P_{h_1(\frac{\cdot}{p}),\frac{m_1}{p},k_1}, P_{h_2,m_2,k_2}) \\ = \;& A + B \end{aligned}$$

Similarly,

$$\begin{aligned} & Q_D(P_{m_1,k_1}, T_pP_{m_2,k_2}) \\ = \;& p^{k_2-1}Q_D(P_{m_1,k_1}, P_{pm_2,k_2}) + Q_D(P_{\frac{m_1}{p},k_1}, P_{\frac{m_2}{p},k_2}) \\ = \;& A_1 + Q_1 \end{aligned}$$

We can check that

$$A(p|d_1) = A_1(p|d_2),$$

$$A(p \nmid d_1) = B_1(p \nmid d_1),$$

$$B(p \nmid d_2) = A_1(p \nmid d_2),$$

$$B(p|d_2) = B_1(p|d_1).$$

Hence, we get (iii).

The proof of (iv) is the most tedious one and we will use the induction to prove that. We have

$$\begin{aligned} & Q_{ND}(T_pP_{m_1,k_1}, P_{m_2,k_2}) \\ = \;& p^{k_1-1}Q_{ND}(P_{pm_1,k_1}, P_{m_2,k_2}) + Q_{ND}(P_{\frac{m_1}{p},k_1}, P_{m_2,k_2}) \end{aligned}$$

From the expression of $Q(P_1, P_2)$, it equals

$$p^{k_1-1}\sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2}\sum_{\substack{d_1|pm_1\\ d_2|m_2}}\sum_{c\geq 1}\int_0^\infty\int_0^\infty\int_0^1\int_0^1 \Im\{\frac{S_c\zeta_8}{c^{\frac{5}{2}}}e_c(\frac{pm_1m_2}{2d_1d_2}-\frac{p^2m_1^2\xi}{4d_1^2\phi}-\frac{m_2^2\phi}{4d_2^2\xi})$$

$$e((d_1d_2)^2\xi\phi c)\}\tilde{h}_1(p\xi_1, m_1, d_1, l_1, \tau_1)\tilde{h}_2(\xi_2, m_2, d_2, l_2, \tau_2)d\tau_i\frac{d\xi_1 d\xi_2}{\xi_1^{3/2-k_1}\xi_2^{3/2-k_2}}$$

$$+\sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2}\sum_{\substack{d_1|m_1/p\\ d_2|m_2}}\sum_{c\geq 1}\int_0^\infty\int_0^\infty\int_0^1\int_0^1 \Im\{\frac{S_c\zeta_8}{c^{\frac{5}{2}}}e_c(\frac{m_1m_2}{2pd_1d_2}-\frac{m_1^2\xi}{4p^2d_1^2\phi}-\frac{m_2^2\phi}{4d_2^2\xi})$$

$$e((d_1d_2)^2\xi\phi c)\}\tilde{h}_1(\xi_1/p, m_1, d_1, l_1, \tau_1)\tilde{h}_2(\xi_2, m_2, d_2, l_2, \tau_2)d\tau_i\frac{d\xi_1 d\xi_2}{\xi_1^{3/2-k_1}\xi_2^{3/2-k_2}}$$

We denote the above sum as $I_1 + I_2$. Similarly,

$$\begin{aligned}
& Q_{ND}(P_{m_1,k_1}, T_pP_{m_2,k_2}) \\
= \;& p^{k_2-1}Q_{ND}(P_{m_1}, P_{pm_2}) + Q_{ND}(P_{m_1}, P_{\frac{m_2}{p}}) \\
= \;& p^{k_2-1}\sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2}\sum_{\substack{d_1|m_1\\ d_2|pm_2}}\sum_{c\geq 1}\int_0^\infty\int_0^\infty\int_0^1\int_0^1 \Im\{\frac{S_c\zeta_8}{c^{\frac{5}{2}}}e_c(\frac{pm_1m_2}{2d_1d_2}-\frac{m_1^2\xi}{4d_1^2\phi}-\frac{p^2m_2^2\phi}{4d_2^2\xi}) \\
& e((d_1d_2)^2\xi\phi c)\}\tilde{h}_1(\xi_1, m_1, d_1, l_1, \tau_1)\tilde{h}_2(p\xi_2, m_2, d_2, l_2, \tau_2)d\tau_i\frac{d\xi_1 d\xi_2}{\xi_1^{3/2-k_1}\xi_2^{3/2-k_2}} \\
& +\sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2}\sum_{\substack{d_1|m_1\\ d_2|m_2/p}}\sum_{c\geq 1}\int_0^\infty\int_0^\infty\int_0^1\int_0^1 \Im\{\frac{S_c\zeta_8}{c^{\frac{5}{2}}}e_c(\frac{m_1m_2}{2pd_1d_2}-\frac{m_1^2\xi}{4d_1^2\phi}-\frac{m_2^2\phi}{4p^2d_2^2\xi}) \\
& e((d_1d_2)^2\xi\phi c)\}\tilde{h}_1(\xi_1, m_1, d_1, l_1, \tau_1)\tilde{h}_2(\xi_2/p, m_2, d_2, l_2, \tau_2)d\tau_i\frac{d\xi_1 d\xi_2}{\xi_1^{3/2-k_1}\xi_2^{3/2-k_2}}
\end{aligned}$$

According to whether or not $p|(c, *, *)$ in $S_{c,*,*}$, we can decompose the above sums I_1, I_2, II_1, II_2 into the following 8 terms

$$I_1 = I_{11} + I_{12}, \quad I_2 = I_{21} + I_{22}, \quad II_1 = II_{11} + II_{12}, \quad II_2 = II_{21} + II_{22}.$$

Note if $p|(c, *, *)$, $S_{c,*,*} = 0$ unless $p^2|c$. Let $c = p^2c_1$, we have

$$S_{c,\frac{|m_1p|}{d_1},\frac{|m_2|}{d_2}} = S_{c_1,\frac{|m_1|}{d_1},\frac{|m_2|}{pd_2}}p^2(1-\frac{\delta(p, c_1)}{p}),$$

where $\delta(p, c_1) = 0$ if $p|c_1$; $\delta(p, c_1) = 1$ if $p \nmid c_1$. Hence we can write $I_{11} = I'_{11} - I''_{11}$ correspondingly.

Similarly we have

$$S_{c,\frac{|m_1|}{pd_1},\frac{|m_2|}{d_2}} = S_{c_1,\frac{|m_1|}{p^2d_1},\frac{|m_2|}{pd_2}}p^2(1-\frac{\delta(p,c_1)}{p}),$$

and write $I_{21} = I'_{21} - I''_{21}$,

$$S_{c,\frac{|m_1|}{d_1},\frac{|m_2p|}{d_2}} = S_{c_1,\frac{|m_1|}{pd_1},\frac{|m_2|}{d_2}}p^2(1-\frac{\delta(p,c_1)}{p}),$$

and write $II_{11} = II'_{11} - II''_{11}$,

$$S_{c,\frac{|m_1|}{d_1},\frac{|m_2|}{pd_2}} = S_{c_1,\frac{|m_1|}{pd_1},\frac{|m_2|}{p^2d_2}}p^2(1-\frac{\delta(p,c_1)}{p}),$$

and write $II_{21} = II'_{21} - II''_{21}$ corresponding $p|c_1$ or not.

By the induction hypothesis on $(\frac{m_1}{p},\frac{m_2}{p})$, we have $I'_{11} + I'_{21} = II'_{11} + II'_{21}$.

We have $S_{cp,a,b} = p^2 S_{c,a,b}$ and $S_{tp^2,ap,b} = 0$ if $p \nmid bc$. Using this and the evaluation of $S_{c,a,b}$ we can verify that

$$I_{12}(p|d_1) = II_{12}(p|d_2),$$

where $I_{12}(p|d_1)$ means the partial sum of I_{12} in which $p|d_1$). Similarly, we have

$$I_{12}(p\nmid d_1, p\nmid d_2, p\nmid c) = II_{12}(p\nmid d_2, p\nmid d_1, p\nmid c),$$
$$I_{12}(p\nmid d_1, p\parallel d_2, p\nmid c) = II_{12}(p\nmid d_2, p\parallel d_1, p\nmid c),$$
$$I_{12}(p\nmid d_1, p^2|d_2, p\nmid c) = I''_{11}(p\nmid d_1, p^2|m_2/d_1),$$
$$I_{12}(p\nmid d_1, p^2|d_2, p\nmid c) = I''_{11}(p\nmid d_1, p\parallel m_2/d_2),$$
$$II''_{11}(p\nmid d_2, p^2|m_1/d_1) = II_{12}(p\nmid d_2, p^2|d_1, p\nmid c),$$
$$II''_{11}(p\nmid d_2, p\parallel m_1/d_1) = II_{12}(p\nmid d_2, p\nmid c),$$
$$I''_{11}(p|d_1) = II''_{11}(p|d_2),$$
$$I_{22}(p|d_2) = II_{22}(p|d_1),$$
$$I_{22}(p\nmid d_2, p\nmid d_1, p\nmid c) = II_{22}(p\nmid d_1, p\nmid d_2, p\nmid c),$$
$$I_{22}(p\nmid d_2, p\parallel d_1, p\nmid c) = II_{22}(p\nmid d_1, p\parallel d_2, p\nmid c),$$
$$I_{22}(p\nmid d_2, p^2|d_1, p\nmid c) = I''_{21}(p\nmid d_2, p^3|m_1/d_1),$$
$$I_{22}(p\nmid d_2, p|c) = I''_{21}(p\nmid d_2, p^2\parallel m_1/d_1),$$
$$II''_{21}(p|d_1) = I''_{21}(p|d_2),$$
$$II_{22}(p\nmid d_1, p^2|d_2, p\nmid c) = II''_{21}(p\nmid d_1, p^3|m_2/d_2),$$
$$II_{22}(p\nmid d_1, p|c) = II''_{21}(p\nmid d_1, p^2\parallel m_2/d_2).$$

Hence we deduce from the above identities that

$$Q_{ND}(T_pP_{m_1,k_1}, P_{m_2,k_2}) = Q_{ND}(P_{m_1,k_1}, T_pP_{m_2,k_2}).$$

This completes the proof of

$$Q(T_pP_{m_1,k_1}, P_{m_2,k_2}) = Q(P_{m_1,k_1}, T_pP_{m_2,k_2})$$

for each T_p, p is a prime. □

For case (b), we need consider $Q(P_{m_1,k_1}, P_{h,m_2})$ and analyze the self-adjointness with Hecke operator in this case. Using the formula of $< Op(P_{m,k})\phi_j, \phi_j >$ which we just evaluated above and the formula of $< Op(P_{h,m})\phi_j, \phi_j >$ in [54], we have

$$\begin{aligned}
&Q(P_{m_1,k_1}, P_{h,m_2})\\
&= \sum_{\frac{m_1}{d_1}=\frac{m_2}{d_2}} \int_0^\infty \int_0^1 \cos(\frac{\pi m_1}{d_1}\xi(2\tau-1)) \exp\left(\frac{-m_1\xi\sqrt{\tau(1-\tau)}}{d_1}\right)(\tau(1-\tau))^{k_1} d\tau\\
&\int_0^1 \frac{\cos(\frac{\pi m_2}{d_2}\xi(2\eta-1)) h\left(\frac{\xi\sqrt{\eta(1-\eta)}}{d_2}\right)}{\eta(1-\eta)} d\eta \frac{\xi^{k_1} d\xi}{\xi^2} +\\
&\sum_{\substack{d_1|m_1\\ d_2|m_2}} \sum_{c\geq 1} \Im(\frac{S_c\zeta_8}{c^{\frac{3}{2}}} \int\int_{\mathrm{R}^2} e_c(\frac{m_1m_2}{2d_1d_2} - \frac{m_1^2\xi}{4d_1^2\phi} - \frac{m_2^2\phi}{4d_2^2\xi}) \frac{2^{\frac{3}{2}}}{(\xi\phi)^{\frac{3}{2}}}\\
&e((d_1d_2)^2\xi\phi c) \int_0^1 \int_0^1 \frac{\cos(\pi m_1 d_2\xi(2\tau-1))\cos(\pi m_2 d_1\phi(2\eta-1))(\tau(1-\tau))^{k_1}}{\eta(1-\eta)}\\
&\exp(-m_1\xi d_2\sqrt{\tau(1-\tau)}) h(\phi d_1\sqrt{\eta(1-\eta)}) d\tau d\eta d\xi d\phi)
\end{aligned}$$

Note that $P_{h,m}$ is a weight 0 Poincare serie and under the Hecke operator, we have

$$T_nP_{h,m}(z) = \sum_{d|(m,n)} (\frac{d^2}{n})^{\frac{1}{2}} P_{h(\frac{ny}{d^2}),\frac{mn}{d^2}}(z).$$

A similar argument about the self-adjointness with respect to Hecke operator works for $Q(P_{m_1,k_1}, P_{h,m_2})$, i.e.

$$Q(T_pP_{m_1,k_1}, P_{h,m_2}) = Q(P_{m_1,k_1}, T_pP_{h,m_2}).$$

For case (c) of ϕ_j and ϕ_k both being Maass forms, it was shown in [54]. Thus, combining these three cases, the Hermitian form $Q(\cdot,\cdot)$ defined on the space spanned by $P_{m,k}$'s is self-adjoint with respect to the Hecke operators T_n, $n \geq 1$. Hence, for the generating vectors ϕ_j, ϕ_k of each irreducible representation, we obtain

Proposition 3.

$$Q(T_n\phi_j, \phi_k) = Q(\phi_j, T_n\phi_k)$$

if ϕ_j, ϕ_k is either weight k holomorphic form or Maass form.

From this, we have

$$\lambda_n(\phi_j)Q(\phi_j, \phi_k) = \lambda_n(\phi_k)Q(\phi_j, \phi_k).$$

Since there is an n such that $\lambda_n(\phi_j) \neq \lambda_n(\phi_k)$ if ϕ_j, ϕ_k are generator vectors of two distinct irreducible representations, we deduce the orthogonality, $Q(\phi_j, \phi_k) = 0$ if ϕ_j, ϕ_k are in distinct eigenspaces of the orthogonal decomposition (1).

In the next section we calculate the eigenvalue of B on such a generating Maass-Hecke cusp form.

4. Eigenvalue of Q

In this section, we shall evaluate the weighted quantum variance on each eigenspace $U_{\pi_j^k}$ by applying Woodbury's explicit formula for the Ichino's trilinear formula with special vectors (see Appendix A), Rankin-Selberg theory, Kuznetsov formula and a principle observed in Luo-Rudnick-Sarnak (Remark 1.4.3 and Prop. 3.1 in [32]).

Proposition 4. *For weight k holomorphic Hecke eigenform f, we have*

$$\lim_{T\to\infty} \frac{1}{T} \sum_{j\geq 1} u\left(\frac{t_j}{T}\right) L(1, \mathrm{sym}^2\varphi_j)|\omega_j(f)|^2 = 2^{k-1}\frac{\Gamma^2(\frac{k}{2})}{\Gamma(k)} L(\frac{1}{2}, f).$$

Proof. Let $\Lambda(s, \varphi_j)$ be the associated completed L-function of φ_j, which admits analytic continuation to the whole complex plane and satisfies the functional equation:

$$\Lambda(s, \varphi_j) := \pi^{-s}\Gamma\left(\frac{s+it_\phi}{2}\right)\Gamma\left(\frac{s-it_\phi}{2}\right)L(s, \varphi_j) = \Lambda(1-s, \varphi_j).$$

Moreover, we have

$$\Lambda(s, \mathrm{sym}^2(\varphi_j) = \pi^{-3s/2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+it_\phi\right)\Gamma\left(\frac{s}{2}-it_\phi\right)L(s, \mathrm{sym}^2\varphi_j).$$

For weight k holomorphic Hecke eigenform f, we have the associated completed L-function,

$$\Lambda(s, f) := \pi^{-s}\Gamma\left(\frac{s+\frac{k-1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{k+1}{2}}{2}\right)L(s, f).$$

Thus, we obtain the Rankin-Selberg L-function,

$$\begin{aligned}\Lambda(s, f\otimes \mathrm{sym}^2\varphi_j) &= \pi^{-3s}\Gamma\left(\frac{s+\frac{k-1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{k-1}{2}}{2}+it_j\right)\Gamma\left(\frac{s+\frac{k-1}{2}}{2}-it_j\right)\\ &\quad \Gamma\left(\frac{s+\frac{k+1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{k+1}{2}}{2}+it_j\right)\Gamma\left(\frac{s+\frac{k+1}{2}}{2}-it_j\right)\\ &\quad L(s, f\otimes \mathrm{sym}^2\varphi_j),\end{aligned}$$

By Ichino's general trilinear formula [18] and its explication in the Appendix with the explicit vectors at hand, we can express the triple product integrals of eigenforms in terms of the Rankin-Selberg L-function $\Lambda(s, f\otimes \mathrm{sym}^2\varphi_j)$as follows;

$$\begin{aligned}| < Op(f)\varphi_j, \varphi_j > |^2 &= \frac{1}{2^3}\cdot \zeta_{\mathbb{R}}(2)^2\cdot \frac{\Lambda(\frac{1}{2}, f\otimes\varphi_j\otimes\varphi_j)}{\Lambda(1, \mathrm{sym}^2\varphi_j)^2\Lambda(1, \mathrm{sym}^2 f)}\\ &= \frac{1}{2^3}\cdot \zeta_{\mathbb{R}}(2)^2\cdot \frac{\Lambda(\frac{1}{2}, f\otimes \mathrm{sym}^2\varphi_j)\Lambda(\frac{1}{2}, f)}{\Lambda(1, \mathrm{sym}^2\varphi_j)^2\Lambda(1, \mathrm{sym}^2 f))}\end{aligned}$$

where $\zeta_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$. The local factors at ∞ place is (Lemma 3 in Woodbury's calculation),

$$\zeta_{\mathbb{R}}(2)^2\cdot\frac{L_\infty(\frac{1}{2}, f\otimes\varphi_j\otimes\varphi_j)}{L_\infty(1, \mathrm{sym}^2\varphi_j)^2 L_\infty(1, \mathrm{sym}^2 f)} = \frac{|\Gamma(\frac{k}{2}+2it_j)|^2|\Gamma(\frac{k}{2})|^2}{2^{k-3}\pi^{k+1}\Gamma(k)|\Gamma(\frac{1}{2}+it_j)|^4}$$

By Stirling formula and the duplication formula of the Gamma factors, it amounts to

$$\begin{aligned}&| < Op(f)\varphi_j, \varphi_j > |^2\\ =\ &\frac{L(\frac{1}{2}, f)L(\frac{1}{2}, f\otimes \mathrm{sym}^2(\varphi_j))\cosh(\pi t_j)|\Gamma(\frac{k}{2})|^2|a_j(1)|^2}{2^k\pi^{k+1}L(1, \mathrm{sym}^2\varphi_j)L(1, \mathrm{sym}^2 f)}(1+O(t_j^{-1}))\end{aligned}$$

Next we apply the approximate functional equation of $L(s, f\otimes\mathrm{sym}^2\varphi_j)$, and Kutznetsov formula to evaluate the variance sum in the Propostition. We compute

$$\sum_{j\geq 1} u\left(\frac{t_j}{T}\right) L(1, \mathrm{sym}^2\phi_j)| < Op(f)\varphi_j, \varphi_j > |^2$$

Let Φ be the cuspidal automorphic form on $GL(3)$ which is the Gelbart-Jacquet lift of the cusp form ϕ, with the Fourier coefficients $a_\Phi(m_1, m_2)$ [5], where

$$a_\Phi(m_1, m_2) = \sum_{d|(m_1, m_2)} \lambda_\Phi(\frac{m_1}{d}, 1)\lambda_\Phi(\frac{m_2}{d}, 1)\mu(d),$$

and

$$\lambda_\Phi(r,1)=\sum_{s^2t=r}\lambda_\phi(t^2).$$

The Rankin-Selberg convolution $L(s,f\otimes\mathrm{sym}^2\varphi_j)$ is represented by the Dirichlet series,

$$L(s,f\otimes\mathrm{sym}^2\varphi_j)=\sum_{m_1,m_2\geq 1}\lambda_f(m_1)a_{\Phi_j}(m_1,m_2)(m_1m_2^2)^{-s},$$

where $\lambda_f(r)$ is the r-th Hecke eigenvalue of f.
Since

$$\Lambda(1/2,f\otimes\mathrm{sym}^2\varphi)=\frac{1}{\pi i}\int_{(2)}\Lambda(s+1/2,f\otimes\mathrm{sym}^2\varphi)\frac{ds}{s}.$$

we have the following approximate functional equation,

$$L(1/2,f\otimes\mathrm{sym}^2\varphi_j)=2\sum_{m_1,m_2\geq 1}\lambda_f(m_1)a_{\Phi_j}(m_1,m_2)(m_1m_2^2)^{-1/2}V(\frac{m_1m_2^2}{t_j^2})$$

where

$$V(y)=\frac{1}{2\pi i}\int_{(2)}y^{-s}\frac{\gamma(1/2+s,f\otimes\mathrm{sym}^2\varphi_j)}{\gamma(1/2,f\otimes\mathrm{sym}^2\varphi_j)}\frac{ds}{s},$$

$$\begin{aligned}\gamma(s,f\otimes\mathrm{sym}^2\varphi_j) &= \pi^{-3s}\Gamma\left(\frac{s+\frac{k-1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{k-1}{2}}{2}+it_j\right)\Gamma\left(\frac{s+\frac{k-1}{2}}{2}-it_j\right)\\ &\quad \Gamma\left(\frac{s+\frac{k+1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{k+1}{2}}{2}+it_j\right)\Gamma\left(\frac{s+\frac{k+1}{2}}{2}-it_j\right)\end{aligned}$$

Thus, by writing

$$\Gamma(\frac{k+1}{2}+it)=\Gamma(\frac{1}{2}+it)(\frac{1}{2}+it)_{\frac{k}{2}}$$

and duplication formula of Gamma functions, we have

$$\begin{aligned}
&\sum_{j\geq 1} u\left(\frac{t_j}{T}\right) L(1,\mathrm{sym}^2\varphi_j)| < Op(f)\varphi_j,\varphi_j > |^2 \\
= \quad & 2^{-k}\pi^{-1-k}L(\frac{1}{2},f)|\Gamma(\frac{k}{2})|^2 \sum_{t_j\geq 1} u\left(\frac{t_j}{T}\right)|a_j(1)|^2 L(1/2, f\otimes \mathrm{sym}^2(\phi_j)) \\
= \quad & 2^{-k}\pi^{-1-k}L(\frac{1}{2},f)|\Gamma(\frac{k}{2})|^2 \sum_{t_j\geq 1} u\left(\frac{t_j}{T}\right)|a_j(1)|^2 \\
& \sum_{m_1,m_2\geq 1} \lambda_f(m_1) a_{\Phi_j}(m_1,m_2)(m_1m_2^2)^{-1/2} V(\frac{m_1m_2^2}{t_j^2}) \\
= \quad & 2^{-k}\pi^{-1-k}L(\frac{1}{2},f)|\Gamma(\frac{k}{2})|^2 \sum_{t_j\geq 1}\sum_{d\geq 1}\frac{\mu(d)}{d^{\frac{3}{2}}}\sum_{n_1,n_2\geq 1}\lambda_f(dn_1)V(\frac{d^3n_1n_2^2}{t_j^2})(n_1n_2^2)^{-1/2} \\
& u\left(\frac{t_j}{T}\right)|a_j(1)|^2 \lambda_{\Phi_j}(n_1,1)\lambda_{\Phi_j}(n_2,1) \\
= \quad & 2^{-k}\pi^{-1-k}L(\frac{1}{2},f)|\Gamma(\frac{k}{2})|^2 \sum_{t_j\geq 1}\sum_{d\geq 1}\frac{\mu(d)}{d^{\frac{3}{2}}}\sum_{s_1,s_2,n_1,n_2\geq 1}\lambda_f(ds_1^2t_1)V(\frac{d^3s_1^2t_1s_2^4t_2^2}{t_j^2})(s_1^2t_1s_2^4t_2^2)^{-1/2} \\
& u\left(\frac{t_j}{T}\right)|a_j(1)|^2 \lambda_j(t_1^2)\lambda_j(t_2^2) \\
= \quad & 2^{-k}\pi^{-1-k}L(\frac{1}{2},f)|\Gamma(\frac{k}{2})|^2 \sum_{d\geq 1}\frac{\mu(d)}{d^{\frac{3}{2}}}\sum_{s_1,s_2,n_1,n_2\geq 1}\lambda_f(ds_1^2t_1)(s_1^2t_1s_2^4t_2^2)^{-1/2} \\
& \sum_{t_j\geq 1} V(\frac{d^3s_1^2t_1s_2^4t_2^2}{t_j^2})u\left(\frac{t_j}{T}\right)|a_j(1)|^2\lambda_j(t_1^2)\lambda_j(t_2^2)
\end{aligned}$$

For the inner sum, by the Kuznetsov formula, we have

$$\begin{aligned}
&\sum_{t_j\geq 1} V(\frac{d^3s_1^2t_1s_2^4t_2^2}{t_j^2})u\left(\frac{t_j}{T}\right)|a_j(1)|^2\lambda_j(t_1^2)\lambda_j(t_2^2) \\
= \quad & \frac{\delta(t_1,t_2)}{\pi^2}\int_{-\infty}^{\infty} V(\frac{d^3s_1^2s_2^4t_1t_2^2}{t^2})u\left(\frac{t}{T}\right)\tanh(\pi t)dt \\
& -\frac{2}{\pi}\int_0^{\infty} V(\frac{d^3s_1^2t_1s_2^4t_2^2}{t^2})\frac{u\left(\frac{t}{T}\right)}{|\zeta(1+2it)|^2}d_{it}(t_1^2)d_{it}(t_2^2)dt \\
& +\sum_{c\geq 1}\frac{S(t_1^2,t_2^2;c)}{c}\int_{-\infty}^{\infty} J_{2it}(\frac{4\pi t_1t_2}{c})V(\frac{d^3s_1^2s_2^4t_1t_2^2}{t^2})u\left(\frac{t}{T}\right)\frac{dt}{\cosh(\pi t)}
\end{aligned}$$

We will estimate the above three sums respectively.

The diagonal term is

$$2^{-k}\pi^{-1-k}L(\frac{1}{2},f)|\Gamma(\frac{k}{2})|^2\int_{-\infty}^{\infty}u\left(\frac{t}{T}\right)\sum_{s_2\geq 1}s_2^{-2}\sum_{s_1\geq 1}s_1^{-1}\lambda_f(s_1^2)V(\frac{s_1^2s_2^4}{T^2})\tanh(\pi t)dt$$

For the sum over s_1, we have

$$\sum_{s_1\geq 1}s_1^{-1}\lambda_f(s_1^2)V(\frac{s_1^2s_2^4}{T^2})=\frac{1}{2\pi i}\int_{(2)}\sum_{s_1\geq 1}\frac{\lambda_f(s_1^2)}{s_1^{2s+1}}U_t(s)(\frac{s_2^4}{T^2})^{-s}\frac{ds}{s},$$

where

$$U_t(s)=(1+P_t(s))\frac{\Gamma\left(\frac{s+\frac{k+1}{2}}{2}\right)\Gamma\left(\frac{s+\frac{k-1}{2}}{2}\right)}{\Gamma\left(\frac{k}{4}\right)\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)},$$

and

$$P_t(s)=\sum_{1\leq r\leq N}\frac{p_{r+1}(s)}{t^r}+O(\frac{|s|^{N+1}}{t^N})$$

is an analytic function in $\mathcal{R}s\geq -2$. $p_{r+1}(s)$ is a polynomial of degree at most $r+1$.
Also, we have

$$\sum_{s_1\geq 1}\frac{\lambda_f(s_1^2)}{s_1^s}=\frac{1}{\zeta(2s)}L(s,\mathrm{sym}^2f).$$

Thus, moving the line of integration in the sum over s_1 to $\mathcal{R}(s)=-1/4+\epsilon$, we get

$$\sum_{s_1\geq 1}s_1^{-1}\lambda_f(s_1^2)V(\frac{s_1^2s_2^4}{T^2})=\frac{1}{\zeta(2)}L(1,\mathrm{sym}^2f)+O(T^{-1/2+\epsilon}).$$

Therefore, we get the diagonal terms contribute

$$2^{-k}\pi^{-k-1}TL(1,\mathrm{sym}^2f)L(\frac{1}{2},f)|\Gamma(\frac{k}{2})|^2+O(T^{1/2+\epsilon}).$$

For the non-diagonal terms

$$\sum_{d\geq 1}\frac{\mu(d)}{d^{\frac{3}{2}}}\sum_{s_1,s_2,t_1,t_2\geq 1}\lambda_f(ds_1^2t_1)(s_1^2t_1s_2^4t_2^2)^{-1/2}\sum_{c\geq 1}\frac{S(t_1^2,t_2^2;c)}{c}$$
$$\int_{-\infty}^{\infty}J_{2it}(\frac{4\pi t_1t_2}{c})V(\frac{d^3s_1^2t_1s_2^4t_2^2}{t^2})u\left(\frac{t}{T}\right)\frac{dt}{\cosh(\pi t)}$$

Let $x=\frac{4\pi t_1t_2}{c}$, the inner integral in the non-diagonal terms is

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{J_{2it}(x)-J_{-2it}(x)}{\sinh\pi t}V(\frac{d^3s_1^2t_1s_2^4t_2^2}{t^2})u\left(\frac{t}{T}\right)\tanh(\pi t)dt.$$

Since $\tanh(\pi t)$ =sgn$(t)+O(e^{-\pi|t|})$ for large $|t|$, we can remove $\tanh(\pi t)$ by getting a negligible term $O(T^{-N})$ for any $N > 0$. Applying the Parseval identity, the Fourier transform in [3],

$$\left(\widehat{\frac{J_{2it}(x) - J_{-2it}(x)}{\sinh(\pi t)}}\right)(y) = -i\cos(x\cosh(\pi y)).$$

and the evaluation of the Fresnel integrals, the integral is

$$\begin{aligned}
&\frac{1}{2}\int_{-\infty}^{\infty}(\frac{J_{2it}(x) - J_{-2it}(x)}{\sinh \pi t})^\wedge(y)(V(\frac{d^3 s_1^2 t_1 s_2^4 t_2^2}{t^2})u\left(\frac{t}{T}\right))^\wedge(y)dy \\
= &\frac{-i}{2}\int_{-\infty}^{\infty}(\cos(x\cosh(\pi y)))(V(\frac{d^3 s_1^2 t_1 s_2^4 t_2^2}{t^2})u\left(\frac{t}{T}\right))^\wedge(y)dy \\
= &\frac{-i}{2}\int_{-\infty}^{\infty}(\cos(x+\frac{1}{2}\pi^2 xy^2))(V(\frac{d^3 s_1^2 t_1 s_2^4 t_2^2}{t^2})u\left(\frac{t}{T}\right))^\wedge(y)dy \\
= &\frac{-i}{2}\int_{0}^{\infty}(\cos(x-y+\frac{\pi}{4}))(V(\frac{d^3 s_1^2 t_1 s_2^4 t_2^2}{t^2})u\left(\frac{t}{T}\right))(\sqrt{\frac{xy}{2}})\frac{dy}{\sqrt{\pi y}} \\
= &\frac{-i}{2}\int_{0}^{\infty}(\cos(x-y+\frac{\pi}{4}))(V(\frac{4d^3 s_1^2 t_1 s_2^4 t_2^2}{xy})u\left(\frac{\sqrt{\frac{xy}{2}}}{T}\right)\frac{dy}{\sqrt{\pi y}} \\
= &\frac{-i}{2}\int_{0}^{\infty}(\cos(4\pi t_1 t_2 c^{-1}-y+\frac{\pi}{4}))(V(\frac{4d^3 s_1^2 t_1 s_2^4 t_2^2}{4\pi t_1 t_2 c^{-1} y})u\left(\frac{\sqrt{\frac{4\pi t_1 t_2 c^{-1} y}{2}}}{T}\right)\frac{dy}{\sqrt{\pi y}}
\end{aligned}$$

Here all the equation is up to an error of $O(T^{-N})$. Thus, the non-diagonal terms is concentrated on

$$T^2 - T^{2-\epsilon} \ll t_1 t_2 c^{-1} y \ll T^2.$$

So, we can assume $d^3 s_1^2 t_1 s_2^4 t_2^2 \ll T^{2+\epsilon}$ since $V(\xi)$ has exponential decay as $\xi \to \infty$. By partial integration, the terms with $c \gg T^\epsilon$ and $t_1 t_2 \ll T^{2-4\epsilon}$ contribute O(1). So we can assume $c \ll T^\epsilon$ and $t_1 t_2 \gg T^{2-4\epsilon}$, therefore we have $t_2 \ll T^{5\epsilon}$, also we have the sum over s_1 and s_2 converges. Let $t = \frac{\sqrt{2\pi t_1 t_2 c^{-1} y}}{T}$, the inner integral is

$$\frac{T\sqrt{c}}{2\pi t_1 t_2}\int_0^\infty u(t)(\cos(4\pi t_1 t_2 c^{-1} - (tT)^2 c/(2\pi t_1 t_2) + \frac{\pi}{4}))V(\frac{4d^3 s_1^2 t_1 s_2^4 t_2^2}{t^2 T^2})dt$$

From Hecke's bound

$$\sum_{r\le R}\lambda_f(r) r^{-1/2} \ll_\epsilon R^\epsilon,$$

where $\alpha \in \mathbf{R}$ and the Hecke relation

$$\lambda_f(r_1 r_2) = \sum_{d|(r_1,r_2)} \mu(d)\lambda_f(r_1/d)(r_2/d);$$

and partial summation, we get the non-diagonal terms contribute $O(T^{5\epsilon})$.

To evaluate the continuous part, we need rewrite

$$\sum_{d\geq 1} \frac{\mu(d)}{d^{\frac{3}{2}}} \sum_{s_1,s_2,t_1,t_2\geq 1} \lambda_f(ds_1^2 t_1)(s_1^2 t_1 s_2^4 t_2^2)^{-1/2}$$
$$\int_0^\infty V(\frac{d^3 s_1^2 t_1 s_2^4 t_2^2}{t^2}) \frac{u\left(\frac{t}{T}\right)}{t|\zeta(1+2it)|^2} d_{it}(t_1^2) d_{it}(t_2^2) dt$$

with respect to L-function and we obtain the continuous part contributes

$$\int_0^\infty u\left(\frac{t}{T}\right) \frac{1}{|\zeta(1+2it)|^2} |L(\frac{1}{2}+it,f)|^2$$
$$\frac{|\Gamma(\frac{1}{4}-\frac{it}{2}-it)\Gamma(\frac{1}{4}+\frac{it}{2}-it)|^2}{|\Gamma(\frac{1}{2}+it)|^4} dt$$

By Stirling formula and the Jutila's bound the subconvex bound

$$L(\tfrac{1}{2}+it, f_j) \ll (\kappa_j + t)^{1/3+\epsilon},$$

we obtain the continuous part contributes $O(T^{\frac{1}{2}+\epsilon})$.

So we conclude that

$$\sum_{j\geq 1} u\left(\frac{t_j}{T}\right) L(1,\mathrm{sym}^2\phi_j)| < Op(f)\varphi_j,\varphi_j > |^2$$
$$= \frac{1}{2^k \pi^{k+1}} T L(1,\mathrm{sym}^2 f) L(\frac{1}{2},f) |\Gamma(\frac{k}{2})|^2 + O(T^{1/2+\epsilon}). \tag{44}$$

Since we normalize f, such that $< f,f >= 1$ and from the fact

$$< f,f >= 2^{1-2k}\pi^{-k-1}\Gamma(k)L(1,\mathrm{sym}^2 f),$$

we obtain the eigenvalue of B at f is

$$L(\frac{1}{2},f)\frac{2^{k-1}|\Gamma(\frac{k}{2})|^2}{\Gamma(k)}.$$

Therefore, we complete the proof of the Proposition 6.

Moreover from [54], we have the following weighted quantum variance for Maass forms,

Proposition 5. *Let $\phi(z)$ be an even Maass-Hecke cuspidal eigenform for Γ, with the Laplacian eigenvalue $\lambda_\phi = \frac{1}{4} + t_\phi^2$, we have*

$$\lim_{T\to\infty} \frac{1}{T} \sum_{j\geq 1} u\left(\frac{t_j}{T}\right) L(1, \mathrm{sym}^2\phi) | < Op(\phi)\varphi_j, \varphi_j > |^2 = L(\frac{1}{2}, \phi) \frac{|\Gamma(\frac{1}{4} - \frac{it_\phi}{2})|^4}{2\pi |\Gamma(\frac{1}{2} - it_\psi)|^2}.$$

Next, we will remove the weights in Proposition 4 and Proposition 5.

5. Removing the Weights

By a simple approximation argument, we can take $u(t)$ in Proposition 4 and Proposition 5 be the characteristic function of an interval, thus we obtain

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t_j \leq T} L(1, \mathrm{sym}^2\varphi_j) |\omega_j(f)|^2 = 2^{k-1} \frac{\Gamma^2(\frac{k}{2})}{\Gamma(k)} L(\frac{1}{2}, f). \tag{45}$$

and

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t_j \leq T} L(1, \mathrm{sym}^2\phi) | < Op(\phi)\varphi_j, \varphi_j > |^2$$

$$= L(\frac{1}{2}, \phi) \frac{|\Gamma(\frac{1}{4} - \frac{it_\phi}{2})|^4}{2\pi |\Gamma(\frac{1}{2} - it_\psi)|^2}. \tag{46}$$

On both left hand sides, the arithmetic weight $L(1, \mathrm{sym}^2\cdot)$ was necessary since we are using Kutznetsov formula. These special values $L(1, \mathrm{sym}^2\cdot)$ do not have much effect since we have the following effective bounds due to Iwaniec and Hoffstein-Lockhart, for any $\epsilon > 0$:

$$\lambda_j^{-\epsilon} \ll_\epsilon L(1, \mathrm{sym}^2\phi_j) \ll_\epsilon \lambda_j^\epsilon.$$

We remove the weights using the mollifier technique as in Iwaniec-Luo-Sarnak [23], Kowalski-Michel [28] and Luo [35].

The symmetric square L-function of ϕ_j is the Dirichlet series $L(s, \mathrm{sym}^2\phi_j)$ defined by

$$L(s, \mathrm{sym}^2\phi_j) = \zeta(2s) \sum_{n\geq 1} \lambda_j(n^2) n^{-s}.$$

We denote $\rho_j(n)$ as the coefficients of this Dirichlet series and we have the following properties of the coefficients $\rho_j(n)$.

Lemma 1. *For any $n \geq 1$, we have*

$$\rho_j(n) = \sum_{ml^2 = n} \lambda_j(m^2),$$

$$\lambda_j(n^2) = \sum_{ml^2=n} \mu(l)\rho_j(m).$$

in particular, $\rho_j(n) = \lambda_j(n^2)$ *if* n *is square-free. The degree 3 L-function* $L(s, \mathrm{sym}^2\phi_j)$ *has Euler product,*

$$L(s, \mathrm{sym}^2\phi_j) = \prod_p (1 - \alpha_p^2 p^{-s})^{-1}(1 - p^{-s})^{-1}(1 - \alpha_p^{-2}p^{-s})^{-1}$$

By Deligne's bound $|\lambda_j(n)| \le \tau(n)$*, we have*

$$|\rho_j(n)| \le \tau(n)^2.$$

Now we define a summation symbol $\sum^h$ by

$$\sum_{t_j}{}^h \alpha_j = \sum_{t_j} \frac{1}{L(1, \mathrm{sym}^2\phi_j)}\alpha_j.$$

Suppose there is a family $\alpha = (\alpha_j)$ of complex numbers for all ϕ_j, if we know the information of the weighted sum $\sum^h \alpha_j$, such as an asymptotic formula in our case, we expect the same formula for the natural unweighted sum $\sum \alpha_j$. One can write the unweighted average as a weighted one with the weight $L(1, \mathrm{sym}^2\phi_j)$, then replace the value of the symmetric square by a short partial sum, say length T^δ of the Dirichlet series.

We approach this by letting $y < T^{10}$, by the approximate functional equation of $L(s, \mathrm{sym}^2\phi_j)$, we have

$$L(1, \mathrm{sym}^2\phi_j) = \sum_{n \le y} \rho_j(n)n^{-1} + O(y^{-1}). \tag{47}$$

with an absolute implied constant. Let $x < y$, we decompose the partial sum as

$$\sum_{n \le y} \rho_j(n)n^{-1} = \sum_{n \le x} \rho_j(n)n^{-1} + \sum_{x < n \le y} \rho_j(n)n^{-1}$$

Now, consider the weighted average from the tail $x < n \le y$, i.e., $\sum^h (\sum_{x<n\le y} \rho_j(n)n^{-1})\alpha_j$. By Holder's inequality, we can separate the sums on the partial sum and α_j. For the weighted sum from the partial sum on $x < n \le y$, it can be estimated by the following lemmas.

Lemma 2. *If* r *is an integer such that* $x^r \ge T^{11}$*, then there is a constant* C*, such that*

$$\sum_{t_j \le T} \Big(\sum_{x<n\le y} \rho_j(n)n^{-1} \Big)^{2r} \ll (\log T)^C.$$

This can be proven by several other lemmas:

Lemma 3. *For $r \geq 1$, we have*

$$(\sum_{x<n\leq y} \rho_j(n)n^{-1})^r = \sum_{x^r<mn\leq y^r} \lambda_j(m^2)\frac{c(m,n)}{mn}.$$

with $c(m,n) = 0$ unless n can be written as $n = dn_1$, with $d|m$, n_1 square-full (if $p|n_1$, then $p^2|n_1$). Moreover there exists $\beta > 0$ such that $c(m,n) \leq \tau(mn)^\beta$.

This can be shown by expanding the formula in Lemma 1 and induction.

Lemma 4. *Let $z \geq 1$, there exists A depending on r, such that*

$$\sum_{x^r<mn\leq y^r, n>z} \lambda_j(m^2)\frac{c(m,n)}{mn} = O(z^{-\frac{1}{2}}(\log TZ)^A).$$

This lemma can be proved by the Deligne's bound on λ_j, Lemma 3 and the following observation

$$\sum_{\text{Square-full} \quad n>z} \frac{1}{n} \ll z^{-\frac{1}{2}}.$$

Lemma 5. *There exists a real number M such that $x^r z^{-1} < M \leq y^r z$ and $c(m)$ and B such that*

$$\begin{aligned}&\sum_{t_j\leq T}(\sum_{x<n\leq y} \rho_j(n)n^{-1})^{2r}\\ \ll\ &(\log Tz)^B \sum_{t_j\leq T} |\sum_{m\sim M} \lambda_j(m^2)\frac{c(m)}{m}|^2 + O(Tz^{-\frac{1}{2}}(\log Tz)^B)\end{aligned}$$

By Lemma 3 and Lemma 4,

$$\begin{aligned}&(\sum_{x<n\leq y} \rho_j(n)n^{-1})^{2r}\\ =\ &\sum_{n\leq z} |\sum_{x^r<mn\leq y^r, n>z} \lambda_j(m^2)\frac{c(m,n)}{mn}| + O(Tz^{-\frac{1}{2}}(\log Tz)^B)\end{aligned}$$

Then dyadic divide the interval $x^r < mn \leq y^r$ and apply Cauchy's inequality, this lemma follows.

Now, let $z = T^2$, so we have $M \geq T^9$, then we can apply the mean value estimate obtained from a property of almost orthogonality of the coefficients of the symmetric square L-functions of the Hecke-Maass forms [22].

Lemma 6. *For $M \geq T^9$ and $a(n) \ll \frac{(\tau(n)\log n)^A}{n}$, there exists a constant D such that*

$$\sum_{t_j \leq T} |\sum_{n \sim M} a(n)\lambda_j(n^2)|^2 \ll (\log M)^D$$

So, by replacing the weight $L(1, \mathrm{sym}^2\phi_j)$ with the short Dirichlet series of length T^δ, for the weight from the partial sum $x = T^\delta < n \leq y < T^{10}$, use Holder's inequality with $(2r)^{-1} + s^{-1} = 1$, where r satisfies $x^r \geq T^{11}$, we have the tail from $x < n \leq y$ contributes $O(T^{-\alpha})$ for $\alpha > 0$, precisely we have

Proposition 6. *There exists an absolute constant $\alpha > 0$, such that*

$$\sum_{t_j \leq T} | < Op(f)\varphi_j, \varphi_j > |^2 = \sum_{t_j \leq T} L(1, \mathrm{sym}^2\phi_j)| < Op(f)\varphi_j, \varphi_j > |^2 + O(T^{1-\alpha})$$

Since

$$< f, f >= 2^{1-2k}\pi^{-k-1}\Gamma(k)L(1, \mathrm{sym}^2 f),$$

we obtain the eigenvalue of Q at f is

$$L(\frac{1}{2}, f)\frac{2^{k-1}|\Gamma(\frac{k}{2})|^2}{\Gamma(k)}.$$

□

Similarly, we remove the arithmetic weight in (40) and have the following

Proposition 7. *Let $\phi(z)$ be an even Maass-Hecke cuspidal eigenform for Γ, with the Laplacian eigenvalue $\lambda_\phi = \frac{1}{4} + t_\phi^2$, we have*

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t_j \leq T} | < Op(\phi)\varphi_j, \varphi_j > |^2 = L(\frac{1}{2}, \phi)\frac{|\Gamma(\frac{1}{4} - \frac{it_\phi}{2})|^4}{2\pi|\Gamma(\frac{1}{2} - it_\psi)|^2}.$$

Hence, combining Propositions 4, 6 and 7, we obtain Theorem 2.

Appendix A. A triple product calculation for $\mathrm{GL}_2(\mathbb{R})$ by Michael Woodbury

Let F be a number field and $\mathbb{A} = \mathbb{A}_F$ the ring of adeles. Let T be the subgroup of GL_2 consisting of diagonal matrices with $Z \subseteq T$ the center. Let $N \subseteq \mathrm{GL}_2$ be the subgroup of upper triangle unipotent matrices so that $P = TN$ the standard Borel.

Given automorphic representations π_1, π_2, π_3 of GL_2 over F such that the product of the central characters is trivial, one can consider the so-called triple product L-function $L(s, \Pi)$ attached to $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$, or the completed L-function $\Lambda(s, \Pi)$. This L-function is closely related to periods of the form

$$I(\varphi) = \int_{[\mathrm{GL}_2]} \varphi_1(g)\varphi_2(g)\varphi(g)dg$$

where $\varphi = \varphi_1 \otimes \varphi_2 \otimes \varphi_3$ with $\varphi_i \in \pi_i$, and $[\mathrm{GL}_2] = \mathbb{A}^\times \mathrm{GL}_2(F)\backslash GL_2(\mathbb{A})$.

One example of this relationship arises in the case that π_1 and π_2 are cupsidal and π_3 is an Eisenstein series. Then $L(s, \Pi)$ is the Rankin-Selberg L-function $L(s, \pi_1 \times \pi_2)$, and for appropriately chosen φ_3, the period I gives an integral representation. Another example occurs when all three representations are cuspidal. In this case, formulas for $L(s, \Pi)$ have been given by Garrett[11], Gross-Kudla[14], Harris-Kudla[15], Watson[49] and Ichino[18].

Let us write $\pi_i = \times_v \pi_{i,v}$ as a (restricted) tensor product over the places v of F, with each $\pi_{i,v}$ an admissible representation of $\mathrm{GL}_2(F_v)$. Let $\langle \cdot, \cdot \rangle_v$ be a (Hermitian) form on π_i. Then, assuming that $\varphi_i = \otimes \varphi_{i,v}$ is factorizable[2], for each v we can consider the matrix coefficient

$$I'(\varphi_v) = \int_{\mathrm{PGL}_2(F_v)} \langle \pi_v(g_v)\varphi_{1,v}, \varphi_{1,v}\rangle_v \langle \pi_v(g_v)\varphi_{2,v}, \varphi_{2,v}\rangle_v \langle \pi_v(g_v)\varphi_{3,v}, \varphi_{3,v}\rangle_v dg_v,$$

and the normalized matrix coefficient

$$I_v(\varphi_v) = \zeta_{F_v}(2)^{-2} \frac{L_v(1, \Pi_v, \mathrm{Ad})}{L_v(1/2, \Pi_v)} I'_v(\varphi_v). \tag{48}$$

When each of the representations π_i is cuspidal, Ichino proved in [18] that there is a constant C (depending only on the choice of measures) such that

$$\frac{|I(\varphi)|^2}{\prod_{j=1}^3 \int_{[\mathrm{GL}_2]} |\varphi_j(g)|^2 \, dg} = \frac{C}{2^3} \cdot \zeta_F(2)^2 \cdot \frac{\Lambda(1/2, \Pi)}{\Lambda(1, \Pi, \mathrm{Ad})} \prod_v \frac{I_v(\varphi_v)}{\langle \varphi_v, \varphi_v \rangle_v} \tag{49}$$

whenever the denominators are nonzero. We remark that, due to the choice of normalizations, the product on the right hand side of (49) is in fact a finite product over some number of "bad" places.

While Ichino's formula is extremely general, for number theoretic applications it is often important to understand well the bad factors. For example, subconvexity for the triple product L-function as proved

[2]As a restricted tensor product, we have chosen vectors $\varphi_{i,v}^0 \in \pi_v$ for almost all places v. We require that the local inner forms must satisfy $\langle \varphi_{i,v}^0, \varphi_{i,v}^0 \rangle_v = 1$ for almost all such v.

by Bernstein-Reznikov in [4] and Venkatesh [48] used, in the former case, Watson's formula from [49] or, in the latter, the present author's paper [50].

In this appendix we calculate I_v in the case that $v \mid \infty$ is a real place, $\pi_{1,v} = \pi^k_{\mathrm{dis}}$ is the discrete series representation of (even) weight k, and $\pi_{v,2} = \pi_{it_2}$ and $\pi_{3,v} = \pi_{it_3}$ are principal series representations where $\pi_{it} = \mathrm{Ind}_P^G(|\cdot|^{it} \otimes |\cdot|^{-it})$ is obtained as the normalized induction of the character

$$|\cdot|^{it} \otimes |\cdot|^{-it} : T(\mathbb{R}) \to \mathbb{C}.$$

Recall that if $f \in \pi_{it}$ then

$$f(\left(\begin{smallmatrix} u & 0 \\ 0 & u \end{smallmatrix}\right)\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right) g) = |y|^{\frac{1}{2}+it} f(g)$$

for all $g \in \mathrm{GL}_2(\mathbb{R})$.

Remark 3. *If π_{it} corresponds to the archimedean component of the automorphic representation associated to a Maass form f of eigenvalue λ under the Laplacian, then $\lambda = \frac{1}{4} + t^2$.*

Let

$$K = O(2) \supseteq \mathrm{SO}(2) = \left\{ \kappa_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \middle| \theta \in \mathbb{R} \right\}.$$

Recall that a function $f_i \in \pi_i$ is said to have *weight* m if $f_i(g\kappa_\theta) = f_i(g)e^{im\theta}$ for all $g \in \mathrm{GL}_2(\mathbb{R})$. As is well known, for each $m \in \mathbb{Z}$ the subspace of π_i consisting of functions of weight m is at most 1-dimensional.

Theorem 3. *Let $f_1 \in \pi^k_{\mathrm{dis}}$ be the vector of weight k, let $f_2 \in \pi_{it_2}$ be the vector of weight zero, and let $f_3 \in \pi_{it_3}$ be the vector of weight $-k$ (each normalized[3] so that $f_i(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)) = 1$.) Then*

$$\begin{aligned} (50) \quad I'_v(f_1 \otimes f_2 \otimes f_3) &= \frac{4\pi}{(k-1)!(\frac{1}{2}+it_3)_{\frac{k}{2}}(\frac{1}{2}-it_3)_{\frac{k}{2}}} \\ &\times \frac{\Gamma(\frac{k}{2}+it_2+it_3)\Gamma(\frac{k}{2}+it_2-it_3)\Gamma(\frac{k}{2}-it_2-it_3)\Gamma(\frac{k}{2}-it_2+it_3)}{\Gamma(\frac{1}{2}+it_2)\Gamma(\frac{1}{2}-it_2)\Gamma(\frac{1}{2}+it_3)\Gamma(\frac{1}{2}-it_3)} \end{aligned}$$

and

$$(51) \qquad I_v(f_1 \otimes f_2 \otimes f_3) = \frac{2^{k-1}\pi^k}{(\frac{1}{2}+it_3)_{\frac{k}{2}}(\frac{1}{2}-it_3)_{\frac{k}{2}}}.$$

where $(z)_m = z(z-1)\cdots(z-m+1)$.

[3] This normalization ensures that $\langle f_i, f_i \rangle = 1$.

A.1. **Real local factors.** For the remainder of this note, we work locally over a real place. Since the place v is assumed fixed, we remove subscripts from the associated L-functions. We trust that no confusion will arise between these and the global L-function considered above. (For example, $L(s,\Pi)$, to be defined below, represents the local L-factor $L_v(s,\Pi)$ appearing in equation (48).)

We will assume, however, that the discrete series π_{it} is unitary. (This is automatically true if π_{it} is the local component of an automorphic representation.) This implies that t is real or that t purely imaginary of absolute value less than 1/2. This requirement will be used implicitly to guarantee that certain integrals converge and that certain functions are real valued. We will use this facts without further mention.

We record the relevant local factors for representations of $\mathrm{GL}_2(\mathbb{R})$. Let

$$\Gamma_{\mathbb{R}}(s)=\pi^{-s/2}\Gamma(s/2), \quad \text{and} \quad \Gamma_{\mathbb{C}}(s)=\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1)=2(2\pi)^{-s}\Gamma(s)$$

where $\Gamma(s)=\int_0^\infty y^s e^{-y} d^\times y$ when $\mathrm{Re}(s)>0$ and is extended by analytic continuation elsewhere. Note that

$$\Gamma_{\mathbb{R}}(1)=1, \qquad \Gamma_{\mathbb{R}}(2)=\frac{1}{\pi}, \qquad \text{and} \qquad \Gamma_{\mathbb{C}}(m)=\frac{(m-1)!}{2^{m-1}\pi^m}. \tag{52}$$

We recall basic facts about the local Langlands correspondence for $\mathrm{GL}_2(\mathbb{R})$ as found in Knapp [27]. The Weil group $W_{\mathbb{R}}=\mathbb{C}^\times \cup j\mathbb{C}^2$ where $j^2=-1$ and $jzj^{-1}=\bar{z}$ for $z\in\mathbb{C}^\times$. The irreducible representations of $W_{\mathbb{R}}$ are all either 1-dimensional or 2-dimensional. The 1-dimensional representations are parametrized by $\delta\in\{0,1\}$ and $t\in\mathbb{C}$:

$$\rho_1(\delta,t):\ \begin{array}{c} z\mapsto |z|^t \\ j\mapsto(-1)^\delta. \end{array}$$

The irreducible 2-dimensional representations are parametrized by positive integers m and $t\in\mathbb{C}$:

$$\rho_2(m,t):\ \begin{array}{c} re^{i\theta}\mapsto\begin{pmatrix} r^{2t}e^{im\theta} & 0 \\ 0 & r^{2t}e^{-im\theta}\end{pmatrix} \\ \\ j\mapsto\begin{pmatrix} 0 & (-1)^m \\ 1 & 0\end{pmatrix}\end{array}$$

Defining $\rho_2(0,t)=\rho_1(0,t)\oplus\rho_1(1,t)$ and $\rho_2(m,t)=\rho_2(|m|,t)$, the following is an elementary exercise.

Lemma 7. *Every (semisimple) finite dimensional representation of $W_{\mathbb{R}}$ is a direct sum of irreducibles each of dimension one or two. Under the operations of direct sum and tensor product, the following is a* *complete set of relations.*

$$\begin{gathered}\rho_2(m,t)\simeq\rho_2(-m,t)\\ \rho_2(0,t)\simeq\rho_1(0,t)\oplus\rho_1(1,t)\\ \rho_1(\delta_1,t_1)\otimes\rho_1(\delta_2,t_2)\simeq\rho_1(\delta,t_1+t_2)\\ \rho_1(\delta,t_1)\otimes\rho_2(m,t_2)\simeq\rho_2(m,t_1+t_2)\\ \rho_2(m_1,t_1)\otimes\rho_2(m_2,t_2)\simeq\rho_2(m_1+m_2,t_1+t_2)\oplus\rho_2(m_1-m_2,t_1+t_2)\end{gathered}$$

In the third line, $\delta=\delta_1+\delta_2 \pmod 2$. *Moreover, if* $\widetilde{\rho}$ *denotes the contragradient of* ρ *then*

$$\widetilde{\rho_1(\delta,t)}\simeq\rho_1(\delta,-t),\quad \textit{and}\quad \widetilde{\rho_2(m,t)}\simeq\rho_1(m,-t).$$

Attached to each irreducible representation ρ of $W_{\mathbb{R}}$ is an L-factor

$$L(s,\rho_1(\delta,t))=\Gamma_{\mathbb{R}}(s+t+\delta),\quad\text{and}\quad L(s,\rho_2(m,t))=\Gamma_{\mathbb{C}}(s+t+\frac{m}{2}).$$

Writing a general representation ρ as a direct sum of irreducibles $\rho_1\oplus\cdots\oplus\rho_r$, we define

$$L(s,\rho)=\prod_{i=1}^{r}L(s,\rho_i).$$

In particular, given ρ, the adjoint representation is

$$\mathrm{Ad}(\rho)\simeq\rho\otimes\widetilde{\rho}\ominus\rho_1(0,0)$$

since $\rho_1(0,0)$ is the trivial representation.

Under the Langlands correspondence, admissible representations π of $\mathrm{GL}_2(\mathbb{R})$ correspond to 2-dimensional representations $\rho=\rho(\pi)$ of $W_{\mathbb{R}}$. For example, $\rho(\pi_{it})=\rho_1(0,it)\oplus\rho_1(0,-it)$ and $\rho(\pi^k_{\mathrm{dis}})=\rho_2(0,k-1)$. Thus the local factors for the discrete series and principal series representations are

$$L(s,\pi^k_{\mathrm{dis}})=\Gamma_{\mathbb{C}}(s+(k-1)/2),\quad\text{and}\quad L(s,\pi_{it})=\Gamma_{\mathbb{R}}(s+it)\Gamma_{\mathbb{R}}(s-it).$$

We define

$$L(s,\Pi)=L(s,\rho(\pi^k_{\mathrm{dis}})\otimes\rho(\pi_{it_2})\otimes\rho(\pi_{it_3}))$$

and

$$L(s,\Pi,\mathrm{Ad})=L(s,\mathrm{Ad}\,\rho(\pi^k_{\mathrm{dis}})\oplus\mathrm{Ad}\,\rho(\pi_{it_2})\oplus\mathrm{Ad}\,\rho(\pi_{it_3})).$$

Lemma 8. *Let $\Pi = \pi_{\text{dis}}^k \otimes \pi_{it_2} \otimes \pi_{it_3}$. The normalizing factor relating I_v and I_v' in (48) at a real place v is*

$$\frac{L(1, \Pi_v, \mathrm{Ad})}{\Gamma_{\mathbb{R}}(2)^2 L(1/2, \Pi_v)} = 2^{k-3}\pi^{k-1}(k-1)! \\ \frac{\Gamma(\frac{1}{2}+it_2)\Gamma(\frac{1}{2}-it_2)\Gamma(\frac{1}{2}+it_3)\Gamma(\frac{1}{2}-it_3)}{\Gamma(\frac{k}{2}+it_2+it_3)\Gamma(\frac{k}{2}-it_2+it_3)\Gamma(\frac{k}{2}+it_2-it_3)\Gamma(\frac{k}{2}-it_2-it_3)}.$$

Proof. Using Lemma 7, one can easily show that

$$\begin{aligned} L(1/2, \Pi) = & \prod\nolimits_{\varepsilon, \varepsilon' \in \{\pm 1\}} \Gamma_{\mathbb{C}}\left(\varepsilon it_2 + \varepsilon' it_3 + \tfrac{k}{2}\right) \\ = & \; 2^4 (2\pi)^{-2k} \prod\nolimits_{\varepsilon, \varepsilon' \in \{\pm 1\}} \Gamma\left(\tfrac{k}{2} + \varepsilon it_2 + \varepsilon' it_3\right) \end{aligned}$$

and, applying (52), $L(1, \Pi, \mathrm{Ad})$ is equal to

$$\big(\Gamma_{\mathbb{C}}(k)\Gamma_{\mathbb{R}}(2)\big)\big(\Gamma_{\mathbb{R}}(1+2it_2)\Gamma_{\mathbb{R}}(1-2it_2)\Gamma_{\mathbb{R}}(1)\big)\big(\Gamma_{\mathbb{R}}(1+2it_3)\Gamma_{\mathbb{R}}(1-2it_3)\Gamma_{\mathbb{R}}(1)\big) \\ = \frac{(k-1)!}{2^{k-1}\pi^{k+3}}\Gamma\left(\frac{1}{2}+it_2\right)\Gamma\left(\frac{1}{2}-it_2\right)\Gamma\left(\frac{1}{2}+it_3\right)\Gamma\left(\frac{1}{2}-it_3\right).$$

Combining these, we arrive at the desired formula. □

A.2. **Whittaker models.** As a matter of notation, set

$$a(y) = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}, \quad z(u) = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \quad n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Let π be an infinite dimensional representation of G with central character ω and $\psi : \mathbb{R} \to \mathbb{C}^\times$ a nontrivial additive character. Then there is a unique space of functions $\mathcal{W}(\pi, \psi)$ isomorphic to π such that

$$W(z(u)n(x)g) = \omega(u)\psi(x)W(g) \tag{53}$$

for all $g \in G$. Recall that the inner product on $\mathcal{W}(\pi, \psi)$ is given by

$$\langle W, W' \rangle = \int_{\mathbb{R}^\times} W(a(y))\overline{W'(a(y))}d^\times y.$$

We fix $\psi : \mathbb{R} \to \mathbb{C}^\times$ once and for all to be the character $\psi(x) = e^{2\pi ix}$.

If the central character of π is trivial, and $W \in \mathcal{W}(\pi, \psi)$ has weight k, (53) becomes

$$W(z(u)n(x)a(y)\kappa_\theta) = e^{2\pi ix}W(a(y))e^{im\theta}. \tag{54}$$

This, by the Iwasawa decomposition, determines W completely provided we can describe $w(y) = W(a(y))$. This can be accomplished for the weight k vector $W_k^k \in \mathcal{W}(\pi_{\text{dis}}^k, \psi)$ by utilizing the fact that W_k^k is annihilated by the lowering operator $X^- \in \mathrm{Lie}(\mathrm{GL}_2(\mathbb{R}))$. Applying X^- to (54), one finds that $w(y)$ satisfies a certain differential equation whose solution is easily obtained. The unique solution with moderate growth is, up to a constant,

$$W_k^k(a(y)) = \begin{cases} y^{k/2}e^{-2\pi y} & \text{if } y \geq 0 \\ 0 & \text{if } y < 0. \end{cases} \tag{55}$$

We calculate directly that

$$\begin{aligned} \int_0^\infty W_k^k(a(y))W_{k'}^{k'}(a(y))y^{s-1}d^\times s &= \int_0^\infty y^{(k+k')/2}e^{-4\pi y}d^\times y \\ &= \frac{\Gamma(s-1+(k+k')/2)}{(4\pi)^{s-1+(k+k')/2}} \end{aligned} \tag{56}$$

By letting $s = 1$ and $k = k'$, this implies that

$$\langle W_k^k, W_k^k \rangle = \frac{(k-1)!}{(4\pi)^k}. \tag{57}$$

Analogously, if $W_m^\lambda \in \mathcal{W}(\pi_{it}, \psi)$ is a weight m-vector which is an eigenvector for the action of the Laplace operator Δ of eigenvalue λ, one can apply Δ to (53) to see that $w(y) = W_m^\lambda(a(y))$ satisfies the confluent geometric differential equation

$$w'' + \left[-\frac{1}{4} + \frac{m}{2y} + \frac{\lambda}{y^2}\right] w = 0. \tag{58}$$

Therefore, $W_m^\lambda(a(y)) = W_{\frac{m}{2},it}(|y|)$ is the unique solution of (58) with exponential decay as $|y| \to \infty$ and $\lambda = \frac{1}{2}+t^2$. (See ...) The weight zero vector W_0^λ can be expressed in terms of the incomplete Bessel function:

$$W_0^\lambda(a(y)) = W_{0,it}(y) = 2\pi^{-1/2}\,|y|^{1/2}\,K_{it}(2\pi\,|y|). \tag{59}$$

By formula (6.8.48) of [8], it follows that

$$\begin{aligned} &\int_0^\infty W_{0,it_1}(a(y))W_{0,it_2}(a(y))y^{s-1}d^\times y \\ &\qquad = \frac{4}{\pi}\int_0^\infty K_{it_1}(2\pi y)K_{it_2}(2\pi y)y^s d^\times y \\ &\qquad = \frac{1}{2\pi^{s+1}} \frac{\Gamma(\frac{s+it_1+it_2}{2})\Gamma(\frac{s-it_1+it_2}{2})\Gamma(\frac{s+it_1-it_2}{2})\Gamma(\frac{s-it_1-it_2}{2})}{\Gamma(s)}. \end{aligned} \tag{60}$$

Evaluating this at $s = 1$ in the case that $t_1 = t_2 = t$, we have that

$$\langle W_0^\lambda, W_0^\lambda \rangle = \frac{\Gamma(\frac{1}{2}+it)\Gamma(\frac{1}{2}-it)}{\pi}. \tag{61}$$

Note that we have used that $W_0^\lambda(a(y))$ is an even function and $\Gamma(1/2) = \sqrt{\pi}$.

Remark 4. *An explicit intertwining map $\pi \to \mathcal{W}(\pi, \psi)$ is given, when the integral is convergent, by*

$$f \mapsto W_f \qquad W_f(g) = \pi^{-1/2} \int_{\mathbb{R}} f(wn(x)g)\psi(x)dx \tag{62}$$

where $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and this can be extended by analytic continuation elsewhere.

As an alternative to the strategy above, one can deduce equations (56) *and* (60) *by working directly from* (62)*. (See* [10]*.) The normalization in* (59) *coincides with this choice of intertwiner.*

A.3. **Proof of Theorem 3.** We are now in a position to prove Theorem 3. Having laid the groundwork above, it is a simple consequence of the following result due to Michel-Venkatesh [38].

Lemma 9 (Michel-Venkatesh)**.** *Let π_1, π_2, π_3 be tempered representations of $\mathrm{GL}_2(\mathbb{R})$ with π_3 a principal series. Fixing an isometry $\pi_i \to \mathcal{W}(\pi_i, \psi)$ for $i = 1, 2$ we may associate for $f_i \in \pi_i$ vectors W_i in the Whittaker model. Then the form $\ell_{\mathrm{RS}} : \pi_1 \otimes \pi_2 \otimes \pi_3 \to \mathbb{C}$ given by*

$$\begin{aligned} &\ell_{\mathrm{RS}}(f_1 \otimes f_2 \otimes f_3) \\ &\qquad = \int_K \int_{\mathbb{R}^\times} W_1(a(y)\kappa)W_2(a(y)\kappa)f_3(a(y)\kappa)\, |y|^{-1}\, d^\times y d\kappa \end{aligned} \tag{63}$$

satisfies $|\ell_{\mathrm{RS}}|^2 = I'(f_1 \otimes f_2 \otimes f_3)$

For $i = 1, 2$ we have $\lambda_i = \frac{1}{4} + t_i^2$. Recall our choice of test functions: $W_1 = W_k^k$, $W_2 = W_0^{\lambda_2}$, and $f_3 \in \pi_{it_3}$ of weight $-k$. Since the sum of the weights of these is zero, the integral over K in (63) is trivial, and

$$\begin{aligned} \ell_{\mathrm{RS}}(W_1 \otimes W_2 \otimes f_3) &= \int_0^\infty W_1(a(y))W_2(a(y))f_3(a(y))\, |y|^{-1}\, d^\times y \\ &= \int_0^\infty e^{-2\pi y} y^{k/2} 2\pi^{-1/2} y^{1/2} K_{it_2}(2\pi y) y^{1/2+it_3} y^{-1} d^\times y \\ &= 2\pi^{-1/2} \int_0^\infty e^{-2\pi y} K_{it_2}(2\pi y) y^{k/2+it_3} d^\times y \\ &= \frac{2}{(4\pi)^{k/2+it_3}} \frac{\Gamma(\frac{k}{2} + it_2 + it_3)\Gamma(\frac{k}{2} - it_2 + it_3)}{\Gamma(\frac{1}{2} + \frac{k}{2} + it_3)} \end{aligned}$$

In the final line we have used equation (6.8.28) from [8]. This simplifies further by using the identity $\Gamma(z + m) = \Gamma(z)(z)_m$.

Recall that we have chosen f_i such that $\langle f_i, f_i\rangle = 1$ for each i. Therefore, in order to apply Lemma 9, we must normalize ℓ_{RS}:

$$\begin{aligned} I'(f_1\otimes f_2\otimes f_3) &= \frac{|\ell_{\mathrm{RS}}(W_1\otimes W_2\otimes f_3)|^2}{\langle W_1, W_2\rangle\langle W_2, W_2\rangle} \\ &= \frac{4\pi}{(k-1)!(\frac{1}{2}-it_3)_{\frac{k}{2}}(\frac{1}{2}+it_3)_{\frac{k}{2}}}\times \\ \times \frac{\Gamma(\frac{k}{2}+it_2+it_3)\Gamma(\frac{k}{2}+it_2-it_3)\Gamma(\frac{k}{2}-it_2-it_3)\Gamma(\frac{k}{2}-it_2+it_3)}{\Gamma(\frac{1}{2}+it_2)\Gamma(\frac{1}{2}-it_2)\Gamma(\frac{1}{2}+it_3)\Gamma(\frac{1}{2}-it_3)} \end{aligned}$$

To complete the proof, we multiply by the normalizing factor of Lemma 8.

Remark 5. *If one or more of the representations π_{it_j} is a complementary series (i.e. if $\lambda_j < \frac{1}{4}$) then the result of Theorem 3 still holds, but the explicit calculation is somewhat different. In this case, it is no longer true that for $r \in \mathbb{R}$*

$$|\Gamma(r+it_j)|^2 = \Gamma(r+it_j)\Gamma(r-it_j),$$

nor is it true that $\langle f_j, f_j\rangle = 1$. Taking into account these differences, however, the final answer ends up agreeing with what has been calculated above.

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