Voevodsky's Lectures on Motivic Cohomology 2000/2001

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Abstract These lectures cover several important topics of motivic homotopy theory which have not been covered elsewhere. These topics include the definition of equivariant motivic homotopy categories, the definition and basic properties of solid and ind-solid sheaves and the proof of the basic properties of the operations of twisted powers and group quotients relative to the $\mathbf{A}^1$-equivalences between ind-solid sheaves.

1 Introduction

The lectures which provided the source for these notes covered several different topics which are related to each other but which do not in any reasonable sense form a coherent whole. As a result, this text is a collection of four parts which refer to each other but otherwise are independent.

In the first part we introduce the motivic homotopy category and connect it with the motivic cohomology theory discussed in [7]. The exposition is a little unusual because we wanted to avoid any references to model structures and still prove the main theorem 2. We were able to do it modulo 6 where we had to refer to the next part.

The second part is about we the motivic homotopy category of G-schemes where G is a finite flat group scheme with respect to an equivariant analog of the Nisnevich topology. Our main result is a description of the class of $\mathbf{A}^1$-equivalences (formerly called $\mathbf{A}^1$-weak equivalences) given in Theorem 4 (also in Theorem 5). For the trivial group G we get a new description of the $\mathbf{A}^1$-equivalences in the non equivariant setting.

In the third part we define a class of sheaves on G-schemes which we call solid sheaves. It contains all representable sheaves and quotients of representable sheaves by subsheaves corresponding to open subschemes. In particular the Thom

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spaces of vector bundles are solid sheaves. The key property of solid sheaves can be expressed by saying that any right exact functor which takes open embeddings to monomorphisms is left exact on solid sheaves. A more precise statement is Theorem 6.

In the fourth part we study two functors. One is the extension to pointed sheaves of the functor from G-schemes to schemes which takes X to X/G. The other one is extension to pointed sheaves of the functor which takes X to X^W where W is a finite flat G-scheme. We show that both functors take solid sheaves to solid sheaves and preserve local and $\mathbf{A}^1$-equivalences between termwise (ind-)solid sheaves.

The material of all the parts of these notes but the first one was originally developed with one particular goal in mind – to extend non-additive functors, such as the symmetric product, from schemes to the motivic homotopy category. More precisely, we were interested in functors given by

$$T : X \mapsto (X^W \times E)/G$$

where G is a finite flat group scheme, W is a finite flat G-scheme and E any G-scheme of finite type. The equivariant motivic homotopy category was introduced to represent T as a composition

$$X \mapsto X^W \mapsto X^W \times E \mapsto (X^W \times E)/G$$

and solid sheaves as a natural class of sheaves on which the derived functor $\mathbf{L}T$ coincides with T.

In the present form these notes are the result of an interactive process which involved all listeners of the lectures. A very special role was played by Pierre Deligne. The text as it is now was completely written by him. He also cleared up a lot of messy parts and simplified the arguments in several important places.

Princeton, NJ, 2001.
Vladimir Voevodsky

2 Motivic Cohomology and Motivic Homotopy Category

We will recall first some of last year results (see [7]).

2.1 Last Year

1.1 We work over a field k which sometimes will have to be assumed to be perfect. The schemes over k we consider will usually be assumes separated and smooth of finite type over k. We note Sm/k their category. Three Grothendieck topologies on Sm/k will be useful: Zariski, Nisnevich and etale. For each of these topologies a

sheaf on (Sm/k) amounts to the data for X smooth over k of a sheaf F_X on the small site X_{Zar} (resp. X_{Nis}, X_{et}) of the open subsets U of X (resp. of $U \to X$ etale), with F_X functorial in X: a map $f : X \to Y$ induces $f^* : f^* F_Y \to F_X$.

1.2 The definition of the motivic cohomology groups of X smooth over k has the following form:

(a) One defines for each $q \in \mathbf{Z}$ a complex of presheaves of abelian groups $\mathbf{Z}(q)$ on Sm/k. It is in fact a complex of sheaves for the etale topology, hence a fortiori for the Nisnevich and Zariski topology. For any abelian group A the same applies to $A(q) := A \otimes \mathbf{Z}(q)$.

(b) The motivic cohomology groups of X with coefficients in A are the hypercohomology groups of the $A(q)$, in the Nisnevich topology:

$$H^{p,q}(X, A) := \mathbf{H}^p(X_{Nis}, A(q))$$

For $A = \mathbf{Z}$ we will write simply $H^{p,q}(X)$.

Motivic cohomology has the following properties:

1. The complex $\mathbf{Z}(q)$ is zero for $q < 0$. For any q it lives in cohomological degree $\leq q$. As a complex of Nisnevich sheaves it is quasi-isomorphic to $\mathbf{Z}$ for $q = 0$ and to $\mathbf{G}_m[-1]$ for $q = 1$.
2. $H^{p,p}(Spec(k)) = K_p^M(k)$ for any $p \geq 0$.
3. For any X in Sm/k one has

$$H^{p,q}(X) = CH^q(X, 2q - p)$$

where $CH^q(X, 2q - p)$ is the $(2q - p)$-th higher Chow group of cycles of codimension q.

4. In the etale topology, for n prime to the characteristic of k, the complex $\mathbf{Z}/n(q)$ is quasi-isomorphic to $\mu_n^{\otimes q}$, giving for the etale analog of $H^{p,q}$ the formula

$$H_{et}^{p,q}(X, \mathbf{Z}/n) := \mathbf{H}^p(X_{et}, \mathbf{Z}/n(q)) = H^p(X_{et}, \mu_n^{\otimes q}).$$

1.3 The category $SmCor(k)$ is the category with objects separated schemes smooth of finite type over k, for which a morphism $Z : X \to Y$ is a cycle $Z = \sum n_i Z_i$ on $X \times Y$ each of whose irreducible components Z_i is finite over X and projects onto a connected component of X. A morphism Z can be thought of as a finitely valued map from X to Y. For $x \in X$, with residue field $k(x)$, it defines a zero cycle $Z(x)$ on $Y_{k(x)}$, and the assumption made on Z implies that the degree of this 0-cycle is locally constant on X.

A morphism of schemes $f : X \to Y$ defines a morphism in $SmCor(k)$: the graph of f. This graph construction defines a faithful functor from Sm/k to the additive category $SmCor(k)$.

A *presheaf with transfers* is a contravariant additive functor from the category $SmCor(k)$ to the category of abelian groups. The embedding of Sm/k in

$SmCor(k)$ allows us to view a presheaf with transfers as a presheaf on Sm/k endowed with an extra structure. A sheaf with transfers (for a given topology on Sm/k, usually the Nisnevich topology) is a presheaf with transfers which, as a presheaf on Sm/k, is a sheaf. The Nisnevich and the etale topologies have the virtue that if F is a presheaf with transfers, the associated sheaf $a(F)$ carries a structure of a sheaf with transfers. This structure is uniquely determined by $F \to a(F)$ being a morphism of presheaves with transfers. For any sheaf with transfers G, one has

$$Hom(a(F), G) \xrightarrow{\sim} Hom(F, G)$$

(Hom of presheaves with transfers). All of this fails for the Zariski topology.

The complexes $\mathbf{Z}(q)$ (or $A(q)$) start life as complexes of sheaves with transfers.

1.4 A presheaf F on Sm/k is called *homotopy invariant* if $F(X) = F(X \times \mathbf{A}^1)$. As the point 0 of $\mathbf{A}^1$ defines a section of the projection of $X \times \mathbf{A}^1$ to X, for any presheaf of abelian groups F, $F(X)$ is naturally a direct factor of $F(X \times \mathbf{A}^1)$; it follows that the condition "homotopy invariant" is stable by kernels, cokernels and extensions of presheaves. The following construction is a derived version of the left adjoint to the inclusion

$$(homotopy \quad invariant \quad presheaves) \subset (all \quad presheaves)$$

(a) For S a finite set, let $A(S)$ be the affine space freely spanned (in the sense of barycentric calculus) by S. Over $\mathbf{C}$, or $\mathbf{R}$, $A(S)$ contains the standard topological simplex spanned by S. The schemes $\Delta^m := A(\{0, \ldots, n\})$ form a cosimplicial scheme.

(b) For F a presheaf, $C_\bullet(F)$ (the "singular complex of F") is the simplicial presheaf $C_n(F) : X \mapsto F(X \times \Delta^n)$.

Arguments imitated from topology show that for F a presheaf of abelian groups, the cohomology presheaves of the complex $C_*(F)$, obtained from $C_\bullet(F)$ by taking alternating sum of the face maps, are homotopy invariant. If F has transfers so do the $C_n(F)$ and hence the $H_n C_*(F)$. A basic theorem proved last year is:

Theorem 1. *Let F be a homotopy invariant presheaf with transfers over a perfect field with the associated Nisnevich sheaf $a_{Nis}(F)$. Then the presheaves with transfers*

$$X \mapsto H^i(X_{Nis}, a_{Nis}(F))$$

are homotopy invariant as well.

The particular case of this theorem for $i = 0$ claims the homotopy invariance of the sheaf with transfers $a_{Nis}(F)$.

Last year, the equivalence of a number of definitions of $\mathbf{Z}(q)$ was proven. Equivalence means: a construction of an isomorphism in a suitable derived category, implying an isomorphism for the corresponding motivic cohomology groups. For our present purpose the most convenient definition is as follows.

Let $\mathbf{Z}_{tr}(X)$ be the sheaf with transfers represented by X (on the category $SmCor(k)$). We set

$$K_q = \begin{cases} 0 & \text{for } q < 0 \\ \mathbf{Z}_{tr}(\mathbf{A}^q)/\mathbf{Z}_{tr}(\mathbf{A}^q - \{0\}) & \text{for } q \geq 0 \end{cases}$$

and $\mathbf{Z}(q) = C_*(K_q)[-q]$.

2.2 Motivic Homotopy Category

The motivic homotopy category $Ho_{\mathbf{A}^1,\bullet}(S)$ (pointed $\mathbf{A}^1$-homotopy category of S), for S a finite dimensional noetherian scheme, will be the category deduced from a category of simplicial sheaves by two successive localizations.[1]

One starts with the category Sm/S of schemes smooth over S, with the Nisnevich topology, and the category of pointed simplicial sheaves on Sm/S. For any site $\mathcal{S}$ (for instance $(Sm/S)_{Nis}$), there is a notion of *local equivalence* of (pointed) simplicial sheaves. It proceeds as follows:

(a) A sheaf G defines a simplicial sheaf G_* with all $G_n = G$ and all simplicial maps the identities. The functor $G \mapsto G_*$ has a left adjoint $F \mapsto \pi_0(F)$:

$$Hom(F_*, G_*) = Hom(\pi_0(F_*), G)$$

The sheaf $\pi_0(F_*)$ can be described as the equalizer of $F_1 \overset{\rightarrow}{\rightarrow} F_0$, as well as the sheaf associated to the presheaf

$$U \mapsto \pi_0(|F_*(U)|)$$

The same holds in the pointed context. We will often write simply G for G_*.

(b) If F_* is a simplicial sheaf, and u a section of F_0 over U, one also disposes of sheaves $\pi_i(F_*, u)$ over U: the sheaves associated to the presheaves

$$V/U \mapsto \pi(|F_*(V)|, u).$$

(c) A morphism $F_* \to G_*$ is a local equivalence, if it induces an isomorphism on π_0 as well as, for any local section u of F_0, an isomorphism on all π_i. This applies also to pointed simplicial sheaves: one just forgets the marked point.

One defines $Ho_\bullet(Sm/S)$ as the category derived from the category of pointed simplicial sheaves on $(Sm/S)_{Nis}$ by formally inverting local equivalences. Until made more concrete, this definition could lead to set-theoretic difficulties, which we leave the reader to solve in its preferred way.

[1] In the Appendix we have assembled the properties of "localization" to be used in this talk and in the next.

For G a pointed sheaf on Sm/S, Proposition 7 applies to G_* and to the localization by local equivalences: one has

$$Hom_{Ho_\bullet}(F_*, G_*) = Hom(F_*, G_*) = Hom(\pi_0(F_*), G) \tag{0.1}$$

Definition 1. An object X of $Ho_\bullet(Sm/S)$ is called $\mathbf{A}^1$-local if for any simplicial sheaf Y, one has

$$Hom_{Ho_\bullet}(Y, X) \xrightarrow{\sim} Hom_{Ho_\bullet}(Y \times \mathbf{A}^1 / * \times \mathbf{A}^1, X)$$

At the right hand side, $/ * \times \mathbf{A}^1$ means that in the product, $* \times \mathbf{A}^1$ is contracted to a point, the new base point.

Proposition 1. *For G a pointed sheaf on Sm/S, the simplicial sheaf G_* is $\mathbf{A}^1$-local if and only if G is homotopy invariant.*

Proof. We have $\pi_0(Y \times \mathbf{A}^1) = \pi_0(Y) \times \mathbf{A}^1$, so that by (0.1) "$\mathbf{A}^1$-local" means that for any pointed sheaf Y, one has

$$Hom(Y, G) = Hom(Y \times \mathbf{A}^1 / * \times \mathbf{A}^1, G)$$

A morphism $Y \to G$ can be viewed as the data, for each $y \in Y(U)$, of $f(y) \in G(U)$, functorial in U and marked point going to marked point. A morphism $g : Y \times \mathbf{A}^1 \to G$ can similarly be described as data for $y \in Y(U)$ of $g(y) \in G(U \times \mathbf{A}^1)$. Homotopy invariance hence implies $\mathbf{A}^1$-locality. The converse is checked by taking for Y the disjoint sum of a representable sheaf and the base point. □

Definition 2. (1) A morphism $f : Y_1 \to Y_2$ in $Ho_\bullet(Sm/S)$ is an $\mathbf{A}^1$-equivalence if for any $\mathbf{A}^1$-local X, one has in $Ho_\bullet(Sm/S)$

$$Hom(Y_2, X) \xrightarrow{\sim} Hom(Y_1, X)$$

(2) The category $Ho_{\mathbf{A}^1,\bullet}(Sm/S)$ is deduced from $Ho_\bullet(Sm/S)$ by formally inverting $\mathbf{A}^1$-equivalences.

Remark 1. If a morphism in $Ho_\bullet(Sm/S)$ becomes an isomorphism in $Ho_{\bullet,\mathbf{A}^1}(Sm/S)$ it is an $\mathbf{A}^1$-equivalence. Indeed, if X in $Ho_\bullet(Sm/S)$ is $\mathbf{A}^1$-local, an application of Proposition 7 shows that for any Y,

$$Hom_{Ho_\bullet}(Y, X) \to Hom_{Ho_{\bullet,\mathbf{A}^1}}(Y, X)$$

is bijective. If $f : Y_1 \to Y_2$ in $Ho_\bullet(Sm/S)$ has an image in $Ho_{\bullet,\mathbf{A}^1}(Sm/S)$ which is an isomorphism, it follows that for any $\mathbf{A}^1$-local X, one has

$$Hom_{Ho_\bullet}(Y_2, X) \xrightarrow{\sim} Hom_{Ho_\bullet}(Y_1, X).$$

Such an f is an $\mathbf{A}^1$-equivalence.

Example 1. Arguments similar to those given before show that if G is a homotopy invariant pointed sheaf, then for any simplicial pointed sheaf F_*, one has

$$Hom_{Ho_{\mathbf{A}^1,\bullet}}(Sm/S)(F_*, G) = Hom(F_*, G) = Hom(\pi_0(F_*), G)$$

in particular, if U is smooth over S and if U_+ is the disjoint union of U and of a base point,

$$Hom_{Ho_{\mathbf{A}^1,\bullet}}(U_+, G) = G(U)$$

2.3 Derived Categories Vs. Homotopy Categories

For any topos T, which for us will be the category of sheaves on some site S, the pointed homotopy category $Ho_\bullet(S)$ as well as the derived category $D(S)$ are obtained by localization. For the derived category, one starts with the category of complexes of abelian sheaves. The subcategory of complexes living in homological degree ≥ 0 is naturally equivalent, by the Dold Puppe construction, to the category of simplicial sheaves of abelian groups. The equivalence is

$$N : (simplicial\ F_*) \mapsto complex(\bigcap_{i \neq 0} ker(\partial_i), \partial_0)$$

We will write K for the inverse equivalence. For S a point, and π an abelian group, $|K(\pi[n])|$ is indeed the Eilenberg–Maclane space $K(\pi, n)$. For a complex C not assumed to live in homological degree ≥ 0, we define

$$K(C) := K(\tau_{\geq 0} C)$$

where $\tau_{\geq n} C$ is the subcomplex in C of the form

$$\ldots C_{n+2} \stackrel{d_{n+1}}{\to} C_{n+1} \to ker(d_n) \to 0$$

Note that $C \mapsto \tau_{\geq 0} C$ is right adjoint to the inclusion functor

$$(\text{complexes in homological degree} \geq 0) \hookrightarrow (\text{all complexes})$$

so that (N, K) form a pair of adjoint functors:

$$(\text{simplicial abelian sheaves}) \rightleftarrows (\text{complexes of sheaves})$$

Theorem 2. *Assume that $S = Spec(k)$ with k perfect. Then, for F a presheaf with transfers, and U_+ as above, and $p \geq 0$*

$$Hom_{Ho_{\mathbf{A}^1,\bullet}}(U_+, K(F[p])) = \mathbf{H}^p(U_{Nis}, C_*(F))$$

In this theorem $K(F[p])$ is the simplicial sheaf of abelian groups whose normalized chain complex is F in homological degree p.

To prove the theorem we establish the chain of equalities,

$$\begin{aligned} \mathbf{H}^p(U_{Nis}, C_*(F)) = Hom_{Ho_\bullet}(U_+, K(C_*(F)[p])) = \\ = Hom_{Ho_{\bullet,\mathbf{A}^1}}(U_+, K(C_*(F)[p])) = Hom_{Ho_{\bullet,\mathbf{A}^1}}(U_+, K(F[p])), \end{aligned} \tag{1.1}$$

the first equality is proved right before Proposition 4, the second right after Proposition 5 and the last one follows from Lemma 1.

Let $forget$ be the forgetting functor from abelian sheaves to sheaves of sets. Its left adjoint is $F \mapsto \mathbf{Z}[F]$: the sheaf associated to the presheaf

$$U \mapsto (\text{abelian group freely generated by } F(U))$$

In the pointed context , the adjoint is

$$(F, *) \mapsto \tilde{\mathbf{Z}}(F) : \mathbf{Z}(F)/\mathbf{Z}(*)$$

We have the same adjunction for (pointed) simplicial objects.

Proposition 2. *On a site with enough points (and presumably always), one has*

(1) The functor $F_* \mapsto N\mathbf{Z}(F_*)$ from pointed simplicial sheaves to complexes of abelian groups transforms local equivalences into quasi-isomorphisms
(2) The right adjoint $C \mapsto forget(K(C))$ transforms quasi-isomorphisms to local equivalences.

The assumption "enough points" applies to Sm/k with the Nisnevich topology: for any U in Sm/k and any point x of U, $F \mapsto F(Spec(\mathcal{O}^h_{U,x}))$ is a point, and they form a conservative system.

Proof. Local equivalence (resp. quasi-isomorphism) can be checked point by point, and the two functors considered commute with passage to the fiber at a point. This reduces our proposition to the case when S is just a point, i.e. to usual homotopy theory. In that case, (1) boils down to the fact that a weak equivalence induces an isomorphism on reduced homology, a theorem of Whitehead, and (1.1) reduces to the fact: for a complex C, $\pi_i(K(C))$, computed using any base point, is $H_i(C)$. The $\pi_i(K(C))$ are easy to compute because $K(C)$ has the Kan property. □

Applying Proposition 8, we deduce from Proposition 2 the following.

Proposition 3. *Under the same assumptions as in Proposition 2, for F_* a pointed simplicial sheaf and C a complex of abelian sheaves, one has*

$$Hom_{Ho_\bullet}(F_*, K(C)) = Hom_D(N\tilde{\mathbf{Z}}(F_*), C) \tag{3.1}$$

Let F be a sheaf and F_+ be deduced from F by adjunction of a base point. We also write F and F_+ for the corresponding "constant" simplicial sheaf. One has

$$N\tilde{\mathbf{Z}}(F_+) = N\mathbf{Z}(F) = (\mathbf{Z}(F) \text{ in degree zero})$$

For the pointed simplicial sheaf F_+, the group $Hom_D(\mathbf{Z}(F), C)$ which now occurs at the right-hand side of (3.1) can be interpreted as hypercohomology of C "over F viewed as a space", i.e. in the topos of sheaves over F. For F defined by an object U of the site S, this is the same as hypercohomology of the site S/U. As we do not want to assume C bounded below (in cohomological numbering), checking this requires a little care.

For a complex of sheaves K over a site S, not necessarily bounded below, $\mathbf{H}^0(S, K)$ can be defined as the Hom group in the derived category $Hom_D(\mathbf{Z}, K)$. For F in a topos T and the topos T/F: "F viewed as a space", besides the morphism of toposes $(T/F) \xrightarrow{j} T$, i.e. the adjoint pair (j^*, j_*), we have for abelian sheaves an adjoint pair $(j_!, j^*)$, with $j_!$ and j^* both exact. By Proposition 8, $(j_!, j^*)$ induce an adjoint pair for the corresponding derived categories. As $j_!\mathbf{Z} = \mathbf{Z}[F]$, we get

$$Hom_D(\mathbf{Z}(F), C) = \mathbf{H}^0(T/F, j^*(C)) \tag{3.2}$$

hence

$$Hom_{Ho_\bullet}(F_+, K(C)) = \mathbf{H}^0(T/F, j^*(C)) \tag{3.3}$$

Let us consider the particular case of Sm/k with the Nisnevich topology. For any complex of sheaves, (3.3) gives for U smooth over k

$$Hom_{Ho_\bullet}(U_+, K(C)) = \mathbf{H}^0(U_{big-Nis}, C) \tag{3.4}$$

Here, $U_{big-Nis}$ is the site $(Sm/S)/U$ with the Nisnevich topology. It has however the same hypercohomology as the small Nisnevich site U_{Nis}. Indeed, one has a morphism $\epsilon : U_{big,Nis} \to U_{Nis}$ and the functors ϵ^* and ϵ_* are exact. One again applies Proposition 8. If we apply (3.4) to a translate (shift) of C, we get

$$Hom_{Ho_\bullet}(U_+, K(C[p])) = \mathbf{H}^p(U_{Nis}, C) \tag{3.5}$$

Applying (3.5) to $C_*(F)$ we get the first equality in (1.1).

Proposition 4. *Let C be a complex of abelian sheaves on Sm/k. The following conditions are equivalent:*

1. *$K(C)$ is $\mathbf{A}^1$-local.*
2. *For $i \leq 0$, the functor $U \mapsto \mathbf{H}^i(U, C)$ is homotopy invariant.*
3. *For any complex L in cohomological degree ≤ 0, one has in the derived category*

$$Hom(L \otimes \mathbf{Z}(\mathbf{A}^1), C) \xrightarrow{\sim} Hom(L, C).$$

Proof. By Proposition 3, condition (1) can be rewritten: for any pointed simplicial sheaf F_*.

$$Hom_D(N\tilde{\mathbf{Z}}(F_*), C) = Hom_D(N\tilde{\mathbf{Z}}(F_* \times \mathbf{A}^1/ * \times \mathbf{A}^1))$$

The operation $F_* \mapsto F_* \times \mathbf{A}^1/ * \times \mathbf{A}^1$ is better written as a smash product $F_* \wedge \mathbf{A}^1_+$ with $\mathbf{A}^1_+$. For pointed sets E and F, $\tilde{\mathbf{Z}}(E \wedge F) = \tilde{\mathbf{Z}}(E) \otimes \tilde{\mathbf{Z}}(F)$. It follows that

$$\tilde{\mathbf{Z}}(F_* \times \mathbf{A}^1/ * \times \mathbf{A}^1) = \tilde{\mathbf{Z}}(F_* \wedge \mathbf{A}^1_+) = \tilde{\mathbf{Z}}(F_*) \otimes \tilde{\mathbf{Z}}(\mathbf{A}^1_+) = \tilde{\mathbf{Z}}(F_*) \otimes \mathbf{Z}(\mathbf{A}^1)$$

(isomorphisms of simplicial sheaves), hence

$$N\tilde{\mathbf{Z}}(F_* \times \mathbf{A}^1/ * \times \mathbf{A}^1) = N\tilde{\mathbf{Z}}(F_*) \otimes \mathbf{Z}(\mathbf{A}^1)$$

It follows that (1) is the particular case of (3) for L of the form $\tilde{\mathbf{Z}}(F_*)$. Similarly, (2) is the particular case of (3) for L of the form $\mathbf{Z}(U)[i]$, with $i \geq 0$.

The suspension $\Sigma^i F_*$ of a simplicial pointed sheaf F_* is its smash product with the simplicial sphere S^i (the i-simplex modulo its boundary). It follows that

$$\tilde{\mathbf{Z}}(\Sigma^i F_*) = \tilde{\mathbf{Z}}(F_*) \otimes \tilde{\mathbf{Z}}(S^i)$$

(isomorphism of simplicial sheaves), and by Eilenberg–Zilber, the normalized complex $N\mathbf{Z}(\Sigma^i F_*)$ is homotopic to the tensor product of the normalized complexes of $\tilde{\mathbf{Z}}(F_*)$ and $\tilde{\mathbf{Z}}(S^i)$. The latter is simply $\mathbf{Z}[i]$:

$$N\tilde{\mathbf{Z}}(\Sigma^i F_*) \cong \tilde{\mathbf{Z}}(F_*)[i]$$

This is just a high-brown way of telling that the reduced homology of a suspension is just a shift of the reduced homology of the space one started with.

Applying this to $F_* = U_+$, one obtains that (1)⇒(3). Indeed, $\tilde{\mathbf{Z}}(\Sigma^i U_+)$ is homotopic to $\tilde{\mathbf{Z}}(U_+)[i] = \mathbf{Z}(U)[i]$.

We now prove that (2)⇒(1). For L a complex, let (*) be the statement that the conclusion of (3) holds for all $L[i]$, $i \geq 0$. The assumption (2) is that (*) holds for L reduced to $\mathbf{Z}(U)$ in degree 0, and we will conclude that it holds for all L in cohomological degree ≤ 0 by "devissage":

(a) The case of a sum of $\mathbf{Z}(U)$, in degree zero, follows from Corollary 10.
(b) Suppose that L is bounded, is in cohomological degree ≤ 0 and that (*) holds for all L^n.

The functors

$$h' : L \mapsto Hom^n(L, C) \tag{3.6}$$
$$h'' : L \mapsto Hom^n(L \otimes \mathbf{Z}(\mathbf{A}^1), C)$$

are contravariant cohomological functors, hence give rise to convergent spectral sequences

$$E_1^{pq} = h^q(L^{-p}) \Rightarrow h^{p+q}(L).$$

One has a morphism of spectral sequences

$$E(\text{for } h') \to E(\text{for } h'')$$

which is an isomorphism for $q \leq 0$, and both E^{pq} vanish for $p < 0$ or p large. It follows that $h'^n(L) \xrightarrow{\sim} h''^n(L)$ for $n \leq 0$, i.e. that L satisfies (*).

The same argument can be expressed as an induction on the number of i such that $L^i \neq 0$. If n is the largest (with $n \leq 0$), the induction assumption applies to $\sigma^{<n}L$, even to $(\sigma^{<n}L)[-1]$, and one concludes by the long exact sequence defined by

$$0 \to L^n[-n] \to L \to \sigma^{<n}L \to 0$$

(c) Expressing L as the inductive limit of the $\sigma^{\geq -n}L$ and using Proposition 11, one sees that we need not assume that L is bounded.

(d) If $L' \to L''$ is a quasi-isomorphism, $L' \otimes \mathbf{Z}(\mathbf{A}^1) \to L'' \otimes \mathbf{Z}(\mathbf{A}^1)$ is one too (flatness of $\mathbf{Z}(\mathbf{A}^1)$), and (*) holds for L' if and only if it holds for L''.

(e) Any abelian sheaf L is a quotient of a direct sum of sheaves $\mathbf{Z}(U)$. For instance, the sum over (U, s), $s \in \Gamma(U, L)$, of $\mathbf{Z}(U)$ mapping to L by s. It follows that L admits a resolution L^* by such sheaves. By (a) and (c), L^* satisfies (*). It follows from (d) that L satisfies (*) and then by (c) that any complex in degree ≤ 0 satisfies (*). □

2.4 Application to Presheaves with Transfers

Let F be a presheaf with transfers. A formal argument [7] shows that the presheaves with transfers $H^i C_*(F)$ are homotopy invariant. By the basic result (Theorem 1) recalled in the first lecture, it follows that for any U, one has

$$\mathbf{H}^*(U, C_*(F)) = \mathbf{H}^*(U \times \mathbf{A}^1, C_*(F)) \tag{4.1}$$

Indeed, as U and $U \times \mathbf{A}^1$ are of finite cohomological dimension, both sides are abutment of a convergent spectral sequence

$$E_2^{pq} = H^p(U, H^q C_*(F)) \Rightarrow \mathbf{H}^{p+q}(U, C_*(F))$$

and the same for $U \times \mathbf{A}^1$. By Theorem 1 applied to $H^q C_*(F)$,

$$H^p(X, H^q C_*(F)) := H^p(X, aH^q C_*(F))$$

is the same for $X = U$ or for $X = U \times \mathbf{A}^1$. Applying (3.5), we conclude from Proposition 4((2)⇒(1)) that

Proposition 5. *For k perfect, if F is a presheaf with transfers, for all p, $K(C_*(F)[p])$ is $\mathbf{A}^1$-local.*

Combining Proposition 5 with Proposition 7 we get the second equality in (1.1).

2.5 End of the Proof of Theorem 2

For any pointed simplicial sheaf $G_\bullet$, $C_\bullet(G_\bullet)$ is a pointed bisimplicial sheaf of which one can take the diagonal $\Delta C_\bullet(G_\bullet)$. For any pointed sheaf G, one has a natural map $G \to C_\bullet(G)$, and for a pointed simplicial sheaf $G_\bullet$, those maps for the G_n induce

$$a : G_\bullet \to \Delta C_\bullet(G_\bullet)$$

Proposition 6. *The morphism $a : G_\bullet \to \Delta C_\bullet(G_\bullet)$ is an $\mathbf{A}^1$-equivalence.*

Proof. We deduce Proposition 6 from Proposition 20.

The two maps $0, 1 : F_\bullet \to F_\bullet \wedge \mathbf{A}^1_+$ are equalized by $F_\bullet \wedge \mathbf{A}^1_+ \to F_\bullet$, hence become equal in the $\mathbf{A}^1$-homotopy category. If two maps of pointed simplicial sheaves $F_\bullet \overset{\to}{\to} G_\bullet$ factor as $F_\bullet \overset{\to}{\to} F_\bullet \wedge \mathbf{A}^1_+ \to G_\bullet$, they also become equal. By the adjunction of $\wedge \mathbf{A}^1_+$ and of $C_1(-) = \underline{Hom}(\mathbf{A}^1_+, -)$, such a factorization can be rewritten as

$$F_\bullet \to C_1(G_\bullet) \to G_\bullet$$

Particular case: the maps $C_1(G_\bullet) \to G_\bullet$, become equal in the homotopy category. Evaluated on X, these maps are the restriction maps $0^*, 1^* : G_\bullet(X \times \mathbf{A}^1) \to G_\bullet(X)$.

The affine space $\mathbf{A}^n$ is homotopic to a point in the sense that $H : \mathbf{A}^1 \times \mathbf{A}^n \to \mathbf{A}^n : (t, x) \mapsto tx$ interpolates between the identity map (for $t = 1$) and the constant map 0 (for $t = 0$). The map H induces

$$G_\bullet(S \times \mathbf{A}^n) \to G_\bullet(S \times \mathbf{A}^n \times \mathbf{A}^1)$$

and, composing with 0, 1 in $\mathbf{A}^1$, we obtain that

$$G_\bullet(S \times \mathbf{A}^n) \overset{\to}{\to} G_\bullet(S \times \mathbf{A}^n)$$

the identity map, and the map induced by $0 : \mathbf{A}^n \to \mathbf{A}^n$, are equal in the $\mathbf{A}^1$-homotopy category. The map of simplicial sheaves $G_\bullet \to C_n G_\bullet$ is hence an $\mathbf{A}^1$-equivalence. It has as inverse in the $\mathbf{A}^1$-homotopy category the map induced by

$0 : Spec(k) \to \mathbf{A}^n$ and one applies Remark 1. We now apply Proposition 20 to the bisimplicial sheaves

$$G_{pq} := G_p$$
$$H_{pq} := C_q G_p : S \mapsto G_p(S \times \Delta^q)$$

and to the natural map $G_{pq} \to H_{pq}$. For fixed q, this is just $G(S) \to G(S \times \mathbf{A}^q)$, and Proposition 20 gives Proposition 6. □

To prove the last equality in (1.1), it suffices to show that:

Lemma 1. *For any abelian sheaf F, $F[p] \to C_*(F)[p]$ induces an $\mathbf{A}^1$-equivalence from $K(F[p])$ to $K(C_*(F)[p])$.*

Proof. For G a monoid (with unit), the pointed simplicial set $B_\bullet G$ is given by

$$B_n G = \left\{ \begin{array}{c} \text{functors from the ordered set } (0, \ldots, n) \text{ viewed as a category} \\ \text{to } G \text{ viewed as a category with one object} \end{array} \right\}$$

This construction can be sheafified, and can be applied termwise to a simplicial sheaf of monoids, leading to a pointed bisimplicial sheaf of which one can take the diagonal

$$BG_\bullet := \Delta B_\bullet(G_\bullet)$$

This construction commutes with the construction $G_\bullet \to \Delta C_\bullet(G_\bullet)$. Indeed, $B_n G_p$ is naturally isomorphic to G_p^n, the operation C_m commutes with products, and $B(\Delta C_\bullet(G_\bullet))$ and $\Delta C_\bullet(BG_\bullet)$ are both diagonals of the trisimplicial pointed sheaf $C_\bullet B_\bullet G_\bullet$.

For abelian simplicial sheaves, the operation B gives again abelian simplicial sheaves, hence can be iterated, and $\Delta C_\bullet$ commutes with B^n.

Via Dold–Puppe construction, B corresponds, up to homotopy, to the shift [1] of complexes:

$$NBG_\bullet \cong (NG_\bullet)[1].$$

This can be viewed as an application of the Eilenberg–Zilber Theorem (see [9, Theorem 8.5.1]): one has

$$NBG_\bullet \cong BG_* \cong Tot B_* G_* \quad \text{(Eilenberg–Zilber)},$$

and for each G_q, the normalization of $B_\bullet G_q$ is just $G_q[1]$, so that the double complexes $B_* G_*$ and $H_{pq} := G_q$ for $p = 1$, 0 otherwise, have homotopic Tot.

If $G_\bullet$ is an abelian simplicial sheaf, applying Proposition 6 to $B^p G_\bullet$, we obtain that

$$B^p G_\bullet \to \Delta C_\bullet B^p G_\bullet = B^p \Delta C_\bullet G_\bullet \tag{6.1}$$

is an $\mathbf{A}^1$-equivalence. The functor K transforms chain homotopy equivalences into simplicial equivalences. For any simplicial abelian group $L_\bullet$ (to be $G_\bullet$ or $\Delta C_\bullet G_\bullet$), we hence have a simplicial homotopy equivalence

$$B^p L_\bullet = KNB^p L_\bullet \cong K((NL_\bullet)[p])$$

Simplicial homotopy equivalences being $\mathbf{A}^1$-equivalences, we conclude that (6.1) induces an $\mathbf{A}^1$-equivalence

$$K((NG_\bullet)[p]) \to K(N(\Delta C_\bullet G_\bullet)[p])$$

□

2.6 Appendix: Localization

Let C be a category and S be a set of morphisms of C. The localized category $C[S^{-1}]$ is deduced from C by "formally inverting all $s \in S$". With this definition, it is clear that one has a natural functor $loc : C \to C[S^{-1}]$, bijective on the set of objects, and that for any category D,

$$F \mapsto F \circ loc : Hom(C[S^{-1}], D) \to Hom(C, D)$$

is a bijection from $Hom(C[S^{-1}], D)$ to the set of functors from C to D transforming morphisms in S into isomorphisms.

If one remembers that the categories form a 2-category, and if one agree with the principle that one should not try to define a category more precisely than up to equivalence (unique up to unique isomorphism), the universal property of $C[S^{-1}]$ given above is doubly unsatisfactory. The easily checked and useful universal property is the following: $F \mapsto F \circ loc$ is an equivalence from the category $Hom(C[S^{-1}], D)$ to the full subcategory of $Hom(C, D)$ consisting of the functors F which map S to isomorphisms.

Proposition 7. *If Y in C is such that the functor*

$$h_Y : C^{op} \to Sets : X \mapsto Hom_C(X, Y)$$

transforms maps in S into bijections, then

$$Hom_C(X, Y) \xrightarrow{\sim} Hom_{C[S^{-1}]}(X, Y)$$

Proof. By Yoneda construction $Y \mapsto h_Y$, C embeds into the category $C^\wedge$ of contravariant functors from C to *Sets*, while $C[S^{-1}]$ embeds into $C[S^{-1}]^\wedge$, identified by (a) with the full subcategory of $C^\wedge$ consisting of F transforming S into bijections. For Y in C, with image $\bar{Y}$ in $C[S^{-1}]$, and for any F in $C(S^{-1})^\wedge \subset C^\wedge$, one has in $C^\wedge$

$$Hom(h_Y, F) = Hom(h_{\bar{Y}}, F).$$

Indeed, by (a) and Yoneda lemma for C and $C[S^{-1}]$ both sides are $F(Y)$. This means that $h_{\bar{Y}}$ is the solution of the universal problem of mapping h_Y into an object of $C[S^{-1}]^\wedge \subset C^\wedge$. In particular, for Y as in (b), i.e. in $C[S^{-1}]^\wedge$, $h_{\bar{Y}}$ coincides with h_Y, as claimed by (b). □

Proposition 8. *Let (L, R) be a pair of adjoint functors between categories C and D. Let S and T be sets of morphisms in C and D. Assume that F maps S to T and that G maps T to S. Then the functors $\bar{L}$, $\bar{R}$ between $C[S^{-1}]$ and $D[T^{-1}]$ induced by L and R again form an adjoint pair.*

Proof. The functors $\bar{L}$ and $\bar{R}$ induced by F and G are characterized by commutative diagrams

$$\begin{array}{ccc} C & \xrightarrow{L} & D \\ \downarrow & & \downarrow \\ C[S^{-1}] & \xrightarrow{\bar{L}} & D[T^{-1}] \end{array} \qquad \begin{array}{ccc} D & \xrightarrow{R} & C \\ \downarrow & & \downarrow \\ D[T^{-1}] & \xrightarrow{\bar{R}} & C[S^{-1}] \end{array}$$

Adjunction can be expressed by the data of $\epsilon : Id \to RL$ and $\eta : LR \to Id$ such that the compositions

$$R \to RLR \to R$$

$$L \to LRL \to L$$

are the identity automorphisms of R and L respectively (see, e.g. [6]).

By the universal property of localization, ϵ induces a morphism $\bar{\epsilon} \to \bar{R}\bar{L}$, indeed, to define such a morphism amounts to defining a morphism $loc \to \bar{R}\bar{L}loc$, and $\bar{R}\bar{L}loc = locRL$. Similarly, η induces $\bar{\eta} : \bar{L}\bar{R} \to Id$. The morphism $\bar{L} \to \bar{L}\bar{R}\bar{L} \to \bar{L}$ is induced by $L \to LRL \to L$, similarly for $\bar{R} \to \bar{R}\bar{L}\bar{R} \to \bar{R}$, and the proposition follows. □

Proposition 9. *Suppose that:*

1. *The localization $C[S^{-1}]$ gives rise to a right calculus of fractions.*
2. *Coproducts exist in C, and S is stable by coproducts.*

Then, a coproduct in C is also a coproduct in $C[S^{-1}]$.

For the definition of "gives rise to a right calculus of fractions" see [10]. It implies that for X in C, the category of $s : X' \to X$ with s in S is filtering, and that

$$Hom_{C[S^{-1}]}(X, Y) = colim_{s:X' \to X} Hom_C(X', Y)$$

Proof. For X in C, let (S/X) be the filtering category of morphisms $X' \to X$ in S. For X the coproduct of X_α, $\alpha \in A$, one has a functor "coproduct":

$$\prod (S/X_\alpha) \to (S/X)$$

It is cofinal: for $s : X' \to X$ in S, one can construct a diagram

$$\begin{array}{ccc} X'_\alpha & \longrightarrow & X_\alpha \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

with $s_\alpha : X'_\alpha \to X$ in S, and $\coprod s_\alpha$ dominates s. For any Y, it follows that

$$Hom_{C[S^{-1}]}(X, Y) = colim_{(S/X)} Hom_C(X', Y) =$$

$$= colim_{\prod (S/X_\alpha)} Hom(\coprod X'_\alpha, Y) = colim_{\prod (S/X_\alpha)} \prod Hom(X'_\alpha, Y) =$$

$$= \prod colim_{S/X_\alpha} Hom(X'_\alpha, Y) = \prod Hom_{C[S^{-1}]}(X_\alpha, Y),$$

meaning that X is also the coproduct of the X_α in $C[S^{-1}]$. □

Corollary 10. *Suppose that in the abelian category A arbitrary direct sums exist and are exact. Then, arbitrary direct sums exist in the derived category $D(A)$, and the localization functor*

$$C(A) \to D(A)$$

commutes with direct sums.

Proof. The functor $C(A) \to D(A)$ factors through the category $K(A)$ of complexes and maps up to homotopy. Direct sums in $C(A)$ are also direct sums in $K(A)$. Indeed,

$$Hom_{K(A)}(\oplus K_\alpha, L) = H^0 Hom^\bullet(\oplus K_\alpha, L) =$$

$$= H^0 \prod Hom^\bullet(K_\alpha, L) = \prod H^0 Hom^\bullet(K_\alpha, L),$$

as $\prod$ is exact for abelian groups. Exactness of $\oplus$ in A ensures that a direct sum of quasi-isomorphisms is again a quasi-isomorphism, and Proposition 9 applies to $K(A)$ and the set S of quasi-isomorphisms, proving the corollary. □

If $A_i, i \geq 0$ is an inductive system of objects of A, the colimit of A_i is the cokernel

$$\oplus A_i \xrightarrow{d} \oplus A_i \to colim\, A_i \to 0$$

of the difference of the identity map and of the sum of the $A_i \to A_{i+1}$. If taking the inductive limit of a sequence is an exact functor, the map d is injective: it is the colimit of the

$$\oplus_{i=0}^{n} A_i \to \oplus_{i=0}^{n+1} A_i$$

each of which is injective, as its graded for the filtration by the $\oplus_{i \geq p} A_i$ is the identity inclusion.

Under the assumptions of Corollary 10 if a complex K is the colimit of an inductive sequence $K_{(i)}$, and if the sequence

$$0 \to \oplus K_{(i)} \stackrel{d}{\to} \oplus K_{(i)} \to K \to 0 \tag{10.1}$$

is exact, then for any L, the long exact sequence of cohomology reads

$$\to Hom(K, L) \to \prod Hom(K_{(i)}, L) \stackrel{d}{\to} \prod Hom(K_{(i)}, L) \to$$

The kernel of d is simply the projective limit of the $Hom(K_{(i)}, L)$. The cokernel is lim^1. One concludes.

Proposition 11. *Suppose that in A countable direct sums exist and are exact. If the complex K is the colimit of the $K_{(i)}$, and if the sequence (10.1) is exact, for instance if either:*

1. *In A inductive limits of sequences are exact.*
2. *In each degree n, each $K^n_{(i)} \to K^n_{(i+1)}$, is the inclusion of a direct factor.*

then, one has a short exact sequence

$$0 \to lim^1 Hom(K_{(i)}, L[-1]) \to Hom(K, L) \to lim Hom(K_{(i)}, L) \to 0$$

Proof. It remains to check that condition (2) implies the exactness of (10.1). This is to be seen degree by degree. By assumption, the $A_i := K^n_{(i)}$, have decompositions compatible with the transition maps $A_i = \oplus^i_{j=0} B_i$. A corresponding decomposition of (10.1) in direct sum follows, and we are reduced to check exactness of the particular case $(10.1)_{(A,n)}$ of (10.1) when $B_i = 0$ for $i \neq n$, i.e. when A_i is a fixed A from $i \geq n$, and is 0 otherwise. Let $(10.1)_0$ be the sequence $(10.1)_{\mathbf{Z},n}$ in Ab for $A = \mathbf{Z}$. It is a split exact sequence of free abelian groups. Because direct sums exist, $L \otimes A$, for L a free abelian group is defined and functorial in L. It is a sum of copies of A, indexed by a basis of L, and is characterized by

$$Hom(L \otimes A, B) = Hom(L, Hom(A, B))$$

(functorial in B). The sequence $(10.1)_{A,n}$ is $(10.1)_0 \otimes A$ and, $(10.1)_0$ being split exact, it splits and in particular is exact. □

The truncation $\sigma_{\leq n}K = \sigma^{\geq -n}K$ of a complex K is the subcomplex which coincides with K in homological degree $\leq n$ and is 0 in homological degree $> n$. For any complex K, one has

$$K = colim \sigma_{\leq n} K$$

and this colimit satisfies condition (2) of Proposition 11. It follows that

Corollary 12. *Under the assumptions of Corollary 10, for any K and L, one has a short exact sequence*

$$0 \to lim^1 Hom(\sigma_{\leq n}K, L[-1]) \to Hom(K, L) \to lim Hom(\sigma_{\leq n}K, L) \to 0$$

3 $\mathbf{A}^1$-Equivalences of Simplicial Sheaves on G-Schemes

3.1 Sheaves on a Site of G-Schemes

We fix a base scheme S, supposed to be separated noetherian and of finite dimension; fiber product $X \times_S Y$ will be written simply as $X \times Y$. We also fix a group scheme G over S, supposed to be finite and flat. We note h_X the representable sheaf defined by X.

Let QP/G be the category of schemes quasi-projective over S, given with an action of G. Any X in QP/G admits an open covering (U_i) by affine open subschemes which are G-stable. This makes it possible to define a reasonable quotient X/G in the category of schemes over S (rather than in the larger category of algebraic spaces). For each U_i, U_i/G is defined as the spectrum of the equalizer

$$\mathcal{O}(U_i) \overset{\to}{\to} \mathcal{O}(U_i \times G),$$

and X/G is obtained by gluing the U_i/G. It is a categorical quotient, i.e. the coequalizer of $G \times X \overset{\to}{\to} X$. The map $X \to X/G$ is finite, open, and the topological space $|X/G|$ is the coequalizer of the map of topological spaces $|G \times X| \overset{\to}{\to} |X|$. One can show that X/G is again quasi-projective. Remark 2 below shows that this fact, while convenient for the exposition, is irrelevant.

One defines on QP/G a pretopology [3, II.1.3] by taking as coverings the family of etale maps $Y_i \to X$ with the following property: X admits a filtration by closed equivariant subschemes $\emptyset = X_n \subset \cdots \subset X_1 \subset X_0 = X$ such that for each j, some map $Y_i \to X$ has a section over $X_j - X_{j+1}$. The *Nisnevich topology* on QP/G is the topology generated by this pretopology. The category QP/G with the Nisnevich topology is the *Nisnevich site* $(QP/G)_{Nis}$.

Remark 1. The corresponding topos is not the classifying topos of [2, IV.2.5]. A morphism $X \to Y$ can become a Nisnevich covering after forgetting the action of G, and not be a Nisnevich covering. Example: $S = Spec(k)$, $G = \mathbf{Z}/2$, $X = S$, $Y = S \coprod S$ and G permutes two copies of S in Y.

Remark 2. Let $(affine/G)_{Nis}$ be the site defined as above, with "quasi-projective" replaced by "affine". It is equivalent to $(QP/G)_{Nis}$, in the sense that restriction to $(affine/G)_{Nis}$ is an equivalence from the category of sheaves on $(QP/G)_{Nis}$ to the category of sheaves on $(affine/G)_{Nis}$.

Remark 3. If G is the trivial group e, the definition given above recovers the usual Nisnevich topology. For $G = e$, the condition usually considered: "every point x of X is the image of a point with the same residue field of some Y_i", is indeed equivalent to the condition imposed above. This is checked by noetherian induction: if a generic point ξ of X can be lifted to Y_i, some open neighborhood $U \subset X$ of ξ can be lifted to Y_i, and one applies the induction hypothesis to $X_1 = (X - U)_{red}$.

We write $(QP)_{Nis}$ for the category of quasi-projective schemes over S, with the Nisnevich topology.

Lemma 2. *If $\mathcal{U} : Y_i \to X \ (i \in I)$ is a covering of X in $(QP/G)_{Nis}$, there is a covering $\mathcal{V}$ of X/G in $(QP)_{Nis}$ whose pull-back to X is finer than $\mathcal{U}$.*

Proof. Fix a filtration $\emptyset = X_n \subset \cdots \subset X_0 = X$ as in the definition of the Nisnevich topology. We write q for the quotient map $X \to X/G$. For x in X/G, $(q^{-1}(x))_{red}$ is in some $X_j - X_{j+1}$, by equivariance of the X_j, and one of the maps $Y_i \to X$ has an equivariant section s over $X_j - X_{j+1}$. Let $(X/G)^h_x$ be the henselization of X/G at x. The map q being finite, the pull-back of $(X/G)^h_x$ to X is the coproduct of the X^h_y for $q(y) = x$. The map from Y_i to X being etale, the section s, restricted to $(q^{-1}(y))_{red}$, extends uniquely to a section (automatically equivariant) of Y_i over $\coprod_{q(y)=x} X^h_y$. Writing $(X/G)^h_x$ as the limit of etale neighborhoods of x, one finds that x has an etale neighborhood $V(x)$ such that Y_i has an equivariant section over $X \times_{X/G} V(x)$. The $V(x)$ form the required covering $\mathcal{V}$. □

We define the G-local henselian schemes to be the schemes Y obtained in the following way. For X in (QP/G), y a point of X/G, and $(X/G)^h_y$ the henselization of X/G at y, take the fiber product $Y := X \times_{X/G} (X/G)^h_y$. As X is finite over X/G, this fiber product is a finite disjoint union of local henselian schemes, and G-local henselian schemes are simply the G-equivariant finite disjoint unions of Y of local henselian schemes, for which Y/G is local.

Proposition 13. *If Y is G-local henselian, the functor $X \mapsto Hom(Y, X)$ is a point of the site $(QP/G)_{Nis}$, i.e. it defines a morphism of the punctual site $(Sets)$ to $(QP/G)_{Nis}$. If $Y = X \times_{X/G} (X/G)^h_y$, the corresponding fiber functor is $F \mapsto colim F(X \times_{X/G} V)$, the colimit being taken over the etale neighborhoods of y in X/G. The collection of fiber functors so obtained is conservative.*

Proof. The functor $X \mapsto Hom(Y, X)$ commutes with finite limits. It follows from Lemma 2 that it transforms coverings into surjective families of maps, hence is a morphism of sites $(Sets) \to (QP/G)_{Nis}$.

To check that the resulting set of fiber functors is conservative, it suffices to check that a family of etale $f_i : U_i \to X$ is a covering if for any G-local henselian Y,

$$\coprod Hom(Y, U_i) \to Hom(Y, X)$$

is onto. The proof, parallel to that of Lemma 2 is left to the reader. □

3.2 The Brown–Gersten Closed Model Structure on Simplicial Sheaves on G-Schemes

We recall that a commutative square of simplicial sets (or pointed simplicial sets)

$$\begin{array}{ccc} K & \longrightarrow & L \\ \downarrow & & \downarrow \\ M & \longrightarrow & N \end{array} \tag{13.1}$$

is *homotopy cartesian* (or a *homotopy pull-back square*) if, when L is replaced by L' weakly equivalent to it and mapping to N by (Kan) fibration: $L \xrightarrow{\cong} L' \to N$, the map from K to $L' \times_N M$ is a weak equivalence.

Definition 3. A simplicial presheaf $F_\bullet$ on $(QP/G)_{Nis}$ is *flasque* if $F(\emptyset)$ is contractible and if for any (upper) distinguished square:

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ A & \xrightarrow{j} & X \end{array}$$

(p – etale, j open embedding, $B = p^{-1}(A)$ and $Y - B \cong X - A$), the square

$$\begin{array}{ccc} F(X) & \longrightarrow & F(Y) \\ \downarrow & & \downarrow p \\ F(A) & \xrightarrow{j} & F(B) \end{array}$$

is homotopy cartesian.

Theorem 3. *Let $f : F_\bullet \to F'_\bullet$ be a morphism of flasque simplicial presheaves. If the induced morphism of simplicial sheaves $aF_\bullet \to aF'_\bullet$ is a local equivalence, then, for any U in QP/G, $F_\bullet(U) \to F'_\bullet(U)$ is a weak equivalence.*

Proof. For a G-scheme X let X_{Nis} be the small Nisnevich site of X and for a presheaf F on (QP/G) let $F_{|X}$ be the restriction of F to X_{Nis}. Our assumption that $aF_\bullet \to aF'_\bullet$ is a local equivalence implies that $aF_{\bullet,|U} \to aF'_{\bullet,|U}$ is a local equivalence. The map $U \to U/G$ defines a morphism of sites $p : U_{Nis} \to (U/G)_{Nis}$ and Lemma 2 implies that the direct image functor p_* commutes with the associated sheaf functor and takes local equivalences to local equivalences. Therefore the morphism $ap_*(F_{\bullet,|U}) \to ap_*(F'_{\bullet,|U})$ is a local equivalence. The presheaves $p_*(F_{\bullet,|U})$ and $p_*(F'_{\bullet,|U})$ are flasque on $(U/G)_{Nis}$ and by [8, Lemma 3.1.18] we conclude that

$$F_\bullet(U) = p_*(F_{\bullet,|U})(U/G) \to p_*(F'_{\bullet,|U})(U/G) = F'_\bullet(U)$$

is a weak equivalence. □

In [1], Brown and Gersten define a simplicial closed model structure on the category of pointed simplicial sheaves on a Noetherian topological space of finite dimension. As in Jardine [11], the equivalences are the local equivalences . The homotopy category is hence the same as Joyal's, but the model structure is different: less cofibrations, more fibrations.

The arguments of [1] work as well in the Nisnevich topology, for the big as well as for the small Nisnevich site, or for $(QP/G)_{Nis}$, once Theorem 3 is available.

We review the basic definitions, working in $(QP/G)_{Nis}$. Let $\Lambda^{n,k}$ be the subsimplicial set of $\partial\Delta^n$, union of all faces but the k-th face. For $n = 0$, $\Lambda^{0,0} = \emptyset$. One takes as *generating trivial cofibrations* the maps of the form (J):

(J_a) $(\Lambda^{n,k} \times h_X)_+ \to (\Delta^n \times h_X)_+$
(J_b) for $U \to X$ an open embedding,

$$(\Delta^n \times h_U \coprod_{\Lambda^{n,k} \times h_U} \Lambda^{n,k} \times h_X)_+ \to (\Delta^n \times h_X)_+$$

One then defines the *fibrations* to be the morphisms p having the right lifting property with respect to generating trivial cofibrations (see, e.g. [5]), the *(weak) equivalences* to be the local equivalences, the trivial fibrations to be fibrations which are also (weak) equivalences, and the *cofibrations* to be the morphisms having the left lifting property with respect to trivial fibrations.

Following [1] and using Theorem 3, one proves that the trivial fibrations can be equivalently described as morphisms having the right lifting property with respect to the following class of morphisms (I):

(I_a) $(\partial\Delta^n \times h_X)_+ \subset (\Delta^n \times h_X)_+$
(I_b) for $U \to X$ open embedding,

$$(\Delta^n \times h_U \coprod_{\partial\Delta^n \times h_U} \partial\Delta^n \times h_X)_+ \to (\Delta^n \times h_X)_+$$

The maps of the form (I) are called *generating cofibrations*.

For X and Y pointed simplicial sheaves, one defines a pointed simplicial set $S(X,Y)$ by

$$S(X,Y)_n = Hom(X \wedge (\Delta^n)_+, Y)$$

Following [1], one sees that the classes of cofibrations, (weak) equivalences, fibrations, and S are a simplicial closed model structure in the sense of []. This has the following consequences.

Corollary 14. *If X is cofibrant and Y fibrant, for any pointed simplicial set K, one has in the relevant homotopy categories*

$$Hom_{Ho}(X \wedge K, Y) = Hom_{Ho}(K, S(X,Y))$$

In particular, taking $k = (\Delta^0)_+$ one gets

$$Hom_{Ho}(X,Y) = \pi_0 S(X,Y)$$

Corollary 15. *If $X \to Y$ is a cofibration and Z a cofibrant object, then $X \wedge Z \to Y \wedge Z$ is a cofibration.*

Corollary 16. *If X is cofibrant and Y is fibrant, then for any Z*

$$Hom_{Ho}(Z, \underline{Hom}(X,Y)) = Hom_{Ho}(Z \wedge X, Y) \tag{16.1}$$

In (16.1), $\underline{Hom}(X,Y)$ is the pointed simplicial sheaf with components the sheaves of homomorphisms from $X \wedge (\Delta^n)_+$ to Y.

We now apply this framework to prove the following criterion for $\mathbf{A}^1$-locality.

Proposition 17. *Let F be a pointed simplicial sheaf on (QP/G). If, as a simplicial presheaf, F is flasque, then F is $\mathbf{A}^1$-local if and only if, for any U in (QP/G),*

$$F(U) \to F(U \times \mathbf{A}^1)$$

is a weak equivalence.

We recall that $\mathbf{A}^1$-*local* means that for any Y one has the following in the homotopy category

$$Hom_{Ho}(Y,F) = Hom_{Ho}(Y \wedge (h_{\mathbf{A}^1})_+, F) \tag{17.1}$$

Lemma 3. *A fibrant pointed simplicial sheaf is flasque.*

Proof. The right lifting property of $F \to *$ relative the morphisms (J_b) means that for $U \subset X$ an open embedding, the morphism $F(X) \to F(U)$ is a Kan fibration. As F is a sheaf, an upper distinguished square

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow \\ A & \longrightarrow & X \end{array}$$

gives rise to a Cartesian square

$$\begin{array}{ccc} F(X) & \longrightarrow & F(Y) \\ \downarrow & & \downarrow \\ F(A) & \longrightarrow & F(B) \end{array}$$

As $F(Y) \to F(B)$ is a Kan fibration, this square is also homotopy Cartesian. □

Lemma 4. *Proposition 17 holds of the assumption "F is flasque" is replaced by the assumption "F is fibrant".*

Proof. "Only if" (I_a) for $n = 0$ says that for any U, $(h_U)_+$ is cofibrant. By Corollary 14, for any pointed simplicial set K, one has

$$Hom_{Ho}((h_U)_+ \wedge K, F) = Hom_{Ho}(K, S((h_U)_+, F))$$

and $S((h_U)_+, F)$ is just $F(U)$. If in (17.1) we take $Y = K \wedge (h_U)_+$, so that $Y \wedge (h_{\mathbf{A}^1})_+ = K \wedge (h_{U \times \mathbf{A}^1})_+$ we get

$$Hom_{Ho}(K, F(U \times \mathbf{A}^1)) = Hom_{Ho}(K, F(U))$$

That this holds for any K means that $F(U) \to F(U \times \mathbf{A}^1)$ becomes an isomorphism in the homotopy category , hence is a weak equivalence.
"If" We apply Corollary 16. As $(h_{\mathbf{A}^1})_+$ is cofibrant and F fibrant,

$$Hom_{Ho}(Y \wedge (h_{\mathbf{A}^1})_+, F) = Hom_{Ho}(Y, \underline{Hom}((h_{\mathbf{A}^1})_+, F))$$

and it suffice to show that

$$F \to \underline{Hom}((h_{\mathbf{A}^1})_+, F)$$

is a local equivalence. This $\underline{Hom}$ is a simplicial sheaf $U \mapsto F(U \times \mathbf{A}^1)$ and the claim follows. □

Proof. We can now finish the proof of Proposition 17. Let $F \to F'$ be a fibrant replacement of F. As F and F' are flasque, $F(U) \to F'(U)$ is a weak equivalence for any U. That all $F(U) \to F(U \times \mathbf{A}^1)$ be weak equivalences is hence equivalent to all $F'(U) \to F'(U \times \mathbf{A}^1)$ be weak equivalences, while F is $\mathbf{A}^1$-local if and only if F' is. □

3.3 $\bar{\Delta}$-Closed Classes

The proof of the main theorem of this section will be postponed.

Definition 4. A class S of morphisms of pointed simplicial sheaves is Δ-closed if:

1. (Simplicial) homotopy equivalences are in S.
2. If two of f, g and fg are in S then so is the third.
3. S is stable by finite coproducts.
4. If $F_{**} \to G_{**}$ is a morphism of pointed bisimplicial sheaves, and if all $F_{*p} \to G_{*p}$ are in S, so is the diagonal $\Delta(F) \to \Delta(G)$.

Definition 5. The class S is $\bar{\Delta}$-closed if, in addition, it is stable by arbitrary coproducts and colimits of sequences $(F_* \to G_*)_n$ with the property that, degree by degree, $(F_k)_n \to (F_k)_{n+1}$ (resp. $(G_k)_n \to (G_k)_{n+1}$) is isomorphic to an embedding $A \subset A \coprod B$ of pointed sheaves.

Theorem 4. *The class of $\mathbf{A}^1$-equivalences is the $\bar{\Delta}$-closure of the union of the classes of:*

1. *Local equivalences*
2. *Morphisms $(U \times \mathbf{A}^1)_+ \to U_+$ for U in Sm/k*

In particular, the class of $\mathbf{A}^1$-equivalences is $\bar{\Delta}$-closed.

3.4 The Class of $\mathbf{A}^1$-Equivalences Is $\bar{\Delta}$-Closed

The properties 4(1), 4(2), 4(3) are clear. The last property is proved in Proposition 20.

Lemma 5. *Let A be a pointed simplicial set and X a pointed simplicial sheaf. If X is fibrant and $\mathbf{A}^1$-local, then X^A is $\mathbf{A}^1$-local.*

Proof. Because X is fibrant, for any Y, one has in the homotopy category

$$Hom_{Ho}(Y, X^A) = Hom_{Ho}(A \wedge Y, X) \tag{17.1}$$

Applying this to Y and $Y \wedge (\mathbf{A}^1_+)$ and using

$$(A \wedge Y) \wedge (\mathbf{A}^1_+) = A \wedge (Y \wedge (\mathbf{A}^1_+))$$

one deduces from the $\mathbf{A}^1$-locality of X that of X^A. □

Lemma 6. *Let $f : K \to L$ be a morphism of pointed simplicial sheaves and A be a pointed simplicial set. If f is an $\mathbf{A}^1$-equivalence, then so is $f \wedge A : X \wedge A \to Y \wedge A$.*

Proof. One has to check that for any $\mathbf{A}^1$-local X one has in the homotopy category

$$Hom_{Ho}(A \wedge L, X) = Hom_{Ho}(A \wedge K, X)$$

Replacing X by a fibrant replacement, one may assume X fibrant. Applying (17.1) one is reduced to Lemma 5. □

Lemma 7. *Let $f : K \to L$ be a morphism of pointed simplicial sheaves. If K and L are cofibrant, then f is a $\mathbf{A}^1$-equivalence if and only if for any fibrant $\mathbf{A}^1$-local X, the morphism of simplicial sets*

$$S(L, X) \to S(K, X)$$

is a weak equivalence.

Proof. "If" Taking π_0 one deduces from the assumptions that

$$Hom_{Ho}(L, X) \stackrel{\to}{\to} Hom_{Ho}(K, X)$$

"Only if" The assumptions imply that $S(K, X)$ and $S(L, X)$ are fibrant. For any pointed simplicial set A one has

$$Hom_{Ho}(A, S(K, X)) = Hom_{Ho}(K \wedge A, X)$$

and similarly for L and one applies Lemma 6. □

Proposition 18. *The coproduct of a family of $\mathbf{A}^1$-equivalences*

$$f_\alpha : X_\alpha \to Y_\alpha$$

is an $\mathbf{A}^1$-equivalence.

Proof. There are commutative diagrams

$$\begin{array}{ccccc} * & \longrightarrow & X'_\alpha & \xrightarrow{f'_\alpha} & Y'_\alpha \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & X_\alpha & \xrightarrow{f_\alpha} & Y_\alpha \end{array}$$

where morphisms on the first line are cofibrations, and where the vertical maps are local equivalences, and similarly for Y. Replacing X_α (resp. Y_α) by X'_α (resp. Y'_α) we may and shall assume that the X_α and Y_α are cofibrant. The coproducts $\coprod X_\alpha$, $\coprod Y_\alpha$ are then cofibrant too. One has

$$S(\coprod X_\alpha, X) = \prod S(X_\alpha, X)$$

and similarly for the Y_α, and one applies Lemma 7, and the fact that a product of a family of weak equivalences of fibrant pointed simplicial sets is a weak equivalence. □

Proposition 19. *The colimit*

$$f : colim F_n \to colim G_n$$

of an inductive sequence of $\mathbf{A}^1$*-equivalences* $f_n : F_n \to F_n$ *is again an* $\mathbf{A}^1$*-equivalence.*

Proof. One inductively constructs an inductive sequence of commutative squares

$$\begin{array}{ccc} F'_n & \xrightarrow{f'_n} & G'_n \\ \downarrow & & \downarrow \\ F_n & \xrightarrow{f_n} & G_n \end{array}$$

in which the vertical maps are local equivalences, the F'_n and G'_n are cofibrant and the transition maps $F'_n \to F'_{n+1}$, $G'_n \to G'_{n+1}$ are cofibrations. A colimit of local equivalences being a local equivalence, it is sufficient to prove the proposition for the sequence (f'_n). We hence may and shall assume that $* \to F_1 \to \ldots \to F_n \to$ is a sequence of cofibrations and similarly for the $* \to G_1 \to \ldots \to G_n \to$. The colimits F and G of those sequences are then cofibrant.

If X is fibrant and $\mathbf{A}^1$-local, $S(G, X) \to S(F, X)$ is the limit of the sequence of weak equivalences

$$S(G_n, X) \to S(F_n, X)$$

In the sequences $S(G_n, X)$ and $S(F_n, X)$ the transition maps are fibrations of fibrant objects. It follows that the limit is again a weak equivalence: the π_i of the limit map onto the limit of π_i, with fibers $(lim^1 \pi_{i+1})$-torsors. It remains to apply Lemma 7. □

Proposition 20. *Let* $F_{**} \to G_{**}$ *be a morphism of pointed bisimplicial sheaves. If all* $F_{p*} \to G_{p*}$ *are* $\mathbf{A}^1$*-equivalences, so is* $\Delta(F) \to \Delta(G)$.

To prove Proposition 20 we will functorially attach to F_{**} an inductive sequence of pointed simplicial sheaves $F^{(n)}$, whose colimit maps to $\Delta(F)$ by a local equivalence. We will then inductively prove that $F^{(n)} \to G^{(n)}$ is an $\mathbf{A}^1$-equivalence, and apply Proposition 19. We begin with preliminaries to the construction of the $F^{(n)}$.

Proposition 21. *Let* Δ_{inj} *be the category of finite ordered sets* $\Delta^n = (0, \ldots, n)$ *and increasing injective maps. For any category* C *with finite coproducts,*

the forgetting functor

$$\omega : \Delta^{op}C \to \Delta^{op}_{inj}C$$

has a left adjoint ω'*: "formally adding degenerate simplicies":* $(\omega' X)_n$ *is the coproduct, over all* p *and all increasing surjective maps* $s : \Delta^n \to \Delta^p$*, of copies of* X_p

$$(\omega' X)_n = \coprod_s X_p$$

We define the wrapping functor $Wr : \Delta^{op}C \to \Delta^{op}C$ *as the composite* $Wr := \omega'\omega$*. For* C *the category of sets or of pointed sets one has the following.*

Lemma 8. *The adjunction map* $a : Wr(X) \to X$ *is a weak equivalence.*

Proof. We will prove it for C the category of sets. The pointed case is similar. The *fundamental groupoid* of X is the category with set of objects X_0, in which all maps are isomorphisms, and universal for the property that:

(1) $\sigma \in X_1$ defines a morphism $f(\sigma) : \partial_1(\sigma) \to \partial_0(\sigma)$.
(2) For $\tau \in X_2$, $f(\partial_1\tau) = f(\partial_0\tau)f(\partial_2\tau)$.

One has $X_0 = Wr(X)_0$. To handle π_0 and π_1 it suffice to show that a induces an isomorphism of fundamental groupoids. For any X and any $p \in X_0$, $f(s_0(p))$ is the identity of p. This results from (2) applied to $s_0 s_0(p)$ which gives

$$f(s_0(p)) = f(s_0(p))f(s_0(p))$$

As generators of the fundamental groupoid, it hence suffices to take non degenerate $\sigma \in X_1$. For relations, it then suffices to take those coming from non degenerate $\tau \in X_2$: the degenerate τ give nothing new.

If we apply this to $Wr(X)$, we find as set of generators X_1, and relations indexed by X_2, the same relations as for X.

The functor Wr commutes with passage to connected components and to passage to a covering. To handle higher π_i, this reduces us to the case where X (and hence $Wr(X)$) is connected and simply connected. In this case it suffices to check that a induces an isomorphism in homology. It does because one has a commutative diagram

$$\begin{array}{ccc} C_*(X) & \xrightarrow{\sim} & C_*(Wr(X))/degeneracies \\ & \searrow & \downarrow \\ & & C_*(X)/degeneracies \end{array}$$

in which the first arrow is an isomorphism, the second the effect of a on homology, and the composite is a homotopy equivalence. □

Proposition 22. *For* X *a pointed simplicial sheaf, let* $sk_n(X)$ *be the n-th skeleton of* X*, i.e. simplicial subsheaf of* X *for which* $(sk_n(X))_p$ *is the union of the images of the degeneracies* $X_q \to X_p$ *for* $q \leq n$*. One has push-out squares*

$$\begin{array}{ccc} X_{n+1} \wedge (\partial \Delta^{n+1})_+ & \longrightarrow & sk_n(Wr(X)) \\ \downarrow & & \downarrow \\ X_{n+1} \wedge (\Delta^{n+1})_+ & \longrightarrow & sk_{n+1}(Wr(X)) \end{array} \tag{22.1}$$

Let now F be bisimplicial. Each $F_{n,\bullet}$ is simplicial, and they form a simplicial system of pointed simplicial sheaves. Let us apply Wr and sk_n to the first variable, i.e. to the simplicial sheaf $F_{\bullet,m}$ for each fixed m. We again have diagrams (22.1) and, taking the diagonal Δ, one obtains push-out squares:

$$\begin{array}{ccc} F_{n+1} \wedge (\partial \Delta^{n+1})_+ & \longrightarrow & \Delta(sk_n(Wr(F))) \\ \downarrow & & \downarrow \\ F_{n+1} \wedge (\Delta^{n+1})_+ & \longrightarrow & \Delta(sk_{n+1}(Wr(F))) \end{array} \tag{22.2}$$

where F_n now stands for the pointed simplicial sheaf $F_{n,\bullet}$. This way the simplicial sheaf $\Delta(Wr(F))$, which by Lemma 8 maps to $\Delta(F)$ by a local equivalence, appears as an inductive limit of (22.2).

Proof of Proposition 20: With the notations of 22 it suffice to show that the

$$\Delta\, sk_n\, Wr(F) \to \Delta\, sk_n\, Wr(G)$$

(with $sk_n\, Wr$ applied in the first variable) are $\mathbf{A}^1$-equivalences. We prove it by induction on n.

For $n = 0$, $\Delta\, sk_0\, Wr(F) = F_0$, and $F_0 \to G_0$ is assumed to be an $\mathbf{A}^1$-equivalence. From n to $n+1$, we have a morphism of push out squares

$$(22.2) \text{ for } F \to (22.2) \text{ for } G$$

As $F_{n+1} \to G_{n+1}$ is an $\mathbf{A}^1$-equivalence, by Lemma 6, so are its smash product with $(\partial \Delta^{n+1})_+$ and $(\Delta^{n+1})_+$. It remain to apply the

Lemma 9. *Suppose given a morphism of push out squares*

$$\begin{array}{ccc} \mathbf{1} & \longrightarrow & \mathbf{2} \\ \downarrow & & \downarrow \\ \mathbf{3} & \longrightarrow & \mathbf{4} \end{array} \quad \longrightarrow \quad \begin{array}{ccc} \mathbf{1}' & \longrightarrow & \mathbf{2}' \\ \downarrow & & \downarrow \\ \mathbf{3}' & \longrightarrow & \mathbf{4}' \end{array}$$

which is an $\mathbf{A}^1$-equivalence in positions $\mathbf{1}, \mathbf{2}$ and $\mathbf{3}$. If in each square the first vertical map is injective, then the morphism of squares is an $\mathbf{A}^1$-equivalence in position $\mathbf{4}$ as well.

Proof. Replacing the push out squares by push out squares of local equivalent objects, we may and shall assume that all objects considered are cofibrant, and that the vertical maps are cofibrant.

If X is fibrant applying $S(-, X)$ to each of the squares we get a morphism of cartesian squares, of pointed simplicial sets in which the vertical maps are cofibrations:

$$\begin{array}{ccc} S(\mathbf{4}, X) & \longrightarrow & S(\mathbf{3}, X) \\ \downarrow & & \downarrow \\ S(\mathbf{2}, X) & \longrightarrow & S(\mathbf{1}, X) \end{array} \quad \longleftarrow \quad \begin{array}{ccc} S(\mathbf{4}', X) & \longrightarrow & S(\mathbf{3}', X) \\ \downarrow & & \downarrow \\ S(\mathbf{2}', X) & \longrightarrow & S(\mathbf{1}', X) \end{array}$$

If X is in addition $\mathbf{A}^1$-local, it is a weak equivalence in positions $\mathbf{1}, \mathbf{2}$ and $\mathbf{3}$, hence also in position $\mathbf{4}$. By Lemma 7, this proves Lemma 9, finishing the proof of Proposition 20 as well as of the claim that the class of $\mathbf{A}^1$-equivalences is $\bar{\Delta}$-closed. □

3.5 The Class of $\mathbf{A}^1$-Equivalences as a $\bar{\Delta}$-Closure

In this section we finish the proof of Theorem 4.

Proposition 23. *The homotopy push-out of a diagram*

$$Q \quad : \quad \begin{array}{ccc} K & \longrightarrow & L \\ \downarrow & & \\ M & & \end{array} \tag{23.1}$$

is the push-out K_Q of

$$Q \quad : \quad \begin{array}{ccc} K \vee K & \longrightarrow & M \vee L \\ \downarrow & & \\ K \wedge (\Delta^1)_+ & & \end{array} \tag{23.2}$$

where the vertical map $K \wedge (\Delta^0 \coprod \Delta^0)_+ \to K \wedge (\Delta^1)_+$ is induced by $\partial_1, \partial_0 : \Delta^0 \to \Delta^1$ mapping Δ^0 to 0 (resp. 1) in Δ^1. In the case of simplicial sets, $|K_Q|$ maps to $|\Delta^1| = [0, 1]$ with fibers $|M|$ above 0, $|L|$ above 1, and $|K|$ above $(0, 1)$.

The homotopy push-out K_Q is the diagonal of the bisimplicial object with columns $M \vee K^{\vee n} \vee L$ obtained by formally adding degeneracies to

$$K \overset{\rightarrow}{\to} M \vee L$$

in $\Delta^{op}_{inj}\Delta^{op}(Sh_\bullet)$ (cf. 21) [∂_0 maps K to L, ∂_1 maps K to M]. If $f : Q \to Q'$ is a morphism of diagrams (23.1), the induced morphism from K_Q to $K_{Q'}$ is hence in

the closure of the three components of f for the operation of finite coproduct and diagonal (4(3), (4)).

A commutative square

$$\begin{array}{ccc} K & \longrightarrow & L \\ \downarrow & & \downarrow \\ M & \longrightarrow & N \end{array} \tag{23.3}$$

induces a morphism $K_Q \to N$.

Example 1. Let $f : K \to L$ be a morphism. The homotopy push-out of

$$\begin{array}{ccc} K & \longrightarrow & L \\ {\scriptstyle id}\downarrow & & \\ K & & \end{array}$$

is the cylinder $cyl(f)$ of f. The morphisms

$$L \to cyl(f) \to L$$

are homotopy equivalences. To check that the composite $cyl(f) \to L \to cyl(f)$ is homotopic to the identity, one observes that $cyl(f)$ is the push-out of

$$\begin{array}{ccc} K & \longrightarrow & L \\ \downarrow & & \\ K \wedge (\Delta^1)_+ & & \end{array}$$

(the vertical map induced by $\partial_0 : \Delta^0 \to \Delta^1$ mapping Δ^0 to 1) and that the composite $cyl(f) \to cyl(f)$ is induced by $\Delta^1 \to \Delta^0 \to \Delta^1$, homotopic to the identity by a homotopy fixing 1.

Similar arguments would show that the homotopy push out $cyl'(f)$ of

$$\begin{array}{ccc} K & \xrightarrow{id} & K \\ \downarrow & & \\ M & & \end{array}$$

is homotopic to L by $L \to cyl'(L) \to L$.

Example 2. In any category with finite coproducts, a *coprojection* is a map isomorphic to the natural map $A \to A \coprod B$ for some A and B. If in a push-out square of pointed simplicial sheaves

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ \downarrow & & \downarrow \\ M & \longrightarrow & N \end{array} \tag{23.4}$$

the morphism f is a coprojection: $L = K \vee A$, the square (23.4) is the coproduct of the squares

$$\begin{array}{ccc} K & \xrightarrow{id} & K \\ \downarrow & & \downarrow \\ M & \xrightarrow{id} & M \end{array} \quad \text{and} \quad \begin{array}{ccc} * & \longrightarrow & A \\ \downarrow & & \downarrow \\ * & \longrightarrow & A \end{array} \tag{23.5}$$

and the resulting morphism $K_Q \to N$ is a homotopy equivalence, being the coproduct of the homotopy equivalences of Example 1 resulting from the two squares (23.5). The same conclusion applies if $K \to M$ is a coprojection.

A morphism of pointed simplicial sheaves $K \to L$ is a termwise coprojection if each $K_n \to L_n$ is a coprojection of pointed sheaves. Example: for any diagram (23.1), the morphisms $L, M \to K_Q$ are termwise coprojections. For any morphism $f : K \to L$, this applies in particular to $K, L \to cyl(f)$.

Proposition 24. *If in a cocartesian square (23.3) either $K \to L$ or $K \to M$ is a termwise coprojection, then the resulting morphism from K_Q to N is in the Δ-closure of the empty set of morphisms.*

Proof. For each n, we have a cocartesian square of pointed sheaves

$$Q_n \quad : \quad \begin{array}{ccc} K_n & \longrightarrow & L_n \\ \downarrow & & \downarrow \\ M_n & \longrightarrow & N_n \end{array}$$

Let us view it as a cocartesian square of pointed simplicial sheaves. By Example 2, it gives rise to a homotopy equivalence $K_{Q_n} \to N_n$. One concludes by observing that $K_Q \to N$ is the diagonal of this simplicial system of morphisms. □

Corollary 25. *If in a cocartesian square (23.3):*

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ M & \xrightarrow{f'} & N \end{array}$$

f or g is a termwise coprojection, then:

1. *f' is in the Δ-closure of $\{f\}$.*
2. *g' is in the Δ-closure of $\{g\}$.*

Proof of (1): The morphism of cocartesian squares

$$Q' \quad : \quad \begin{array}{ccc} K & \xrightarrow{id} & K \\ \downarrow & & \downarrow \\ M & \xrightarrow{id} & M \end{array} \longrightarrow \quad Q \quad : \quad \begin{array}{ccc} K & \longrightarrow & L \\ \downarrow & & \downarrow \\ M & \longrightarrow & N \end{array}$$

defines a commutative square

$$\begin{array}{ccc} K_{Q'} & \longrightarrow & K_Q \\ \downarrow & & \downarrow \\ M & \longrightarrow & N \end{array}$$

in which the vertical maps are in the Δ-closure of the empty set by Proposition 24, while the first horizontal map is in the Δ-closure of f by 8.

Proof of (2): One similarly uses

$$Q' \quad : \quad \begin{array}{ccc} K & \longrightarrow & L \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle id} \\ K & \xrightarrow{i} & L \end{array} \longrightarrow \quad Q \quad : \quad \begin{array}{ccc} K & \longrightarrow & L \\ \downarrow & & \downarrow \\ M & \longrightarrow & N. \end{array}$$

Proposition 26. *A pointed simplicial sheaf $F_\bullet$ is* reliably compact *if it coincides with its n-skeleton for some n and each F_i is compact in the sense that the functor $Hom(F_i, -)$ commutes with filtering colimits. This property is stable by $F_\bullet \to F_\bullet \wedge K$ for K a finite pointed simplicial set (finite number of non degenerate simplices) and implies that $F_\bullet$ is compact.*

Construction 27. Let E and N be classes of morphisms such that:

(a) Sources and targets are reliably compact.
(b) Each f in N is a termwise coprojection.

We will construct a functor Ex from pointed simplicial sheaves to pointed simplicial sheaves and a morphism $Id \to Ex$ such that:

(i) For any F, $F \to Ex(F)$ is in the $\bar{\Delta}$-closure of E
(ii) If $f : K \to L$ is in E, the morphism

$$S(L, Ex(X)) \to S(K, Ex(X)) \tag{27.1}$$

is a weak equivalence.
(iii) If $f : K \to L$ is in N, the morphism (27.1) is a Kan fibration.

Let us factorize $f : K \to L$ as $K \to cyl(f) \to L$. As the second map is a homotopy equivalence, the first is in the Δ-closure of E. In the corresponding factorization of (27.1):

$$S(L, Ex(F)) \to S(cyl(f), Ex(F)) \to S(K, Ex(F))$$

the first map is a homotopy equivalence. To obtain (ii), it hence suffices that $S(cyl(f), Ex(F)) \to S(K, Ex(F))$ be a weak equivalence.

Replacing each $f : K \to L$ in E by the corresponding $K \to cyl(f)$, this reduces us to the case where

(c) each f in E is a termwise coprojection,

and we will construct in this case a functor Ex such that

(iii)* for f in E, (27.1) is a trivial fibration.

The conditions (ii), (iii)* are lifting properties:

for f in E, in squares:

$$\begin{array}{ccc} \partial\Delta^n_+ & \longrightarrow & S(L, Ex(F)) \\ \downarrow & & \downarrow \\ \Delta^n_+ & \longrightarrow & S(K, Ex(F)) \end{array}$$

for f in N, in squares:

$$\begin{array}{ccc} (\Lambda^n_k)_+ & \longrightarrow & S(L, Ex(F)) \\ \downarrow & & \downarrow \\ \Delta^n_+ & \longrightarrow & S(K, Ex(F)) \end{array}$$

In the first case, the data are morphisms $\Delta^n_+ \wedge K \to Ex(F)$ and $\partial\Delta^n_+ \wedge L \to Ex(F)$ agreeing on $\partial\Delta^n_+ \wedge K$, i.e. a morphism

$$(\Delta^n_+ \wedge K) \coprod_{\partial\Delta^n_+ \wedge K} (\partial\Delta^n_+ \wedge L) \to Ex(F)$$

and we want it to extend to $\Delta^n_+ \wedge L$. Similarly in the second case, with $\partial\Delta^n$ replaced by Λ^n_k:

for f in E:

$$\begin{array}{ccc} (\Delta^n_+ \wedge K) \coprod_{\partial\Delta^n_+ \wedge K} (\partial\Delta^n_+ \wedge L) & \longrightarrow & Ex(F) \\ \downarrow & & \downarrow \\ \Delta^n_+ \wedge L & \longrightarrow & * \end{array} \qquad (27.2)$$

for f in N:

$$\begin{array}{ccc} (\Delta^n_+ \wedge K) \coprod_{(\Lambda^n_k)_+ \wedge K} ((\Lambda^n_k)_+ \wedge L) & \longrightarrow & Ex(F) \\ \downarrow & & \downarrow \\ \Delta^n_+ \wedge L & \longrightarrow & * \end{array} \tag{27.3}$$

The left vertical maps are termwise coprojections, and their sources are compact. One now uses the standard trick of defining $Ex(F)$ as the inductive limit of the iterates of functors $F \to T(F)$, where $T(F)$ is deduced from F by push out, simultaneously for all

$$(\Delta^n_+ \wedge K) \coprod_{\partial\Delta^n_+ \wedge K} (\partial\Delta^n_+ \wedge L) \to Ex(F) \quad (f : K \to L \text{ in } E)$$

and

$$(\Delta^n_+ \wedge K) \coprod_{(\Lambda^n_k)_+ \wedge K} ((\Lambda^n_k)_+ \wedge L) \to Ex(F) \quad (f : K \to L \text{ in } N)$$

The push out is by

$$\bigvee(\text{sources}) \to \bigvee(\Delta^n_+ \wedge L)$$

a morphism which is a termwise coprojection. By 26, to check that the resulting $F \to Ex(F)$ is in the $\bar{\Delta}$-closure of E, it suffices to check that the left vertical morphism in (27.2) (resp. (27.3)) is in the Δ-closure of E (resp. of the empty set).

For (27.2), this is the map marked **3** in

$$\begin{array}{ccccc} \partial\Delta^n_+ \wedge K & \xrightarrow{\mathbf{1}} & \partial\Delta^n_+ \wedge L & & \\ \downarrow & & \downarrow & & \\ \Delta^n_+ \wedge K & \xrightarrow{\mathbf{2}} & \ldots & \xrightarrow{\mathbf{3}} & \Delta^n_+ \wedge L \end{array}$$

The morphisms **1** and **3** $\circ$ **2** are in the Δ-closure of E. So is **2** by 26 and one applies the 2 out of 3 property.

For (27.3), the diagram is

$$\begin{array}{ccccc} (\Lambda^n_k)_+ \wedge K & \longrightarrow & (\Lambda^n_k)_+ \wedge L & & \\ \downarrow \mathbf{1} & & \downarrow \mathbf{2} & & \\ \Delta^n_+ \wedge K & \longrightarrow & \ldots & \xrightarrow{\mathbf{3}} & \Delta^n_+ \wedge L \end{array}$$

with **1** and **3** $\circ$ **2** in the Δ-closure of the empty set. Indeed, Λ^n_k and Δ^n are both contractible.

Remark 4. Let P be a property of pointed simplicial schemes stable by coproduct, and suppose that:

(a) For $f : K \to L$ in E, the K_n and L_n have property P.
(b) For $f : K \to L$ in N, f is in degree n isomorphic to the natural map $K_n \to K_n \vee A$ for some A having property P.

The functor Ex constructed in Construction 27 is then such that for any K, each morphism $K_n \to Ex(K)_n$ is isomorphic to some $K_n \to K_n \vee A$ where A has property P. In particular, if the K_n have property P, so have the $Ex(K)_n$.

Proposition 28. *(Proof of Theorem 4)* We apply Construction 11, on the site QP/G, taking for E and N the following classes.

E: For any X in the site, the morphism

$$(X \times \mathbf{A}^1)_+ \to X_+ \tag{28.1}$$

and for any upper distinguished square

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow \\ A & \longrightarrow & X, \end{array} \tag{28.2}$$

the morphism

$$(K_Q)_+ \to X_+ \tag{28.3}$$

N: For any X in the site,

$$(\emptyset)_+ \to X_+ \tag{28.4}$$

If a pointed simplicial sheaf G is of the form $Ex(F)$, that (28.4) is in N ensures that each $G(X)$ is Kan. That (28.3) is in E ensures that for each upper distinguished square (28.2), the morphism

$$G(X) \to S((K_Q)_+, G)$$

is a weak equivalence. As each $G(Y)$ is Kan, $S((K_Q)_+, G)$ is the homotopy fiber product of $G(A)$ over $G(B)$, and

$$\begin{array}{ccc} G(X) & \longrightarrow & G(A) \\ \downarrow & & \downarrow \\ G(Y) & \longrightarrow & G(B) \end{array}$$

is homotopy cartesian: G is flasque.

Further, as (28.1) is in E, for each X,

$$G(X) \to G(X \times \mathbf{A}^1)$$

is a weak equivalence: by Theorem 3, G is $\mathbf{A}^1$-local.

Suppose now that $f : F \to G$ is a $\mathbf{A}^1$-equivalence. In the commutative diagram

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ Ex(F) & \longrightarrow & Ex(G) \end{array}$$

the vertical maps are in the $\bar{\Delta}$-closure of the morphisms (28.1) and (28.2), the later being local equivalences. In particular, they are $\mathbf{A}^1$-equivalences and $Ex(F) \to Ex(G)$ is an $\mathbf{A}^1$-equivalence between $\mathbf{A}^1$-local objects , hence is a local equivalence. It follows that f is in the required $\bar{\Delta}$-closure, proving Theorem 4.

The functor Ex used introduced in 28 can also be used to prove the following lemma.

Lemma 10. *If $F^{(i)} \to F^{(j)}$ is a filtering system of $\mathbf{A}^1$-equivalences, then $F^{(n)} \to colim_i F^{(i)}$ is again an $\mathbf{A}^1$-equivalence.*

Proof. Consider the square:

$$\begin{array}{ccc} F^{(n)} & \longrightarrow & colim_i F^{(i)} \\ \downarrow & & \downarrow \\ Ex(F^{(n)}) & \longrightarrow & Ex(colim_i F^{(i)}). \end{array}$$

Since the functor Ex commutes with filtering colimits, the bottom arrow is a filtering colimit of local equivalences, hence a local equivalence. The vertical maps are $\mathbf{A}^1$-equivalences, hence the top map is an $\mathbf{A}^1$-equivalence. □

3.6 One More Characterization of Equivalences

Denote by $[QP/G]_+$ the full subcategory in the category of pointed sheaves on QP/G generated by all coproducts of sheaves of the form $(h_X)_+$.

Theorem 5. *The class of local equivalences (resp. $\mathbf{A}^1$-equivalences) in $\Delta^{op}[QP/G]_+$ is the smallest class W which contains morphisms $(K_Q)_+ \to X_+$ for Q upper distinguished and has the following properties:*

1. *Simplicial homotopy equivalences (resp. and $\mathbf{A}^1$-homotopy equivalences) are in W.*

2. *If two of f, g and fg are in W then so is the third.*
3. *If $F^{(i)} \to F^{(j)}$ is a filtering system of termwise coprojections in W, then $F^{(n)} \to colim_i F^{(i)}$ is again in W.*
4. *If $F_{**} \to F'_{**}$ is a morphism of bisimplicial objects, and if all $F_{*p} \to F'_{*p}$ are in W, so is the diagonal $\Delta(F) \to \Delta(F')$.*

The proof is given in 32.

Lemma 11. *If the morphism $f : F \to G$ is such that, for each U, $F(U) \to G(U)$ is a weak equivalence, and if the F_n and G_n are all of the form $(\coprod h_{U_i})_+$, then f is in the $\bar{\Delta}$-closure of the empty set.*

The proof will use the following construction.

Construction 29. Let C be a category, and let C_0 be a set of objects of C, such that any isomorphism class has a representative in C_0. Let i_* be the functor which to a presheaf of pointed sets on C attaches the family of pointed sets $(F(U))_{U \in C_0}$. It has a left adjoint i^*:

$$\text{family } (A_U)_{U \in C_0} \mapsto \bigvee_U ((h_U)_+ \wedge A_U) =$$
$$= (\text{disjoint sum over the } U \in C_0 \text{ and } (A_U - *) \text{ of } h_U)_+$$

If C_0 is viewed as a category whose only morphisms are identities, the natural functor

$$i : C_0 \to C$$

defines a morphism of sites $C \to C_0$, both endowed with the trivial topology (any presheaf a sheaf), and i_*, i^* are the corresponding direct and inverse image of pointed sheaves.

By a general story valid for any pair of adjoint functors, for any pointed presheaf F on C, the $(i^*i_*)^{n+1}(F)$ form a pointed simplicial presheaf $R(F)$ augmented to F:

$$a : R(F) \to F$$

Further:

(a) If F is of the form i^*A, i.e. of the form $(\coprod h_{U_i})_+$, a is a homotopy equivalence.
(b) For any F, $i^*(a)$ is a homotopy equivalence: for each U in C, $R(F)(U) \to F(U)$ is a homotopy equivalence.

For a simplicial presheaf F we define

$$R(F) = \Delta(\text{simplicial system of } R(F_p))$$

Proposition 30. *(Proof of Lemma 11)* Let us say that X in QP/G is connected if it is not empty and is not a disjoint union: G should act transitively on the set of connected components of X. Let $C \subset QP/G$ be the full subcategory of connected objects. A sheaf F on QP/G is determined by its restriction to C. Indeed, $F(\coprod X_i) = \prod F(X_i)$. To apply Construction 29, we will use this remark to identify the category of sheaves on QP/G to a full subcategory of the category of presheaves on C. For any C_0 as in Construction 29, the functor i^* takes values in sheaves, that is in the restriction of sheaves to C. Indeed, for X connected, $(\coprod h_{U_i})_+(X)$ is the same, whether $\coprod$ and $+$ are taken in the sheaf or in the presheaf sense.

Fix $f : F \to G$ as in Lemma 11. For each n, the assumption on F_n ensures that $R(F_n) \to F_n$ is in the Δ-closure of the empty set, and similarly for G.

For each (connected) U, the morphism of pointed simplicial sets $F(U) \to G(U)$ is a weak equivalence, hence in the $\bar{\Delta}$-closure of the empty set. It follows that $i^*i_*(F) \to i^*i_*(G)$: the $\vee$ over C_0 of the

$$(h_U)_+ \wedge F(U) \to (h_U)_+ \wedge G(U)$$

is in the $\bar{\Delta}$-closure of the empty set. Iterating one finds the same for $(i^*i_*)^n(F) \to (i^*i_*)^n(G)$, and $R(F) \to R(G)$ is in this $\bar{\Delta}$-closure too. It remains to apply the two out of three property to

$$\begin{array}{ccc} R(F) & \longrightarrow & R(G) \\ \downarrow & & \downarrow \\ F & \longrightarrow & G \end{array}$$

Lemma 12. *If $f : F \to G$ is a local equivalence and if the F_n and G_n are all of the form $(\coprod h_{U_i})_+$, then f is in the $\bar{\Delta}$-closure of the $(K_Q)_+ \to X_+$ for Q upper distinguished.*

Proof. We will use the Construction 27 for E the class of morphisms $(K_Q)_+ \to X_+$ for Q upper distinguished, and for N the class of morphisms $* \to X_+$. By Remark 4, if the F_n are of the form $(\coprod h_{U_i})_+$, so are the $Ex(F)_n$. In the commutative diagram

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ Ex(F) & \longrightarrow & Ex(G) \end{array}$$

the vertical maps are in the required Δ-closure. They are in particular local equivalences and so is $Ex(f)$. One verifies as in 28 that $Ex(F)$ and $Ex(G)$ are flasque. By Construction 27(ii), for each U, $Ex(f)(U)$ is a weak equivalence, and it remains to apply Lemma 11 to Ex(f). □

Lemma 13. *If $f : F \to G$ is an $\mathbf{A}^1$-equivalence and if the F_n and G_n are all of the form $(\coprod h_{U_i})_+$, then f is in the $\bar{\Delta}$-closure of the $(K_Q)_+ \to X_+$ for Q upper distinguished and $(X \times \mathbf{A}^1 \to X)_+$ for $X \in QP$.*

Proof. Similar to the proof of Lemma 12. □

Proposition 31. *Since for any simplicial sheaf F the map $R(F) \to F$ is a local equivalence Lemmas 12 and 13 imply that for any local (resp. $\mathbf{A}^1$-) equivalence $f : F \to G$, the morphism $R(f)$ belongs to the $\bar{\Delta}$-closure of the $(K_Q)_+ \to X_+$ for Q upper distinguished (resp. the $(K_Q)_+ \to X_+$ for Q upper distinguished and $(X \times \mathbf{A}^1 \to X)_+$ for $X \in QP$).*

Proposition 32. *Proof of Theorem 5*: We consider only the case of $\mathbf{A}^1$-equivalences. Proposition 20 and Lemma 10 imply that $\mathbf{A}^1$-equivalences contain the class W. In view of Lemma 13 it remains to see that W is $\bar{\Delta}$-closed. The only condition to check is that it is closed under coproducts. Let

$$f_\alpha : F^{(\alpha)} \to H^{(\alpha)}, \quad \alpha \in A$$

be a family of morphisms in W. For a finite subset I in A set

$$\Phi_I = (\coprod_{\alpha \in I} H^{(\alpha)}) \coprod (\coprod_{\alpha \in A - I} F^{(\alpha)})$$

For $I \in J$ we have a morphism $\Phi_I \to \Phi_J$ and the map $\coprod_{f_\alpha}$ is isomorphic to the map

$$\Phi_\emptyset \to colim_{I \subset A} \Phi_I$$

It remains to show that $\Phi_I \to \Phi_{I \cup \{\alpha}$ is in W. This morphism is of the form $Id_H \coprod (f : F \to F')$ where f is in W. Using the fact that W is closed for diagonals we reduce to the case $H = \coprod (h_U)_+$. Using the same reasoning as above we further reduce to the case $H = (h_U)_+$.

Consider the class of f such that $Id_{(h_U)_+} \coprod f$ is in W. This class clearly contains morphisms $(K_Q)_+ \to X_+$, has the two out of three property and is closed under filtering colimits. It also contains simplicial homotopy equivalences. It contains morphisms of the form $p_+ : (X \times \mathbf{A}^1)_+ \to X_+$ because such morphisms are $\mathbf{A}^1$-homotopy equivalences.

4 Solid Sheaves

4.1 Open Morphisms and Solid Morphisms of Sheaves

We fix S and G as in Sect. 3.1, and will work in $(QP/G)_{Nis}$. The story could be repeated in $(Sm/S)_{Nis}$.

Definition 6. A morphism of sheaves $f : F \to G$ is open if it is relatively representable by open embeddings, i.e. if for any morphism $u : h_X \to G$ (that is,

$u \in G(X)$, X in QP/G), the fiber product $F \times_G h_X$ mapping to h_X is isomorphic to $h_U \to h_X$ for U a G-stable open subset of X.

In other words: f should identify F with a subsheaf of G and, for any $s \in G(X)$, there is U open in X and G-stable such that the pull-back of s with respect to $Y \to X$ is in $F(Y)$ if and only if Y maps to U.

The property "open" is stable under composition. It is also stable by pull-back: if in a cartesian square

$$\begin{array}{ccc} F' & \xrightarrow{f'} & G' \\ \downarrow & & \downarrow u \\ F & \xrightarrow{f} & G \end{array} \tag{32.1}$$

f is open, then f' is open. This follows from transitivity of pull-backs. Conversely, if f' is open and u is an epimorphism, then f is open. Indeed,

Lemma 14. *For $F \to h_X$ a morphism, the property that F is represented by U open in X is local on X (for the Nisnevich topology).*

Proof. Suppose that the X_α cover X, and that each $F_\alpha = F \times_{h_X} h_{X_\alpha}$ is represented by $U_\alpha \subset X_\alpha$. For $Y \to X_\alpha$, $F_Y := F \times_{h_X} h_Y$ is then represented by $U_Y \subset Y$ with U_Y the inverse image of U_α. By descent for open embedding, the U_α come from some $U \subset X$, we have locally on X an isomorphism $F \xrightarrow{\sim} h_U$ and by descent for isomorphisms of sheaves one has $F \xrightarrow{\sim} h_U$. □

Given a square of the form (32.1) with f' open and u an epimorphism, if s is in $G(X)$, s can locally be lifted to a section of G'. As f' is open, it follows that locally on X, $F \times_G h_X$ is represented by an open subset. Applying Lemma 14 one concludes that f is open. The same argument shows that if we have cartesian diagrams

$$\begin{array}{ccc} F'_\alpha & \xrightarrow{f'_\alpha} & G'_\alpha \\ \downarrow & & \downarrow u_\alpha \\ F & \xrightarrow{f} & G \end{array}$$

with each f'_α open and $\coprod u_\alpha : \coprod G'_\alpha \to G$ onto, then f is open.

Proposition 33. *The property "open" is stable by push-outs.*

Proof. Suppose

$$\begin{array}{ccc} F & \xrightarrow{f} & G \\ \downarrow & & \downarrow \\ F' & \xrightarrow{f'} & G' \end{array}$$

is a cocartesian diagram, with f open. In particular f is a monomorphism, and it follows that f' is a monomorphism and that the square is cartesian as well. The morphism $F' \coprod G \to G'$ is onto. The pull-back of f' by $f' : F' \to G'$ is an isomorphism (f' being a monomorphism) hence open. The pull-back of f' by $G \to G'$ is just f, open by assumption. It follows that f' is open. □

We now fix a class C of open embeddings $U \to V$ in (QP/G). We require the following stabilities

Conditions 34. 1. If $U \to U' \to V$ are open embeddings and if $U \to V$ is in C, so is $U' \to V$.

2. If $U \to V$ is an open embedding in C, and if $f : V' \to V$ is etale, with $f^{-1}(Y) \subset Y$ for Y the complement of U in V, then $f^{-1}(U) \to V'$ is in C.

The classes C we will have to consider are the following:

1. The open embeddings $U \to V$ with V smooth.
2. The open embeddings $U \to V$ with V smooth such that the action of G is free on $V - U$. Equivalently: V is the union of U and the open subset on which the action of G is free.
3. When working in (Sm/S): all open embeddings.

Definition 7. A morphism $f : F \to G$ is C-solid if it is a composite $F = F_0 \to F_1 \to \ldots \to F_n = G$ where each $F_i \to F_{i+1}$ is deduced by push-out from some $h_U \to h_X$, $U \subset X$ in C.

A sheaf F is solid if $\emptyset \to F$ is C-solid.

In the pointed context, a pointed sheaf is (pointed) C-solid if the morphism $pt \to F$ is C-solid.

Example 1. For U open in X, let $h_{X/U}$ be the sheaf h_X, with the subsheaf h_U contracted to a point p. If $U \to X$ is in C, then $p : pt \to h_{X/U}$ is solid: it is the push-out of $h_U \to h_X$ by $h_U \to pt$. Thom spaces are of this form: starting from a vector bundle V on Y, one contracts, in the total space of this vector bundle, the complement of the zero section to a point.

The class of solid morphisms is the smallest class closed by compositions and push-outs which contains all $h_U \to h_X$ for $U \subset X$ in C. By Proposition 33 solid morphisms are open.

For $F \to G$ a monomorphism of sheaves, define G/F to be the pointed sheaf obtained by contracting F to a point: one has a cocartesian square

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ pt & \longrightarrow & F/G \end{array}$$

By transitivity of push-out, any cocartesian diagram

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ F' & \longrightarrow & G' \end{array}$$

induces an isomorphism $G/F \to G'/F'$.

Proposition 35. *A morphism of sheaves $f : F \to G$ is C-solid if and only if it is a composite $F = F_0 \to F_1 \to \ldots \to F_n = G$ of monomorphisms where each F_i/F_{i+1} is isomorphic to some $h_{V/U} = h_V/h_U$ for $U \subset V$ in C.*

Proof. If a morphism $F \to G$ is deduced by push-out from $U \to V$, G/F is isomorphic to $h_{V/U}$. From this, "only if" results. Conversely, if we have

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ * & \longrightarrow & h_{V/U} \end{array} \quad , \quad \begin{array}{ccc} h_U & \longrightarrow & h_V \\ \downarrow & & \downarrow \\ * & \longrightarrow & h_{V/U} \end{array} \tag{35.1}$$

cocartesian, and if $h_V \to h_{V/U}$ lifts to G, then $F \to G$ is deduced by push-out from $h_U \to h_V$. Indeed, the diagrams (35.1) being cartesian as well as cocartesian, we have a cartesian

$$\begin{array}{ccc} h_U & \longrightarrow & h_V \\ \downarrow & & \downarrow \\ F & \longrightarrow & G \end{array}$$

If G_1 is deduced from $h_U \to h_V$ by push-out:

$$\begin{array}{ccccc} h_U & \longrightarrow & h_V & \overset{=}{\longrightarrow} & h_V \\ \downarrow & & \downarrow & & \downarrow \\ F & \longrightarrow & G_1 & \longrightarrow & G \end{array}$$

then $G_1/F \cong G/F$ and it follows that $G_1 = G$.

Let us suppose only that we have $v : \tilde{V} \to V$ etale, inducing an isomorphism from $\tilde{V} - v^{-1}(U)$ to $V - U$ and a lifting of $h_{\tilde{V}} \to h_{V/U}$ to G. If $\tilde{U} := v^{-1}(U)$, $h_{\tilde{V}/\tilde{U}} \to h_{V/U}$ is an isomorphism. This is most easily checked by applying the fiber functors defined by a G-local henselian Y: a morphism $Y \to V$, if it does not map to U, lifts uniquely to a morphism to $\tilde{V}$. The assumptions made hence imply that $F \to G$ is a push-out of $h_{\tilde{U}} \to h_{\tilde{V}}$. Note that by the second stability property of C, $\tilde{U} \to \tilde{V}$ is in C.

We will reduce the proof of "if" to that case. We have to show that if a monomorphism $f : F \to G$ is such that $G/F \cong h_{V/U}$ with $U \to V$ in C, then f is C solid. The cocartesian square

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ pt & \longrightarrow & F/G \end{array} \tag{35.2}$$

induces an epimorphism $pt \coprod G \to h_{V/U}$. The natural section of $h_{V/U}$ on V can hence locally be lifted to pt or to G: for some filtration $\emptyset = Z_n \subset \cdots \subset Z_1 \subset Z_0 = V$ of V by closed equivariant subschemes, we have etale maps $\phi_i : Y_i \to V$ with a (equivariant) section over $Z_i - Z_{i+1}$, and a lifting of $h_{Y_i} \to h_{V/U}$ to pt or to G. Note $V_i := V - Z_{i+1}$. We may:

1. Start with $V_0 = U$, taking $Y_0 = U$: here the lifting is to pt
2. Assume $V_i \neq V_{i+1}$; the succeeding liftings then cannot be to pt: they must be to G
3. Shrink Y_i, first so that it maps to V_i, next so that it induces an isomorphism from $Y_i - \phi_i(V_{i-1})$ to $Z_i - Z_{i+1}$

As $F \to G$ is a monomorphism the cocartesian (35.2) is cartesian as well. The composition

$$pt = h_{V_0/U} \to h_{V_1/U} \to \ldots \to h_{V/U}$$

gives by pull-back a factorization of $F \to G$ as

$$F \to F_1 \to \ldots \to G$$

with each

$$\begin{array}{ccc} F_i & \longrightarrow & F_{i+1} \\ \downarrow & & \downarrow \\ h_{V_i/U} & \longrightarrow & h_{V_{i+1}}/U \end{array}$$

cartesian and cocartesian, hence $F_{i+1}/F_i \cong h_{V_{i+1}}/h_{V_i}$. Further, the morphism $\phi_{i+1} : h_{Y_{i+1}} \to h_{V_{i+1}} \to h_{V/U}$ lifts to G, hence $h_{Y_{i+1}} \to h_{V_{i+1}/U}$ lifts to F_{i+1}. It follows that $F_i \to F_{i+1}$ is a push-out of $\phi_{i+1}^{-1}(V_i) \to Y_{i+1}$, which is in C, and solidity follows. □

Remark 1. Another formulation of Proposition 35 is: a morphism $F \to G$ is C-solid if and only if the pointed sheaf G/F is an iterated extension of $h_{V/U}$'s with $U \to V$ in C, in the sense that there are morphisms

$$pt = H_0 \to \ldots \to H_n = G/F$$

with each H_{i+1}/H_i of the form $h_{V/U}$.

Proposition 36. *If $f : F \to G$ is open and G is C-solid, then f is C-solid.*

Proof. In the proof we say "solid" instead of "C-solid". Let (*) be the property of a sheaf G that any open $f : F \to G$ is solid. If G is solid, G sits at the end of a

chain $\emptyset = G_0 \to G_1 \to \ldots \to G_n = G$ with each $G_i \to G_{i+1}$ push out of some $h_U \to h_X$ for $U \to X$ in C. We prove by induction on i that G_i satisfies (*).

For $i = 1$, $G_1 = h_X$ is representable and $\emptyset \to X$ is in C. If $f : F \to G_1$ is open, it is of the form $h_U \to h_X$ for U open in X, hence solid by Condition 34(1). It remains to check that if in a cocartesian square

$$\begin{array}{ccc} h_U & \longrightarrow & h_X \\ \downarrow & & \downarrow \\ G' & \longrightarrow & G \end{array} \tag{36.1}$$

the sheaf G' satisfies (*), so does G. In (36.1), $h_U \to h_X$ is a monomorphism and the square (36.1) hence cartesian as well as cocartesian.

Fix $f : F \to G$ open, and take the pull-back of (36.1) by f. It is again a cartesian and cocartesian square and, f being open, it is of the form

$$\begin{array}{ccc} h_V & \longrightarrow & h_Y \\ \downarrow & & \downarrow \\ F' & \longrightarrow & F \end{array} \tag{36.2}$$

where Y is open in X and $V = U \cap Y$. The diagram

$$\begin{array}{ccccc} F' & \longrightarrow & F & \longrightarrow & h_{Y/V} \\ \downarrow & & \downarrow & & \downarrow \\ G' & \longrightarrow & G & \longrightarrow & h_{X/U} \\ \downarrow & & \downarrow & & \downarrow \\ G'/F' & \longrightarrow & G/F & \longrightarrow & h_{X/(U\cup V)} \end{array}$$

expresses G/F as an extension of $h_{X/(U\cup Y)}$ by G'/F' and one concludes by Remark 1 using (*) for G' and the fact that $U \cup Y \to X$ is in C. □

Proposition 37. *The pull-back of a solid morphism f by an open morphism s is solid. In particular, if $g : F \to G$ is open and if G is solid, so is F.*

Proof. Since the pull-back of an open morphism is open, it suffices to check the proposition for f a push-out of $h_U \subset h_X$ for U open in X:

$$\begin{array}{ccc} h_U & \longrightarrow & h_X \\ \downarrow & & \downarrow \\ G' & \xrightarrow{f} & G \end{array}$$

Pulling back by g, we obtain a cocartesian square

$$\begin{array}{ccc} h_{U'} & \longrightarrow & h_{X'} \\ \downarrow & & \downarrow \\ F' & \longrightarrow & F \end{array}$$

with U' open in U and X' open in X. This shows that $F' \to F$ is solid. □

Suppose now that we are given two classes C and C' of open embeddings satisfying conditions 34. We define $C \times C'$ as the smallest class stable by Condition 34 containing the

$$(U \times V') \cup (U' \times V) \subset V \times V'$$

for $U \subset V$ in C and $U' \subset V'$ in C'.

Example 2. If C is a class of all open embeddings and C' is the class of those $U' \subset V'$ for which G acts freely outside U', then $C \times C' = C'$.

Proposition 38. *If the pointed sheaves F and F' are respectively C and C'-solid, the $F \wedge F'$ is $C \times C'$-solid.*

Proof. By assumption, F is an iterated extension in the sense of Remark 1 of pointed sheaves h_{V_i/U_i} with $U_i \to V_i$ in C. Similarly for F', with $U'_j \to V'_j$ in C'. The smash product $F \wedge F'$ is then an iterated extension of the

$$h_{V_i/U_i} \wedge h_{V'_j/U'_j} = h_{V_i \times V'_j/((U_i \times V'_j) \cup (V_i \times U'_j))},$$

taken for instance in the lexicographical order, hence it is $C \times C'$ solid. □

Definition 8. A morphism is called ind-solid relative to C if it is a filtering colimit of C-solid morphisms.

Exercise 39. We take G to be the trivial group. A section on Y of a push-out

$$\begin{array}{ccc} h_U & \longrightarrow & h_X \\ {\scriptstyle\psi}\downarrow & & \downarrow \\ F & \longrightarrow & G \end{array}$$

can be described as follows. For an open subset V of Y and a section ϕ of F on V consider on the small Nisnevich site Y_{Nis} of Y the presheaf $\Phi(V, \phi)$ which sends $a : W \to Y$ to the set of morphisms $f : W \to X$ such that $f^{-1}(U) = a^{-1}(V)$ and $\phi_{|a^{-1}(V)} = f^*(\psi)$. A section of G on Y is given by data:

1. An open subset V of Y.
2. A section ϕ of F on V.

3. A section of $i^*(a_{Nis}\Phi(V,\phi))$ on $Y - V$ where i is the closed embedding $Y - V \to Y$ and a_{Nis} denotes the associated Nisnevich sheaf.

Exercise 40. In the notations of Exercise 39, if F is a sheaf for the etale topology, so is G. For any Y, the (V,ϕ) as in (1),(2) above form a sheaf for the etale topology. It hence suffices to prove that for (V,ϕ) fixed, the datum (3) forms a sheaf for the etale topology. This is checked by using the following criterion to check if a Nisnevich sheaf is etale. For $y \in Y$, and for L a finite separable extension of k_y, let $\mathcal{O}^h_{L,y}$ be deduced by "extension of the residue field" from the henselization $\mathcal{O}^h_y$ of Y at y. The criterion is that $Spec(L) \mapsto F(Spec(\mathcal{O}^h_{L,y}))$ should be an etale sheaf on $Spec(k_y)_{et}$.

Exercise 41. It follows from Exercises 39 and 40 that if $f : F \to G$ is ind solid, and if F is etale, then G is etale. In particular, a solid sheaf, as well as a pointed solid sheaf, are etale sheaves.

Remark 2. The same formalism of open and solid morphisms holds in the site of all schemes of finite type over S with the etale topology.

4.2 A Criterion for Preservation of Local Equivalences

We work with pointed sheaves on QP/G. Our aim in this section is to prove the following result

Theorem 6. *Let Φ be a functor from pointed sheaves to pointed sets. Suppose that Φ commutes with all colimits, and that for any open embedding $U \to X$, $\Phi((h_U)_+) \to \Phi((h_X)_+)$ is a monomorphism. Then if $f : F_\bullet \to G_\bullet$ is a local equivalence and if F_n and G_n are (pointed) ind-solid, then $\Phi(f)$ is a weak equivalence.*

Suppose that

$$Q \quad : \quad \begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow \\ A & \longrightarrow & X \end{array}$$

is an upper distinguished square. Adding a base point, we obtain Q_+. The morphism $K_{Q_+} \to X_+$ is then a local equivalence. Let us check that $\Phi(K_{Q_+}) \to \Phi(X_+)$ is a weak equivalence. As Φ commutes with coproducts, this morphism is deduced from the commutative square

$$\begin{array}{ccc} \Phi((h_B)_+) & \longrightarrow & \Phi((h_Y)_+) \\ \downarrow & & \downarrow \\ \Phi((h_A)_+) & \longrightarrow & \Phi((h_X)_+) \end{array}$$

by applying the same construction (23.3). This square is cocartesian because Q is. The top horizontal line being a monomorphism, it is homotopy cocartesian, and the claim follows. As Φ commutes with colimits, this special case implies that more generally one has

Lemma 15. *If f is in the $\bar{\Delta}$-closure of the $(K_Q)_+ \to X_+$ as above, then $\Phi(f)$ is a weak equivalence.*

Proposition 42. *(Proof of Theorem 6)* For any pointed sheaf F, $R(F) \to F$ is a local equivalence. Indeed for any connected X in QP/G, $R(F)(X) \to F(X)$ is a weak equivalence by Construction 29. It follows that for $f : F \to G$ a local equivalence,

$$\begin{array}{ccc} R(F) & \longrightarrow & R(G) \\ \downarrow & & \downarrow \\ F & \longrightarrow & G \end{array}$$

is a commutative square of local equivalences. By Lemmas 15 and 12, $\Phi(R(f))$ is a weak equivalence. It remains to see that $\Phi(R(F)) \to \Phi(F)$ is a weak equivalence – and the same for G. For this it suffices to see that for a pointed ind-solid sheaf F, $\Phi(R(F)) \to \Phi(F)$ is a weak equivalence. As Φ and R commute with filtering colimits, the ind-solid reduces to solid, and by the inductive definition of solid, it suffices to prove the following lemma.

Lemma 16. *Let $U \to X$ be an open embedding. If in a cartesian square of pointed sheaves*

$$Q \quad : \quad \begin{array}{ccc} (h_U)_+ & \longrightarrow & (h_X)_+ \\ \downarrow & & \downarrow \\ F & \longrightarrow & G \end{array}$$

F is such that $\Phi R(F) \to \Phi(F)$ is a weak equivalence, the same holds for G.

Proof. Consider the cocartesian square

$$Q' \quad : \quad \begin{array}{ccc} R((h_U)_+) & \longrightarrow & R((h_X)_+) \\ \downarrow & & \downarrow \\ R(F) & \longrightarrow & R \end{array}$$

One can easily see that the top morphism is a monomorphism. It follows that Q' is point by point homotopy cocartesian, and $R \to R(G)$ is a local equivalence. The functor i^*i_* of Construction 29 transforms a monomorphism into a coprojection of the form $A \to A \vee ((\coprod h_{U_i})_+)$. It follows that each R_n is of the form $(\coprod h_{U_i})_+$ and, by Lemmas 15 and 12, $\Phi(R) \to \Phi(R(G))$ is a weak equivalence. It remains to show that $\Phi(R) \to \Phi(G)$ is a weak equivalence.

Let us apply Φ to the morphism of cocartesian squares $Q' \to Q$. By Construction 29 both $R((h_U)_+) \to (h_U)_+$ and $R((h_X)_+) \to (h_X)_+$ are homotopy equivalences, and remain so by applying Φ. We assumed $\Phi R(F) \to \Phi(F)$ to be a weak equivalence. As $\Phi(Q')$ and $\Phi(Q)$ are cocartesian with a top morphism which is a monomorphism (by the assumption on Φ, for Q), it follows that $\Phi(R) \to \Phi(G)$ is a weak equivalence. Hence so is $\Phi(R(G)) \to \Phi(G)$. □

5 Two Functors

5.1 The Functor $X \mapsto X/G$

One has a natural morphism of sites

$$\eta : (QP/G)_{Nis} \to (QP)_{Nis}$$

given by the functor

$$\eta^f : QP \to QP/G : X \mapsto (X \text{ with the trivial } G\text{-action})$$

Indeed, the functor η^f commutes with finite limits and transforms covering families into covering families.

In particular the functor η^f is continuous: if F is a sheaf on $(QP/G)_{Nis}$, the presheaf

$$X \mapsto F(X \text{ with the trivial } G\text{-action})$$

is a sheaf on $(QP)_{Nis}$. The functor η^f has a left adjoint $\lambda^f : X \mapsto X/G$. As η^f is continuous, the functor λ^f is cocontinuous, and the functor η^* from sheaves on $(QP)_{Nis}$ to sheaves on $(QP/G)_{Nis}$ is

$$F \mapsto (\text{sheaf associated to the presheaf } X \mapsto F(X/G))$$

Proposition 43. *The cocontinuous functor $\lambda^f : X \mapsto X/G$ is also continuous, that is, if F is a sheaf on $(QP)_{Nis}$, the presheaf $X \mapsto F(X/G)$ on $(QP/G)_{Nis}$ is a sheaf.*

Proof. By Lemma 2 it is sufficient to test the sheaf property of $X \mapsto F(X/G)$ for a covering of X deduced by pull-back from a Nisnevich covering $V_i \to X/G$ of X/G. Passage to quotient commutes with flat base change. Taking as base X/G, this gives that

$$X \times_{X/G} V_i \to V_i$$

identifies V_i with the quotient of $X \times_{X/G} V_i$ by G. Similarly, if $V_{ij} = V_i \times_{X/G} V_j$, the quotient by G of the pull-back to X of V_{ij} is V_{ij} again. This reduces the sheaf property of $X \mapsto F(X/G)$, for the covering of X by the $X \times_{X/G} V_i$, to the sheaf property of F for the covering (V_i) of X/G. □

The functor $\lambda^f : X \mapsto X/G$ gives rise to a pair of adjoint functors (λ_*, λ^*) between the categories of presheaves on (QP/G) and (QP), with $\lambda_*(F)$ being $X \mapsto F(X/G)$. As λ^f is continuous , it induces a similar pair of adjoint functors between the categories of sheaves. This pair is

$$(\eta_\# := (\text{associated sheaf}) \circ \lambda^*, \eta^* = \lambda_*),$$

so that one has a sequence of adjunctions $(\eta_\#, \eta^*, \eta_*)$. If F on (QP/G) is representable: $F = h_X$, then $\eta_\#(F) = h_{X/G}$. In particular, $\eta_\#$ transforms the final sheaf h_S on $(QP/G)_{Nis}$, also called "point", into the final sheaf on $(QP)_{Nis}$, and $(\eta_\#, \eta^*)$ is a pair of adjoint functors in the category of pointed sheaves as well. It is clear that $\eta_\#$ takes solid sheaves to solid sheaves. We also have the following.

Proposition 44. *Let F be a pointed sheaf solid with respect to open embeddings $U \subset V$ of smooth schemes such that the action of G on V is free outside U. Then $\eta_\#(F)$ is solid with respect to open embeddings of smooth schemes.*

Proof. If V' is the open subset of V where the action of G is free, then $U \cup V' = V$ and if $U' := U \cap V'$, a push-out of $U \to V$ is also a push-out of $U' \to V'$: we gained that the action is free everywhere. The next step is applying $\eta_\#$, from pointed sheaves on (QP/G) to pointed sheaves on (QP). This functor is a left adjoint, hence respects colimits and in particular push-outs. It transforms h_U to $h_{U/G}$, and in particular, for $U = S$, the final object into the final object. To check that it respects solidity it is hence sufficient to apply:

Lemma 17. *If G acts freely on U smooth over S, then U/G is smooth.*

Proof. If G is finite etale, for instance S_n, the case which most interests us, this is clear, resulting from $U \to U/G$ being etale. In general one proceeds as follows. The assumption that G acts freely on U implies that U is a G-torsor over U/G. In particular, $U \to U/G$ is faithfully flat. As U is flat over S, this forces U/G to be flat over S. To check smoothness of U/G over S it is hence sufficient to check it geometric fiber by geometric fiber. For $\bar{s}$ a geometric point of S, smoothness of $(U/G)_{\bar{s}}$ amounts to regularity. As $U_{\bar{s}}$ is smooth over $\bar{s}$, hence regular, and $U_{\bar{s}} \to (U/G)_{\bar{s}}$ is faithfully flat, this is [4] (an application of Serre's cohomological criterion for regularity). □

Proposition 45. *The functor $\eta_\#$ respects local (resp. $\mathbf{A}^1$-) equivalences between termwise ind-solid simplicial sheaves.*

Proof. Let $f : F \to F'$ be a local equivalence between termwise ind-solid simplicial sheaves on QP/G. To verify that $\eta_\#(f)$ is a local equivalence it is sufficient to check that for any X in QP and $x \in X$ the map

$$\eta_\#(F)(Spec\mathcal{O}^n_{X,x}) \to \eta_\#(F')(Spec\mathcal{O}^n_{X,x})$$

is a weak equivalence of simplicial sets. Since $\eta_\#$ is a left adjoint, the functor

$$F \mapsto \eta_\#(F)(Spec\mathcal{O}^n_{X,x}) \tag{45.1}$$

commutes with colimits. For an open embedding $U \to V$ in QP/G, $U/G \to V/G$ is again an open embedding and we can apply to (45.1) Theorem 6.

Let $f : F \to F'$ be an $\mathbf{A}^1$-equivalence. Consider the square

$$\begin{array}{ccc} R(F) & \xrightarrow{R(f)} & R(F') \\ \downarrow & & \downarrow \\ F & \xrightarrow{f} & F' \end{array}$$

By the first part of proposition $\eta_\#$ takes the vertical maps to local equivalences. Since $\eta_\#$ commutes with colimits, Lemma 13 implies that $\eta_\#(R(f))$ is in the $\bar{\Delta}$-closure of the class which contains $\eta_\#((K_Q)_+ \to X_+)$ for Q upper distinguished and $\eta_\#((X \times \mathbf{A}^1)_+ \to X_+)$ for X in QP/G. By Theorem 4 it suffice to prove that morphisms of these two types are $\mathbf{A}^1$-equivalences. For morphisms of the first type it follows from the first half of the proposition. For the morphism of the second type it follows from the fact that morphisms $\eta_\#((X \times \mathbf{A}^1)_+ \to X_+)$ and $\eta_\#(X_+ \overset{Id\times\{0\}}{\to} (X \times \mathbf{A}^1)_+)$ are mutually inverse $\mathbf{A}^1$-homotopy equivalences. □

Define $\mathbf{L}\eta_\# : Ho_\bullet \to Ho_\bullet$ (and similarly on $Ho_{\mathbf{A}^1,\bullet}$) setting

$$\mathbf{L}\eta_\#(F) := \eta_\#(R(F))$$

where $R(F)$ is defined in Construction 29. Proposition 45 shows that $\mathbf{L}\eta_\#$ is well defined and that for a termwise ind-solid F one has $\mathbf{L}\eta_\#(F) \cong \eta_\#(F)$.

5.2 The Functor $X \mapsto X^W$

As in Sect. 3.1, we fix G and S. We also fix W in QP/G which is finite and flat over S.

For F a presheaf on QP/G, we define F^W to be the internal hom object $\underline{Hom}(h_W, F)$. Its value on U is $F(U \times_S W)$. If F is a sheaf, so is F^W.

Example 6. Take G and W deduced from the finite group S_n acting on $\{1, \ldots, n\}$ by permutations. In that case, if F is represented by X, with a trivial action of S_n, then F^W is represented by X^n, on which S_n acts by permutation of the factors.

Remark 1. If F is representable (resp. and represented by X smooth over S), so is F^W. More precisely, if F is represented by X in QP/G, consider the contravariant functor on Sch/S of morphisms of schemes from W to X, that is the functor

$$U \mapsto Hom_U(W \times_S U, X \times_S U)$$

This functor is representable, represented by some Y quasi-projective over S (resp. and smooth). This Y carries an obvious action ρ of G, and (Y, ρ) in QP/G represents F^W. Proof: by attaching to a morphism $W \to X$ its graph, one maps the functor considered into the functor of finite subschemes of $W \times_S X$, of the same rank as W, that is the functor

$$U \mapsto \left\{ \begin{array}{c} \text{subschemes of } (W \times_S X) \times_S U \text{ finite and flat over } U, \\ \text{with the same rank as } W \times_S U \text{ over } U. \end{array} \right\}$$

The later functor is represented by a quasi-projective scheme *Hilb*, by the theory of Hilbert schemes. The condition that $\Gamma \subset W \times_S X$ be the graph of a morphism from W to X is an open condition. This means: let $\Gamma \subset (W \times_S X) \times_S U$ be a subscheme finite and flat over U. There is U' open in U such that for any base change $V \to U$, the pull-back Γ_V of Γ is the graph of some V-morphism from $W \times_S V$ to $W \times_S X$ if and only if V maps to U'. This gives the existence of the required Y, and that it is open in *Hilb*. If X is smooth the smoothness of Y follows from the infinitesimal lifting criterion. The quasi-projectivity follows from that of *Hilb*. On the functors represented, the action $g(y) = gyg^{-1}$ of G is clear. For T in QP/G, one has

$$Hom_{QP/G}(T, Y) = Hom_G(T, \underline{Hom}(W, X)) = Hom_G(T \times_S W, X) =$$

$$= Hom_{QP/G}(T \times_S W, X) = F^W(T)$$

Let C be a class of open embeddings in $(QP/G)_{Nis}$. We will simply say "solid" for "C-solid".

Theorem 7. *If F is a solid sheaf on $(QP/G)_{Nis}$, so is F^W.*

If a morphism of sheaves $A \to F$ is open, i.e. relatively representable by open (equivariant) embeddings, there is a natural sequence of sheaves intermediate between A^W and F^W. In the case considered in Example 6, and for $h_U \to h_X$, they are represented by the open equivariant subschemes $(X, U)^n_k$ of X^n consisting of those n-uples $(x_1, \ldots, x_n)$ for which at least k of the x_i are in U. The formal definition is as follows.

A section of F^W over T is a section s of F over $W \times_S T$. Let $U(s)$ be the equivariant open subscheme of $W \times_S T$ on which s is in A. The sheaf $(F, A)^W_k$ is the subsheaf of F^W consisting of those s such that all fibers $U(s)_t$ of $U(s)$ over T are of degree at least k. The condition that the fiber at k be of degree $\geq k$ is open in t, and it follows that the inclusion of $(F, A)^W_k$ in F^W is open. For $k = 0$, $(F, A)^W_k$ is simply F^W. For k large, it is A^W.

Lemma 18. *Suppose that $A \to F$ is deduced by push-out from an open map $B \to G$, so that we have a cocartesian square*

$$\begin{array}{ccc} B & \longrightarrow & G \\ \downarrow & & \downarrow \\ A & \longrightarrow & F \end{array} \tag{45.2}$$

Then, for each k, the cartesian square

$$\begin{array}{ccc} (\text{fiber product}) & \longrightarrow & (A \coprod G, A)_k^W \\ \downarrow & & \downarrow \\ (F, A)_{k+1}^W & \longrightarrow & (F, A)_k^W \end{array} \tag{45.3}$$

is cocartesian as well.

Proof. The site $(QP/G)_{Nis}$ has enough points: as a consequence of Lemma 2, for each X in QP/G and $x \in X/G$, the functor

$$F \mapsto colim \quad F(X \times_{X/G} V),$$

the limit being taken over the Nisnevich neighborhoods of x in X/G, is a point (= a fiber functor). The class of all such functors is clearly conservative. Such a functor depends only on $Y := X \times_{X/G} (X/G)_x^h$, where $(X/G)_x^h$ is the henselization of X/G at x, and Y can be any equivariant S-scheme which is a finite disjoint union of local henselian schemes essentially of finite type over S, and for which Y/G is local. We call such a scheme G-local henselian, and write $F \mapsto F(Y)$ for the corresponding fiber functor.

We will show that (45.3) becomes cocartesian after application of any of the fiber functors $F \mapsto F(Y)$ defined above. It suffices to show that for any s in $(F, A)_k^W(Y)$, the fiber of (45.3)(Y) above s is cocartesian in Set. This fiber is of the form

$$\begin{array}{ccc} K \times L & \longrightarrow & K \\ \downarrow & & \downarrow \\ L & \longrightarrow & \{s\} \end{array}$$

and such a square is cocartesian if and only if whenever K or L is empty, the other is reduced to one element. Here, we also know that $L \to \{s\}$ is a injective. It hence suffice to check that if L is empty, then K is reduced to one element. Fix s in $(F, A)_k^W(Y)$, and view it as a section of F over $W \times_S Y$. Let $U \subset W \times_S Y$ be the open equivariant subset where it is in A. The assumption that s be in $(F, A)_k^W$ means that the degree of the fiber U_g of $U \to Y$ at a closed point y of Y is at least k. By G-equivariance of U, this degree is independent of the chosen y. We have to show that if s is not in $(F, A)_{k+1}^W(Y)$, that is if this degree is exactly k, then s is the image of a unique element of $(A \coprod G, A)_k^W$.

The scheme $W \times_S Y$ is a disjoint union of G-local henselian schemes $(W \times_S Y)_i$. By assumption, (45.2)$((W \times_S Y)_i)$ is cocartesian, hence if s is not in A on $(W \times_S Y)_i$, then on $(W \times_S Y)_i$ it comes from a unique $\tilde{s}_i$ in G. Let $(W \times_S Y)'$ be the union of those $(W \times_S Y)_i$ on which s is in A, and $(W \times_S Y)''$ be the union of $(W \times_S Y)_i$ on which it is not. That s is in $(F, A)^W_k$ but not in $(F, A)^W_{k+1}$, means that $(W \times_S Y)'$ is of degree $d = k$ over Y. On $(W \times_S Y)'$, s is in A. On $(W \times_S Y)''$, it comes from a unique $\tilde{s}$ in G. The section

$$s_1 := (s \text{ in } A \text{ on } (W \times_S Y)', \tilde{s} \text{ on } (W \times_S Y)'')$$

of $A \coprod G$ over $W \times_S Y$ is a section of $(A \coprod G, A)^W_k$ on Y lifting s. It is the unique such lifting: any other lifting s_2, viewed as a section of $A \coprod G$ on $W \times_S Y$, can be in A at most on $(W \times_S Y)'$, hence must be in A on the whole of $(W \times_S Y)'$ which has just the required degree over Y. This determines s_2 uniquely on $(W \times_S Y)'$, where it is in A, as well as on $(W \times_S Y)''$, where it is the unique lifting of s to G. □

Proof of Theorem 7: The induction which works to prove Theorem 7 is the following. As F is solid, it sits at the end of a sequence

$$\emptyset \to F_1 \to \ldots \to F_n = F$$

where each $F_i \to F_{i+1}$ is a push-out of some open embedding in QP/G. We prove by induction on i that for any Y, $(F_i \coprod h_Y)^W$ is solid. For $i = 1$, F_1 is representable, hence so is $(F_1 \coprod h_Y)^W$ (1). A fortiori, $(F_1 \coprod h_Y)^W$ is solid. For the induction step one applies the following lemma to $F_i \coprod h_Y \to F_{i+1} \coprod h_Y$.

Lemma 19. *Let*

$$\begin{array}{ccc} h_U & \longrightarrow & h_X \\ \downarrow & & \downarrow \\ F & \longrightarrow & G \end{array}$$

be a cocartesian square with U open in X. Suppose that for any Z, $(F \coprod h_Z)^W$ is solid. Then $F^W \to G^W$ is solid.

Proof. As $F \to G$ is open by Proposition 33, the $(G, F)^W_j$ are defined. It suffices to prove that for each j, the open morphism $(G, F)^W_{j+1} \to (G, F)^W_j$ is solid.

By Lemma 18, this morphism sits in a cartesian and cocartesian square

$$\begin{array}{ccc} (\text{fiber product}) & \xrightarrow{[2]} & (F \coprod h_X, F)^W_k \\ \downarrow & & \downarrow \\ (G, F)^W_{j+1} & \xrightarrow{[1]} & (G, F)^W_j \end{array} \tag{45.4}$$

By assumption, $(F \coprod h_X)^W$ is solid. It follows that $(F \coprod h_X)^W_k$ is solid too (apply Proposition 37 to the open morphism $(F \coprod h_X)^W_k \to (F \coprod h_X)^W$). As [1] is open,

so is [2], and by Proposition 36, [2] is solid. The map [1] is then solid as a push-out of a solid map. □

Example 7. It is not always true that if $f : A \to B$ is a solid morphism, so is f^W. Take G the trivial group and W two points (i.e. $S \coprod S$). Then $F^W = F^2$. For any sheaf F, the inclusion f of F in $F \coprod pt$ is solid (deduced by push-out from $\emptyset \to pt$), and applying $(-)^W$, we obtain $F^2 \to F^2 \coprod F \coprod F \coprod pt$. Pulling back by the natural map from F to one of the summands F (an open map), we see that if f^W is solid, so is F.

Corollary 46. *If $f : F \to G$ is open and G solid, then $f^W : F^W \to G^W$ is solid. In particular, if G is pointed solid, so is G^W.*

Proof. f^W is open and one applies Theorem 7 and Proposition 36. □

We now define for pointed sheaves on $(QP/G)_{Nis}$ a "smash" variant of the construction $F \mapsto F^W$. If we assume that the marked point $pt \to F$ is open, it is

$$F^{\wedge W} := F^W/(F, pt)_1^W$$

that is F^W with $(F, pt)_1^W$ contracted to the new base point. This definition can be repeated for any pointed sheaf if, for any monomorphism $A \to F$, we define$(F, A)_1^W$ to be the following subsheaf of F^W: a section s of $F^W(X) = F(X \times W)$ is in $(F, A)_1^W$ if for any non empty $X' \to X$, there exists a commutative diagram

$$\begin{array}{ccc} X'' & \longrightarrow & X \times W \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

with X'' non empty and s in A on X''.

Example 8. Under the assumptions of Example 6, if U is open in X with complement Z and if $F = h_X/h_U$, then $F^{\wedge W}$ is $h_{X^n}/h_{X^n - Z^n}$, where X^n has the natural action of S_n. In particular, if F is the Thom space of a vector bundle E over Y (that is, h_E with $h_{E - s_0(Y)}$ contracted to the base point), then $F^{\wedge W}$ is the Thom space of the S_n-equivariant vector bundle $\oplus pr_i^* E$ on Y^n.

Proposition 47. *If a pointed sheaf F is pointed solid, that is if $pt \to F$ is solid, then so is $F^{\wedge W}$.*

Proof. As F^W is solid, the open morphism $(F, pt)_1^W \to F^W$ is solid too (Proposition 36), and $pt \to F^{\wedge W}$ is deduced from it by push-out. □

The definition of $F^{\wedge W}$ immediately implies the following:

Lemma 20. *Let Y be a G-local henselian scheme. Then*

$$F^{\wedge W}(Y) = \bigwedge_i F((W \times Y)_i)$$

where $(W \times Y)_i$ *are* G*-local henselian schemes such that*

$$W \times Y = \coprod_i (W \times Y)_i$$

Proposition 48. *The functor* $F^{\wedge W}$ *respects local and* $\mathbf{A}^1$*-equivalences.*

Proof. Let $f : F \to H$ be a local equivalence. To check that $F^W \to H^W$ is a local equivalence it is enough to show that for any G-local henselian Y, the map

$$F^W(Y) = F(Y \times W) \to H(Y \times W) = H^W(Y)$$

is a weak equivalence of simplicial sets. This follows from Lemma 20.

Let f be an $\mathbf{A}^1$-equivalence. By the first part it is sufficient to show that $R(f)^W : R(F)^W \to R(H)^W$ is an $\mathbf{A}^1$-equivalence. We use the characterization of $\mathbf{A}^1$-equivalences given in Theorem 5. Since $F \mapsto F^{\wedge W}$ commutes with filtering colimits and preserves local equivalences it suffices to check that it takes $\mathbf{A}^1$-homotopy equivalences to $\mathbf{A}^1$-homotopy equivalences. This is seen using the natural map

$$F^{\wedge W} \wedge (h_{\mathbf{A}^1})_+ \to (F \wedge (h_{\mathbf{A}^1})_+)^{\wedge W}.$$

□

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References

1. K.S. Brown and S.M. Gersten. *Algebraic K-theory and generalized sheaf cohomology*. Lecture Notes in Mathematics, vol. 341. Springer, Heidelberg, 1973, pp. 266–292.
2. A. Grothendieck, M. Artin, and J.-L. Verdier. *Theorie des topos et cohomologie etale des schemas (SGA 4)*. Lecture Notes in Mathematics, vols. 269, 270, 305. Springer, Heidelberg, 1972–1973.
3. A. Grothendieck and M. Demazure. *Schemas en groupes (SGA 3)*. Lecture Notes in Mathematics, vols. 151, 152, 153. Springer, Heidelberg, 1970.
4. A. Grothendieck and J. Dieudonne. *Etude Locale des Schemas et des Morphismes de Schemas (EGA 4). Publ. Math. IHES* 20,24,28,32, 1964–1967.
5. Mark Hovey. *Model categories*. AMS, Providence, 1999.
6. S. MacLane. *Categories for the working mathematician*. Graduate texts in Mathematics, vol. 5. Springer, Berlin, 1971.
7. Carlo Mazza, Vladimir Voevodsky, and Charles Weibel. *Lecture notes on motivic cohomology*. Clay Mathematics Monographs, vol. 2. American Mathematical Society, Providence, RI, 2006.
8. Fabien Morel and Vladimir Voevodsky. $\mathbf{A}^1$-homotopy theory of schemes. *Publ. Math. IHES*, 90:45–143, 1999.
9. Charles A. Weibel. *An introduction to homological algebra*. Cambridge University Press, Cambridge, 1994.
10. P. Gabriel and M. Zisman Calculus of fractions and homotopy theory, Springer-Verlag, Berlin, 1967.
11. J.F. Jardine Simplicial presheaves, *J. Pure Appl. Algebra*, 47, 1987, 35–87.