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Abel's Theorem and Webs

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Dedicated to Professors
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Index of notations and list of structure equations

General

$\boldsymbol{P}^n$ = projective space
$\boldsymbol{G}(k,n)$ = Grassmann manifold of projective k-planes in $\boldsymbol{P}^n$
$\boldsymbol{P}^{n*}$ = dual projective space of hyperplanes in $\boldsymbol{P}^n$

§ I

p = point in $\boldsymbol{P}^n$ with homogeneous coordinates $[z_0,\dots,z_n]$;
$\{p_1,\dots,p_d\}$ = linear span of $p_1,\dots,p_d \in \boldsymbol{P}^n$;
$S \oplus S'$ = span in $\boldsymbol{P}^n$ of linear subspaces S and S';
ξ = point in $\boldsymbol{P}^{n*}$ defining a hyperplane $\sum_{a=0}^{n} \xi_a z_a = 0$;
$r(Q)$ = rank of a quadric Q in $\boldsymbol{P}^n$;
$\pi = \pi(C)$ = genus of algebraic curve C;
$h^i(\ )$, $\barwedge$, $|V|$, Ω^1, $[D]$, $\mathfrak{O}(D)$, K_M are standard notations from algebraic geometry explained in § I;
$\varkappa\colon C \to \boldsymbol{P}^{\pi-1}$ canonical mapping with image $C_\varkappa$.

§ II

$x \in U$ an open set in $\boldsymbol{R}^n$ or $\boldsymbol{C}^n$;
$\{u_i(x) = \text{constant}\}$ defines a d-web in U;
$\boldsymbol{P}(T_x^*)$ = projectivized cotangent space at $x \in U$;
$\omega^i(x) \in \boldsymbol{P}(T_x^*)$ denotes the i^{th} web normal;
$\mathrm{d}\omega^i = \pi^i \wedge \omega^i$ (no summation) defines π^i;
rank of a d-web is $r \leqq \pi(d,n)$ = Castelnuovo's number;
$Z_i(x) \in \boldsymbol{P}^{d-n-1}(x) \subset \boldsymbol{P}^{r-1}$;
$\{Z_1(x),\dots,Z_d(x)\} = \boldsymbol{P}^{d-n-1}(x)$

§ III

$D_x \subset \boldsymbol{P}(T_x^*)$ and $E_x \subset \boldsymbol{P}^{d-n-1}(x)$ are rational normal curves;
$D_x \barwedge E_x$ projectivity carrying $\omega^i(x)$ to $Z_i(x)$;
$\phi^1,\dots,\phi^n$ = moving coframe for $D_x \subset \boldsymbol{P}(T_x^*)$;
$\mathrm{d}\phi^\alpha = \sum_\beta \phi^\beta \wedge \phi_\beta^\alpha$ is a symmetric connection;

$P^{n-2}(x,\sigma) = \{Z \in P^{d-n-1}(x) : \mathrm{d}Z/\mathrm{d}\sigma \in P^{d-n-1}(x)\}$
(*Note:* Suppose $F : U \to G(k,n)$ is a smooth mapping given by $x \to P^k(x) \subset P^n$. For a point $Z \in P^k(x)$ and tangent direction $\sigma \in P(T_x^*)$ we choose a curve $x(t)$ with $x(0) = x$ and with tangent $x'(0)$ in the direction σ and $Z(t) \in P^k(x(t))$ with $Z(0) = Z$. For a lifting $\tilde{Z}(t)$ of $Z(t)$ to R^{n+1} we let

$$\mathrm{d}Z/\mathrm{d}\sigma = \text{projection to } P^n \text{ of } \tilde{Z}'(0),$$

and note that $\mathrm{d}Z/\mathrm{d}\sigma$ is well-defined modulo $P^k(x)$. In particular the condition

$$\mathrm{d}Z/\mathrm{d}\sigma \in P^k(x)$$

is intrinsic. Geometrically, this means that as we move along the curve $x(t)$ the linear space $P^k(x(t)) \subset P^k(x)$ contains Z up to 2nd order.)
$\mathrm{d}Z_i + \pi^i Z_i = Z_i'\omega^i$ (no summation);
$\omega^i(x) = \sum_\alpha t_i^\alpha(x)\phi^\alpha(x)$; and
$\mathrm{d}t_i^\alpha = \pi^i t_i^\alpha + \sum_\beta t_i^\beta \phi_\alpha^\beta + \sum_\beta t_{i\alpha\beta}\phi^\beta$ where $t_{i\alpha\beta} = t_{i\beta\alpha}$

§ *IV*

$\mathscr{F}(P^n)$ = manifold of frames $F = \{Z_0,\dots,Z_n\}$, $Z_a \in R^{n+1}$;
$\{\theta^\alpha, \phi_\beta^\alpha, \phi_\alpha^0\}$ = projective connection matrices;
$\Theta^\alpha = \mathrm{d}\theta^\alpha - \theta^\beta \wedge \phi_\beta^\alpha$ = projective torsion;
$\Phi_\beta^\alpha = \mathrm{d}\phi_\beta^\alpha - \phi_\beta^\gamma \wedge \phi_\gamma^\alpha - \theta_\beta^0 \wedge \theta^\alpha + \delta_\beta^\alpha \theta^\gamma \wedge \theta_\gamma^0$
$= 1/2\{R_{\beta\gamma\lambda}^\alpha \theta^\gamma \wedge \theta^\lambda\}$, $R_{\beta\gamma\lambda}^\alpha + R_{\beta\lambda\gamma}^\alpha = 0$, denotes the projective curvature;
$\Theta_\beta^0 = \mathrm{d}\theta_\beta^0 - \phi_\beta^\alpha \wedge \phi_\alpha^0$;
$\Theta^\alpha = 0$ and $R_{\alpha\beta\gamma}^\alpha = 0$ define normal a projective connection.

Finally, we shall use the following

Ranges of indices

$1 \leqq i \leqq d,$
$1 \leqq \alpha,\beta,\gamma \leqq n,$
$1 \leqq \varrho,\sigma,\tau \leqq 2n,$
$n+1 \leqq s \leqq d,$
$0 \leqq a,b,c \leqq n.$

Introduction

In recent years there have been important developments in the study of the global properties of foliations. A foliation is, briefly speaking, a local slicing of a manifold M (supposed to be C^∞). If $x^1,\dots,x^n$ are local coordinates on M, a foliation of dimension $n-k$, or codimension k, has leaves defined by equations of the form

(1) $$F_1(x) = \text{const},\dots,\ F_k(x) = \text{const},\ x = (x^1,\dots,x^n),$$

where $F_1,\dots,F_k$ are smooth functions such that the Jacobian matrix

$$(2) \qquad (\partial F_i/\partial x^\alpha), \ 1 \leqq i \leqq k, \ 1 \leqq \alpha \leqq n,$$

is of rank k everywhere. The functions $F_1,\dots,F_k$ are defined up to an arbitrary C^∞ transformation. A *d-web* consists of d foliations. Throughout we make the additional assumption that the leaves are everywhere in general position.

It was Blaschke who began in the thirties a systematic study of webs. Web geometry has intimate contacts with the foundations of geometry, differential geometry, and algebraic geometry. From 1927 to 1938 Blaschke and his co-workers and students published 66 papers under the general heading "Topologische Fragen der Differentialgeometrie". These and other results were given a unified account in the book "Geometrie der Gewebe", by W. Blaschke and G. Bol [1].

From a projective variety a web can be constructed as follows: Let V^k be an algebraic variety of dimension k and degree d in a projective space $\boldsymbol{P}^n$ of dimension n. Then V^k meets a linear space $\boldsymbol{P}^{n-k}$ of dimension $n-k$ in d points. Consider the Grassmann manifold $\boldsymbol{G}(n-k,n)$ of all $\boldsymbol{P}^{n-k}$'s in $\boldsymbol{P}^n$; its dimension is $k(n-k+1)$. The $\boldsymbol{P}^{n-k}$'s through a point of $\boldsymbol{P}^n$ form a submanifold of dimension $k(n-k)$. Thus the d points of intersection of V^k with a given $\boldsymbol{P}^{n-k}$ associate to $\boldsymbol{P}^{n-k}$ d submanifolds of $\boldsymbol{G}(n-k,n)$, of dimension $k(n-k)$ or codimension k, which contain $\boldsymbol{P}^{n-k}$. This shows that V^k defines in $\boldsymbol{G}(n-k,n)$, or at least in a neighborhood of it, a d-web of codimension k. In particular, for $k=1$, an algebraic curve of degree d in $\boldsymbol{P}^n$ defines in the dual space $\boldsymbol{P}^{n*}$ ($=\boldsymbol{G}(n-1,n)$) a d-web of codimension 1. We will call *algebraic* the web defined from V^k by the above construction. In this sense web geometry generalizes the geometry of projective varieties.

The relationship between web geometry and algebraic geometry goes much deeper. In this paper we will restrict ourselves to the study of webs of codimension one, and mostly to their local properties. The d foliations defining the web will each be defined by an equation

$$(3) \qquad u_i(x^1,\dots,x^n) = \text{const}, \quad 1 \leqq i \leqq d,$$

where we suppose u_i to be a smooth function, with gradient $\neq 0$. The function u_i can be replaced by a function $v_i(u_i)$ with $v_i' \neq 0$ without changing the i^{th} foliation.

An equation of the form

$$(4) \qquad \sum_i f_i(u_i)\,\mathrm{d}u_i = 0$$

is called an *abelian* equation. An abelian equation remains an abelian equation under the above change $u_i \to v_i(u_i)$. The validity of an abelian equation is a strong property on the web. The number of linearly independent abelian equations (over the constants) is called the *rank* of the web. We have the following theorem[0]):

Let r be the rank of a d-web of codimension 1 *in* $\boldsymbol{R}^n$, $d \geqq n+1$. *Then*

$$r \leqq \pi(d,n), \tag{5}$$

where

$$\pi(d,n) = 1/2(n-1)\{(d-1)(d-n) + s(n-s-1)\}, \tag{6}$$

s being defined by

$$s \equiv (-d+1)\operatorname{mod}(n-1), \quad 0 \leqq s \leqq n-2. \tag{7}$$

$\pi(d,n)$ *is an integer.*

The integer $\pi(d,n)$ plays a role in the theory of algebraic curves. In 1889 Castelnuovo proved that a non-degenerate algebraic curve of degree d in a complex projective space $\boldsymbol{P}^n$ of dimension n has a genus $\leqq \pi(d,n)$. (A curve is non-degenerate if it does not lie in a lower dimensional projective space.) Those for which the maximum genus is attained are called *extremal curves*. They were investigated by Castelnuovo. In particular, they lie on special ruled surfaces, the Castelnuovo surfaces.

The algebraic web constructed from a Castelnuovo extremal curve C is of maximum rank $\pi(d,n)$. For let ω_λ, $1 \leqq \lambda \leqq \pi = \pi(d,n)$, be the linearly independent abelian differentials on C. By Abel's theorem, we have

$$\sum_i \int_{P_0}^{P_i} \omega_\lambda = \text{const}, \tag{8}$$

where P_0 is a fixed point on C and P_i are the points of intersection of C by a hyperplane. In differentials this relation can be written

$$\sum_i \omega_\lambda(P_i) = 0, \quad 1 \leqq \lambda \leqq \pi, \tag{9}$$

which are the π abelian equations of the web. Needless to say, this argument applies to the complex domain. A careful analysis gives a d-web of codimension 1 in a neighborhood of $\boldsymbol{R}^n$ with rank $\pi(d,n)$. (Cf. [3].)

A fundamental problem in web geometry is whether a d-web of codimension 1 in $\boldsymbol{R}^n$ of rank $\pi(d,n)$ is necessarily algebraic and is obtained from an extremal curve by the above construction. The problem can be separated into two parts:

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[0]) The result appears in the paper [3] by the first author. It is number **T 60** in the series "Topologische Fragen der Differentialgeometrie" mentioned above, and constituted part of the first author's dissertation written in Hamburg under the supervision of Blaschke.

a) Linearization problem. Can all the leaves of the web become hyperplanes under a change of coordinates?

b) Algebraization problem. If all the leaves of a web are hyperplanes, under what conditions will they belong to an algebraic variety?

In the plane ($n = 2$) the answer to the linearization problem is negative. In fact, Bol gave an example of a plane 5-web of maximum rank 6 whose leaves cannot be mapped into straight lines by a change of coordinates. It is of interest to remark that one of the abelian equations in Bol's example involves Euler's dilogarithm, which plays a role in several problems of current interest: the volume of an odd-dimensional simplex in a non-euclidean space, the combinatorial formula for the first Pontrjagin number of a compact oriented 4-manifold, etc.

The book of Blaschke-Bol quoted above culminates with the following theorem of Bol:

A d-web of codimension 1 *in* $\boldsymbol{R}^3$, $d \neq 5$, *of maximum rank* $\pi(d,3)$ *is equivalent to a web with plane leaves. The leaves belong to an algebraic curve of degree d in the dual projective space* $\boldsymbol{P}^{3*}$.

By "equivalence" is meant "local equivalence", i.e., change of coordinates in a sufficiently small neighborhood. The same is understood in the following theorem, which is the main result of this paper:

Theorem. *Consider a d-web of codimension* 1 *in* $\boldsymbol{R}^n$ *of maximum rank* $\pi(d,n)$. *Suppose that* $n \geqq 3$, $d \geqq 2n$. *Then the web is linearizable, i.e., equivalent to a web whose leaves are hyperplanes.*

The idea of the proof is to compare the geometry in a neighborhood $U \subset \boldsymbol{R}^n$ where the web is given with that of an auxiliary projective space. In fact, let

$$\sum_i f_i^\lambda(u_i)\,du_i = 0\,, \quad 1 \leqq \lambda \leqq \pi = \pi(d,n) \tag{10}$$

be the linearly independent abelian equations. By interpreting

$$Z_i(x) = [f_i^1(u_i),\ldots,f_i^\pi(u_i)]\,, \quad 1 \leqq i \leqq d \tag{11}$$

as the homogeneous coordinates in an auxiliary projective space $\boldsymbol{P}^{\pi-1}$, we obtain a mapping $U \to \boldsymbol{G}(d-n-1,\pi-1)$ with $x \to \{Z_1(x),\ldots,Z_d(x)\} = \boldsymbol{P}^{d-n-1}(x)$, to be called the *Poincaré mapping*. The d points $Z_i(x), x \in U$, determine a normal curve $E(x)$ in the linear space $\boldsymbol{P}^{d-n-1}(x)$ of $\boldsymbol{P}^{\pi-1}$. To a curve in U corresponds ∞^1 normal curves $E(x)$. Our main lemma is to show that the curves in U whose corresponding normal curves pass through $n-1$ fixed points are the integral curves of a system of ordinary differential equations of the second order. This defines intrinsically a family of paths.

On the other hand, in the projectivized cotangent space $\boldsymbol{P}(T^*)$ (of dimension $n-1$) at every point $x \in U$ the web of maximum rank defines a normal curve C_x, which contains the tangent hyperplanes of the leaves. (The points of $\boldsymbol{P}(T^*)$ and the hyperplanes of the tangent space T_x can be identified by the pairing of T_x and T_x^*.) The curves C_x and $E(x)$ are in a projective correspondence. When $n = 3$, the normal curves C_x are conics and the field of C_x for $x \in U$ leads intrinsically to a Weyl geometry. The general case is considerably more complicated. We believe we are justified in saying that it is Elie Cartan's method of moving frames that effectively leads to the goal. The computations involve some mysterious and unexpected simplifications, to be expected of a good geometrical problem.

The path geometry so introduced has ∞^2 totally geodesic hypersurfaces which include the leaves of the web. The presence of these totally geodesic hypersurfaces implies that the normal projective connection associated to the path geometry is flat, and that the paths can be mapped into straight lines by a local change of coordinates. This mapping carries the totally geodesic hypersurfaces into hyperplanes.

Our proof contains also a simplification of Bol's proof in the case $n = 3$, by proving a sufficient condition for the flatness of a projective connection. (Cf. § IV. B.) Bol had to resort to a theorem of Enriques in algebraic geometry to complete his proof. Our proof of the main theorem is purely differential-geometric.

Once the linearization problem is solved, the affirmative answer to the algebraization problem is given by a converse of Abel's theorem. For details we refer the reader to a recent paper by the second author [4]. The relevant theorem goes back to Sophus Lie and can be stated as follows:

Suppose A_0 is a hyperplane in $\boldsymbol{P}^n$ and $C_1, \ldots, C_d$ are arcs each meeting A_0 in a single point. Suppose there are abelian forms $\omega_i \not\equiv 0$ on C_i such that

$$\sum_i \omega_i(A \cdot C_i) = 0, \quad 1 \leqq i \leqq d, \tag{12}$$

is valid for hyperplanes A varying in a neighborhood of A_0. Then there is an algebraic curve C in $\boldsymbol{P}^n$ and an abelian form ω on C such that $C_i \subset C$, $\omega | C_i = \omega_i$.

By duality in the projective space $\boldsymbol{P}^n$ this theorem can be translated to the following theorem on webs:

Consider a d-web in a neighborhood of $\boldsymbol{R}^n$ whose leaves are hyperplanes such that an abelian equation (4) *holds, with $f_i(u_i) \not\equiv 0$. Then the leaves belong to an algebraic curve of degree d in the dual projective space.*

Observe that in this theorem it is sufficient to have one abelian equation.

Added in proof: A summary of the proof of our main result appears in the paper "Linearization of Webs of Codimension One and Maximum Rank", to appear in Proc. of the Int. Symp. on Alg. Geom., Kyoto, Japan (1977).

I. Extremal algebraic curves

A. Rational normal curves

i. *Definition and basic properties.* An *algebraic curve* is given by a holomorphic mapping

$$f : C \to \boldsymbol{P}^n$$

from a compact Riemann surface C into the complex projective space of dimension n. The curve is *non-degenerate* in case the image does not lie in a lower dimensional linear space, and it is *normal* in case the image is not the projection of a curve in some $\boldsymbol{P}^{n'}$ for $n' > n$. In this paper it will usually be the case that f is a smooth embedding, and we shall identify C with its image $f(C)$ when there is no ambiguity.

In order to facilitate our discussion of algebraic curves it will be convenient to use the language of line bundles and Chern classes – cf. Chapters 0 and I of [5].

Over a compact complex manifold M we consider a holomorphic line bundle $L \to M$. We denote its *Chern class* by $c_1(L) \in H^2(M,\boldsymbol{Z})$. The two most important cases for us are when M is $\boldsymbol{P}^n$ or a compact Riemann surface; then $H^2(M,\boldsymbol{Z}) \cong \boldsymbol{Z}$ and the Chern class will be an integer called the *degree* and denoted by deg (L). The space $H^0(M,\mathfrak{O}(L))$ of global holomorphic sections is a finite dimensional vector space, frequently denoted by $H^0(L)$. Two sections s,s' in $H^0(L)$ have the same zero divisors if and only if $s = \lambda s'$ $(\lambda \in \boldsymbol{C})$ is a constant multiple of s'. A *linear system* is the projective space $\boldsymbol{P}(V)$ of lines through the origin in a linear subspace $V \subset H^0(L)$; thus $\boldsymbol{P}(V)$ may be identified with the divisors of sections $s \in V$. We shall sometimes use the classical notation $|V|$ instead of $\boldsymbol{P}(V)$. The linear system is *complete* in case $V = H^0(L)$. A *base point* is a point $p \in M$ where all sections $s \in V$ are zero. A linear system with no base points defines a holomorphic mapping

$$f : M \to \boldsymbol{P}(V^*)$$

by

$$f(p) = \{s \in \boldsymbol{P}(V) : s(p) = 0\}.$$

In other words, $p \in M$ is mapped to the hyperplane in $\boldsymbol{P}(V)$ consisting of all divisors in the linear system which pass through p.

The unique line bundle of degree $+1$ over $\boldsymbol{P}^n$ will be denoted by $H \to \boldsymbol{P}^n$ and called the *hyperplane line bundle*. Using homogeneous coordinates $z = [z_0,\ldots,z_n]$ the space $H^0(\boldsymbol{P}^n,\mathfrak{O}(H))$ of global holomorphic sections is naturally identified with the linear functions

$$\xi(z) = \sum_{\nu=0}^{n} \xi_\nu z_\nu$$

on $\boldsymbol{P}^n$. The divisor ξ of this section is the linear hyperplane $\xi(z) = 0$, and consequently the complete linear system is the dual projective space $\boldsymbol{P}^{n*}$ of hyperplanes in $\boldsymbol{P}^n$. This linear system is base point free, and using *projective duality*

$$(\boldsymbol{P}^{n*})^* \cong \boldsymbol{P}^n$$

to identify the hyperplanes passing through $p \in \boldsymbol{P}^n$ with p itself makes the corresponding map $\boldsymbol{P}^n \to \boldsymbol{P}(H^0(\boldsymbol{P}^n, \mathfrak{O}(H))^*)$ the identity.

Given any non-degenerate holomorphic mapping $f : M \to \boldsymbol{P}^n$, we may set $L = f^*H$ and $V = f^*H^0(\boldsymbol{P}^n, \mathfrak{O}(H))$ to obtain a holomorphic line bundle $L \to M$ and base point free linear system $V \subset H^0(L)$ whose corresponding map is just f. The "dictionary"

$$\left\{\begin{array}{l}\text{non-degenerate}\\ \text{holomorphic mappings}\\ f : M \to \boldsymbol{P}^n\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\text{holomorphic line bundles}\\ L \to M \text{ and base point free}\\ \text{linear subspaces } V \subset H^0(L)\end{array}\right\}$$

will be used throughout.

In case $M = C$ is a compact Riemann surface and $f : C \to \boldsymbol{P}^n$ a non-degenerate algebraic curve $-$ we always assume that f is generically one to one $-$ then the *degree* $\deg(C)$ is defined to be $\deg(f^*H)$. It is the number of points in which the image meets a hyperplane. The algebraic curve is normal if and only if the corresponding linear system is complete.

In order to illustrate the above dictionary, let us prove that a non-degenerate curve in $\boldsymbol{P}^n$ has degree $\geqq n$. Using the language of line bundles, we set $L = f^*H$ and let $\mathfrak{O}(L - p) \subset \mathfrak{O}(L)$ be the subsheaf of sections vanishing at a point $p \in C$. Then $\mathfrak{O}(L - p)$ is the sheaf of the line bundle $L \otimes [-p]$, and the exact sheaf sequence

$$0 \to \mathfrak{O}(L - p) \to \mathfrak{O}(L) \to L_p \to 0$$

is valid. From the exact cohomology sequence we obtain

$$h^0(L) \leqq h^0(L - p) + 1$$

where the notation

$$h^i(L) = \dim H^i(M, \mathfrak{O}(L))$$

will be used consistently. Applying this inequality recursively gives

$$h^0(L) \leqq \deg(L) + 1$$

since $\deg(L \otimes [-p]) = \deg L - 1$ and $h^0(L') = 0$ in case $\deg L' < 0$. In particular, since $h^0(f^*H) \geqq n + 1$ we obtain $\deg(C) \geqq n$.

The corresponding geometric argument is the following: Linear projection from the point $p \in f(C)$ gives

$$\begin{tikzcd} & \boldsymbol{P}^n \arrow[dd] \\ C \arrow[ur, "f"] \arrow[dr, "f'"'] & \\ & \boldsymbol{P}^{n-1} \end{tikzcd}$$

where $f'(C)$ has degree at least one less than $f(C)$. Since an algebraic curve of degree one is a straight line it follows that $\deg(C) \geqq n$.

The line bundle corresponding to f' is $L \otimes [-p]$, and the associated linear system is just $H^0(C, \mathfrak{O}(L - p)) \subset H^0(C, \mathfrak{O}(L))$. So the two arguments bounding the degree correspond under our dictionary.

The general result is this: A non-degenerate algebraic variety $V \subset \boldsymbol{P}^n$ of dimension k has degree at least $n - k + 1$. Thus a surface has degree $\geqq n - 1$, etc. Varieties of minimal degree are quite interesting, and will play an important role for us. We begin with the case of curves.

Definition. *A rational normal curve is a non-degenerate curve of degree n in* $\boldsymbol{P}^n$.

We shall prove that a rational normal curve is rational — i.e., that C is the Riemann sphere $\boldsymbol{P}^1$ — and that f is a smooth embedding. The curve is normal because it has degree n.

The second geometric argument used to bound the degree shows that the projection of a rational normal curve in $\boldsymbol{P}^n$ is a rational normal curve in $\boldsymbol{P}^{n-1}$. Iterating this we obtain

$$\begin{tikzcd} & \boldsymbol{P}^n \arrow[dd] \\ C \arrow[ur, "f"] \arrow[dr, "F"'] & \\ & \boldsymbol{P}^{2} \end{tikzcd}$$

where $F(C)$ has degree two. It follows that $F(C)$ is first of all a plane conic, and secondly is smooth since otherwise it would consist of two straight lines. By projecting once more (stereographic projection), we find that $F(C)$ is isomorphic to $\boldsymbol{P}^1$. This proves that C is rational and f is a smooth embedding. The fact that *rational normal curves project onto plane conics* will be important for us.

For a rational normal curve $f : \boldsymbol{P}^1 \to \boldsymbol{P}^n$ the corresponding line bundle L has degree n. It follows that $L \cong H^n$ where $H \to \boldsymbol{P}^1$ is the hyperplane bundle. The inequalities

$$n + 1 \leqq h^0(\boldsymbol{P}^1, \mathfrak{O}(L)) \leqq \deg(L) + 1 = n + 1$$

give

$$h^0(\boldsymbol{P}^1, \mathfrak{O}(H^n)) = n + 1 .$$

This implies:
A rational normal curve is the mapping

$$f : \boldsymbol{P}^1 \to \boldsymbol{P}^n$$

given by choosing a basis $s_0, \ldots, s_n$ *for* $H^0(\boldsymbol{P}^1, \mathfrak{O}(H^n))$ *and setting*

$$f(p) = [s_0(p), \ldots, s_n(p)] .$$

For example, using homogeneous coordinates $z = [z_0, z_1]$ the sections

$$z_0^n, z_0^{n-1} z_1, \ldots, z_0 z_1^{n-1}, z_1^n$$

give such a basis. The image under this choice of basis will be called the *standard curve*. Setting $t = z_1/z_0$, it is given parametrically by

$$t \to [1, t, t^2, \ldots, t^n] .$$

We note that the two points $[1,0,\ldots,0]$ and $[0,0,\ldots,1]$ lie on the standard rational normal curve, and that the projection of this curve in $\boldsymbol{P}^{n-1}$ from either of these points is again a standard rational normal curve. This geometric property will underlie several inductive arguments later on in the paper.

Since any two bases of $H^0(\boldsymbol{P}^1, \mathfrak{O}(H^n))$ are related by a linear transformation, any two rational normal curves are projectively equivalent. This allows us to deduce properties of a general rational normal curve from those of the standard curve. Thus, e.g., distinct points $t_0, \ldots, t_N$ map to points in general position in $\boldsymbol{P}^n$ by noting the Vandermonde identity

$$\begin{vmatrix} 1 & t_0 & \ldots & t_0^k \\ 1 & t_1 & \ldots & t_1^k \\ \vdots & & & \\ 1 & t_k & \ldots & t_k^k \end{vmatrix} = n \prod_{i \neq j} (t_i - t_j)$$

to deduce that any $k + 1 \leqq n + 1$ of these points are linearly independent. Another property is that there is a 3-parameter subgroup of the projective linear group $\mathrm{PGL}_{n+1} \cong \mathrm{Aut}(\boldsymbol{P}^n)$ taking any rational normal curve into itself. For the standard curve these ∞^3 automorphisms of $\boldsymbol{P}^n$ induce the linear fractional group $t \to (at + b)/(ct + d)$ $(ad - bc = 1)$ on $\boldsymbol{P}^1$.

Using these remarks we may count the number of rational normal curves to be

$$\underbrace{(n+1)^2-1}_{\dim(\mathrm{PGL}_{n+1})} - \underbrace{3}_{\dim(\mathrm{PGL}_2)} = (n-1)(n+3)$$

This suggests that:

(1.1) *There is a unique rational normal curve through any $n+3$ points in general position in $\boldsymbol{P}^n$.*

To prove this we let $p_0, \ldots, p_n, q, r$ be our $n+3$ points in general position, and choose a homogeneous coordinate system having $p_0 = [1,0,\ldots,0], \ldots, p_n = [0,\ldots,0,1]$ as vertices of the fundamental simplex. The parametric representation

$$z_i = a_i/(t-b_i) \quad a_i, b_i \neq 0$$

gives, by finding a least common denominator, the rational normal curve

$$t \to \left[\prod_{i \neq 0}(t-b_i)a_0, \ldots, \prod_{i \neq n}(t-b_i)a_n\right]$$

sending $t = b_i$ to $p_i = [0,\ldots,1_i,\ldots,0]$. The points p and q have all coordinates non-zero, and consequently the inversion

$$z_i' = 1/z_i$$

maps p and q to distinct points p' and q' having all z_i'-coordinates non-zero. The above rational normal curve is mapped to the line

$$z_i' = (t-b_i)/a_i,$$

and we may uniquely choose the a_i and b_i so that this line passes through q' and r'. Since any rational normal curve containing the vertices of the coordinate simplex has the parametric form given above, this proves that there is a unique such curve passing through $n+3$ points in general position.

ii) *Quadrics and rational normal curves.* Although simple to describe parametrically, the rational normal curves have a somewhat complicated set of defining equations since they are not complete intersections. The previously noted fact that rational normal curves project onto conics in the plane suggests that we study the quadrics containing them, which we now proceed to do.

A *quadric Q* in $\boldsymbol{P}^n$ is the hypersurface given by a quadratic equation

$$q(z) = \sum_{i,j} q_{ij} z_i z_j = 0 \quad (q_{ij} = q_{ji}).$$

In a suitable coordinate system, we will have

$$q(z') = \sum_{i=0}^{r(Q)} z_i'^2$$

where $r(Q)$ is defined to be the *rank* of the quadric Q. Thus $r(Q) = n$ if and only if $\det(q_{ij}) \neq 0$, or equivalently Q is non-singular. At the other extreme, if $r(Q) = 0$ then Q is a hyperplane ($z_0' = 0$) counted twice, and if $r(Q) = 1$ then

$$Q = \boldsymbol{P}^{n-1} + \boldsymbol{P}'^{n-1}$$

is the union of distinct linear spaces. In general a quadric of rank $r \geqq 2$ is the cone over a non-singular quadric in $\boldsymbol{P}^r$.

The quadratic polynomials $q(z)$ are just the sections of the line bundle $H^2 \to \boldsymbol{P}^n$, and thus

$$h^0(\boldsymbol{P}^n, \mathfrak{O}(H^2)) = (n+1)(n+2)/2\,.$$

A linear subspace $V \subset H^0(\boldsymbol{P}^n, \mathfrak{O}(H^2))$ defines a linear system $\boldsymbol{P}(V)$ of quadrics corresponding to sections $q \in V$. Given a rational normal curve $f: \boldsymbol{P}^1 \to \boldsymbol{P}^n$, we set $C = f(\boldsymbol{P}^1)$ and denote by $V(C)$ the quadratic polynomials which vanish on C. Using the notation $|V(C)|$ for the linear system $\boldsymbol{P}(V(C))$, we may think of $|V(C)|$ as the space of quadrics passing through C. We will now derive the classical result:

(1.2) *The linear system* $|V(C)|$ *has dimension given by*

$$\dim V(C) = n(n-1)/2\,.$$

The curve C *is the ideal-theoretic intersection of the quadrics* $Q \in |V(C)|$. *Finally, the linear system* $|V(C)|$ *is spanned by quadrics of rank two.*

Proof. Suppose that we choose coordinates such that C has a parametric representation

$$t \to [1, t, \ldots, t^n]\,.$$

Under the substitution $z_i = t^i$ each quadric $q(z)$ goes into a polynomial $g(t)$ in t of degree $\leqq 2n$, and every such $g(t)$ appears in this way. Thus

$$\dim V(C) = (n+1)(n+2)/2 - (2n+1) = n(n-1)/2\,.$$

By projecting C onto a plane conic C' and taking the cone in $\boldsymbol{P}^n$ over C' we may find a rank two quadric containing C and not passing through a preassigned point in $\boldsymbol{P}^n - C$. Thus C is the set-theoretic intersection of the rank two quadrics which contain it.

To check that the quadrics in $V(C)$ generate the ideal of C, it will suffice to do this at one point p_0 and then use the fact there is a transitive group of projective transformations acting on the configuration $C \subset \boldsymbol{P}^n$. We may suppose that C is the standard curve, $p_0 = [1,0,\ldots,0]$, and take $x = [1,x_1,\ldots,x_n]$ as affine coordinates around p_0. The $n-1$ quadrics

$$q_k(x) = x_{k+1} - x_1 x_k = 0 \quad (k = 1,\ldots,n-1)$$

all vanish on C and satisfy the Jacobian condition

$$\mathrm{d}q_1 \wedge \cdots \wedge \mathrm{d}q_{n-1} \neq 0$$

at p_0.

Finally, it is not difficult to write down a basis for $V(C)$ consisting of rank two quadrics. Q.E.D.

There is a lovely synthetic construction of rational normal curves and the rank two quadrics which contain them which will prove useful in a little while. Before going into it we recall that the hyperplanes in $\boldsymbol{P}^n$ which contain a fixed $\boldsymbol{P}^{n-2}$ form a *pencil* $\boldsymbol{P}^{n-1}(t)$ – here t denotes a linear coordinate on the $\boldsymbol{P}^1$ of hyperplanes containing $\boldsymbol{P}^{n-2}$. To obtain this pencil we take a $\boldsymbol{P}^2$ meeting $\boldsymbol{P}^{n-2}$ in a point p_0. The lines in $\boldsymbol{P}^2$ passing through p_0 form a pencil $\boldsymbol{P}^1(t)$ and the linear span $\boldsymbol{P}^1(t) \oplus \boldsymbol{P}^{n-2} = \boldsymbol{P}^{n-1}(t)$ traces out the pencil of hyperplanes with axis $\boldsymbol{P}^{n-2}$. Two such pencils are in *correspondence*, written

$$\boldsymbol{P}^{n-1}(t) \barwedge \boldsymbol{P}^{n-1}(t'),$$

if we are given a projective isomorphism

$$t' = (at + b)/(ct + d)$$

between their parameter spaces. Such an isomorphism is uniquely specified by giving three distinct values

$$t_i' = (at_i + b)/(ct_i + d) \quad (i = 1,2,3).$$

Now let p and p' be two points in the plane and $\boldsymbol{P}^1(t)$ and $\boldsymbol{P}^1(t')$ the respective pencils of lines through them. A correspondence $\boldsymbol{P}^1(t) \barwedge \boldsymbol{P}^1(t')$ is uniquely determined by requiring that corresponding lines pass through three non-colinear points r,r',r'' (Fig. 1).

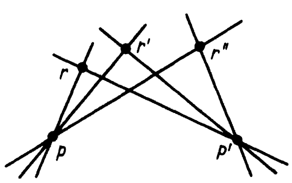

Fig. 1

The variable point of intersection $p(t) = \boldsymbol{P}^1(t) \cap \boldsymbol{P}^1(t'(t))$ will then trace out a conic passing through p, p', r, r', r''. This is Steiner's synthetic construction of a conic through five points in general position in the plane.

Before giving the generalizations of this, we need one remark concerning quadrics. Given points $p_1, p_2, \ldots, p_N$ in $\boldsymbol{P}^n$ we denote by $V(p_1, \ldots, p_N)$ the linear space of quadratic polynomials vanishing at the p_i. Then we have:

(1.3) *codim* $V(p_1, \ldots, p_N) = N$ *provided that* $N \leqq 2n + 1$ *and the* p_i *are in general position.*

Proof. By adding additional points it will suffice to do the case $N = 2n + 1$. We must show for each i there is a quadric Q passing through the p_j for $j \neq i$ but not containing p_i. Relabelling we may assume that $i = 2n + 1$. By general position,

$$\overline{\{p_1, \ldots, p_n\}} = \boldsymbol{P}^{n-1} \text{ and } \overline{\{p_{n+1}, \ldots, p_{2n}\}} = \boldsymbol{P}'^{n-1}.$$

We may take

$$Q = \boldsymbol{P}^{n-1} + \boldsymbol{P}'^{n-1}. \qquad \text{Q.E.D.}$$

We shall refer to what we just proved by saying that $N \leqq 2n + 1$ *points in general position impose independent conditions on the linear system of quadrics containing them.*

Steiner's construction generalizes in two ways. The first gives a synthetic method for finding rank two quadrics containing a rational normal curve $C \subset \boldsymbol{P}^n$. Let $p_0, \ldots, p_{n-2}$; $p'_0, \ldots, p'_{n-2}$; r, r', r'' be $2n + 1$ distinct points on our curve. Then these points are in general position, and since $\dim V(C) = 2n + 1$ and, as just proved, these points impose independent conditions on the quadrics containing them, we deduce that: *any quadric passing through* $2n + 1$ *distinct points on* C *must contain* C *entirely.* To find Q passing through these points, we set $\boldsymbol{P}^{n-2} = \overline{\{p_0, \ldots, p_{n-2}\}}$ and $\boldsymbol{P}'^{n-2} = \overline{\{p'_0, \ldots, p'_{n-2}\}}$, and consider the pencils $\boldsymbol{P}^{n-1}(t)$ and $\boldsymbol{P}^{n-1}(t')$ of hyperplanes having $\boldsymbol{P}^{n-2}$ and $\boldsymbol{P}'^{n-2}$ as respective axes. A correspondence $\boldsymbol{P}^{n-1}(t) \barwedge \boldsymbol{P}'^{n-1}(t')$ is set up by requiring that corresponding hyperplanes should pass through each of r, r', r''. Then the variable intersection

$$\boldsymbol{P}^{n-1}(t) \cap \boldsymbol{P}^{n-1}(t'(t))$$

traces out a quadric Q, which is of rank two since it contains $\boldsymbol{P}^{n-2}$'s but no $\boldsymbol{P}^{n-1}$'s, and which passes through the $2n + 1$ given points on C and hence contains the curve by our previous remark.

A second generalization of Steiner's construction gives a synthetic method for tracing out a rational normal curve through $n + 3$ points

$p_1,\dots,p_n,r_1,r_2,r_3$ in general position in $\boldsymbol{P}^n$. Let

$$\boldsymbol{P}_i^{n-2} = \{p_1,\dots,\hat{p}_i,\dots,p_n\}$$

be the edges of the simplex determined by the p_i's and $\boldsymbol{P}^{n-1}(t_i)$ the corresponding pencil of hyperplanes with axis $\boldsymbol{P}_i^{n-2}$. For a fixed pencil $\boldsymbol{P}^{n-1}(t)$ of hyperplanes we establish a correspondence

$$\boldsymbol{P}^{n-1}(t) \barwedge \boldsymbol{P}^{n-1}(t_i)$$

by requiring that corresponding hyperplanes pass through r_1,r_2,r_3. Thus $r_i \in \boldsymbol{P}^{n-1}(t_i(t))\,(= 1,2,3)$ for each i. The variable point of intersection

$$p(t) = \boldsymbol{P}_1^{n-1}(t_1(t)) \cap \cdots \cap \boldsymbol{P}_n^{n-1}(t_n(t))$$

traces out a rational normal curve thorugh our given $n + 3$ points (Fig. 2).

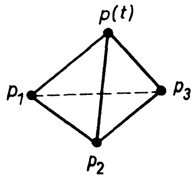

Fig. 2

We have been leading up to the following lemma, which appears implicitly in the classical literature, and will play a crucial role for us:

(1.4) *Let $p_1,\dots,p_N$ be points in general position in $\boldsymbol{P}^n$ and assume that*

$$N > 2n + 2$$

$$\dim V(p_1,\dots,p_N) \geqq n(n-1)/2\,.$$

Then these points lie on a unique rational normal curve, and the second inequality is an equality[1]).

Proof. Let $C = \bigcap_{Q \in |V(\Gamma)|} Q$ be the intersection of all the quadrics containing our set $\Gamma = p_1,\dots,p_N$. Then C is an algebraic variety and we will first prove that

$$\dim C \leqq 1\,.$$

If, on the contrary, this dimension is $\geqq 2$ then C will meet *every* $\boldsymbol{P}^{n-2}$ in a non-empty set Z. We will choose $\boldsymbol{P}^{n-2}$ such that the quadrics in $V(\Gamma)$ remain linearly independent when restricted to this subspace.

[1]) Briefly, we may say that a set of $N > 2n + 2$ points in general position lying on $n(n-1)/2$ quadrics must lie on a rational normal curve. Since $(n+1)(n+2)/2 = 2n + 1 + n(n-1)/2$, the second inequality is equivalent to codim $V(p_1,\dots,p_N) \leqq 2n + 1$.

For this we let $\boldsymbol{P}^{n-1} = \{p_1,\ldots,p_n\}$ and choose a $\boldsymbol{P}^{n-2} \subset \boldsymbol{P}^{n-1}$ not containing any point of Γ. If some $Q \in |V(\Gamma)|$ contains $\boldsymbol{P}^{n-2}$, then $Q|\boldsymbol{P}^{n-1} = \boldsymbol{P}^{n-2} + \boldsymbol{P}'^{n-2}$ since this restriction has rank one. But $Q|\boldsymbol{P}^{n-1}$ must pass through $p_1,\ldots,p_n$ and hence these points lie on a $\boldsymbol{P}'^{n-2}$ contradicting general position. This proves that the restriction

$$V(\Gamma) \to H^0(\boldsymbol{P}^{n-2},\mathfrak{O}(H^2))$$

is an injection. But $h^0(\boldsymbol{P}^{n-2},\mathfrak{O}(H^2)) = n(n-1)/2$, and so all quadrics in $\boldsymbol{P}^{n-2}$ must pass through Z, which is not the case. This contradiction shows that $\dim C \leqq 1$.

We now use Steiner's method to construct C. Label our points as

$$p_1,\ldots,p_{2n+1}\,;\; r_1,\ldots,r_{N-2n-1} \quad (N - 2n - 1 \geqq 2)\,,$$

and set

$$\boldsymbol{P}_i^{n-2} = \{p_1,\ldots,\hat{p}_i,\ldots,p_n\},\; \boldsymbol{P}^{n-2} = \{p_{n+2},\ldots,p_{2n}\}\,.$$

Denote by $\boldsymbol{P}^{n-1}(t_i)$ and $\boldsymbol{P}^{n-1}(t)$ the pencils of hyperplanes with axes $\boldsymbol{P}_i^{n-2}$ and $\boldsymbol{P}^{n-2}$, and establish a correspondence

$$\boldsymbol{P}^{n-1}(t) \barwedge \boldsymbol{P}^{n-1}(t_i)$$

by requiring that corresponding hyperplanes pass through p_i, p_{n+1}, p_{2n+1}. Then the quadric Q_i traced out by the intersection $\boldsymbol{P}^{n-1}(t) \cap \boldsymbol{P}^{n-1}(t_i(t))$ passes through $p_1,\ldots,p_{2n+1}$ and hence must contain the points r_α as well. In particular, for suitable values $t_{n+1}, t_{2n+1}, t_\alpha$ $(\alpha = 1,\ldots,N-2n-1)$ of t we will have, *for all i*,

$$\begin{aligned} &p_{n+1} \in \boldsymbol{P}_i^{n-1}(t_i(t_{n+1})) \\ &p_{2n+1} \in \boldsymbol{P}_i^{n-1}(t_i(t_{2n+1})) \\ &r_\alpha \in \boldsymbol{P}_i^{n-1}(t_i(t_\alpha))\,. \end{aligned}$$

The rational normal curve traced out by the variable intersection

$$p(t) = \boldsymbol{P}_1^{n-1}(t_1(t)) \cap \cdots \cap \boldsymbol{P}_n^{n-1}(t_n(t))$$

then passes through $p_1,\ldots,p_n,p_{n+1},p_{2n+1},r_1,\ldots,r_{N-2n-1}$. Put another way, *the unique rational normal curve C' through $p_1,\ldots,p_{n+1},r_1,r_2$ will then contain p_{2n+1} and the remaining r_α's*. It follows that C' passes through Γ. Now every quadric in $|V(\Gamma)|$ meets C' in $\geqq 2n+3$ points and hence contains C' entirely. Thus $V(\Gamma) \subset V(C')$, and since $\operatorname{codim} V(\Gamma) = 2n+1 = \operatorname{codim} V(C')$ we must have $V(\Gamma) = V(C')$ and $C = C'$. Q.E.D.

B. Extremal algebraic curves of positive genus

i. *Castelnuovo's bound.* Let C be a compact Riemann surface of genus π. For a holomorphic line bundle $L \to C$ we denote by

$$h^0(L) = \dim H^0(C, \mathfrak{O}(L))$$

the number of linearly independent holomorphic sections of the bundle. Following general conventions L^* will be the line bundle dual to L, and K will denote the *canonical line bundle* whose associated sheaf $\mathfrak{O}(K) = \Omega^1$ is the 1-forms. The basic fact in the study of curves is the *Riemann-Roch theorem*

$$h^0(L) = \deg L - \pi + h^0(K \otimes L^*) + 1 . \tag{1.5}$$

Especially useful is the case when $L = [D]$ is the line bundle associated to an effective divisor $D = p_1 + \cdots + p_d$ of degree d. Then $\mathfrak{O}([D])$, or frequently just $\mathfrak{O}(D)$, is the sheaf of meromorphic functions having poles no worse than D, and one traditionally sets

$$l(D) = h^0([D]) .$$

On the other hand, $\mathfrak{O}(K \otimes [D]^*)$ is naturally identified with the sheaf $\Omega^1(-D)$ of holomorphic differentials which vanish on D, and here one traditionally sets

$$i(D) = h^0(K \otimes [D]^*)$$

and says that the divisor D is *special* in case the *index of speciality* $i(D) > 0$. With these notations the Riemann-Roch becomes

$$l(D) = d - \pi + i(D) + 1 . \tag{1.6}$$

Finally, if we let $|D|$ denote the complete linear system associated to $H^0(C, \mathfrak{O}(D))$, then $|D|$ is the projective space of all divisors $D' = p_1' + \cdots + p_d'$ linearly equivalent to D in the sense that

$$D' - D = (\phi)$$

for some meromorphic function ϕ on C. With this notation (1.5) assumes its most symmetric form

$$\dim |D| = d - \pi + i(d) . \tag{1.7}$$

Now suppose that $f : C \to \boldsymbol{P}^n$ is a non-degenerate algebraic curve of degree d. We assume that f is generically one-to-one and set $L = f^* H$. We have proved that $d \geqq n$, and that if $d = n$ then $\pi = 0$ and C is a rational normal curve. In 1889 Castelnuovo found a general bound on π in terms of d and n, and was moreover able to determine the structure of the curves of maximum genus[2]). Because of the central role which these *extremal algebraic curves* play in the theory of webs we shall sketch Castelnuovo's argument here – a complete discussion and details may be found in Chapters II and IV of [5].

The bound is based on a lemma which will now be explained. We denote by V_k the image of the mapping

$$H^0(\boldsymbol{P}^n, \mathfrak{O}(H^k)) \to H^0(C, \mathfrak{O}(L^k)).$$

Since $H^0(\boldsymbol{P}^n, \mathfrak{O}(H^k))$ is just the vector space of homogeneous polynomials of degree k, the linear system $|V_k|$ consists of the divisors on $f(C)$ cut out by the hypersurfaces of degree k in $\boldsymbol{P}^n$. We will prove the

(1.8) **Lemma.** $\dim V_k - \dim V_{k-1} = \begin{cases} \geqq k(n-1)+1 & \text{if } k \leqq (d-1)/(n-1) \\ = d & \text{if } k > (d-1)/(n-1) \end{cases}$

Proof. For $k = 1$ the bound is

$$\dim V_1 \geqq n + 1,$$

which is equivalent to the curve being non-degenerate. For $k > 1$ we shall use the following (loc. cit)

General position principle: A generic hyperplane $\xi \in \boldsymbol{P}^{n*}$ cuts the curve C in d points

$$\xi \cdot C = p_1 + \cdots + p_d,$$

any n of which are linearly independent.

Now assume that $k(n-1)+1 \leqq d$ and break the points up into k groups of $n-1$ each plus a remainder as follows:

$$\underbrace{p_1,\ldots,p_{n-1}}_{\substack{\| \\ D_1}};\ \underbrace{p_n,\ldots,p_{2(n-1)}}_{\substack{\| \\ D_2}};\ \cdots;\ \underbrace{p_{k(n-2)+1},\ldots,p_{k(n-1)}}_{\substack{\| \\ D_k}};\ \ldots,p_d.$$

By the general position principle we may find hyperplanes $\xi_1,\ldots,\xi_k$ such that each ξ_i contains D_i but not $p_{k(n-1)+1}$. The hypersurface $h = \xi_1 + \cdots + \xi_k$ in $\boldsymbol{P}^n$ has degree k and passes through $p_1,\ldots,p_{k(n-1)}$ but not $p_{k(n-1)+1}$. On the other hand, any hypersurface of the form $h' = \xi + h''$ where h'' has degree $k-1$ passes through all the p_i. There are clearly at least $k(n-1)+1+\dim V_{k-1}$ linearly independent hypersurfaces of the form

$$h + h',$$

[2]) For $n = 2$ the bound was classical. For $n = 3$ it was found by Clifford and Max Noether.

which proves the lemma in case $k(n-1)+1 \leqq d$. The other case is similar and easier. Q.E.D.

Now we set $m = [(d-1)/(n-1)]$ and successively apply the lemma for $k = 0, 1, \dots, m+m'$ to obtain

$$\dim V_{m+m'} \geq (1/2m(m+1))(n-1) + m + 1 + m'd\,.$$

On the other hand, obviously

$$h^0(L^k) \geqq \dim V_k$$

for any k, while the index of speciality

$$i(L^k) = 0$$

for large k[3]. Applying (1.5) we obtain

$$d(m+m') - \pi + 1 = h^0(L^{m+m'}) \geqq (1/2m(m+1))(n-1) + (m+1) + m'd$$

or

$$\pi \leqq \pi(d,n)$$

where

$$\begin{cases} \pi(d,n) = m[d - 1/2(m+1)(n-1) - 1] & \text{and} \\ m = [(d-1)/(n-1)] \end{cases}$$

This is Castelnuovo's bound on the genus.

We now make some brief remarks about the various cases.

Case i: For an extremal algebraic curve C of degree d with $n \leqq d \leqq 2n-1$, *we have*

$$\pi = d - n$$

$$h^0(L) = n+1, \quad i(L) = 0$$

The bound on π follows immediately from Castelnuovo. In the extremal case when $\pi = \pi(d,n) = d - n$ we use the form (1.5) of Riemann-Roch to obtain

$$h^0(L) = \deg L - \pi + i(L) + 1 = n + 1 + i(L)$$

$$h^0(L) \geqq n+1\,,$$

which together imply the second statements.

It can be shown that for any compact Riemann surface C of genus π and integers d, n with

$$\pi = d - n$$

$$n \leqq d \leqq 2n-1\,,$$

[3]) Specifically, since $i(L^k) = h^0(K \otimes L^{-k})$ we will have $i(L^k) = 0$ as soon as $\deg(K \otimes L^k) = 2\pi - 2 - kd < 0$.

there is a holomorphic embedding $f: C \to \boldsymbol{P}^n$ of C as a curve of degree d. Consequently, there is nothing special about those Riemann surfaces which appear as extremal curves when $n \leqq d \leqq 2n - 1$.

Case ii: For an extremal algebraic curve of degree $d = 2n$, the line bundle L is the canonical bundle and C is a canonical curve. By Castelnuovo's bound

$$\pi \leqq \pi(2n, n) = n + 1 .$$

Arguing as before, in case $\pi = n + 1$

$$h^0(L) = n + h^0(K \otimes L^*)$$
$$h^0(L) \geqq n + 1$$

which together imply that $h^0(L) = n + 1$ and $h^0(K \otimes L^*) = 1$. Now then

$$\deg(K \otimes L^*) = \deg K - \deg L = 2\pi - 2 - 2n = 0 ,$$

and $h^0(K \otimes L^*) \neq 0$ implies that $K = L$ since any section of $K \otimes L^*$ can have no zeroes.

By definition, a *canonical curve* is $f: C \to \boldsymbol{P}^n$ where f is generically one-to-one and $f^* H = K$ is the canonical line bundle (it then turns out that f is a smooth embedding). We shall say more about these curves in section I-B iii.

Case iii. These are the curves where $d > 2n$, and are the ones in which we are most interested. They will be described in more detail in the next sections.

We conclude this discussion with three observations. The first is that for line bundles $L \to C$ whose associated complete linear system gives a generically one-to-one mapping $f: C \to \boldsymbol{P}^n$ we have proved *Clifford's theorem:*

$$\begin{cases} i(L) \neq 0 \Rightarrow h^0(L) \leqq \deg L/2 + 1 \\ \text{with equality} \Leftrightarrow L = K . \end{cases}$$

Proof. Setting $d = \deg L$ and $h^0(L) = n + 1$, we assume

$$n + 1 > d/2 + 1$$

or $d < 2n$. Then $\pi \leqq d - n$ and

$$n + 1 = h^0(L) = d - \pi + 1 + i(L) \geqq n + 1 + i(L) ,$$

which implies that $i(L) = 0$. If $n + 1 = d/2 + 1$ then $d = 2n$ and we have proved that $L = K$. Q.E.D.

The second observation is that in case C is extremal, then all the inequalities in the proof of Castelnuovo's bound must be equalities and so

$$h^0(L^k) = \dim V_k \quad k = 1,2,\dots,^4).$$

In particular

$$h^0(L^2) = 3n \quad \text{in case} \quad d \geqq 2n - 1 .$$

Since $h^0(\boldsymbol{P}^n, \mathfrak{O}(H^2)) = (n+1)(n+2)/2$ and $(n+1)(n+2)/2 - 3n = (n-1)(n-2)/2$, this implies that:

(1.9) *An extremal algebraic curve C of degree $d \geqq 2n - 1$ lies on $\infty^{(n-1)(n-2)/2}$ quadrics in $\boldsymbol{P}^n$.*

Recently, Joe Harris took up the question of maximizing the genus π of a curve C of degree d in $\boldsymbol{P}^3$ which lies on a surface S of degree k but not on one of degree $k - 1$. Asymptotically he found the bound

$$\pi(C) \leqq d^2/2(k+1).$$

When $k = 3$ the precise bound is

$$\pi(C) \leqq \begin{cases} (d^2 - 3d + 2)/6 & \text{for} \quad d \equiv 1,2,(3) \\ (d^2 - 3d + 6)/6 & \text{for} \quad d \equiv 0\,(3) \end{cases}$$

As a consequence,

If $C \subset \boldsymbol{P}^3$ is a non-degenerate algebraic curve of degree d and genus π with

$$\pi > (d^2 - 3d)/6 + 1,$$

then C lies on a quadric surface.

In this case Castelnuovo's bound is

$$\pi \leqq \begin{cases} (d-2)^2/4 & d \equiv 0\,(2) \\ (d-1)(d-3)/2 & d \equiv 1\,(2) \end{cases}$$

Asymptotically we have

$$\pi \leqq d^2/4 \text{ for any non-degenerate curve}$$
$$\pi > d^2/6 \Rightarrow C \text{ lies on a quadric surface}.$$

There are also results of a similar nature for non-degenerate curves in $\boldsymbol{P}^n$.

ii. *Extremal curves of degree $d > 2n$.* We now consider an extremal algebraic curve $f: C \to \boldsymbol{P}^n$ of degree d where $d > 2n$ and $n \geqq 3$. It will turn out that f is generically an embedding, and we shall identify C with its image.

---

^{4}) This means that the hypersurfaces of degree k cut out *complete* linear systems on the curve for all k; such curves are called *arithmetically normal*. That canonical curves are arithmetically normal is a classical result of Max Noether.

The basic fact is

(1.10) *C lies on a surface S of degree* $n-1$ [5]).

Proof. Denote by $|V(C)|$ the linear system of $\infty^{(n-1)(n-2)/2}$ quadrics which contain C – cf. (1.9) above. Since C is non-degenerate, no quadric $Q \in |V(C)|$ has rank one. Consequently the restriction

$$\text{quadrics} \quad Q \in |V(C)| \to Q|\boldsymbol{P}^{n-1}$$

is injective for any hyperplane $\boldsymbol{P}^{n-1} \subset \boldsymbol{P}^n$. It follows that all the hyperplane sections $\xi \cdot C$ consists of

$$d > 2(n-1) + 2$$

points lying on $\infty^{(n-1)(n-2)/2}$ quadrics. For generic ξ these points are in general position, and we deduce from (1.4) that

$$\xi \cdot \bigcap_{Q \in |V(C)|} Q = D_\xi$$

is a rational normal curve. It follows that

$$S = \bigcap_{Q \in |V(C)|} Q$$

is a surface of degree $n-1$ containing the curve C. Q.E.D.

Now then the non-degenerate surfaces of degree $n-1$ in $\boldsymbol{P}^n$ were classified by del Pezzo and may be described as follows (cf. Chapter IV of [5] for proofs):

First, a cone over a rational normal curve is clearly a surface of minimal degree having a singularity at the vertex of the cone.

Secondly, the embedding $\boldsymbol{P}^2 \to \boldsymbol{P}^5$ defined by the complete linear system $|H^0(\boldsymbol{P}^2, \mathfrak{O}(H^2))|$ of quadrics in $\boldsymbol{P}^2$ defines a smooth surface, the *Veronese surface*. Since any two conics meet in 4 points the degree of this surface is the minimal number four. The Veronese surface is the unique non-degenerate surface in $\boldsymbol{P}^5$ whose chordal variety has dimension four, rather than five as one would generally expect.

In general a minimal surface[6]) turns out to be rational, and so $h^1(S,\mathfrak{O}) = h^2(S,\mathfrak{O}) = 0$. It follows that the group of line bundles

$$\operatorname{Pic}(S) \cong H^2(S, \boldsymbol{Z}),$$

the isomorphism being via the Chern class. The remaining minimal surfaces

$$S = S(k,l)$$

---

[5]) We recall that this is the minimal degree of a non-degenerate surface in $\boldsymbol{P}^n$.

[6]) Minimal here means having no exceptional curves of the 1st kind. This is the only time we shall use this terminology.

are described as follows: Over $\boldsymbol{P}^1$ we consider the standard line bundle $H^k \to \boldsymbol{P}^1$ of degree $k \geqq 0$. Adding a point at infinity to each fibre we obtain a surface $S(k)$ which is a $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^1$. The zero section of $H^k \to \boldsymbol{P}^1$ is a curve on $S(k)$ having self-intersection number k. We denote by $e \in \mathrm{Pic}(S(k)) \cong H^2(S(k), \boldsymbol{Z})$ the class of the curve at infinity, so that

$$e \cdot e = -k .$$

We may characterize $S(k)$ as being the unique minimal rational surface having a curve of self-intersection number $-k$. Let $f \in \mathrm{Pic}(S(k))$ be the class of a fibre so that

$$\begin{cases} f \cdot f = 0 \\ f \cdot e = 1 . \end{cases}$$

The curves e and f give an integral basis for $\mathrm{Pic}(S(k))$. Therefore, any line bundle is a linear combination $\alpha e + \beta f\,(\alpha, \beta \in \boldsymbol{Z})$. In particular, the canonical bundle is of this form and using the *adjunction formula*

$$\pi(C) = 1/2(C \cdot C + C \cdot K) + 1$$

for the genus of a smooth curve C applied to e and f we obtain

$$K = -2b + (-k - 2)f .$$

We now consider the line bundle $L(k,l) \to S$ whose Chern class

$$c_1(L(k,l)) = e + (k + l + 1)f .$$

In [5] it is proved that

$$h^0(S(k), \mathfrak{O}(L(k,l))) = k + 2l + 4$$

and that the complete linear system $|H^0(S(k), \mathfrak{O}(L(k,l)))|$ gives an embedding for $l \geqq 0$. We identify $S(k)$ with its image $S(k,l)$ to obtain a surface in $\boldsymbol{P}^n$ where $n = k + 2l + 3$. The degree of $S(k,l)$ is

$$(e + (k + l + 1)f) \cdot (e + (k + l + 1)f) = -k + 2k + 2l + 2 = n - 1 .$$

We note that since

$$(e + (k + l + 1)f) \cdot f = e \cdot f = 1$$

the fibres of $S(k) \to \boldsymbol{P}^1$ map into straight lines in $\boldsymbol{P}^n$. Thus the $S(k,l)$ are *rational ruled surfaces*, sometimes referred to as *scrolls. Together with the cones over rational normal curves and Veronese surface, the surfaces $S(k,l)$ give all non-degenerate surfaces of degree $n - 1$ in $\boldsymbol{P}^n$.*

Moreover, the genus of a curve on $S(k,l)$ may be easily computed by the adjunction formula with the following conclusion:

(1.11) *For each $n \geqq 3$ and $d > 2n$, curves C of maximum genus $\pi(d,n)$ exist. These curves are non-singular and lie either on a cone over a rational normal curve or on a surface $S(k,l)$. In all cases the surface S is the intersection of the quadrics containing C. The hyperplane sections $\xi \cdot C(\xi \in \boldsymbol{P}^{n*})$ are d points on a rational normal curve $D_\xi = \xi \cdot S$. Finally, extremal curves exist over $\mathbb{R}$ as well as $\mathbb{C}$.*

Given an extremal algebraic curve C, the unique ruled surface S which contains the curve will be referred to as *Castelnuovo's ruled surface*. When $n = 3$, S is the standard doubly ruled quadric. An extremal algebraic curve has either type $(k,k-1)$ or (k,k) depending on whether its degree is odd or even. In the latter case C has degree $2k$ and is a complete intersection of S with a hypersurface of degree k; in the former case C + (line) is a complete intersection. This property of extremal algebraic curves to be as close as possible to complete intersections persists in higher dimensions.

iii. *The canonical curve and Poincaré mapping*. We have now discussed extremal algebraic curves of degree d in $\boldsymbol{P}^n$ when $n \leqq d \leqq 2n - 1$ and $d > 2n$, and have mentioned in passing that those of degree $d = 2n$ are canonical curves. These will now be described in more detail, and then we shall relate those of degree $d > 2n$ to the canonical curve.

Suppose that C is a compact Riemann surface of genus $\pi \geqq 2$. The space $H^0(C,\mathfrak{O}(K)) = H^0(C,\Omega^1)$ of holomorphic 1-forms on C has dimension π and the associated complete linear system is base point free – i.e., for every point $p \in C$ there is an $\omega \in H^0(C,\Omega^1)$ with $\omega(p) \neq 0$. Choosing a basis $\omega_1,\ldots,\omega_\pi$ for the holomorphic differentials we obtain the *canonical mapping*

$$\varkappa : C \to \boldsymbol{P}^{\pi-1}$$

defined by

$$\varkappa(p) = [\omega_1(p),\ldots,\omega_\pi(p)] .$$

We note that

$$\boldsymbol{P}^{*\pi-1} \cong \boldsymbol{P}(H^0(C,\Omega^1)) ;$$

i.e., the hyperplane sections are just the divisors of holomorphic differentials. In case C is non-hyperelliptic, which is the general case when the genus $\pi \geqq 3$, the canonical mapping is a one-to-one embedding. The image $C_\varkappa$ is a *canonical curve*. It has degree

$$2\pi - 2 = 2(\dim \boldsymbol{P}^{\pi-1}) ,$$

is intrinsically attached to the compact Riemann surface C and, as we shall now discuss, the geometry of $C_\varkappa$ reflects many properties of the special divisors on C.

The basic observation is this: Suppose that C is non-hyperelliptic and

$$D = p_1 + \cdots + p_d$$

is a divisor of degree d. Denote by $\{D\} = \{p_1, \ldots, p_d\}$ the linear subspace of $\boldsymbol{P}^{\pi-1}$ spanned by the points p_i on the canonical curve. Then, since the index of speciality is just the number of linearly independent hyperplanes in $\boldsymbol{P}^{\pi-1}$ which contain D, by elementary linear algebra

$$\dim\{D\} = \pi - 1 - i(D).$$

Comparing this with the third form (1.7) of the Riemann-Roch gives

$$\dim\{D\} = (d - 1) - \dim|D|.$$

Consequently, $\dim|D|$ *exactly measures the extent to which the points* p_i *fail to be linearly independent on the canonical curve* [7]).

Put geometrically, suppose we say that a linear subspace $\boldsymbol{P}^{d-n-1} (n > 0)$ of $\boldsymbol{P}^{\pi-1}$ is a *multisecant plane* in case it is the span of d points on the canonical curve. For example, when $d = 3$ and $n = 1$ we have a trisecant line, and so forth. Then the study of special divisors is equivalent to the study of multisecant planes on the canonical curve. *This study is governed by the remarkable fact that if there is one multisecant* $\boldsymbol{P}^{d-n-1} = \{p_1, \ldots, p_d\} (p_i \in C_\kappa)$, *then there are at least* ∞^n *such multisecant planes*, as follows from the Rieman-Roch. Although it does not seem to have been made rigorous, this is perhaps the distinguishing characteristic of the canonical curve — at least provided we assume there is a non-empty set of special divisors.

Now, suppose $f : C \to \boldsymbol{P}^n$ is a normal algebraic curve of degree d with typical hyperplane section

$$D_\xi = \xi \cdot f(C) = p_1(\xi) + \cdots + p_d(\xi).$$

By the Riemann-Roch

$$n + \pi = d + i(D_\xi).$$

Therefore, fixing n and d and maximizing the genus π is equivalent to maximizing the index of speciality $i(D_\xi)$. By the preceding remark this is the same as finding extremal multisecant planes, and what is suggested is that we investigate how the multisecant plane $\{D_\xi\}$ varies with $\xi \in \boldsymbol{P}^{n*}$.

) We are primarily interested in divisors of degree $d \leqq \pi - 1$. For a generic D of this degree $i(D) = \pi - d$. Consequently

$$\dim|D| = 0$$

for a generic such D, hence the source of the name *special divisors*.

So, given any normal algebraic curve $f: C \to \boldsymbol{P}^n$ we define the *Poincaré mapping*[8])

$$F: \boldsymbol{P}^{n^*} \to \boldsymbol{G}(d-n-1, r-1)$$

by

$$F(\xi) = \{p_1(\xi), \ldots, p_d(\xi)\}.$$

This allows us to study our original algebraic curve by investigating a mapping between two familiar spaces.

The understanding of F requires analyzing its infinitesimal structure. To see what is involved, we denote by $p_i'(\xi)$ a point on the tangent line to the canonical curve at $p_i(\xi)$. Then the infinitesimal structure of the Poincaré mapping is reflected in the linear space

$$\{p_1(\xi), \ldots, p_d(\xi); p_1'(\xi), \ldots, p_d'(\xi)\} = \{p_i(\xi); p_i'(\xi)\}$$

spanned by the points $p_i(\xi)$ and $p_i'(\xi)$. This clearly relates to the divisor $2D_\xi$, and in fact

$$\dim \{p_i(\xi); p_i'(\xi)\} = 2d - \pi + i(2D_\xi).$$

Referring to the lemma (1.8) in § I B i,

$$3n - 1 \leqq \dim |2D|$$

while

$$\dim |2D_\xi| = 2d - \pi + i(2D_\xi)$$

by the Riemann-Roch. These combine to yield

$$\dim \{p_i(\xi), p_i'(\xi)\} \leqq 2d - 3n,$$

with equality holding exactly when $f(C)$ lies on $\infty^{(n-1)(n-2)/2}$ quadrics. *In general, we may say that the infinitesimal structure of the Poincaré mapping reflects the quadrics containing the curve $f(C)$.*

Suppose in particular that $f: C \to \boldsymbol{P}^n$ is an extremal algebraic curve of degree $d > 2n, n \geqq 3$. Let S be *Castelnuovo's ruled surface* on which the curve lies. We will prove that:

(1.12) *The canonical mapping $\varkappa: C \to \boldsymbol{P}^{\pi-1}$ extends uniquely to a mapping $\varkappa: S \to \boldsymbol{P}^{\pi-1}$. Consequently the canonical curve C lies on a rational surface $S_\varkappa$ in $\boldsymbol{P}^{\pi-1}$.*

Proof. From surface theory we recall the adjunction formula now in the form

$$K_C = K_S \otimes [C]_C$$

[8]) In a somewhat different context this method was used by Poincaré in his paper *Sur les surfaces de translation et les fonctions Abeliennes, Bull. Soc. Math. France* vol. **29** (1901) 61 – 86 on the Sophus Lie theorem. We introduce the Poincaré mapping here because it will make sense in a purely web-theoretic context, and in this form will play a crucial role in our discussions.

where, as usual, K_X denotes the canonical bundle of a variety X. The adjunction formula yields the exact sheaf sequence

$$0 \to \Omega_S^2 \to \Omega_S^2(C) \to \Omega_C^1 \to 0$$

where

$$\Omega_S^2(C) \cong \mathfrak{O}(K_S \otimes [C]).$$

Since S is rational

$$h^0(S,\Omega_S^2) = 0 = h^1(S,\Omega_S^2),$$

and we obtain an isomorphism

$$H^0(S,\Omega^2(C)) \simeq H^0(C,\Omega_C^1).$$

If we think of the canonical curve $\varkappa : C \to \boldsymbol{P}^{\pi-1}$ as being given by the line bundle $K_C \to C$ together with its complete linear system, then we have just shown that there is a unique line bundle $J \to S$ with $J|C = K_C$, and that $H^0(S,(J)) \simeq H^0(C,\mathfrak{O}(K_C))$ is an isomorphism.

The linear system $|H^0(S,\mathfrak{O}(J))|$ has no base points and gives a one-to-one mapping $S \to \boldsymbol{P}^{\pi-1}$ along the curve C. By allowing C to vary on S we may conclude that the above linear system is entirely base point free and gives an embedding $\varkappa : S \to \boldsymbol{P}^{\pi-1}$. To carry this out it is only necessary to know that C varies in a large linear system on S, and this is proved in Chapter IV of [5]. Q.E.D.

We shall use this to study the Poincaré mapping associated to the extremal algebraic curve. The notation

$$\{p_1(\xi),\ldots,p_d(\xi)\} = \boldsymbol{P}^{d-n-1}(\xi)$$

will be used in lieu of $F(\xi)$ — this has the advantage of emphasizing not only the linear space $F(\xi)$, but also the set of d points $p_i(\xi)$ generating this multisecant $\boldsymbol{P}^{d-n-1}(\xi)$. A first property is

(1.13) *The points $p_i(\xi)$ lie on a rational normal curve E_ξ in $\boldsymbol{P}^{d-n-1}(\xi)$.*

Proof. Of course we will take

$$E_\xi = \varkappa(\xi \cdot S)$$

as the image under $\varkappa : S \to \boldsymbol{P}^{\pi-1}$ of a hyperplane section $\xi \cdot S$ of Castelnuovo's ruled surface. Clearly E_ξ is rational, and we must prove that

$$\deg E_\xi = d - n - 1 .$$

By the intersection properties derived in the previous section

$$\begin{aligned}\deg(E_\xi) &= [\xi \cdot S] \cdot (K_S \otimes [C]) \\ &= E \cdot K_S + E \cdot C \\ &= d - n - 1 .\end{aligned}$$

Q.E.D.

Now we shall prove that:

(1.14) *Any two of the rational normal curves E_ξ and $E_{\xi'}$ have $(n-1)$ points in common. Consequently, as ξ varies the ∞^n rational normal curves E_ξ trace out the surface $S_\varkappa$, the image under the canonical mapping of Castelnuovo's ruled surface.*

Proof. Two hyperplanes ξ and ξ' meet in a $\boldsymbol{P}^{n-2}$; we let $\sigma = \xi \circ \xi'$. Then

$$\sigma \cdot S = \xi \cdot \xi' \cdot S$$

is just $(n-1)$ points, and under $\varkappa$ these go into $E_\xi \cdot E_{\xi'}$. Q.E.D.

Note that as we move along the line[9])

$$\xi_t = \xi + t\xi'$$

in $\boldsymbol{P}^{n*}$, the corresponding points

$$\boldsymbol{P}^{d-n-1}(\xi(t))$$

will contain the $(n-2)$-fold secant $\boldsymbol{P}^{n-2}(\sigma)$ spanned by the $(n-1)$ points in $E_\xi \cdot E_{\xi'}$.

Summarizing, *we are able to reconstruct Castelnuovo's ruled surface in the space $\boldsymbol{P}^{\pi-1}$ of the canonical curve as being the locus of the rational normal curves E_ξ lying on the multisecant $\boldsymbol{P}^{d-n-1}(\xi)$'s containing a fixed $\boldsymbol{P}^{n-2}(\sigma)$ as σ runs over the hyperplanes in ξ.*

Actually, we have only proved that the intersection $\boldsymbol{P}^{d-n-1}(\xi) \cdot \boldsymbol{P}^{d-n-1}(\xi')$ contains the $\boldsymbol{P}^{n-2}(\sigma)$ spanned by the $n-1$ points on $E_\xi \cdot E_{\xi'}$. But the linear space $\{p_i(\xi), p_i'(\xi)\}$ spanned by the points $p_i(\xi)$ of $\xi \cdot C$ and the tangent lines to the canonical curve at these points is a $\boldsymbol{P}^{2d-3n}$ containing $\boldsymbol{P}^{d-n-1}(\xi)$ and $\boldsymbol{P}^{d-n-1}(\xi')$. Since $2(d-n-1)-(2d-3n) = n-2$ we have the equality $\boldsymbol{P}^{d-n-1}(\xi) \cdot \boldsymbol{P}^{d-n-1}(\xi') = \boldsymbol{P}^{n-2}(\sigma)$.

These special properties of the Poincaré map associated to an extremal algebraic curve will be important when we prove our main result characterizing the webs defined by such curves. These properties may all be stated purely in terms of the web defined by an extremal curve. This will be donne in § II A ii., and in § III we shall reprove them in a purely web-theoretic context.

II. Webs and abelian equations

A. Basic definitions and examples

i. *Definitions and the first non-trivial example.* Our study will be local, and we shall work in an open set U in $\boldsymbol{R}^n$ or $\boldsymbol{C}^n$ with coordinates denoted

[9]) This line is the pencil of hyperplanes with axis $\sigma = \xi \cdot \xi'$. We may identify the lines in $\boldsymbol{P}^{n*}$ through ξ with the dual space ξ^* of all such $\boldsymbol{P}^{n-2}$'s σ contained in ξ.

$x = (x_1, \ldots, x_n)$ in either case. The open set U may be thought of as a sufficiently small neighborhood of some point x_0, and it may be shrunk a finite number of times during the discussion. We shall consider functions $u(x)$, differential forms $\omega(x) = \sum_{\alpha=1}^{n} f_\alpha(x)\,dx_\alpha$, etc. In the real case these will be real-valued and C^∞; in the complex case they will be holomorphic, i.e., convergent power series in the x_α's. Since no use will be made of the conjugate variables $\bar{x}_\alpha$ or Cauchy-Riemann equations $\partial u/\partial \bar{x}_\alpha = 0$, it will not be necessary to specify whether we are in the real or complex case.

Definition. *A d-web is given by d codimension-one foliations in U. The leaves of the foliations will be called the web hypersurfaces, and we shall always assume that the tangent hyperplanes to the web hypersurfaces through a point in U are in general position.*

In general we may define a d-web of codimension k by requiring that the leaves of the d foliations should be submanifolds of codimension k whose tangent spaces are in general position. The study of such webs will almost certainly be quite interesting, but in this paper we shall be concerned exclusively with the codimension one case. Some remarks pertaining to the higher codimension case are given in § 3 of the second author's paper *On Abel's Differential Equations*, to appear in Amer. J. Math. The Blaschke-Bol book [1] contains an extensive discussion of the origins of the study of webs and of the simplest special cases, such as a 3-web of curves in plane. It will be our main reference.

In general we may say that *the study of webs is concerned with the local invariants of a set of d foliations in general position.* Invariance here means under the group of diffeomorphisms. As an example of an invariant, suppose we are given a 4-web of curves in the plane. At each point p the tangent lines to the four curves through p will define four points in the projectivized tangent space $\boldsymbol{P}(T_p) \cong \boldsymbol{P}^1$, and the *cross ratio* of these is invariant under diffeomorphisms.

One of the main interests in the theory is to find conditions under which a web may be put in a standard local form. For example, suppose we make the

Definition. *A d-web is linear in case the web hypersurfaces are (pieces of) linear hyperplanes in $\boldsymbol{R}^n$ or $\boldsymbol{C}^n$. A web is linearizable in case it is equivalent to a linear web under a change of coordinates.*

The main result of this paper is to give sufficient conditions under which a web is linearizable.

We first remark that when $d \leqq n$ any d-web is linearizable. Indeed, a general d-web may be given by the level sets

$$u_i(x) = \text{constant} \quad (i = 1,\dots,d) \tag{2.1}$$

of functions $u_i(x)$. These defining functions are unique up to a change

$$U_i(x) = v_i(u_i(x)) \tag{2.2}$$

where $v_i(u)$ is a function of a single variable with derivative $v_i'(u) \neq 0$. The non-degeneracy condition

$$\mathrm{d}u_{i_1} \wedge \cdots \wedge \mathrm{d}u_{i_k} \neq 0 \quad (k \leqq n)$$

implies that when $d \leqq n$ we may take $u_1,\dots,u_d$ as part of a coordinate system in which the web is linear[10]).

Consequently the first non-trivial case is when $d = n + 1$. Before discussing this we remark that it is frequently convenient to give the i^{th} foliation in our web by

$$\omega^i(x) = 0 \tag{2.3}$$

where ω^i is a 1-form satisfying the *integrability condition*

$$\mathrm{d}\omega^i \wedge \omega^i = 0\,. \tag{2.4}$$

Here the equation (2.3) means that the tangent hyperplane to the i^{th} web hypersurface is defined by

$$\langle \omega^i(x), \xi \rangle = 0\,, \quad \xi \in T_x.$$

The form ω^i is well-defined up to a change

$$\tilde{\omega}^i = \varrho^i \omega^i$$

where ϱ^i is a non-zero function. Consequently the points

$$\omega^i(x) \in \boldsymbol{P}(T_x^*)$$

in the projectivized cotangent spaces are well-defined; these will be called the web normals. The non-degeneracy condition is just that the web normals should be d points in general position in $\boldsymbol{P}(T_x^*) \cong \boldsymbol{P}^{n-1}$.

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[10]) This contrasts sharply with the higher codimension case. For example, suppose we have a 2-web of codimension 2 in $U \subset \boldsymbol{R}^3$. Thru each point x there will pass two curves whose normal spaces will have a line $L_x \subset T_x^*$ in common. Taking $\omega(x)$ to be a 1-form generating L_x, the condition

$$\mathrm{d}\omega \wedge \omega \not\equiv 0$$

is independent of the choice of ω and gives the obstruction to linearization.

The relation between these two ways of defining a web is

$$(2.5) \qquad du_i = \varrho^i \omega^i .$$

Now suppose that $d = n + 1$. There will be a linear relation

$$\alpha_1 \tilde{\omega}^1 + \cdots + \alpha_{n+1} \tilde{\omega}^{n+1} = 0$$

among the web normals $\tilde{\omega}^i(x)$. The coefficients are all non-zero, and this is the unique such relation up to multiplication by non-zero homogeneity factor. Setting $\omega^i = \alpha_i \tilde{\omega}^i$ we thus have

$$(2.6) \qquad \omega^1 + \cdots + \omega^{n+1} = 0$$

where the ω^i are unique up to

$$\tilde{\omega}^i = \varrho\, \omega^i$$

with the same ϱ.

The integrability condition (2.4) is equivalent to

$$d\omega^i = \pi^i \wedge \omega^i$$

where π^i is determined up to

$$\pi^i \to \pi^i + \beta^i \omega^i .$$

The form $\tilde{\pi}^i$ corresponding to $\tilde{\omega}^i = \varrho\,\omega^i$ is

$$\tilde{\pi}^i = \pi^i + d\varrho/\varrho$$

Consequently the condition

There exist functions β^i such that

$$(2.7) \qquad \pi^i + \beta^i \omega^i = \pi$$

is the same for all i

has intrinsic meaning. We will show that

(2.8) *If $n \geqq 3$ and if* (2.7) *is satisfied, then the web is linearizable.*

Proof. We may assume that

$$d\omega^i = \pi \wedge \omega^i$$

with the same π for all i. Taking exterior derivatives gives

$$0 = d\pi \wedge \omega^i ,$$

so that $d\pi$ is a multiple of ω^i for all i. Since $n \geqq 3$ it follows that $d\pi = 0$. Locally we may find a function ϱ with $d\varrho = \pi$, and then

$$\tilde{\omega}^i = e^{-\varrho} \omega^i$$

satisfies $d\tilde{\omega}^i = 0$. If $u_i = \int \tilde{\omega}^i$ is a function with $du_i = \tilde{\omega}^i$, then by (2.6)

$$u_1(x) + \cdots + u_{n+1}(x) = \text{constant}.$$

Taking $u_1,\ldots,u_n$ as a coordinate system, the web is equivalent to one formed by $n + 1$ families of parallel hyperplanes. In particular, it is linearizable. Q.E.D.

Definition. *A web is octahedral*[11]*) in case it is equivalent to one formed by* $(n + 1)$ *families of parallel hyperplanes.*

An octahedral web is linearizable, but not conversely. Since an octahedral web is, in a suitable coordinate system, defined by

$$\begin{cases} x_\alpha = \text{constant} & \alpha = 1,\ldots,n \\ x_1 + \cdots + x_n = \text{constant}, \end{cases}$$

the converse of (2.8) is obvious. Thus:

(2.9) *The condition* (2.7) *is equivalent to the* $(n + 1)$*-web being octahedral.*

We now wish to reinterpret the conditions (2.6) and (2.7) for a web given by $\{u_i(x) = \text{constant}\}$. If (2.7) is satisfied then we may multiply (2.6) by a non-zero factor to have $d\omega^i = 0$. It follows that

$$\omega^i(x) = f_i(u_i(x))\,du_i(x)$$

and (2.6) assumes the form

$$\sum f_i(u_i(x))\,du_i(x) \equiv 0. \tag{2.11}$$

Definition. *For a general d-web an equation* (2.11) *is called an abelian equation. The maximum number of linearly independent abelian equations is called the rank r of the web.*

In other words, *an abelian equation is a relation among the web normals* $du_i(x)$ *whose coefficients* $f_i(u_i(x))$ *are constant along the web hypersurfaces.* The terminology abelian equation will be explained in the next section, and in § II B we will prove that the rank satisfies

$$r \leqq \pi(d,n)$$

where $\pi(d,n)$ is Castelnuovo's bound on the genus of a non-degenerate curve of degree d in $\boldsymbol{P}^n$.

For the moment we note that the rank $r = 0$ in case $d \leqq n$ and that $r \leqq 1$ for $d = n + 1$. Then (2.8) may be rephrased as:

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[11]) For $n = 2$ we say that the web is *hexagonal*. The terminology is explained in Blaschke-Bol [1].

(2.12) *An* $(n+1)$*-web of maximum rank* $r = 1$ *is linearizable*[13]).

The main theorem of this paper will be a generalization of this to d-webs of maximal rank $r = \pi(d,n)$ when $d \geqq 2n$ and $n \geqq 3$. In fact, the result will be similar to (2.9) in that a stronger assertion than just linearizability will be proved. The problem of finding necessary and sufficient conditions for just linearizability seems difficult – cf. Blaschke-Bol [1].

ii. *Webs defined by algebraic varieties; relation to Abel's theorem.* We first recall the

Principle of projective duality: The mapping

$$\boldsymbol{P}^n \to (\boldsymbol{P}^{n*})^*$$

given by sending a point $p \in \boldsymbol{P}^n$ *to the set* $\boldsymbol{P}_p^{n-1}$ *of hyperplanes passing through* p *is an isomorphism.*

Now suppose that $f: C \to \boldsymbol{P}^n$ is a non-degenerate algebraic curve of degree d. For simplicity of notation we identify C with its image $f(C)$. Since the curve is non-degenerate, a general hyperplane ξ will meet C in d points in general position in $\xi \cong \boldsymbol{P}^{n-1}$. We write

$$\xi \cdot C = p_1(\xi) + \cdots + p_d(\xi).$$

In a neighborhood $U \subset \boldsymbol{P}^{n*}$ of such a general point we define a d-web by letting the i[th] web hypersurface passing through ξ be $\boldsymbol{P}_{p_i(\xi)}^{n-1}$ (Fig. 3).

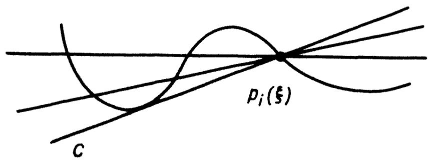

Fig. 3

Thus, a non-degenerate algebraic curve of degree d defines a linearizable d-web in sufficiently small open sets $U \subset \boldsymbol{P}^{n*}$ [14]).

For these webs defined by an algebraic curve the web normals have an especially nice description. To give it we recall that: The projectivized

[13]) Our discussion above pertained to the case $n \geqq 3$. However, it is immediate that a 3-web in the plane which satisfies one abelian equation (2.11) is linearizable.

[14]) When $d \geqq n + 2$, this web is not in general equivalent to one given by d families of parallel hyperplanes. For example, suppose that $n = 2$ and $d = 4$. Then the four lines through each point have a cross-ratio which is invariant under diffeomorphisms and which is a constant in $x \in U$ for 4 families of parallel lines. It is, however, not constant for the web defined by a quartic curve as may be seen by letting ξ tend to a tangent line to the curve.

cotangent space $P(T_\xi^*)$ at a point $\xi \in P^{n^*}$ is naturally identified with ξ itself, i.e.,

$$P(T_\xi^*) \cong \xi .$$

Indeed, the projectivized tangent space is just the set of directions emanating from ξ, and this has a natural identification with the set of lines in P^{n^*} passing through ξ. Such a line is a pencil $|\xi + t\xi'|_{t \in P^1}$ of hyperplanes containing ξ, and this pencil is uniquely determined by its axis $\xi \cap \xi' \cong P^{n-2}$, which is a hyperplane in $\xi \cong P^{n-1}$. Dualizing this natural identification

$$P(T_\xi) \cong \xi^*$$

gives (2.13).

Now a moment's reflection shows that:

(2.14) *Under the identification* (2.13), *the web normals* $\omega^i(\xi) \in P(T_\xi^*)$ *are just the points* $p_i(\xi) \in \xi$ [15]).

If it happens to be the case that the curve lies on a surface S in P^n, then the hyperplane section $\xi \cdot S$ will be an algebraic curve $D_\xi \subset P(T_\xi^*)$ on which the web normals $\omega^i(\xi)$ will lie. This will be the case for Castelnuovo's extremal curves of degree $d > 2n$ when $n \geqq 3$.

We now wish to relate the classical Abel theorem to the abelian equations (2.11) of a web defined by an algebraic curve. Although not strictly necessary it will simplify our explanations if we assume that $f: C \to P^n$ is a smooth embedding – cf. [4] for a discussion of Abel's theorem for possibly singular curves.

Suppose that C has genus π and let $\omega \in H^0(C, \Omega^1)$ be a holomorphic differential. The indefinite integral

$$u(p) = \int_{p_0}^{p} \omega \tag{2.15}$$

is defined modulo the periods of ω and is called an *abelian integral*. A special case of the classical form of *Abel's theorem* states that the *abelian sum*

$$u(p_1(\xi)) + \cdots + u(p_d(\xi)) = \text{const}$$

associated to variable points of intersection of a hyperplane ξ with the curve C is constant. Differentiation of (2.15) gives the equivalent form

$$\omega(p_1(\xi)) + \cdots + \omega(p_d(\xi)) \equiv 0 \tag{2.16}$$

of Abel's theorem. Here $\omega(p_i(\xi))$ denotes the pullback to $U \subset P^{n^*}$ of ω under the mapping $\xi \to p_i(\xi)$. Since the i$^{\text{th}}$ web hypersurface through ξ defined by

[15]) Referring to the preceding footnote, this identification makes it apparent that the cross-ratio of the four points $\omega^i(\xi) = p_i(\xi)$ is not constant in ξ.

the algebraic curve is obtained by holding $p_i(\xi)$ fixed, we see that (2.16) gives an abelian equation associated to the web. It is not difficult to verify that *any* abelian equation arises in this manner, and consequently:

For the web defined by an algebraic curve C in $\mathbf{P}^n$, the abelian equations are given by the holomorphic differentials ω via Abel's theorem in the form (2.16). *In particular, the rank of the web is equal to the genus of the curve.*

In this case maximizing the rank of the d-web is the same as maximizing the genus of the curve of fixed degree d, and this should help explain the bound mentioned in the previous section.

There is also a relationship between the linearization of a web and the classical converse to Abel's theorem on a compact Riemann surface C. Let $M = C^{(d)}$ be the *d-fold symmetric product* of C. M is a smooth compact, complex manifold of dimension d which we may think of as the set of effective divisors

$$D = p_1 + \cdots + p_d \quad (p_i \in C)$$

of degree d on C. The holomorphic differentials $\omega \in H^0(C, \Omega^1)$ induce holomorphic differentials $\boldsymbol{\omega} \in H^0(M, \Omega^1)$ by

$$\boldsymbol{\omega}(D) = \omega(p_1) + \cdots + \omega(p_d),$$

and the mapping $\omega \to \boldsymbol{\omega}$ is an isomorphism.

An *integral variety* $U \subset M$ is a local complex-analytic subvariety of M such that all

$$\boldsymbol{\omega} | U \equiv 0.$$

If x denotes a coordinate on U and $D(x) = p_1(x) + \cdots + p_d(x)$ the corresponding point in U, then we may define a d-web in U by requiring that the i^{th} web hypersurface passing through x_0 is given by

$$p_i(x) = p_i(x_0).$$

The web is non-degenerate in case each $p_i(x)$ varies in an open set on C, and since

$$\boldsymbol{\omega}(x) = \omega(p_1(x)) + \cdots + \omega(p_d(x)) \equiv 0$$

for any $\omega \in H^0(C, \Omega^1)$ it has rank $r = \pi(C)$. Linearizing this web is analogous to proving that $D(x)$ varies in a linear equivalence class on C, which is a consequence of the classical converse to Abel's theorem.

B. Bound on the rank of a web

i. *Proof of the bound.* We consider a d-web $\{u_i(x) = \text{constant}\}$ and recall that the rank is the maximum number of linearly independent abelian equations

(2.11) $$\sum_i f_i(u_i(x))\,du_i(x) \equiv 0\,.$$

We set
$$m = [(d-1)/(n-1)]$$

and recall Castelnuovo's number
$$\pi(d,n) = m[d - 1/2(m+1)(n-1) - 1]$$

giving the maximum genus of a non-degenerate curve of degree d in $\boldsymbol{P}^n$. The following bound was proved for $n = 2,3$ by Blaschke-Bol [1] and for general n by Chern [3]:

(2.17) **Proposition.** *The rank r of a d-web in n-space satisfies*
$$r \leqq \pi(d,n)\,.$$

Webs of maximum rank $r = \pi(d,n)$ exist by taking the web associated to an extremal algebraic curve.

The method of proof, which will play a central role in this paper, originated with Poincaré. Briefly stated, the idea is *to mimic for general webs the construction of the canonical curve associated to an algebraic curve in $\boldsymbol{P}^n$*. We shall now explain this. Suppose that C is a smooth non-degenerate algebraic curve of degree d in $\boldsymbol{P}^n$ and having genus π. Consider the canonical mapping
$$\varkappa : C \to \boldsymbol{P}^{\pi-1}$$
defined by
$$\varkappa(p) = [\omega_1(p),\ldots,\omega_\pi(p)]$$

where $\omega_1,\ldots,\omega_\pi$ are a basis for the holomorphic differentials on the curve. For a variable hyperplane $\xi \in \boldsymbol{P}^{n*}$ we write
$$\xi \cdot C = p_1(\xi) + \cdots + p_d(\xi)$$
and set
$$\begin{aligned} Z_i(\xi) &= \varkappa(p_i(\xi)) \\ &= [\omega_1(p_i(\xi)),\ldots,\omega_\pi(p_i(\xi))]\,. \end{aligned}$$

The $Z_i(\xi)$ give d points in $\boldsymbol{P}^{\pi-1}$, and Abel's theorem in differential form
$$\omega(p_1(\xi)) + \cdots + \omega(p_d(\xi)) \equiv 0\,, \quad \omega \in H^0(C,\Omega^1)\,,$$

says that the $Z_i(\xi)$ span at most a $\boldsymbol{P}^{d-n-1}(\xi)$ in $\boldsymbol{P}^{\pi-1}$ [16]. A basic observation

[16]) We also encountered this statement in the discussion of the Riemann-Roch theorem in § I B i. This suggests a link between Abel's theorem and the Riemann-Roch theorem. We shall attempt to clarify this point at the end of this section.

is that *since* $Z_i(\xi)$ *depends only on the* i^{th} *web hypersurface passing through* ξ, *the Poincaré mapping*

$$\xi \to \boldsymbol{P}^{d-n-1}(\xi)$$

can be expressed purely in terms of the web defined by the given algebraic curve.

In general, suppose that we have r independent abelian equations

$$\sum_i f_i^\lambda(u_i(x))\,du_i(x) \equiv 0 \quad (\lambda = 1,\ldots,r)\,. \tag{2.18}$$

We may assume that the coefficient matrix has no column ${}^t(f_i^1,\ldots,f_i^r)$ identically zero since otherwise we would be reduced to a $(d-1)$-web. If we set

$$Z_i(x) = [f_i^1\,(u_i(x)),\ldots,f_i^r(u_i(x))]\,, \tag{2.19}$$

Then the points $Z_i(x)$ *are intrinsically defined by the web and lie in a* $\boldsymbol{P}^{d-n-1}$ *in* $\boldsymbol{P}^{r-1}$. Indeed, under a change

$$\tilde{u}_i = v_i(u_i(x))$$

of defining function for the i^{th} hypersurface

$$\tilde{f}_i^\lambda(u_i(x)) = v_i'(u_i(x))^{-1} f_i^\lambda(u_i(x))$$

so that the homogeneous coordinate vector (2.19) has intrinsic meaning. Setting $u_{i\alpha} = \partial u_i/\partial x_\alpha$ we may rewrite the abelian equations (2.18) as

$$\sum Z_i(x) u_{i\alpha}(x) \equiv 0 \tag{2.20}$$

where the coefficient matrix $(u_{i\alpha}) = \partial(u_1,\ldots,u_d)/\partial(x_1,\ldots,x_n)$ has rank n. Consequently the points span at most a $\boldsymbol{P}^{d-n-1}$.

Now we set out to bound the rank of a general web. The abelian equations (2.18) impose at least n independent conditions on the points $Z_i(x)$ in $\boldsymbol{P}^{\pi-1}$. We shall assume that these are all the conditions, and at the end of the proof it will be apparent that if there were more conditions then the estimate on the rank would be improved[17]).

We note that as x varies the points $Z_i(x) = Z_i(u_i(x))$ will trace out a piece of an arc C_i in $\boldsymbol{P}^{r-1}$. Moreover, these arcs span $\boldsymbol{P}^{r-1}$ since otherwise the abelian equations (2.18) would not be independent. Each x determines d points, one $Z_i(x)$ on each arc C_i — we may say that there is a correspondence $x \to Z_1(x),\ldots,Z_d(x)$ — where the linear span

$$\{Z_1(x),\ldots,Z_d(x)\} = \boldsymbol{P}^{d-n-1}(x)\,.$$

[17]) This is the web-theoretic analogue of the observation that the curves of maximal genus of fixed degree d in $\boldsymbol{P}^n$ are necessarily normal; i.e., the linear system of hyperplane sections is complete.

The mapping $x \to \{Z_1(x), \dots, Z_d(x)\}$ is the *Poincaré mapping* for the abelian equations (2.18) associated to a general web. Here is an attempt at the picture for $n = 2$, $d = 4$, $r = 3$ (Fig. 4).

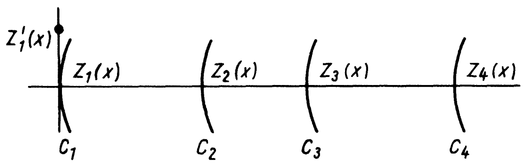

Fig. 4

We denote by $Z_i'(x)$ a point on the tangent line to C_i at $Z_i(x)$, by $Z_i''(x)$ a point such that $\{Z_i(x), Z_i'(x), Z_i''(x)\}$ is the osculating 2-plane to C_i at $Z_i(x)$, and so forth. In general we let

$$\begin{aligned} \boldsymbol{P}^{N(k)}(x) &= \{Z_1(x), \dots, Z_d(x); Z_1'(x), \dots, Z_d'(x); \dots; Z_1^{(k)}(x), \dots, Z_d^{(k)}(x)\} \\ &= \{Z_i(x); Z_i'(x); \dots; Z_i^{(k)}(x)\} \qquad i = 1, \dots, d \end{aligned}$$

denote the span in $\boldsymbol{P}^{r-1}$ of the k$^{\text{th}}$ osculating spaces to the arcs C_i at corresponding points $Z_i(x)$. As before, the rank will turn out to be maximized when the dimensions $N(k)$ are maximal subject to the conditions obtained by successively differentiating (2.20), and so we assume this to be the case. There are obvious inclusions

$$\boldsymbol{P}^{N(0)}(x) \subset \boldsymbol{P}^{N(1)}(x) \subset \cdots \subset \boldsymbol{P}^{r-1}.$$

Lemma: *If* $N(l) = N(l+1)$ *for some* l, *then* $N(l) = r - 1$.

Proof. We may assume that $\partial u_i/\partial x_\alpha \neq 0$. With the notations

$$f_i^{\lambda,(k)}(u) = \mathrm{d}^k f_i^\lambda(u)/\mathrm{d}u^k,$$

we have

$$\begin{aligned} \partial Z_i^{(k)}(x)/\partial x_\alpha &= [\partial f_i^{1,(k)}(u_i(x))/\partial x_\alpha, \dots, \partial f_i^{r,(k)}(u_i(x))/\partial x_\alpha] \\ &= [f_i^{1,(k+1)}(u_i(x)), \dots, f_i^{r,(k+1)}(u_i(x))] \\ &= Z_i^{(k+1)}(x). \end{aligned}$$

If $N(l) = N(l+1)$ then $Z_i^{(l+1)}(x)$ lies in $\boldsymbol{P}^{N(l)}(x)$, and consequently for any $Z(x) \in \boldsymbol{P}^{N(l)}(x)$ the derivative

$$\partial Z(x)/\partial x_\alpha \in \boldsymbol{P}^{N(l)}(x).$$

This says that the subspace $\boldsymbol{P}^{N(l)}(x)$ is a constant independent of x. Since this fixed subspace contains all arcs C_i we conclude that $N(l) = r - 1$. Q.E.D.

Now to the proof of the bound (2.17). We break the integers $1,\ldots,d-1$ up into $m = [(d-1)/(n-1)]$ disjoint subsets of $n-1$ elements each plus a remainder labelled as follows:

$$\underbrace{\overbrace{1,\ldots,n-1}; \overbrace{n,\ldots,2n-2}; \ldots; \underbrace{(m-1)(n-1)+1,\ldots,m(n-1); \underbrace{m(n-1)+1,\ldots,d-1}_{I_{m-1}}}_{I_{m-2}}}_{I_0}$$

We will inductively estimate the dimensions $N(k)$ using the following device: For any $k < m$ the functions

$$u_{k(n-1)+1},\ldots,u_{(k+1)(n-1)},u_d$$

form a coordinate system in which

$$\partial u_{k(n-1)+1}/\partial u_d = \cdots = \partial u_{(k+1)(n-1)}/\partial u_d = 0\,. \tag{2.21}$$

We will successively differentiate the abelian equation (2.20) using (2.21). For the first step we take $(u_1,\ldots,u_{n-1},u_d)$ as coordinate system, and then by (2.21) for $\alpha = d$ the equation (2.20) becomes

$$Z_n \partial u_n/\partial u_d + \cdots + Z_{d-1}\partial u_{d-1}/\partial u_d + Z_d = 0\,. \tag{2.22}$$

Thus Z_d is a linear combination of $Z_n,\ldots,Z_{d-1}$, and using the evident symmetry of the indices

$$P^{N(0)} = \{Z_i(x) : i \in I_0\}\,.$$

Next, choose $(u_n,\ldots,u_{2n-2},u_d)$ as coordinate system and apply $\partial/\partial u_d$ to (2.22) to obtain

$$\varrho_{2n-1}Z'_{2n-1} + \cdots + \varrho_{d-1}Z'_{d-1} + Z'_d \equiv 0 \text{ modulo } P^{N(0)}(x)\,.$$

Again, by symmetry of the indices

$$P^{N(1)}(x) = \{Z'_i(x) : i \in I_1\} \oplus P^{N(0)}(x)$$

where "$\oplus$" denotes the linear span of the two subspaces of P^{r-1}. Iterating the argument gives

$$P^{N(k)}(x) = \{Z_i^{(k)}(x) : i \in I_k\} \oplus P^{N(k-1)}(x) \tag{2.23}$$

where $k \leqq m-1$ and

$$N(k) = N(k-1) + (d - 1 - (k+1)(n-1))\,, \quad N(-1) = -1\,. \tag{2.24}$$

The first two dimensions are

$$\begin{cases} N(1) = d - n - 1 \\ N(2) = 2d - 3n. \end{cases} \tag{2.25}$$

At the last step

$$P^{N(m-1)}(x) = \{Z_i^{(m-1)}(x) : i \in I_{m-1}\} \oplus P^{N(m-2)}(x),$$

so that in particular

$$Z_d^{(m-1)}(x) \equiv \sum_{j \in I_{m-1}} \sigma_j Z_j^{(m-1)}(x) \text{ modulo } P^{N(m-2)}(x).$$

Choose $(u_i : i \in I_{m-1}, u_d)$ as part of a coordinate system and apply $\partial/\partial u_d$ to this relation to obtain

$$Z_d^{(m)}(x) \equiv 0 \text{ modulo } P^{N(m-1)}(x),$$

i.e.,

$$N(m) = N(m-1).$$

By the lemma

$$r - 1 = N(m-1) = m(d-1) - \sum_{k=1}^{m} k(n-1) - 1$$

or

$$r = m(d-1) - 1/2m(m+1)(n-1) = m[d - 1/2(m+1)(n-1) - 1]$$

as desired. Q.E.D.

We note the similarity between the argument just given and the algebro-geometric proof bounding the genus of a non-degenerate curve of degree d in P^n. Both relied on a combinatorial argument decomposing the integers $1,\dots,d$ into $m = [(d-1)/(n-1)]$ disjoint subsets, and in both instances the k^{th} step had to do with the k^{th} osculating curves. On the other hand, whereas Castelnuovo's argument was based on the Riemann-Roch the web-theoretic proof relied on Abel's theorem. To explain the connection we shall prove:

For an effective divisor D on a compact Riemann surface C, Abel's theorem (2.16) *implies the inequality*

$$\dim|D| \leqq \deg D - \pi + i(D)$$

in the Riemann-Roch theorem [18]).

We shall assume that C is non-hyperelliptic and that the complete linear system $|D|$ gives a projective embedding $C \to P^n$ ($n = \dim|D|$). This case

[18]) This inequality is all that is needed to establish Castelnuovo's bound. The converse of Abel's theorem yields the opposite inequality, and hence the full Riemann-Roch.

will cover the comparison we are trying to draw between the two proofs of Castelnuovo's bound and the argument we are about to give may be modified to cover the general situation. For a hyperplane $\xi \in \boldsymbol{P}^{n*}$ we write

$$\xi \cdot C = p_1(\xi) + \cdots + p_d(\xi) \in |D|$$

and set

$$Z_i(\xi) = \varkappa(p_i(\xi))$$

as above. Then by Abel's theorem (2.16) the $Z_i(\xi)$ span at most a $\boldsymbol{P}^{d-n-1}(\xi)$ in the space $P^{\pi-1}$ of the canonical curve. On the other hand $\dim\{Z_1(\xi), \ldots, Z_d(\xi)\} = \pi - 1 - i(D)$. Combining we obtain

$$d - n - 1 \leqq \pi - 1 - i(D)$$

which implies (2.26).

Finally, if we go back to the proof of (2.17) we see that the first two steps (2.25) remain valid under the assumption

$$r > \mu(d,n)$$

for some $\mu(d,n)$ strictly less than Castelnuovo's number $\pi(d,n)$. For example, when $n = 3$ one finds (2.25) under the assumption

$$r > (d^2 - 3d)/6 + 1$$

in agreement with the bound obtained by Joe Harris for curves in $\boldsymbol{P}^3$ which we mentioned at the end of § I B i. We have not determined the best exact value for $\mu(d,n)$, but the asymptotic estimates are

$$\begin{cases} r < d^2/2(n-1) \text{ for any } d\text{-web in } n\text{-space} \\ r > d^2/3(n-1) \Rightarrow \text{the bound (2.25)} \end{cases}$$

ii. *Statement and discussion of the main theorem*

Definition. *A web has maximum rank in case equality holds in* (2.17).

As mentioned before webs of maximum rank may be found in either the real or complex case, by taking the web in $U \subset \boldsymbol{P}^{n*}$ associated to an extremal algebraic curve $C \subset \boldsymbol{P}^n$. By construction these webs are linear. The principal result of this paper is a converse

(2.27) **Linearization theorem.** *A d-web in n-space of maximal rank $r = \pi(d,n)$ is linearizable provided that*

$$\begin{cases} d = n + 1, 2n & \textit{for any } n \text{[19]} \\ d > 2n & \text{for } n \geqq 3 \end{cases}$$

[19]) The index restrictions will turn out to be sharp. Cf. the discussion following Corollary (2.28).

Combining this with the main theorem proved in [4][20]) we deduce the

(2.28) **Corollary.** *Under the conditions in the main theorem, the given web is equivalent to one defined by an extremal algebraic curve C.*

We emphasize that there are *two* steps in constructing the algebraic curve C. The deeper local step consists of linearizing the web; the globalization of a linear web is of a different and more analytic character.

Regarding the index restrictions in the main theorem, recall that

$$\pi(d,n) = d - n \quad \text{for} \quad n \leqq d \leqq 2n - 1 .$$

The case $d = n + 1$ where $\pi(d,n) = 1$ has been taken up in § II A i and the main theorem proved there – cf. (2.12). If, on the other hand $n + 2 \leqq d \leqq 2n - 1$, we may construct non-linearizable d-webs of maximal rank as follows: Define the web by level sets $\{u_i(x) = \text{constant}\}$ with

$$\left\{\begin{aligned} u_1 &= x_1 \\ &\vdots \\ u_n &= x_n \\ u_{n+1} &= U_{1,1}(x_1) + \cdots + U_{1,n}(x_n) \\ &\vdots \\ u_{n+r} &= U_{r,1}(x_1) + \cdots + U_{r,n}(x_n) \end{aligned}\right. \tag{2.29}$$

where the $U_{s,\alpha}(x_\alpha)$ $(1 \leqq s \leqq r = d - n,\ 1 \leqq \alpha \leqq n)$ are functions of a single variable. A typical n-fold Jacobian is

$$\begin{aligned} &\mathrm{d}u_{k+1} \wedge \cdots \wedge \mathrm{d}u_n \wedge \mathrm{d}u_{n+1} \wedge \cdots \wedge \mathrm{d}u_{n+k} \\ &\quad = \pm \begin{vmatrix} U'_{1,1} \cdots U'_{1,k} \\ \vdots \\ U'_{k,1} \cdots U'_{k,k} \end{vmatrix} \mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_n , \end{aligned}$$

involving the Wronskians of the functions $U_{s,\alpha}$. For generic choice of such functions the web defined by the functions (2.29) will therefore be non-degenerate. Again, because Wronskians appear as coefficient matrices the abelian equations

$$\mathrm{d}u_{n+s} - \sum_\alpha U'_{s,\alpha}(u_\alpha)\mathrm{d}u_\alpha = 0 \quad (s = 1,\ldots,r)$$

will generically be linearly independent so that the web has maximal rank $r = d - n$. If this web were linearizable, then by the Sophus Lie theorem (cf. footnote 20) there would be an algebraic curve C in $\boldsymbol{P}^n$ of degree $n + r$ defining an equivalent web. This curve depends on only a finite number

[20]) The result referred to implies that a *linear* d-web of rank $r \geqq 1$ is defined by an algebraic curve C of degree d. We shall call this the *generalized Sophus Lie theorem.*

of constants (Chow variety), and moreover since any local diffeomorphism preserving a linear d-web for $d > n + 1$ is a projective transformation we see again that a general web (2.29) is not linearizable.

A further remark is that the proof of the linearization theorem will primarily depend on showing that under the assumptions

$$\begin{cases} r = \pi(d,n) \\ d > 2n \end{cases}$$

the web normals lie on a rational normal curve in the projectivized cotangent spaces. This follows from (2.25) and will be proved in § III A i; an outline of how this fits into the overall proof is discussed in § III A ii. On the other hand, as remarked at the end of the preceding section, the step (2.25) necessary to prove that the web normals lie on a rational normal curve will still be true provided $d > 2n$ and the rank

$$r > \mu(d,n) .$$

The upshot is that, by an posteriori analysis of the proof, our linearization theorem may be extended to d-webs ($d > 2n$) whose rank

$$r > \mu(d,n) ,$$

where

$$\mu(d,3) = (d^2 - 3d)/6 + 1 ,$$

and in general

$$\mu(d,n) \sim d^2/3(n - 1)$$

is strictly less than Castelnuovo's number $\pi(d,n) \sim d^2/2(n - 1)$.

Now we shall prove the linearization theorem in the case

$$d = 2n, \pi(d,n) = n + 1$$

corresponding to the canonical curve. The Poincaré map is

$$x \to \{Z_1(x),\dots,Z_{2n}(x)\} = \boldsymbol{P}^{n-1}(x) \subset \boldsymbol{P}^n ,$$

which is consequently an *equidimensional* map

$$F : U \to \boldsymbol{P}^{n*} .$$

It is easy to check that as a consequence of the non-degeneracy F has non-zero Jacobian, and since the i[th] web hypersurface is defined by

$$Z_i(x) = \text{constant} ,$$

this hypersurface corresponds under F to the hyperplane

$$\boldsymbol{P}^{n-1}_{Z_i(x)} \in (\boldsymbol{P}^{n*})^{*\ 21)} .$$

[21]) The notation is explained in the beginning of § II A ii where projective duality is formally stated.

Consequently, the Poincaré mapping F serves to linearize the web, and the theorem is proved in this case.

When $d > 2n, n \geqq 3$ the Poincaré mapping

$$x \to \{Z_1(x),\dots,Z_d(x)\} = \boldsymbol{P}^{d-n-1}(x) \subset \boldsymbol{P}^{r-1}$$

associated to our web of maximal rank $r = \pi(d,n)$ will induce

$$F: U \to \boldsymbol{G}(d-n-1, r-1).$$

The Poincaré space $\boldsymbol{P}^{r-1}$ carries a natural linear structure, but F is no longer equidimensional and so is of a more complicated nature.

In fact, suppose for each point $Z \in \boldsymbol{P}^{r-1}$ we let $\boldsymbol{G}_Z(d-n-1,r-1)$ be the Schubert cycle in $\boldsymbol{G}(d-n-1,r-1)$ of all $\boldsymbol{P}^{d-n-1}$'s passing through Z. Then

$$\begin{cases} \dim \boldsymbol{G}(d-n-1,r-1) = (d-n)(r-d+n) \\ \dim \boldsymbol{G}_Z(d-n-1,r-1) = (d-n-1)(r-d+n) \end{cases}$$

and consequently

$$\operatorname{codim} \boldsymbol{G}_Z(d-n-1,r-1) = r-d+n.$$

Now the i^{th} web hypersurface passing through $x \in U$ is, as in the equidimensional case,

$$F^{-1}(\boldsymbol{G}_{Z_i(x)}(d-n-1,r-1)).$$

If $F(U)$ and $\boldsymbol{G}_{Z_i(x)}(d-n-1,r-1)$ are in general position then they should meet in a variety of dimension $d-r$. Since for large d, $r = \pi(d,n) \sim d^2/2(n-1)$ this number will eventually be negative. In fact the right number

$$d - r = n - 1$$

occurs exactly when $d = 2n$ so that $r = n + 1$ and we are in the case of the canonical curve. As a result we see that *when $d > 2n$ the Poincaré mapping F is highly non-generic, and we are in a situation somewhat analogous to an overdetermined system of equations for which the abelian relations* (2.20) *serve as compatibility conditions.*

iii. *Properties of webs defined by extremal algebraic curves.* From now on we will assume that $d > 2n$ and $n \geqq 3$. Let $C \subset \boldsymbol{P}^n$ be an extremal algebraic curve of degree d and genus $\pi = \pi(d,n)$. We denote by $C_\kappa \subset \boldsymbol{P}^{\pi-1}$ the canonical curve, and recall that the original curve C defined a d-web of maximal rank π in suitable open sets $U \subset \boldsymbol{P}^{n*}$ according to the prescription: *If, for a hyperplane ξ we write*

$$\xi \cdot C = p_1(\xi) + \cdots + p_d(\xi),$$

then the i^{th} web hypersurface passing through ξ is the $\mathbf{P}^{n-1}_{p_i(\xi)}$ of hyperplanes ξ' with $p_i(\xi') = p_i(\xi)$. We restate what is perhaps the basic underlying principle in our study of maximal rank webs: *Any property of the curve C in $\mathbf{P}^n$ or of its canonical image C_κ in $\mathbf{P}^{\pi-1}$ which is expressible purely in terms of the web defined by C may be expected to hold for a general d-web of maximal rank.* For example, the Poincaré mapping given by

$$\xi \to \{Z_1(\xi),\dots,Z_d(\xi)\} = \mathbf{P}^{d-n-1}(\xi) \subset \mathbf{P}^{\pi-1}$$

makes sense for a general web of maximal rank. Moreover, the Poincaré space $\mathbf{P}^{\pi-1}$ and projectivized cotangent spaces $\mathbf{P}(T_\xi^*)$ have intrinsic linear structures, even though this will not be true of the manifold U in which a general web is given.

We shall now list some properties of webs defined by extremal algebraic curves, and then the proof of our main theorem will proceed by showing that these same properties hold for general webs of maximal rank and may be used to interrelate the intrinsic linear structures in the Poincaré spaces and projectivized cotangent spaces in a sufficiently tight manner as to eventually yield the linearization theorem. These properties were either proved or are consequences of the discussion in § I d iii.

The first property of the web defined by C is (cf. (2.14))

(2.30) *The web-normals $\omega^i(\xi) \in \mathbf{P}(T_\xi^*)$ lie on a rational normal curve D_ξ.*

We shall be able to prove that, in general, a maximal rank web defines a field of rational normal curves D_x in the projectivized cotangent spaces $\mathbf{P}(T_x^*(U))$, and that the web normals $\omega^i(x)$ give completely integrable cross-sections of this structure.

The second property is (cf. (1.13))

(2.31) *The points $Z_i(\xi)$ lie on a rational normal curve E_ξ in $\mathbf{P}^{d-n-1}(\xi)$. The curves D_ξ and E_ξ are in projective correspondence in such a way that $\omega^i(\xi)$ corresponds to $Z_i(\xi)$.*

Again this property will be proved to remain valid for a general web of maximal rank.

The third property (cf. (1.14) and the end of § I d iii) refines the link established by the projectivity $D_\xi \barwedge E_\xi$ in (2.31). In fact it tells us how to define the straight lines in the linear structure on $U \subset \mathbf{P}^{n^*}$ purely in terms of the web associated to our given curve C.

(2.32) *A tangent direction $\sigma \in \mathbf{P}(T_\xi)$ gives a hyperplane $\mathbf{P}^{n-2}(\sigma)$ in $\mathbf{P}(T_\xi^*)$. This hyperplane meets D_ξ in $n - 1$ points, and the corresponding $n - 1$ points on E_ξ span a secant plane $\mathbf{P}^{n-2}(\xi,\sigma)$ to C_ξ. In fact $\mathbf{P}^{n-2}(\xi,\sigma)$ is defined by the* *condition (cf. the explanation given in the list of notations)*

(2.33) $$\mathrm{d}Z/\mathrm{d}\sigma \in \boldsymbol{P}^{d-n-1}(\xi), \quad Z \in \boldsymbol{P}^{d-n-1}(\xi).$$

The straight line through $\xi \in U$ in the direction σ is given by all ξ' such that $\boldsymbol{P}^{d-n-1}(\xi')$ contains $\boldsymbol{P}^{n-2}(\xi,\sigma)$.

For a general web of maximal rank we will be able to show that the correspondence $D_x \barwedge E_x$ takes the hyperplanes $\boldsymbol{P}^{n-2}(\sigma)$ in $\boldsymbol{P}(T_x^*)$ bijectively onto the $(n-2)$-fold secants $\boldsymbol{P}^{n-2}(x,\sigma)$ to the rational normal curve E_x in $\boldsymbol{P}^{d-n-1}(x)$, and moreover this $\boldsymbol{P}^{n-2}(x,\sigma)$ will be defined by the equation

$$\mathrm{d}Z/\mathrm{d}\sigma \in \boldsymbol{P}^{d-n-1}(x)$$

analogous to (2.33). The condition

$$\boldsymbol{P}^{d-n-1}(x') \quad \text{contains} \quad \boldsymbol{P}^{n-2}(x,\sigma)$$

will then be proved to define a path $x(t)$ in U passing through x and with tangent direction σ there. In this way the maximal rank web will induce a path geometry in U such that the linearization theorem will be true if, and only if, there is a change of coordinates in U transforming the paths into straight lines.

Finally to show that there is such a change of coordinates, we will use the property for general webs of maximal rank analogous to the following property of webs defined on extremal algebraic curve C in $\boldsymbol{P}^n$:

(2.34) *Consider Castelnuovo's ruled surface S in $\boldsymbol{P}^n$ on which the curve C lies. For a hyperplane ξ the intersection $\xi \cdot S$ is the curve D_ξ under the identification $\xi \cong \boldsymbol{P}(T_\xi^*)$. Consequently, for each point $p \in \xi \cdot S$ there is a hypersurface $\boldsymbol{P}_p^{n-1}$ in U passing through ξ and whose normal corresponds to p under the above identification. In other words, the original web can be embedded in a larger family of ∞^2 hypersurfaces defined by the condition that their normals should lie on the field of rational normal curves $D_\xi \subset \boldsymbol{P}(T_\xi^*)$. Going over to the Poincaré space, if $\omega \in D_\xi$ corresponds to $Z \in E_\xi$, then the hypersurface passing through ξ and with normal ω is given by ξ' satisfying*

(2.35) $$Z \in \boldsymbol{P}^{d-n-1}(\xi').$$

Again we shall prove the analogous property for general webs of maximal rank. The ∞^2 hypersurfaces will turn out to be totally geodesic for the path geometry defined by (2.33), and the existence of this large number of totally geodesic hypersurfaces will imply projective flatness.

At this juncture it is likely that the complexity of the structure of webs arising from extremal algebraic curves is confusing, and perhaps the following correspondence table will help clarify the situation.

(1) given space U $\longleftrightarrow$ Poincaré space $\boldsymbol{P}^{n-1}$

(2) point $\xi \in U$ $\longleftrightarrow$ subspace $\boldsymbol{P}^{d-n-1}(\xi)$ in $\boldsymbol{P}^{r-1}$

(3) $\begin{Bmatrix} \text{web normals} \\ \omega^i(\xi) \in \boldsymbol{P}(T_\xi^*) \end{Bmatrix} \leftrightarrow \begin{Bmatrix} \text{points } Z_i(\xi) \text{ with} \\ \{Z_1(\xi), \ldots, Z_d(\xi)\} = \boldsymbol{P}^{d-n-1}(\xi) \end{Bmatrix}$

(4) $\begin{Bmatrix} \text{rational normal} \\ \text{curve } D_\xi \subset \boldsymbol{P}(T_\xi^*) \end{Bmatrix} \longleftrightarrow \begin{Bmatrix} \text{rational normal curve} \\ E_\xi \subset \boldsymbol{P}^{d-n-1}(\xi) \end{Bmatrix}$

Under the projectivity $D_\xi \barwedge E_\xi$ the web normals $\omega^i(\xi)$ correspond to the $Z_i(\xi)$

(5) $\begin{Bmatrix} \text{tangent direction} \\ \sigma \in \boldsymbol{P}(T_\xi) \end{Bmatrix} \longleftrightarrow \begin{Bmatrix} \text{secant plane} \\ \boldsymbol{P}^{n-2}(\xi, \sigma) \text{ to } E_\xi \end{Bmatrix}$

Here, σ defines a hyperplane in $\boldsymbol{P}(T_\xi^*)$ which meets D_ξ in $n-1$ points, and the secant plane $\boldsymbol{P}^{n-2}(\xi, \sigma)$ is spanned by the corresponding points on E_ξ

(6) $\begin{Bmatrix} \text{line in } U \text{ through} \\ \xi \text{ and with tangent } \sigma \end{Bmatrix} \longleftrightarrow \begin{Bmatrix} \text{set of } \xi' \text{ such that} \\ \boldsymbol{P}^{n-2}(\xi, \sigma) \subset \boldsymbol{P}^{d-n-1}(\xi') \end{Bmatrix}$

We may think of this condition as defining a path geometry in U

(7) $\begin{Bmatrix} \text{totally geodesic} \\ \text{hypersurface through } \xi \\ \text{and with normal } \omega \in D_\xi \end{Bmatrix} \longleftrightarrow \begin{Bmatrix} \text{set of } \xi' \text{ satisfying} \\ Z \in \boldsymbol{P}^{d-n-1}(\xi') \\ \text{where } Z \in E_\xi \text{ corresponds} \\ \text{to } \omega \text{ under } D_\xi \barwedge E_\xi \end{Bmatrix}$

Provided we replace the word "line" with "path" in (6), all the statements in this dictionary make sense for general webs of maximal rank, and beginning in the next section we shall prove them in this context. Once this has been done the main theorem will follow from some rather general results about projective differential geometry.

III. Path geometry associated to maximal rank webs

A. Abelian equations and rational normal curves

i. *Properties of the Poincaré map for maximal rank webs.* Let $\{u_i(x) = \text{constant}\}$ define a d-web of maximal rank r in an open set U in n-space. Suppose that

$$\sum_i f_i^\lambda(u_i(x))du_i(x) = 0 \quad (\lambda = 1, \ldots, r) \tag{3.1}$$

give a basis for the abelian equations associated to the web, and define

$$Z_i(x) = [f_i^1(u_i(x)), \ldots, f_i^r(u_i(x))] \in \boldsymbol{P}^{r-1}.$$

As x varies the points $Z_i(x)$ trace out an arc C_i in P^{r-1}. Each $x \in U$ has associated to it points $Z_i(x) \in C_i$ – one may think of the assignment

$$x \to Z_1(x), \ldots, Z_d(x)$$

as a *correspondence*. We denote by $Z_i'(x)$ a point on the tangent line to C_i at $Z_i(x)$. From (2.25) we have:

If $d \geqq n+1$ then the $Z_i(x)$ span a $P^{d-n-1}(x)$ in P^{r-1}; If $d \geqq 2n$ then the $Z_i(x)$ and $Z_i'(x)$ together span a $P^{2d-3n}(x)$ in P^{r-1}.

We shall abbreviate these statements by writing

$$\{Z_1(x), \ldots, Z_d(x)\} = P^{d-n-1}(x) \tag{3.2}$$

$$\{Z_1(x), \ldots, Z_d(x); Z_1'(x), \ldots, Z_d'(x)\} = P^{2d-3n}(x). \tag{3.3}$$

The mapping (3.2)

$$F: U \to G(d-n-1, r-1)$$

is the *Poincaré mapping*. We will see that (3.3) gives the infinitesimal structure of F.

The purpose of the present section is to prove analogues for general d-webs of maximal rank of the properties (2.30) – (2.34) of the webs defined by extremal algebraic curves. We shall begin with the two statements:

(3.4) *Under the assumption $d > 2n, n \geqq 3$ the web normals $\omega^i(x)$ lie on a unique rational curve D_x in the projectivized cotangent space $P(T_x^*) \cong P^{n-1}$.*

(3.5) *With the same assumptions as in* (3.4), *the points $Z_i(x)$ lie on a rational normal curve E_x in $P^{d-n-1}(x)$. There is a projectivity*

$$D_x \barwedge E_x$$

taking $\omega^i(x)$ to $Z_i(x)$.

Proof of (3.4): Using the notations

$$u_{i\alpha} = \partial u_i / \partial x_\alpha, \quad u_{i\alpha\beta} = \partial^2 u_i / \partial x_\alpha \partial x_\beta$$

the abelian equations (3.1) may be written

$$\sum_i Z_i(x) u_{i\alpha}(x) \equiv 0 \quad \alpha = 1, \ldots, n. \tag{3.6}$$

Moreover, *any linear relation among the $Z_i(x)$ is a combination of the equations* (3.6). Applying $\partial / \partial x_\beta$ to (3.6) gives

$$\sum_i Z_i'(x) u_{i\alpha}(x) u_{i\beta}(x) + \sum_i Z_i(x) u_{i\alpha\beta}(x) = 0. \tag{3.7}$$

Taken together, (3.6) and (3.7) yield

$$n + n(n+1)/2 = n(n+3)/2$$

relations among the $Z_i(x)$ and $Z_i'(x)$. According to (3.3) only $3n - 1$ of these can be independent, and so there must be

$$n(n+3)/2 - (3n-1) = (n-1)(n-2)/2$$

relations among the relations. This implies that there will be equations

$$\sum_{\alpha,\beta} k^{\alpha\beta} u_{i\alpha}(x) u_{i\beta}(x) = 0 \tag{3.8}$$

$$\sum_{\alpha,\beta} k^{\alpha\beta} u_{i\alpha\beta}(x) = \sum_{\gamma} m^{\gamma} u_{i\gamma}(x) \tag{3.9}$$

where $k^{\alpha\beta} = k^{\beta\alpha}$ varies over an $(n-1)(n-2)/2$-dimensional linear space of quadrics.

The first equation (3.8) says that the web normals

$$\omega^i(x) = [u_{i1}(x), \ldots, u_{in}(x)] \in \boldsymbol{P}(T_x^*)$$

lie on $\infty^{(n-1)(n-2)/2}$ independent quadrics in $\boldsymbol{P}(T_x^*) \cong \boldsymbol{P}^{n-1}$. Since the $\omega^i(x)$ are in general position, if

$$d > 2(n-1) + 2 = 2n$$

then (1.4) implies that there is a unique rational normal curve D_x passing through the $\omega^i(x)$. Q.E.D. for (3.4).

We note that the restrictions $d > 2n$, $n \geqq 3$ appear naturally in this proof.

Also, (3.9) may be interpreted as stating that *the defining functions $u_i(x)$ for the web satisfy inhomogenous Laplace equations relative to the quadrics containing the web normals*. This will be of crucial importance in our later work.

Proof of (3.5): Choose $u_1, \ldots, u_n$ as coordinate system. Then D_x is a rational normal curve passing through the vertices $[1,0,\ldots,0], \ldots, [0,\ldots,0,1]$ of the coordinate simplex in $\boldsymbol{P}^{n-1}$. Using $\xi = [\xi_1, \ldots, \xi_n]$ as homogeneous coordinates, according to the proof of (1.1) D_x is given parametrically by

$$\varrho \xi_\alpha = a_\alpha/(t - b_\alpha) \quad \alpha = 1, \ldots, n \tag{3.10}$$

where ϱ is a homogeneity factor and a_α, b_α are functions of x. The n points $t = b_\alpha$ correspond to the vertices ω^α.

Since the web normals lie on D_x

$$\varrho_i u_{i\alpha} = a_\alpha/(t_i - b_\alpha) \quad \alpha = 1, \ldots, n$$

for $i = 1,\dots,d$. We set $\omega_{i\alpha} = \varrho_i u_{i\alpha}$ and write the abelian equations (3.6) in the form

$$Z_\alpha + \sum_{s=n+1}^{d} Z_s \omega_{s\alpha} = 0 \quad \alpha = 1,\dots,n. \tag{3.11}$$

This shows that $Z_{n+1}(x),\dots,Z_d(x)$ give a basis for $\boldsymbol{P}^{d-n-1}(x)$. In terms of the homogeneous coordinates corresponding to this basis, (3.10) and (3.11) imply that

$$Z_\alpha = [1/(t_{n+1} - b_\alpha),\dots,1/(t_d - b_\alpha)] \quad \alpha = 1,\dots,n.$$

Consequently, the points $Z_i(x)$ all lie on the rational normal curve E_x in $\boldsymbol{P}^{d-n-1}(x)$ given parametrically by

$$b \to [1/(t_{n+1} - b),\dots,1/(t_d - b)].$$

The rational normal curves D_x and E_x have respective linear parameters t and b. Setting $b = t$ gives a projectivity $D_x \barwedge E_x$ under which the corresponding points are

$$\begin{cases} t = b_\alpha \leftrightarrow \omega^\alpha \\ b = b_\alpha \leftrightarrow Z_\alpha \quad 1 \leqq \alpha \leqq n \end{cases}$$

$$\begin{cases} t = t_s \leftrightarrow \omega^s \\ b = t_s \leftrightarrow Z_s \quad n + 1 \leqq s \leqq d. \end{cases}$$

Hence the projectivity takes ω^i to Z_i. Q.E.D. for (3.5).

When $n = 2$ the $Z_i(x)$ are d points on a $\boldsymbol{P}^{d-3}(x)$, and these will always lie on a unique rational curve. Consequently this part of (3.5) remains valid for webs in the plane but imposes no restriction on the Z_i.

Next we shall prove an analogue of part of (2.32):

(3.12) *For a tangent direction $\sigma \in \boldsymbol{P}(T_x)$, the set of $Z \in \boldsymbol{P}^{d-n-1}(x)$ satisfying*

$$\mathrm{d}Z/\mathrm{d}\sigma \in \boldsymbol{P}^{d-n-1}(x) \tag{3.13}$$

constitutes a $\boldsymbol{P}^{n-2}(x,\sigma)$ [22]. *This $\boldsymbol{P}^{n-2}(x,\sigma)$ is obtained by first considering σ as a hyperplane in $\boldsymbol{P}(T_x^*)$ meeting D_x in $n - 1$ points and then taking the $(n - 2)$-fold secant plane spanned by the corresponding points on E_x.*

Proof. We may assume that

$$Z_{n+1}(x),\dots,Z_d(x) \quad \text{span}\, \boldsymbol{P}^{d-n-1}(x).$$

[22]) Geometrically, $\boldsymbol{P}^{n-2}(x,\sigma)$ is the intersection of $\boldsymbol{P}^{d-n-1}(x)$ with the infinitely nearly linear space $\boldsymbol{P}^{d-n-1}(x + \varepsilon\sigma)$. The notation (3.13) is explained in index of notations in the introduction.

On account of (3.3) there will be $(n-1)$ independent relations among the points $Z'_s(x)(n+1 \leqq s \leqq d)$. We write these as

$$\sum_s A_{\mu s} Z'_s \equiv 0 \text{ modulo } \boldsymbol{P}^{d-n-1} \quad (\mu = 1,\dots,n-1)\,. \tag{3.14}$$

A general point Z in $\boldsymbol{P}^{d-n-1}$ is written

$$Z = \sum_s p_s Z_s\,,$$

and

$$\mathrm{d}Z/\mathrm{d}\sigma \equiv \sum_s p_s \mathrm{d}u_s/\mathrm{d}\sigma\, Z'_s \text{ modulo } \boldsymbol{P}^{d-n-1} \tag{3.15}$$

where we are thinking of σ as a non-zero tangent vector and have set $\mathrm{d}u_s/\mathrm{d}\sigma = \langle \mathrm{d}u_s, \sigma\rangle$. Comparing (3.14) and (3.15), the condition (3.13) is equivalent to

$$p_s \mathrm{d}u_s/\mathrm{d}\sigma = \sum_{\mu=1}^{n} c_\mu A_{\mu s}\,. \tag{3.16}$$

This shows that the solutions to (3.13) constitute a $\boldsymbol{P}^{n-2}(x,\sigma)$ which is the image of a standard $\boldsymbol{P}^{n-2}$ with homogeneous coordinates $[c_1,\dots,c_{n-1}]$ under a linear map

$$\boldsymbol{P}^{n-2} \to \boldsymbol{P}^{d-n-1}(x)$$

as prescribed in (3.16)[23]).

We now want to prove that this $\boldsymbol{P}^{n-2}(x,\sigma)$ is given by the $(n-2)$-fold secant plane description. If ω lies on the rational normal curve D_x in $\boldsymbol{P}(T_x^*)$, then the tangent directions σ satisfying

$$\langle \omega, \sigma\rangle = 0$$

constitute a $\boldsymbol{P}^{n-2}(\omega)$ in $\boldsymbol{P}(T_x) \cong \boldsymbol{P}^{n-1}$. Let $\sigma_1,\dots,\sigma_{n-1}$ be a basis for $\boldsymbol{P}^{n-2}(\omega)$, and consider the intersection

$$\boldsymbol{P}^{n-2}(x,\sigma_1) \cap \cdots \cap \boldsymbol{P}^{n-2}(x,\sigma_{n-1}) \tag{3.17}$$

in $\boldsymbol{P}^{d-n-1}(x)$. The crucial observation is that *if* (3.12) *is to be true, then this intersection must be the point $Z \in E_x$ which corresponds to ω under the projectivity $D_x \barwedge E_x$*. We shall prove – somewhat indirectly – that this is the case.

Now suppose that ω is *any* point in $\boldsymbol{P}(T_x^*)$. If $\sigma_1,\dots,\sigma_{n-1}$ are a basis for $\boldsymbol{P}^{n-2}(\omega)$, we shall call a point Z lying on the intersection (3.17) a *knotpoint*. For example in case $\omega = \omega^i$ is a web normal, then for any vector σ which is tangent to the i[th] web hypersurface

$$\mathrm{d}Z_i/\mathrm{d}\sigma = Z'_i \mathrm{d}u_i/\mathrm{d}\sigma = 0\,,$$

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[23]) If $\mathrm{d}u_s/\mathrm{d}\sigma = 0$ then we should take $p_s = 1$ in (3.16); the justification for this will emerge below. The clearest picture of the infinitely near $\boldsymbol{P}^{n-2}(x,\sigma)$ is obtained by using moving frames – cf. the discussion at the end of § III B i, especially equations (3.57) and (3.58).

and consequently Z_i is a knotpoint. In general, suppose we can prove that:

(3.18) *The set of all knotpoints – i.e., points lying on* $(n-1)$*-fold intersections* (3.17) *– is a rational normal curve in* $\boldsymbol{P}^{d-n-1}(x)$.

Then we may complete the proof of (3.12) as follows: Since it contains the $Z_i(x)$, by uniqueness the rational normal curve of all knotpoints must be E_x. Moreover, since we have just proved the knotpoint corresponding to the web normal ω^i is Z_i, it follows that the projectivity

$$D_x \barwedge E_x$$

takes the normal curve D_x bijectively onto the set of knotpoints. A general tangent direction $\sigma \in \boldsymbol{P}(T_x)$ defines a hyperplane in $\boldsymbol{P}(T_x^*)$ meeting D_x in $(n-1)$ points $\omega_1(\sigma), \ldots, \omega_{n-1}(\sigma)$, and what we have just said proves that $\boldsymbol{P}^{n-2}(x,\sigma)$ is the $(n-2)$-fold secant plane spanned by the knotpoints $Z(\omega_1(\sigma)), \ldots, Z(\omega_{n-1}(\sigma))$.

So all that remains is to prove (3.18). We set

$$\begin{cases} \mathrm{d}u_s/\mathrm{d}\sigma = \sum\limits_{\alpha} u_{s\alpha}\sigma^\alpha \\ \varrho_s = -1/p_s \end{cases}$$

and rewrite (3.16) in the form

$$\sum_{\alpha} u_{s\alpha}\sigma^\alpha + \sum_{\mu} c^\mu A_{\mu s}\varrho_s = 0 \qquad s = n+1, \ldots, d. \tag{3.19}$$

We consider (3.19) as a set of $(d-n)$ homogeneous linear equations in the $2n-1$ unknowns $(\sigma^1, \ldots, \sigma^n; c^1, \ldots, c^{n-1})$. The condition that

$$Z = \sum_{s} p_s Z_s$$

be a knotpoint is that (3.19) should have $(n-1)$ linearly independent solutions. If we consider the coefficient matrix

$$\begin{pmatrix} u_{n+1,1} \cdots u_{n+1,n} & \overbrace{A_{1,n+1}\varrho_{n+1} \cdots A_{n-1,n+1}\varrho_{n+1}} \\ \vdots & \vdots \\ u_{d,1} \quad \cdots u_{d,n} & A_{1,d}\varrho_d \quad \cdots A_{n-1,d}\varrho_d \end{pmatrix}$$

as a linear map from $\boldsymbol{R}^{2n-1}$ to $\boldsymbol{R}^{d-n}$, the condition is that the image have dimension $\leqq n+1$. Since the first n columns in this matrix are linearly independent, this in turn is equivalent to saying that the last $n-1$ columns – i.e., those with the bracket over them – be linear combinations of the first n columns. Equivalently, any $(n+1) \times (n+1)$ minor

$$\begin{vmatrix} u_{i_1,1} & \cdots u_{i_1,n} & A_{\mu,i_1}\varrho_{i_1} \\ \vdots & & \\ u_{i_{n+1},1} & \cdots u_{i_{n+1},n} & A_{\mu,i_{n+1}}\varrho_{i_{n+1}} \end{vmatrix} = 0. \tag{3.20}$$

The equations (3.20) are linear in the ρ_s, and so may be given parametrically by $\varrho_s = \varrho_s(t)$ where $t = (t_1,\dots,t_k)$ are linear parameters. It is easy to see that since any $n \times n$ minor from the Jacobian matrix $\partial(u_{n+1},\dots,u_d)/\partial(x_1,\dots,x_n)$ is non-zero, the number of linear parameters is *at most* one. On the other hand, it must be *at least* one since we have already found the d knotpoints Z_i. It follows that

$$\varrho_s = -(\alpha_s t + \beta_s),$$

and consequently

$$Z(t) = 1/(\alpha_s t + \beta_s) Z_s$$

gives a parametric representation of the knotpoints. It is now clear that these constitute a rational normal curve. Q.E.D. for (3.12).

The proof of (3.18) is similar to the argument in pages 266–272 in Blaschke-Bol [1], from which the word knotpoint was taken. We observe that the projectivity

$$D_x \barwedge E_x$$

is characterized by: *A point* $\omega \in D_x \subset \boldsymbol{P}(T_x^*)$ *corresponds to* $Z \in E_x \subset \boldsymbol{P}^{d-n-1}(x)$ *if and only if*

$$\mathrm{d}Z/\mathrm{d}\sigma \in \boldsymbol{P}^{d-n-1}(x)$$

for all σ *satisfying* $\langle \omega, \sigma \rangle = 0$.

ii. *Strategy of the proof.*

Definition. *A path geometry is the system of curves defined by* 2^{nd} *order differential equations*

$$(3.21)\qquad \begin{aligned} &\mathrm{d}x_\beta/\mathrm{d}t(\mathrm{d}^2x_\alpha/\mathrm{d}t^2 + \textstyle\sum \Gamma_\alpha^{\lambda\mu}(\mathrm{d}x_\lambda(t)/\mathrm{d}t)(\mathrm{d}x_\mu/\mathrm{d}t)) \\ &\quad = \mathrm{d}x_\alpha/\mathrm{d}t(\mathrm{d}^2x_\beta/\mathrm{d}t^2 + \textstyle\sum \Gamma_\beta^{\lambda\mu}(\mathrm{d}x_\lambda(t)/\mathrm{d}t)(\mathrm{d}x_\mu(t)/\mathrm{d}t)). \end{aligned}$$

Through each point x_0 and in each tangent direction $\sigma \in \boldsymbol{P}(T_{x_0})$ there is a unique solution curve $x(t)$ of the system (3.21) having the initial data $x(0) = x_0$ and $x'(0) = \sigma$ in $\boldsymbol{P}(T_{x_0})$ – these are the *paths*. It is important to note that there is no distinguished parameter, such as arc length, for the paths. In fact, the form of the equations (3.21) is invariant under arbitrary changes of coordinates

$$(3.22)\qquad y_\alpha = y_\alpha(x_1,\dots,x_n)$$

and changes of parameter

$$(3.23)\qquad s = s(t).$$

This is not true of the corresponding system of homogeneous equations.

Definition. *The path geometry is flat in case there is a change of coordinates* (3.22) *and change of parameter* (3.23) *transforming* (3.21) *into the differential equations*

$$d^2 y_\alpha / ds^2 = 0$$

characteristic of straight lines in Euclidean space.

In section IV we shall discuss how a path geometry leads to an intrinsically defined *projective connection* whose associated projective curvature tensor is zero if and only if the path geometry is flat. The situation is somewhat analogous to the manner in which a Riemannian metric leads to an intrinsic Riemannian connection whose curvature is zero exactly when the original structure is equivalent to the standard Euclidean one. The difference is that the flat model space is $\mathbf{P}^n$ with the projective group operating in the first case and $\mathbf{R}^n$ with the Euclidean group in the second.

Now if our linearization theorem is true then, according to (2.33) and the analogous property (3.12) for general webs of maximal rank, the straight line through x_0 and in the direction σ is characterized as

$$\{x : \mathbf{P}^{n-2}(x_0, \sigma) \subset \mathbf{P}^{d-n-1}(x)\}. \tag{3.24}$$

Equivalently, as we move along $x(t)$ the secant plane $\mathbf{P}^{n-2}(x(t), x'(t))$ should be the fixed $\mathbf{P}^{n-2}(x_0, \sigma)$ where $x(0) = x_0, x'(0) = \sigma$, i.e., $x(t)$ is a solution curve of the differential equation

$$d/dt(\mathbf{P}^{n-2}(x(t), x'(t))) \subset \mathbf{P}^{n-2}(x(t), x'(t)). \tag{3.25}$$

Now (3.25) may be written out as a system of 2nd order equations which, for the same reasons as those discussed at the end of § II B ii, is overdetermined. The main step in our proof of the linearization theorem will be to show that the compatibility conditions in (3.25) are automatically satisfied, i.e., that:

(3.26) *The equations* (3.25) *define a path geometry* (3.21).

Once (3.26) has been established, the proof of the main theorem is reduced to showing that the associated projective connection is flat. To explain how this is done, we need the

Definition. *A totally geodesic submanifold S for a path geometry given by* (3.21) *is characterized by the property that a path $x(t)$ lies entirely in S in case, for some t_0, $x(t_0) \in S$ and $x'(t_0)$ is tangent to S.*

It is a classical theorem that if $n \geqq 3$, and if for every point x and normal $\omega \in \boldsymbol{P}(T_x^*)$ there is a totally geodesic hypersurface passing through x with normal ω, then the path geometry is flat. We are able to refine this to

(3.27) *Given a path geometry* (3.21) *and field $D_x \subset \boldsymbol{P}(T_x^*)$ of rational normal curves, if for every x and $\omega \in \boldsymbol{P}(T_x^*)$ there is a totally geodesic hypersurface passing through x and with normal ω, then the path geometry is flat.*

Assuming (3.27) the proof of the main theorem is completed as follows: According to (3.5) and (3.12) a point $\omega \in D_x$ corresponds to $Z \in \boldsymbol{P}^{d-n-1}(x)$ characterized by (c.f. the end of § III A (i)).

$$dZ/d\sigma \in \boldsymbol{P}^{d-n-1}(x) \Leftrightarrow \langle \omega, \sigma \rangle = 0\,. \tag{3.28}$$

The equations $\langle \omega, \sigma \rangle = 0$ define a $\boldsymbol{P}^{n-2}(\omega)$ in $\boldsymbol{P}(T_x)$, and the paths emanating from x and with tangent direction $\sigma \in \boldsymbol{P}^{n-2}(\omega)$ fill out a hypersurface passing through x and with normal ω. Points x' in this hypersurface satisfy (cf. (2.35))

$$Z \in \boldsymbol{P}^{d-n-1}(x')\,. \tag{3.29}$$

On the other hand, the conditions (3.29) can define at most a hypersurface in U since not all $\boldsymbol{P}^{d-n-1}(x)$'s pass through any one point of $\boldsymbol{P}^{r-1}$. Consequently, (3.29) defines a totally geodesic hypersurface for our path geometry, and in this way we have embedded our original web in a larger family of ∞^2 hypersurfaces whose normals fill out the rational normal curves $D_x \subset \boldsymbol{P}(T_x^*)$. Our result then follows from (3.27).

One way of summarizing the proof is this: *The maximal rank web defines a field of rational normal curves D_x in the projectivized cotangent spaces $\boldsymbol{P}(T_x^*)$. We want to construct Castelnuovo's ruled surface S, and the D_x constitute the disjoint union of the hyperplane sections of S. So, in order to find the identifications necessary to obtain S we go to the Poincaré space $\boldsymbol{P}^{r-1}$ and field of rational normal curves $E_x \subset \boldsymbol{P}^{d-n-1}(x)$ in projective correspondence with D_x. Our theorem is true exactly when the E_x lie on a 2-dimensional surface S in $\boldsymbol{P}^{r-1}$, and since there are ∞^n curves E_x this will be the case when two infinitely nearby curves E_x and $E_{x+\varepsilon\sigma}$ meet in $(n-1)$ points. Thinking of σ as a hyperplane in $\boldsymbol{P}(T_x^*)$, these $(n-1)$-points are the images of the hyperplane section $\sigma \cap D_x$ of D_x under the projectivity $D_x \barwedge E_x$. In fact, the lines in the $\boldsymbol{P}^n$ in which Castelnuovo's surface S is to lie are characterized by letting $x \in U$ vary subject to the condition that the $\boldsymbol{P}^{d-n-1}(x)$ have a fixed $\boldsymbol{P}^{n-2}(x_0, \sigma)$ as axis. These lines are described by an — a priori overdetermined — system of O.D.E.'s; the necessary compatibility conditions are a reflection of E_x and $E_{x+\varepsilon\sigma}$ meeting in $(n-1)$-points.*

We conclude this section by discussing informally what is involved in the proof of the central result (3.26). A first remark is that the group G of projective transformations leaving fixed a rational normal curve $D \subset \boldsymbol{P}^{n-1}$ induces the full transitive group of projectivities on the curve $D \cong \boldsymbol{P}^1$. Since any two rational normal curves are projectively equivalent, the structure of a field $D_x \subset \boldsymbol{P}(T_x^*)$ of rational curves given by (3.4) is a *G-structure*. When $n = 3$, which was the case considered by Blaschke-Bol [1], D_x is a conic in the plane $\boldsymbol{P}(T_x^*) \cong \boldsymbol{P}^2$ and the G-structure is a *conformal Riemannian structure*. In particular, there exist torsion-free G-connections in the tangent bundle (Weyl connections) to which one may apply existing formalism in differential geometry. It was in this setting that Bol gave his proof of the $n = 3$ case of the theorem.

Now, and this was the principal technical difficulty we encountered, when $n \geqq 4$ the group G is relatively small and there need not exist torsion-free connections leaving fixed a general field of rational normal curves[24]). However, we have two additional pieces of information:

(3.30) the field of rational normal curves has a large number d of *completely integrable cross-sections* $\omega^i(x)$; and

(3.31) the defining functions for the foliations given by $\omega^i(x) = 0$ have the *harmonic property* (3.9).

So our G-structure has rather special properties which will have to come into play if we are to be able to prove (3.26). In fact, one might hope that the properties (3.30) and (3.31) might imply the existence of a torsion-free connection in the tangent bundle which leaves invariant the curves $D_x \subset \boldsymbol{P}(T_x^*)$, and for a while we thought this would be the case. However, this hope turned out to be naive, and in § III B iv we have given the structure of the "best" connection possible for the problem; it is a torsion-free connection leaving invariant the D_x only when $n = 3$.

So in this paper we show by direct computation that the desired path geometry can be introduced. This is done in § III B ii, and constitutes the essential step in the proof. Then a continuation of this computation leads to the best connection in the following section.

B. Introduction of the path geometry

i. *Structure equations for maximal rank webs.* We begin by considering a general d-web in n-space given by a Pfaffian system

$$\omega^i(x) = 0 \qquad i = 1, \ldots, d$$

[24]) This reflects the overdetermined character of the equations (3.25).

satisfying the complete integrability condition

$$d\omega^i = \pi^i \wedge \omega^i \tag{3.33}$$

(no summation of the index i). The form ω^i is non-vanishing and is determined up to multiplication by a non-zero function. In particular we will have

$$\omega^i = e^{\varrho_i} du_i$$

where $u_i(x)$ is a function whose level sets define the i^{th} foliation in the web.

Now suppose the web has maximal rank r with basis

$$\sum_i f_i^\lambda(u_i(x))\,du_i(x) \equiv 0 \qquad \lambda = 1, \ldots, r$$

for its abelian equations. The point in the Poincaré space $\boldsymbol{P}^{r-1}$ corresponding to $x \in U$ is given by

$$Z_i(x) = [f_i^1(u_i(x)), \ldots, f_i^r(u_i(x))] .$$

In this section we shall slightly abuse notation and denote by

$$Z_i(x) = e^{-\varrho_i}(f_i^1(u_i(x)), \ldots, f_i^r(u_i(x)))$$

the designated vector lying over the point in $\boldsymbol{P}^{r-1}$. When this is done the condition that $Z_i(x) \in \boldsymbol{P}^{r-1}$ should depend only on the i^{th} web hypersurface through x is expressed by

$$d(Z_i \omega^i) = 0$$

(no summation again). This inplies

$$(dZ_i + \pi^i Z_i) \wedge \omega^i = 0 ,$$

and so we may define the point Z_i' on the tangent line to the arc C_i traced out at Z_i by

$$dZ_i + \pi^i Z_i = Z_i' \omega^i . \tag{3.34}$$

Now we suppose that $d > 2n$ and $n \geqq 3$. Then the web normals $\omega^i(x)$ lie on a rational normal curve D_x in the projectivized cotangent space $\boldsymbol{P}(T_x^*)$. This in turn defines a G-structure, to which one may seek to apply standard techniques in differential geometry – in particular the method of moving frames – to find the invariants. Carrying this out to arrive at the path geometry will be the backbone of the proof of our main theorem.

We recall that for any closed subgroup $G \subset \mathrm{GL}_n$ a G-structure on a manifold M is given relative to an open covering $U, V, \ldots$ of M by bases $\omega_U^\alpha, \omega_V^\beta, \ldots$ for the 1-forms in the respective open sets such that in intersections $U \cap V$

$$\omega_U^\alpha = \sum_\beta \omega_V^\beta (g_{UV})_\beta^\alpha$$

where $g_{UV} = \{(g_{UV})^\alpha_\beta\}$ is a G-valued matrix. In our case we may use the fact that any two rational normal curves in $\boldsymbol{P}^{n-1}$ are projectively equivalent and that such a curve has a transitive group of automorphisms induced by projective transformations of the ambient space to deduce the existence of a G-structure where G is the group leaving fixed the standard rational normal curve

$$t \to [t, t^2, \ldots, t^n]\ ^{25)}.$$

Definition. *A moving frame is given by a coframe $\{\phi^\alpha\}$, or basis for the cotangent bundle, such that the rational normal curve C_x is given parametrically by*

$$e^{\varrho(x)}\Phi(x,t) = \sum_{\alpha=1}^{n} t^\alpha \phi^\alpha(x). \tag{3.35}$$

In other words a moving frame gives a homogeneous coordinate system for $\boldsymbol{P}(T_x^*) \cong \boldsymbol{P}^{n-1}$ in which C_x is the standard rational normal curve. To determine the group G, it is more convenient to set $t = t_1/t_0$ and use the homogeneous coordinate representation

$$e^\varrho \Phi = \sum_{\alpha=1}^{n} t_0^{\alpha-1} t_1^{n-\alpha} \phi^\alpha \tag{3.36}$$

for the standard rational normal curve. If we make a change of homogeneous variables

$$\begin{cases} t_0 = a_{00} t_0^* + a_{01} t_1^* \\ t_1 = a_{10} t_0^* + a_{11} t_1^* \end{cases}$$

and substitute in (3.36), then

$$e^\varrho \Phi = \sum_\alpha t_0^{*\alpha-1} t_1^{*n-\alpha} \phi^{*\alpha}$$

where

$$\phi^{*\alpha} = \sum_\beta \phi^\beta g^\alpha_\beta$$

and $g(x) = (g^\alpha_\beta(x))$ describes a general element in the subgroup $G \subset \mathrm{GL}_n$. We note that G has 4 parameters rather than the usual 3 when GL_n is considered as acting on $\boldsymbol{P}^{n-1}$. This is because we are in effect considering the *cone* in T_x^* lying over the rational normal curve C_x in $\boldsymbol{P}(T_x^*)$.

Now the equations of G are somewhat messy, but those for the Lie algebra of $\mathfrak{g} \subset \mathfrak{gl}(n)$ are controllable. To derive them we set

$$\begin{aligned} t_0 &= (1 + E_{00}) t_0^* + E_{01} t_1^* \\ t_1 &= E_{10} t_0^* + (1 + E_{11}) t_1^*. \end{aligned}$$

[25]) We shall use this parametric representation rather than the usual $t \to [1, t, \ldots, t^{n-1}]$ so as to allow all indices to run from 1 to n.

Then, modulo higher order terms in the $E_{\mu\nu}$,

$$\begin{aligned} t_0^{\alpha-1} t_1^{n-\alpha} = {} & (1 + (\alpha - 1)E_{00} + (n - \alpha)E_{11}) t_0^{*\alpha-1} t_1^{*n-\alpha} \\ & + E_{10}(n-2) t_0^{*\alpha} t_1^{*n-\alpha-1} + E_{01}(\alpha - 1) t_0^{*\alpha-2} t_1^{*n-\alpha+1} . \end{aligned}$$

It follows that $\mathfrak{g}$ is generated by the transformations

$$\begin{cases} \phi^{\alpha *} = (\alpha - 1)\phi^{\alpha} \\ \phi^{\alpha *} = (n - \alpha)\phi^{\alpha} \\ \phi^{\alpha *} = (n - \alpha + 1)\phi^{\alpha - 1} \\ \phi^{\alpha *} = \alpha\phi^{\alpha + 1} \end{cases}$$

corresponding to the matrices

$$(3.37) \qquad \begin{aligned} h_{11} &= \begin{pmatrix} n-1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & 0 \end{pmatrix} & h_{22} &= \begin{pmatrix} 0 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & n-1 \end{pmatrix} \\ h_{12} &= \begin{pmatrix} 0 & n-1 & & 0 \\ & & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{pmatrix}, & h_{21} &= \begin{pmatrix} 0 & & & 0 \\ 1 & & & \\ & \ddots & & \\ 0 & & n-1 & 0 \end{pmatrix} \end{aligned}$$

The indices are meant to suggest that G is the image of GL_2 under the standard representation $\varrho : GL_2 \to GL_n$ corresponding to the $(n-1)^{st}$ symmetric power.

The property which is special about our G-structure in the presence of the large number of completely integrable cross-sections $\omega^i(x)$. If $\{\phi^\alpha\}$ is a moving frame, then after multiplying $\omega^i(x)$ by a scalar factor if necessary, we will have an equation

$$(3.38) \qquad \omega^i(x) = \sum_{\alpha} t_i^{\alpha}(x)\phi^{\alpha}(x) .$$

In order to conveniently express the complete integrability condition (3.33) we need to have a formula for $d\phi^\alpha$; i.e., a connection in the tangent bundle. Moreover, since equality of mixed partials will be involved it is desirable that this connection be *symmetric*[26]). Therefore, we recall that such a symmetric connection is given by a matrix $\{\phi_\beta^\alpha\}$ of 1-forms satisfying

$$(3.39) \qquad d\phi^{\alpha} = \sum_{\beta} \phi^{\beta} \wedge \phi_{\beta}^{\alpha} .$$

---

[26]) We are using symmetric rather than the more common *torsion free*, since the G-structure in question has its own well-defined torsion.

Once we have chosen a symmetric connection, any other one is given by

$$\phi_\beta^{*\alpha} = \phi_\beta^\alpha + \sum_\gamma h_{\alpha\beta\gamma}\phi^\gamma, \quad h_{\alpha\beta\gamma} = h_{\alpha\gamma\beta}. \tag{3.40}$$

Under a change of coframe

$$\phi^{\alpha *} = \sum_\beta \phi^\beta g_\beta^\alpha$$

the connection matrix transforms according to the usual equation

$$\phi^* = g^{-1}\phi g - g^{-1}\mathrm{d}g. \tag{3.41}$$

Definition. *The G-structure is torsion-free is there exists a symmetric connection* $\{\phi_\beta^\alpha\}$ *with values in the Lie algebra* $\mathfrak{g} \subset \mathfrak{gl}(n)$.

Note that this definition makes sense for any G-structure, and that by (3.41) the condition $\{\phi_\beta^\alpha\} \in \mathfrak{g}$ is independent of the choice of moving frame.

For example, suppose that we have a Riemannian structure corresponding to $G = 0(n)$. A moving coframe is characterized by

$$\mathrm{d}s^2 = \sum_\alpha (\phi^\alpha)^2.$$

Proving that this G-structure has no torsion is equivalent to first choosing an arbitrary symmetric connection $\{\phi_\beta^\alpha\}$ and then modifying it according to (3.40) so that

$$\phi_\beta^{*\alpha} + \phi_\alpha^{*\beta} = 0 \quad (\alpha < \beta). \tag{3.42}$$

The number of functions $h_{\alpha\beta\gamma}$ is $1/2(n^2(n+1))$, and the number of equations (3.42) is $1/2(n^2(n-1))$. Since $\{\phi_\beta^\alpha\}$ has n^3 entries and

$$n^3 - 1/2(n^2(n+1)) = 1/2(n^2(n-1)),$$

what is suggested is that a Riemannian structure is *uniquely* torsion free. This is well known to be the case (Levi-Civita connection).

We shall now derive the equations analogous to (3.42) that $\{\phi_\beta^\alpha\}$ have values in $\mathfrak{g} \subset \mathfrak{gl}(n)$ for the G-structure defined by a field of rational normal curves. Referring to (3.37) a $\mathfrak{g}$-valued 1-form may be written as

$$h_{11}\omega_{11} + h_{22}\omega_{22} + h_{12}\omega_{12} + h_{21}\omega_{21} \tag{3.43}$$

where the $h_{\mu\nu}$ are as in (3.37) and the $\omega_{\mu\nu}$ are 1-forms. The conditions that $\{\phi_\beta^\alpha\}$ have the form (3.43) are

$$\begin{cases} \phi_\beta^\alpha = 0 & |\alpha - \beta| \geqq 2 \\ \phi_{\alpha+1}^\alpha = (n-\alpha)\omega_{12} & \\ \phi_{\alpha-1}^\alpha = (\alpha-1)\omega_{21} & \\ \phi_\alpha^\alpha = (n-\alpha)\omega_{11} + (\alpha-1)\omega_{22}. & \end{cases} \tag{3.44}$$

If we eliminate the $\omega_{\mu\nu}$'s we obtain

$$(3.45)\qquad \begin{cases} \phi^\alpha_\beta = 0 \quad |\alpha - \beta| \geqq 2 \\ (n-\alpha-1)\phi^\alpha_{\alpha+1} = (n-\alpha)\phi^{\alpha+1}_{\alpha+2} \\ (\alpha+1)\phi^{\alpha+1}_\alpha = \alpha\phi^{\alpha+2}_{\alpha+1} \\ \phi^{\alpha+1}_{\alpha+1} + \phi^{\alpha-1}_{\alpha-1} = 2\phi^\alpha_\alpha . \end{cases}$$

The number of these equations is

$$2((n-1)(n-2)/2) + 2(n-2) + (n-2) = n^2 - 4\,,$$

which is correct since the codimension of $\mathfrak{g}$ in $\mathfrak{gl}(n)$ is $n^2 - 4$. Now $\{\phi^\alpha_\beta\}$ has n^3 entries and may be modified by the $1/2n^2(n+1)$ functions $h_{\alpha\beta\gamma}$. To be in the Lie algebra $\mathfrak{g}$ imposes $n^3 - 4n$ conditions, so that in general our G-structure will be torsion-free only when

$$n^3 - 1/2n^2(n+1) \geqq n^3 - 4n\,.$$

This is turn occurs only when $n = 3$. In this case the G-structure is equivalent to a *conformal pseudo-Riemannian structure*, and one has the Weyl geometry. Bol's proof of the theorem when $n = 3$ was based on this fortunate occurrence.

When $n \geqq 4$ the G-structure given by a field of rational normal curves will in general *not* be torsion-free. This caused us considerable difficulty, and meant that if our theorem were true then the restrictions on the G-structure imposed by the completely integrable cross-sections $\omega^i(x)$ would have to play a crucial role — cf. (3.30) and (3.31). This turns out to be so, and also the "harmonic" property (3.9) of the defining equations of the web comes essentially into the picture. Before taking up these matters, we want to give in a convenient manner the analogue of the equations (3.42) for our G-structure.

Recall that the algebro-geometric study of rational normal curves in $\boldsymbol{P}^n$ could, by successively projecting from points on the curve, be reduced to the study of conics in the plane. For somewhat similar reasons there will turn out to be an asymmetry involving the indices 1 and 2 in the structure equations of a field of rational normal curves. This will now appear when we write the connection matrix $\phi = \{\phi^\alpha_\beta\}$ in the form

$$\phi = \eta + \psi$$

where η has values in the Lie algebra $\mathfrak{g}$, and where the equation

$$\psi = 0$$

expresses the condition that ϕ have values in $\mathfrak{g}$. To do this we observe that, if this latter is the case, then the whole matrix ϕ is uniquely determined by the four entries $\phi_1^1, \phi_2^2, \phi_2^1, \phi_1^2$. Indeed, according to (3.44)

$$\begin{aligned} \phi_1^1 &= (n-1)\omega_{11} & \phi_2^1 &= (n-1)\omega_{12} \\ \phi_2^2 &= (n-2)\omega_{11} + \omega_{22} & \phi_1^2 &= \omega_{21} \end{aligned}$$

uniquely determines the $\omega_{\mu\nu}$ in terms of the ϕ_ν^μ. We then write

(3.46) $$\phi_\beta^\alpha = \eta_\beta^\alpha + \psi_\beta^\alpha$$

where

(3.47) $$\begin{aligned} \eta_\beta^\alpha = {} & \delta_\beta^{\alpha+1}((n-\alpha)/(n-1))\phi_2^1 + \delta_\beta^{\alpha-1}(\alpha-1)\phi_1^2 \\ & + \delta_\beta^\alpha[(\alpha-1)\phi_2^2 - (\alpha-2)\phi_1^1] \end{aligned}$$

has values in $\mathfrak{g}$. The forms ψ_β^α defined by (3.46) and (3.47) have the properties:

i. $\psi_1^1 = \psi_2^1 = \phi_1^2 = \psi_2^2 = 0$;
ii. the $n^2 - 4$ equations $\psi_\beta^\alpha = 0$ express the condition that ϕ have values in $\mathfrak{g}$; and
iii. the forms ψ_β^α are *horizontal* for any choice of symmetric connection ϕ_β^α.

Here, horizontal means that if we consider the $(n+4)$-dimensional principal bundle B_G of all coframes for the G-structure, then the ϕ^α are intrinsically defined on B_G and

$$\psi_\beta^\alpha \equiv 0 \text{ modulo} \{\phi^1, \ldots, \phi^n\}.$$

The special role played by the indices 1 and 2 will be particularly apparent in the next section.

We now come to expressing the complete integrability property (3.33) for the web normal $\omega^i(x)$ given by (3.38). By (3.39)

$$d\omega^i = \sum_\alpha dt_i^\alpha \wedge \phi^\alpha + \sum_{\alpha,\beta} t_i^\beta \phi^\alpha \wedge \phi_\alpha^\beta.$$

Comparison with (3.33) yields

$$\sum_\alpha (dt_i^\alpha - \pi^i t_i^\alpha - \sum_\beta t_i^\beta \phi_\alpha^\beta) \wedge \phi^\alpha = 0.$$

By the *Cartan lemma*

(3.48) $$dt_i^\alpha - \pi^i t_i^\alpha - \sum_\beta t_i^\beta \phi_\alpha^\beta = \sum_\beta t_{i\alpha\beta}\phi^\beta \quad \text{where} \quad t_{i\alpha\beta} = t_{i\beta\alpha}.$$

The symmetry $t_{i\alpha\beta} = t_{i\beta\alpha}$ *exactly expresses the complete integrability condition* (3.33).

In the notation of moving frames the abelian equation (3.1) becomes

(3.49) $$\sum_i Z_i t_i^\alpha = 0 \quad (\alpha = 1, \ldots, n).$$

Taking the exterior derivative we obtain

$$0 = \sum_i (dZ_i t_i^\alpha + Z_i dt_i^\alpha) = \sum_\beta \left(\sum_i Z_i' t_i^{\alpha+\beta} + Z_i t_{i\alpha\beta} \right) \phi^\beta$$

by (3.34), (3.48), and (3.49). This implies that

$$\sum_i Z_i' t_i^{\alpha+\beta} + Z_i t_{i\alpha\beta} = 0\,, \quad 1 \leqq \alpha,\beta \leqq n\,. \tag{3.50}$$

This beautiful relation is the intrinsic analogue of (3.7). One advantage of using moving frames, as we shall now see, is that this method renders most visible the quadrics containing the rational normal curves $D_x \subset \boldsymbol{P}(T_x^*)$.

The condition that a general quadric $\sum_{\alpha,\beta} k^{\alpha\beta}\zeta^\alpha\zeta^\beta = 0$ in $\boldsymbol{P}^{n-1}$ should pass through the standard rational normal curve $\zeta^\alpha = t^\alpha$ is clearly

$$\sum_{\alpha+\beta=\lambda} k^{\alpha\beta} = 0 \qquad \lambda = 2,\ldots,2n\,. \tag{3.51}$$

This gives $2n - 1$ independent equations, and so, as previously noted, there are

$$1/2(n(n+1)) - (2n-1) = (n-1)(n-2)/2$$

independent quadrics containing a rational normal curve. If we multiply (3.50) by $k^{\alpha\beta}$ satisfying (3.51) and sum over α and β we obtain

$$\sum_i \left(\sum_{\alpha,\beta} k^{\alpha\beta} t_{i\alpha\beta} \right) Z_i = 0\,.$$

Since the equations (3.49) are a basis, we will obtain the intrinsic analogue of (3.9)

$$\sum_{\alpha,\beta} k^{\alpha\beta} t_{i\alpha\beta} = \sum_\gamma m_\gamma t_i^\gamma \tag{3.52}$$

whenever $k^{\alpha\beta}$ satisfies (3.51). As mentioned before, these are sort of inhomogeneous Laplace equations, and the simplest situation would be if the right hand side of (3.52) were zero.

Definition. *We shall call the symmetric connection $\{\phi_\beta^\alpha\}$ harmonic in case*

$$\begin{cases} \sum_{\alpha,\beta} k^{\alpha\beta} t_{i\alpha\beta} = 0 & \textit{whenever} \\ \sum_{\alpha+\beta=\lambda} k^{\alpha\beta} = 0\,. & \end{cases} \tag{3.53}$$

Lemma. *The connection is harmonic if, and only if,*

$$t_{i\alpha\beta} = t_{i\alpha-1,\beta+1} = \cdots \tag{3.54}$$

depends only on the sum $\alpha + \beta$. *Harmonic connections exist, and once* ϕ^α_β *gives a harmonic connection then any other is of the form* (3.40) *where*

$$h_{\alpha\beta\gamma} = h_{\alpha,\beta+1,\gamma-1} \tag{3.55}$$

depends only on the sum $\beta + \gamma$ *and on* α.

Before proving the lemma we remark that we may now define

$$t_{i\alpha\beta} = t_{i,\alpha+\beta}$$

for any α,β with $2 \leqq \alpha + \beta \leqq 2n$. When this is done, equations (3.49) and (3.50) assume the particularly symmetric form

$$\begin{cases} \sum_i Z_i t_i^\alpha = 0 & \alpha = 1,\dots,n \\ \sum_i Z_i' t_i^\varrho + Z_i t_{i,\varrho} = 0 & \varrho = 2,\dots,2n\,. \end{cases} \tag{3.56}$$

We note that there are here just the correct number $n + (2n - 1) = 3n - 1$ of these equations.

Proof of lemma (3.54): We let

$$\phi^{*\alpha}_\beta = \phi^\alpha_\beta + \sum_\gamma h_{\alpha\beta\gamma}\phi^\gamma$$

where

$$h_{\alpha\beta\gamma} = h_{\alpha\gamma\beta}\,.$$

Referring to (3.48)

$$t^*_{i\alpha\beta} = t_{i\alpha\beta} - \sum_\gamma h_{\gamma\alpha\beta} t_i^\gamma\,.$$

For each quadric $k^{\alpha\beta}$ satisfying

$$\sum_{\alpha+\beta=\lambda} k^{\alpha\beta} = 0\,, \quad 2 \leqq \lambda \leqq 2n\,,$$

we have for all i

$$\sum_{\alpha,\beta} k^{\alpha\beta} t_{i\alpha\beta} = \sum_\gamma m_\gamma(k) t_i^\gamma$$

where $m_\gamma(k)$ depends on the quadric. It is required to determine $h_{\gamma\alpha\beta}$ so that

$$\sum_{\alpha,\beta} k^{\alpha\beta} t^*_{i\alpha\beta} = 0\,.$$

This is equivalent to

$$\sum_{\alpha,\beta} k^{\alpha\beta} h_{\gamma\alpha\beta} = m_\gamma(k)\,.$$

Fix λ with $4 \leqq \lambda \leqq 2n - 2$. We suppose first that $\lambda = 2\mu$ is even, and let $\mu + \eta = \min(\lambda - 1, n)$. We consider quadrics $k^{\alpha\beta}$ whose only non-zero entries are when $\alpha + \beta = \lambda$ and which satisfy $\sum k^{\alpha\beta} = 0$. We represent such a quadric by the quadratic form

$$\sum_{\alpha,\beta} k^{\alpha\beta} \xi^\alpha \xi^\beta = 0 .$$

A basis for these is

$$\begin{aligned}
&\xi^{\mu+1}\xi^{\mu-1} - (\xi^\mu)^2 = 0 \\
&\xi^{\mu+2}\xi^{\mu-2} - \xi^{\mu+1}\xi^{\mu-1} = 0 \\
&\quad\vdots \\
&\xi^{\mu+\eta}\xi^{\mu-\eta} - \xi^{\mu+\eta-1}\xi^{\mu-\eta+1} = 0 .
\end{aligned}$$

We label the corresponding quadrics as $k_1, \ldots, k_\eta$. The equations we must solve are

$$\begin{aligned}
h_{\gamma,\mu+1,\mu-1} &= h_{\gamma,\mu,\mu} + m_\gamma(k_1) \\
h_{\gamma,\mu+2,\mu-2} &= h_{\gamma,\mu+1,\mu-1} + m_\gamma(k_2) \\
&\vdots \\
h_{\gamma,\mu+\eta,\mu-\eta} &= h_{\gamma,\mu+\eta-1,\mu-\eta+1} + m_\gamma(k_\eta) .
\end{aligned}$$

Setting $h_{\gamma,\mu,\mu} = 0$ for instance, the recursive solution

$$h_{\gamma,\mu+\nu,\mu-\nu} = m_\gamma(k_1) + \cdots + m_\gamma(k_\nu)$$

established the lemma for λ even. The case λ odd is similar. Q.E.D.

As another illustration of the use of moving frames, and also as preparation for the computation in the next section, let us re-examine the proof of (3.12) – especially the equation (3.13). A point $Z \in \boldsymbol{P}^{d-n-1}(x)$ is written

$$Z = \sum_i p^i Z_i .$$

For a non-zero tangent vector σ we set $\sigma^\alpha = \langle \phi^\alpha, \sigma \rangle$, and use the notation "$\equiv$" to mean "congruent modulo $\boldsymbol{P}^{d-n-1}(x)$". By definition

$$\boldsymbol{P}^{n-2}(x,\sigma) = \{Z : \mathrm{d}Z/\mathrm{d}\sigma \equiv 0\} .$$

Using (3.34) and (3.38)

$$\begin{aligned}
\mathrm{d}Z/\mathrm{d}\sigma &\equiv \sum_i p^i \,\mathrm{d}Z_i/\mathrm{d}\sigma \\
&\equiv \sum_i p^i \langle \omega^i, \sigma \rangle Z'_i \\
&\equiv \sum_{i,\alpha} p^i t_i^\alpha \sigma^\alpha Z'_i .
\end{aligned}$$

According to (3.50),

$$Z \in \boldsymbol{P}^{n-2}(x,\sigma) \Leftrightarrow p^i = \left(\sum_{\varrho=2}^{2n} k^\varrho t_i^\varrho\right) \Bigg/ \left(\sum_{\alpha=1}^{n} t_i^\alpha \sigma^\alpha\right). \tag{3.57}$$

The number of parameters k^ϱ is $2n - 1$. However, if p^i is given by (3.57) then p^i is only determined modulo the abelian equations (3.49). In other words the k^ϱ of the form

$$k^\varrho = \sum_{\alpha+\beta=\varrho} l^\alpha \sigma^\beta \tag{3.58}$$

give trivial solutions to the left hand side of (3.57), and so the number of essential parameters k^ϱ is

$$2n - 1 - n = n - 1,$$

which by homogeneity gives a $\boldsymbol{P}^{n-2}$. The equations (3.58) illustrate quite clearly how $\boldsymbol{P}^{n-2}(x,\sigma)$ varies with σ, and one may use them to give an alternate proof of (3.12).

ii. *The main computation*. We want to prove (3.26). Changing the notation for the path parameter from t to s, this amounts to showing that the condition

$$\mathrm{d}/\mathrm{d}s(\boldsymbol{P}^{n-2}(x(s),x'(s))) \subset \boldsymbol{P}^{n-2}(x(s),x'(s)) \tag{3.59}$$

is expressed by an equation of the type (3.21). Along the path $x(s)$ we set

$$\begin{cases} \phi^\alpha = y^\alpha \mathrm{d}s \\ \mathrm{d}y^\alpha + y^\beta \phi_\beta^\alpha = z^\alpha \mathrm{d}s \\ x = x(s) \text{ and } \quad y = x'(s). \end{cases}$$

Here, as throughout this section, repeated Greek indices are summed and we shall use the index ranges

$$\begin{cases} 1 \leqq \alpha, \beta, \gamma, \delta \leqq n \\ 2 \leqq \varrho, \sigma, \tau \leqq 2n. \end{cases}$$

With these notations (3.21) is a system of second order O.D.E.'s

$$y^\beta(z^\alpha + \Gamma^\alpha_{\gamma\delta} y^\gamma y^\delta) = y^\alpha(z^\beta + \Gamma^\beta_{\gamma\delta} y^\gamma y^\delta) \tag{3.60}$$

for all α and β and where $\Gamma^\alpha_{\gamma\delta} = \Gamma^\alpha_{\delta\gamma}$.

Now, according to (3.57), points in $\boldsymbol{P}^{n-2}(x,y)$ are given parametrically by

$$Z = \sum_i (k^\varrho t_i^\varrho / y^\alpha t_i^\alpha) Z_i. \tag{3.61}$$

For such a Z (3.59) is equivalent to

$$\mathrm{d}Z/\mathrm{d}s = \sum_i (K^\varrho t_i^\varrho / y^\alpha t_i^\alpha) Z_i. \tag{3.62}$$

The left hand side of (3.62) will involve y^α and z^α, but it is not clear that the compatibilities conditions necessary for it to have the form (3.60) will be satisfied. The computation we are about to give will show that these compatibilities are a consequence of the integrability conditions (3.48) and harmonic property in the form (3.54). It will show, moreover, that along the path we may choose $k^\varrho(x(s))$ with given initial value and with $K^\varrho = 0$ in (3.62).

The first step is to compute $\mathrm{d}Z$. For this we use the notations

$$(3.63)\qquad \begin{cases} \Delta = y^\alpha t_i^\alpha \\ D k^\varrho t_i^\varrho = \mathrm{d}k^\varrho t_i^\varrho + k^{\alpha+\beta}(t_i^{\gamma+\beta}\phi_\gamma^\alpha + t_i^{\gamma+\alpha}\phi_\gamma^\beta). \end{cases}$$

Recall that repeated Greek indices are summed. In the second equation Dk^ϱ is defined to be the coefficient of t_i^ϱ on the right hand side. With these conventions, the formula is

$$(3.64)\qquad \begin{aligned} \mathrm{d}Z/\mathrm{d}s = \sum_i (&-k^\varrho t_{i,\varrho} + 1/\Delta((Dk^\varrho/\mathrm{d}s)t_i^\varrho) \\ &+ 1/\Delta\{k^{\alpha+\beta}(t_i^\beta t_{i\alpha\gamma}y^\gamma + t_i^\alpha t_{i\beta\gamma}y^\gamma \\ &- 1/\Delta^2 k^\varrho t_i^\varrho(z^\alpha t_i^\alpha + t_{i\alpha\beta}y^\alpha y^\beta)\})Z_i\,. \end{aligned}$$

Proof. By (3.34)

$$\mathrm{d}Z_i = -\pi^i Z_i + Z_i' t_i^\gamma y^\gamma \mathrm{d}s$$

and so

$$\begin{aligned} \sum_i (k^\varrho t_i^\varrho / y^\alpha t_i^\alpha)\mathrm{d}Z_i &= -\sum_i (k^\varrho t_i^\varrho/\Delta)\pi^i Z_i + \sum_i (k^\varrho t_i^\varrho) Z_i' \mathrm{d}s \\ &= -\sum_i (k^\varrho t_i^\varrho/\Delta)\pi^i Z_i - \sum_i (k^\sigma t_{i,\sigma}) Z_i \mathrm{d}s \end{aligned}$$

by (3.56). Next, by the definition of z^α

$$\begin{aligned} \mathrm{d}(y^\alpha t_i^\alpha) &= \mathrm{d}y^\alpha t_i^\alpha + y^\alpha(\pi^i t_i^\alpha + t_i^\beta \phi_\alpha^\beta + t_{i\alpha\beta}y^\beta \mathrm{d}s) \\ &= z^\alpha t_i^\alpha + y^\alpha \pi^i t_i^\alpha + t_{i\alpha\beta}y^\beta \mathrm{d}s\,. \end{aligned}$$

Finally, setting $\varrho = \alpha + \beta$ and using (3.48)

$$\begin{aligned} \mathrm{d}(k^\varrho t_i^\varrho) &= \mathrm{d}k^\varrho t_i^\varrho + k^{\alpha+\beta}(\mathrm{d}t_i^\alpha t_i^\beta + t_i^\alpha \mathrm{d}t_i^\beta) \\ &= Dk^\varrho t_i^\varrho + 2k^\varrho t_i^\varrho \pi^i + k^{\alpha+\beta}(t_i^\beta t_{i\alpha\gamma}y^\gamma + t_i^\alpha t_{i\beta\gamma}y^\gamma)\mathrm{d}s\,. \end{aligned}$$

Putting everything together

$$\begin{aligned} \mathrm{d}Z/\mathrm{d}s &= \sum_i ((k^\varrho t_i^\varrho)/\Delta)\mathrm{d}Z_i - \sum_i ((k^\varrho t_i^\varrho)/\Delta^2)\mathrm{d}\Delta\, Z_i \\ &\quad + \sum_i (\mathrm{d}(k^\varrho t_i^\varrho)/\Delta) Z_i \\ &= (3.64) \times \mathrm{d}s \end{aligned}$$

since the four terms containing π^i cancel. Q.E.D.

Now the term containing Dk^ϱ has the desired form (3.62). In fact, along the paths $x(s)$ whose equation we shall derive we may determine $k^\varrho(x(s))$ with given initial value and satisfying $Dk^\varrho(x(s)) = 0$. So we may as well set $Dk^\varrho = 0$, and then (3.64) becomes

$$\text{(3.65)} \qquad \begin{aligned} \mathrm{d}Z/\mathrm{d}s = \sum_i (-k^\varrho t_{i,\varrho} &+ 1/\Delta\{k^{\alpha+\beta}(t_i^\beta t_{i\alpha\gamma} y^\gamma + t_i^\alpha t_{i\beta\gamma} y^\gamma)\} \\ &- 1/\Delta^2\{k^\varrho t_i^\varrho (z^\alpha t_i^\alpha + t_{i\alpha\beta} y^\alpha y^\beta)\}) Z_i . \end{aligned}$$

There are four terms involving the 2$^{\text{nd}}$ derivatives $t_{i\alpha\beta} = t_{i,\alpha+\beta}$ (harmonic property of the connection). *Of these, two have plus and two have minus signs.* It turns out that if our connection matrix $\{\phi_\beta^\alpha\}$ had values in the Lie algebra $\mathfrak{g}$, then these four terms cancel out and the paths are given by

$$z^\alpha = 0 ,$$

i.e., they are the geodesics for this connection. Since we don't know this good property of the connection we must set about simplifying the terms containing the $t_{i\alpha\beta}$, essentially by expressing them in powers of t_i.

As a preliminary we will prove the

(3.66) **Lemma.** *Suppose that, for some* σ_i, l, m,

$$t_{i\alpha\beta} = \sigma_i(\alpha + \beta - l) t_i^{\alpha+\beta-m} + T_{i\alpha\beta}$$

then $\mathrm{d}Z/\mathrm{d}s$ *is given by the same formula* (3.65) *with* $T_{i\alpha\beta}$ *replacing* $t_{i\alpha\beta}$.

Proof. Moving minus signs across we must prove that

$$\begin{aligned} &k^\varrho(\rho - l) t_i^{\varrho-m} \Delta^2 + k^\varrho t_i^\varrho (\alpha + \beta - l) t_i^{\alpha+\beta-m} y^\alpha y^\beta \\ &\quad = \Delta\{k^{\alpha+\beta}((\alpha + \gamma - l) t_i^{\alpha+\beta+\gamma-m} y^\alpha + (\beta + \gamma - l) t_i^{\alpha+\beta+\gamma-m} y^\gamma)\}. \end{aligned}$$

The left hand side is

$$\text{(3.67)} \qquad k^\varrho(\varrho + \alpha + \beta - 2l) t_i^{\varrho+\alpha+\beta-m} y^\alpha y^\beta .$$

The right hand side is

$$\begin{aligned} &\Delta\{k^{\alpha+\beta}((\alpha + \beta + 2\gamma - 2l) t_i^{\alpha+\beta+\gamma-m} y^\gamma)\} \\ &\quad = \Delta\{k^\varrho((\varrho + \alpha - 2l) t_i^{\varrho+\alpha-m} y^\alpha + \beta t_i^{\varrho+\beta-m} y^\beta) \\ &\quad = k^\varrho(\varrho + \alpha + \beta - 2l) t_i^{\varrho+\alpha+\beta-m} y^\alpha y^\beta \\ &\quad = (3.67) . \end{aligned}$$

Q.E.D.

We now turn to the problem of simplifying the $t_{i\alpha\beta}$. Recall the horizontal forms ψ_β^α defined by (3.46). We first will establish the formula:

$$\text{(3.68)} \qquad \begin{aligned} &t_i^\beta \phi_\alpha^\beta - (\alpha - 1) t_i^{\alpha+\beta-2} \phi_2^\beta + (\alpha - 2) t_i^{\alpha+\beta-1} \phi_1^\beta \\ &\quad = t_i^\beta \psi_\alpha^\beta - (\alpha - 1) t_i^{\alpha+\beta-2} \psi_2^\beta + (\alpha - 2) t_i^{\alpha+\beta-1} \psi_1^\beta . \end{aligned}$$

Proof. Using (3.47) we must show that the η^α_β terms drop out of the left hand side of (3.66). This is equivalent to the vanishing of the coefficients of ϕ^1_1, ϕ^1_2, ϕ^2_1, ϕ^2_2 under the substitution (3.47). The coefficient of ϕ^1_1 is

$$-(\alpha - 2)t_i^\alpha + (\alpha - 2)t_i^\alpha = 0 .$$

The coefficient of ϕ_{12} is

$$((n-\alpha)/(n-1))t_i^{\alpha+1} - (\alpha - 1)((n-2)(n-1))t_i^{\alpha+1} + (\alpha - 2)t_i^{\alpha+1} = 0 .$$

Similarly the coefficients of ϕ^2_1 and ϕ^2_2 are zero. Q.E.D.

The meaning of this relation is that for any choice of symmetric connection $\{\phi^\alpha_\beta\}$ the combinations on the left hand side of (3.68) are horizontal. Moreover, they are zero in case $\{\phi^\alpha_\beta\}$ has values in the Lie algebra $\mathfrak{g}$ [27]). The symmetry arising from the complete integrability of $\omega^i = t_i^\alpha \phi^\alpha$ is not present in (3.68), but does appear in the following alternate expression for the left hand side of (3.68):

$$\begin{aligned} t_i^\beta \phi^\beta_\alpha - (\alpha - 1)t_i^{\alpha+\beta-2}\phi^\beta_2 + (\alpha - 2)t_i^{\alpha+\beta-1}\phi^\beta_1 \\ = -t_{i\alpha\gamma}\phi^\gamma + (\alpha - 1)t_i^{\alpha-2}t_{i2\gamma}\phi^\gamma - (\alpha - 2)t_i^{\alpha-1}t_{i1\gamma}\phi^\gamma . \end{aligned} \tag{3.69}$$

Proof. Write down the equations (3.48) for 1, 2, and α to obtain

(1) $$dt_i - \pi^i t_i - t_i^\beta \phi^\beta_1 = t_{i1\gamma}\phi^\gamma$$

(2) $$2t_i dt_i - \pi^i t_i^2 - t_i^\beta \phi^\beta_2 = t_{i2\gamma}\phi^\gamma$$

(3) $$\alpha t_i^{\alpha-1} dt_i - \pi^i t_i^\alpha - t_i^\beta \phi^\beta_\alpha = t_{i\alpha\gamma}\phi^\gamma .$$

Then the linear combination

$$-((\alpha - 2)t_i^{\alpha-1})(1) + ((\alpha - 1)t_i^{\alpha-2})(2) - (3)$$

eliminates the coefficients of dt_i and π^i, and gives our desired formula. Q.E.D.

The left hand side of (3.69) is a polynomial in t_i of degree $\leqq \alpha + n - 1$. We set it equal to

$$\sum_{1 \leqq A \leqq \alpha+n-1} t_i^A a_{A\alpha\gamma}\phi^\gamma = A_{i\alpha\gamma}\phi^\gamma \tag{3.70}$$

and note that

$$A_{i1\gamma} = A_{i2\gamma} = 0 . \tag{3.71}$$

The symmetry

$$t_{i\alpha\gamma} = t_{i\gamma\alpha} \tag{3.72}$$

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[27]) The converse is also true, but we won't need this here.

and harmonic property

$$t_{i\alpha\gamma} = t_{i\alpha-1,\gamma+1} = \cdots \tag{3.73}$$

are by no means apparent from the left hand side of (3.69), and by (3.68) this imposes conditions on the ψ^α_β's. We proceed to exploit this, first by showing that

$$t_{i\alpha\gamma} = t_{i11}(-\alpha - \gamma + 3)t_i^{\alpha+\gamma-2} + t_{i12}(\alpha + \gamma - 2)t_i^{\alpha+\gamma-3} + T_{i\alpha\gamma} \tag{3.74}$$

where

$$\begin{aligned} T_{i\alpha\gamma} = & -(\alpha - 1)(\gamma - 1)t_i^{\alpha+\gamma-4}A_{i31} - A_{i\alpha\gamma} \\ & -(\alpha - 1)t_i^{\alpha-2}A_{i\gamma 2} + (\alpha - 2)t_i^{\alpha-1}A_{i\gamma 1} \text{ [28]}. \end{aligned} \tag{3.75}$$

Proof. By (3.69) and (3.70)

$$t_{i\alpha\gamma} - (\alpha - 1)t_i^{\alpha-2}t_{i2\gamma} + (\alpha - 2)t_i^{\alpha-1}t_{i1\gamma} + A_{i\alpha\gamma} = 0 \tag{3.76}$$

$$t_{i\gamma\alpha} - (\gamma - 1)t_i^{\gamma-2}t_{i2\alpha} + (\gamma - 2)t_i^{\gamma-1}t_{i1\alpha} + A_{i\gamma\alpha} = 0\,. \tag{3.77}$$

We put $\alpha = 1, 2$ in the second equation and use (3.72) to obtain

$$\begin{cases} t_{i1\gamma} - (\gamma - 1)t_i^{\gamma-2}t_{i21} + (\gamma - 2)t_i^{\gamma-1}t_{i11} + A_{i\gamma 1} = 0 \\ t_{i2\gamma} - (\gamma - 1)t_i^{\gamma-2}t_{i22} + (\gamma - 2)t_i^{\gamma-1}t_{i12} + A_{i\gamma 2} = 0\,. \end{cases} \tag{3.78}$$

By (3.73) and (3.69)

$$t_{i22} = t_{i13} = -t_i^2 t_{i11} + 2t_i t_{i12} - A_{i31}\,.$$

If we substitute this in the second equation in (3.78), then we will have expressed $t_{i1\gamma}$ and $t_{i2\gamma}$ in terms of t_{i11}, t_{i12}, $A_{i\gamma 1}$, $A_{i\gamma 2}$, and A_{i31}. Plugging these into the first equation in (3.76) gives (3.74) and (3.75). Q.E.D.

According to lemma (3.66), dZ/ds is given by (3.65) with $T_{i\alpha\beta}$ in (3.75) replacing $t_{i\alpha\beta}$. We note that by (3.72), (3.73), and (3.74)

$$\begin{aligned} T_{i\alpha\gamma} &= T_{i\gamma\alpha} \\ T_{i\alpha\gamma} &= T_{i\alpha-1,\gamma+1} = \cdots = T_{i\varrho}, \quad \varrho = \alpha + \gamma\,, \end{aligned}$$

and shall now use these relations to express $T_{i\alpha\gamma}$ entirely in terms of $A_{i3\,\beta}$'s. For $\alpha + \gamma \leqq n + 1$ the desired formula is

$$T_{i\varrho} = -\sum_{3 \leqq \sigma \leqq \varrho - 1} (\varrho - \sigma)t_i^{\varrho-\sigma-1}A_{i3,\sigma-2} \qquad 4 \leqq \varrho \leqq n + 1\,. \tag{3.79}$$

[28] Here we can make an interesting observation. If ϕ^α_β is a symmetric connection with values on the Lie algebra $\mathfrak{g}$, then by (3.46), (3.68), and the definition (3.70) $A_{i\alpha\gamma} = 0$. Consequently, $T_{i\alpha\gamma} = 0$ and $t_{i\alpha\gamma}$ is a linear combination of terms $\sigma_i(\alpha + \gamma - m)t_i^{\alpha+\gamma-l}$. According to lemma (3.66)

$$dZ/ds = -\sum_i ((k^\varrho t_i^\varrho)(t_i^\alpha z^\alpha)/\Delta^2)Z_i\,,$$

and so the paths are defined by $z^\alpha = 0$.

Proof. By (3.71)

$$T_{i2} = T_{i3} = 0\,, \quad T_{i4} = -A_{i31}\,.$$

For $\gamma \geqq 4$

$$\begin{aligned} T_{i1\gamma} &= -A_{i\gamma 1} \\ T_{i2,\gamma-1} &= -(\gamma-2)t_i^{\gamma-3}A_{i31} - A_{i\gamma-1,2} \\ T_{i3,\gamma-2} &= -2(\gamma-3)t_i^{\gamma-3}A_{i31} - A_{i3,\gamma-2} \\ &\quad -2t_iA_{i\gamma-2,2} + t_i^2A_{i\gamma-2,1}\,. \end{aligned} \tag{3.80}$$

These expressions are all equal. Thus equating the second and third

$$\begin{aligned} &-(\gamma-2)t_i^{\gamma-3}A_{i31} - A_{i\gamma-1,2} \\ &\quad = -2(\gamma-3)t_i^{\gamma-3}A_{i31} - A_{i3,\gamma-2} - 2t_iA_{i\gamma-2,2} \\ &\qquad + t_i^2\{(\gamma-4)t_i^{\gamma-5}A_{i31} + A_{i\gamma-3,2}\}\,, \end{aligned}$$

where the term in curly brackets came from equating the first two equations for $\gamma - 2$. The terms involving A_{i31} cancel and we obtain

$$-A_{i3,\gamma-2} + A_{i\gamma-1,2} - 2t_iA_{i\gamma-2,2} + t_i^2A_{i\gamma-3,2} = 0\,.$$

We multiply by $(\alpha - \gamma + 1)t_i^{\alpha-\gamma}$ and sum

$$\begin{aligned} \sum_{4\leqq\gamma\leqq\alpha}(\alpha-\gamma+1)t_i^{\alpha-\gamma}A_{i3,\gamma-2} &= \sum_{4\leqq\gamma\leqq\alpha}(\alpha-\gamma+1)t_i^{\alpha-\gamma}A_{i\gamma-1,2} \\ &\quad - \sum_{4\leqq\gamma\leqq\alpha}2(\alpha-\gamma+1)t_i^{\alpha-\gamma+1}A_{i\gamma-2,2} \\ &\quad + \sum_{4\leqq\gamma\leqq\alpha}(\alpha-\gamma+1)t_i^{\alpha-\gamma+2}A_{i\gamma-3,2}\,. \end{aligned}$$

Telescoping occurs with the result that

$$A_{i\alpha-1,2} = \sum_{4\leqq\gamma\leqq\alpha}(\alpha-\gamma+1)t_i^{\alpha-\gamma}A_{i3,\gamma-2}\,.$$

Similarly

$$A_{i\alpha 1} = \sum_{3\leqq\gamma\leqq\alpha}(\alpha-\gamma+1)t_i^{\alpha-\gamma}A_{i3,\gamma-2}\,.$$

By the first of these and middle equation in (3.80), for $4 \leqq \varrho \leqq n+1$

$$\begin{aligned} T_{i\varrho} = T_{i2,\varrho-2} &= -(\varrho-3)t_i^{\varrho-4}A_{i31} \\ &\quad - \sum_{4\leqq\sigma\leqq\varrho-1}(\varrho-\sigma)t_i^{\varrho-\sigma-1}A_{i3,\sigma-2} \\ &= (3.79)\,. \end{aligned}$$

Q.E.D.

By using

$$T_{i,n+2} = T_{i3,n-1}\,, \quad T_{i,n+3} = T_{i3,n}$$

this formula holds for $\rho = n + 2, n + 3$. For $\alpha \geqq 4$

$$\begin{aligned} T_{i\alpha n} = &-(\alpha - 1)(n - 1)t_i^{\alpha+n-4}A_{i31} - A_{i\alpha n} \\ &-(\alpha - 1)t_i^{\alpha-2}\Bigg(\sum_{4 \leqq \sigma \leqq n+1}(n - \sigma + 2)t_i^{n-\sigma+1}A_{i3,\sigma-2}\Bigg) \\ &+(\alpha - 2)t_i^{\alpha-1}\Bigg(\sum_{3 \leqq \sigma \leqq n}(n - \sigma + 1)t_i^{n-\sigma}A_{i3,\sigma-2}\Bigg) \\ = &-\sum_{3 \leqq \sigma \leqq n+1}(n + \alpha - \sigma)t_i^{n+\alpha-\sigma-1}A_{i3,\sigma-2} - A_{i\alpha n}. \end{aligned}$$

We may then introduce $A_{i3,n+1},\ldots,A_{i3,2n-3}$ so that (3.79) holds for all ϱ; i.e.,

$$T_{i\varrho} = -\sum_{3 \leqq \sigma \leqq \varrho-1}(\varrho - \sigma)t_i^{\varrho-\sigma-1}A_{i3,\sigma-2}, \quad 4 \leqq \varrho \leqq 2n. \tag{3.81}$$

Now we may complete the proof. By (3.65) and (3.66)

$$\begin{aligned} dZ/ds = \sum_i(-k^\varrho T_{i,\varrho} &+ 1/\Delta\{k^{\alpha+\beta}(t_i^\beta T_{i\alpha\gamma}y^\gamma + t_i^\alpha T_{i\beta\gamma}y^\gamma) \\ &-1/\Delta^2 k^\varrho t_i^\varrho(z^\alpha t_i^\alpha + T_{i\alpha\beta}y^\alpha y^\beta)\})Z_i. \end{aligned} \tag{3.82}$$

Since we have now expressed $T_{i\alpha\beta} = T_{i,\alpha+\beta}$ as a polynomial in t_i, the last term may be simplified. Specifically, we have the

Lemma. *Define*

$$P_\alpha = -\sum_{\substack{3 \leqq \varrho \leqq \alpha+n \\ \alpha-\varrho-1 \geqq 0}}(2\alpha - \varrho)t_i^{\alpha-\varrho-1}A_{i3,\varrho-2}.$$

Then

$$T_{i\alpha\beta}y^\alpha y^\beta = (P_\alpha y^\alpha)\Delta + t_i^\gamma Q_{\gamma\alpha\beta}y^\alpha y^\beta. \tag{3.83}$$

Proof. This is equivalent to the congruence

$$T_{i\alpha\beta}y^\alpha y^\beta \equiv (P_\alpha y^\alpha)(t_i^\beta y^\beta) \text{ modulo } t_i, t_i^2, \ldots, t_i^n.$$

The coefficient of $t_i^{n+\gamma}y^\alpha y^\beta$ on the left is obtained by setting $\varrho = \alpha + \beta$ and $\sigma = \alpha + \beta - n - \gamma - 1$ in (3.81) to obtain

$$-2(n + \gamma + 1)A_{i3,\alpha+\beta-n-\gamma-3}.$$

The coefficient of $t_i^{n+\gamma}y^\alpha y^\beta$ on the right is the coefficient of $t_i^{n+\gamma}$ in

$$P_\alpha t_i^\beta + P_\beta t_i^\alpha,$$

which by the definition of the P_α's is equal to

$$-((n + \gamma + 1 + \beta - \alpha) + (n + \gamma - 1 + \alpha - \beta))A_{i3,\alpha+\beta-n-\gamma-3}.$$

Similarly the constant terms are equal. Q.E.D.

Having made the division (3.83), we shift attention to the middle term in (3.82) and have the

Lemma. *For P_α as in the previous lemma*

(3.84) $-(P_\alpha y^\alpha)(k^\varrho t_i^\varrho) - \Delta(k^\varrho T_{i,\varrho}) + k^{\alpha+\beta}(t_i^\beta T_{i\alpha\gamma} y^\gamma + t_i^\alpha T_{i\beta\gamma} y^\gamma) = 0\,.$

Proof. This may be verified by straightforward substitution, as in the preceding lemma.

Taken together these lemmas imply the somewhat miraculous divisibility properties:

First, $T_{i\alpha\beta} y^\alpha y^\beta$ is divisible by $t_i^\alpha y^\alpha$ modulo terms containing $t_i, \ldots, t_i^n$. When this is done and the quotient $P_\alpha y^\alpha$ moved to the middle term in (3.82), *we again obtain an expression which is divisible by $\Delta = t_i^\alpha y^\alpha$ modulo $t_i, \ldots, t_i^n$. In this case the quotient exactly cancels $k^\varrho T_{i,\varrho}$.*

Combining (3.82), (3.83), and (3.84) we obtain our final formula

(3.85) $$\mathrm{d}Z/\mathrm{d}s = -\sum_i (k^\varrho t_i^\varrho/\Delta^2\{t_i^\alpha(z^\alpha + Q_{\alpha\beta\gamma} y^\beta y^\gamma)\})Z_i\,.$$

The equation of the paths is therefore

$$z^\alpha + Q_{\alpha\beta\gamma} y^\beta y^\gamma = 0\,.$$

This completes the proof of (3.26), and shows that in case $d > 2n$ and $n \geqq 3$ a d-web of maximal rank has associated to it a path geometry in the sense of the definition (3.21) where the paths are themselves characterized by the geometric property (3.24). This path geometry has ∞^2-totally geodesic hypersurfaces, one passing through each point x and with given normal $\omega \in D_x \subset \boldsymbol{P}(T_x^*)$, and the final step in our proof will be to show (3.27) that such a path geometry is flat.

iii. *The best harmonic connection.* We retain the notations from the preceeding section. As was mentioned on several occasions such as in footnote[28]), if there were a symmetric torsion-free connection for the G-structure in our problem then the main computation could be considerably simplified. Moreover, these is such a connection in Bol's case $n = 3$ – in fact there is a unique one with the harmonic property (3.53). So it is of interest to generally determine the optimal connection for our problem.

If $\{\phi^\alpha\}$ is any moving frame for the G-structure in our problem, then a symmetric connection is given by a matrix of 1-forms $\{\phi_\beta^\alpha\}$ satisfying

$$\mathrm{d}\phi^\alpha = \sum_\beta \phi^\beta \wedge \phi_\beta^\alpha\,.$$

We assume the connection to be harmonic, and then the most general one such is given by

(3.86) $$\begin{cases} \phi_\beta^{*\alpha} = \phi_\beta^\alpha + \sum_\gamma h_{\alpha\beta\gamma}\phi^\gamma \\ h_{\alpha\beta\gamma} = h_{\alpha,\beta+\gamma} \end{cases} \qquad \text{where}$$

depends only on α and the sum $\beta + \gamma$ (cf. (3.55)). There are $n(2n-1)$ such functions, and we want to choose them so as to annihilate as much as possible of the torsion form $\{\psi^\alpha_\beta\}$ in (3.46); here we remember that $\psi^\alpha_\beta = 0 \Leftrightarrow \phi^\alpha_\beta$ has values in the Lie algebra $\mathfrak{g}$ of the group in our G-structure. We also recall from (3.68) and (3.69) that this is equivalent to annihilating as much as possible of the tensor $A_{i\alpha\gamma}$ introduced in (3.70). The result is the

(3.87) **Proposition.** (Best Harmonic Connection): *There is a unique symmetric, harmonic connection such that*

$$A_{i\alpha\gamma} = 0 \quad \textit{for} \quad \alpha + \gamma \leqq n + 3, \tag{3.88}$$

and where $A_{i\alpha\gamma} = A_{i,\alpha+\gamma}$ depends only on i and the sum $\alpha + \gamma = \lambda$ (say) and is a linear combination of $t_i^{n+1},\ldots,t_i^{\lambda-2}$.

(3.89) **Corollary.** *When $n = 3$ there is a unique symmetric, torsion-free, and harmonic connection. However, when $n \geqq 4$ this "best connection" given by* (3.88) *is in general not torsion free.*

So this makes even more intriguing the divisibility which occurred in the proof of (3.26). Even for webs arising from extremal algebraic curves we have no explanation for the distinction between $n = 3$ and $n \geqq 4$, other than the obvious one that a rational normal curve in $\boldsymbol{P}^{n-1}$ is given by $(n-1)(n-2)/2$ quadratic equations, and this number is 1 when $n = 3$.

The remainder of this section will be devoted to the proof of (3.87).

Lemma. *We have*

$$\left\{\begin{aligned} A_{i\alpha\gamma} &= \sum_{\gamma+2 \leqq \sigma \leqq \lambda-1} (\lambda - \sigma) t_i^{\lambda-\sigma-1} A_{i3,\sigma-2}, \qquad \lambda = \alpha + \gamma \\ &= \sum_{\substack{\gamma+2 \leqq \sigma \leqq \lambda-1 \\ 1 \leqq D \leqq n+2}} (\lambda - \sigma) t_i^{\lambda-\sigma-1+D} a_{D,3,\sigma-2}\,. \end{aligned}\right. \tag{3.90}$$

Proof. Combining equation (3.75) with (3.79), we obtain

$$\begin{aligned} A_{i\alpha\gamma} = {} & -(\alpha-1)(\gamma-1)t_i^{\alpha+\gamma-4}A_{i31} + \sum_{3 \leqq \sigma \leqq \alpha+\gamma-1} (\alpha+\gamma-\sigma)A_{i3,\sigma-2} \\ & -(\alpha-1)t_i^{\alpha-2}\left(\sum_{4 \leqq \sigma \leqq \gamma+1} (\gamma-\sigma+2)t_i^{\gamma-\sigma+1}A_{i3,\sigma-2}\right) \\ & +(\alpha-2)t_i^{\alpha-1}\left(\sum_{3 \leqq \sigma \leqq \gamma+1} (\gamma-\sigma+1)t_i^{\gamma-\sigma}A_{i3,\sigma-2}\right). \end{aligned}$$

Here we have substituted for $A_{i\gamma 2}$ and $A_{i\gamma 1}$ the expressions appearing just at the end of the proof of (3.79). The coefficient of A_{i31} is

$$t_i^{\alpha+\gamma-4}(-(\alpha-1)(\gamma-1) + (\alpha+\gamma-3) + (\alpha-2)(\gamma-2)) = 0\,.$$

For $\sigma \geqq 4$ the coefficient of $A_{i3,\sigma-2}$ is

$$t_i^{\alpha+\gamma-\sigma-1}((\alpha+\gamma-\sigma)-(\alpha-1)(\gamma-\sigma+2)+(\alpha-2)(\gamma-\sigma+1))=0\,.$$

This is valid for $\sigma \leqq \gamma + 1$ and gives the first equation in (3.90). The second follows from the definition (3.70). Q.E.D.

By setting $\lambda = \alpha + \gamma$ and

$$v = \lambda - \sigma - 1 + D \Leftrightarrow D = \sigma + v - \lambda + 1$$

so that

$$\gamma + 2 \leqq \sigma \leqq \lambda - 1 \Leftrightarrow v - \alpha + 3 \leqq D \leqq v$$

we rewrite (3.90) as

$$(3.91)\qquad A_{i\alpha\gamma} = \sum_{1 \leqq v \leqq \alpha+n-1} t_i^v \left(\sum_{\gamma+2 \leqq \sigma \leqq \lambda-1} (\lambda-\sigma) a_{v-\lambda+1+\sigma,3,\sigma-2} \right).$$

The following is a direct consequence of the definition (3.69) and (3.70).

Lemma. *Under a change of connection* (3.86)

$$(3.92)\quad \begin{cases} A_{i\alpha\gamma}^* = A_{i\alpha\gamma} + t_i^\beta h_{\beta\alpha\gamma} - (\alpha-1)t_i^{\alpha+\beta-2}h_{\beta\gamma2} + (\alpha-2)t_i^{\alpha+\beta-1}h_{\beta\gamma1} \\ \quad = \displaystyle\sum_{1 \leqq v \leqq \alpha+n-1} t_i^v \left(\sum_{\gamma+2 \leqq \sigma \leqq \lambda-1} (\lambda-\sigma) a_{v-\lambda+1+\sigma,3,\sigma-2} \right) + t_i^\beta h_{\beta\alpha\gamma} \\ \quad\quad - (\alpha-1)t_i^{\alpha+\beta-2}h_{\beta\gamma2} + (\alpha-2)t_i^{\alpha+\beta-1}h_{\beta1\gamma}\,. \end{cases}$$

The second equation here follows from (3.91).

Now according to (3.91) the basic quantities are $A_{i3,\gamma}$; this is also clear from (3.79). So we consider the equations

$$(3.93)\qquad A_{i3\gamma}^* = 0\,.$$

(3.94) **Lemma.** *These equations have a solution which uniquely determines the functions*

$$h_{\beta,\lambda} \quad \textit{for} \quad 2 \leqq \lambda \leqq n+3\,.$$

In case $n = 3$ this gives the unique torsion-free harmonic connection.

Proof. By (3.92) the equations (3.93) have the following expansion in powers of t_i:

$$\begin{aligned} & t_i(h_{1,3+\gamma} + a_{13\gamma}) \\ & + t_i^2(h_{2,3+\gamma} - 2h_{1,\gamma+2} + a_{23\gamma}) \\ & + t_i^3(h_{3,3+\gamma} - 2h_{2,\gamma+2} + h_{1,\gamma+1} + a_{33\gamma}) \\ & + \cdots \\ & + t_i^n(h_{n,3+\gamma} - 2h_{n-1,\gamma+2} + h_{n-2,\gamma+1} + a_{n3\gamma}) \\ & + t_i^{n+1}(-2h_{n,\gamma+2} + h_{n-1,\gamma+1} + a_{n+1,3\gamma}) \\ & + t_i^{n+2}(h_{n,\gamma+1} + a_{n+2,3\gamma})\,. \end{aligned}$$

Equating to zero the powers of t_i gives $n(n+2)$ equations which clearly determine uniquely the $h_{\beta,\lambda}$ for $1 \leqq \beta \leqq n$, $2 \leqq \lambda \leqq n+3$. Explicitly, we obtain

$$(3.95\text{a}) \qquad -h_{\beta,\lambda} = \sum_{0 \leqq \sigma \leqq \beta-1} (\beta - \sigma) a_{\sigma+1,3,\lambda-\beta-2+\sigma}, \quad \beta + 3 \leqq \lambda \leqq n+3$$

$$(3.95\text{b}) \qquad -h_{\beta,\lambda} = \sum_{0 \leqq \sigma \leqq n-\beta} (n+1-\beta-\sigma) a_{n+2-\sigma,3,\lambda+n-1-\beta-\sigma}, \quad \lambda \leqq \beta + 1$$

$$(3.95\text{c}) \qquad -h_{\beta,\beta+2} = -\beta h_{1,3} + \sum_{2 \leqq \sigma \leqq \beta} (\beta + 1 - \sigma) a_{\sigma,3,\sigma-1}$$

$$(3.95\text{d}) \qquad (n+1) h_{1,3} = \sum_{\beta} (n - \beta + 1) a_{\beta+1,3,\beta}. \qquad \text{Q.E.D.}$$

According to the expansion of $A_{i\alpha\gamma}$ in terms of $A_{i3\gamma}$, the equations (3.93) will imply relations on $A^*_{i\alpha\gamma}$. Moreover, the $n(n-3)$ functions $h_{\beta,\lambda}$, $n+4 \leqq \lambda \leqq 2n$ are still at our disposal. To see what happens we set

$$(3.96) \qquad \begin{aligned} B_{i\alpha\gamma} &= A_{i\alpha\gamma} - (\alpha - 1) t_i^{\alpha+\beta-2} h_{\beta,2+\gamma} + (\alpha - 2) t_i^{\alpha+\beta-1} h_{\beta,1+\gamma} \\ &= \sum_{1 \leqq \nu \leqq n+\alpha-1} p_{\alpha\gamma\nu} t_i^\nu \end{aligned}$$

where the second equation defines the $p_{\alpha\gamma\nu}$. The reason for considering this $B_{i\alpha\gamma}$ is that the $h_{\beta,\lambda}$ appearing on the right are already determined by (3.95). Moreover, for $\lambda = \alpha + \gamma \geqq n + 4$

$$(3.97) \qquad A^*_{i\alpha\gamma} = t_i^\beta h_{\beta,\lambda} + B_{i\alpha\gamma}.$$

Now the $p_{\alpha\gamma\nu}$ may according to (3.92) and (3.95) be expressed in terms of the $a_{\varrho 3\sigma}$. To do this we separate into cases.

Case i) $\nu \geqq \lambda - 1$ where $\lambda = \alpha + \gamma$. Then

$$\begin{aligned} p_{\alpha\gamma\nu} = &- \sum_{0 \leqq \varrho \leqq n-\nu+\alpha-1} (-n + \nu + \varrho - 1) a_{n+2-\varrho,3,n-\nu-1+\lambda-\varrho} \\ &+ \sum_{\gamma+2 \leqq \varrho \leqq \lambda-1} (\lambda - \sigma) a_{\nu-\lambda+1+\sigma,3,\sigma-2}. \end{aligned}$$

When $n + 2 - \varrho = \nu - \lambda + 1 + \sigma$ we have

$$\begin{cases} -n + \nu + \varrho - 1 = \lambda - \sigma \\ n - \nu - 1 + \lambda - \varrho = \sigma - 2 \end{cases}$$

and so $p_{\alpha\gamma\nu} = 0$ for $\nu \geqq n+2$ since $\lambda - 1 \geqq \sigma \Leftrightarrow \nu + \varrho - n - 2 \geqq 0$. When $\nu = n + 1$ we also find $p_{\alpha\gamma\nu} = 0$. If $\nu \leqq n$, say $\nu = \beta$, then

$$p_{\alpha\gamma\beta} = a_{\beta+2,3,\lambda-1} + 2 a_{\beta+3,3,\lambda} + \cdots + (n + 1 - \beta) a_{n+2,3,\beta-\lambda+1-n}.$$

By (3.97) and (3.95b)

$$\begin{aligned} A^*_{i\alpha\gamma} &= t_i^\beta h_{\beta\lambda} + B_{i\alpha\gamma} \\ &= 0. \end{aligned}$$

We conclude that:

A_i^* is a linear combination of $t_i^{\lambda-2}, t_i^{\lambda-3}, \ldots, t_i$, which we may describe as saying that the equations (3.93) eliminate the high powers of t_i from $A^*_{i\alpha\gamma}$ in (3.92). The remaining choices for $h_{\beta\lambda}$ will eliminate low powers, as will now be seen

Case ii) $\nu = \lambda - 2$. Then by (3.91) and (3.95)

$$\begin{aligned} p_{\alpha\gamma\nu} &= -(\lambda-2)h_{13} + (\alpha-1)\Big(\sum_{2\leq\varrho\leq\gamma}(\gamma+1-\varrho)a_{\varrho,3,\varrho-1}\Big) \\ &\quad -(\alpha-2)\Big(\sum_{2\leq\varrho\leq\gamma-1}(\gamma-\varrho)a_{\varrho,3,\varrho-1}\Big) + \sum_{\gamma+2\leq\sigma\leq\lambda-1}(\lambda-\sigma)a_{\sigma-1,3,\sigma-2} \\ &= -(\lambda-2)h_{13} + \sum_{2\leq\varrho\leq\lambda-2}(\lambda-\varrho-1)a_{\varrho,3,\varrho-1}\,. \end{aligned}$$

Case iii) $\nu \leqq \lambda - 3$. Then, as before

$$\begin{aligned} p_{\alpha\gamma\nu} &= (\alpha-1)\Big(\sum_{0\leq\varrho\leq\nu-\alpha+1}(\nu-\alpha+2-\varrho)a_{\varrho+1,3,\lambda-\nu-2+\varrho}\Big) \\ &\quad -(\alpha-2)\Big(\sum_{0\leq\varrho\leq\nu-\alpha}(\nu-\alpha+1-\varrho)a_{\varrho+1,3,\lambda-\nu-2+\varrho}\Big) \\ &\quad + \sum_{\gamma+2\leq\sigma\leq\lambda-1}(\lambda-\sigma)a_{\nu-\lambda+1+\sigma,3,\sigma-2} \\ &= \sum_{0\leq\varrho\leq\nu-\alpha+1}(\nu-\varrho)a_{\varrho+1,3,\lambda-\nu-2+\varrho} + \sum_{\gamma+2\leq\sigma\leq\lambda-1}(\lambda-\sigma)a_{\nu-\lambda+1+\sigma,3,\sigma-2} \\ &= \sum_{0\leq\varrho\leq\nu-1}(\nu-\varrho)a_{\varrho+1,3,\lambda-\nu-2+\varrho}\,. \end{aligned}$$

Now if $\nu = \beta \leqq n$ we set

$$A^*_{i\alpha\gamma} = t_i^\beta h_{\beta\lambda} + B_{i\alpha\gamma} \equiv 0 \text{ modulo } t_i^{n+1}$$

by choosing

$$\text{(3.95e)}\quad -h_{\beta\lambda} = \sum_{0\leq\varrho\leq\beta-1}(\beta-\varrho)a_{\varrho+1,3,\lambda-\beta-2+\varrho}, \quad \beta+3 \leqq \lambda \leqq 2n$$

as prescribed by cases ii) and iii). We note that (3.95e) are the same equations as (3.95a) only extended now to the full index range. Continuing this with case i) we deduce that *$A^*_{i\alpha\gamma}$ depends only on i, $\alpha+\gamma=\lambda$, and is a linear combination of $t_i^{\lambda-2}, \ldots, t_i^{n+1}$. In particular,*

$$A^*_{i\alpha\gamma} = 0 \quad \text{for} \quad \alpha+\gamma \leqq n+2\,.$$

Moreover, at this point the connection $\phi_\beta^{*\alpha}$ is uniquely determined, and it remains only to examine the case $\lambda = n+3$. By (3.98) $A^*_{i\alpha\gamma}(\alpha+\gamma = n+3)$ is a multiple of t_i^{n+1}. By case ii) when $\nu = n+1$ the coefficient is

$$-(n+1)h_{13} + \sum_{2\leq\varrho\leq n+1}(n+2-\varrho)a_{\varrho,3,\varrho-1} = 0$$

by (3.95d).

IV. Projective differential geometry and completion of the proof

A. Projective connections and path geometry

i. *Basic definitions.* We first give the structure equations for $\boldsymbol{P}^n$, as this provides the model space for projective differential geometry. The index ranges

$$\begin{cases} 0 \leqq a, b, c \leqq n \\ 1 \leqq \alpha, \beta, \gamma \leqq n \end{cases}$$

will be used, and repeated indices are summed. We shall work with real projective space – the discussion carries over with the same notation to the complex case.

Definition. *A frame for $\boldsymbol{P}^n$ is given by*

$$F = \{Z_0, Z_1, \ldots, Z_n\}$$

where the Z_a form a basis for $\boldsymbol{R}^{n+1}$.

The manifold of all frames will be denoted by $\mathscr{F}(\boldsymbol{R}^n)$. It may be identified with GL_{n+1}. Occasionally, we shall speak of *normalized frames* defined by

$$Z_0 \wedge Z_1 \wedge \cdots \wedge Z_n = 1$$

the set of which may be identified with SL_{n+1}. There is a projection $\pi : \mathscr{F}(\boldsymbol{P}^n) \to \boldsymbol{P}^n$ given by

$$\pi(F) = Z_0 \text{ [29]},$$

and the fibre $\pi^{-1}(Z_0)$ consists of all frames $F^* = \{Z_0^*, Z_1^*, \ldots, Z_n^*\}$ where

$$(4.1) \qquad \begin{cases} Z_0^* = A_0^0 Z_0 \\ Z_1^* = A_1^0 Z_0 + A_1^1 Z_1 + \cdots + A_1^n Z_n \\ \vdots \\ Z_n^* = A_n^0 Z_0 + A_n^1 Z_1 + \cdots + A_n^n Z_n \, . \end{cases}$$

This equation may be abbreviated by writing F as a column vector and (A_b^a) as a matrix A whereby (4.1) becomes

$$(4.2) \qquad F^* = A \cdot F \, .$$

Each frame F gives a coordinate simplex (Fig. 5) in $\boldsymbol{P}_n$, and up to the homogeneity factor A_0^0, $\pi^{-1}(Z_0)$ consists of all coordinate simplices whose first point is Z_0. The linear structure on $\boldsymbol{P}^n$ is given by identifying each

[29]) We shall frequently abuse notation and denote by Z_0 the point in $\boldsymbol{P}^n$ defined by the non-zero vector Z_0 in $\boldsymbol{R}^{n+1}$.

line $\overrightarrow{Z_0 Z}$ with a tangent vector to $\boldsymbol{P}^n$ at Z_0, and this should be kept in mind in the following discussion.

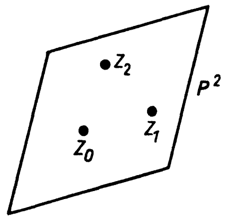

Fig. 5

The frame entries Z_a may be considered as vector-valued functions

$$Z_a : \mathscr{F}(\boldsymbol{P}^n) \to \boldsymbol{R}^{n+1}.$$

Expanding the exterior derivative $\mathrm{d}Z_a$ at $F \in \mathscr{F}(\boldsymbol{P}^n)$ in terms of the basis determined by the frame F we obtain

$$\mathrm{d}Z_a = \theta_a^b Z_b. \tag{4.3}$$

The matrix $\theta = (\theta_a^b)$ gives the coefficients of infinitesimal displacement of the moving frame F. It is the Maurer-Cartan matrix on GL_{n+1}. Equation (4.3) may be abbreviated as

$$\mathrm{d}F = \theta \cdot F. \tag{4.4}$$

Under a change of frame (4.2)

$$\theta^* = \mathrm{d}A \cdot A^{-1} + A \theta A^{-1}. \tag{4.5}$$

The integrability relation, or Maurer Cartan equation,

$$\mathrm{d}\theta = \theta \wedge \theta \tag{4.6}$$

follows by taking the exterior derivative of (4.4).

It will be useful to write (4.5) out making use of the special block from (4.1) of the matrix A. For this we introduce the notations

$$\begin{cases} \theta^\alpha = \theta_0^\alpha \\ \phi_\beta^\alpha = \theta_\beta^\alpha - \delta_\beta^\alpha \theta_0^0, \end{cases}$$

which will be motivated in a little while. Then (4.5) becomes

$$\begin{cases} \theta^{*\alpha} = A_0^0 \theta^\beta (A^{-1})_\beta^\alpha \\ \phi^{*\alpha}_{\ \beta} = \mathrm{d}A_\beta^\gamma (A^{-1})_\gamma^\alpha - \delta_\beta^\alpha \mathrm{d}\log A_0^0 + A_\beta^\gamma \phi_\gamma^\lambda (A^{-1})_\lambda^\alpha \\ \qquad\quad + A_\beta^0 \theta^\lambda (A^{-1})_\lambda^\alpha \\ \theta^{*0}_{\ \alpha} = \mathrm{d}A_\alpha^0 (A^{-1})_0^0 + \mathrm{d}A_\alpha^\beta (A^{-1})_\beta^0 + A_\alpha^\beta \theta_\beta^0 (A^{-1})_0^0 \\ \qquad\quad + A_\alpha^\gamma \theta_\gamma^\lambda (A^{-1})_\lambda^0 + A_\alpha^0 \theta^\beta (A^{-1})_\beta^0. \end{cases} \tag{4.7}$$

The integrability conditions (4.6) are

$$(4.8)\qquad \begin{cases} d\theta^\gamma = \theta^\beta \wedge \phi^\gamma_\beta \\ d\phi^\alpha_\beta = \phi^\gamma_\beta \wedge \phi^\alpha_\gamma + \theta^0_\beta \wedge \theta^\alpha - \delta^\alpha_\beta \theta^\gamma \wedge \theta^0_\gamma \\ d\theta^0_\beta = \phi^\alpha_\beta \wedge \theta^0_\alpha . \end{cases}$$

From the first equations in (4.7) and (4.8) it follows that on $\mathscr{F}(\boldsymbol{P}^n)$

$$\theta^\alpha = 0$$

defines a completely integrable Pfaffian system, one whose integral manifolds are the fibres of $\mathscr{F}(\boldsymbol{P}^n) \xrightarrow{\pi} \boldsymbol{P}^n$.

To interpret the second and third equations, we denote by $H^* \to \boldsymbol{P}^n$ the universal bundle whose fibre over $Z_0 \in \boldsymbol{P}^n$ is the line $\lambda \cdot Z_0$ in $\boldsymbol{R}^{n+1}$. Letting

$$Q = \boldsymbol{P}^n \times \boldsymbol{R}^{n+1}/H^*$$

be the quotient of the trivial bundle by H^*, the identification of lines with tangent vectors gives an isomorphism

$$T \cong \mathrm{Hom}(H^*,Q) = H \otimes Q .$$

More precisely, sections of $H \otimes \boldsymbol{R}^{n+1}$ may be interpreted as vector fields of the form

$$\theta = \sum_{a,b} \theta^a_b(z_1/z_0,\ldots,z_n/z_0) z^b \partial/\partial z^a .$$

The inclusion $H^* \to \boldsymbol{P}^n \times \boldsymbol{R}^{n+1}$ induces an inclusion of the trivial bundle in $H \otimes \boldsymbol{R}^{n+1}$ with the generator 1 going into $\sum_a z^a \partial/\partial z^a$. By Euler's theorem $\pi(\theta) = 0$ in T_{Z_0} if and only if

$$\theta = \lambda(\sum_a z^a \partial/\partial z^a) \quad (Z_0 = [1,z_1/z_0,\ldots,z_n/z_0]) .$$

Summarizing, the exact sequence

$$(4.9)\qquad 0 \to H^* \to \boldsymbol{P}^n \times \boldsymbol{R}^{n+1} \to Q \to 0$$

and isomorphism

$$(4.10)\qquad T \cong H \otimes Q$$

embody the linear structure on $\boldsymbol{P}^n$. $\mathscr{F}(\boldsymbol{P}^n)$ is the manifold of frames for the trivial bundle in the middle of (4.9), and $\phi^\alpha_\beta = \theta^\alpha_\beta - \delta^\alpha_\beta \theta^0_0$ looks something like a connection in $T \cong Q \otimes H$. This is not quite correct, since the flat connection on $\boldsymbol{P}^n \times \boldsymbol{R}^{n+1}$ does not leave H^* invariant, or equivalently since the extension in (4.9) is non-trivial relative to the connection in $\boldsymbol{P}^n \times \boldsymbol{R}^{n+1}$.

In general, a projective connection on a manifold will be given by a vector bundle sequence

$$0 \to L^* \to E \to Q \to 0$$

with isomorphism

$$T(M) \cong L \otimes Q$$

and with a connection in E. Rather than try to formalize this, we shall adopt a "working definition". So, given a covering $\{U, V, \ldots, \ldots\}$ of M we assume given in each open set U an $(n + 1) \times (n + 1)$ matrix of 1-forms θ_U satisfying

$$\theta_U = \mathrm{d}A_{UV} A_{UV}^{-1} + A_{UV} \theta_V A_{UV}^{-1}$$

in $U \cap V$ where A_{UV} has the block form

$$A_{UV} = \begin{pmatrix} * & 0 & \cdots & 0 \\ * & * & \cdots & * \\ \vdots & & & \\ * & * & \cdots & * \end{pmatrix}.$$

We assume that the entries

$$(\theta_U)_0^1, \ldots, (\theta_U)_0^n$$

are linearly independent, and shall admit over U all connection matrices

$$\theta = \mathrm{d}A \cdot A^{-1} + A \theta_U A^{-1}$$

where A has the block form given above. This will be our definition of a *projective connection.*

For example, suppose that $M_n \subset \boldsymbol{P}^{n+k}$ is a submanifold and consider the *Darboux frames* $F = \{Z_0; Z_1, \ldots, Z_n; Z_{n+1}, \ldots, Z_{n+k}\} \in \mathscr{F}(\boldsymbol{P}^n)$ where $Z_0 \in M$ and $Z_0, Z_1, \ldots, Z_n$ span the projective tangent space to M at Z_0. The connection matrix for $\boldsymbol{P}^{n+k}$ has the block form

$$\theta_{\boldsymbol{P}^{n+k}} = \left(\begin{array}{ccc|ccc} * & \cdots & * & 0 & \cdots & 0 \\ & & & * & \cdots & * \\ \vdots & & \vdots & \vdots & & \vdots \\ * & \cdots & * & * & \cdots & * \\ \hline * & \cdots & * & * & \cdots & * \\ \vdots & & \vdots & \vdots & & \vdots \\ * & \cdots & * & * & \cdots & * \end{array}\right) \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} n+1 \\ \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} k \end{array}$$

on the Darboux frames, and the upper left hand block defines a projective connection on M.

At this point, since our discussion is local it will simplify matters if we assume that our projective connection is *special* in the sense that the trace

$$\sum_a (\theta_{UV})^a_a = 0. \tag{4.11}$$

It follows that $\det(A_{UV}) = $ constant, and we shall assume this constant is $+1$. Although somewhat unnatural, the use of special projective connections will alleviate the necessity of saying when two projective connections are equivalent, a notion which takes into account the fact that GL_{n+1} does not act effectively on $\boldsymbol{P}^n$. There is a natural special projective connection on the manifold of normalized frames on $\boldsymbol{P}^n$. There are $(n+1)^2 - 1 = n(n+2)$ forms in the connection matrix of a special projective connection, and θ^0_0 is then determined by the ϕ^α_α according to

$$-(n+1)\theta^0_0 = \phi^1_1 + \cdots + \phi^n_n. \tag{4.12}$$

Given a special projective connection with connection matrix $\theta = (\theta^b_a)$ (we omit reference to the open set U), set

$$\begin{cases} \theta^\alpha = \theta^\alpha_0 \\ \phi^\alpha_\beta = \theta^\alpha_\beta - \delta^\alpha_\beta \theta^0_0. \end{cases} \tag{4.13}$$

The *projective torsion* Θ^α and *projective curvature* Φ^α_β are defined by

$$\Theta^\alpha = d\theta^\alpha - \theta^\beta \wedge \phi^\alpha_\beta \tag{4.14}$$

$$\Phi^\alpha_\beta = d\phi^\alpha_\beta - \phi^\gamma_\beta \wedge \phi^\alpha_\gamma - \theta^0_\beta \wedge \theta^\alpha + \delta^\alpha_\beta \theta^\gamma \wedge \theta^0_\gamma. \tag{4.15}$$

By the same computations as led to (4.7) and (4.8) it follows that Θ^α and Φ^α_β are tensors. In particular, they are horizontal and we set

$$\Phi^\alpha_\beta = 1/2(R^\alpha_{\beta\gamma\lambda}\theta^\gamma \wedge \theta^\lambda), \quad R^\alpha_{\beta\gamma\lambda} + R^\alpha_{\beta\lambda\gamma} = 0.$$

Definition. *The special projective connection is normal in case*

$$\begin{cases} \Theta^\alpha = 0 \\ \sum_\alpha R^\alpha_{\beta\gamma\alpha} = 0. \end{cases} \tag{4.16}$$

For a torsion-free projective connection there is the *Bianchi identity*

$$R^\alpha_{\beta\gamma\lambda} + R^\alpha_{\lambda\beta\gamma} + R^\alpha_{\gamma\lambda\beta} = 0. \tag{4.17}$$

Proof. By (4.14)

$$\begin{aligned} 0 = d^2\theta^\alpha &= d\theta^\gamma \wedge \phi^\alpha_\gamma - \phi^\beta \wedge d\phi^\alpha_\beta \\ &= -\theta^\beta \wedge (d\phi^\alpha_\beta - \phi^\gamma_\beta \wedge \phi^\alpha_\gamma) \\ &= -\theta^\beta \wedge (\Phi^\alpha_\beta + \theta^0_\beta \wedge \theta^\alpha - \delta^\alpha_\beta \theta^\gamma \wedge \theta^0_\gamma) \end{aligned}$$

by (4.15)

$$= -\theta^{\beta} \wedge \Phi^{\alpha}_{\beta},$$

which implies (4.17). Q.E.D.

We note that (4.17) is vacuous unless $n \geqq 3$. Somewhat deeper is the following: Define (cf. the 3rd equation in (4.8))

$$\Theta^0_\beta = d\theta^0_\beta - \phi^\alpha_\beta \wedge \phi^0_\alpha . \tag{4.18}$$

Then, in case $n \geqq 3$,

$$\Theta^\alpha = 0 \quad \text{and} \quad \Phi^\alpha_\beta = 0 \Rightarrow \Theta^0_\beta = 0 . \tag{4.19}$$

Proof. Applying exterior differentiation to (4.15) gives

$$\begin{aligned} 0 = & -d\phi^\gamma_\beta \wedge \phi^\alpha_\gamma + \phi^\gamma_\beta \wedge d\phi^\alpha_\gamma - d\theta^0_\beta \wedge \theta^\alpha + \theta^0_\beta \wedge d\theta^\alpha \\ & + \delta^\alpha_\beta d\theta^\gamma \wedge \theta^0_\gamma - \delta^\alpha_\beta \theta^\gamma \wedge d\theta^0_\gamma \\ = & -(\phi^\lambda_\beta \wedge \phi^\gamma_\lambda + \theta^0_\beta \wedge \theta^\gamma - \delta^\gamma_\beta \theta^\lambda \wedge \theta^0_\lambda) \wedge \theta^\alpha_\gamma \\ & + \phi^\gamma_\beta \wedge (\phi^\lambda_\gamma \wedge \phi^\alpha_\lambda + \theta^0_\gamma \wedge \theta^\alpha - \delta^\alpha_\gamma \theta^\lambda \wedge \theta^0_\lambda) \\ & - (\Theta^0_\beta + \phi^\gamma_\beta \wedge \theta^0_\gamma) \wedge \theta^\alpha + \theta^0_\beta \wedge \theta^\gamma \wedge \phi^\alpha_\gamma \\ & + \delta^\alpha_\beta \theta^\lambda \wedge \phi^\gamma_\lambda \wedge \theta^0_\gamma - \delta^\alpha_\beta \theta^\gamma \wedge (\Theta^0_\gamma + \phi^\lambda_\gamma \wedge \theta^0_\lambda) \\ = & -\Theta^0_\beta \wedge \theta^\alpha - \delta^\alpha_\beta \theta^\gamma \wedge \Theta^0_\gamma . \end{aligned}$$

Suppose that $n = 3$ and take $\beta = 1$ and $\alpha = 2,3$ to obtain

$$\Theta^0_1 \wedge \theta^2 = 0 = \Theta^0_1 \wedge \theta^3 .$$

This implies that

$$\Theta^0_1 = \varrho_1 \theta^2 \wedge \theta^3 .$$

Similarly

$$\Theta^0_2 = -\varrho_2 \theta^1 \wedge \theta^3 , \quad \Theta^0_3 = \varrho_3 \theta^1 \wedge \theta^2 .$$

Taking $\alpha = \beta = 1$ gives

$$(-2\varrho_1 - \varrho_2 - \varrho_3)\theta^1 \wedge \theta^2 \wedge \theta^3 = 0 ,$$

and consequently

$$\begin{aligned} -2\varrho_1 &= \varrho_2 + \varrho_3 \\ -2\varrho_2 &= \varrho_1 + \varrho_3 \\ -2\varrho_3 &= \varrho_1 + \varrho_2 \end{aligned}$$

which implies

$$-2(\varrho_1 + \varrho_2 + \varrho_3) = 2(\varrho_1 + \varrho_2 + \varrho_3)$$

or $\varrho_1 + \varrho_2 + \varrho_3 = 0$. But then $\varrho_1 = \varrho_2 = \varrho_3 = 0$.

The case $n \geqq 4$ is even easier. Q.E.D.

Definition. *The connection is projectively flat in case* $\Theta^\alpha = \Phi^\alpha_\beta = \Theta^0_\beta = 0$.

Assuming projective flatness, the projective connection matrix $\theta = (\theta^a_b)$ satisfies

$$\begin{cases} \mathrm{d}\theta = \theta \wedge \theta \\ \sum\limits_a \theta^a_a = 0\,. \end{cases}$$

According to a standard application of the Frobenius theorem there are locally maps $f: M \to \mathrm{SL}_{n+1}$ inducing θ from the Maurer-Cartan matrix on SL_{n+1}. In this sense connections which are projectively flat are equivalent to the standard projective connection on $\boldsymbol{P}^n$. We shall say more about this in the next section.

In concluding this section, we assume that the projective torsion $\Theta^\alpha = 0$ so that the Bianchi identity (4.17) holds. Then normality implies

$$\sum_\alpha R^\alpha_{\alpha\beta\gamma} = 0\,. \tag{4.20}$$

ii. *Fundamental theorem in local projective differential geometry; the Beltrami theorem.* We shall discuss the relationship between projective connections and path geometry as defined in § III A ii, beginning with the case of the straight lines in the model space $\boldsymbol{P}^n$. Following the notations in the preceding section, let $\{Z_0(t), Z_1(t), \ldots, Z_n(t)\}$ be a curve in the frame manifold lying over the curve $Z_0(t)$ in $\boldsymbol{P}^n$. We denote t-derivatives by a dot, and for a 1-form ψ the notation $\dot{\psi}$ will be used for $\psi/\mathrm{d}t$ along the curve. Thus, by (4.3)

$$\dot{Z}_0 = \dot{\theta}^a_0 Z_a$$
$$\ddot{Z}_0 = (\ddot{\theta}^a_0 + \dot{\theta}^b_0 \dot{\theta}^a_b) Z_0$$

and

$$Z_0 \wedge \dot{Z}_0 \wedge \ddot{Z}_0 = \sum_{\alpha<\beta} \{\dot{\theta}^\alpha(\ddot{\theta}^\beta + \dot{\theta}^\gamma \dot{\phi}^\beta_\gamma + 2\dot{\theta}^\beta \dot{\theta}^0_0) - \dot{\theta}^\beta(\ddot{\theta}^\alpha + \dot{\theta}^\gamma \dot{\phi}^\alpha_\gamma + 2\dot{\theta}^\alpha \ddot{\theta}^0_0)\} Z_0 \wedge Z_\alpha \wedge Z_\beta\,.$$

Since the equation

$$Z_0 \wedge \dot{Z}_0 \wedge \ddot{Z}_0 = 0$$

characterizes the straight lines, it follows that these lines are given in terms of the standard projective connection on $\boldsymbol{P}^n$ by the O.D.E. system

$$(\ddot{\theta}^\beta + \dot{\theta}^\gamma \dot{\phi}^\beta_\gamma)/\dot{\theta}^\beta = (\ddot{\theta}^\alpha + \dot{\theta}^\gamma \dot{\phi}^\alpha_\gamma)/\dot{\theta}^\alpha\,. \tag{4.21}$$

We recognize (4.21) as being of the form (3.21), and the resulting path geometry is the geometry of lines in $\boldsymbol{P}^n$.

Now suppose that we are given a special projective connection on a manifold M. If θ_U is the connection matrix in the open set U, then we are permitted to use any other connection matrix

$$\theta = \mathrm{d}A \cdot A^{-1} + A \cdot \theta_U A^{-1}$$

where A has the block form given by (4.1) and $\det A = +1$. In particular, since $(\theta_U)_0^1 \wedge \cdots \wedge (\theta_U)_0^n \neq 0$ we may choose a coordinate system $x = (x^1,\ldots,x^n)$ and A such that for the new connection matrix

$$\theta^\alpha = \mathrm{d}x^\alpha \text{ [30]}. \tag{4.22}$$

If we think of the various choices of A as giving the possible frames for the projective connection, then we may refer to a frame which satisfies (4.22) as a *coordinate frame*. We write

$$\begin{aligned}\phi_\beta^\alpha &= \Gamma_{\beta\lambda}^\alpha \mathrm{d}x^\lambda \\ \theta_\beta^0 &= \Gamma_{\beta\lambda}^0 \mathrm{d}x^\lambda ,\end{aligned}$$

and determine θ_0^0 by (4.12) so that $\theta_\beta^\alpha = \phi_\beta^\alpha + \delta_\beta^\alpha \theta_0^0$. By the definition (4.14), the torsion

$$\Theta^\alpha = -1/2\left\{\sum_{\beta,\lambda}(\Gamma_{\beta\lambda}^\alpha - \Gamma_{\lambda\beta}^\alpha)\mathrm{d}x^\beta \wedge \mathrm{d}x^\lambda\right\},$$

so that for a coordinate frame the torsion being zero is equivalent to the symmetry

$$\Gamma_{\beta\lambda}^\alpha = \Gamma_{\lambda\beta}^\alpha . \tag{4.23}$$

Henceforth, although not strictly necessary, we shall suppose that the torsion is zero. Equations (4.21) are of the form

$$(\ddot{x}^\beta + \Gamma_{\gamma\lambda}^\beta \dot{x}^\gamma \dot{x}^\lambda)/\dot{x}^\beta = (\ddot{x}^\alpha + \Gamma_{\gamma\lambda}^\alpha \dot{x}^\gamma \dot{x}^\lambda)/\dot{x}^\alpha,\ \Gamma_{\gamma\lambda}^\beta = \Gamma_{\lambda\gamma}^\beta , \tag{4.24}$$

and define a path geometry in the sense of § III A ii. A basic remark is that *the form of the equations* (4.24) *is invariant under an arbitrary change of coordinates* $x^\alpha = x^\alpha(y^1,\ldots,y^n)$ *and change of parameter* $t = t(s)$.

For example, suppose we change parameter and denote by a prime the s-derivatives. Then

$$\begin{cases} x^{\alpha\prime}(t(s)) = \dot{x}^\alpha t' \\ x^{\alpha\prime\prime}(t(s)) = \ddot{x}^\alpha t'^2 + \dot{x}^\alpha t'' \end{cases}$$

and so

$$\begin{aligned}(\ddot{x}^\alpha + \Gamma_{\gamma\lambda}^\gamma \dot{x}^\gamma \dot{x}^\lambda)/\dot{x}^\alpha &= ((x^{\alpha\prime\prime}/t'^2) - (\dot{x}^\alpha t''/t'^2) \\ &\quad + (\Gamma_{\gamma\lambda}^\alpha x^{\gamma\prime} x^{\lambda\prime}/t'^2))/(x^{\alpha\prime}/t') \\ &= 1/t'((x^{\alpha\prime\prime} + \Gamma_{\gamma\lambda}^\alpha x^{\gamma\prime} x^{\lambda\prime})/x^{\alpha\prime}) - 1/t'^2 ,\end{aligned}$$

which implies the invariance of (4.24) under a change of parameter. The behaviour of (4.24) under a change of coordinates is done by a similar

[30]) Recall that under the above transformation $\theta^\alpha = A_0^0(\theta_U)^\beta(A^{-1})_\beta^\alpha$.

calculation, and taken together these show that *the path geometry defined by* (4.24) *is intrinsically attached to the projective connection.*

The fundamental theorem of local projective differential geometry is the converse:

(4.25) **Proposition.** *A path geometry* (4.24) *intrinsically defines a unique special, normal projective connection.*

Proof. We write (4.24) in the form

$$(4.26)\qquad (\mathrm{d}^2x^\beta + \Gamma^\beta_{\gamma\lambda}\mathrm{d}x^\gamma\mathrm{d}x^\lambda)/\mathrm{d}x^\beta = (\mathrm{d}^2x^\alpha + \Gamma^\alpha_{\gamma\lambda}\mathrm{d}x^\gamma\mathrm{d}x^\lambda)/\mathrm{d}x^\alpha,$$

and set

$$Q^\alpha(\mathrm{d}x) = \Gamma^\alpha_{\gamma\lambda}\mathrm{d}x^\gamma\mathrm{d}x^\lambda$$

considered as a quadratic polynomial in the $\mathrm{d}x^\gamma$'s. The most general change preserving (4.26) is

$$\tilde{Q}^\alpha(\mathrm{d}x) = Q^\alpha(\mathrm{d}x) + \mathrm{d}x^\alpha P(\mathrm{d}x)$$

where P is linear. We will uniquely determine P so that

$$(4.27)\qquad \sum_\alpha \partial\tilde{Q}^\alpha(\xi)/\partial\xi^\alpha = 0\,.$$

This is equivalent to

$$\begin{aligned} 0 &= \sum_\alpha \partial/\partial\xi^\alpha(Q^\alpha(\xi) + \xi^\alpha P(\xi)) \\ &= \sum_\alpha \partial Q^\alpha(\xi)/\partial\xi^\alpha + 2P(\xi) \end{aligned}$$

so that $P = -1/2(\sum_\alpha \partial Q^\alpha/\partial\xi^\alpha)$ gives the unique solution. Changing notation, we assume (4.27) for the $Q^\alpha(\xi)$, or equivalently that

$$\sum_\alpha \Gamma^\alpha_{\alpha\lambda} = 0\,.$$

This equation is the same as

$$(4.28)\qquad \sum_\alpha \phi^\alpha_\alpha = 0\,.$$

Now to prove (4.25) we must show how to uniquely determine the connection matrix (θ^a_b) subject to certain properties. We set

$$\begin{cases} \theta^\alpha_0 = \theta^\alpha = \mathrm{d}x^\alpha \\ \theta^\alpha_\beta = \phi^\alpha_\beta = \Gamma^\alpha_{\beta\lambda}\mathrm{d}x^\lambda\,, \quad \Gamma^\alpha_{\beta\lambda} = \Gamma^\alpha_{\lambda\beta}\,. \end{cases}$$

Then by (4.12) and (4.28) we must have $\theta^0_0 = 0$, and it remains to determine

$$\theta^0_\alpha = \Gamma^0_{\alpha\beta}\mathrm{d}x^\beta$$

by the normality condition (cf. (4.16))

$$\sum_{\alpha} R^{\alpha}_{\beta\gamma\alpha} = 0. \tag{4.29}$$

These are n^2 equations in the n^2 unknowns $\Gamma^0_{\alpha\beta}$, and it must be proved that they are linear equations whose coefficient matrix is non-singular.

We define $S^{\alpha}_{\beta\gamma\lambda} = -S^{\alpha}_{\beta\lambda\gamma}$ by

$$-1/2\left\{\sum_{\gamma,\lambda} S^{\alpha}_{\beta\gamma\lambda}\mathrm{d}x^{\gamma} \wedge \mathrm{d}x^{\lambda}\right\} = \mathrm{d}\phi^{\alpha}_{\beta} - \phi^{\gamma}_{\beta} \wedge \phi^{\alpha}_{\gamma}.$$

These are known functions in terms of the $\Gamma^{\alpha}_{\gamma\lambda}$'s and their 1st derivatives. By the definition (4.15) of the projective curvature tensor

$$\begin{aligned} 1/2\{R^{\alpha}_{\beta\gamma\lambda}\mathrm{d}x^{\gamma} \wedge \mathrm{d}x^{\lambda}\} + 1/2\{S^{\alpha}_{\beta\gamma\lambda}\mathrm{d}x^{\gamma} \wedge \mathrm{d}x^{\lambda}\} &= -\theta^0_{\beta} \wedge \theta^{\alpha} + \delta^{\alpha}_{\beta}\theta^{\gamma} \wedge \theta^0_{\gamma} \\ &= (\delta^{\alpha}_{\gamma}\Gamma^0_{\beta\lambda} + \delta^{\alpha}_{\beta}\Gamma^0_{\gamma\lambda})\mathrm{d}x^{\gamma} \wedge \mathrm{d}x^{\lambda}. \end{aligned}$$

Consequently

$$R^{\alpha}_{\beta\gamma\lambda} + S^{\alpha}_{\beta\gamma\lambda} = (\delta^{\alpha}_{\gamma}\Gamma^0_{\beta\lambda} - \delta^{\alpha}_{\lambda}\Gamma^0_{\beta\gamma}) + \delta^{\alpha}_{\beta}(\Gamma^0_{\gamma\lambda} - \Gamma^0_{\lambda\gamma}).$$

Setting $\alpha = \lambda$ and summing gives, by (4.29),

$$\sum_{\alpha} S^{\alpha}_{\beta\gamma\alpha} = S_{\beta\gamma} = -n\Gamma^0_{\beta\gamma} + \Gamma^0_{\gamma\beta}.$$

When $n > 1$ the coefficient matrix of this linear system of n^2 equations in n^2 unknowns has a non-zero determinant. Q.E.D.

We recall from section III A ii that the path geometry defined by (4.24) is *flat* in case there is locally a diffeomorphism taking the paths onto straight lines in $\boldsymbol{P}^n$. Combining (4.25) with (4.19) we obtain the

(4.30) **Corollary.** *When* $n \geqq 3$ *the path geometry* (4.24) *is flat if, and only if, the associated projective connection has projective curvature*

$$\Phi^{\alpha}_{\beta} = 0.$$

While we are discussing flatness we want to prove the beautiful

Beltrami theorem. *The path geometry defined by the geodesics of a Riemannian metric is flat if, and only if, the metric has constant sectional curvatures.*

Proof. We choose an orthonormal coframe ω^{α} for the metric – thus

$$\mathrm{d}s^2 = \sum_{\alpha}(\omega^{\alpha})^2.$$

The structure equations for the Riemannian connection matrix ω^{α}_{β} and curvature matrix $\Omega^{\alpha}_{\beta} = 1/2\{T^{\alpha}_{\beta\gamma\lambda}\omega^{\gamma} \wedge \omega^{\lambda}\}$ are

$$\begin{cases} \mathrm{d}\omega^{\alpha} = \omega^{\beta} \wedge \omega^{\alpha}_{\beta}, \quad \omega^{\alpha}_{\beta} + \omega^{\beta}_{\alpha} = 0 \\ \Omega^{\alpha}_{\beta} = \mathrm{d}\omega^{\alpha}_{\beta} - \omega^{\gamma}_{\beta} \wedge \omega^{\alpha}_{\gamma}. \end{cases} \tag{4.31}$$

Taking exterior derivatives gives the two Bianchi identities

$$\text{(4.32)} \qquad \begin{cases} T^{\alpha}_{\beta\gamma\lambda}\omega^{\beta} \wedge \omega^{\gamma} \wedge \omega^{\lambda} = 0 \\ T^{\alpha}_{\beta\gamma\lambda|\mu}\omega^{\gamma} \wedge \omega^{\lambda} \wedge \omega^{\mu} = 0 . \end{cases}$$

The geodesics of the Riemannian metric are the solution curves to the O.D.E. system

$$\text{(4.33)} \qquad \ddot{\omega}^{\alpha} + \dot{\omega}^{\beta}\dot{\omega}^{\alpha}_{\beta} = 0 .$$

According to the proof of (4.25), the projective connection matrix $\{\theta^{\alpha}, \phi^{\alpha}_{\beta}, \theta^{0}_{\beta}\}$ associated to the path geometry given by the geodesics (4.33) is uniquely determined by the equations

$$\begin{cases} \theta^{\alpha} = \omega^{\alpha} \\ \phi^{\alpha}_{\beta} = \omega^{\alpha}_{\beta} \qquad ^{31)} \\ \theta^{0}_{\beta} = \Gamma^{0}_{\beta\gamma}\omega^{\gamma} \end{cases}$$

where

$$\text{(4.33)} \qquad \sum_{\alpha} T^{\alpha}_{\beta\gamma\alpha} = T_{\beta\gamma} = n\Gamma^{0}_{\beta\gamma} - \Gamma^{0}_{\gamma\beta} .$$

Consider first the case $n = 2$. The only non-zero component of the Riemannian curvature tensor is

$$K = T^{1}_{212} \; ^{32)} .$$

Taking $\beta = \gamma = 1$ in (4.33) gives

$$-K = T^{2}_{112} = T_{11} = 2\Gamma^{0}_{11} - \Gamma^{0}_{11} = \Gamma^{0}_{11} ,$$

and similarly $\Gamma^{0}_{22} = -K$. Taking $\beta = 1$, $\gamma = 2$ and then interchanging β and γ gives

$$\begin{aligned} 0 &= T_{12} = 2\Gamma^{0}_{12} - \Gamma^{0}_{21} \\ 0 &= T_{21} = 2\Gamma^{0}_{21} - \Gamma^{0}_{12} , \end{aligned}$$

which implies that

$$\begin{aligned} \Gamma^{0}_{12} &= \Gamma^{0}_{21} = 0 , \\ \Gamma^{0}_{\beta\gamma} &= -\delta^{\gamma}_{\beta}K . \end{aligned}$$

It follows that the projective curvature tensor

$$R^{\alpha}_{\beta\gamma\lambda} = T^{\alpha}_{\beta\gamma\lambda} - K(\delta^{\alpha}_{\gamma}\delta^{\beta}_{\lambda} - \delta^{\alpha}_{\lambda}\delta^{\beta}_{\gamma}) .$$

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[31]) Note that trivially $\sum_{\alpha} \phi^{\alpha}_{\alpha} = 0$, so that $\theta^{0}_{0} = 0$.

[32]) Up to the symmetries $T^{\alpha}_{\beta\gamma\lambda} = -T^{\beta}_{\alpha\gamma\lambda} = -T^{\alpha}_{\beta\lambda\gamma}$.

Since $R^{\alpha}_{\beta\gamma\lambda} = 0$ if $\alpha = \beta$, the only possible non-zero components of this tensor are

$$R^{1}_{212} = K - K = 0$$
$$R^{2}_{112} = -K - K(-1) = 0\,.$$

So, when $n = 2$ the projective torsion and curvature are always zero, and the obstruction to projective flatness is entirely measured by

$$\begin{aligned}\Theta^0_\beta &= \mathrm{d}\theta^0_\beta - \theta^\alpha_\beta \wedge \theta^0_\alpha \\ &= -\mathrm{d}K \wedge \omega^\beta\end{aligned}$$

since
$$\theta^0_\beta = -K\omega^\beta\,.$$

It follows that

$$\Theta^0_\beta = 0 \Leftrightarrow \mathrm{d}K = 0\,,$$

which proves the result in this case.

Now suppose that $n \geqq 3$. According to (4.30) the geodesic path geometry is flat if, and only if, $R^{\alpha}_{\beta\gamma\lambda} = 0$. By the symmetries of $T^{\alpha}_{\beta\gamma\lambda}$,

$$T_{\beta\gamma} = T_{\gamma\beta}$$

in (4.33). We may choose the coframe ω^α to diagonalize $T_{\beta\gamma}$ at a point x_0. Call the eigenvalues $(n-1)T_\beta$. Then, by (4.33)

$$n\Gamma^0_{\beta\gamma} - \Gamma^0_{\gamma\beta} = (n-1)\delta^\beta_\gamma T_\beta$$
$$n\Gamma^0_{\gamma\beta} - \Gamma^0_{\beta\gamma} = (n-1)\delta^\gamma_\beta T_\gamma\,,$$

which gives
$$\Gamma^0_{\beta\gamma} = \Gamma^0_{\gamma\beta} = \delta^\beta_\gamma T_\beta\,.$$

Suppose that $R^{\alpha}_{\beta\gamma\lambda} = 0$. Then, by (4.15), at x_0

$$\begin{cases} 0 = T^{\alpha}_{\beta\gamma\lambda} + (\delta^\alpha_\gamma\delta^\beta_\lambda - \delta^\alpha_\lambda\delta^\beta_\gamma)T_\beta \\ 0 = T^{\beta}_{\alpha\gamma\lambda} + (\delta^\beta_\gamma\delta^\alpha_\lambda - \delta^\beta_\lambda\delta^\alpha_\gamma)T_\alpha\,. \end{cases}$$

Adding these equations and using $T^{\alpha}_{\beta\gamma\lambda} + T^{\beta}_{\alpha\gamma\lambda} = 0$ gives

$$(T_\alpha - T_\beta)\delta^\alpha_\lambda\delta^\beta_\gamma = (T_\alpha - T_\beta)\delta^\alpha_\gamma\delta^\beta_\lambda\,.$$

Taking $\alpha = \lambda$, $\beta = \gamma$ but $\alpha \neq \beta$ we find

$$T_\alpha - T_\beta = 0\,.$$

It follows that, at x_0, all T_α are the same. Since x_0 was any point,

$$T^{\alpha}_{\beta\gamma\lambda}(x) = (\delta^\alpha_\gamma\delta^\beta_\lambda - \delta^\alpha_\lambda\delta^\beta_\gamma)K(x)\,.$$

By the second Bianchi identity in (4.32),

$$K_\mu\omega^\alpha \wedge \omega^\beta \wedge \omega^\mu = 0\,.$$

Thus $K_\mu = 0$ for $\mu \neq \alpha, \beta$. Since $n \geqq 3$ this implies that all $K_\mu = 0$, and so ds^2 has constant sectional curvature.

If, conversely, the sectional curvatures are constant, then we may reverse the calculation to conclude that $R^\alpha_{\beta\gamma\lambda} = 0$.

When the sectional curvatures are a constant $K > 0$, we may argue geometrically as follows: Represent the manifold as a portion of the sphere S^n in $\mathbf{R}^{n+1}$. The geodesics are the intersections of planes through the origin with S^n. Intersecting these planes with the $\mathbf{P}^n$ at infinity in $\mathbf{R}^{n+1}$ maps the geodesics to lines.

B. Totally geodesic hypersurfaces and completion of the proof

We consider a path geometry given by an O.D.E. system (4.24). Recall that a submanifold is *totally geodesic* if any path which is tangent to the submanifold lies entirely in it. We will be mainly concerned with totally geodesic hypersurfaces, and will prove:

(4.34) *Suppose that $n \geqq 3$ and we are given a field of rational normal curves $D_x \subset \mathbf{P}(T_x^*)$ such that each $\omega \in D_x$ is the normal to a totally geodesic hypersurface through x. Then the path geometry is flat.*

The proof requires some preliminary discussions, and we begin by deriving the equations for a totally geodesic hypersurface. We write (4.24) in the form

$$\dot{x}^\gamma(\ddot{x}^n + \Gamma^n_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta) = \dot{x}^n(\ddot{x}^\gamma + \Gamma^\gamma_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta)\,. \tag{4.35}$$

The condition that the family of hypersurfaces $\{x^n = \text{constant}\}$ all be totally geodesic may be expressed as follows: Suppose that

$$x(t) = x_0 + x_1 t + x_2 t^2 + \cdots$$

is a curve such that (4.35) is satisfied when t is set equal to zero. Equivalently, $x(t)$ is a path up to 3rd order around $t = 0$. Then the totally geodesic condition is

$$\dot{x}^n(0) = 0 \Rightarrow \ddot{x}^n(0) = 0\,.$$

This is the same as

$$\sum_{\lambda,\mu=1}^{n-1} \Gamma^n_{\lambda\mu}\xi^\lambda\xi^\mu = 0\,. \tag{4.36}$$

The single hypersurface $\{x^n = 0\}$ is totally geodesic in case (4.36) is satisfied at points $(x^1, \dots, x^{n-1}, 0)$.

In terms of the connection matrix $\{\theta^\alpha, \phi^\alpha_\beta, \theta^0_\beta\}$ for the canonical connection (4.25) associated to the path geometry we may restate (4.36) in the form:

The conditions that the differential system

$$\theta^n = 0$$

define a family of totally geodesic hypersurfaces for the path geometry (4.35) *are*

$$\text{(4.37)} \qquad \begin{cases} d\theta^n \equiv 0 \text{ modulo } \theta^n \\ \phi^n_\mu \equiv 0 \text{ modulo } \theta^n, \quad 1 \leqq \mu \leqq n-1 \end{cases}$$

Here is a sketch of an alternate method for arriving at (4.37). Suppose that $\theta^n = 0$ defines a codimension one foliation, and denote by $\bar{\psi}$ the restriction of a differential form ψ to the leaves of this foliation. Since the projective connection is torsion free

$$0 = d\bar{\theta}^n = \sum_{\mu=1}^{n-1} \bar{\theta}^\mu \wedge \bar{\psi}^n_\mu,$$

which by the Cartan lemma implies that

$$\bar{\phi}^n_\mu = \sum_{\nu=1}^{n-1} h_{\mu\nu}\theta^\nu, \qquad h_{\mu\nu} = h_{\nu\mu}.$$

The quadratic differential form

$$\text{II} = \sum_{\mu,\nu=1}^{n-1} h_{\mu\nu}\bar{\theta}^\mu\bar{\theta}^\nu$$

is the *second fundamental form* of the hypersurfaces $\theta^n = 0$, and just as in the Riemannian case one may show that the condition for being totally geodesic is

$$\text{II} = 0.$$

This is the same as

$$\bar{\phi}^n_\mu = 0 \Leftrightarrow \phi^n_\mu \equiv 0 \text{ modulo } \theta^n,$$

which is (4.37).

We shall find the implication of (4.37) on the projective curvature tensor. By (4.15), for $1 \leqq \mu \leqq n-1$

$$\begin{aligned} 0 &= \theta^n \wedge d\phi^n_\mu \\ &= \theta^n \wedge (\phi^\gamma_\mu \wedge \phi^n_\gamma + \theta^0_\mu \wedge \theta^n - \delta^n_\mu \theta^\gamma \wedge \theta^0_\gamma + \Phi^n_\mu) \\ &= \theta^n \wedge \Phi^n_\mu. \end{aligned}$$

Removing the distinguished role of the index n we obtain the conclusion:

If the hyperplane

$$\sum_\alpha u_\alpha \theta^\alpha = 0$$

is tangent to a totally geodesic hypersurface, then

$$\text{(4.38)}\qquad \begin{cases} \sum_{\alpha,\beta} v^\beta u_\alpha R^\alpha_{\beta\gamma\lambda}\theta^\gamma \wedge \theta^\lambda \equiv 0 \ \textit{modulo} \ \sum_\alpha u_\alpha \theta^\alpha \\ \textit{whenever} \ \sum_\beta v^\beta u_\beta = 0\,. \end{cases}$$

Indeed, the condition (4.38) is equivalent to

$$\Big(\sum_\alpha u_\alpha \theta^\alpha\Big) \wedge \Big(\sum_{\alpha,\beta} v^\beta u_\alpha R^\alpha_{\beta\gamma\lambda}\theta^\gamma \wedge \theta^\lambda\Big) = 0$$

whenever $\sum_\alpha v^\alpha u_\alpha = 0$.

As a preliminary to the proof of (4.34) we shall show that

(4.39) *If $n \geqq 3$ and if every plane $\sum_\alpha u_\alpha \theta^\alpha = 0$ is tangent to a totally geodesic hypersurface, then the path geometry is flat.*

Proof. Setting

$$\Phi^\alpha_\beta = 1/2\{R^\alpha_{\beta\gamma\lambda}\theta^\gamma \wedge \theta^\lambda\}\,,$$

we are assuming that

$$\text{(4.40)}\qquad v^\beta u_\alpha \Phi^\alpha_\beta \equiv 0(u_\alpha \theta^\alpha)$$

for all u_α, v^β subject to

$$\text{(4.41)}\qquad v^\alpha u_\alpha = 0\,.$$

The most obvious way (4.40) can hold is if

$$\text{(4.42)}\qquad \Phi^\alpha_\beta = \delta^\alpha_\beta \Psi + \eta_\beta \wedge \theta^\alpha$$

for a 2-form Ψ and 1-form η_β. We first show that

(4.43) **Lemma.** *If Φ^α_β has the form* (4.42), *then*

$$\Phi^\alpha_\beta = 0\,.$$

Proof. The curvature tensor satisfies

$$\begin{cases} R^\alpha_{\beta\gamma\lambda} = -R^\alpha_{\beta\lambda\gamma} \\ R^\alpha_{\beta\gamma\lambda} + R^\alpha_{\lambda\beta\gamma} + R^\alpha_{\gamma\lambda\beta} = 0 & (\textit{Bianchi}) \\ R^\alpha_{\beta\gamma\alpha} = 0 & (\textit{normality}) \end{cases}$$

As noted in (4.20) these imply

$$R^\alpha_{\alpha\gamma\lambda} = 0\,.$$

Set

$$\eta_\beta = S_{\beta\lambda}\theta^\lambda\,.$$

Since $\Phi^\alpha_\alpha = 0$,

$$n\Psi + S_{\beta\lambda}\theta^\lambda \wedge \theta^\beta = 0\,.$$

Substituting in (4.42)

$$\begin{aligned} 1/2\{R^\alpha_{\beta\gamma\lambda}\theta^\gamma \wedge \theta^\lambda\} &= -1/n(\delta^\alpha_\beta S_{\lambda\gamma}\theta^\gamma \wedge \theta^\lambda) + S_{\beta\lambda}\theta^\lambda \wedge \theta^\alpha \\ &= -1/n(\delta^\alpha_\beta S_{\lambda\gamma}\theta^\gamma \wedge \theta^\lambda) - \delta^\alpha_\gamma S_{\beta\lambda}\theta^\gamma \wedge \theta^\lambda \\ \Rightarrow R^\alpha_{\beta\gamma\lambda} &= -1/n(\delta^\alpha_\beta(S_{\lambda\gamma} - S_{\gamma\lambda})) - \delta^\alpha_\gamma S_{\beta\lambda} + \delta^\alpha_\lambda S_{\beta\gamma}, \end{aligned}$$

Now by normality

$$\begin{aligned} &R^\alpha_{\beta\alpha\lambda} = 0 \\ &\Rightarrow 0 = -1/n(S_{\lambda\beta} - S_{\beta\lambda}) - nS_{\beta\lambda} + S_{\beta\lambda} \\ &\Rightarrow (-n^2 + n + 1)S_{\beta\lambda} = S_{\lambda\beta} \\ &\qquad\qquad = (1/(-n^2 + n + 1))S_{\beta\lambda}\,. \end{aligned}$$

When $n = 2$, this gives

$$-S_{\beta\lambda} = -S_{\beta\lambda},$$

but for $n \geqq 3$ we obtain

$$\begin{aligned} &S_{\beta\lambda} = 0 \\ \Rightarrow\ &R^\alpha_{\beta\gamma\lambda} = 0\,. \end{aligned}$$

Q.E.D.

(4.44) **Lemma.** *If* (4.40) *holds for all* (u_α), (v^β) *subject to* (4.41), *then* Φ^α_β *has the form* (4.42).

Proof. Fix β and let

$$\begin{aligned} v &= (0,\ldots,\overset{\beta}{1},\ldots,0) \\ u &= (u_1,\ldots,u_{\beta-1},0,u_{\beta+1},\ldots,u_n)\,. \end{aligned}$$

Then (4.41) is automatically satisfied and we claim that:

$$\begin{aligned} &\sum_{\alpha\neq\beta} u_\alpha \Phi^\alpha_\beta \equiv 0\Big(\sum_{\alpha\neq\beta} u_\alpha\theta^\alpha\Big) \\ &\Rightarrow \sum_{\alpha\neq\beta} u_\alpha \Phi^\alpha_\beta = \eta_\beta \wedge \Big(\sum_{\alpha\neq\beta} u_\alpha\theta^\alpha\Big) \end{aligned} \tag{4.45}$$

where η_β *is independent of* (u_α).

Assuming $\beta \neq 1$, $u_1 \neq 0$, we take a new basis

$$\bar\theta^1 = \sum_{\alpha\neq\beta} u_\alpha\theta^\alpha,\quad \bar\theta^2 = \theta^2,\ldots,\ \bar\theta^n = \theta^n\,.$$

Then $\Big(\sum_{\alpha\neq\beta} u_\alpha\Phi^\alpha_\beta\Big) \wedge \bar\theta^1 = 0$ gives

$$\sum_{\alpha\neq\beta} u_\alpha\Phi^\alpha_\beta = \bar\theta^1 \wedge (\varrho_{2\beta}(u)\bar\theta^2 + \cdots + \varrho_{n\beta}(u)\bar\theta^n)\,.$$

The coefficients $\varrho_{\nu\beta}(u)$ are homogeneous of degree 0 in u_α. So (4.45) follows by taking

$$\eta_\beta = -\sum_{\nu=2}^{n} \varrho_{\nu\beta}(0)\theta^\nu,$$

independent of u_α.

Thus we have

$$\Phi_\beta^\alpha = \eta_\beta \wedge \theta^\alpha \quad (\alpha \neq \beta)$$

which gives (4.42) for $\alpha \neq \beta$. In general the left hand side of (4.40) is

$$\begin{aligned}\sum_{\alpha,\beta} v^\beta u_\alpha \Phi_\beta^\alpha &= \sum_{\beta\neq\alpha} v^\beta u_\alpha \Phi_\beta^\alpha + \sum v^\alpha u_\alpha \Phi_\alpha^\alpha \\ &= \sum_{\beta,\alpha} v^\beta u_\alpha \eta_\beta \wedge \theta^\alpha + \sum_\alpha v^\alpha u_\alpha(\Phi_\alpha^\alpha - \eta_\alpha \wedge \theta^\alpha).\end{aligned}$$

Assuming (4.41) and setting $u_n = -1$

$$\begin{aligned}&\sum_{\alpha=1}^{n} v^\alpha u_\alpha(\Phi_\alpha^\alpha - \eta_\alpha \wedge \theta^\alpha) \\ &\quad = \sum_{k=1}^{n-1} v^k u_k(\Phi_k^k - \eta_k \wedge \theta^k - \Phi_n^n + \eta_n \wedge \theta^n) \\ &\quad \equiv 0(u_\alpha \theta^\alpha).\end{aligned}$$

This is true for all $v^1,\dots,v^{n-1}$. Consequently

$$u_k(\Phi_k^k - \eta_k \wedge \theta^k - \Phi_n^n + \eta_n \wedge \theta^n) \equiv 0(u_\alpha\theta^\alpha)$$

for each k and all $u_1,\dots,u_{n-1}$. This is possible only if

$$\Phi_k^k - \eta_k \wedge \theta^k = \Phi_n^n - \eta_n \wedge \theta^n = \Psi$$

for each k, in which case

$$\Phi_\beta^\alpha = \delta_\beta^\alpha \Psi + \eta_\beta \wedge \theta^\alpha$$

as desired. Q.E.D.

As an application, if we combine what we just proved with the Beltrami theorem we obtain the

Corollary. *If a Riemannian manifold M_n of dimension $n \geqq 3$ has ∞^{n-1} totally geodesic hypersurfaces passing through each point, then it has constant sectional curvature.*

We shall now establish (4.34) by using lemma (4.43) and by proving

(4.46) **Lemma.** *Suppose that*

$$v^\beta t^\alpha \Phi_\beta^\alpha \equiv 0(t^\alpha \theta^\alpha)$$

for all $(v^\beta), t$ satisfying

$$v^\beta t^\beta = 0. \tag{4.47}$$

Then the representation (4.42) *is valid*[33]).

Proof. The argument is by induction on n, treating (Φ^α_β) as an $(n \times n)$ matrix of 2-forms. We shall denote by $\bar{\psi}$ the restriction of a form ψ to the hyperplane $\theta^n = 0$, and shall use the additional index range $1 \leqq i,j,k \leqq n-1$. Letting $t \to \infty$ we have

$$v^i \bar{\Phi}^n_i = 0\,. \tag{4.48}$$

What this means is the following: By hypothesis

$$v^\beta t^\alpha \Phi^\alpha_\beta = 0$$

in the tangent hyperplane

$$t^\alpha \theta^\alpha = 0$$

whenever

$$t^\alpha v^\alpha = 0\,.$$

The tangent hyperplane is the same as that defined by

$$\theta^1/t^{n-1} + \theta^2/t^{n-2} + \cdots + \theta^{n-1}/t + \theta^n = 0\,,$$

which tends to $\theta^n = 0$ as $t \to \infty$. Similarly the linear condition on the (v^α) tends to $v^n = 0$ as $t \to \infty$. Dividing the relations through by t^n gives

$$v^\beta \Phi^\alpha_\beta/t^{n-\alpha} = 0 \quad \text{in} \quad \theta^\alpha/t^{n-\alpha} = 0$$

whenever

$$v^\alpha/t^{n-\alpha} = 0\,.$$

Letting $t \to \infty$ the (v^α) tend to any preassigned vector (v^i), so that in the limit

$$v^i \Phi^n_i = 0 \quad \text{in} \quad \theta^n = 0$$

for all (v^i) [34]). Thus

$$\bar{\Phi}^n_i = 0\,. \tag{4.49}$$

Keeping $v^n = 0$, we deduce from (4.49) that

$$v^i t^j \bar{\Phi}^j_i \equiv 0(t^j \bar{\theta}^j)$$

whenever

$$t^i v^i = 0\,.$$

By induction hypothesis

$$\bar{\Phi}^i_k = \delta^i_k \bar{\Psi}' + \eta'_k \wedge \bar{\theta}^i\,,$$

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[33]) Here, t^α is the variable t raised to the α^{th} power.

[34]) Geometrically, $v^\alpha t^\alpha = 0$ means that $(v^\alpha) \in P^{*n-1}$ is in the hyperplane $t^\perp$ orthogonal to $(t^\alpha) \in P^{n-1}$. As $t \to \infty$ this hyperplane tends to $v^n = 0$, so that in the limit $(v^i) \in P^{*n-2}$ is arbitrary.

which implies

$$\begin{cases} \Phi^i_k = \delta^i_k \bar{\Psi}' + \eta'_k \wedge \theta^i + \theta^n \wedge \phi^i_k \\ \Phi^n_i = \eta''_i \wedge \theta^n \end{cases} \tag{4.50}$$

We now return to the full n-dimensional space. Assuming (4.47) and using (4.50),

$$\begin{aligned} v^\beta t^\alpha \Phi^\alpha_\beta &= v^k t^i (\delta^i_k \bar{\Psi}' + \eta'_k \wedge \theta^i + \theta^n \wedge \phi^i_k) \\ &\quad + v^n t^i \Phi^i_n + v^k t^n \eta''_k \wedge \theta^n + v^n t^n \Phi^n_n \\ &\equiv v^n t^n (\Phi^n_n - \Psi') + v^k t^n ((\eta''_k - \eta'_k) \wedge \theta^n) \\ &\quad + \theta^n \wedge v^k t^i \phi^i_k + v^n t^i \Phi^i_n \\ &\equiv 0(t^\alpha \theta^\alpha)\,. \end{aligned}$$

Setting $v^n = 0$ we obtain

$$t^n (v^k (\eta''_k - \eta_k)) \wedge \theta^n + t^i v^k \theta^n \wedge \phi^i_k \equiv 0(t^\alpha \theta^\alpha)$$

whenever $t^i v^i = 0$. This implies that

$$t^n v^k (\eta''_k - \eta_k) + t^i v^k \phi^i_k \equiv 0(t^i \theta^i, \theta^n) \tag{4.51}$$

whenever $t^i v^i = 0$. Letting $t \to \infty$ we conclude as before that

$$\bar{\eta}''_k = \bar{\eta}_k\,.$$

We may choose $\eta''_k = \eta'_k$ in (4.50). Then by (4.51)

$$t^i v^k \bar{\phi}^i_k \equiv 0(t^i \bar{\theta}^i)$$

whenever $t^i u_i = 0$. This is a situation similar to our induction hypothesis, only simpler in that ϕ^i_k is a matrix of 1-forms. We may assume then that

$$\begin{aligned} &\bar{\phi}^i_k = \delta^i_k \bar{\Lambda} + b_k \bar{\theta}^i \\ \Rightarrow\ &\Phi^i_k = \delta^i_k \Psi + \eta_k \wedge \theta^i \end{aligned}$$

where

$$\begin{cases} \Psi = \Psi' + \theta^n \wedge \Lambda \\ \eta_k = \eta''_k + b_k \theta^n\,. \end{cases}$$

Summarizing, we have obtained

$$\begin{cases} \Phi^i_k = \delta^i_k \Psi + \eta_k \wedge \theta^i \\ \Phi^n_k = \eta_k \wedge \theta^n\,. \end{cases} \tag{4.52}$$

Now then, assuming (4.47) and using (4.52)

$$\begin{aligned} v^\beta t^\alpha \Phi^\alpha_\beta &= v^k t^\alpha (\delta^\alpha_k \Psi + \eta_k \wedge \theta^\alpha) + v^n t^\alpha \Phi^\alpha_n \\ &\equiv 0(t^\alpha \theta^\alpha)\,. \end{aligned}$$

This gives

$$-v^n t^n \Psi - v^k t^n \eta_k \wedge \theta^n + v^n t^\alpha \Phi_n^\alpha \equiv 0(t^\alpha \theta^\alpha) \tag{4.53}$$

which implies

$$-t^n \Psi + t^\alpha \Phi_n^\alpha \equiv 0(t^\alpha \theta^\alpha).$$

Letting $t \to \infty$ gives

$$\bar{\Psi} = \bar{\Phi}_n^n$$

so

$$\Phi_n^n = \Psi - \eta_n \wedge \theta^n.$$

By (4.53)

$$-v^\alpha t^n \eta_\alpha \wedge \theta^n + v^n t^k \Phi_n^k \equiv 0(t^\alpha \theta^\alpha).$$

This implies that

$$\Phi_n^k = \eta_n \wedge \theta^k,$$

completing the induction step. Q.E.D.

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