ALGEBRAIC CYCLES AND SINGULARITIES OF NORMAL FUNCTIONS, II

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Abstract. In our previous paper [14], denoted by "I", and whose notations we shall follow here, we proposed a definition of extended normal functions (ENF), and for an ENF ν we defined its singular locus $\operatorname{sing}\nu$. There is a reciprocal relationship between primitive Hodge classes $\zeta \in Hg^n(X)_{\text{prim}}$ and the corresponding ENF ν_ζ. Moreover, algebraic cycles Z with $[Z] = \zeta$ give rise to singularities of ν_ζ and vice versa.

In this paper we shall discuss universally defined rational maps ρ to partially compactified classifying spaces for polarized Hodge structures of odd weight and paritally compactified universal families of intermediate Jacobians over these spaces. These maps have the property that $\operatorname{sing}\nu_\zeta$ appears as a component of the inverse image of certain boundary components $\mathcal{B}$; i.e.,

$$\operatorname{sing}\nu_\zeta \subseteq \rho^{-1}(\mathcal{B}) \ .$$

Since the Hodge conjecture (HC) is equivalent to

$$\operatorname{sing}\nu_\zeta \neq \emptyset \ \text{ for } \ L \gg 0 \ ,$$

and since for the fundamental class $[\mathcal{B}]$ of $\mathcal{B}$

$$\rho^*([\mathcal{B}]) \neq 0 \Rightarrow \rho^{-1}(\mathcal{B}) \neq \emptyset \ ,$$

a natural question is to investigate $\rho^*([\mathcal{B}])$. We will see that this leads to attempting to understand $\dim(\operatorname{sing}\nu_\zeta)$, and this in turn leads to some to us very beautiful interplay between algebraic geometry and Hodge theory "at the boundary." Among other things, we find that there is transversality for $L \gg 0$ in the classical case (Lefschetz $(1,1)$ theorem), but assuming the HC there cannot be transversality in the higher codimensional situation. In part this lack of transversality is Hodge-theoretic, due to the infinitesimal period relation and with a very nice geometric interpretation. Additionally there is a non-zero algebro-geometric contribution to non-transversality, a contribution that we do not yet understand geometrically.

Outline

I. INTRODUCTION

I(i). *Notations*

This is a continuation of [14], referred to below as "I". We shall retain the notations introduced there, and which will be recalled as needed. In addition we shall use the following notation:

- $\mathcal{M}_g$ is the moduli space of stable curves of genus g and $\overline{\mathcal{M}}_g$ is the Deligne-Mumford compactification.
- $\mathcal{C}_g \to \mathcal{M}_g$ is the universal family of genus g curves with compactification $\overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$, with the usual understanding that these spaces must be interpreted as stacks.
- $\mathcal{A}_{\mathbf{h}}$ is the classifying space for polarized Hodge structures of weight $2n-1$ and with given Hodge numbers $\mathbf{h} = (h^{2n-1,0}, \dots, h^{n,n-1})$. When $n = 1$ and $h^{1,0} = g$, we assume that the polarization is principal and use the customary notation $\mathcal{A}_g$.

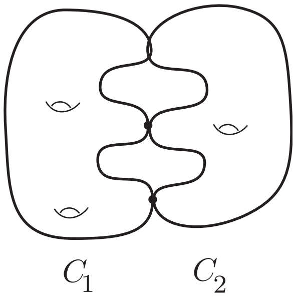
- $\mathcal{J}_{\mathbf{h}} \to \mathcal{A}_{\mathbf{h}}$ is the associated universal family of polarized complex tori, again with the stack interpretation. When $n = 1$ we shall write $\mathcal{J}_g \to \mathcal{A}_g$.
- For a choice of fan Σ, $\overline{\mathcal{A}}_{\mathbf{h},\Sigma}$ will denote the corresponding partial compactification (cf. Kato-Usui [17] and for $n = 1$ cf. Carlslon-Cattani-Kaplan [4], Cattani [5], Alexeev-Nakamura [2], Alexeev [1] and the references cited there; cf. also the note at the end of (I(i)).
- For $\mathbf{g} = (g_1, g_2)$ where

$$g = g_1 + g_2 + l - 1 \qquad\qquad (l \geqq 1)$$

  we shall denote by $\Sigma_{\mathbf{g}} \subset \overline{\mathcal{M}}_g$ the boundary component of reducible curves

$$C = C_1 \cup C_2$$

  where $g_i = g(C_i)$ and $l = \#(C_1 \cap C_2)$



- For a rational nilpotent cone $\sigma \in \Sigma$ we shall denote by

$$\mathcal{B}_\sigma \subset \overline{\mathcal{A}}_{\mathbf{h},\Sigma}$$

  the corresponding boundary component;
- We shall anticipate further work in progress by Kato-Usui and denote by

$$\overline{\mathcal{J}}_{\mathbf{h},\Sigma} \to \overline{\mathcal{A}}_{\mathbf{h},\Sigma}$$

  a partial compactification of the universal family $\mathcal{J}_{\mathbf{h}} \to \mathcal{A}_{\mathbf{h}}$ of polarized complex tori (cf. the note below).

- Although the boundary component structure of $\overline{\mathcal{J}}_{\mathbf{h},\Sigma}$ is, at this time, work in progress there will be particular boundary components $\Gamma_{d,\sigma}$ of geometric interest and which will be described below.
- We shall denote by $\mathrm{Pic}^\circ(\mathcal{C}_g) \to \mathcal{M}_g$ the universal family of Jacobians and by
$$\overline{\mathrm{Pic}^\circ(\mathcal{C}_g)} \to \overline{\mathcal{M}}_g$$
a compactification, also to be described below (cf. Caparosa [3]).
- Given (X, L) where X is a smooth projective variety of dimension $2n$ and $L \to X$ is a very ample line bundle, we set $|L| = \mathbb{P}H^0(\mathcal{O}_X(L))$ and shall denote by
$$\begin{tikzcd} \mathcal{X}_L \subset X \times |L| \arrow[d] \\ |L| \end{tikzcd}$$
the universal family $\{X_s\}_{s\in|L|}$ of hyperplane sections relative to the embedding $X \hookrightarrow \mathbb{P}\check{H}^0(\mathcal{O}_X(L))$.
- For (X, L) as above, we shall denote by $\mathcal{M}_X$ the moduli space, assumed to exist, and by $\mathcal{X}_\mathcal{M} \to \mathcal{M}_X$ the corresponding family $\{X_t, L_t\}_{t\in\mathcal{M}_X}$ of polarized varieties.
- Finally, we set
$$\mathcal{L} = \bigcup_{t\in\mathcal{M}_X} |L_t|$$
and denote by
$$\begin{tikzcd} \mathcal{X}_\mathcal{L} \subset \mathcal{X}_\mathcal{M} \times |\mathcal{L}| \arrow[d] \\ |\mathcal{L}| \end{tikzcd}$$
the family of hyperplane sections of all the X_t, $t \in \mathcal{M}_X$.

We shall always assume that $L \to X$ has been chosen sufficiently ample so as to have properties such as

$$h^q(\mathcal{O}_{X_t}(L_t)) = 0, \qquad q > 0$$

and projective normality of the embedding given by $|L_t|$.

Note: The paper by Alexeev gives a canonical compactification $\overline{\mathcal{A}}_g$ of $\mathcal{A}_g$, one not requiring an a priori choice of a fan. Moreover, Alexeev's $\overline{\mathcal{A}}_g$ coincides with the compactification associated to the second Voronoi fan. Now a fan Σ is composed of rational nilpotent cones σ in $\mathcal{G}_{\mathbb{Q}}$ satisfying certain conditions, and in this work the essential point is that Σ contain those σ's that arise from several variable degenerations of Hodge structures in the families that we shall be considering. The reason is that certain of these cones gives rise to the boundary components $\mathcal{B}(\sigma)$ that correspond to singularities of extended normal functions. For this reason we shall in this paper work with $\overline{\mathcal{A}}_{g,\Sigma}$. At a later time we hope to more directly relate this work to Alexeev's $\overline{\mathcal{A}}_g$.

Of course, a natural question is whether for the odd weight principally polarized case the Kato-Usui work may be extended to give a *canonical* partial compactification $\overline{\mathcal{A}}_{\mathbf{h}}$, together with a partially compactified universal family $\overline{\mathcal{J}}_h \to \overline{\mathcal{A}}_{\mathbf{h}}$ for which the Alexeev-Nakamura result [2] holds.

As noted above, in this paper we will anticipate some consequences of the work [17] by Kato-Usui on partial compactifications $\overline{\mathcal{A}}_{\mathbf{h},\Sigma}$ of classifying spaces for polarized Hodge structure of odd weight. Boundary components $\mathcal{B}(\sigma(\mathbf{N}))$ of $\overline{\mathcal{A}}_{\mathbf{h},\Sigma}$ are constructed from certain rational cones $\sigma(\mathbf{N})$ associated to commuting nilpotent elements $N_1, \dots, N_l$ of $\mathcal{G}_{\mathbb{Q}}$. The boundary component $\mathcal{B}(\sigma(l,1))$ corresponds to the particular situation (4) in section I(ii) below. In general one may imagine a local system

$$\mathcal{H}(\mathcal{I}) \to \overline{\mathcal{A}}_{\mathbf{h},\Sigma} \tag{1}$$

given over the interior of $\mathcal{B}(\sigma(\mathbf{N}))$ by the 1st cohomology group $H^1(B^\bullet(\mathbf{N}))$ where $B^\bullet(\mathbf{N})$ is the Koszul-like complex constructed from the N_i (cf. section I.C in "I"), modulo the action of the discrete "monodromy group" associated to $\mathcal{B}(\sigma(\mathbf{N}))$. We feel that the study of the local system (1) should be interesting and could be of importance for the study of cycles.

I(ii). *Informal statements of results*

In this paper we shall study the geometry and topology of a number of rational maps arising from the linear systems $|L|$ (X fixed), $|\mathcal{L}|$ (X variable), and $\mathcal{X}_{\mathcal{L}}$. We are especially interested in these maps near a point s_0 where

(2) *X_{s_0} has ordinary double points (nodes) p_i with one relation among them.*

The motivation for studying this situation is explained in "I"; cf. especially section 4.2.4. Briefly, given a primitive Hodge class $\zeta \in Hg^n(X)_{\mathrm{prim}}$ and assuming the Hodge conjecture (HC), we will have for L sufficiently ample (written $L \gg 0$) and some non-zero integer k_0

$$k_0\zeta = [W - H], \tag{3}$$

where W is a smooth n-dimensional subvariety and H is a complete intersection. Moreover, a general

$$X_{s_0} \in |\mathcal{I}_W(L)| \qquad (\mathcal{I}_W(L) = \mathcal{O}_X(L) \underset{\mathcal{O}_X}{\otimes} \mathcal{I}_W)$$

will have the property (1) (cf. loc. cit. and section IV(i) below for various constructions of W). There are several equivalent formulations of the condition that there be one relation among the nodes. For us the one we shall use is

(4) *For X_s smooth with s close to s_0, let $\delta_i \in H_{2n-1}(X_s, \mathbb{Z})$ be the vanishing cycle, defined up to ± 1, associated to p_i. Then there is one generating relation*

$$\sum_{i=1}^{l} \delta_i \sim 0 \ .$$

If ν_ζ is the extended normal function (ENF) associated to ζ with singular locus $\operatorname{sing}\nu_\zeta$ (loc. cit.), then (3) implies that

$$s_0 \in \operatorname{sing}\nu_\zeta \ . \tag{5}$$

The basic observation will be the following result:

> *There are rational maps, a general one of which we denote here by ρ, to compactified moduli/classifying spaces, a general one of which we again here denote by $\overline{\mathcal{P}}$, such that near s_0 and with Δ denoting a suitable boundary compact of $\overline{\mathcal{P}}$ we have*

$$\rho^{-1}(\Delta) = \operatorname{sing} \nu_\zeta \ . \tag{6}$$

Some of these maps depend on ζ and some do not. Those which do not will give

$$|L| \overset{\rho}{\dashrightarrow} \overline{\mathcal{P}} \qquad (X \text{ fixed})$$

and

$$|\mathcal{L}| \overset{\boldsymbol{\rho}}{\dashrightarrow} \overline{\mathcal{P}} \qquad (X \text{ variable}) \ .$$

In general, we will use boldface notation to signal that X is allowed to vary. If $\rho = \rho_\zeta$ does depend on ζ, then denoting by

$$\mathcal{M}_{X,\zeta} \subset \mathcal{M}_X$$

the *Noether-Lefschetz locus* where ζ remains a Hodge class (this is an algebraic subvariety by Cattani-Deligne-Kaplan [7]) and by $|\mathcal{L}|_\zeta$ the subvariety of $|\mathcal{L}|$ lying over $\mathcal{M}_{X,\zeta}$, we will have again near s_0

$$\boldsymbol{\rho}_\zeta^{-1}(\Delta) = \operatorname{sing} \boldsymbol{\nu}_\zeta \ . \tag{7}$$

Thus, restricting to the nodal locus

$$\begin{cases} \text{(i)} & \rho^{-1}(\Delta) \neq \emptyset \Rightarrow \operatorname{sing} \nu_\zeta \neq \emptyset \\ \text{(ii)} & \boldsymbol{\rho}^{-1}(\Delta) \neq \emptyset \Rightarrow \operatorname{sing} \boldsymbol{\nu}_\zeta \neq \emptyset \end{cases} \tag{8}$$

and either of these gives, by "I", the existence of an algebraic cycles Z with

$$\langle \zeta, [Z] \rangle \neq 0 \ ,$$

where in (ii) the cycles exist on the non-empty set of $X_t \in \mathcal{M}_X$ where

$$|L_t| \cap \boldsymbol{\rho}^{-1}(\Delta) \neq \emptyset \ .$$

Next we assume the "generally true" statements

$$\begin{cases} \rho^*([\Delta]) \neq 0 \Rightarrow \rho^{-1}(\Delta) \neq \emptyset \\ \boldsymbol{\rho}^*([\Delta]) \neq 0 \Rightarrow \boldsymbol{\rho}^{-1}(\Delta) \neq \emptyset \end{cases} \tag{9}$$

where here we are leaving aside the technical issues dealing with the non-compactness of $\mathcal{M}_X$. The point here is that

> *A topological result gives an existence result*,

here in the context of the induced map on cohomology of maps to certain universal spaces. This leads to the central question of this paper:

> *Assuming the Hodge conjecture, for at least some of the maps* ρ *do we have*

$$\begin{cases} \text{(i)} \qquad \rho^*([\Delta]) \neq 0 \\ \text{(ii)} \qquad \boldsymbol{\rho}^*_\zeta([\Delta]) \neq 0 \; . \end{cases}$$

We shall see that this question leads to some to us very beautiful and unexpected geometry. Our results are very preliminary and may be informally summarized as follows:

(10) *In the case* $n = 1$ *when* X *is an algebraic surface, if* ρ *does not depend on* ζ *we have for essentially trivial reasons that* $\rho^*([\Delta]) = 0$*, but for* $L \gg 0$ *we have with certain technical assumptions that*

$$\boldsymbol{\rho}^*_\zeta([\Delta]) \neq 0 \; .$$

We expect that these technical assumptions, the main one of which is

$$\operatorname{codim}_{\mathcal{M}_X}(\mathcal{M}_{X,\zeta}) = h^{2,0}(X) \; , \tag{11}$$

are not necessary.

Underlying the analysis of $\rho, \boldsymbol{\rho}$ and $\boldsymbol{\rho}_\zeta$ are transversality results. In the $n = 1$ case we shall prove that these always hold along components of $\operatorname{sing} \nu_\zeta$ and $\operatorname{sing} \boldsymbol{\nu}_\zeta$ whose general point corresponds to a nodal curve with one generating relation among the vanishing cycles (we may think of these as "principal components"). We shall also show that this transversality is not always true for the maps that do not depend on ζ, but is true for those which do provided that $L \gg 0$.

When we turn to the higher codimensional case, which in this paper will mean the case $n = 2$ where we are studying codimension 2 cycles in a fourfold, our principal finding is

(12) *the situation for $n = 2$ is completely different from the classical case.*

Some of what turns up is not unexpected. For example, concerning the Noether-Lefschetz locus corresponding to a Hodge class $\zeta \in Hg^2(X)_{\text{prim}}$ we find that we have (cf. section IV(iii))

$$\operatorname{codim}_{\mathcal{M}_X}(\mathcal{M}_{X,\zeta}) \leqq h^{3,1}(X) - \sigma_\zeta \tag{13}$$

where

$$\begin{cases} T_\zeta = \{\theta \in H^1(\Theta_X) : \theta \cdot \zeta = 0 \text{ in } H^3(\Omega^1_X)\} \\ \sigma_\zeta = \dim(\text{Image}\{H^0(\Omega^4_X) \otimes T_\zeta \to H^1(\Omega^3_X)\}. \end{cases} \tag{14}$$

At first glance one has

$$\operatorname{codim}_{\mathcal{M}_X}(\mathcal{M}_{X,\zeta}) \leqq h^{4,0}(X) + h^{3,1}(X)$$

since the RHS is the apparent number of conditions for a class $\zeta \in H^4(X, \mathbb{Q})$ to be of Hodge type $(2, 2)$. The refinement (13) is a consequence of the infinitesimal period relations and their integrability conditions.

As will be seen in section IV(iv) the inequality (13) is an equality in significant examples.

Somewhat more subtle, although not unexpected when one thinks about it, is that again in contrast to the $n = 1$ case

(15) *even assuming the Hodge conjecture, there are torsion and "layers of Chow group" obstructions to deforming a subvariety W along with (X, ζ) where $[W]_{\text{prim}} = \zeta$, no matter how "ample" one makes W*

(cf. section IV(ii)).

Perhaps most unexpected is the following: In the $n = 1$ case for $L \gg 0$ we have

$$\begin{cases} \text{(i)} \quad \operatorname{codim}_{|L|}(\operatorname{sing} \nu_\zeta)_{s_0} = l - h^{2,0} & (X \text{ fixed}) \\ \text{(ii)} \quad \operatorname{codim}_{|\mathcal{L}|}(\operatorname{sing} \boldsymbol{\nu}_\zeta)_{s_0} = l & (X \text{ variable}) \end{cases} \tag{16}$$

where in the second statement we have assumed (11). This statement has the implication (cf. section III(ii))

(17) *the rational mapping*

$$|\mathcal{L}| \overset{\Psi}{-\!\!-\!\!\rightarrow} \overline{\mathcal{A}}_{g,\Sigma}$$

obtained sending a smooth curve to its Jacobian is transverse along the boundary component $\mathcal{B}(\sigma(l,1))$ *corresponding to rational nilpotent cone generated by the logarithms of Picard-Lefschetz transformations corresponding to cycles* δ_i *satisfying (4).*

Here we have (6) above where $\rho = \Psi$ and $\Delta = \mathcal{B}(\sigma(l,1))$, which using

$$\operatorname{codim}_{\overline{\mathcal{J}}_{g,\Sigma}}(\mathcal{B}(\sigma,1)) = l \tag{18}$$

leads to the conclusion (9) in this case (cf. (10)).

In contrast, for $n = 2$ we have, still assuming the Hodge conjecture,

$$\begin{cases} \text{(i)} \quad \operatorname{codim}_{|L|}(\operatorname{sing}\nu_\zeta)_{s_0} \leqq l - (h^{3,1} - \sigma_\zeta) - \delta_W & (X \text{ fixed}) \\ \text{(ii)} \quad \operatorname{codim}_{|\mathcal{L}|}(\operatorname{sing}\boldsymbol{\nu}_\zeta)_{s_0} \leqq l - \delta_W & (X \text{ variable}) \end{cases} \tag{19}$$

where

$$\delta_W > 0 \tag{20}$$

is an algebro-geometric correction term depending on the particular subvariety W with $W \subset X_{s_0}$ and where δ_W becomes large as $L \gg 0$. This will imply that, assuming the Hodge conjecture,

(21) *the rational mapping*

$$|\mathcal{L}| \overset{\Psi}{-\!\!-\!\!\rightarrow} \overline{\mathcal{A}}_{\mathbf{h},\Sigma}$$

obtained by sending a smooth threefold to its intermediate Jacobian will, for $L \gg 0$, *never be* I*-transverse to the boundary component* $\mathcal{B}(\sigma(l,1))$.

Here the concept of I-transversality will be explained; for reasons arising from the infinitesimal period relation it is the only form of transversality to have meaning for higher weight Hodge structures. That the Hodge conjecture implies a *generic non-transversality* result (cf. section V(ii)) stands in contrast to what one might naively expect and was to us a striking and thus far non-understood phenomenon.

The result and (21) depends on what to us is a very beautiful new phenomenon. Namely, in the $n = 2$ case we will show that (cf. section V(i))

$$\operatorname{codim}_{\overline{\mathcal{A}}_{\mathbf{h},\Sigma}}(\mathcal{B}(\sigma(l,1))) = l + h^{3,0}(l-1)\ . \tag{22}$$

For a map such as

$$|\mathcal{L}| \overset{\Psi}{\dashrightarrow} \overline{\mathcal{A}}_{\mathbf{h},\Sigma}$$

that is regular near a point s_0, restricting to a neighborhood of s_0 one may conclude from (22) that

$$\operatorname{codim}_{|\mathcal{L}|}\left(\Psi^{-1}(\mathcal{B}(\sigma(l,1))\right) \leqq l + h^{3,0}(l-1)\ . \tag{23}$$

Eliminating the $h^{3,0}(l-1)$ results from the infinitesimal period relation in the boundary as a special case of the following observation: Given a diagram of maps of smooth varieties

$$\begin{tikzcd} A \arrow[r, "F"] & M \\ \cup & \cup \\ B \arrow[r] & N \end{tikzcd}$$

where

$$F^{-1}(N) = B$$

then

$$\operatorname{codim}_A(B) \leqq \operatorname{codim}_M(N)\ . \tag{24}$$

Under suitable assumptions on the differential F_* this may be rewritten as

$$\operatorname{codim}_A(B) \leqq \operatorname{rank}\ (TM/TN)\ . \tag{25}$$

Now suppose there is a sub-bundle

$$I \subset TM$$

such that $F_* : TA \to I \subset TM$ and I meets TN transversely. Then (25) may be refined to

$$\operatorname{codim}_A(B) \leqq \operatorname{rank}\ (I/I \cap TN)\ . \tag{26}$$

When applied to

$$\begin{tikzcd}
|\mathcal{L}| \arrow[r, dashed, "\Psi"] & \overline{\mathcal{A}}_{\mathbf{h},\Sigma} \\
\cup & \cup \\
|\mathcal{L}|_\zeta \arrow[r, dashed] & \mathcal{B}(\sigma(l,1))
\end{tikzcd}$$

where $|\mathcal{L}|_\zeta$ is the subvariety of $|\mathcal{L}|$ lying over $\mathcal{M}_{X,\zeta}$ and I is the infinitesimal period relation, this leads to the elimination of the $h^{3,0}(l-1)$ term in (23). We shall refer to this phenomenon as *I-transversality.*

The result (10) follows from an excess intersection formula applied to the mapping in (17), which gives

$$\Psi_\zeta^*\left([\mathcal{B}(\sigma(l,1))]\right) = \left[\Psi_\zeta^{-1}(\mathcal{B}(\sigma(l,1)))\right] \wedge c_{\mathrm{top}}(\mathcal{H}^{0,2}) \tag{27}$$

where Ψ_ζ is the restriction of Ψ to $|\mathcal{L}|_\zeta$. By the Lefschetz $(1,1)$ theorem

$$\Psi_\zeta^{-1}(\mathcal{B}(\sigma(l,1))) \neq \emptyset$$

and this leads to (10) in this case.

When $n = 2$ the formula (27) holds where the "correction term" on the far right is replaced by $c_{\mathrm{top}}(V)$ where V is a vector bundle that has three components:

(i) a bundle of rank $h^{3,1} - \sigma_\zeta$ that arises for the same reason as $\mathcal{H}^{0,2}$ in the $n = 1$ case;

(ii) a bundle of rank $h^{3,0}(l-1)$ that arises from the infinitesimal period relation in the boundary of $\overline{\mathcal{A}}_{\mathbf{h},\Sigma}$; and

(iii) a bundle of rank δ_W.

The situations (i) and (ii) are Hodge-theoretic and do not depend on the particular W, provided that a certain technical condition that holds in many examples is satisfied. The situation (iii) is algebro-geometric and represents a phenomenon that is a consequence of the HC and does not appear in the classical $n = 1$ case.

All of the dimension count phenomena described just now are quite visible and explicit in the case of the example

$$\Lambda \subset X \subset \mathbb{P}^5$$

where Λ is a $\mathbb{P}^2$, X is a smooth fourfold of degree $d \geqq 6$, and

$$\zeta = [\Lambda]_{\mathrm{prim}} \; .$$

The calculations have the by now familiar polynomial flavor, with there being a few new subtleties (cf. section IV(iv)).

To conclude this introduction we remark that to us one of the more surprising and gratifying aspects of this study turned out to be the very elegant interplay between Hodge theory and geometry near and at the boundary of the partially compactified classifying spaces for Hodge structures. As is well known, in algebraic geometry a major tool is to study the degenerations of an algebraic variety, these degenerations frequently being "simpler" than the smooth variety. What we had not expected was the nice geometry that arises when these degenerations are viewed as pullbacks of the aforementioned boundary components.

II. The classical case of curves on a surface, Part A

II(i). *Maps to moduli when X is fixed*

Let (X, L, ζ) be as in the introduction where

- X is a smooth algebraic surface and $L \to X$ is a very ample line bundle;
- $\zeta \in Hg^1(X)_{\text{prim}}$ is a primitive Hodge class.

We shall write

$$L_m = mH$$

where H is a fixed ample line bundle and the positive integer m varies. Then for $m_1 \gg 0$ we may find a smooth curve

$$C_1 \in |\zeta + m_1 H| \ ,$$

and for $m = m_1 + m_2$ with $m_2 \gg 0$ we may find a smooth curve

$$C_2 \in | -\zeta + m_2 H|$$

and where

$$X_{s_0} = C_1 \cup C_2 \in |L_m|$$

has ordinary nodes so that C_1, C_2 meet transversely in

$$l = \#(C_1 \cap C_2) = -\zeta^2 + m_1 m_2 h, \qquad h = H^2 \tag{1}$$

points. Denoting by ν_ζ the extended normal function associated to ζ (cf. "I"), we have

(2) **Theorem:** *For* $m_2 \gg 0$*, there is an irreducible component* $(\operatorname{sing} \nu_\zeta)_{s_0}$ *of the singular locus of* $\operatorname{sing}_{\nu_\zeta}$ *passing through* $s_0 \in |L|$ *and*

$$\operatorname{codim}_{|L|}(\operatorname{sing} \nu_\zeta)_{s_0} = l - h^{2,0} \tag{3}$$

where $h^{2,0} = h^{2,0}(X)$.

The LHS of (3) may be thought of as an algebro-geometric quantity, since it is

$$\dim \left\{ \begin{array}{c} \text{deformation space} \\ \text{of } C_1 \text{ in } X \end{array} \right\} + \dim \left\{ \begin{array}{c} \text{hypersurfaces in } |L_m| \\ \text{passing through } C_1 \end{array} \right\} . \tag{4}$$

The RHS of (3) is Hodge-theoretic, since the number l of nodes may be read off from the local monodromy around s_0.

The above interpretation of (3) will be seen to hold for $n = 2$ — i.e., when X is a fourfold — as well as the present case, where of course writing as above

$$X_{s_0} = C = C_1 + C_2$$

the second term is

$$\{\text{deformations of } C_2 \text{ in } X\} .$$

As noted in the introduction, in contrast to the $n = 1$ case, in the $n = 2$ case there will always be a non-zero algebro-geometric "correction term."

Proof of #1*:* For simplicity we assume that $h^{1,0}(X) = 0$. In the general case the same argument with slightly more complicated computations will apply.

From the cohomology sequence of

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C_i) \to N_{C_i/X} \to 0 ,$$

and using the assumption $h^1(\mathcal{O}_X) = 0$ and

$$h^q\left(\mathcal{O}_X(C_i)\right) = 0 , \qquad q > 0$$

which holds for $m_1 \gg 0$, $m_2 \gg 0$, we have (cf. the remark below following Proof #2)

$$\begin{aligned}\dim(\operatorname{sing}\nu_\zeta)_{s_0} &= h^0\left(N_{C_1/X}\right) + h^0\left(N_{C_2/X}\right)\\ &= h^0\left(\mathcal{O}_X(C_1)\right) + h^0\left(\mathcal{O}_X(C_2)\right) - 2\ .\end{aligned}$$

Now using

$$\mathcal{X}\left(\mathcal{O}_X(C_i)\right) = \frac{1}{2}\left(C_i^2 + C_i\cdot K_X\right) + \mathcal{X}(\mathcal{O}_X)$$

we have

$$h^0\left(\mathcal{O}_X(C_i)\right) = \frac{1}{2}\left(C_i^2 + C_i\cdot K_X\right) + 1 + h^{2,0}\ ,$$

which gives

$$\begin{aligned}\dim(\operatorname{sing}\nu_\zeta)_{s_0} &= \frac{1}{2}\left(C_1^2 + C_2^2 + C\cdot K_X\right) + 2h^{2,0}\\ &= \frac{1}{2}\left(C^2 + C\cdot K_X\right) - l + 2h^{2,0}\end{aligned}$$

since $C_1\cdot C_2 = l$. Then since $\mathcal{O}_X(C)\cong L$

$$\begin{aligned}\dim|L| &= h^0\left(\mathcal{O}_X(C)\right) - 1\\ &= \frac{1}{2}\left(C^2 + C\cdot K_X\right) + h^{2,0}\end{aligned}$$

where we have used $h^q(\mathcal{O}_X(C)) = 0$ for $q > 0$. It follows that

$$\operatorname{codim}(\operatorname{sing}\nu_\zeta)_{s_0} = l - h^{2,0}. \qquad \square$$

Proof #2: We set

$$\Delta = C_1 \cap C_2$$

and note that near s_0

$$(\operatorname{sing}\nu_\zeta)_{s_0} = \left\{\begin{array}{c}\text{curves } C'\in|L| \text{ near } C \text{ such that}\\ \Delta \text{ deforms to nodes } \Delta'\subset C'\end{array}\right\}.$$

The reason is that for topological reasons, there must be one relation among the nodes Δ', and hence

$$C' = C_1' \cup C_2' \in \Sigma_{\mathbf{g}}\ .$$

Thus (cf. the remark below)

$$T(\operatorname{sing}\nu_\zeta)_{s_0} \cong H^0(\mathcal{I}_\Delta\otimes L)\ ,$$

so that from the cohomology sequence of

$$0 \to \mathcal{I}_\Delta \otimes L \to L \to L_\Delta \to 0$$

and $h^q(L) = 0$ for $q > 0$, we have

$$\begin{aligned} \operatorname{codim}(\operatorname{sing} \nu_\zeta)_{s_0} &= \dim \left(\text{Image}\left\{H^0(L) \to H^0(L_\Delta)\right\}\right) \\ &= \deg \Delta - h^1(\mathcal{I}_\Delta \otimes L)\ . \end{aligned}$$

Now there is the usual Koszul sequence

$$0 \to \mathcal{O}_X \to \bigoplus_{i=1}^{2} \mathcal{O}_X(L - C_i) \to \mathcal{I}_\Delta \otimes L \to 0$$

and since $C_1 + C_2 \in |L|$ the middle term is $\mathcal{O}_X(C_1) \oplus \mathcal{O}_X(C_2)$. Then from $h^q(\mathcal{O}_X(C_i)) = 0$ for $q > 0$ we have

$$\operatorname{codim}(\operatorname{sing} \nu_\zeta)_{s_0} = l - h^{2,0}. \qquad \square$$

(5) *Remark:* Above we have used

$$\dim(\operatorname{sing} \nu_\zeta)_{s_0} = \dim T(\operatorname{sing} \nu_\zeta)_{s_0}$$

where the RHS is the Zariski tangent space. In fact, from the above discussion we see that $(\operatorname{sing} \nu_\zeta)_{s_0}$ *is smooth and is equal to the Veronese variety*

$$|\zeta + m_1 H| \times |-\zeta + m_2 H| \subset |mH|, \qquad m = m_1 + m_2\ .$$

We shall now discuss one interpretation of Theorem (2). For this we set $L = mH$ where m is sufficiently large as above and consider the rational map

$$|L| \overset{\varphi}{\dashrightarrow} \overline{\mathcal{M}}_g \tag{6}$$

defined for a general point s by

$$\varphi(s) = X_s \in \overline{\mathcal{M}}_g$$

where as usual

$$g = \frac{1}{2}\left(L^2 + L \cdot K_X\right) + 1\ .$$

The map φ is regular in a neighborhood of s_0 where

$$X_{s_0} = C_1 \cup C_2 \tag{7}$$

as above.

We denote by

$$\Sigma_{\mathbf{g}} \subset \overline{\mathcal{M}}_g, \qquad \mathbf{g} = (g_1, g_2)$$

the boundary component of all curves of the type (7) where $g_i = g(C_i)$. Then we have

$$(\operatorname{sing} \nu_\zeta)_{s_0} = \varphi^{-1}(\Sigma_{\mathbf{g}})_{s_0} \ . \tag{8}$$

where both sides denote the irreducible components passing through s_0. From

$$g = g_1 + g_2 + l - 1$$

we have

$$3g - 3 = (3g_1 - 3) + (3g_2 - 3) + l$$

which gives the well-known result that

$$\operatorname{codim}_{\overline{\mathcal{M}}_g}(\Sigma_{\mathbf{g}}) = l \ . \tag{9}$$

Geometrically, the l nodes of $C_1 \cup C_2$ may be smoothed independently in $\overline{\mathcal{M}}_g$. From (3) we conclude the

(10) **Corollary:** *The rational mapping (6) fails by* $h^{2,0}$ *to be tranvserse at* s_0.

From (8) and (9) we have trivially

$$\operatorname{codim}(\operatorname{sing} \nu_\zeta)_{s_0} \leqq \operatorname{codim}_{\overline{\mathcal{M}}_g}(\Sigma_{\mathbf{g}}) = l \ .$$

As noted above, the LHS is an algebro-geometric term while the RHS is a Hodge-theoretic term. We shall write it as

$$AG \leqq HT \ , \tag{11}$$

and by the corollary the correction term needed to make this inequality into an equality is also a Hodge-theoretic term.

Assuming the Hodge conjecture, we will find an analogue of the inequality (11) for fourfolds, and the investigation of the correction term will lead to new and to us very interesting and only partially understood phenomena. In particular, the correction term will turn out to *not* be purely Hodge theoretic, one of a number of significant differences between the classical and higher codimensional cases.

Any rational map (or correspondence for that matter) induces in the usual way a map on cohomology by considering the fundamental class of the graph of the map and using the Künneth decomposition. In the next section we shall prove the

(12) **Proposition:** *For the induced mapping on cohomology associated to the rational mapping (6) we have*

$$\varphi^*([\Sigma_{\mathbf{g}}]) = 0\ .$$

Intuitively this has to be true. If we let X vary in moduli, then a general nearby X' will have no Hodge class ζ' corresponding to ζ and therefore no curves of the type (7). The mapping φ does not depend on ζ, and if $\varphi^*([\Sigma_{\mathbf{g}}]) \neq 0$ then this would be true for the φ' corresponding to (X', L') and

$$\varphi'^*([\Sigma_{\mathbf{g}}]) \neq 0 \Rightarrow \varphi'^{-1}(\Sigma_{\mathbf{g}}) \neq \emptyset\ .$$

This reasoning is of course heuristic, but it does explain (12).

II(ii). *Maps to moduli when X varies*

We recall the notation $\{X_t\}_{t\in\mathcal{M}_X}$ for the moduli space of X, which is assumed to exist and which for simplicity we assume is irreducible, and

$$|\mathcal{L}| = \bigcup_{t\in\mathcal{M}_X} |L_t|$$

for the family of hyperplane sections $|L_t|$ of X_t. We will consider the rational map

$$|\mathcal{L}| \overset{\Phi}{\dashrightarrow} \overline{\mathcal{M}}_g\ , \tag{13}$$

which we think of as the rational map (6) when X varies.

We also recall the notation

$$\mathcal{M}_{X,\zeta} \subset \mathcal{M}_X$$

for the Noether-Lefshetz locus where ζ remains a Hodge class $\zeta_t \in Hg^1(X_t)_{\mathrm{prim}}$. Now $\mathcal{M}_{X,\zeta}$ is defined by the condition

$$(\zeta_t)^{0,2} = 0 \tag{14}$$

where $\zeta_t \in H^2(X_t, \mathbb{Z})$. The class ζ_t is uniquely defined for X_t close to X, and if (14) is satisfied locally then it is invariant under monodromy and therefore satisfied globally (cf. Cattani-Deligne-Kaplan [7]).

Because of (14) we may say that

$$\text{“expected” } \operatorname{codim}_{\mathcal{M}_X}(\mathcal{M}_{X,\zeta}) = h^{2,0} \, . \tag{15}$$

It is known that (cf. [14]): *In general*

$$\operatorname{codim}_{\mathcal{M}_X}(\mathcal{M}_{X,\zeta}) = h^{2,0} \, , \tag{16}$$

but there are exceptional cases, such as when

$$\zeta = [\Lambda]_{\mathrm{prim}}$$

where $\Lambda \subset X \subset \mathbb{P}^3$ is a line on a smooth surface in $\mathbb{P}^3$ of degree at least five. In this case

$$\operatorname{codim}_{\mathcal{M}_X}(\mathcal{M}_{X,\zeta}) = h^{2,0} - h^{2,0}(-\Lambda)$$

where

$$h^{2,0}(-\Lambda) = \dim H^0\left(\Omega^2_X(-\Lambda)\right) \, .$$

For simplicity of exposition we will assume (16) and remark that the modifications necessary in the general case may be made by considerations similar to this example.

We denote by

$$|\mathcal{L}|_\zeta \subset |\mathcal{L}|$$

the part of $|\mathcal{L}|$ lying over $\mathcal{M}_{X,\zeta}$, which we think of as complete linear systems of hyperplane sections of X_t having $\zeta_t \in Hg^1(X_t)$. By our assumption

$$\operatorname{codim}_{|\mathcal{L}|}(|\mathcal{L}|_\zeta) = h^{2,0} \, .$$

We shall denote by

$$|\mathcal{L}|_\zeta \overset{\Phi_\zeta}{\dashrightarrow} \overline{\mathcal{M}}_g$$

the restriction to $|\mathcal{L}|_\zeta$ of the mapping Φ in (13). We also recall our notation $\boldsymbol{\nu}_\zeta$ for the family of ν_{ζ_t}'s over $\mathcal{M}_{X,\zeta}$. Let

$$s_0 \in \operatorname{sing} \nu_\zeta$$

correspond to a nodal $X_{s_0} = C_1 \cup C_2$ as above. Then $s_0 \in \operatorname{sing} \boldsymbol{\nu}_\zeta$ and from Theorem (2) and its proof

$$(17) \qquad \begin{cases} \text{(i)} & (\operatorname{sing} \boldsymbol{\nu}_\zeta)_{s_0} \subset |\mathcal{L}|_\zeta \\ \text{(ii)} & \operatorname{codim}_{|\mathcal{L}|_\zeta} (\operatorname{sing} \boldsymbol{\nu}_\zeta)_{s_0} = l - h^{2,0} \\ \text{(iii)} & \operatorname{codim}_{|\mathcal{L}|} (\operatorname{sing} \boldsymbol{\nu}_\zeta)_{s_0} = l \, . \end{cases}$$

From this we may draw the following

(18) **Conclusion:** *For the mapping* Φ *in (13) we have*

$$\Phi^{-1}(\Sigma_{\mathbf{g}})_{s_0} = \Phi_\zeta^{-1}(\Sigma_{\mathbf{g}})_{s_0} = (\operatorname{sing} \boldsymbol{\nu}_\zeta)_{s_0}$$

and Φ *is transverse relative to* $\Sigma_{\mathbf{g}}$ *in a neighborhood of* s_0.

The infinitesimal calculation needed to establish transversality may be established by a standard sheaf cohomological computation which we shall not give here.

In the following it will be understood that we are considering irreducible components passing through s_0. We also denote by

$$\mathcal{H}^{0,2} \to |\mathcal{L}|$$

the pullback of the Hodge bundle with fibres $H^{0,2}(X_t)$ over $\mathcal{M}_X$ and we set

$$h = h^{2,0} \, .$$

(19) **Theorem:** *With the above assumptions and notations we have in* $H^{2l}(|\mathcal{L}_\zeta|)$

$$(20) \qquad \left[\Phi_\zeta^{-1}(\Sigma_{\mathbf{g}})\right] \wedge c_h(\mathcal{H}^{0,2}) = \Phi_\zeta^*([\Sigma_{\mathbf{g}}]) \, .$$

Proof: We first observe that the normal bundle of $|\mathcal{L}|_\zeta$ in $|\mathcal{L}|$ is $\mathcal{H}^{0,2}$. Here we recall our assumption that in a neighborhood of (X, ζ) the Noether-Lefschetz locus is smooth of comdimension h and defining equations (14). The result then follows from standard excess intersection formula considerations (cf. Fulton [11] — in the special case one may rely on relatively elementary considerations using the Gysin map on smooth compactifications; cf. the remark below). □

We remark that

$$\Phi^*([\Sigma_{\mathbf{g}}]) \neq 0 \tag{21}$$

in the following sense: Let $\overline{\mathcal{M}}_X$ be *any* smooth compactification of $\overline{\mathcal{M}}_X$ and $\overline{\Phi}$ the extension of the rational map (13) to $\overline{\mathcal{M}}_X$. Then

$$\overline{\Phi}^*([\Sigma_{\mathbf{g}}]) = [\overline{\Phi}^{-1}(\Sigma_{\mathbf{g}})]$$

where the RHS is the fundamental class of the closure in $\overline{\mathcal{M}}_X$ of the irreducible, codimension h subvariety

$$\Phi^{-1}(\Sigma_{\mathbf{g}})_{s_0} \ .$$

The LHS is therefore non-zero, and this is what is meant by (21).

What is obviously of more interest is to know that $\Phi^*_\zeta([\Sigma_{\mathbf{g}}]) \neq 0$, and for this there are various assumptions that will imply that the LHS of (20) is non-zero. For example, if there is another Hodge class $\zeta' \in Hg^2(X)_{\text{prim}}$ such that

$$\mathcal{M}_{X,\zeta'} \ \text{ meets } \ \mathcal{M}_{X,\zeta} \ \text{ transversely,} \tag{22}$$

then the LHS of (20) is the fundamental class

$$\left[\Phi_\zeta^{-1}(\Sigma_{\mathbf{g}}) \cap \mathcal{M}_{X,\zeta'}\right] \ ,$$

which is non-zero in the sense explained above. Thus we have the

(23) **Corollary:** *With the asssumption (22),*

$$\Phi^*_\zeta([\Sigma_{\mathbf{g}}]) \neq 0 \ .$$

Remark that this result is illustrative and certainly not definitive.

Finally, since for the rational mapping φ in (6) we have

$$\varphi = \Phi_\zeta \ |_{|L|}$$

where $|L| \subset |\mathcal{L}|_\zeta$, we have

$$\varphi^*([\Sigma_{\mathbf{g}}]) = 0$$

since the LHS of (20) restricted to $|L|$ is zero. This proves Corollary (23).

Remark: As noted above, (20) is an elementary formula. The general situation is where we first have a diagram of maps

$$\begin{tikzcd} A \arrow[r, "F"] & M \\ \cup & \cup \\ B \arrow[r] & N \end{tikzcd} \quad , \quad B = F^{-1}(N)$$

where

$$\operatorname{codim}_A(B) = \operatorname{codim}_M(N) \ ,$$

and where for simplicity of explanation we assume that everything is smooth. Then

$$F^*([N]) = [F^{-1}(N)] = [B] \ . \tag{24}$$

Next suppose we have

$$B \subset C \subset A \ .$$

Denoting by $j : C \to A$ the inclusion and by $[B]_C \in H^*(C)$ the fundamental class of B in C we have

$$[B]_C = j^*([B])_{\wedge} c_{\text{top}}(N_{C/A}) \ .$$

Setting $F_C = F|_C$ and combining this with the formula above gives the elementary excess intersection formula

$$F_C^*([N]) = [F_C^{-1}(N)] \wedge c_{\text{top}}(N_{C/A}) \ . \tag{25}$$

The result (20) is the special case of this formula where

$$A = |\mathcal{L}|, \quad B = \Phi^{-1}(\Sigma_{\mathbf{g}}), \quad C = |\mathcal{L}|_\zeta$$
$$M = \overline{\mathcal{A}}_{\mathbf{h},\Sigma}, \quad N = \Sigma_{\mathbf{g}} \ .$$

The formula in the $n = 2$ case that will by way of contrast be given in section V(iii) will be a true excess intersection formula. To anticipate it we have in general

$$F_* : N_{B/A} \to N_{N/M}$$

and assuming F_* is injective we have the excess normal bundle

$$Q = F^{-1}(N_{N/M})/N_{B/A}$$

and then the excess intersection formula gives

$$F^*([N]) = Gy_{B/A}(c_{\text{top}}(Q)) \tag{26}$$

where

$$Gy_{B/A} : H^*(B) \to H^{*+\operatorname{codim}_A(B)}(A)$$

denotes the Gysin map. In the situation of formula (20) we have that Q is trivial and (26) gives

$$[B] = Gy_{B/A}(1_B) \tag{27}$$

where $1_B \in H^0(B)$. The formula (24) then is a consequence of (27) when we insert C between B and A.

II(iii). *Maps to* $\overline{\mathcal{C}}_g$

Keeping the above notations, let us define an *interesting curve* to be an irreducible curve $C \subset X$ with

$$\langle \zeta, [C] \rangle = \zeta^2\ . \tag{28}$$

By the $(1,1)$ theorem interesting curves exist in $|\zeta + m_1 H|$ for $m_1 \gg 0$. The presence of interesting curves also implies

$$\begin{array}{ll} \text{(i)} & \varphi^{-1}(\Sigma_{\mathbf{g}}) \neq \emptyset \\ \text{(ii)} & \Phi_\zeta^{-1}(\Sigma_{\mathbf{g}}) \neq \emptyset\ . \end{array} \tag{29}$$

We have seen above that $\varphi^*([\Sigma_{\mathbf{g}}]) = 0$ but under mild technical assumptions we have

$$\Phi_\zeta^*([\Sigma_{\mathbf{g}}]) \neq 0\ , \tag{30}$$

and this implies that

$$\Phi_\zeta^{-1}(\Sigma_{\mathbf{g}}) \neq \emptyset\ . \tag{31}$$

This in turn implies that there exists $(X_t, \zeta_t) \in \mathcal{M}_{X,\zeta}$ such that

$$\Phi_\zeta^{-1}(\Sigma_{\mathbf{g}}) \cap |L_t| \neq \emptyset\ .$$

Relabelling (X_t, ζ_t) to be just (X, ζ) we have

$$\varphi^{-1}(\Sigma_{\mathbf{g}}) \neq \emptyset \; . \tag{32}$$

Theorem: *The condition (30) implies that there exist interesting curves on X.*

In other words,

> *The topological condition (30) leads to an existence theorem.*

Proof: The argument will be indirect, and we shall first explain the difficulty in giving a direct proof. We are grateful to Mark de Cataldo and Luca Migliorini for pointing this difficulty out to us.

Since Φ_ζ is only a rational map, (30) only implies the existence of a family

$$\{X_s\}_{s\in\Delta} \subset |L| \tag{33}$$

with Δ the disc, where X_s is smooth for $s \neq 0$, and where semi-stable reduction (SSR) applied to the family (33) only produces an interesting curve on a blownup branched covering $\tilde{X}$ of X. This curve upstairs may then either contract or map to an uninteresting curve under the projection $\tilde{X} \to X$.

For the proof we shall first assume that

$$h^{2,0} = 0 \tag{34}$$

and then indicate how the argument may be modified in the general case. We consider the universal family of hyperplane sections relative to $|L|$

$$\begin{tikzcd} \mathcal{X}_{|L|} \subset X \times |L| \arrow[d] \\ |L| \; . \end{tikzcd}$$

There is then the following diagram of mappings where $\overline{\mathcal{C}}_g$ is the compactified universal curve of genus g and the dotted arrows are rational maps

$$\begin{tikzcd} X & \mathcal{X}_L \arrow[l] \arrow[r, dashed, "\lambda"] \arrow[d] & \overline{\mathcal{C}}_g \arrow[d] \\ & {|L|} \arrow[r, dashed, "\varphi"] & \overline{\mathcal{M}}_g . \end{tikzcd} \tag{35}$$

We denote by $\mathcal{C}_{g_1}$ in the diagram

$$\begin{tikzcd}[column sep=small] \mathcal{C}_{g_1} \arrow[d] & \subset & \overline{\mathcal{C}}_g \arrow[d] \\ \Sigma_{\mathbf{g}} & \subset & \overline{\mathcal{M}}_g \end{tikzcd}$$

the family of curves over $\Sigma_{\mathbf{g}}$ where $\mathbf{g} = (g_1, g_2)$ and whose general member is the curve component of genus g_1 lying over a general point of $\Sigma_{\mathbf{g}}$. Then

$$\operatorname{codim}_{\overline{\mathcal{C}}_g}(\mathcal{C}_{g_1}) = l\ . \tag{36}$$

Taking components of $\lambda^{-1}(\mathcal{C}_{g_1})$ lying over a point $s_0 \in |L|$ as in the proof of Theorem (2), we have by the assumption (34)

$$\operatorname{codim}_{\mathcal{X}_L}\left(\lambda^{-1}(\mathcal{C}_{g_1})\right) = l\ . \tag{37}$$

Assuming for the moment that λ and φ are regular maps, we will show that

$$\pi^*(\zeta) \cdot \lambda^*\left([\mathcal{C}_{g_1}]\right) \neq 0 \tag{38}$$

in $H^{2l+2}(\mathcal{X}_L)$. This is the main geometric step in the argument and the reason for it is as follows: Because of (36) and (37)

$$\lambda^*\left([\mathcal{C}_{g_1}]\right) = \left[\lambda^{-1}(\mathcal{C}_{g_1})\right]\ . \tag{39}$$

Now $\lambda^{-1}(\mathcal{C}_{g_1})$ fibres over $\varphi^{-1}(\Sigma_{\mathbf{g}})$ whose general fibre is the curve C_1 in the above picture. Then

$$\langle \pi^*(\zeta), [C_1] \rangle = \zeta^2 \neq 0\ ,$$

from which it follows that class (39) is Poincaré dual to a subvariety $Z \subset \lambda^{-1}(\mathcal{C}_{g_1})$ that maps to $\varphi^{-1}(\Sigma_{\mathbf{g}})$ as a generically finite map of degree ζ^2. Then

$$\pi^*(\zeta) \cdot \lambda^*([\mathcal{C}_{g_1}]) = [Z] \neq 0$$

which is (38).

Now from the argument used to prove (38) we see that a general fibre of $\lambda^{-1}(\mathcal{C}_{g_1}) \to \varphi^{-1}(\Sigma_{\mathbf{g}})$ maps to X onto a curve $\pi(C_1)$ with

$$\langle \zeta, [\pi(C_1)] \rangle = \zeta^2 \neq 0 \ ,$$

which then gives an interesting curve on X.

We now remove the assumption that λ and φ are regular maps. We let

$$\begin{tikzcd} & \tilde{\mathcal{X}}_L \arrow[dl, "\tilde{\pi}"'] \arrow[d] \arrow[dr, "\tilde{\lambda}"] & \\ X & \mathcal{X}_L \arrow[l, "\pi"'] \arrow[r, dashed, "\lambda"] & \overline{\mathcal{C}}_g \end{tikzcd}$$

be a resolution of the rational maps λ and φ, so that $\tilde{\lambda}$ and $\tilde{\varphi}$ are now regular maps. Then the above argument applies to give

$$\tilde{\pi}(\zeta) \cdot \tilde{\lambda}^* \left([C_{g_1}] \right) \neq 0 \ ,$$

which then gives as before an interesting curve $\tilde{\pi}(C_1)$ on X.

We finally remove the assumption (34). This was made in order to have the dimension count (36) leading to (39). If (34) is not satisfied then as we saw in section II(ii) we need to let (X, ζ) vary over $\mathcal{M}_{X,\zeta}$ in order to have the correct dimension counts. The argument given also then extends with the same underlying geometric idea to produce an $(X_t, \zeta_t) \in \mathcal{M}_{X,\zeta}$ such that there is an interesting curve on X_t. $\square$

II(iv). *Maps to* $\overline{\mathrm{Pic}^\circ(\mathcal{C}_g)}$

The rational maps

$$|L| \overset{\varphi}{\dashrightarrow} \overline{\mathcal{M}}_g$$

$$|L| \overset{\lambda}{\dashrightarrow} \overline{\mathcal{C}}_g$$

(cf. (6) and (35)), and their analogues when X varies do not depend on a Hodge class $\zeta \in Hg^1(X)_{\mathrm{prim}}$. What is the case is that given a ζ

and at a point s_0 where

$$X_{s_0} = C_1 \cup C_2 \tag{40}$$

$$\begin{cases} C_1 \in |\zeta + m_1 H|, & m_1 \gg 0 \\ C_2 \in |-\zeta + m_2 H|, & m_2 \gg 0 \end{cases}$$

we have

$$\varphi^{-1}(\Sigma_{\mathbf{g}})_{s_0} = (\operatorname{sing} \nu_\zeta)_{s_0} \tag{41}$$

where each side denotes the component of an irreducible variety through s_0.

This is for X fixed. Letting X vary in $\mathcal{M}_X$ we have seen that

$$\Phi^{-1}(\Sigma_{\mathbf{g}})_{s_0} \ \textit{projects onto a component of} \ \mathcal{M}_{X,\zeta}$$

and

$$\Phi^{-1}(\Sigma_{\mathbf{g}})_{s_0} = (\operatorname{sing} \boldsymbol{\nu}_\zeta)_{s_0} \ ,$$

which in particular implies that the analogue of (41) holds for all (X_t, ζ_t) near (X, ζ).

Without assuming the existence of ζ suppose we have a point s_0 such that φ is defined and regular in a neighborhood and

$$\varphi^{-1}(\Sigma_{\mathbf{g}})_{s_0} \neq \emptyset \ .$$

Then there exists a curve (7) where $g_i(C_i) = g_i$ are given with

$$g = g_1 + g_2 + l - 1 \ .$$

For suitable m_1 (which we may have to take to lie in $\mathbb{Q}$) we will have

$$(C_1 - m_1 H) \cdot H = 0 \ ,$$

so that ζ defined by

$$[C_1] = \zeta + [m_1 H] \tag{42}$$

is primitive. We note that, setting $h = H^2$,

$$\begin{cases} g_1 = \frac{1}{2}\,(\zeta^2 + m_1^2 h + \zeta \cdot K_X) + 1 \\ g_2 = \frac{1}{2}(\zeta^2 + m_2^2 h - \zeta \cdot K_X) + 1 \ . \end{cases}$$

(43) **Conclusion:** *If $\varphi^{-1}(\Sigma_{\mathbf{g}}) \neq \emptyset$, then there are Hodge classes $\zeta \in Hg^1(X)_{\mathrm{prim}}$ with given ζ^2 and $\zeta \cdot K_X$.*

In particular, there could be several ζ's corresponding to different components of $\varphi^{-1}(\Sigma_{\mathbf{g}})$.

We shall now discuss a rational map that does depend on ζ. This map is just the extended normal function viewed as a rational map

$$|L| \overset{\nu_\zeta}{-\!\!-\!\!\!\longrightarrow} \overline{\operatorname{Pic}^\circ(\mathcal{C}_g)}\ . \tag{44}$$

For the RHS of (44) we shall take the compactification defined by Caparoso [3]. Actually the compactification in her work is $\overline{\operatorname{Pic}^k(\mathcal{C}_g)}$ where k is large relative to g. For our purposes we can set $H_s = H\mid_{X_s}$ and use the rational map

$$s \to \nu_\zeta(s) + nH_s \in \operatorname{Pic}^k(X_s)$$

for large n where $X_s \in |mH|$ and $k = mnH^2$. With this understood, for simplicity of notation and exposition we shall just consider (44). The point will be to show how the image of (44) meets certain boundary components to be described below and whose relation to (44) will be clear.

We are interested in the behaviour of $\operatorname{Pic}^\circ(X_s)$ under a specialization

$$X_s \to X_{s_0} \in \Sigma_{\mathbf{g}}$$

of a smooth curve X_s to a curve of the type (7). A line bundle $M_s \to X_s$ of degree zero will specialize to a line bundle $M_{s_0} \to X_{s_0}$ of total degree zero with

$$\begin{cases} \deg_{C_1}(M_{s_0}) = d \\ \deg_{C_2}(M_{s_0}) = -d\ . \end{cases} \tag{45}$$

From loc. cit. there are bounds on d in terms of g and the integer k above, but these need not concern us here. We may think of $M_{s_0} \to X_{s_0}$ as given by line bundles

$$\begin{cases} M_1 \to C_1 \\ M_2 \to C_2 \end{cases} \tag{46}$$

with $\deg_{C_1}(M_1) = d$, $\deg_{C_2}(M_2) = -d$ and with isomorphisms (gluing data)

$$M_{1,p_i} \cong M_{2,p_i} \tag{47}$$

at the nodes p_i. The number of parameters in (46) and (47) is

$$g_1 + g_2 + l - 1 = g$$

where the "-1" on the LHS comes from the independent scalings of M_1 and M_2. We denote by

$$\Gamma_{d,\mathbf{g}} \subset \overline{\mathrm{Pic}^\circ(\mathcal{C}_g)}$$

the boundary component lying over $\Sigma_{\mathbf{g}}$ whose inverse image over a curve (7) is given by the data (46) and (47).

The basic observation is

(48) *For the extended normal function* ν_ζ *we have*

$$\nu_\zeta(s_0) \in \Gamma_{d,\mathbf{g}}\ , \quad d = \zeta^2\ .$$

In fact

$$\nu_\zeta^{-1}(\Gamma_{d,\mathbf{g}})_{s_0} = (\operatorname{sing} \nu_\zeta)_{s_0}\ . \tag{49}$$

As in section II(ii) we may prove that for $d = \zeta^2$

$$\begin{array}{ll} \nu_\zeta^*([\Gamma_{d,\mathbf{g}}]) = 0 & (X \text{ fixed}) \\ \boldsymbol{\nu}_\zeta^*([\Gamma_{d,\mathbf{g}}]) \neq 0 & (X \text{ variable})\ , \end{array} \tag{50}$$

where (50) holds under the same assumption as in Corollary (23).

Although ν_ζ does depend on ζ, the topological information in $[\nu_\zeta^{-1}(\Gamma_{d,\mathbf{g}})]$ only involves ζ^2. As will be discussed on a later occasion, we feel that ultimately one needs to understand

$$\nu_{\zeta\times\zeta'}^{-1}(\mathcal{P})$$

and its Chern classes, where $\zeta, \zeta' \in Hg^1(X)_{\mathrm{prim}}$ and $\mathcal{P}$ is an extension of the Poincaré line bundle to $\overline{\mathrm{Pic}^\circ(\mathcal{C}_g)}$ (cf. section V(D) in "I").

III. The classical case, Part B

III(i). *Hodge-theoretic description of* $\overline{\mathcal{A}}_{g,\Sigma}$

Thus far we have studied rational maps to compactifications of algebro-geometric moduli spaces and "universal" families over such. Now a normal function is a Hodge-theoretic object, as is an extended normal function. Moreover, although the algebro-geometric compactifications arising from algebraic curves are not generally available in the higher-dimensional case, there are recently available (partial) compactifications of many of the Hodge-theoretic players in our story.[1] This suggests studying rational maps to these Hodge-theoretic objects. In this section we shall begin this study again in the classical case of curves on a surface.

We shall use the standard notation

$$\mathcal{A}_g = \left\{ \begin{array}{l} \text{moduli space of principally polarized abelian} \\ \text{varieties (PPAV's) of dimension } g \end{array} \right\} .$$

For our purposes, $\mathcal{A}_g$ and its compactifications $\overline{\mathcal{A}}_{g,\Sigma}$ will be constructed Hodge-theoretically and we briefly review this, referring to Cattani [5], Carlson-Cattani-Kaplan [4], Alexeev-Nakamura [2], Alexeev [1] and the references cited there for details.

For the construction we assume given a pair $(H_{\mathbb{Z}}, Q)$ where $H_{\mathbb{Z}}$ is a lattice of rank $2g$ and

$$Q : H_{\mathbb{Z}} \otimes H_{\mathbb{Z}} \to \mathbb{Z}$$

is a unimodular symplectic form. A *symplectic basis* will as usual be a basis $\alpha_1, \dots, \alpha_g; \beta_1, \dots, \beta_g$ for $H_{\mathbb{Z}}$ relative to which

$$Q = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} .$$

Setting $H = H_{\mathbb{Z}} \otimes \mathbb{C}$, a (*principally*) *polarized Hodge structure of weight one*[2] is given by a filtration

$$H = F^0 \supset F^1 \supset \{0\}$$

---

[1] These partial compactifications are not compact as spaces, but they are compact relative to Hodge-theoretically defined maps to them, in the sense that maps of punctured discs to them extend across the origin. For this reason, for simplicity of terminology we shall simply refer to these partial compactifications as simply compactifications.

[2] Since the polarized Hodge structures we shall consider will all be principally polarized, we shall drop the adjective "principally" in section III.

where $\dim F^1 = g$ and where

$$\left\{\begin{array}{ll} Q(u,v) = 0 & u, v \in F^1 \\ \sqrt{-1}Q(u,\overline{u}) > 0 & 0 \neq u \in F^1 \ . \end{array}\right. \tag{1}$$

Relative to a symplectic basis as above there is a unique basis for F^1 given by the row vectors in a $g \times 2g$ matrix

$$\Omega = (I, Z)$$

where the bilinear relations (1) are

$$\left\{\begin{array}{l} Z = {}^tZ \\ \operatorname{Im} Z > 0 \ . \end{array}\right. \tag{2}$$

This period matrix representation will be useful in computation of examples.

We denote by D_g the set of polarized Hodge structures of weight one and set

$$\left\{\begin{array}{ll} \Gamma_g & = \operatorname{Aut}(H_{\mathbb{Z}}, Q) \\ G & = \operatorname{Aut}(H_{\mathbb{R}}, Q) \end{array}\right.$$

where $H_{\mathbb{R}} = H_{\mathbb{Z}} \otimes R$. Then G acts transitively on D_g and the quotient

$$D_g/\Gamma_g \simeq \mathcal{A}_g \tag{3}$$

of equivalence classes of polarized weight one Hodge structures may be identified with the moduli space of PPAV's.

Algebraic families of PPAV's over a curve may be localized to give a variation of the polarized Hodge structures of weight one (VHS)

$$f : \Delta^* \to D_g/\Gamma_g \tag{4}$$

over the punctured disc $\Delta^* = \{t : 0 < |t| < 1\}$. Denoting $\mathcal{U} = \{z : \operatorname{Im} z > 0\}$ the upper-half-plane we may lift (4) to

$$\begin{tikzcd} \mathcal{U} \arrow[r, "\tilde{f}"] \arrow[d] & D_g \arrow[d] \\ \Delta^* \arrow[r, "f"] & D_g/\Gamma_g \end{tikzcd}$$

where

$$\tilde{f}(z+1) = \tilde{f}(z)T$$

with $T \in \Gamma_g$ being the monodromy transformation. It is well-known that, replacing Δ^* by a finite covering by setting $t = s^k$, we will have

$$T = I + N$$

where

$$N = \log T \in \mathcal{G}_{\mathbb{Q}}$$

with $\mathcal{G}_{\mathbb{Q}}$ denoting the rational subspace of the Lie algebra $\mathcal{G} \subset \mathrm{Hom}(H_{\mathbb{R}}, H_{\mathbb{R}})$ of G, and where

$$N^2 = 0\ . \tag{5}$$

Using this we may define the *monodromy weight filtration*

$$\{0\} \subset W_0 \subset W_1 \subset W_2 = H \tag{6}$$

where

$$\begin{cases} W_0 = \mathrm{Im}(N) \\ W_1 = \mathrm{Ker}(N)\ . \end{cases} \tag{7}$$

One central reason for the importance of monodromy is that the VHS (4) is asymptotic to a nilpotent orbit, as follows: Let $\check{D}_g$ be the g planes in H satisfying the first bilinear relation in (1), and define

$$\tilde{h} : \mathcal{U} \to \check{D}_g$$

by

$$\tilde{h}(z) = f(z) \cdot \exp(-zN)\ .$$

Then $\tilde{h}(z+1) = \tilde{h}(z)$ so that $\tilde{h}$ descends to a map

$$h : \Delta^* \to \check{D}_g\ .$$

The main results are

$$\begin{cases} \text{(i)} & \textit{h extends across the origin, and we set } F_0 = h(0); \\ \text{(ii)} & \textit{for } \mathrm{Im}\, z \gg 0,\ F_0 \cdot \exp(zN) \in D_g; \textit{ and} \\ \text{(iii)} & \textit{in a precise sense, the nilpotent orbit} \\ & \qquad F_0 \cdot \exp\left(\frac{\log t}{2\pi\sqrt{-1}} N\right) \in D_g/\{T^n\} \\ & \textit{is asymptotic as } t \to 0 \textit{ to (4).} \end{cases} \tag{8}$$

Remark: We may choose a symplectic basis that *over* $\mathbb{Q}$ spans W_0; say

$$W_0 = \operatorname{span}_{\mathbb{Q}}\{\alpha_1, \dots, \alpha_l\} \ .$$

Then

$$N = \begin{pmatrix} 0 & \eta \\ 0 & 0 \end{pmatrix} \qquad\qquad (g \times g \ \text{ blocks})$$

where

$$\eta = \begin{pmatrix} \eta_{11} & 0 \\ 0 & 0 \\ \underbrace{\phantom{\eta_{11}}}_{l} & \underbrace{\phantom{\eta_{11}}}_{g-l} \end{pmatrix} \begin{matrix} \}l \\ \}g-l \\ \end{matrix}$$

where $\eta_{11} = {}^t\eta_{11}$. Then $\tilde{h}(z)$ has normalized period matrix

$$\tilde{Z}(z) - z\eta =: \tilde{W}(z) \ .$$

The nilpotent orbit then has normalized period matrix

$$\left(\frac{\log t}{2\pi\sqrt{-1}}\right) N + W(t)$$

which is positive definite for $0 < |t| < \epsilon$. It follows that

$$\eta_{11} > 0 \ . \tag{9}$$

The above discussion extends to a localized several variable VHS

$$f : (\Delta^*)^k \to D_g/\Gamma_g \tag{10}$$

(one may also include a Δ^m factor as parameters) with commuting monodromies $T_1, \dots, T_k$, which again going to finite coverings may be assumed to satisfy

$$\begin{cases} T_i = I + N_i \\ N_i^2 = 0 \\ [N_i, N_j] = 0 \ . \end{cases} \tag{11}$$

This then leads to a several variable nilpotent orbit

$$F_0 \cdot \exp\left(\sum_{i=1}^{k} \frac{\log t_i}{2\pi\sqrt{-1}} N_i\right) \ . \tag{12}$$

Under a rescaling

$$t_i \to \exp\left(\frac{u_i}{2\pi\sqrt{-1}}\right) t_i \qquad\qquad (u_i \in \mathbb{C})$$

we have

$$F_0 \to F_0 \cdot \exp\left(\sum_{i=1}^{k} u_i N_i\right) \ . \tag{13}$$

We set

$$\begin{cases} \mathbf{N} = (N_1, \dots, N_k) \\ N_\lambda = \displaystyle\sum_{i=1}^{i} \lambda_i N_i \ , \end{cases}$$

and then by (11) we have for all λ

$$N_\lambda^2 = 0 \ .$$

A crucial fact is (cf. Cattani, loc. cit.):

(14) *For $\lambda_i \neq 0$ the monodromy weight filtration $W_k(N_\lambda)$ defined by N_λ is independent of λ.*

From that reference one also has

(i) *The filtration F_0 defines a Hodge structure of weight p on*

$$Gr_p W(N_\lambda) = W_p(N_\lambda)/W_{p-1}(N_\lambda), \qquad\qquad p = 0, 1, 2 \ .$$

(ii) *The Hodge structure on $Gr_p W(N_\lambda)$ is polarized by quadratic forms constructed from Q and N_λ.*

(15) *Remark:* From (13) and the definitions it follows that the Hodge structure on $Gr_\bullet W(N_\lambda)$ is well defined, as in the limit mixed Hodge structure on $W_1(N_\lambda)$.

We are now ready to define the boundary components that arise in the compactifications $\overline{\mathcal{A}}_{g,\Sigma}$. Let

$$W_0 \subset H$$

be an isotropic subspace, $W_1 = W_0^\perp$ and set

$$\eta(W_0) = \{N \in \mathcal{G} : \operatorname{Im} N \subset W_0\}.$$

Then $\ker(N) \supset W_1$ and we set

$$\eta^+(W_0) = \{N \in \eta(W_0) : Q_N > 0\}$$

where Q_N is the symmetric form on H/W_1 defined for $u, v \in H$ by

$$Q_N(u,v) = Q(u, Nv) \ .$$

The filtration $\{0\} \subset W_0 \subset W_1 \subset H$ is then the monodromy weight filtration of any $N \in \eta^+(W_0)$ as well as that of any rational cone

$$\sigma = \{\sum_{i=1}^{k} \lambda_i N_i : \lambda_i \in \mathbb{R}^+, N_i \in \overline{\eta^+(W_0)}\} \ ,$$

where we assume that σ contains an element in $\eta^+(W_0)$ and that the N_i are in $\mathcal{G}_{\mathbb{Q}}$. We set

$$B(\sigma) = \left\{ \begin{array}{c} F^1 \in \check{D}_g : (F^1, W_0) \text{ defines a mixed Hodge} \\ \text{structure polarized for every } N \in \operatorname{Int}\sigma. \end{array} \right\}$$

(16) **Definition:** *The boundary component associated to* (W_0, σ) *is defined by*

$$\mathcal{B}(\sigma) = B(\sigma)/\exp\sigma_{\mathbb{C}} = \exp\sigma_{\mathbb{C}} \cdot D_g/\exp\sigma_{\mathbb{C}} \ .$$

The equality in the definition is a result; cf. Cattani, loc. cit.

The compactifications $\overline{\mathcal{A}}_{g,\Sigma}$ are constructed using a fan Σ, which is a collection of rational cones as above having certain incidence properties and where the isotropic subspace is replaced by a flag

$$\{0\} \subset S_1 \subset S_2 \subset \cdots \subset S_g$$

of isotropic subspaces where $\dim S_i = i$. The details of this construction, for which we refer to Cattani and the references cited there, are not necessary for the discussion in this paper.

We conclude this section with two examples.

Example 1: We use the period matrix notation above and set

$$\eta_i = \left(\begin{array}{cc} \delta_{ij} & 0 \\ 0 & 0 \\ \underbrace{\phantom{\delta_{ij}}}_{l} & \underbrace{}_{g-l} \end{array} \right) \begin{array}{l} \}l \\ \}g-l. \\ \end{array}$$

The corresponding nilpotent orbit is

$$\left(\begin{array}{ccc|c} \frac{\log t_i}{2\pi\sqrt{-1}} & & 0 & \\ & \ddots & & 0 \\ 0 & & \frac{\log t_l}{2\pi\sqrt{-1}} & \\ \hline & 0 & & 0 \end{array}\right) + \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{\lambda 1} & Z_{22} \end{pmatrix} \tag{17}$$

where

$$\begin{cases} Z_{11} = {}^tZ_{11} \\ Z_{21} = {}^tZ_{12} \\ Z_{22} = {}^tZ_{22}, \quad \operatorname{Im} Z_{22} > 0 \, . \end{cases} \tag{18}$$

Under a rescaling as above we have

$$Z_{11} \to Z_{11} + \begin{pmatrix} u_1 & & 0 \\ & \ddots & \\ 0 & & u_l \end{pmatrix} .$$

Denoting this rational cone by σ_l, it follows that we may choose a unique *normalized representative* of each point in $\mathcal{B}(\sigma_l)$ to be the period matrix given by the conditions (18) together with

$$(Z_{11})_{ii} = 0 \qquad\qquad 1 \leqq i \leqq l \, .$$

In particular,

$$\operatorname{codim}_{\overline{\mathcal{A}}_{g,\Sigma}}(\mathcal{B}(\sigma_l)) = l \, . \tag{19}$$

Below we shall see that this boundary component corresponds under the Torelli map to the boundary component in $\overline{\mathcal{M}}_g$ given by the image of $\mathcal{M}_{g-l,2l}$.

Example 2: We define $N_1, \ldots, N_l$ by

$$\eta_i = \begin{pmatrix} \delta_{ij} & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} \}l-1 \\ \}g-l+1 \end{matrix}$$
$$\qquad \underbrace{\phantom{\delta_{ij}}}_{l-1} \quad \underbrace{}_{g-l+1}$$

$$\eta_l = \begin{pmatrix} 1 \ldots 1 & \\ 1 \ldots 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} \}l-1 \\ \}g-l+1. \end{matrix}$$
$$\qquad \underbrace{}_{l-1} \quad \underbrace{}_{g-l+1}$$

For the corresponding nilpotent orbit of the form (12)

$$\sum_{i=1}^{l} \frac{\log t_i}{2\pi\sqrt{-1}} \eta_i + \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{matrix} \}l-1 \\ \}g-l+1 \end{matrix}$$
$$\qquad \underbrace{\phantom{Z_{11}}}_{l-1} \quad \underbrace{\phantom{Z_{12}}}_{g-l+1}$$

we have under rescaling that

$$Z_{11} \to Z_{11} + \begin{pmatrix} u_1 & & 0 \\ & \ddots & \\ 0 & & u_{l-1} \end{pmatrix} + \begin{pmatrix} u_l & \ldots & u_l \\ \vdots & & \vdots \\ u_l & \ldots & u_l \end{pmatrix} .$$

Denoting this rational cone by $\sigma_{l,1}$, we may as above choose a unique normalized representative of each point in $\mathcal{B}(\sigma_{l,1})$ to be the period matrix given by the conditions (18) (where the block sizes are as in this example) together with

$$(20) \qquad \begin{cases} (Z_{11})_{ii} = 0 & 1 \leqq i \leqq l-1 \\ (Z_{11})_{1,l-1} = 0 \, . \end{cases}$$

In particular

$$(21) \qquad \operatorname{codim}_{\overline{\mathcal{A}}_{g,\Sigma}}(\mathcal{B}(\sigma_{l,1})) = l \, .$$

Below we shall see that this boundary component corresponds under the Torelli map to the images of all inclusions

$$\Sigma_{\mathbf{g}} \hookrightarrow \overline{\mathcal{M}}_g, \qquad g = g_1 + g_2 + l - 1 \, .$$

III(ii). *Maps to* $\overline{\mathcal{A}}_{g,\Sigma}$

Keeping the previous notation, assigning to a general curve $X_s \in |L|$ the polarized Hodge structure on $H^1(X_s)$ defines rational maps

$$|L| \overset{\Psi}{\dashrightarrow} \overline{\mathcal{A}}_{g,\Sigma} \tag{22}$$

$$|L|_\zeta \ \subset \ |\mathcal{L}| \overset{\Psi}{\dashrightarrow} \overline{\mathcal{A}}_{g,\Sigma}\ .$$

We assume given $s_0 \in \operatorname{sing} \nu_\zeta$ where as above

$$X_{s_0} = C_1 \cup C_2 \in \Sigma_{\mathbf{g}}\ .$$

(23) **Theorem:** *Working in a neighborhood of* s_0*, in the diagram*

$$\begin{tikzcd} |\mathcal{L}| \arrow[r, dashed, "\Psi"] & \overline{\mathcal{A}}_{g,\Sigma} \\ \cup & \cup \\ \operatorname{sing} \boldsymbol{\nu}_\zeta \arrow[r, dashed] & \mathcal{B}(\sigma(l,1)) \end{tikzcd} \tag{24}$$

the mapping Ψ *is generically transverse along* $\operatorname{sing} \boldsymbol{\nu}_\zeta$.

This means that

$$\begin{cases} \operatorname{codim}_{\mathcal{A}_{g,\Sigma}}(\mathcal{B}(\sigma(l,1))) = l & \text{(cf. section III(i))} \\ \operatorname{codim}_{|\mathcal{L}|}(\operatorname{sing} \nu_\zeta) = l & \text{(cf. section II(ii))} \\ \Psi^{-1}(\mathcal{B}(\sigma(l,1)) = \operatorname{sing} \boldsymbol{\nu}_\zeta & \end{cases} \tag{25}$$

and that an injectivety condition on the differential of Ψ is satisfied. As in section II(ii) the above result and its proof will give an excess intersection formula

(26) **Corollary:** $\Psi^{-1}(\mathcal{B}(\sigma(l,1)) \subset |\mathcal{L}|_\zeta$ *and*

$$\Psi^*_\zeta([\mathcal{B}(\sigma(l,1))]) = [\Psi^{-1}(\mathcal{B}(\sigma(l,1))]_\wedge c_h(\mathcal{H}^{0,2})$$

where $\Psi_\zeta = \Psi \,|_{|\mathcal{L}|_\zeta}$.

The proof of theorem (23) and its corollary will occupy the rest of this section. Although it could be given directly, since the mapping Ψ factors

$$\begin{tikzcd} |\mathcal{L}| \arrow[rr, dashed, "\Psi"] \arrow[dr, dashed, "\Phi"'] & & \overline{\mathcal{A}}_{g,\Sigma} \\ & \overline{\mathcal{M}}_g \arrow[ur, dashed, "\tau"'] & \end{tikzcd}$$

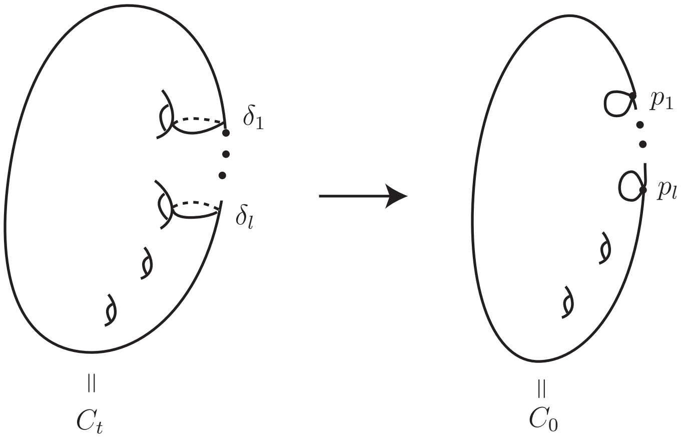
where τ is the *Torelli map* and Φ is the map previously studied in section II, we shall study τ and this will lead to the proof. The result that τ is a regular map is due to Mumford (cf. Alexeev [1] and the references cited therein). For our purposes we will need the explicit description of τ in the case of interest here.

We being by analyzing an l-parameter curve degeneration around a general point

$$C_0 \in \mathcal{M}_{g-l,2l} \subset \overline{\mathcal{M}}_g$$

cf. Cattani [5] and Friedman [10].

The picture is



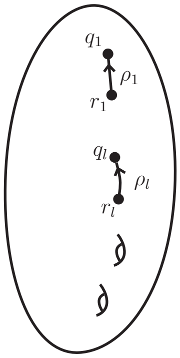
where $t = (t_1, \dots, t_l) \in \Delta^l$ and as $t_i \to 0$ the vanishing cycle δ_i shrinks to a node p_i. We assume the family is smooth over $(\Delta^*)^l$ and the monodromy around $t_i = 0$ is the Picard-Lefschetz transformation T_{δ_i} corresponding to the vanishing cycle δ_i. We set

$$N_i = \log T_{\delta_i} = T_{\delta_i} - I \ .$$

We denote by

$$\hat{C}_0 \xrightarrow{\pi} C_0$$

the normalization with $\pi^{-1}(p_i) = q_i + r_i$ and denote by ρ_i a path connecting q_i and r_i in $\hat{C}_0$



Let $\omega_1(t), \dots, \omega_g(t)$ be a basis for $H^0(\Omega^1_{C_t})$ normalized so that the period matrix $Z(t)$ of C_t is normalized where the symplectic basis $\alpha_1, \dots, \alpha_g; \beta_1, \dots, \beta_g$ for $H_1(C_t, \mathbb{Z}) \cong H^1(C_t, \mathbb{Z})$ is chosen so that

$$\alpha_1 = \delta_1, \dots, \alpha_l = \delta_l \ .$$

Then, again following the notations in section III(i) above, it is well known that

$$Z(t) = \begin{pmatrix} \sum_i \frac{\log t_i}{2\pi\sqrt{-1}} \eta_i + W(t) & Z_{12}(t) \\ Z_{21}(t) & Z_{22}(t) \end{pmatrix}$$

where $W(t)$, $Z_{12}(t) = {}^tZ_{21}(t)$, and $Z_{22}(t)$ are holomorphic in $t_1, \dots, t_l$.

It is also well-known that as $t \to 0$ and for $1 \leqq i \leqq 1$

$$\omega_i(t) \longrightarrow \eta_{q_i, r_i}$$

where η_{q_i,r_i} is a differential of the 3rd kind (dtk) on $\hat{C}_0$ with polar divisor at $q_i + r_i$ and residues ±1. We then have the following interpretation:

(i) $Z_{22}(0)$ is the period matrix of C_0;

(ii) using the bilinear relations between differentials of 1st and 3rd kinds, the entries in $Z_{12}(0)$ are interpreted as the Abel-Jacobi images

$$AJ_{\hat{C}_0}(q_i - r_i) \in J(\hat{C}_0)$$

of the divisors $q_i - r_i$ on $\hat{C}_0$;

(iii) the off-diagonal entries in $W(0)$ are the integrals

$$\int_{\rho_i} \eta_{q_j,r_j} = \int_{\rho_j} \eta_{q_i,r_i} \ ,$$

where the equality comes from the bilinear relations for dtk's.

It may now be checked that this implies that in the diagram

$$\begin{tikzcd}
\overline{\mathcal{M}}_g \arrow[r, "\tau"] & \overline{\mathcal{A}}_{g,\Sigma} \\
\mathcal{M}_{g-l,2l} \arrow[u, phantom, "\cup"] \arrow[r, "\tau"] & \mathcal{B}(\sigma(l)) \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

the Torelli map is transverse along a general l-nodal irreducible curve.

A more conceptual argument may be given as follows: The nilpotent orbit

$$(u_1, \dots, u_l) \to \begin{pmatrix} \sum_{i=1}^{l} u_i \eta_i & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \tag{27}$$

is linear in $u_1, \dots, u_l$. Let

$$\sum_{i,j=1}^{l} a_{ij}^{\lambda} Z_{ij}, \qquad Z_{ij} = Z_{ji} \text{ and } \lambda = 1, \dots, l(l-1)/2$$

be a basis for the linear functions on $l \times l$ symmetric matrices that annihilate the image of the top left hand block in (27). Recalling our notation $\alpha_1, \dots, \alpha_l; \beta_1, \dots \beta_l$ for the symplectic basis for $H_1(C_t, \mathbb{Z})$, we have

As $t \to 0$ the integrals

$$\sum_{i,j} a_{ij}^{\lambda} \int_{\beta_j} \omega_i(t)$$

remain finite and tend to the $l(l-1)/2$ quantities

$$\sum_{i,j} a_{ij}^{\lambda} \int_{\rho_j} \eta_{q_i,r_i} \ .$$

The quantities give the upper left hand block of

$$\lim_{t \to u} \tau(C_t) \ ,$$

the other parts of the period matrix being holomorphic in $t_1, \dots, t_l$.

The point of the more conceptual argument is that it applies to a degeneration $C_t \to C_0$ where $C_0 \in \Sigma_{\mathbf{g}}$ and the degeneration arises by smoothing the l nodes on C_0. The linear combination of the $\int_{\beta_j} \omega_i(t)$ that remain finite are not quite as obvious as in the first case, but it is not necessary to conclude that again the Torelli map is transverse in a neighborhood of C_0.

Theorem (23) and its corollary (26) now follow from this discussion and that in section II(ii).

III(iii). *Maps to* $\overline{\mathcal{J}}_{g,\Sigma}$

In this section we shall give a preliminary study of the map $\mathcal{V}_\zeta$ in the situations

$$\begin{tikzcd} & \overline{\mathcal{J}}_{g,\Sigma} \arrow[d] \arrow[r, phantom, "\supset"] & \Theta(\zeta^2;l,1) \arrow[d] \\ {|L|} \arrow[ur, dashed, "\mathcal{V}_\zeta"] \arrow[r, dashed, "\Psi"] & \overline{\mathcal{A}}_{g,\Sigma} \arrow[r, phantom, "\supset"] & \mathcal{B}(\sigma(l,1)) \end{tikzcd} \qquad (X \text{ fixed}) . \tag{28}$$

and

$$\begin{tikzcd} & \overline{\mathcal{J}}_{g,\Sigma} \arrow[d] \arrow[r, phantom, "\supset"] & \Theta(\zeta^2;l,1) \arrow[d] \\ {|\mathcal{L}|_\zeta} \arrow[ur, dashed, "\boldsymbol{\mathcal{V}}_\zeta"] \arrow[r, dashed, "\Psi_\zeta"] & \overline{\mathcal{A}}_{g,\Sigma} \arrow[r, phantom, "\supset"] & \mathcal{B}(\sigma(l,1)) \end{tikzcd} \qquad (X \text{ variable}). \tag{29}$$

Here $\overline{\mathcal{J}}_{g,\Sigma}$ is a compactification of $\mathcal{J}_g \to \mathcal{A}_g$ corresponding to a choice of fan Σ (cf. Alexeev [1] and the references cited therein), and $\Theta(\zeta^2; l, 1)$ is a boundary component to be specified below. The objective will be to give a generic transversality result and subsequent excess intersection formula analogous to theorem (23) and corollary (26) above.

The reasons for studying (28) and (29) are

(i) the maps $\mathcal{V}_\zeta$ and $\boldsymbol{\mathcal{V}}_\zeta$ are defined purely Hodge theoretically, in contrast to the maps ν_ζ and $\boldsymbol{\nu}_\zeta$ in section II(iv) above; and

(ii) these maps depend on ζ, in contrast to the maps ψ and Ψ that were studied in section III(ii) just above.

The study is preliminary in that the boundary component $\Theta(\zeta^2; l, 1)$ will here be defined only indirectly.

Following the discussion of (28) and (29) we shall give some general remarks concerning the codimension of *any* component of $\rho^{-1}(\mathcal{B})$, where ρ is one of the rational maps studied above and $\mathcal{B}$ is the corresponding boundary component.

Rather than study the maps ν_ζ and $\boldsymbol{\nu}_\zeta$ directly, we shall study the composition of the map $\mathcal{V}_\zeta$ and $\boldsymbol{\mathcal{V}}_\zeta$ in section II(iv) with the maps in the following diagram:

(30)
$$\begin{tikzcd}
\overline{\mathrm{Pic}^\circ(\mathcal{C})} \arrow[r, "\mu"] \arrow[d, "\pi"] & \overline{\mathcal{J}}_{g,\Sigma} \arrow[d, "\widetilde{\omega}"] \\
\overline{\mathcal{M}}_g \arrow[r, "\tau"] & \overline{\mathcal{A}}_{g,\Sigma}.
\end{tikzcd}$$

We note that for the Torelli map τ we have

(31)
$$\Sigma_{\mathbf{g}} \xrightarrow{\tau} \mathcal{B}(\sigma(l, 1)) \ .$$

The main step in our discussion is the following

(32) **Proposition:** *(i) The number of components of* $\pi^{-1}(\Sigma_{\mathbf{g}})$ *is* l*; and (ii) similarly, the number of components of* $\widetilde{\omega}^{-1}(\mathcal{B}(\sigma(l, 1))$ *is* l*.*

In both cases the components have dimension g, and we shall see that distinct components of $\pi^{-1}(\Sigma_{\mathbf{g}})$ map to distinct components of $\widetilde{\omega}^{-1}(\mathcal{B}(\sigma(l, 1)))$. We shall define $\Theta(\zeta^2; l, 1)$ to be the component of the boundary of $\overline{\mathcal{J}}_{g,\Sigma}$ to which μ maps $\Gamma(\zeta^2, \mathbf{g})$; i.e. to have

(33)
$$\Gamma(\zeta^2; \mathbf{g}) \xrightarrow{\mu} \Theta(\zeta^2; l, 1) \ .$$

The transversality and excess intersection formula will then follow from the results in sections II(iv) and III(ii).

Proof of the proposition: Part (i) is in Caparosa [3]. Turning to part (ii), we consider the situation of a family

(34)
$$\begin{tikzcd}
A \arrow[d, "p"] \\
\Delta
\end{tikzcd}$$

of principally polarized abelian varieties $A_t = p^{-1}(t)$, $t \neq 0$ where the total space A is smooth, p is proper, the monodromy T on $H^1(A_t, \mathbb{Z})$ is unipotent; i.e.

$$(T - I)^2 = 0 \ ,$$

and the singular fibre A_{t_0} has local normal crossings with all components having multiplicity one and which is the compactification of a connected semi-abelian variety $A^0_{t_0}$ (cf. Alexeev-Nakamura [2]). We may think of (34) as $\widetilde{\omega}^{-1}(\Delta)$ where $\Delta \subset \overline{\mathcal{A}}_{g,\Sigma}$ is a general disc meeting $\mathcal{B}(\sigma(l,1))$ transversely at a general point t_0. Let $\mathbf{t} \in \Delta^*$ be a base point and

$$H^1(A_{\mathbf{t}}, \mathbb{Z})_{\mathrm{ev}} \subset H^1(A_{\mathbf{t}}, \mathbb{Z})$$

the space of vanishing cycles. Then

$$(T - I)H^1(A_{\mathbf{t}}, \mathbb{Z}) \subset H^1(A_{\mathbf{t}}, \mathbb{Z})_{\mathrm{ev}} \ ,$$

and by Clemens-Schmid [9] the quotient

$$G = H^1(A_{\mathbf{t}}, \mathbb{Z})_{\mathrm{ev}}/(T - I)H^1(A_{\mathbf{t}}, \mathbb{Z})$$

is a finite group. There is then a *Néron model* $\tilde{A}_{t_0}$ (cf. Clemens [8] and M. Saito [20]) which is an extension of $A^0_{t_0}$ by G

$$0 \to A^0_{t_0} \to \tilde{A}_{t_0} \to G \to 0 \ .$$

(35) **Lemma:** *(i)* $\#G = l$.

(ii) $\#$ *(components of* $\widetilde{\omega}^{-1}(t_0)) = l$.

Proof: The second statement is a consequence of the first statement together with the general results in the literature.

For the second result we may as above choose a symplectic basis $\delta_1, \dots, \delta_g, \gamma_1, \dots, \gamma_g$ for $H^1(A_{\mathbf{t}}, \mathbb{Z})$ such that

$$T = T_{\delta_1} + \cdots + T_{\delta_{l-1}} + T_\delta$$

where the T_{δ_i} and T_δ are Picard-Lefschetz transformations and where

$$\delta = \delta_1 + \cdots + \delta_{l-1} \ .$$

Setting

$$\lambda = \delta_1 + \cdots + \delta_{l-1}$$

we have

$$\begin{cases} H^1(A_{\mathbf{t}}, \mathbb{Z})_{\mathrm{ev}} = \mathrm{span}_{\mathbb{Z}}\{\delta_1, \dots, \delta_{l-1}\} \\ (T-I)H^1(A_{\mathbf{t}}, \mathbb{Z}) = \mathrm{span}_{\mathbb{Z}}\{\delta_1 + \lambda, \dots, \delta_{l-1} + \lambda\} \ . \end{cases}$$

In general, if we have a lattice Λ with basis γ_i and sub-lattice $\Lambda' \subset \Lambda$ with basis γ_i' where

$$\gamma_i' = \Sigma a_i^j \gamma_j, \qquad a_i^j \in \mathbb{Z} \ ,$$

then it is well-known that

$$\#(\Lambda/\Lambda') = \det \|a_i^j\| \ .$$

Taking $\Lambda = H^1(A_t, \mathbb{Z})_w$ and $\Lambda' = (T-I)H^1(A_t, \mathbb{Z})$, our result follows from

$$\begin{aligned} (\delta_1 + \lambda) \wedge \cdots \wedge (\delta_{l-1} + \lambda) &= \delta_1 \wedge \cdots \wedge \delta_{l-1} \\ &\quad + \sum_{i=1}^{l} \delta_1 \wedge \cdots \wedge \lambda^i \wedge \cdots \wedge \delta_{l-1} \\ &= (1 + (l-1))\delta_1 \wedge \cdots \wedge \delta_{l-1} \\ &= l\delta_1 \wedge \cdots \wedge \delta_{l-1} \ . \end{aligned}$$ □

We now turn to the question: Let

$$|L| \overset{\rho}{\dashrightarrow} \overline{\mathcal{P}} \qquad (X \text{ fixed})$$

be one of the rational maps $\varphi, \psi, \nu_\zeta, \mathcal{V}_\zeta$ considered above, $\Delta \subset \overline{\mathcal{P}}$ the corresponding boundary component. Then, recalling that $L = mH$,

Does there exist m_0 and l such that

(36) $$\mathrm{codim}_{|L|}(\rho^{-1}(\Delta)) = l - h^{2,0}$$

for **all** *components and for $m \geqq m_0$?*

Here we are taking X to be fixed — there is a corresponding question without the $h^{2,0}$ term for X variable. We shall not treat this question in full generality (non-reduced components, etc.), but shall assume that

$$C_0 = C_1 \cup C_2 \in \rho^{-1}(\Delta)$$

is a nodal curve with C_1, C_2 smooth, and ask the related question

(37) *Is there an m_0 such that for $m \geqq m_0$ the deformation space of $C_0 \in |L|$ keeping the $l = C_1 \cdot C_2$ nodes is smooth and of codimension $l - h^{2,0}$?*

(38) **Theorem:** *(i) The answer to question (37) is in general* **no** *for the maps φ, ψ that do not depend on ζ. (ii) If*

$$\dim Hg^1(X)_{\text{prim}} = 1 \ ,$$

i.e. $Hg^1(X, \mathbb{Q})_{\text{prim}} = \mathbb{Q} \cdot \zeta$, the answer is **yes** *for ν_ζ and $\boldsymbol{\nu}_\zeta$.*

Proof: As in section II(ii), to have an affirmative answer to the question (37) we must in general have

$$h^q(\mathcal{O}_X(C_i)) = 0 \ , \qquad q > 0 \text{ and } m \geqq m_0 \ . \tag{39}$$

Suppose that

$$C_1 \in |\lambda + m_1 H|$$

where $\lambda \in Hg^1(X)_{\text{prim}}$. Then $C_2 \in |-\lambda + (m - m_1)H|$. We shall deal with (39) in the case of C_1, the case of C_2 being similar. For simplicity we shall assume that

$$K_X = H \ ; \tag{40}$$

the general case may be done by a similar but slightly more complicated argument.

The numerical information at hand for the mapping φ is

$$\begin{cases} g_i = \frac{1}{2}(\lambda^2 + m_i(m_1 + 1)H^2) + 1 \\ m = m_1 + m_2 \\ l = C_1 C_2 = -\lambda^2 + m_1 m_2 H^2 \\ g = g_1 + g_2 + l - 1 \ . \end{cases} \tag{41}$$

We set

$$C_1(k) = \lambda + kH \ .$$

According to Castelnuovo-Mumford regularity, we will have

$$h^q(\mathcal{O}_X(C_1(k))) = 0, \qquad q > 0 \text{ and } k \geqq k_0 \tag{42}$$

where k_0 is expressed in terms of the Hilbert polynomial

$$\chi(\mathcal{O}_X(C_1(k))) \ .$$

With our assumption (40), k_0 is then determined by the quantities

$$\left\{\begin{array}{l} C_1(0)^2 = \lambda^2 \\ \chi(\mathcal{O}_X)\ . \end{array}\right. \tag{43}$$

The numerical quantities we are given are

$$g, m, l\ .$$

We want to have the existence of $m_0 = m_0(g_1, g_2, l)$ such that

$$m_1 \geqq k_0 \ \text{ for } \ m \geqq m_0\ .$$

From (41), (43) we see that this is not the case without an a priori bound on λ^2.

Next we consider the map ν_ζ and write

$$C_1 \in |a\zeta + \zeta' + m_1 H|$$

where $a \in \mathbb{Z}$, $\zeta' \in Hg^1(X)_{\text{prim}}$ and $\zeta \cdot \zeta' = 0$. With the assumption in (ii) we have $\zeta' = 0$. Then

$$\nu_\zeta(s_0) \in \overline{\operatorname{Pic}^\circ(C_0)}$$

and the additional information is

$$\text{bidegree } (\nu_\zeta(s_0)) = (a\zeta^2, -a\zeta^2)\ ,$$

so that we now have an upper bound on $-\lambda^2 = -a^2\zeta^2 > 0$ and can therefore determine the k_0 for which (42) holds. We then observe from (41) that we will have an m_0 such that

$$m_1 \geqq k_0 \ \text{ for } \ k_0 \geqq k_0(m, l) \ \text{ and } \ m \geqq m_0$$

which was to be proved. More precisely, from

$$\left\{\begin{array}{c} m_1 m_2 \sim l \\ m_1 + m_2 = m \end{array}\right.$$

where "$\sim$" means modulo constants, we have

$$m_i \sim m_i(m, l) \qquad\qquad i = 1, 2\ ,$$

which implies the result. $\square$

IV. Analysis of $\operatorname{sing}\nu_\zeta$ on fourfolds

IV(i). *Structure of the space of cycles*

In the classical case $n = 1$ of curves on a surface, given

$$\zeta \in Hg'(X)_{\text{prim}}$$

then the *principal components* of $\operatorname{sing}\nu_\zeta$, i.e. by definition those corresponding to nodal curves X_{s_0} with two components C_1, C_1, where

$$\begin{aligned} [C_1] &= \zeta + m_1 H \\ [C_2] &= -\zeta + m_2 H \\ L &= (m_1 + m_2)H \end{aligned}$$

are well-behaved provided

$$m_1 \gg 0,\ \ m_2 \gg 0 \ \text{ relative to } \ \zeta\ .$$

In this case

$$H^q(\mathcal{O}_X(C_i)) = 0 \ \text{ for } \ q \gg 1, \quad i = 1, 2\ .$$

From

$$0 \to \mathcal{O}_X \to \overset{2}{\underset{i=1}{\oplus}} \mathcal{O}_X(C_i) \to \mathcal{O}_X(L) \to \mathcal{O}_X(\Delta) \to 0$$

where

$$\Delta = C_1 \cdot C_2 \quad \text{is the the singular locus of } \ X_{s_0} = C_1 \cup C_2$$

and unobstructedness of $|C_1|, |C_2|$, we conclude

$$\begin{aligned} \left\{ \begin{array}{c} \text{codim in } |L| \text{ of deformations} \\ \text{of } X_{s_0} \text{ remaining reducible} \end{array} \right\} &= \dim|L| - \dim|C_1| - \dim|C_2| \\ &= \dim \operatorname{Im}(H^0(\mathcal{O}_X(L)) \to H^0(\mathcal{O}(\Delta))) \\ &= l - h^2(\mathcal{O}_X) \\ &= l - h^{2,0}(X) \end{aligned}$$

where

$$l = \ \text{ number of nodes.}$$

Thus

$$\operatorname{codim}_{|L|}(\operatorname{sing}\nu_\zeta)_{s_0} = l - h^{2,0}(X)\ ,$$

a Hodge-theoretic quantity. Note, however, that we may expect there to be other components of sing (ν_ζ) where the codimension comes out differently (cf. section III(iii) above).

Note that

$$h^{2,0}(X) = \text{“expected”}\ \operatorname{codim}_{\mathcal{M}_X}(\mathcal{M}_{X,\zeta})$$

although it is only inequality the actual codimension. If equality does hold, then

$$\operatorname{codim}_{|\mathcal{L}|}(\operatorname{sing}\nu_\zeta)_{s_0} = l\ .$$

In this section, *assuming the Hodge conjecture* we shall investigate the analogous questions in the $n = 2$ case of surfaces on a fourfold, both for X fixed and for (X, ζ) varying. Informally stated what we shall perphaps not non-expectedly find is

> *The situation for $n = 2$ is completely different than the classical case.*

For example, as will be seen in the next section, one's first guess for the Noether-Lefschetz loci is

$$\text{“expected”}\operatorname{codim}_{\mathcal{M}_X}(\mathcal{M}_{X,\zeta}) = h^{4,0}(X) + h^{3,1}(X)\ .$$

Because of the infinitesimal period relations this is first corrected to

$$\text{“expected”}\ \operatorname{codim}_{\mathcal{M}_X}(\mathcal{M}_{X,\zeta}) = h^{3,1}(X)\ .$$

But the integrability conditions associated to the infinitesimal period relations give an addition correction term

$$\text{(1)}\qquad \text{“expected”}\ \operatorname{codim}_{\mathcal{M}_X}(\mathcal{M}_{X,\zeta}) = h^{3,1}(X) - \sigma_\zeta$$

where σ_ζ is a Hodge-theoretic term depending on ζ. We shall also see in §IV(iv) that (1) is sharp in an interesting set of examples.

The main general results on the structure of the space of cycles hold for all n and date to the work of Kleiman [18]:[3]

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[3] We would like to especially thank Rob Lazarsfeld for discussions related to the work of Kleiman.

(2) *Assuming the Hodge conjecture, given* $\zeta \in Hg^n(X)_{\text{prim}}$ *we have for some* $k \in \mathbb{Z}^*$

$$k\zeta = [W - H]$$

where W *is a smooth, codimension* n *subvariety and* H *is a smooth complete intersection of hypersurface sections;*

In (2) we may assume that W *is the degeneracy locus of general sections* $\sigma_1, \dots, \sigma_{r-1}$ *of a rank* r *vector bundle* $F \to X$*;*

Under some mild assumptions we shall also give the operations on pairs (F, S) where $S \in \text{Grass}(r-1, H^0(F))$ that correspond to addition of cycles, modulo complete intersections.

(3) *Denoting by* $\text{def}(*)_1$ *the* 1st *order deformations of* $*$*, assuming that* F *is chosen sufficiently ample and that*

$$h^{1,0}(X) = h^{2,0}(X) = 0$$

we have

$$\dim(\text{def}_1(W)) = \dim(\text{def}_1(F)) + \dim\ \text{Grass}(r-1, H^0(F));$$

The assumptions are not essential, and we have given them for purposes of illustrating the essential geometric point.

(4) *For* $L \gg 0$ *relative to* W*, we may find nodal* $X_{s_0} \in |L|$ *with*

$$\begin{aligned} &\text{(i)} \qquad W \subset X_{s_0} \\ &\text{(ii)} \qquad \zeta \neq 0 \ \textit{in} \ H^{2n}(\widehat{X}_{s_0}) \end{aligned}$$

where $\widehat{X}_{s_0}$ *is the standard desingularization of* X_{s_0} *and cohomology is with* $\mathbb{Q}$ *coefficients;*

(5) *Again for* $L \gg 0$ *and* $W \subset X_{s_0}$ *as above, to* 1st *order any deformation of* X_{s_0} *preserving the nodes carries* W *along with it; we write this as*

$$\text{def}_1 \begin{pmatrix} X_{s_0} \text{ preserving nodes} \\ \text{and modulo } H^0(\mathfrak{I}_W(L)) \end{pmatrix} = \text{def}_1(W)\ .$$

We shall also give cohomological expressions for dim and codim of $\mathrm{def}_1(*)$ above (in fact, this is how the various results are proved). One such is

$$\text{codim}\left\{\mathrm{def}_1\left(\begin{array}{c} X_{s_0} \text{ preserving} \\ \text{the nodes}\end{array}\right)\right\} = l - h^1(\mathcal{I}_{\Delta_{s_0}/X}(L)) \tag{6}$$

where

$$\left\{\begin{array}{l} l = \#\text{ nodes} \\ \Delta_{s_0} \text{ is the set of nodes}\end{array}\right. \tag{7}$$

and $\mathcal{I}_{\Delta_{s_0}/X}(L) \subset \mathcal{O}_X(L)$ is the ideal sheaf of Δ_{s_0} in X.

All of the above is for fixed X and any n. For $n = 1$, from the second proof of Theorem (2) in section II(i) we have in the circumstances at the beginning of this section

$$h^1(\mathcal{I}_{\Delta_{s_0}/X}(L)) = h^{2,0}(X)\ . \tag{8}$$

The RHS of this equality is a Hodge-theoretic term with the geometric interpretation

$$h^{2,0}(X) = \text{“expected” } \operatorname{codim}_{\mathcal{M}_X}(\mathcal{M}_{X,\zeta})\ .$$

When $n = 2$ we will see that

(9)

$$h^1(\mathcal{I}_{\Delta_{s_0}/X}(L)) = \text{“expected” } \operatorname{codim}_{\mathcal{M}_X}(\mathcal{M}_{X,\zeta}) + \left(\begin{array}{c}\text{algebro-geometric} \\ \text{correction term} \\ \text{depending on } W\end{array}\right)$$

where for $n = 2$ the correction term is the image of the mapping σ in the exact sequence

$$0 \to H^1(N_{W/X}) \to H^1(\mathcal{I}_{\Delta_{s_0}/X}(L)) \xrightarrow{\sigma} H^2(\Lambda^2 N_{W/X} \otimes L^*)$$

to be derived below, and which will be seen to be non-zero in typical examples.

This discussion refers to the situation when X is fixed. When X varies, and using the notation

$$\Sigma_{X,L} = \left\{\begin{array}{c}\text{sheaf of } 1^{\text{st}} \text{ order linear} \\ \text{differential operators on } \mathcal{O}_X(L)\end{array}\right\}$$

we will see below that there is a map

$$H^1(\Sigma_{X,L}) \to H^1(\mathcal{I}_{\Delta_{s_0}/X}(L)) \tag{10}$$

such that

$$\operatorname{codim}\operatorname{def}_1\begin{pmatrix}\text{pairs } (X, X_{s_0})\\ \text{preserving}\\ \text{the nodes}\end{pmatrix} = \dim(\text{image of the map } (10))$$

Proof of (4). This is standard. The point is that for a section $s_0 \in H^0(L)$ such that

$$W \subset X_{s_0},$$

i.e., $s_0 = 0$ on W, the differential ds_0 is well-defined along W and gives a section

$$ds_0 \in H^0(N^*_{W/X}(L))\ .$$

By choosing $L \gg 0$ relative to W, we may insure that there exists such an s_0 such that ds_0 has isolated, non-degenerate zeroes. This is equivalent to X_{s_0} having ordinary double points (nodes).

There is a standard desingularization

$$\widehat{X}_{s_0} \to X_{s_0}$$

where each node p_i is blown up to a smooth quadratic surface $Q_i \subset \widehat{X}_i$. The proper transform $\hat{W}$ of W is smooth and the induced map $H_4(\hat{W}) \to H_4(W)$ is an isomorphism. Since

$$\langle \zeta, [W] \rangle = k\zeta^2 \neq 0$$

it follows that the pullback

$$\hat{\zeta} \in H^4(\widehat{X}_{s_0})$$

is non-zero.

Proofs of (2) and (3). Although we shall give the following arguments in the $n = 2$ case they work in general. A general algebraic cycle may be written

$$Z = Z' - Z''$$

where Z' and Z'' are effective. By passing a complete intersection H of high degree through Z'' we will have

$$Z'' + Z''' = H$$

where Z''' is effective. Setting $V = Z' + Z'''$ we then have

$$Z = V - H$$

where V is effective and H is a complete intersection, which by moving in a rational equivalence class may be assumed to be smooth. We may also assume that V is reduced, since by a similar argument any multiple component of V may be moved by a rational equivalence into a sum of components of multiplicity one.

Following Kleiman [18] there is a resolution (here H is an ample line bundle)

$$0 \to F \to E_3 \to E_2 \to E_1 \to E_0 \to \mathcal{O}_V \to 0 \tag{11}$$

where

$$E_i \cong \bigoplus_j \mathcal{O}_X(-m_{ij}H)$$

where F is locally free of rank r (we identify vector bundles and locally free sheaves). It follows (cf. Fulton [11, Ex. 3.2.3]) that

$$\begin{cases} c_1(\mathcal{O}_V) = 0 \\ c_2(\mathcal{O}_V) = \pm[V] \equiv_H c_2(F) \end{cases}$$

where $\equiv_H$ denotes "congruent modulo complete intersections." Still working modulo complete intersections, we may replace F by $F(k)$ for large k and then it is well-known that

> $c_2(F)$ *is represented by the degeneracy locus of* $r-1$ *general sections* $\sigma_1, \dots, \sigma_{r-1} \in H^0(F)$.

Moreover, the subvariety where the degeneracy locus drops rank by two has codimension six, so that for generic choice of $\sigma_1, \dots \sigma_{r-1}$ we may assume that

$$W = \{x \in X : \sigma_1(x) \wedge \cdots \wedge \sigma_{r-1}(x) = 0\}$$

will be a smooth surface in X. This establishes (2).

Next there is an exact Eagon-Northcut complex

$$0 \to \overset{r-1}{\oplus} \wedge^r F^* \xrightarrow{\oplus\sigma_i} \wedge^{r-1} F^* \xrightarrow{\sigma_1\wedge\cdots\wedge\sigma_{r-1}} \mathcal{O}_X \to \mathcal{O}_W \to 0\,. \tag{12}$$

It is a general fact that for any resolution

$$0 \to E_2 \to E_1 \to E_0 \to \mathcal{O}_W \to 0 \tag{13}$$

of $\mathcal{O}_W$ by vector bundles, when restricted to W the homology sheaf of (13) is

$$\wedge^2 N^*_{W/X}, N^*_{W/X}, \mathcal{O}_W \ .$$

Applied to (12) this gives, setting $F^*_W = F^*|_W$,

$$0 \to \wedge^2 N^*_{W/X} \to \overset{r-1}{\oplus} \wedge^r F^*_W \to \wedge^{r-1} F^*_W \to N^*_{W/X} \to 0$$

or dually

$$0 \to N_{W/X} \to \wedge^{r-1} F_W \to \overset{r-1}{\oplus} \wedge^r F_W \to \wedge^2 N_{W/X} \to 0 \ . \tag{14}$$

Our goal is to compute $H^0(N_{W/X})$ and $H^1(N_{W/X})$; the results are:

$$\begin{aligned} 0 \to T \text{ Grass}(r-1, H^0(F)) \to H^0(N_{W/X})/T \text{ Aut}(F) \\ \to H^1(F \otimes F^*) \to 0 \end{aligned} \tag{15}$$

if

$$h^{1,0}(X) = h^{2,0}(X) = 0 \ , \tag{16}$$

and

$$H^1(N_{W/X}) \cong H^2(F \otimes F^*) \tag{17}$$

if

$$h^{1,0}(X) = h^{2,0}(X) = h^{3,0}(X) = 0 \ . \tag{18}$$

There are somewhat more elaborate statements that hold without assuming (16) or (18); the point is to have an illustrative result that is valid in interesting examples. We note that (15) gives (3).

We may rephrase (15) and (17) as

(19) *Under the stated assumptions and to* 1st *order, the deformations of* W *in* X *are obtained by deforming* F *and by varying the sections* $\sigma_1, \dots, \sigma_{r-1}$*;*

(20) *The obstruction space to deforming* W *is isomoprphic to the obstruction space to deforming* F*.*

Proofs of (15) and (17). From (14) we have

$$H^0(N_{W/X}) \cong \ker\left\{H^0\left(\wedge^{r-1}(F_W) \xrightarrow{\oplus\sigma_i} H^0(\oplus \wedge^r F_W\right)\right\} . \tag{21}$$

Tensoring (12) with $\wedge^{r-1}F$ gives

$$\begin{tikzcd} 0 \arrow[r] & \overset{r-1}{\oplus} F^* \arrow[r] & \wedge^{r-1}F^* \otimes \wedge^{r-1}F \arrow[r] \arrow[d, phantom, "\wr\|"] & \wedge^{r-1}F \arrow[r] & \wedge^{r-1}F_W \arrow[r] & 0 \\ & & F \otimes F^* . & & & \end{tikzcd}$$

Using the spectral sequence in cohomology and

$$\begin{cases} H^q(F^*) = 0, & q < 4 \\ H^p(\wedge^{r-1}F) = 0, & p > 0 . \end{cases}$$

We have

$$0 \to \frac{H^0(\wedge^{r-1}F)}{H^0(F\otimes F^*)} \to H^0(\wedge^{r-1}F_W) \to H^1(F\otimes F^*) \to 0 . \tag{22}$$

Tensoring (12) with $\wedge^r F$ gives

$$0 \to \overset{r-1}{\oplus} \mathcal{O}_X \to F \to \wedge^r F \to \wedge^r F_W \to 0 . \tag{23}$$

We shall use the notations

$$\begin{cases} V = H^0(F) \\ S = \operatorname{Im}\left\{\overset{r-1}{\oplus} H^0(\mathcal{O}_X) \xrightarrow{\oplus\sigma_i} H^0(F)\right\} \\ Q = V/S \end{cases}$$

so that

$$\mathrm{T}\,\mathrm{Grass}(r-1, H^0(F)) \cong \mathrm{Hom}(S,Q) .$$

From the cohomology sequence of (23) we have, using the assumption (16),

$$0 \to S \to V \to H^0(\wedge^r F) \to H^0(\wedge^r F_W) \to 0$$

which gives

$$0 \to Q \to H^0(\wedge^r F) \to H^0(\wedge^r F_W) \to 0 . \tag{24}$$

The dual of (12) is

$$0 \to \mathcal{O}_X \to \wedge^{r-1}F \to \overset{r-1}{\oplus} \wedge^r F \to \wedge^2 N_{W/X} \to 0 \tag{25}$$

which gives

$$(26)\ \ 0 \to H^0(\mathcal{O}_X) \to H^0(\wedge^{r-1}F) \to H^0\left(\overset{r-1}{\oplus} \wedge^r F\right) \to H^0(\wedge^2 N_{W/X})\ .$$

Using this and (21) gives the commutative diagram with rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & H^0(F^* \otimes F) \arrow[d] & S^* \otimes Q \arrow[d] & \\
0 \arrow[r] & H^0(\mathcal{O}_X) \arrow[r] \arrow[ur] & H^0(\wedge^{r-1}F) \arrow[r] \arrow[d] & H^0(\overset{r-1}{\oplus} \wedge^r F) \arrow[r] \arrow[d] & H^0(\wedge^2 N_{W/X}) \\
0 \arrow[r] & H^0(N_{W/X}) \arrow[r] & H^0(\wedge^{r-1}F_W) \arrow[r] \arrow[d] & H^0(\overset{r-1}{\oplus} \wedge^r F_W) \arrow[d] & \\
 & & H^1(F^* \otimes F) \arrow[d] & 0 & \\
 & & 0 & &
\end{tikzcd}$$

From

$$\begin{tikzcd}
S^* \otimes Q \arrow[d] & \\
H^0(\overset{r-1}{\oplus} \wedge^r F) \arrow[r] \arrow[d] & H^0(\wedge^2 N_{W/X}) \arrow[d, equal] \\
H^0(\overset{r-1}{\oplus} \wedge^r F_W) \arrow[r] & H^0(\wedge^2 N_{W/X})
\end{tikzcd}$$

we have that

$$S^* \otimes Q \to H^0(\wedge^2 N_{W/X})$$

is the zero map. This leads to the diagram with exact rows and columns

(27)

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] \\
 & & H^0(F^* \otimes F)/H^0(\mathcal{O}_X) \arrow[r] \arrow[d] & S^* \otimes Q \arrow[d] \\
 & 0 \arrow[r] & H^0(\wedge^{r-1} F)/H^0(\mathcal{O}_X) \arrow[r, "\alpha"] \arrow[d] & S^* \otimes H^0(\wedge^r F) \arrow[d] \\
0 \arrow[r] & H^0(N_{W/X}) \arrow[r] & H^0(\wedge^{r-1} F_W) \arrow[r] \arrow[d] & S^* \otimes H^0(\wedge^r F_W) \arrow[d] \\
 & & H^1(F^* \otimes F) \arrow[d] & 0 \\
 & & 0\ . &
\end{tikzcd}$$

The diagram (27) gives

(28)

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & H^0(F^* \otimes F)/H^0(\mathcal{O}_X) \arrow[r] \arrow[d] & S^* \otimes Q \arrow[d] \arrow[dr, "0"] & \\
 & 0 \arrow[r] & H^0(\wedge^{r-1} F)/H^0(\mathcal{O}_X) \arrow[r, "\alpha"] \arrow[d, "r_w"] & S^* \otimes H^0(\wedge^r F) \arrow[r] \arrow[d, "r_w"] & H^0(\wedge^2 N_{W/X}) \arrow[d, equal] \\
0 \arrow[r] & H^0(N_{W/X}) \arrow[r] & H^0(\wedge^{r-1} F_W) \arrow[r, "\beta"] \arrow[d, "p"] & S^* \otimes H^0(\wedge^r F_W) \arrow[r] \arrow[d] & H^0(\wedge^2 N_{W/X}) \\
 & & H^1(F^* \otimes F) \arrow[d] & 0 & \\
 & & 0\ . & &
\end{tikzcd}$$

The row with $H^0(N_{W/X})$ in it is only exact for the first two terms. It follows that

$$\alpha^{-1}(S^* \otimes Q) \cong S^* \otimes Q$$

and that

$$\operatorname{im}(\beta) \subseteq r_W(\operatorname{im} \alpha)$$

It follows that the natural map $(\beta \circ p^{-1})$

$$H^1(F^* \otimes F) \to \frac{S^* \otimes H^0(\wedge^{r-1}/W)}{\operatorname{im} \beta \circ r_W}$$

is zero, because

$$\operatorname{im}(\beta \circ r_W) = \operatorname{im}(r_W \circ \alpha) \supseteq \operatorname{im} \beta \, .$$

Thus

$$p^{-1}(H^1(F^* \otimes F)) \subseteq \operatorname{im} r_W + \ker \beta = \operatorname{im} r_W + \operatorname{im} H^0(N_{W/X})$$

which implies the surjectivity of the map

$$H^0(N_{W/X}) \to H^1(F^* \otimes F) \, .$$

Now

$$\ker(H^0(N_{W/X}) \to H^1(F^* \otimes F))$$

injects to Image r_W in $H^0(\wedge^{r-1}F_W)$, and chasing the diagram leads to

$$\ker(H^0(N_{W/X}) \to H^1(F^* \otimes F)) \to \frac{S^* \otimes Q}{\operatorname{im} H^0(F^* \otimes F)} \, . \tag{29}$$

Surjectivity comes because

$$S^* \otimes Q \mapsto 0$$

$\operatorname{im} H^0(\wedge^2 N_{W/X})$ by commutativity of the diagram (28) so

$$S^* \otimes Q \mapsto \operatorname{im} \alpha$$

and under r_W on the left,

$$\begin{aligned} \frac{S^* \otimes Q}{H^0(F^* \otimes F)} \mapsto \frac{\ker \beta}{\ker p} &= \operatorname{im} H^0(N_{W/X}) \cap \ker p \\ &= \ker(H^0(N_{W/X}) \to H^1(F^* \otimes F)) \end{aligned}$$

from which we get injectivity of (29). ☐

We have now proved (3) and (4). We shall next prove

$$\operatorname{def}_1(X_{s_0} \text{ preserving the nodes } \Delta_{s_0}) \cong H^0(\mathcal{I}_{\Delta_s/X}(L)) \, . \tag{30}$$

Proof: This is a general fact. If locally

$$s(p(t), t) = 0, \quad \frac{\partial s}{\partial z_i}(p(t), t) = 0, \qquad \text{all } i$$

where z_i are local coordinates on X, then we have

$$\dot{s}(p(0),0) + (ds(p(0),0))\rfloor\dot{p}(0) = 0$$

where $\dot{p}(0) \in T_{p(0)}X$. Since $ds(p(0),0) = 0$ we obtain $\dot{s}(p(0),0) = 0$ which gives

$$\dot{s} \in H^0(\mathcal{I}_{\Delta_{s_0}}(L)) \ .$$

This is a 1st order condition. The 2nd order condition may be written as

$$d\dot{s}(p(0),0) + \big(d^2s(p(0),0)\big)\rfloor\dot{p}(0) = 0$$

where $d^2s \in S^2T^*_{p(0)}X$ is the Hessian, which is non-degenerate. For any $d\dot{s}(p(0),0)$ this gives a unique $\dot{p}(0)$. ☐

We next note the standard interpretation

$$\text{(31)} \qquad \begin{cases} H^1(\mathcal{I}_{\Delta_{s_0/X}}(L)) \ \textit{is the obstruction space to} \\ \mathrm{def}_1(X_{s_0} \ \text{ preserving } \ \Delta_{s_0}) \ . \end{cases}$$

From the cohomology sequence of

$$0 \to \mathcal{I}_{\Delta_{s_0}/X}(L) \to \mathcal{O}_X(L) \to \mathcal{O}_{\Delta_{s_0}}(L) \to 0$$

we obtain (9) above.

$$\text{(32)} \qquad \operatorname{codim} \mathrm{def}_1(X_{s_0} \ \text{ preserving } \ \Delta_{s_0}) = l - h^1(\mathcal{I}_{\Delta_{s_0}/X}(L))$$

where $l = h^0(\mathcal{O}_{\Delta_{s_0}}(L))$ is the number of nodes.

We will now show that

$$\text{(33)} \qquad H^0(N_{W/X}) \cong H^0(\mathcal{I}_{\Delta_{s_0}/X}(L)/H^0(\mathcal{I}_{W/X}(L)) \ .$$

Proof. From the cohomology sequence of

$$0 \to \mathcal{I}_{W/X}(L) \to \mathcal{I}_{\Delta_{s_0}/X}(L) \to \mathcal{I}_{\Delta_{s_0}/W}(L) \to 0 \ ,$$

where $\mathcal{I}_{W/X}(L)$ is the ideal sheaf of W tensored with $\mathcal{O}_X(L)$ and the assumption $L \gg 0$ relative to W, we obtain

$$\text{(34)} \qquad H^0(\mathcal{I}_{\Delta_{s_0}}(L)) \cong H^0(\mathcal{I}_{\Delta_{s_0}/X}(L))/H^0(\mathcal{I}_{W/X}(L)) \ .$$

Next, since the p_i are ordinary nodes, the Koszul complex associated to

$$ds_0|_W \in H^0(N^*_{W/X}(L))$$

gives the exact sheaf sequence

$$0 \to \wedge^2 N_{W/X}(L^*) \to N_{W/X} \to \mathcal{I}_{\Delta_{s_0}/W}(L) \to 0\ . \tag{35}$$

Again using the assumption $L \gg 0$ relative to W we obtain

$$H^0(N_{W/X}) \cong H^0(\mathcal{I}_{\Delta_{s_0}/W}(L))\ . \tag{36}$$

Comparing with (34) gives (33). $\square$

At this stage we have proved (5).

We next observe the exact cohomology sequence of (35) gives

$$0 \to H^1(N_{W/X}) \to H^1(\mathcal{I}_{\Delta_{s_0}/X}(L)) \to H^2(\wedge^2 N_{W/X} \otimes L^*)$$

where we have used

$$H^1(\mathcal{I}_{\Delta_{s_0}/W}(L)) \cong H^1(\mathcal{I}_{\Delta_{s_0}/X}(L))$$

for $L \gg 0$ relative to W. This is the sequence referred to above and may be interpreted as saying

$$\left\{ \begin{array}{c} \textit{the obstruction space} \\ \textit{to } \mathrm{def}_1(W \subset X) \end{array} \right\} \subset \left\{ \begin{array}{c} \textit{the obstructive space to} \\ \mathrm{def}_1 \left(\begin{array}{c} X_{s_0} \text{ preserving} \\ \text{the nodes} \end{array} \right) \end{array} \right\} .$$

All of the above discussion is for X fixed. We now turn to the situation when X varies leading to (10) and the statement just below. Let then

$$\widehat{X} \xrightarrow{\epsilon} X$$

be the blowup of X at the nodes $p_i \in X_{s_0}$. Then each $\epsilon^{-1}(p_i)$ is a $\mathbb{P}^3$ that we label E_i and that meets the proper transform $\widehat{X}_{s_0}$ of X_{s_0} in a smooth quadric Q_i. We will show that

$$\mathrm{def}_1(\widehat{X}, \widehat{X}_{s_0}) \cong \mathbb{H}^1(\Theta_X \to \mathcal{I}_{\Delta_{s_0}/X}(L))\ , \tag{37}$$

where the LHS denotes the 1st order deformations of the pair $\widehat{X}_{s_0} \subset \widehat{X}$.

Proof. We have

$$0 \to \epsilon_* \Theta_{\widehat{X}} \to \Theta_X \to \underset{i}{\oplus} N_{p_i/X} \to 0 \ .$$

From the diagram whose 1st column is this sequence

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
\epsilon_* \Theta_{\widehat{X}} \arrow[r] \arrow[d] & \epsilon_*(\epsilon^* L[-2\sum_i E_i]\big|_{\widehat{X}_{s_0}}) \arrow[d] \arrow[r, phantom, "\cong"] & \mathcal{J}^2_{\Delta_{s_0}/X}(L) \\
\Theta_X \arrow[d] & \mathcal{J}_{\Delta_{s_0}/X}(L) \arrow[d] & \\
\underset{i}{\oplus} N_{p_i/X} \arrow[r] \arrow[d] & \epsilon_*(\epsilon^* L[-\sum_i E_i]\big|_{\widehat{X}_{s_0}}) \arrow[r, phantom, "\cong"] & L \otimes (\underset{i}{\oplus} N^*_{p_i/X}) \\
0 & &
\end{tikzcd}$$

we infer that

$$\begin{aligned}
(38) \qquad 0 \to \mathbb{H}^0 \left(\underset{i}{\oplus} N_{p_i/X} \xrightarrow{d^2 s_0} \oplus N^*_{p_i/X}(L) \right) &\to \mathbb{H}^1(\Theta_{\widehat{X}} \to N_{\widehat{X}_{s_0}/\widehat{X}}) \\
&\to \mathbb{H}^1(\Theta_X \to \mathcal{J}_{\Delta_{s_0}/X_{s_0}}(L)) \to 0 \ .
\end{aligned}$$

Since each p_i is a node

$$N_{p_i/X} \xrightarrow{d^2 s_0} N^*_{p_i/X}(L)$$

is an isomorphism, so the first term in (18) is zero. Since

$$\mathrm{def}_1(\widehat{X}, \widehat{X}_{s_0}) \cong \mathbb{H}^1(\Theta_{\widehat{X}} \to N_{\widehat{X}_{s_0/\widehat{X}}})$$

we may infer (17). ☐

IV(ii). *The "layers of Chow" obstruction in higher codimension*

In the classical case of curves on a surface the following is well-known (and comes out of the proof of Theorem (2) in section II(iii)): Keeping the notations from that section, let (X_t, ζ_t) be a variation of (X, ζ) over an analytic neighborhood of $(X, \zeta) \in \mathcal{M}_{X,\zeta}$. Then there exists an m_0 with the following property:

(39) *For* $m \geqq m_0$ *and* **any** *smooth curve* $W \subset X$ *with*

$$[W] = \zeta + m[H] \ ,$$

there exists a smooth deformation W_t *with*

$$[W_t] = \zeta_t + m[H_t] \ .$$

Here, H and H_t are hyperplane sections of X and X_t, respectively. In words we may say

(40) *Any smooth representative of a sufficiently ample Hodge class deforms with the Hodge class.*

The example of Soulé-Voisin (cf. section 4.4.1 in "I") shows that, because of torsion phenomena, in higher codimension (42) will in general be false even in situations where the HC holds. So minimally the statement might be:

(41) *There exists* m_0, k_0 *such that for* $m \geqq m_0$ *and* any *smooth* n*-fold* W *with*

$$[W] = k_0\zeta + m[H]$$

there exists a smooth deformation W_t *with*

$$[W_t] = k_0\zeta_t + m[H_t] \ .$$

In fact, the HC implies that there exists k_0 and a W as above that deforms with (X_t, ζ_t). But as we shall now see

(42) **Theorem:** *Assuming the HC and for any* k_0*, there exist examples such that no matter how ample we make* W *it will not deform with* (X_t, ζ_t)*.*

In other words, assuming the HC we will have (40) for **some**, but not **any**, W.

To construct examples we first observe: To prove theorem (42), it will suffice to find an algebraic cycle $Z \in Z^n(X)$ and a direction $\theta \in \mathrm{T}_X\mathcal{M}_{X,\zeta}$ such that

$$(43) \qquad \begin{cases} \text{(i)} & [Z] = 0 \\ \text{(ii)} & \textit{for any } m,\ Z + mH \textit{ does not deform} \\ & \textit{to } 1^{\text{st}} \textit{ order in the direction } \theta. \end{cases}$$

The reason is this: Suppose that W deforms with (X_t, ζ_t), and for any k_0 consider

$$W' = k_0 W + Z \ .$$

Suppose that the rational equivalence class of W'

$$\{W'\} \in CH^n(X) \otimes \mathbb{Q}$$

deforms to 1^{st} order in the direction θ. Then the same would be true of

$$\{Z\} \in CH^n(X) \otimes \mathbb{Q}\ .$$

In the examples to be constructed

(44) $\{Z\}$ *will not deform to* 1^{st} *order in the direction* θ*.*

It follows that $\{W'\}$ will also not deform to 1^{st} order in the direction θ. But by the proof of (3) in section IV(i), for $m \gg 0$ we see that $\{W'\}$ will contain a smooth subvariety W'' with

$$[W''] = k_0\zeta + m[H]$$

and which does not deform to 1^{st} order in the direction θ.

The geometric idea behind the construction of cycles Z as in (43) is the following:

Denoting by $CH^p(X)$ the Chow group *considered modulo torsion*, according to the conjecture of Bloch and Beilinson there exists a filtration $F^kCH^p(X)$ such that

(i) $Gr^kCH^p(X)$ has a Hodge-theoretic description;
(ii) $F^0CH^p(X) = CH^p(X)$ and $F^1CH^p(X)$ is the kernel of the fundamental class mapping;
(iii) $F^2CH^p(X)$ is the kernel of the Abel-Jacobi mapping;
(iv) $F^{p+1}CH^p(X) = 0$; and
(v) if X is defined over $\overline{\mathbb{Q}}$, then

$$F^2CH^p(X(\overline{\mathbb{Q}})) = 0\ .$$

In [12] we have given a Hodge-theoretic construction of such a filtration, assumming (v) and the GHC. Other proposed constructions are given in [16] and [19].

In our paper [13], we have given a construction of the formal tangent spaces $TF^kCH^p(X)$ with the infinitesimal version of the properties (i)–(v), and that is what we use here. Specifically, given a class

$$\{Z\} \in F^kCH^p(X)$$

and a tangent to moduli $\theta \in H^1(\Theta_X)$, we constructed an obstruction class

$$\mathcal{O}_k(\{Z\},\theta) \in H^{p-k-1,p+1}(X) \otimes \Omega^k_{\mathbb{C}/\mathbb{Q}} \tag{45}$$

to formally deforming the graded piece

$$\{Z\}_k \in Gr^kCH^p(X)$$

to 1st order in the direction θ. In particular:

(46) *If* $\mathcal{O}_k(\{Z\},\theta) \neq 0$, *then* $\{Z\}$ *does not deform to* 1st *order in the direction* θ.

Note that for $k = p$, for degree reasons

$$\mathcal{O}_p(\{Z\},\theta) = 0$$

so that in the $p = 1$ case the only obstruction to deforming $\{Z\}$ to 1st order is that it remain of Hodge type $(1,1)$. More generally, there are p independent conditions — the first being to remain of Hodge type (p,p) — to deforming $\{Z\} \in CH^p(X)$ to first order.

For $p = 2$ and $\{Z\} \in F^1CH^2(X)$ the obstruction

$$\mathcal{O}_1(\{Z\},\theta) \in H^{0,3}(X) \otimes \Omega^1_{\mathbb{C}/\mathbb{Q}} ,$$

which suggests that we look for examples when $n = 2$, $\{Z\} \in F^1CH^2(X)$ and $H^3(\mathcal{O}_X) \neq 0$.

In general, one has the *arithmetic cycle class* (cf. loc. cit.)

$$[Z]_a \in H^p(\Omega^p_{X/\mathbb{Q}}) .$$

There is a mapping

$$\Theta_{X,\mathbb{C}} \to \Theta_{X/\mathbb{Q}}$$

which gives a pairing (setting $\Theta_X = \Theta_{X/\mathbb{C}}$)

$$H^1(\Theta_X) \otimes H^p(\Omega^p_{X/\mathbb{Q}}) \to H^{p+1}(\Omega^{p-1}_{X/\mathbb{Q}}) \tag{47}$$

and in loc. cit. it is proved that

(48) *If* $\{Z\}$ *deforms to* 1^{st} *order in the direction* θ*, then*

$$\theta \cdot [Z]_a = 0$$

in $H^{p+1}(\Omega^{p-1}_{X/\mathbb{Q}})$.

At the infinitesimal level one has that the formal tangent space to the Chow groups is given by

$$TCH^p(X) \cong H^{p+1}(\Omega^{p-1}_{X/\mathbb{Q}}) \ .$$

The filtration $F^kTCH^p(X)$ results from the filtration induced on $TCH^p(X)$ by the filtration

$$F^k\Omega^{p-1}_{X/\mathbb{Q}} = \text{image}\left\{\Omega^k_{\mathbb{C}/\mathbb{Q}} \otimes \Omega^{p-1-k}_{X/\mathbb{Q}} \to \Omega^{p-1}_{X/\mathbb{Q}}\right\} \ ,$$

and this is the basis for (45).

In the $p = 2, k = 1$ case the pairing (46) is (omitting the $\Omega^1_{\mathbb{C}/\mathbb{Q}}$ factor and setting $\Omega^1_X = \Omega^1_{X/\mathbb{C}}$)

$$H^1(\Theta_X) \otimes H^2(\Omega^1_X) \to H^3(\mathcal{O}_X) \ .$$

Suppose that X is defined over $\overline{\mathbb{Q}}$, that the algebraic part $J^2_{\text{alg}}(X)$ of the intermediate Jacobian is non-zero and parameterized by algebraic cycles, and that for

$$TJ^2_{\text{alg}}(X) \subset H^{1,2}(X)$$

the pairing

$$H^1(\Theta_X) \otimes TJ^2_{\text{alg}}(X) \to H^3(\mathcal{O}_X)$$

is non-zero. Then there will be directions $\theta \in H^1(\Theta_X)$ and non-zero classes

$$\{Z\} \in F^1CH^2(X)$$

that do not deform to 1^{st} order in the direction θ.

Explicit examples may be constructed from Fermat hypersurfaces, as e.g. in the various works of Shioda [21].

Summary:

(i) *The HC implies that there exists* $k_0 \in \mathbb{Z}^*$ *and a smooth subvariety* W *with*

$$[W] = k_0\zeta + mH, \qquad m \gg 0 \; ; \tag{49}$$

which deforms with (X, ζ)*;*

(ii) *Still assuming the HC, there exist smooth* W *satisfying (48) and which do not deform with* (X, ζ)*, no matter what* k_0 *is and how large we take* m*.*

In (ii), there are (at least) two types of obstructions, *torsion obstructions* as in the Kollár-Soulé-Voisin example (cf. section 4.4.1 of "I"), and *layers of Chow obstructions* as described above.

IV(iii). *Codimension of Noether-Lefschetz loci for fourfolds*

As noted above, for an algebraic surface X and Hodge class $\zeta \in Hg^1(X)$, the Noether-Lefschetz locus $\mathcal{M}_{X,\zeta}$ in the moduli space $\mathcal{M}_X$ where ζ remains a Hodge class is an algebraic variety with

$$\operatorname{codim}_{\mathcal{M}_X}(\mathcal{M}_{X,\zeta}) \leqq h^{2,0} \; . \tag{50}$$

As also noted above the inequality (50) is equality in many examples and one may informally say that

$$\text{“expected” } \operatorname{codim}_{\mathcal{M}_X}(\mathcal{M}_{X,\zeta}) = h^{2,0} \; . \tag{51}$$

For X a fourfold, as will now be explained the situation is quite different. For $\zeta \in Hg^2(X)$ we have for the Zariski tangent space

$$T_X\mathcal{M}_{X,\zeta} \overset{\text{def}}{=} T_\zeta = \ker\{H^1(\Theta_X) \xrightarrow{\zeta} H^{1,3}(X)\} \; . \tag{52}$$

(53) **Definition:** *We set*

$$\sigma_\zeta = \dim\left(\text{Image}\{T_\zeta \otimes H^{4,0}(X) \to H^{3,1}(X)\}\right) \; .$$

(54) **Theorem:** *For the Noether-Lefschetz locus we have*

$$\operatorname{codim}_{\mathcal{M}_X}(\mathcal{M}_{X,\zeta}) \leqq h^{3,1} - \sigma_\zeta \; .$$

We shall prove this statement for the Zariski tangent space, which is sufficient. As will be seen in the next section, we will see that equality holds in (52) in significant examples. For this reason we propose to use the terminology

$$\text{“expected”}\ \ \operatorname{codim}_{\mathcal{M}_X}(\mathcal{M}_{X,\zeta}) = h^{3,1} - \sigma_\zeta\ .$$

Proof: Working in an analytic neighborhood of $X \in \mathcal{M}_X$, the Noether-Lefschetz locus is defined by the equations

$$(\zeta_t)^{(1,3)+(0,4)} = 0\ .$$

Thus it would seem at first glance that

$$\text{“expected”}\ \ \operatorname{codim}_{\mathcal{M}_X}(\mathcal{M}_{X,\zeta}) = h^{4,0} + h^{3,1}\ . \tag{55}$$

However, if $X = X_{t_0}$ since $\zeta_{t_0} \in H^{2,2}(X_{t_0})$ we have from the infinitesimal period relation

$$\text{to } 1^{\text{st}} \text{ order in any direction, } (\zeta'_{t_0})^{0,4} = 0\ . \tag{56}$$

Thus (55) might seem to need to be amended to

$$\text{“expected”}\ \ \operatorname{codim}_{\mathcal{M}_X}(\mathcal{M}_{X,\zeta}) = h^{3,1}\ . \tag{57}$$

Now (56) is based on the infinitesimal period relation, which may be viewed as an exterior differential system (EDS) on the classifying space for polarized Hodge structures. As for any EDS there are integrability conditions, and these need to be taken into account along integral manifolds, such as $\mathcal{M}_{X,\zeta}$ for the EDS (56). This is what we shall now do.

Denoting as usual the Gauss-Manin correction by ∇, for any

$$\begin{cases} \theta \in H^1(\Theta_X) \\ \omega \in H^{4,0}(X) \end{cases}$$

we may write (56) as

$$\langle \omega, \theta \cdot \zeta \rangle = 0$$

where the pairing is

$$H^{p,q}(X) \otimes H^{4-p,4-q}(X) \to \mathbb{C}\ .$$

Now suppose that

$$\theta \in T_\zeta \subset H^1(\Theta_X)$$

and let $\theta' \in H^1(\Theta_X)$. Then

$$\begin{aligned} \langle \theta \cdot \omega, \theta' \cdot \zeta \rangle &= \langle \omega, \theta\theta'\zeta \rangle \\ &= \langle \omega, \theta'\theta\zeta \rangle \\ &= 0 \end{aligned}$$

since $\theta\zeta \in H^{2,2}(X)$. □

The crucial step in this calculation is commuting the cup-product actions of θ and θ', which is a reflection of the integrability.

Discussion: Let $D_{\mathbf{h}}$ be the classifying space for the set of equivalence classes of polarized Hodge structures on a lattice $H_{\mathbb{Z}}$ with a nondegenerate symmetric form Q and with Hodge numbers

$$\mathbf{h} = (h^{4,0}, h^{3,1})$$

where $h^{1,3} = h^{3,1}$, $h^{0,4} = h^{4,0}$, $h^{2,2} = \text{rank } H_{\mathbb{Z}} - 2(h^{4,0} + h^{3,1})$ and

$$\text{signature } Q = 2h^{4,0} + h^{2,2} - 2h^{1,3} .$$

For $\zeta \in H_{\mathbb{Z}}$ denote by

$$D_{\mathbf{h},\zeta} \subset D_{\mathbf{h}}$$

the subvariety where ζ is of type $(2,2)$ (cf. the heuristic argument for (4.17) on page 29 in "I"). We have the period mapping

$$\begin{tikzcd} \mathcal{M}_X \arrow[r, "P"] & D_{\mathbf{h}} \\ \mathcal{M}_{X,\zeta} \arrow[u, phantom, "\cup"] \arrow[r, "P_*"] & D_{\mathbf{h},\zeta} \arrow[u, phantom, "\cup"] \end{tikzcd} \quad , \quad \mathcal{M}_{X,\zeta} = P^{-1}(D_{\mathbf{h},\zeta})$$

and $P_\zeta = P|_{\mathcal{M}_{X,\zeta}}$. Now

$$\text{codim}_{D_{\mathbf{h}}}(D_{\mathbf{h},\zeta}) = h^{4,0} + h^{3,1}$$

so that

$P(\mathcal{M}_X)$ does not meet $D_{\mathbf{h},\zeta}$ transversely.

If we let $I \subset TD_{\mathbf{h},\zeta}$ be the exterior differential system (EDS) given by the infinitesimal period relation, then the differential

$$P_* : T\mathcal{M}_X \to I \subset TD_{\mathbf{h}} .$$

Now I meets $TD_{\mathbf{h},\zeta}$ transversely along $D_{\mathbf{h},\zeta}$, and one might wonder if the period mapping is *I-transversal* along $\mathcal{M}_{X,\zeta}$ in the sense that

$$\operatorname{codim}_{\mathcal{M}_X}(\mathcal{M}_{X,\zeta}) = \operatorname{rank}(I_\zeta/I_\zeta \cap TD_{\mathbf{h},\zeta})$$

where $I_\zeta \subset TD_{\mathbf{h}}|_{D_{\mathbf{h},\zeta}}$ is the restriction of I to $D_{\mathbf{h},\zeta}$? In fact, this does not hold due to the integrability conditions, which contribute the σ_ζ term.

It is plausible that there is more sophisticated a concept of I-transversality based on the subvariety

$$V(I) \subset \prod_k \operatorname{Grass}(k, TD_{\mathbf{h}})$$

of integral elements of various dimensions, but we shall not pursue this here.

We do note that the notion of I-transversality appeared in the introduction in the discussion of (19) there, where one does in fact have such a result "at the boundary."

IV(iv). *Analysis of an example*

In this section we shall study the example

(58) $$\Lambda \subset X \subset \mathbb{P}^5$$

where X is a smooth hypersurface of degree $d \geqq 6$ and Λ is a 2-plane. We set

(59) $$\zeta = [\Lambda]_{\text{prim}}$$

and denote by

(60) $$\mathcal{M}_{X,\Lambda} \subset \mathcal{M}_X$$

the local moduli space of deformations X' of X for which Λ moves to a (unique) 2-plane $\Lambda' \subset X'$.[4] We note that since $h^0(N_{\Lambda/X}) = 0$

$$\mathcal{M}_{X,\Lambda} \subseteq \mathcal{M}_{X,\zeta}\ .$$

---

[4] More properly, one could define $\mathcal{M}_{X,\Lambda}$ to be the moduli space of pairs (X, Λ) as in (58) and then there is an étalè map $\mathcal{M}_{X,\Lambda} \to \mathcal{M}_X$. For the purpose of dimension counts we shall just work locally in moduli where we have (60).

In this section we shall prove the following statements (61)–(66):

$$\mathcal{M}_{X,\Lambda} = \mathcal{M}_{X,\zeta} \; . \tag{61}$$

Geometrically, *any deformation of X preserving the Hodge class ζ is obtained by deforming the pair (X,Λ) where (58) holds.*

$$\operatorname{codim}_{\mathcal{M}_X}(\mathcal{M}_{X,\zeta}) = h^{3,1} - \sigma_\zeta \; . \tag{62}$$

Thus, in this case the inequality in Theorem (54) is sharp. The result (62) will be proved by showing:

For $\varphi \in H^{3,1}(X)$,

$$\big\langle \varphi, \zeta \cdot H^1(\Theta_X)\big\rangle = 0 \Leftrightarrow \varphi \in T_\zeta \cdot H^{4,0}(X) \; . \tag{63}$$

The inequality in theorem (54)results from the implication "$\Leftarrow$", which always holds. Below we shall by dimension count prove "$\Rightarrow$". We shall also show that

$$h^{3,1} - \sigma_\zeta = h^1(N_{\Lambda/X}) \; . \tag{64}$$

Next, setting $L = \mathcal{O}_X(1)$ and denoting by l the number of nodes in general with $\Lambda \subset X_{s_0}$ we shall show that

$$\operatorname{codim}_{|L|}\big(\operatorname{sing}\nu_\zeta\big) = l - (h^{3,1} - \sigma_\zeta) - h^2(\Lambda^2 N_{\Lambda/X}(-1)\big) \; . \tag{65}$$

This is for fixed X. Now letting (X,ζ) vary — which by (61) is the same as letting (X,Λ) vary — we shall show that

$$\operatorname{codim}_{|\mathcal{L}|}(\operatorname{sing}\boldsymbol{\nu}_\zeta) = l - h^2\left(\Lambda^2 N_{\Lambda/X}(-1)\right) \; . \tag{66}$$

We note that

$$h^2\left(\Lambda^2_{\Lambda/X}(-1)\right) = h^0\left(\mathcal{O}_\Lambda(d-5)\right) = \binom{d-3}{2} \neq 0 \; . \tag{67}$$

Discussion: This example (for which the VHC holds) illustrates two fundamental differences between the classical $n = 1$ case and curves on a surface and the $n = 2$ case (surfaces on a fourfold). Namely

A. *Both the infinitesimal period relations and its integrability conditions enter in the codimension of the Noether-Lefschetz locus, which in general* (not just in special cases) *depends on the particular Hodge class.*

B. *There is a non-zero correction term* $h^2(\Lambda^2 N_{\Lambda/X}(-1))$ *to the codimension of* $\operatorname{sing}\boldsymbol{\nu}_\zeta$*. In particular, the to be defined rational map*

$$|\mathcal{L}| \overset{\Psi}{\dashrightarrow} \overline{\mathcal{A}}_{\mathbf{h},\Sigma}$$

always fails to be transverse along the boundary component $\mathcal{B}(\sigma(l,1))$ *with*

$$\Psi^{-1}(\mathcal{B}(\sigma(l,1)) = \operatorname{sing}\boldsymbol{\nu}_\zeta \ .$$

Assuming the VHC we have noted the phenomenon B in general but with the assumption that $L \gg 0$ relative to W. Here of course we see it in a concrete special case.

We now turn to the proofs of the assertions (61)–(66) above. These are based on explicit computations based on the well-known polynomial description of the relevant cohomology groups (cf. [15]). We suppose that

$$\mathbb{P}^5 = \mathbb{P}V^*$$

where $\dim V = 6$ and set $V^k = \operatorname{Sym}^k V$. Then X is given by

$$F = 0 \ , \qquad F \in V^d \ .$$

We denote by

$$\mathcal{J}^k \subset V^k$$

for the k^{th} homogeneous piece of the *Jacobian ideal* $\mathcal{J}_\bullet = \{F_{z_0}, \dots, F_{z_5}\}$ where $z_0, \dots z_5$ give a basis for V and $F_{z_i} = \partial F/\partial z_i$, and we set

$$R^k = V^k/\mathcal{J}^k \ . \tag{68}$$

Then it is well known that

$$H^1(\Theta_X) \cong R^d \tag{69}$$

$$H^q(\Omega^p_X)_{\text{prim}} \cong R^{(q+1)d-6} \qquad (p+q=4) \ . \tag{70}$$

We may suppose that

$$\Lambda = \{z_0 = z_1 = z_2 = 0\}$$

so that from $\Lambda \subset X$ we have

$$F = z_0G_0 + z_1G_1 + z_2G_2 \ .$$

Then denoting by $V^{\cdot}_{\Lambda}, \mathcal{J}^{\cdot}_{\Lambda}, R^{\cdot}_{\Lambda}$ the restrictions of $V^{\cdot}, \mathcal{J}^{\cdot}, R^{\cdot}$ to Λ we have

$$\mathcal{J}^{\bullet}_{\Lambda} = \{G_0, G_1, G_2\} \ .$$

For later use we note the identifications given by the vertical maps in the diagram

$$\begin{tikzcd}
0 \arrow[r] & N_{\Lambda/X} \arrow[r] \arrow[d, equal] & N_{\Lambda/\mathbb{P}^5} \arrow[r] \arrow[d, equal, "\wr"'] & N_{X/\mathbb{P}^5}\,|_{\Lambda} \arrow[r] \arrow[d, equal, "\wr"] & 0 \\
0 \arrow[r] & N_{\Lambda/X} \arrow[r] & \overset{3}{\oplus}\, \mathcal{O}_{\Lambda}(1) \arrow[r] & \mathcal{O}_{\Lambda(d)} \arrow[r] & 0 \ .
\end{tikzcd} \tag{71}$$

We shall now show that:

$$R^{3d-6}_{\Lambda} = V^{3d-6}_{\Lambda}/(G_0, G_1, G_2) \cong \mathbb{C} \tag{72}$$

and the surjective map

$$R^{3d-6} \to R^{3d-6}_{\Lambda}$$

gives a natural element of $H^2(\Omega^2_X)_{\mathrm{prim}}$ *which is just*

$$\zeta = [\Lambda]_{\mathrm{prim}} \ .$$

Up to a non-zero constant this is

$$\det\|\partial G_i/\partial z_j\| \qquad\qquad 0 \leqq i, \ \ j \leqq 2 \ .$$

For simplicity of notation we set

$$P = \det\|\partial G_i/\partial z_j\| \in R^{3d-6} \ .$$

Proof: From the cohomology arising from the Koszul complex for $\{G_0, G_2, G_2\}$

$$O \to \mathcal{O}_{\Lambda}(-3) \to \overset{3}{\oplus}\, \mathcal{O}_{\Lambda}(d-4) \to \overset{3}{\oplus}\, \mathcal{O}_{\Lambda}(2d-5) \to \mathcal{O}_{\Lambda}(3d-6) \to 0$$

and

$$H^2(\mathcal{O}_{\Lambda}(-3)) \cong H^2(\Omega^2_{\Lambda}) \cong \mathbb{C}$$

we conclude (72). The various identifications may be verified by standard compuations — what they amount to is that for $H \in R^{3d-6} \cong H^2(\Omega^2_{\Lambda})_{\mathrm{prim}}$

$$\int_{\Lambda} H = \text{ image of } H \text{ in } H^2(\Omega^2_{\Lambda}) \ . \qquad\qquad \square$$

For later use we note that

$$H^1(N_{\Lambda/X}) \cong V^d_\Lambda/(G_0,G_1,G_2) \tag{73}$$

$$H^1(N_{\Lambda/X}\otimes K_X) \cong V^{2d-6}_\Lambda/(G_0,G_1,G_2)\ . \tag{74}$$

The first follows from the cohomology sequence in the bottom row of (71), and the second from the cohomology sequence arising from tensoring the bottom row with $K_X \cong \mathcal{O}_X(d-6)$.

We now turn to the proof of (61). The 1st order deformation theory of pair (X,W) is given by the hypercohomology group $\mathbb{H}^1(\Theta_X \to N_{W/X})$, from which we conclude

$$\begin{pmatrix} \text{deformations of } X \\ \text{preserving } W \end{pmatrix} = \ker\left\{H^1(\Theta_X)\to H^1/N_{W/X}\right\} \tag{75}$$

In the case at hand, using the identifications above the RHS of (75) is the kernel of

$$V^d/\mathcal{J}^d \to V^d_\Lambda/\mathcal{J}^d_\Lambda \cong V^d_\Lambda/(G_0,G_1,G_2)$$

This kernel is just

$$I^d_\Lambda/I^d_\Lambda\cap\mathcal{J}^d \tag{76}$$

where $I^\bullet_\Lambda=\{z_0,z_1,z_2\}$ is the ideal of Λ.

From the commutative diagram

$$\begin{tikzcd} H^1(\Theta_X) \arrow[r, "\zeta"] \arrow[d, phantom, "\wr\|"] & H^3(\Omega^1_X) \arrow[d, phantom, "\wr\|"] \\ R^d \arrow[r, "P"] & R^{3d-6} \end{tikzcd}$$

we have

$$T_\zeta = [\mathcal{J}^\bullet : P]^d = \{R\in R^d : RP\in\mathcal{J}^{3d-6}\}\ .$$

It also follows from the discussion above that we have a factorization

$$\begin{tikzcd} R^{2d-6} \arrow[r, "P"] \arrow[dr] & R^{4d-12}\cong\mathbb{C} \\ & R^{2d-6}_\Lambda\ . \arrow[u] \end{tikzcd}$$

The basic diagram in the story is

(77)

$$\begin{tikzcd}
H^1(\Theta_X)\otimes H^1(\Omega^3_X) \arrow[r] \arrow[d] & H^2(\Omega^2_X) \arrow[d] \arrow[ddrr, "\zeta"] & & \\
H^1(\Theta_X|_\Lambda)\otimes H^1(\Omega^3_X|_\Lambda) \arrow[r] \arrow[d] & H^2(\Omega^2_X|_\Lambda) \arrow[d] & & \\
H^1(N_{\Lambda/X})\otimes H^1(N_{\Lambda/X}\otimes K_X) \arrow[r] & H^2(\Lambda^2 N_{\Lambda/X}\otimes K_X) \arrow[r, phantom, "\cong"] & H^2(K_\Lambda) \arrow[r, phantom, "\cong"] & \mathbb{C}
\end{tikzcd}$$

where we have used

$$\begin{cases} \Omega^3_X|_\Lambda \cong \quad K_X\otimes\Theta_X|_\Lambda \quad \to \quad K_X\otimes N_{\Lambda/X} \\ \Omega^2_X|_\Lambda \cong K_X\otimes\Lambda^2\Theta_X|_\Lambda \to K_X\otimes\Lambda^2 N_{\Lambda/X}\ . \end{cases}$$

From this diagram we have that

$$\left\langle \zeta\cdot\ker\left\{H^1(\Theta_X)\to H^1(N_{\Lambda/X}), H^1(\Omega^3_X)\right\}\right\rangle = 0$$

and thus

$$T_\zeta\subseteq\ker\left\{H^1(\Theta_X)\to H^1(N_{\Lambda/X})\right\}\ . \tag{78}$$

Geometrically this is clear: If Λ deforms to 1st order in the direction $\theta\in H^1(\Theta_X)$, then so does its fundamental class. It is the converse, which is equality in (61), that we want to establish.

The basic observation is that the map

$$H^1(\Omega^3_X)\to H^1(N_{\Lambda/X}\otimes K_X)\cong H^1(N^*_{\Lambda/X}\otimes K_\Lambda)\cong H^1(N_{\Lambda/X})^* \tag{79}$$

is surjective; this follows from the above identifications and the surjectivity of

$$R^{2d-6}\to R^{2d-6}_\Lambda\ .$$

It follows from this surjectivity that the left kernel T_ζ in the map

$$H^1(\Theta_X)\otimes H^1(\Omega^3_X)\to H^2(\Omega^2_X)\xrightarrow{\zeta}\mathbb{C}$$

(77) is equal to the left kernel of the vertical column in (77), which is just

$$\ker\left\{H^1(\Theta_X)\to H^1(N_{\Lambda/X})\right\}\ .$$

This establishes equality in the inclusion (78), and with it proves (61).

We next turn to (63). Working on the other side of the $\otimes$ in (77) we have

$$\begin{aligned}(80)\qquad (\mathrm{Im}\ \zeta)^{\perp} &= \ker\left\{H^1(\Omega^3_X)\to H^1(N_{\Lambda/X}\otimes K_X)\right\}\\ &= I^{2d-6}_{\Lambda}/R^{2d-6}_{\Lambda}\cap \mathcal{J}^{2d-6}\ .\end{aligned}$$

On the other hand we have seen that

$$T_{\zeta} = \mathrm{image}\left\{I^d_{\Lambda}/R^d_{\Lambda}\cap\mathcal{J}^d \to V^d/\mathcal{J}^d\right\}\ .$$

From the diagram

$$\begin{tikzcd}
T_{\zeta}\otimes H^0(\Omega^4_X) \arrow[r] \arrow[d, phantom, "\wr\|"] & (\mathrm{Image}\ \zeta)^{\perp} \arrow[d, phantom, "\wr\|"] \\
(I^d_{\Lambda}/I^d_{\Lambda}\cap\mathcal{J}^2)\otimes V^{d-6} \arrow[r] & I^{2d-6}_{\Lambda}/I^{2d-6}_{\Lambda}\cap\mathcal{J}^{2d-6}
\end{tikzcd}$$

in which the bottom row is surjective we conclude the surjectivity of the top row, which is equivalent to (63).

We now also have (62), since

$$\sigma_{\zeta} = \dim\left(\mathrm{coker}\left\{H^1(\Theta_X)\xrightarrow{\zeta}H^3(\Omega^1_X)\right\}\right)$$

which implies that

$$\mathrm{codim}\, T_{\zeta} = h^{3,1} - \sigma_{\zeta}\ ,$$

while we have seen above that also

$$\mathrm{codim}\, T_{\zeta} = h^1(N_{\Lambda/X})\ .$$

Turning to (65) we have from the Koszul resolution of the double point locus restricted to Λ the sequence

$$(81)\qquad 0\to\Lambda^2 N_{\Lambda/X}(-1)\to N_{\Lambda/X}\to\mathcal{O}_{\Lambda}(1)\to\mathcal{O}_{\Delta}(1)\to 0\ .$$

From the resulting spectral sequence and $h^0(\mathcal{O}_{\Delta}(1)) = l$ we obtain

$$l = \underbrace{h^0(\mathcal{O}_{\Lambda}(1)) + h^1(N_{\Lambda/X})} + h^2(\Lambda^2 N_{\Lambda/X}(-1))$$

and the term over the brackets is just

$$\mathrm{codim}_{|\mathcal{L}|}(\mathrm{sing}\,\boldsymbol{\nu}_{\zeta})\ .$$

This gives (65), and (66) is clear since in this case

$$\mathrm{codim}_{|L|}(\mathrm{sing}\,\nu_{\zeta}) = h^0(\mathcal{O}_{\Lambda}(1))\ .$$

V. MAPS IN THE CASE OF FOURFOLDS

V(i). *Structure of nodal boundary components in* $\overline{\mathcal{A}}_{\mathbf{h},\Sigma}$ *when* $\mathbf{h} = (h^{3,0}, h^{2,1})$ *with* $h^{3,0} \neq 0$

Kato–Usui [17] have defined a partial compactification for general variations of Hodge structures. Although not an analytic variety in the usual sense, these spaces do have a "logarithmic structure" and much of the classical theory when the classifying space is a bounded symmetric domain seems to extend in a useable manner. For us the relevant fact is that variations of Hodge structures defined over punctured polycylinders and with unipotent monodromies give maps that extend to the Kato-Usui partial compactifications.

As in the classical case boundary components are associated to certain rational cones σ of commuting nilpotent elements in a Lie algebra. Here we will describe two boundary components

$$\text{(1)} \qquad \mathcal{B}(\sigma(l)) \leftrightarrow \left\{ \begin{array}{c} \text{cone generated by logarithms } N_i \\ \text{of Picard-Lefschetz transformation} \\ \text{corresponding to } l \text{ independent nodes} \end{array} \right\}$$

and

$$\text{(2)} \quad \mathcal{B}(\sigma(l,1)) \leftrightarrow \left\{ \begin{array}{c} \text{cone generated by logarithms } N_i \\ \text{of Picard-Lefschetz transformations} \\ \text{corresponding to } l \text{ nodes with one relation} \end{array} \right\} .$$

We shall elucidate the structure of these boundary components in a way that is amenable to algebro-geometric calculations and shall show that

$$\text{(3)} \qquad \operatorname{codim}_{\overline{\mathcal{A}}_{h,\Sigma}}(\mathcal{B}(\sigma(l)) = l + l \cdot h^{3,0}$$

and

$$\text{(4)} \qquad \operatorname{codim}_{\overline{\mathcal{A}}_{\mathbf{h},\Sigma}}(\mathcal{B}(\sigma(l,1)) = l + (l-1)h^{3,0} .$$

These are at first glance surprising, since for example in case (1) when $l = 1$, acquiring a single node generally imposes one condition on a family of varieties. What the proofs of (3) and (4) will show is that, because of the infinitesimal period relation, when $h^{3,0} \neq 0$ period mappings in a family of threefolds acquiring nodes that are either independent or have one relation can never be transverse in the usual to the corresponding boundary component. However, in the next section we shall show that because of refined notion of dimension counts arising where one is considering integral manifolds of an exterior differential system, and may drop the terms $lh^{3,0}$ and $(l-1)h^{3,0}$ in (3) and (4) when estimating codimensions of the pullbacks of these boundary components under period mappings.

We shall now discuss the boundary component $\mathcal{B}(\sigma(l))$ in (1). For this we assume given $(H_{\mathbb{Z}}, Q)$ where $H_{\mathbb{Z}}$ is a lattice of rank $2h$ and Q is a unimodular symplectic form on $H_{\mathbb{Z}}$. We may describe $\mathcal{A}_{\mathbf{h}}$ as the set of polarized Hodge structures on $(H_{\mathbb{Z}}, Q)$ of weight three and with given Hodge numbers $h^{3,0}$ and $h^{2,1}$ where $h^{3,0}+h^{2,1}=h$. As usual we have

$$\mathcal{A}_{\mathbf{h}} \subset \check{\mathcal{A}}_{\mathbf{h}}$$

where the dual classifying space $\check{\mathcal{A}}_{\mathbf{h}}$ is the set of filtrations $F^3 \supset F^2$ with

$$\begin{cases} \dim F^3 = h^{3,0} \\ \dim F^2/F^3 = h^{2,1} \\ Q(F^2, F^2) = 0\ . \end{cases}$$

We set $F^1 = F^{3\perp}$.

Setting $G_{\mathbb{C}} = \mathrm{Aut}(H_{\mathbb{C}}, Q)$, in general boundary components are described as a quotient of several variable nilpotent orbits. Specifically we assume given

$$N_1, \dots, N_l = \mathcal{G}_{\mathbb{Q}}$$

satisfying

$$\begin{cases} N_i^4 = 0 \\ [N_i, N_j] = 0\ , \end{cases}$$

together with certain other conditions discussed in [7] and Kato-Usui (loc. cit.). Let

$$\sigma_{\mathbb{C}} = \left\{ \sum_i \lambda_i N_i : \lambda = (\lambda_1, \dots, \lambda_l) \in \mathbb{C}^l \right\}$$

be the nilpotent subalgebra of $\mathcal{G}_{\mathbb{C}}$ generated by the N_i. We set

$$\sigma = \left\{ \exp\left(\sum_i \lambda_i N_i \right) : \lambda_i > 0 \right\}$$

and then as a set the corresponding boundary component is

$$\mathcal{B}(\alpha) = \exp(\sigma_{\mathbb{C}}) \cdot \mathcal{A}_{\mathbf{h}} / \exp(\sigma_{\mathbb{C}}) \; .$$

As will be explained below, the necessity for passing to the quotient is because of rescaling.

We choose a symplectic basis $\delta_1, \dots, \delta_h; \gamma_1, \dots, \gamma_h$ for $H^*_{\mathbb{Z}}$, and shall write elements of H^*_Z relative to this basis as column vectors. Thus

$$\delta_1 = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \\ \\ 0 \end{pmatrix}, \dots, \gamma_h = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \\ 0 \\ 1 \end{pmatrix} \; .$$

Elements of $H_{\mathbb{C}}$ will then be written as row vectors

$$\begin{gathered} \delta_1^* = (1, 0, \dots, 0) \\ \vdots \\ \gamma_h^* = (0, \dots, 0, 1) \; , \end{gathered}$$

and transformation $N \in \mathcal{G}_{\mathbb{C}}$ will be realized as $2h \times 2h$ matrices multiplying row vectors on the right. Points $F^\bullet \in \check{\mathcal{A}}_{\mathbf{h}}$ will be written as *period matrices* obtained by choosing adapted bases for $F^3 \subset F^2$; pictorially

$$\Omega \;=\; \underbrace{\begin{pmatrix} * & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & * \end{pmatrix}}_{2h} \begin{matrix} \}h^{3,0} \\ \\ \}h^{2,1} \end{matrix}$$

Changing the adapted basis occurs by left multiplication by a nonsingular matrix of the form

$$\begin{array}{ccc} \left(\begin{matrix} * \\ 0 \end{matrix}\right. & \left.\begin{matrix} * \\ * \end{matrix}\right) & \begin{matrix} \}h^{3,0} \\ \}h^{2,1} \end{matrix} \\ \underbrace{\qquad}_{h^{3,0}} & \underbrace{\qquad}_{h^{2,1}} & \end{array} \tag{5}$$

Specifically, in case (1) above we may choose our symplectic basis so that

$$\delta_i^* N_j = \delta_i^j \gamma_j^* \qquad\qquad 1 \leqq i, j \leqq l$$

and all other terms are zero. Thus

$$N_i = \begin{array}{ccc} \left(\begin{matrix} 0 \\ 0 \end{matrix}\right. & \left.\begin{matrix} \eta_i \\ 0 \end{matrix}\right) & \begin{matrix} \}h \\ \}h \end{matrix} \\ \underbrace{\quad}_{h} & \underbrace{\quad}_{h} & \end{array}$$

where

$$\eta_i = \left(\begin{array}{ccc|c} \multicolumn{3}{c|}{\overbrace{\qquad\qquad}^{l}} & \\ 0 & & & 0 \\ \cdot & 1 & & \vdots \\ \cdot & & \cdot \ 0 & \vdots \\ \cline{1-3} 0 & & & 0 \end{array}\right) \Big\}\, l$$

has 1 in the (i,i) spot and is zero elsewhere. Below we shall show that

$$l \leqq h^{2,1} \tag{6}$$

and we shall correspondingly write period matrices in the block form

$$\Omega = \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} = \begin{array}{ccccc} \left(\begin{matrix} \Omega_{11} \\ \Omega_{21} \\ \Omega_{31} \end{matrix}\right. & \begin{matrix} \Omega_{12} \\ \Omega_{22} \\ \Omega_{32} \end{matrix} & \begin{matrix} \Omega_{13} \\ \Omega_{23} \\ \Omega_{33} \end{matrix} & \left.\begin{matrix} \Omega_{14} \\ \Omega_{24} \\ \Omega_{34} \end{matrix}\right) & \begin{matrix} \}h^{3,0} \\ \}l \\ \}\hat{h}^{2,1} \end{matrix} \\ \underbrace{\quad}_{l} & \underbrace{\quad}_{\hat{h}} & \underbrace{\quad}_{l} & \underbrace{\quad}_{\hat{h}} & \end{array} \tag{7}$$

where we have set

$$\begin{aligned} \hat{h} &= h - l \\ \hat{h}^{2,1} &= h^{2,1} - l\,. \end{aligned}$$

We shall now use period matrices to describe a general VHS $\Omega(t)$ with parameters $(t_1,\dots,t_l)\in(\Delta^*)^l$ and whose monodromy logarithm around $t_i=0$ is N_i. This monodromy assumption implies that

$$\tilde{\Omega}(t)\overset{\text{def}}{=}\Omega(t)\exp\left(-\sum_i\frac{\log t_i}{2\pi\sqrt{-1}}N_i\right)$$

is single-valued in $(t_1,\dots t_n)$. We may assume that $\tilde{\Omega}(t)$ is holomorphic in Δ^l and that $\tilde{\Omega}(0)\in\check{\mathcal{A}}_{\mathbf{h}}$. We then may write

$$\Omega(t)=\begin{pmatrix}\Omega_{11}(t)&\Omega_{12}(t)&L(\Omega_{11}(t))+H_1(t)&\Omega_{14}(t)\\ \Omega_{21}(t)&\Omega_{22}(t)&L(\Omega_{21}(t))+H_2(t)&\Omega_{24}(t)\\ \Omega_{31}(t)&\Omega_{32}(t)&L(\Omega_{31}(t))+H_3(t)&\Omega_{34}(t)\end{pmatrix}\tag{8}$$

and where

$$L(\Omega_{21}(t))=\Omega_{21}(t)\left(\sum_{i=1}^{l}\frac{\log t_i}{2\pi\sqrt{-1}}\eta_i\right)\tag{9}$$

where we are here considering η_i as an $l\times l$ matrix. The reason for (9) is that analytic contribution of $\Omega_{23}(t)$ around $t_i=0$ is given by

$$\Omega_{23}(t)\to\Omega_{23}(t)+\Omega_{21}(t)\ ,$$

and therefore

$$\Omega_{23}(t)-\Omega_{21}\left(\sum_{i=1}^{l}\frac{\log t_i}{2\pi\sqrt{-1}}\eta_i\right)=H_2(t)$$

is single-valued and holomorphic.

The main step in the argument is the following

(10) **Lemma:** $\Omega_{11}(t)=0 \mod \{t_1,\dots,t_l\}$.

Proof: The idea is to use the infinitesimal period relation

$$\Omega_1'(t)=A(t)\Omega_1(t)+B(t)\Omega_2(t)+C(t)\Omega_3(t)\tag{11}$$

where " $'$ " is any derivative and A,B,C are holomorphic matrices of the appropriate size. To see the essential point suppose first that $l=1$ and $h^{3,0}=1$. Then

$$\Omega_{11}(t)=(2\pi\sqrt{-1})\big(c+tf(t)\big)$$

where c is a constant and $f(t)$ is a holomorphic function. From the proof of (8) we have

$$\Omega_{13}(t) = c \log t + t f(t) \log t + h(t)$$

where $h(t)$ is again a holomorphic function. Then

$$\Omega'_{13}(t) = \frac{c}{t} + f(t) + f'(t) t \log t + h'(t)$$

from which we conclude that the constant $c = 0$, i.e.

$$\Omega_{11}(0) = 0 \ .$$

This argument extends as follows: Any non-zero constant term in an entry in $\Omega_{11}(t)$ will produce a term

$$c \log t_i, \qquad c \neq 0$$

in $L(\Omega_{11})$ and hence a non-zero polar term in Ω'_{13}. $\square$

This lemma has the inequality (6) as a corollary, since at $t_1 = \cdots = t_l = 0$ the first column

$$\begin{pmatrix} 0 \\ \Omega_{21}(0) \\ \Omega_{31}(0) \end{pmatrix}$$

of $\Omega(0)$ must have rank l. By a suitable basis change (5) we may make

$$\Omega_{21}(t) = I_l \ , \tag{12}$$

and then by a further basis change we may arrange that

$$\Omega_{31}(t) = 0 \ . \tag{13}$$

It is known that any VHS over $(\Delta^*)^l$ and with unipotent mondromy may be approximated by a nilpotent orbit. In this case the nilpotent orbit is

$$\tilde{\Omega}(t) \overset{\text{def}}{=} \tilde{\Omega} \cdot \exp\left(\sum_i \frac{\log t_i}{2\pi\sqrt{-1}} N_i \right)$$

where

$$\tilde{\Omega} = \tilde{\Omega}(0) \ .$$

The boundary component $\mathcal{B}(\sigma(l))$ may be intuitively thought of as all "limits of VHS's $\Omega(t)$" as above, modulo the ambiguity resulting from rescaling the t_i. Since

$$\text{“}\lim \Omega(t)\text{”} = \text{“}\lim \tilde{\Omega}(t)\text{”}$$

our strategy to compute $\dim \mathcal{B}(\sigma(l))$ will be to count the number of parameters in $\tilde{\Omega}$ and subract off the number resulting from rescaling.

Now using (13)

$$\tilde{\Omega}(t) = \begin{pmatrix} \Omega_{11}(t) & \Omega_{12}(t) & H_1(t) & \Omega_{14}(t) \\ I & \Omega_{22}(t) & H_2(t) & \Omega_{24}(t) \\ 0 & \Omega_{32}(t) & H_3(t) & \Omega_{34}(t) \end{pmatrix}$$

where we note that in (8)

$$\begin{cases} L(\Omega_{21}(t)) = \sum_i \frac{\log t_i}{2\pi\sqrt{-1}} \eta_i \\ L(\Omega_{31}(t)) = 0 \ . \end{cases}$$

From (10) we have at $t = 0$

$$\tilde{\Omega} = \begin{pmatrix} 0 & \Omega_{12} & H_1 & \Omega_{11} \\ I & \Omega_{22} & H_2 & \Omega_{21} \\ 0 & \Omega_{32} & H_3 & \Omega_{31} \end{pmatrix} \ . \tag{14}$$

Restricting the VHS to a disc $\Delta^* \subset (\Delta^*)^l$ with "slope" $\lambda_1, \dots, \lambda_l$ where all $\lambda_i \neq 0$ gives a 1-parameter VHS with a monodromy weight filtration, which by [5] is independent of the slope, and limiting MHS given by (14). Denoting the logarithm of monodromy by N_λ, the monodromy weight filtration is

$$\{0\} = W_0 = W_1 \subset \underset{\substack{\| \\ \operatorname{Im} N_\lambda}}{W_2} \subset \underset{\substack{\| \\ \operatorname{Ker} N_\lambda}}{W_3} \subset W_4 = W_5 = H \ .$$

The weight three Hodge structure on $Gr_3 W_0 = W_3/W_2$ corresponds to a polarized complex torus $\hat{\mathcal{J}}$ with Hodge numbers

$$\begin{cases} \hat{h}^{3,0} = h^{3,0} \\ \hat{h}^{2,1} = h^{2,1} - l \ . \end{cases}$$

To see this we note that the monodromy weight filtration and limiting Hodge filtration may be pictured as

$$\tilde{\Omega} \;=\; \left.\begin{array}{c} \begin{pmatrix} 0 & \Omega_{12} & H_1 & \Omega_{14} \\ I & \Omega_{22} & H_2 & \Omega_{24} \\ 0 & \Omega_{32} & H_3 & \Omega_{34} \end{pmatrix} \begin{array}{l} \}\, F^3 \\ \\ \end{array} \end{array}\right\} F^2 \qquad \begin{array}{c} \underbrace{\;H_3\;}_{W_2} \\ \underbrace{\qquad\qquad\quad}_{W_3} \end{array} \tag{15}$$

Note that

$$F^3 \subset W_3 \ ,$$

i.e., under a VHS of the type given above

> *the Hodge number $H^{3,0}$ does not change in the limit, and the Hodge number $h^{3,1}$ drops by* $\dim Gr_4 = \dim Gr_2$.

In addition we have

> *for the limiting MHS we have $F^2 \cap W_{2,\mathbb{C}} = 0$; i.e., the Hodge structure on W_2 is "of Tate type".*

Degeneration of this type, which are very special, have been studied by Clemens [8]. We are indebted to Gregory Pearlstein for showing us interesting examples where $N^2 = 0$ but $F^2 \cap W_{2,\mathbb{C}} \neq 0$.

The mixed Hodge structure on W_3 gives rise to an extension

$$0 \to (\mathbb{C}^*)^l \to \mathfrak{J}_e \to \hat{\mathfrak{J}} \to 0 \ . \tag{16}$$

where $\mathfrak{J}_e$ is a complex Lie group of dimension h, which may be thought of as the intermediate Jacobian analogue of a quasi-abelian variety, and $\hat{\mathfrak{J}}$ is a polarized complex torus of dimension $\hat{h}$. The blocks in the period matrix (15) have the following interpretations:

$$\begin{pmatrix} \Omega_{12} & \Omega_{14} \\ \Omega_{32} & \Omega_{34} \end{pmatrix} \text{ is the period matrix of } \hat{\mathfrak{J}} \tag{17}$$

$$\Omega_{24} \text{ gives the extension data for (16).} \tag{18}$$

We note that

$$\text{(19)} \qquad \#\ \text{parameters in } \hat{\jmath} = \frac{\hat{h}(\hat{h}+1)}{2} + \hat{h}^{3,0}\hat{h}^{2,1}$$

$$\text{(20)} \qquad \#\ \text{parameters in } \Omega_{24} = l\hat{h}\ .$$

Thus far we have not used the Riemann-Hodge bilinear relations, which it seems must enter in any significant result. The first bilinear relations are

$$\begin{array}{rl}
\text{(i)} & \Omega_{12}{}^t\Omega_{14} = \Omega_{14}{}^t\Omega_{12} \\
\text{(ii)} & {}^tH_2 + \Omega_{22}{}^t\Omega_{24} = H_2 + \Omega_{24}{}^t\Omega_{22} \Rightarrow H_2 = {}^tH_2 \\
\text{(iii)} & \Omega_{32}{}^t\Omega_{34} = \Omega_{34}{}^t\Omega_{32} \\
\text{(iv)} & \Omega_{12}{}^t\Omega_{24} = H_1 + \Omega_{34}{}^t\Omega_{22} \\
\text{(v)} & \Omega_{12}{}^t\Omega_{34} = \Omega_{14}{}^t\Omega_{32} \\
\text{(vi)} & \Omega_{24}{}^t\Omega_{32} = H_3 + {}^t\Omega_{34}\Omega_{22}.
\end{array}$$

We note that

> (i), (iii), (v) *give the* 1^{st} *bilinear relations for the polarized Hodge structure on* Gr_3W.

Next we turn to rescaling . Under

$$t_i \to \exp(2\pi\sqrt{-1}u_i)t_i$$

we see from (8) that all the block entries in $\tilde{\Omega}$ are unchanged except that

$$H_2 \to H_2 + \begin{pmatrix} u_1 & & 0 \\ & \ddots & \\ 0 & & u_l \end{pmatrix}.$$

Since from the bilinear relation (ii) we have that H_2 is symmetric, we conclude that

$$\text{(21)} \qquad \#\ \text{essential parameters in } H_2 = l(l-1)/2\ .$$

For the last step we will use the remaining basis change (5) that preserves the normalization made thus far. We note that

$$\operatorname{rank}\begin{pmatrix}\Omega_{12}\\ \Omega_{32}\end{pmatrix} = \hat{h}$$

and that we may add linear combinations of the rows of Ω_{12} to the rows in Ω_{22} and Ω_{32}, and similarly we may add linear combinations of the rows in Ω_{32} to those in Ω_{22}. Write

$$\begin{aligned} \Omega_{22} &= \Big(\underbrace{\Omega'_{12}}_{\hat{h}^{3,0}}, \underbrace{\Omega''_{22}}_{\hat{h}_{2,1}} \Big) \Big\} \hat{h}^{3,0} \\ \Omega_{32} &= \Big(\underbrace{\Omega'_{32}}_{\hat{h}^{3,0}}, \underbrace{\Omega''_{32}}_{\hat{h}^{2,1}} \Big) \Big\} \hat{h}^{2,1} \ . \end{aligned}$$

An open set in $\check{\mathcal{A}}_{\mathbf{h}}$ is given by the conditions

$$\det \Omega'_{12} \neq 0, \ \ \det \Omega''_{32} \neq 0 \ ,$$

and for the purposes of parameter counts it will suffice to work in that set. Then we may normalize to have

$$\left\{ \begin{array}{l} \Omega'_{12} = I_{\hat{h}^{3,0}} \\ \Omega'_{32} = 0, \ \Omega''_{32} = I_{\hat{h}^{2,1}} \ . \end{array} \right. \tag{22}$$

Then by a transformation as described above we may arrange that

$$\Omega_{22} = 0 \ . \tag{23}$$

From the bilinear relations (iv) and (vi) we then find

H_1 and H_3 are determined by the data (17) and (18).

We are now ready for the parameter count:

$$\begin{aligned} \dim \mathcal{A}_{\mathbf{h}} &= \frac{h(h+1)}{2} + h^{3,0}h^{2,1} \\ &= \frac{(\hat{h}+l)(\hat{h}+l+1)}{2} + \hat{h}^{3,0}(l+\hat{h}^{2,1}) \\ &= \underbrace{\frac{\hat{h}(\hat{h}+1)}{2} + \hat{h}^{3,0}\hat{h}^{2,1}}_{\substack{\| \\ (19)}} + \underbrace{\ \hat{h}l\ }_{\substack{\| \\ (20)}} + \underbrace{\frac{l(l-1}{2})}_{\substack{\| \\ (21)}} + (l + lh^{3,0}) \ . \end{aligned}$$

The last term gives the codimension of the boundary component $\mathcal{B}(\sigma(l))$, thus establishing (3). ☐

We now turn to the boundary component $\mathcal{B}(\sigma(l,1))$ and the proof of (4). The argument will be analogous to that given above for the proof of (3). In this case we choose a symplectic basis

$$\delta_1, \dots, \delta_{l-1}, \delta_l, \dots \delta_h, \quad \gamma_1, \dots, \gamma_{l-1}, \gamma_l, \dots, \gamma_h$$

for $H^*_{\mathbb{Z}}$ and set

$$\delta = \delta_1 + \cdots + \delta_{l-1} \,.$$

We define N_i for $1 \leqq i \leqq l$ by

$$\left\{ \begin{array}{ll} \delta^*_i N_j = \delta^j_i \gamma^*_j & \qquad\qquad 1 \leqq i,\ j \leqq l-1 \\ \delta^*_i N_l = \gamma^*_1 + \cdots + \gamma^*_{l-1} \,, & \end{array} \right. \tag{24}$$

where N_l is modeled on a Picard-Lefschetz transformation corresponding to the vanishing cycle δ. In analogy to (6) above we will see that

$$l - 1 \leqq h^{2,1} \,. \tag{25}$$

Using the above notations we have

$$N_i = \left(\begin{array}{cc} 0 & \eta_i \\ \underbrace{0}_{h} & \underbrace{0}_{h} \end{array} \right) \begin{array}{l} \}h \\ \}h \end{array}$$

where

$$\eta_i = \left(\begin{array}{ccc|c} 0 & \cdots & 0 & 0 \\ \vdots & 1 & \vdots & \cdot \\ 0 & \cdots & 0 & \cdot \\ \hline 0 & \cdot & \cdot & 0 \end{array} \right) \begin{array}{l} \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} {\scriptstyle l-1} \\ \}{\scriptstyle \hat{h} = h-(l-1)} \end{array} \qquad\qquad 1 \leqq i \leqq l-1$$

$$\eta_l = \left(\begin{array}{ccc|c} 1 & \cdots & 1 & 0 \\ \vdots & & \vdots & \cdot \\ 1 & \cdots & 1 & \cdot \\ \hline 0 & \cdot & \cdot & 0 \end{array} \right) \begin{array}{l} \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} {\scriptstyle l-1} \\ \}{\scriptstyle \hat{h} = h-(l-1)} \end{array}$$

The nilpotent orbits will arise from

$$\sigma_{\mathbb{C}} = \left\{ \sum_i \lambda_i N_i : \lambda = (\lambda_1, \dots, \lambda_l) \in \mathbb{C}^l \right\} \,.$$

The same arguments as above, especially lemma (10), lead to the partially normalized period matrix

(26)

$$\Omega(t) \;=\; \begin{array}{c} \begin{pmatrix} \Omega_{11}(t) & \Omega_{12}(t) & L(\Omega_{11}(t))+H_1(t) & \Omega_{14}(t) \\ I & \Omega_{22}(t) & L(t)+H_2(t) & \Omega_{24}(t) \\ 0 & \Omega_{32}(t) & H_3(t) & \end{pmatrix} \\ \underbrace{\hphantom{\Omega_{11}}}_{l-1}\;\underbrace{\hphantom{\Omega_{12}(t)}}_{\hat h=h-(l-1)}\;\underbrace{\hphantom{L(\Omega_{11}(t))+H}}_{l-1}\;\underbrace{\hphantom{\Omega_{14}(t)}}_{\hat h=h-(l-1)} \end{array} \begin{matrix} \}\hat h^{3,0} \\ \}l-1 \\ \}\hat h^{2,1}=h^{2,1}-(l-1) \\ \\ \end{matrix}$$

where

$$(27) \qquad L(t) = \begin{pmatrix} \frac{\log t_1}{2\pi\sqrt{-1}} & & 0 \\ & \ddots & \\ 0 & & \frac{\log t_{l-1}}{2\pi\sqrt{-1}} \end{pmatrix} + \begin{pmatrix} \frac{\log t_l}{2\pi\sqrt{-1}} & \cdots & \frac{\log t_l}{2\pi\sqrt{-1}} \\ \vdots & & \vdots \\ \frac{\log t_l}{2\pi\sqrt{-1}} & \cdots & \frac{\log t_l}{2\pi\sqrt{-1}} \end{pmatrix}$$

and where

$$\Omega_{11}(0) = 0\ .$$

For $\tilde\Omega$ given by $\Omega(t)\exp\left(-\sum_{i=1}^{1}\frac{\log t_i}{2\pi\sqrt{-1}}N_i\right)$ at $t=0$ we then have

$$\tilde\Omega = \begin{pmatrix} 0 & \Omega_{12} & H_1 & \Omega_{14} \\ I & \Omega_{22} & H_2 & \Omega_{24} \\ 0 & \Omega_{32} & H_3 & \Omega_{34} \end{pmatrix}\ .$$

As before we may further normalize so as to have

$$\tilde\Omega \;=\; \begin{array}{c} \begin{pmatrix} 0 & I & \Omega''_{12} & H_1 & \Omega_{14} \\ I & 0 & 0 & H_2 & \Omega_{24} \\ 0 & 0 & I & H_3 & \Omega_{34} \end{pmatrix} \\ \underbrace{\hphantom{00}}_{l-1}\;\underbrace{\hphantom{00}}_{\hat h^{3,0}}\;\underbrace{\hphantom{00}}_{\hat h^{2,1}}\;\underbrace{\hphantom{00}}_{l-1}\;\underbrace{\hphantom{00}}_{\hat h} \end{array} \begin{matrix} \}\hat h^{3,0}=h^{3,0} \\ \}l-1 \\ \}\hat h^{2,1}=h^{2,1}-(l-1) \\ \\ \end{matrix}$$

where

$$(28) \qquad \begin{matrix} \hat h^{3,0}\{ \\ \hat h^{2,1}\{ \\ \\ \end{matrix} \begin{array}{c} \begin{pmatrix} I & \Omega''_{12} & \Omega_{14} \\ 0 & I & \Omega_{34} \end{pmatrix} \\ \underbrace{\hphantom{00}}_{\hat h^{3,0}}\;\underbrace{\hphantom{00}}_{\hat h^{2,1}}\;\underbrace{\hphantom{00}}_{\hat h} \end{array} \quad \leftrightarrow \quad \hat{\mathcal{J}} \in \mathcal{A}_{\hat{\mathbf{h}}}$$

$$(29) \qquad \Omega_{24} \qquad \leftrightarrow \quad e \in \mathrm{Ext}^1(\hat{\mathcal{J}}, (\mathbb{C}^*)^{l-1})\ .$$

Also as before H_1 and H_3 are uniquely determined by the $1^{\rm st}$ bilinear relation from the data (28), (29) which also gives

$$H_2 = {}^tH_2\ .$$

The main new point is that under rescaling

$$H_2 \to H_2 + \begin{pmatrix} u_1 & & 0 \\ & \ddots & \\ 0 & & u_{l-1} \end{pmatrix} + \begin{pmatrix} u_1 & \cdots & u_l \\ \vdots & & \vdots \\ u_1 & \cdots & u_l \end{pmatrix}\ . \tag{30}$$

Thus

(i) # of parameters in (28) $= \frac{\hat h(\hat h+1)}{2} + \hat h^{3,0}\hat h^{2,1}$

(ii) # of parameters in (29) $= (l-1)\hat h$

and by (30)

(iii) # of parameters in $H_2 = \frac{(l-1)(l)}{2} - l = \frac{l(l-3)}{2}$.

Then

$$\begin{aligned}\dim \mathcal{A}_{\mathbf{h}} &= \frac{h(h+1)}{2} + h^{3,0}h^{2,1} \\ &= \frac{(\hat h + l - 1)(\hat h + l)}{2} + \hat h^{3,0}(\hat h^{2,1} + l - 1) \\ &= \underbrace{\frac{\hat h(\hat h+1)}{2} + \hat h^{3,0}\hat h^{2,1}}_{\text{(i)}} + \underbrace{\hat h(l-1)}_{\text{(ii)}} + \underbrace{\frac{l(l-3)}{2}}_{\text{(iii)}} + l + h^{3,0}(l-1)\ .\end{aligned}$$

where the last term on the right results from

$$\frac{l(l-1)}{2} = \frac{l(l-3)}{2} + l\ .$$

In conclusion, since $\dim \mathcal{B}(\sigma(l,1)) = \text{(i)} + \text{(ii)} + \text{(iii)}$

$$\dim \mathcal{A}_{\mathbf{h}} = \dim \mathcal{B}(\sigma(l,1)) + l + h^{3,0}(l-1)$$

as desired. □

V(ii). *Non-transversality of the VHS at the boundary components $\mathcal{B}(\sigma(l))$ and $\mathcal{B}(\sigma(l,1))$*

For the situation

$$|L| \overset{\psi}{-\!\!-\!\!\!\longrightarrow} \overline{\mathcal{A}}_{\mathbf{h},\Sigma}$$

of interest in this paper, all we can a priori conclude from (3) in §V(i) is that

$$\text{(31)} \qquad \operatorname{codim}_{|L|}\left(\psi^{-1}(\mathcal{B}(\sigma(l))\right)_{s_0} \leqq l + lh^{3,0}$$

near where X_{s_0} has l independent nodes. But we know that, e.g. when $l = 1$, we have

$$\text{(32)} \qquad \operatorname{codim}_{|L|}\left(\psi^{-1}(\mathcal{B}(\sigma(l))\right)_{s_0} \leqq l\ ,$$

with equality frequently holding at least for small l. Geometrically, for $L \gg 0$ the first few nodes impose independent conditions on $|L|$. In fact, we shall now show

The inequality (32) always holds.

In other words,

(33) *The mapping ψ always fails, by at least in dimension equal to $lh^{3,0}$, to be transverse along $\mathcal{B}(\sigma(l))$.*

We shall see that (33) is a general VHS result. For the proof it will be notationally convenient to interchange the 2nd and 3rd row and column blocks in the period matrix (14) and write it as follows

$$\text{(34)} \qquad \tilde{\Omega} = \begin{array}{l} \left(\begin{array}{cccccc} I & \omega_{12} & 0 & \omega_{14} & \omega_{15} & K_1 \\ 0 & I & 0 & \omega_{24} & \omega_{25} & K_2 \\ 0 & 0 & I & \omega_{34} & \omega_{35} & K_3 \end{array}\right) \begin{array}{l} \}\hat{h}^{3,0} \\ \}\hat{h}^{2,1} \\ \}l \end{array} \\ \begin{array}{cccccc} \underbrace{\quad}_{\hat{h}^{3,0}} & \underbrace{\quad}_{\hat{h}^{2,1}} & \underbrace{\quad}_{l} & \underbrace{\quad}_{\hat{h}^{3,0}} & \underbrace{\quad}_{\hat{h}^{2,1}} & \underbrace{\quad}_{l} \end{array} \end{array}$$

Then

$$\begin{pmatrix} I & \omega_{12} & \omega_{14} & \omega_{15} \\ 0 & I & \omega_{24} & \omega_{24} \end{pmatrix} \text{ corresponds to } \hat{\mathcal{J}} \in \mathcal{A}_{\hat{\mathbf{h}}}$$

and

$$\left(\omega_{34}, \omega_{35}\right) \text{ corresponds to the extension class } e \in \operatorname{Ext}^1(\hat{\mathcal{J}}, (\mathbb{C}^*)^l)$$

for the MHS given by $(W_3, F^\bullet)$.

The 1st bilinear relations give

$$\begin{aligned} &\text{(i)} && \omega_{14}+\omega_{15}{}^t\omega_{12}={}^t\omega_{14}+\omega_{12}{}^t\omega_{15}\\ &\text{(ii)} && \omega_{15}={}^t\omega_{24}+\omega_{12}{}^t\omega_{25}\\ &\text{(iii)} && \omega_{25}={}^t\omega_{25}\\ &\text{(iv)} && K_1={}^t\omega_{34}+\omega_{12}{}^t\omega_{35}\\ &\text{(v)} && K_2={}^t\omega_{35}\\ &\text{(vi)} && K_3={}^tK_3. \end{aligned}$$

We note that (i), (ii), (iii) are the 1st blinear relations for $\hat{\mathcal{J}}$, and (iv), (v) show that K_1 and K_2 are determined by $(e, \hat{\mathcal{J}})$. (vi) corresponds to the symmetry (ii) in section V(i).

We write the new blocks of $\tilde{\Omega}$ as $\omega_1, \omega_2, \omega_3$

$$\tilde{\Omega}=\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} .$$

Then for a VHS given by a map

$$S \to \overline{\mathcal{A}}_{\mathbf{h},\Sigma} \tag{35}$$

we have along the inverse image of $\mathcal{B}(\sigma(l))$ that

$$\omega_1' = A\omega_1 + B\omega_2 + C\omega_3 \tag{36}$$

where $'$ is any derivative along a direction mapping to $T(\overline{\mathcal{B}}(\sigma(l))$. From (34) and $\omega_{11}'=0$ we infer that

$$A = 0 .$$

From $\omega_{13}'=0$ we obtain

$$C = 0 .$$

Then (35) gives

$$B = \omega_{12}' . \tag{37}$$

The formulas for ω_{14}' and ω_{15}' simply say that W_3 gives an admissable VMHS (cf. [20]). The remaining relations in (36) together with (37) give

$$\left({}^t\omega_{34}+\omega_{12}{}^t\omega_{25}\right)' = \omega_{12}'{}^t\omega_{35} . \tag{38}$$

This is a matrix equation of size $\hat{h}^{3,0} \times l$ and may be interpreted as measuring the amount of *non*-transversality in the intersection of the image of TS and $T\mathcal{B}(\sigma(l))$. Put another way

> $\omega_{12}, \omega_{34}, \omega_{35}$ *are free to move arbitrarily in the image of a general map (35), but are subject to (38) for a VHS*

which is what was to be shown. □

The same argument just given, when directly adapted to calculations given above in section V(i) for the proof of (4) there, give the conclusion

(39) *We have*

$$\operatorname{codim}_{|L|} \psi^{-1}(\mathcal{B}(\sigma(l,1)) \leqq l\ .$$

> *In words, the mapping ψ always fails, by at least in dimension equal to $h^{3,0}(l-1)$, to be transverse along $\mathcal{B}(\sigma(l,1))$.*

The general geometric reason for this argument was explained in the introduction. The above are the specific calculations that apply in the two cases of interest here.

V(iii). *The excess intersection formula in the $n = 2$ case*

We will apply the considerations of the remark at the end of the section II(ii) where we had the general setup

$$\begin{tikzcd} A \arrow[r, "F"] & M \\ C \arrow[u, phantom, "\cup"] & \\ B \arrow[u, phantom, "\cup"] \arrow[r, "F"] & N \end{tikzcd} \qquad F^{-1}(N) = B \tag{40}$$

to the case where

$$A = |\mathcal{L}|, \quad C = |\mathcal{L}|_\zeta, \quad B = \operatorname{sing} \boldsymbol{\nu}_\zeta$$
$$M = \overline{\mathcal{A}}_{\mathbf{h},\Sigma}, \quad N = \mathcal{B}(\sigma(l,1))$$

and

$$\begin{cases} F = \Psi \\ \Psi_\zeta = \Psi|_{|\mathcal{L}|_\zeta} \ . \end{cases}$$

For simplicity of notation we set

$$\begin{cases} \mathcal{B} = \mathcal{B}(\sigma(l,1)) \\ \mathcal{S}_\zeta = (\operatorname{sing} \boldsymbol{\nu}_\zeta)_{s_0} \ . \end{cases}$$

The diagram (40) is then

$$\begin{tikzcd} {|\mathcal{L}|} \arrow[r, dashed, "\Psi"] & \overline{\mathcal{A}}_{\mathbf{h},\Sigma} \\ \cup & \| \\ {|\mathcal{L}|_\zeta} \arrow[r, dashed, "\Psi_\zeta"] & \overline{\mathcal{A}}_{\mathbf{h},\Sigma} \\ \cup & \cup \\ \mathcal{S}_\zeta \arrow[r, dashed] & \mathcal{B} \end{tikzcd} \qquad \mathcal{S}_\zeta = \Psi^{-1}(\mathcal{B}) \ . \tag{41}$$

As before we are considering only the component $(\operatorname{sing} \boldsymbol{\nu}_\zeta)_{s_0}$ of $\Psi^{-1}(\mathcal{B})$. Keeping the notations from section IV(i), we make the additional assumptions

$$\begin{cases} \operatorname{codim}_{\mathcal{M}_X}(\mathcal{M}_{X,\zeta}) = h^{3,1} - \sigma_\zeta \\ H^1(\Theta_X) \twoheadrightarrow H^1(N_{W/X}) \ . \end{cases} \tag{42}$$

These assumptions are satisfied in our basic example (section IV(iv)) and conceivably are satisfied "in general". With these assumptions the codimensions in (41) are

$$\begin{cases} \text{(i)} & \operatorname{codim}_{|\mathcal{L}|}\left(\Psi^{-1}(\mathcal{B})\right) = l - \delta \\ \text{(ii)} & \operatorname{codim}_{|\mathcal{L}|_\zeta}\left(\Psi_\zeta^{-1}(\mathcal{B})\right) = l - \delta - (h^{3,1} - \sigma_\zeta) \\ \text{(iii)} & \operatorname{codim}_{\overline{\mathcal{A}}_{\mathbf{h},\Sigma}}(\mathcal{B}) = l + h^{3,0}(l-1) \end{cases} \tag{43}$$

where

$$\delta = \dim\left(\ker\left\{H^2(\Lambda^2 N_{W/X}(L^*)) \xrightarrow{ds_0} H^2(N_{W/X})\right\}\right)$$

so that $\delta \gg 0$ for $L \gg 0$ relative to W.

There are two excess intersection formulas, one for

$$\Psi^*([\mathcal{B}]) \in H^{2(l+h^{3,0}(l-1))}(|\mathcal{L}|) \ , \tag{44}$$

and one for

$$\Psi_\zeta^*([\mathcal{B}]) \in H^{2(l+h^{3,0}(l-1))}(|\mathcal{L}|_\zeta) \ . \tag{45}$$

For (44) the excess normal bundle

$$\mathcal{E} \to \mathcal{S}_\zeta$$

has two components which we denote by $\mathcal{E}'$ and $\mathcal{E}''$ (this means that topologically $\mathcal{E} = \mathcal{E}' \oplus \mathcal{E}''$). Their fibres are given by

$$\mathcal{E}'_{s_0} = I_{\Psi(s_0)}/(I \cap T\mathcal{B})_{\Psi(s_0)} \tag{46}$$

where $I \subset T\overline{\mathcal{A}}_{\mathbf{h},\Sigma}$ is the infinitesimal period relation, and

$$\mathcal{E}''_{s_0} = \ker\left\{H^2(\Lambda^2 N_{W/X}(L^*)) \xrightarrow{ds_0} H^2(N_{W/X})\right\} \ . \tag{47}$$

Here, we are *not* claiming that $I \subset T\overline{\mathcal{A}}_{\mathbf{h},\Sigma}$ has been defined in general, but only that it is defined along the locus in which we are interested, if one wishes by the explicit computations in section V(i). With this understood we have the

(48) **Theorem:** *With the above notations and assumptions we have*

$$\Psi^*([\mathcal{B}]) = Gy_{\mathcal{S}_\zeta/|\mathcal{L}|}\left(c_{\text{top}}(\mathcal{E}') \cdot c_{\text{top}}(\mathcal{E}'')\right) \ .$$

For the excess intersection formula corresponding to (45), we have an additional component $\mathcal{E}'''$ to the excess normal bundle whose fibres are

$$\mathcal{E}'''_{s_0} = N_{\mathcal{M}_{X,\zeta}/\mathcal{M}_X} \ . \tag{49}$$

(50) **Theorem:** *With again keeping our notations and assumptions we have*

$$\Psi^*_\zeta([\mathcal{B}]) = Gy_{\mathcal{S}_\zeta/|\mathcal{L}|_\zeta}\left(c_{\text{top}}(\mathcal{E}') \cdot c_{\text{top}}(\mathcal{E}'') \cdot c_{\text{top}}(\mathcal{E}''')\right)$$

where the fibres of $\mathcal{E}', \mathcal{E}'', \mathcal{E}'''$ are given by (46), (47), and (49) respectively.

The ranks of $\mathcal{E}', \mathcal{E}'', \mathcal{E}'''$ are given by

$$\begin{cases} \text{rank } \mathcal{E}' = h^{3,0}(l-1) \\ \text{rank } \mathcal{E}'' = \delta \\ \text{rank } \mathcal{E}''' = h^{3,1} - \sigma_\zeta \ , \end{cases}$$

and using (43) one may verify that both sides of the formulas (48) and (49) are in $H^{2(l+h^{3,0}(l-1))}(|\mathcal{L}|)$ and $H^{2(l+h^{3,0}(l-1))}(|\mathcal{L}|_\zeta)$ respectively.

Our story is not complete in that we have not described $\mathcal{E}', \mathcal{E}'', \mathcal{E}'''$ globally over $\mathcal{S}_\zeta$; all we have done is give their fibre at a general point. We expect to take up this consideration together with the computation of an example in a subsequent work.

VI. Conclusions

A first conclusion is that

(1) *When considering questions of Hodge classes and extended normal functions one should let* (X, ζ) *vary in moduli.*

The basic diagram governing the situation is

(2)
$$\begin{tikzcd} & \overline{\mathcal{J}}_{\mathbf{h},\Sigma} \arrow[d] \\ |\mathcal{L}| \arrow[ur, dashed, "\boldsymbol{\nu}_\zeta"] \arrow[r, dashed, "\Psi"] & \overline{\mathcal{A}}_{\mathbf{h},\Sigma} \end{tikzcd}$$

Here, $\mathcal{J}_{\mathbf{h},\Sigma}$ is a partial compactification of the universal family of intermediate Jacobians. An issue is

(3) *The family* $\overline{\mathcal{J}}_{\mathbf{h},\Sigma}$ *has not yet been defined in general.*

In this paper we have explicitly given the set-theoretic boundary components of the partial compactification $\overline{\mathcal{A}}_{\mathbf{h},\Sigma}$ by describing the limiting MHS's in the nodal case where there is at most one relation among the vanishing cycles, and Kato-Usui have some preliminary handwritten notes that describe the part of $\overline{\mathcal{J}}_{\mathbf{h},\Sigma}$ lying over the components of $\overline{\mathcal{A}}_{\mathbf{h},\Sigma}$ around which the monodromy satisfies $(T-I)^2 = 0$, which includes the nodal case mentioned above. Of special interest may be the boundary components where the local systems $\mathcal{H}(\mathcal{I})$ with "fibres" $H^1(B^\bullet(\mathbf{N}))$ are supported (cf. the end of section I(ii)).

From sections V(i), V(ii) we would suggest that the geometry of the infinitesimal period relations and resulting "I-transversality" may be of particular interest.

Restricting to nodal loci, one has the basic observation

(4) $\operatorname{sing} \boldsymbol{\nu}_\zeta$ *is equal to components of* $\Psi^{-1}(\mathcal{B}(\sigma(l, 1)))$.

This suggests that the study

$$\operatorname{codim}_{|\mathcal{L}|}(\operatorname{sing}\nu_\zeta) \tag{5}$$

and

$$\Psi^*([\mathcal{B}(\sigma(l,1)]) \ .$$

From (4) and general excess intersection formula considerations one expects a relation of the form

$$[\operatorname{sing}_{|\mathcal{L}|}(\boldsymbol{\nu}_\zeta)] = \Psi^*_\zeta([\mathcal{B}(\sigma(l,1)])_\wedge K \tag{6}$$

where K is a "correction" term. Much of this paper has been devoted to an initial study of (5) and (6), and perhaps the major issue raised in this work is to

(7) *Define* $\overline{\mathcal{J}}_{\mathbf{h},\Sigma}$ *and a set of boundary components* $\mathcal{B}_\lambda \subset \overline{\mathcal{J}}_{\mathbf{h},\Sigma}$ *such that for* $L \gg 0$

$$\operatorname{sing}\boldsymbol{\nu}_\zeta = \underset{\lambda}{\cap}\boldsymbol{\nu}_\zeta^{-1}(\mathcal{B}_\lambda). \tag{8}$$

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