PAGE 13 NONMOTIVATED EXAMPLES OF PRIME GEODESICS:

THERE IS A VERY MOTIVATED VARIANT ALREADY POSED BY GAUSS. THE PRIME GEODESICS $\mathcal{P}$ COME WITH TWO INVARIANTS, $p = \{\gamma\}_{\Gamma}$ HAS ITS TRACE $t(p)$ AND DISCRIMINAT $d(p)$ (ACCORDING TO THE DESCRIPTION [SA]).

$t(p)$ IS DETERMINED BY $d(p)$ VIA THE PELL EQUATION $t^2 - du^2 = 4$ WHOSE BEHAVIOR IS A NOTORIOUSLY DIFFICULT PROBLEM. DOES CHEBOTAREV HOLD WHEN THE p'S ARE ORDERED BY $d(p)$? FOR $\rho \in Epi(\Gamma, G)$

$$\sum_{\substack{d(p) \le x \\ B_G^{|p|}(p) = C}} 1 \sim \frac{|C|}{|G|} \sum_{d(p) \le x}{}' 1 \quad \text{AS } x \to \infty ? \qquad (5)$$

A CONJECTURE [HOO] GIVES A PROPOSED ASYMPTOTIC FOR THE RHS OF (5) AND THIS HAS BEEN EXAMINED NUMERICALLY (KWON, PETROW P.U. SENIOR THESES)

PAGE 17: TWISTED SPECTRAL COUNTING FOR LATTICES

FOR $\Gamma \leq G$ A LATTICE IN A REAL SEMISIMPLE MATRIX GROUP G, AND GROWING "ROUND" SUBSETS Ω_T OF G, ON CAN ESTIMATE

$$N(T) := |\Omega_T \cap \Gamma| \sim \frac{\mu(\Omega_T)}{\mu(G/\Gamma)} \quad \text{AS } T \to \infty$$

μ BEING HAAR MEASURE ON G.
([DEL], [SEL]) FOR $G = SL_2(\mathbb{R})$ AND [D-R-S]
MORE GENERALLY – WE WILL USE THE GENERAL RESULTS OF GORODNIK-NEVO [G-N].

LET X BE A COMPACT METRIC SPACE AND $\sigma : \Gamma \to \mathrm{HOMEO}(X)$ A MORPHISM GIVING AN ACTION OF Γ ON X. WE ARE INTERESTED IN THE DISTRIBUTION OF THE POINTS $\sigma(\gamma)\xi$, $\gamma \in \Omega_T \cap \Gamma$ IN X. THAT IS OF THE PROBABILITY MEASURES ON X GIVEN BY

$$\nu_{\xi,T} = \frac{1}{N(T)} \sum_{\gamma \in \Omega_T \cap \Gamma} \delta_{\sigma(\gamma)\xi}, \quad \text{WHERE } \xi \in X$$

AND δ_v A POINT MASS AT v.

IF ν IS A Γ-INVARIANT ERGODIC PROBABILITY MEASURE ON X THEN ONE CAN STUDY THESE $\nu_{\xi,T}$'s QUITE GENERALLY AS $T \to \infty$ AND FOR ν-ALMOST ALL ξ.
OUR INTEREST IS UNDERSTAND THE LIMITS FOR EVERY ξ WHICH IS DIFFICULT IN GENERAL. HOWEVER IF $\sigma(\Gamma)$ ACTS ISOMETRICALLY ON X THEN WE HAVE

THEOREM [G-N CHAPT 6]; IF $\sigma(\Gamma)$ ACTS ISOMETRICALLY ON X AND IS ERGODIC W.R.T ν AND SUPPORT $\nu = X$ THEN

$$\frac{1}{N_\Gamma(T)} \sum_{\gamma \in \Omega_T \cap \Gamma}' \delta_{\sigma(\gamma)\xi} \longrightarrow \nu \quad \text{AS } T \to \infty$$

IE FOR ALL $f \in C(X)$, $\xi \in X$

$$\frac{1}{N_\Gamma(T)} \sum_{\gamma \in \Omega_T \cap \Gamma}' f(\sigma(\gamma)\xi) \longrightarrow \int_X f(x)\, d\nu(x).$$

- ONE CAN GIVE A RATE OF CONVERGENCE IN THE ABOVE WHICH DEPENDS ON THE SPECTRAL GAP FOR THE ACTION OF G ON $L^2(G \times X/\Gamma)$ WITH Γ ACTING DIAGONALLY ON THE PRODUCT.

EISENSTEIN CASE:

WE NEED A VARIANT OF THE ABOVE WITH $\Gamma = SL_2(\mathbb{Z})$ (OR $PGL_2(\mathbb{Z})$) AND $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z} \right\}$ THE UNIPOTENT SUBGROUP AND WHERE WE COUNT COSETS.

FOR THE GROWING ROUND SETS IN $\mathbb{R}^2$ WE TAKE $\{ \|v\| \le T \}$ WHERE $\| \ \|$ CORRESPONDS TO A (CONTINUOUS) BOUNDED CONVEX SYMMETRIC (ABOUT 0) BODY. LET $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ AND

$$N_T = \left| \{ \gamma e_1 ; \gamma \in \Gamma/\Gamma_\infty, \|\gamma e_1\| \le T \} \right|$$

$$\sim \frac{6}{\pi^2} \operatorname{AREA}(\Omega_T) \sim \frac{6}{\pi^2} T^2 \quad \text{AS } T \to \infty.$$

LET X BE A COMPACT METRIC SPACE AND σ AN ACTION OF Γ BY HOMEOS OF X.

LET $\pi : X \to Z$ BE CONTINUOUS AND Γ_∞ INVARIANT; IE.

$$\pi(\sigma(m)x) = \pi(x) \quad \text{FOR } x \in X \text{ AND } m \in \Gamma_\infty.$$

THE PROBABILITY MEASURES* ON Z FOR $x \in X$

$$\mu_{T,x} = \frac{1}{N_T} \sum_{\substack{\gamma \in \Gamma/\Gamma_\infty \\ \|\gamma e_1\| \leq T}} \delta_{\pi(\sigma x)}$$

ARE WELL DEFINED AND DEPEND ONLY ON $\sigma x \in Z$.

- ONE CAN SHOW THAT UNDER THE SAME ASSUMPTIONS : ν ERGODIC AND OF FULL SUPPORT ON X AND Γ ACTING ISOMETRICALLY, THAT FOR ANY x;

$$\mu_{T,x} \longrightarrow \pi^{\#}(\nu), \quad \text{ON } Z$$

($\pi^{\#}$ IS THE PUSH FORWARD)

AGAIN THE RATE IS CONTROLLED BY THE SAME SPECTRAL GAP (BUT ONLY FOR THE EISENSTEIN PART OF THE SPECTRUM ON $G \times X / \Gamma$).

WE APPLY THIS TO STUDY THE ACTION OF THE MAPPING CLASS GROUP $\Gamma \cong PGL_2(\mathbb{Z})$ ~~OF~~ THE ONCE PUNCTURED TORUS $\Sigma_{1,1}$ ~~FOR~~, ~~R~~ACTING ON CHARACTER VARIETIES OF FINITE AND p-ADIC GROUPS. IN THE CASE OF A FINITE GROUP THE ACTION IS BY PERMUTATIONS OF A FINITE SET X AND HENCE BY ISOMETRIES OF X WITH THE DISCRETE METRIC. FOR THE p-ADIC CASE THE ACTION OF Γ ON $X(\mathbb{Z}_p)$ IS BY ISOMETRIES W.R.T. THE p-ADIC METRIC.

APPLICATION:

$$\pi_1(\Sigma_{1,1}) = F_2 = \langle A, B \rangle$$ FREE GROUP ON A, B.

G A COMPACT GROUP WITH HAAR MEASURE μ.

$G^\#$ THE CONJUGACY CLASSES OF G

$\pi : G \to G^\#$ TAKE $g \to \{g\}_G$.

$\pi^\#(\mu)$ IS THE CHEBOTAREV MEASURE ON $G^\#$.

G-FINITE:

$Epi(F_2, G)$ THE EPIMORPHISMS ρ OF F_2 TO G.
THESE ARE PARAMETRIZED BY PAIRS (C, D) IN $G \times G$ WHICH GENERATE G WITH $\rho(A) = C$, $\rho(B) = D$.

$X = Epi(F_2, G)/G$ THE CHARACTER CLASSES

$(C, D) \sim (C', D')$ IF $(gCg^{-1}, gDg^{-1}) = (C', D')$ SOME $g \in G$.

- THE MAPPING CLASS GROUP $\Gamma \cong PGL_2(\mathbb{Z}) \cong OUT(F_2)$ ACTS AS PERMUTATIONS OF X BY NIELSEN MOVES.

- IT PRESERVES THE HIGMAN INVARIANT $t_\rho \in G^{\#}$ OF $\bar{\rho}$ IN X; NAMELY THE CONJUGACY CLASS t_ρ OF $\rho(A)\rho(B)\rho(A^{-1})\rho(B^{-1})$.

- LET $Y_1 \sqcup Y_2 \ldots \sqcup Y_\nu = X$ BE THE DISTINCT ORBITS OF Γ ON X. EACH Y_j HAS A COMMON HIGMAN INVARIANT t_j AND Γ ACTS TRANSITIVELY ON Y_j. THESE ARE THE t-SYSTEMS OF THE NEUMANS.

THE PROJECTION $\pi : X \to G^{\#}$ ON THE FIRST COORDINATE

$$\overline{(C, D)} \to \{C\}_G$$

IS WELL DEFINED AND Γ_∞ INVARIANT ($\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ IS THE CORRESPONDIN MOVE)

FOR EACH $j = 1, \ldots, \nu$

$$\pi|_{Y_j} : Y_j \to G^{\#}$$

AND ν_{Y_j} THE COUNTING MEASURE ON Y_j IS THE UNIQUE Γ INVARIANT PROB. MEASURE.

THEN

$$\pi_{Y_j}^{\#}(\nu_{Y_j})(\{g\}_G) = \frac{\# \pi_{Y_j}^{-1}(\{g\})}{\# Y}$$

WE CAN NOW STATE THE CHEBOTAREV THEOREM FOR SIMPLE CLOSED GEODESICS ON $\Sigma_{1,1}$ WITH A FIXED HYPERBOLIC METRIC.

LET $\rho \in Epi(\pi_1(\Sigma_{1,1}), G)$ IF THE IMAGE OF ρ IN X LIES IN Y_j THEN

FOR $c \in G^{\#}$

$$\sum_{\substack{p \text{ simple} \\ \text{length}(p) \leq T \\ B(p) = c}} 1 \sim \frac{|\pi_j^{-1}(c)|}{|Y|} \sum_{\substack{p \text{ simple} \\ \ell(p) \leq T}} 1.$$

AS $T \to \infty$.

<u>PROOF</u>: APPLY THE DISCUSSION FROM THE TWISTED LATTICE COUNT FOR $\Gamma = PGL_2(\mathbb{Z})$ TO THE DETERMINATION OF THE SIMPLE CLOSED GEODESICS ON $\Sigma_{1,1}$ DUE TO McSHANE-RIVIN [M-S].

EXAMPLES:

$G = SL_2(\mathbb{F}_p)$, p A LARGE PRIME.

THE CONJUGACY CLASSES $G^{\#}$ ARE PARAMETRIZED BY THE TRACE $t \in \mathbb{F}_p$ (EXCEPT FOR $t = \pm 2$). THE TRANSITIVITY PROPERTIES OF Γ ON X ARE EXPECTED TO BE ESSENTIALLY DETERMINED BY THE THE HIGMAN INVARIANT (CONJECTURES OF [M-W] WHICH ARE PROVED FOR ~~ALL~~ MOST PRIMES, ALL BUT A POSSIBLE SMALL SET OF EXCEPTIONS [B-G-S]).

THUS FOR ρ HAVING HIGMAN INVARIANT $t_\rho \in \mathbb{F}_p$, THE MEASURE $\Pi_y(\nu_y)$ ~~BE~~ AT t IS

$$\frac{\#\{t^2 + y^2 + z^2 - tyz = t_\rho + 2\}}{\#\{x^2 + y^2 + z^2 - xyz = t_\rho + 2\}}$$

SOLUTIONS IN $\mathbb{F}_p$.

(13)

- IF G IS COMPACT THE SET UP IS SIMILAR.

FOR EXAMPLE IN THE INTERESTING CASE OF $G = SU(2)$.

IF $\rho \in \mathrm{Epi}(F, SU(2))$ ITS HIGMAN INVARIANT IS $t_\rho = \mathrm{TRACE}(\rho(A)\rho(B)\rho(A^{-1})\rho(B^{-1})) \in [-2,2]$.

THE CORRESPONDING (REAL) CHARACTER VARIETY IS

$$X_k : \quad x^2+y^2+z^2+xyz = k = t_\rho + 2 \in [0,4].$$
$$|x|, |y|, |z| \leq 2.$$

THE MEASURE ν_k ON X_k

$$\nu_k = \frac{dx\,dy}{2z+xy}$$

IS Γ INVARIANT (GOLDMAN) AND ERGODIC.

$\pi^{\#}(\nu_k)$ IS SUPPORTED ON; $|\cos\theta| \leq \frac{\sqrt{k}}{2}$

AND IS A MULTIPLE OF $d\theta$ ON THIS INTERVAL.

GIVEN THAT ν_k IS THE UNIQUE Γ-INVARIANT PROBABILITY MEASURE ON X_k [C-D-MB] WE EXPECT THAT THE CHEBOTAREV THEOREM FOR $\rho : F \longrightarrow SU(2)$ HOLDS ~~WITH~~ FOR SIMPLE CLOSED GEODESICS ON $\Sigma_{1,1}$ WITH THE EQUIDISTRIBUTION MEASURE $\pi^{\#}(\nu_k)$ AS ABOVE, AND $k = t_\rho + 2$. ONE CANNOT APPLY THE RESULT ON PAGE 7 SINCE THE ACTION OF Γ ON X_k IS NOT ISOMETRIC. WHAT WOULD SUFFICE, WHAT WOULD SUFFICE IS TO SHOW THAT ANY LIMIT OF $\mu_{T,x}$ IS Γ-INVARIANT.