AHLFORS LECTURE I

APPLICATIONS OF POINTS ON SUBVARIETIES OF TORI

PETER SARNAK

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LANG'S $\mathbb{G}_m$ CONJECTURES (1970's)

COMES IN TWO FLAVORS
VERTICAL AND HORIZONTAL.

$\mathbb{G}_m$ THE MULTIPLICATIVE GROUP $\mathbb{C}^*$ OF NON-ZERO COMPLEX NUMBERS.

$T = (\mathbb{C}^*)^N$ IS AN N-TORUS, AN ABELIAN ALGEBRAIC GROUP UNDER COORD MULTIPLICATION!

- $V \subset (\mathbb{C}^*)^N$ AN ALGEBRAIC SUBVARIETY GIVEN BY THE ZERO SET OF LAURENT POLYNOMIALS.

FOR EXAMPLE

$$V: \quad z_1 + z_1^{-1} + z_2 + z_2^{-1} + z_3 + z_3^{-1} + z_4 + z_4^{-1} = 0 \quad \text{IN } (\mathbb{C}^*)^4$$

- $\mathrm{tor}(T) = \{(z_1, z_2, .., z_N) : z_j \text{ IS A ROOT OF ONE FOR } j = 1, 2, \dots N.\}$

$\mathrm{tor}(T)$ CONSISTS OF ALL OF THE POINTS IN T OF FINITE ORDER.

VERTICAL CONJECTURE:

GIVEN $V \subset T$ AS ABOVE, THERE ARE FINITELY MANY SUBTORI (OR TRANSLATES THEREOF BY TORSION POINTS) $T_1, T_2, \ldots, T_\nu$ CONTAINED IN V SUCH THAT

$$\mathrm{tor}(T) \cap V = \mathrm{tor}(T) \cap (T_1 \cup T_2 \cup \cdots \cup T_\nu).$$

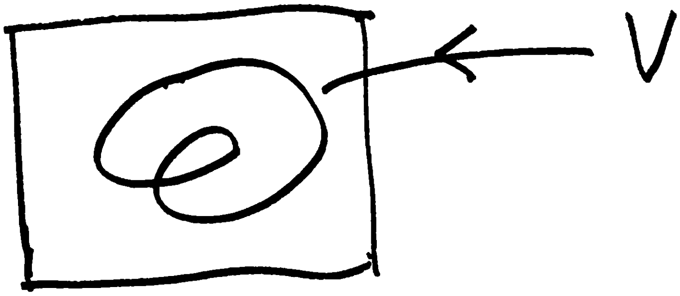
SO WHAT APPEARS TO BE A NONLINEAR VERY COMPLICATED PROBLEM IS IN FACT STRUCTURED IN THAT TORSION POINTS LIE ON FINITELY MANY COSETS OF SUBGROUPS IN V.

- NOTE THE T_j'S MAY BE ZERO DIMENSIONAL IN WHICH CASE THEY ARE TORSION POINTS.

- THERE ARE A NUMBER OF PROOFS OF THIS VERTICAL CASE AND SOME OF THESE ARE EFFECTIVE IN DETERMINING THE T_j'S. THIS IS RELEVANT IN APPLICATIONS.

ONE PROOF PROCEEDS AS FOLLOWS:

$N=2$: $\quad V \cap (S^1 \times S^1) \subset V \cap T \qquad S^1 = \{z \mid |z|=1\}$



IF $\zeta = (\zeta_1, \zeta_2) \in tor(T) \cap V$, $\zeta_1^{m_1} = \zeta_2^{m_2} = 1$.
IF $\sigma \in GAL(K(\zeta_1,\zeta_2)/K)$ WHERE K IS THE FIELD OF DEFINITION OF V, THEN

$$\sigma((\zeta_1,\zeta_2)) \in tor(T) \cap V.$$

NOW THESE GALOIS ORBITS GROW FAST WITH THE ORDER OF ζ SINCE

$$\deg[\mathbb{Q}(\zeta_m):\mathbb{Q}] = \phi(m) \gg_\varepsilon m^{1-\varepsilon}.$$

HENCE IF ONE CAN ESTABLISH A SUITABLE NON-TRIVIAL UPPER BOUND FOR THE NUMBER OF TORSION POINTS ON V (ASSUMING V CONTAINS NO SUBTORI) THEN ONE IS LED TO THERE BEING NO SUCH POINTS OF LARGE ORDER.

• SUCH UPPER BOUNDS CAN BE GIVEN IN THIS TORUS CASE BY ELEMENTARY METHODS.

EG: JARNIK: A STRICTLY CONVEX CURVE IN THE PLANE OF LENGTH L HAS AT MOST $L^{2/3}$ LATTICE POINTS.

HERON

THIS UPPER BOUND VS SIZE OF GALOIS ORBIT
METHOD HAS RECENTLY PROVEN TO BE ROBUST
FOR VERTICAL PROBLEMS:

- BOMBIERI-PILA : GIVE UPPER BOUNDS, SHARP UP TO
THE EXPONENT, FOR RATIONAL POINTS ON
TRANSCENDENTAL CURVES IN THE PLANE.

- PILA-WILKIE : GIVE SHARP UPPER BOUNDS FOR
RATIONAL POINTS ON THE TRANSCENDENTAL
PARTS OF DEFINABL SETS IN o-MINIMAL STRUCTURES
IN $\mathbb{R}^n$.

- PILA-ZANNIER : PROVE THE VERTICAL TORSION
POINTS ON AN ABELIAN VARIETY VERSION OF
LANG ; ALSO KNOWN AS THE MANIN-MUMFORD CONJ.

- THE VERTICAL ANALOGUE IN SHIMURA
VARIETIES OF TORSION POINTS ARE "CM-
POINTS" AND THESE SHOULD LIE ON FINITELY
MANY SHIMURA SUBVARIETIES ; "ANDRE-OORT CONJ".

- PROVED FOR PRODUCTS OF MODULAR CURVES;
(PILA)

- PROVED FOR $\mathcal{A}_g$ BY PILA-TSIMERMAN.
A KEY FURTHER INGREDIENT IS AN AX-LINDEMANN
THEOREM.

HORIZONTAL LANG CONJECTURE:

IF $V \subset T$ AS ABOVE AND Γ IS A FINITELY GENERATED SUBGROUP OF T; THERE ARE FINITELY MANY TRANSLATES OF SUBTORI $T_1, T_2, \dots, T_\nu$ CONTAINED IN V SUCH THAT

$$\Gamma \cap V = \Gamma \cap (T_1 \cup T_2 \dots \cup T_\nu) .$$

THIS LIES DEEPER AND WAS PROVEN BY M. LAURENT. THE KEY INPUT IS THE SCHMIDT SUBSPACE THEOREM WHICH IS A STRIKING HIGHER DIMENSIONAL VERSION OF THE THUE-SIEGEL-ROTH THEOREM.

SIMPLEST VERSION (SCHMIDT):

LET $L_1(x), L_2(x), \dots L_n(x)$ BE n LINEARLY INDEPENDENT FORMS IN $x = (x_1, \dots, x_n)$ WITH REAL ALGEBRAIC NUMBER COEFFICIENTS, THEN FOR $\varepsilon > 0$ THE SET OF $x \in \mathbb{Z}^n$ SATISFYING

$$|L_1(x) L_2(x) \dots L_n(x)| < \|x\|^{-\varepsilon}$$

LIE IN FINITELY MANY PROPER $\mathbb{Q}$ LINEAR SUBSPACES OF $\mathbb{Q}^n$.

NOTE: THE PROOF YIELDS AN EFFECTIVE BOUND FOR THE NUMBER OF SUBSPACES BUT NOT THEIR DETERMINATION.

VERTICAL AND HORIZONAL

ONE CAN COMBINE THESE

LET $\overline{\Gamma}$ BE THE DIVISION GROUP OF Γ

$$\overline{\Gamma} = \{ z \in T : z^{\ell} \in \Gamma \text{ FOR SOME } \ell \geq 1 \}$$

$$\overline{1} = \mathrm{tor}(T).$$

THE ULTIMATE VERSION WHICH IS ALSO UNIFORM OVER DEFINING FIELDS AND QUANTITATIVE IN THE RANK r OF Γ IS DUE TO

EVERTSE / SCHLICKEWEI / SCHMIDT

<u>THEOREM</u>: $V \subset (\mathbb{C}^*)^N$, Γ A FINITELY GENERATED SUBGROUP OF RANK r, THERE ARE $T_1, T_2, \ldots T_\nu$ TRANSLATES OF SUBTORI CONTAINED IN V SUCH THAT

$$\overline{\Gamma} \cap V = \overline{\Gamma} \cap (T_1 \cup T_2 \cup \ldots T_\nu)$$

AND $\quad \nu \leq (C(V))^{r}.$

<u>REMARK</u>: THE CONSTANT $C(V)$ CAN BE GIVEN EXPLICITLY, HOWEVER THE ACTUAL SAY ZERO DIMENSIONAL T_j'S CANNOT IN GENERAL BE DETERMINED BY THIS PROOF.

THE PROOF INVOLVES SPECIALIZATION ARGUMENT & REDUCING TO $\Gamma \subset T(\overline{\mathbb{Q}})$ AND ABSOLUTE HEIGHT VERSIONS OF THE SCHMIDT SUBSPACE THEOREM AS WELL AS A STUDY OF POINTS OF SMALL HEIGHT.

A SPECIAL ROLE IS PLAYED BY

$$V: \quad a_1 x_1 + a_2 x_2 + \cdots + a_N x_N = 1,$$

IN $(\mathbb{C}^*)^N$.

THREE APPLICATIONS:

(A) (VERTICAL) BETTI NUMBERS IN ABELIAN COVERS

LET X BE A TOPOLOGICAL SPACE
$\pi_1(X)$ ITS FUNDAMENTAL GROUP
$H_1(X,\mathbb{Z})$ FIRST HOMOLOGY GROUP (ASSUME INFINITE)

FOR $N \geq 1$ LET ρ_N BE

$$\begin{tikzcd} \pi_1(X) \arrow[r, two heads] \arrow[rr, bend right, "\rho_N"'] & H_1(X,\mathbb{Z}) \arrow[r, two heads] & H_1(X, \mathbb{Z}/N\mathbb{Z}) \end{tikzcd}$$

ρ_N DEFINES A FINITE ABELIAN COVERING $Y(N)$ OF X WITH DECK GROUP $\pi_1(X)/\ker \rho_N$.

<u>QUESTION</u> HOW DOES THE TOPOLOGY OF $Y(N)$ DEPEND ON N?

THE EULER NUMBERS AND CHERN NUMBERS VARY MULTIPLICATIVELY IN N (UNRAMIFIED) BUT THE BETTI NUMBERS ARE MORE SUBTLE.

• LANG'S VERTICAL $\mathbb{G}_m$ IMPLIES THAT THE BETTI NUMBERS ARE <u>POLYNOMIAL PERIODIC</u> IN N. THAT IS THERE IS $q \geq 1$ SUCH THAT $b_j(Y(N))$ IS A POLYNOMIAL IN N ALONG ARITHMETIC PROGRESSIONS MOD q.

THIS WAS EXPLOITED BY E. HIRONAKA TO STUDY THE IRREGULARITIES (FIRST BETTI NO) OF RAMIFIED ABELIAN COVERS OF SURFACES.

X A (COMPLEX) SURFACE, $C_1, C_2, \ldots, C_\ell$ DISTINCT IRREDUCIBLE CURVES IN X.

- THE UNRAMIFIED ABELIAN COVERS $Y(N)$ OF $X \setminus C$ $(C = C_1 \cup C_2 \cdots \cup C_\ell)$ CONSTRUCTED ABOVE CAN BE RESOLVED TO COMPLETE SURFACES $\tilde{Y}(N)$ BRANCHED OVER C.

THEY YIELD INTERESTING SURFACES OF GENERAL TYPE (ZARISKI, HIRZEBRUCH, LIVNE, ...).
FOR EXAMPLE IF $C_1, C_2, \ldots C_\ell$ ARE LINES IN $\mathbb{P}^2$ THESE CAN YIELD EXAMPLES WITH EQUALITY $C_1^2 = 3C_2$ IN THE BOGOMOLOV–MIYAOKA-YAU INEQUALITY.

<u>E. HIRONAKA</u> THE IRREGULARITIES OF $\tilde{Y}(N)$ ARE <u>POLYNOMIAL PERIODIC</u> IN N.

SO THE IRREGULARITIES ARE NOT SO IRREGULAR IN ABELIAN COVERS.

THE REDUCTION TO LANG INVOLVES UNDERSTANDING THE JUMP AND NORMALIZATION LOCI IN THE CHARACTER VARIETY

$$CH(X \setminus C) = \left\{ \chi : H_1(X \setminus C, \mathbb{Z}) \longrightarrow \mathbb{C}^* \atop \chi \text{ A CHARACTER} \right\}$$

WHICH IS A TORUS.

THE JUMP LOCI ARE THE POINTS AT WHICH THE DIMENSIONS OF THE χ-TWISTED COHOMOLOGY GROUPS JUMP. THESE AND THE NORMALIZATION LOCI ARE ALGEBRAIC SUBVARIETIES AND BETTI NUMBERS ARE DETERMINED BY THE INTERSECTION OF THE N-TORSION POINTS WITH THESE LOCI.

<u>REMARK</u> IN MANY OF THESE ALGEBRO GEOMETRIC SETTINGS THE JUMP LOCI OF TWISTED RANK 1 (LINE) BUNDLES ARE THEMSELVES TRANSLATES (UNIONS) OF SUBTORI BY TORSION POINTS (SIMPSON, ...)

HIS PROOF USES RESULTS FROM TRANSCENDENTAL NUMBER THEORY.

(B) (VERTICAL) ALGEBRAIC PAINLEVÉ VI

DUBROVIN / MAZZOCCO AND LISOVYY / TIKHYY CLASSIFIED THE SOLUTIONS $W(t)$ TO PAINLEVÉ VI EQUATIONS WHICH ARE ALGEBRAIC FUNCTIONS OF t.

$$\frac{d^2W}{dt^2} = \frac{1}{2}\left(\frac{1}{W} + \frac{1}{W-1} + \frac{1}{W-t}\right)\left(\frac{dW}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{W-t}\right)\frac{dW}{dt}$$

$$+ \frac{W(W-1)(W-t)}{2t^2(t-1)^2}\left((\theta_\infty - 1)^2 - \frac{\theta_x^2 t}{W^2} + \frac{\theta_y^2(t-1)}{(W-1)^2} + \frac{(1-\theta_z^2)t(t-1)}{(W-t)^2}\right)$$

HERE $\theta_\infty, \theta_x, \theta_y, \theta_z$ ARE PARAMETERS DEFINING THE P-VI. USING APPROPRIATE CO-ORDINATES THE NON-LINEAR MONODROMY OF SOLUTIONS REDUCES TO THE PROBLEM OF DETERMINING THE FINITE ORBITS OF A NONLINEAR ACTION ON A^3.

FOR EXAMPLE IF $\theta_x = \theta_y = \theta_z$ THIS GROUP G IS GENERATED BY INVOLUTIONS R_1, R_2, R_3 AND PERMUTATIONS OF THE CO-ORDS

$$R_3 : (x_1, x_2, x_3) \longrightarrow (x_1, x_2, x_1 x_2 - x_3)$$

AND SIMILARLY FOR R_1 AND R_2.

THE TRANSFORATION

$$\sigma_{2,3} R_3 : (x_1, x_2, x_3) \longrightarrow (x_1, x_1 x_2 - x_3, x_2)$$

IS IN G AND IT FIXES x_1 AND INDUCES THE LINEAR TRANSF A ON (x_2, x_3).

$$A = \begin{bmatrix} x_1 & -1 \\ 1 & 0 \end{bmatrix}$$

SO IF (x_1, x_2, x_3) IS ON A FINITE ORBIT THEN

$$x_1 = \text{TRACE}(A) = 2\cos(2\pi t_1) \quad \text{WITH } t_1 \in \mathbb{Q}.$$

SIMILARLY FOR x_2 AND x_3 AND WITH A SUITABLY CHOSEN ELEMENT OF G ONE FINDS THAT IF (x_1, x_2, x_3) IS IN A FINITE ORBIT IT GIVES RISE TO A SOLUTION OF

$$\cos 2\pi\phi_1 + \cos 2\pi\phi_2 + \cos 2\pi\phi_3 + \cos 2\pi\phi_4 = 0 \qquad (*)$$

WITH THE ϕ_j'S IN $\mathbb{Q}$ AND RELATED TO t_1, t_2, t_3.

BY LANG'S G_m WE CAN PARAMETRIZE ALL THE SOLUTIONS OF $(*)$ AND THEN CHECK DIRECTLY WHICH CORRESPOND TO FINITE ORBITS OF G ACTING ON A^3.

(C) (HORIZONTAL AND VERTICAL) QUANTUM GRAPHS
JOINT WORK WITH P. KURASOV.

THE ONE DIMENSIONAL COMPACT RIEMANNIAN MANIFOLDS X ARE CIRCLES AND DETERMINED BY THEIR LENGTH ℓ.

$$X = \mathbb{R}/\mathbb{Z}$$

THE k-SPECTRUM OF $\Delta = \frac{d^2}{dx^2}$ ON FUNCTIONS ON X ; $\Delta\phi + k^2\phi = 0$: $e^{2\pi i m x}$

THE k-SPECTRUM IS AN ARITHMETIC PROGRESSION AS IS ITS FOURIER TRANSFORM

POISSON SUMMATION:

IF $\mu_X = \sum_{k \in \mathrm{SPEC}(X)} \delta_k = \sum_{k \in \mathbb{Z}} \delta_k$

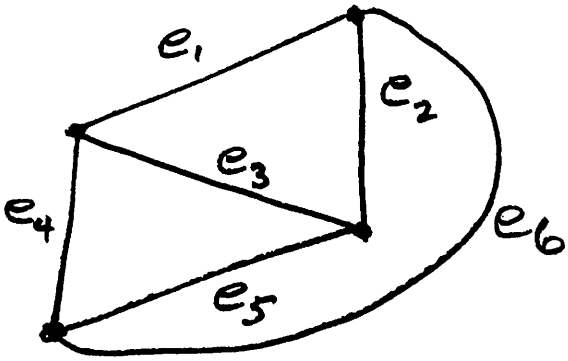
THEN $\hat{\mu}_X(\xi) = \mathrm{TRACE}\left(2\cos(\sqrt{\Delta}\,\xi)\right)$

$$= \sum_{m \in \mathbb{Z}} \delta_m(\xi).$$

AN EXAMPLE OF A CRYSTALLINE MEASURE (MEYER)

WE ALLOW OUR 1-DIMENSIONAL X TO HAVE A FINITE NUMBER OF BRANCH SINGULARITIES.

METRIC OR QUANTUM GRAPHS:



G COMBINATORIAL CONNECTED GRAPH

N EDGES e_j

M VERTICES v_k

EQUIP THE EDGES WITH LENGTHS ℓ_j, $j = 1, 2, \ldots N$

TO GET A METRIC GRAPH X WHICH IS SMOOTH ON THE EDGES (INTERIOR) SINGULAR AT THE VERTICES.

$$\Delta = \frac{d^2}{dx_j^2}$$ ON FUNCTIONS ϕ ON THE EDGES W.R.T x_j

FOR THE BOUNDARY CONDITIONS AT THE VERTICES WE CHOOSE THE NEUMANN OR KIRCHOFF CONDITION:

- ϕ IS CONTINUOUS AT THE v'S
- $$\sum_{e} \partial_e \phi(v) = 0$$ FOR EACH VERTEX v AND e IS INWARD EDGE TO v.

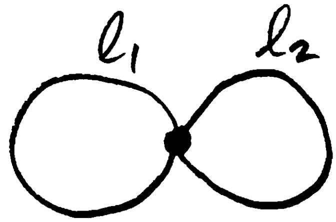
WITH THIS A DEGREE ONE VERTEX  CORRESPONDS TO THE USUAL NEUMANN CONDITION.

A DEGREE TWO VERTEX  HAS A REMOVABLE SINGULARITY; SO ASSUME THERE ARE NO DEGREE TWO VERTICES.

Δ IS SELF ADJOINT AND HAS DISCRETE k SPECTRUM IN $\mathbb{R}$.

- IT IS CONVENIENT TO DEFINE <u>SPEC(X)</u> TO TO BE THE NON-ZERO 'k-SPECTRUM OF Δ' AND TO INCLUDE 0 WITH MULTIPLICITY $2+N-M$.

<u>EXAMPLE:</u>

$X =$ 

; FIGURE EIGHT , $N=2, M=1$

$$\operatorname{Spec}(X) = \left\{ \frac{2\pi k_1}{\ell_1}, \frac{2\pi k_2}{\ell_2}, \frac{2\pi k_3}{\ell_1+\ell_2} : k_1, k_2, k_3 \in \mathbb{Z} \right\}$$

<u>WEYL LAW:</u> FOR ANY X AS ABOVE

$$\#\left\{ \operatorname{Spec}(X) \cap [-T, T] \right\} = \frac{2(\ell_1+\ell_2+\cdots+\ell_N)}{\pi} T + O(1)$$

AS $T \to \infty$.

SO SPEC(X) HAS A DENSITY IN $\mathbb{R}$ WHICH IS THAT OF AN ARITHMETIC PROGRESSION AND μ_X IS LOCALLY UNIFORMLY BOUNDED — THE NUMBER OF ATOMS IN AN INTERVAL OF FIXED LENGTH IS BOUNDED FROM ABOVE.

COMPUTING SPEC(X):

ON THE EDGES AN EIGENCTION TAKES THE FORM $\phi(x_j) = a e^{kx_j} + b e^{-kx_j}$; OUR BOUNDARY CONDITIONS LEAD TO THE SECULAR DETERMINANT (KOTTOS/SMILANSKY)

. GIVEN THE UNDERLYING GRAPH G DEFINE THE THE 2N BY 2N MATRICES INDEXED BY THE ORIENTED EDGES $e_1, \hat{e}_1, e_2, \hat{e}_2, \ldots, e_N, \hat{e}_N$

$$U(z_1, \ldots, z_N) = (u_{fg}) \quad ; \quad u_{fg} = z_f \delta_{fg}$$

AND THE SCATTERING MATRIX

$$S = (s_{f,g}) \quad ; \quad s_{fg} = \begin{cases} -\delta_{fg} + \dfrac{2}{\deg(v)} & \text{if } g \text{ follows } f \text{ through } v \\ 0 & \text{otherwise} \end{cases}$$

HERE $\deg(v)$ is its degree.

S IS UNITARY.

SPECTRAL OR SECULAR POLYNOMIAL OF G:

$$P_G(z_1, z_2, \ldots, z_N) := \det\left(I - U(z_1, \ldots, z_N) S\right).$$

WHICH WE CONSIDER AS A LAURENT POLYNOMIAL IN $z_1, \ldots, z_N$.

IMMEDIATE PROPERTIES OF P_G:

(i) $P_G(z)$ IS DEGREE $2N$ AND IS OF DEGREE TWO IN EACH z_j.

(ii) LET $P^{\iota}(z_1, \dots, z_N) = P(1/z_1, 1/z_2, \dots, 1/z_N)$.
THEN BOTH P_G AND P_G^{ι} ARE "$D = \{z : |z| < 1\}$ STABLE" THAT IS THEY DON'T VANISH FOR ~~z~~ z WITH $z_j \in D$, FOR ALL j (FOLLOWS FROM THE UNITARITY OF S).

- THE CONNECTION TO COMPUTING SPEC(X) IS:
(BARRA/GASPARD)

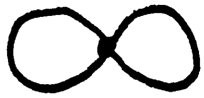
$$\mathrm{spec}(X) = \{ \text{ZEROS WITH MULTIPLICITY OF } k \longrightarrow P_G(e^{ik\ell_1}, e^{ik\ell_2}, \dots, e^{ik\ell_N}) \}.$$

CLEARLY THE ALGEBRAIC VARIETY

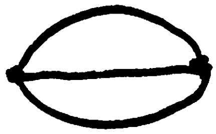
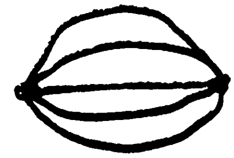
$$Z_G = \{ z : P_G(z) = 0 \} \subset (\mathbb{C}^*)^N$$

PLAYS A CENTRAL ROLE AND IN PARTICULAR THE QUESTION OF ~~ITS~~ THE FACTORIZATION OF P_G (OVER $\mathbb{C}$).

SPECIAL EXAMPLES:

$G =$  FIGURE EIGHT; $P_G(z_1, z_2) = (z_1 - 1)(z_2 - 1)(z_1 z_2 - 1)$

Z_G IS A UNION OF THREE SUBTORI.

$G =$  W_3 ; OR MORE GENERALLY W_N: 

$$P_G(z_1, z_2, z_3) = \left(z_1 z_2 z_3 + \tfrac{1}{3}(z_1 z_2 + z_1 z_3 + z_2 z_3) - \tfrac{1}{3}(z_1 + z_2 + z_3) - 1\right)\left(z_1 z_2 z_3 - \tfrac{1}{3}(z_1 z_2 + z_1 z_3 + z_2 z_3) - \tfrac{1}{3}(z_1 + z_2 + z_3) + 1\right)$$

FACTORIZATION CORRESPONDS TO THE SYMMETRY: REFLECTION THRU THE MIDPOINT OF EACH EDGE.

THEOREM 1 (KURASOV-S):
ASSUME THAT G IS NOT W_N THEN

(i) $$P_G(z) = Q_G(z) . \prod_{e \text{ A LOOP}} (z_e - 1)$$

WHERE THE PRODUCT IS OVER ALL LOOP EDGES IN G, AND $Q_G(z)$ IS ABSOLUTELY IRREDUCIBLE.

(ii) Z_{Q_G} DOES NOT CONTAIN AN $N-1$ DIMENSIONAL SUBTORUS OR TRANSLATE THEREOF UNLESS G IS THE FIGURE EIGHT.

REMARK: PART (i) WAS CONJECTURED BY COLIN DE VERDIERE.

THEOREM 2 (K–S) ADDITIVE STRUCTURE OF SPEC(X)

X IS A METRIC GRAPH ON G

(i) $\mathrm{Spec}(X) = L_1(X) \sqcup L_2(X) \cdots L_\nu(X) \sqcup N(X)$ (WITH MULT)

WHERE $L_j(X)$ IS A FULL INFINITE ARITHMETIC PROGRESSION AND THE <u>NON-STRUCTURED</u> PART, IF NON-EMPTY SATISFIES

- $\#(N(X) \cap [-T,T]) = \alpha T + O(1)$ AS $T \to \infty$

  WITH $\alpha = \frac{2}{\pi}(\ell_1 + \cdots + \ell_N) - \left(\frac{1}{d_1} + \cdots \frac{1}{d_\nu}\right)$; d_j's THE COMMON DIFF.

  AND $\alpha > 0$.

- THERE IS $c = c(G) < \infty$ SUCH THAT FOR ANY ARITHMETIC PROGRESSION $P \subset \mathbb{R}$

  $$\#(N(X) \cap P) \leq c(G)$$

- IN PARTICULAR $N(X)$ CONTAINS NO ARITHMETIC PROGRESSION LONGER THAN $c(G)$.

- $\dim_{\mathbb{Q}} \mathrm{span}(N(X)) = \infty$.

(ii) IF $\ell_1, \ell_2, \ldots \ell_N \in \mathbb{Q}$ (PROJECTIVELY) THEN $N(X) = \phi$.

IF $\ell_1, \ell_2, \ldots \ell_N$ ARE LINEARLY INDEPENDENT OVER $\mathbb{Q}$, THEN EXCEPT FOR THE FIGURE EIGHT ν IS EQUAL TO THE NUMBER OF LOOPS IN G, $\dim_{\mathbb{Q}}(\mathrm{spec}\, X) = \infty$, AND IF G HAS NO LOOPS, $\mathrm{spec}(X) = N(X)$.

<u>REMARK:</u> THE STRUCTURED PART $L_j(X)$ $j = 1, 2, \ldots, \nu$ AND $c(G)$ CAN BE DETERMINED EFFECTIVELY.

SUMMATION FORMULA FOR X

FOR METRIC GRAPHS THE SUMMATION FORMULA TAKES AN EXACT FORM (ROTH, KOTTOS/SMILANSKY, KURASOV)

$$\widehat{\sum_{k \in \mathrm{spec}(X)} \delta_k} = \frac{2(\ell_1 + \cdots \ell_n)}{\pi} \delta_0 + \frac{1}{\pi} \sum_{p \in \mathcal{P}} \ell(\mathrm{prim}\, p) \left[S_v(p) \delta_{\ell(p)} + \overline{S_v(p)} \delta_{-\ell(p)} \right]$$

WHERE:

- $\mathcal{P}$ IS THE SET OF ORIENTED PERIODIC PATHS IN G UP TO CYCLIC EQUIVALENCE (BACKTRACKING ALLOWED)
- $\ell(p)$ IS THE LENGTH OF THE PATH
- $\mathrm{prim}(p)$ IS THE PRIMITIVE PART OF p (GOING AROUND ONCE)
- $S_v(p)$ IS THE PRODUCT OF THE SCATTERING COEFF ~~OF~~ THE VERTICES ENCOUNTERED ON TRAVERSING p.

$\widehat{\mu_X}$ IS SUPPORTED IN

$$\Lambda = \{ m_1 \ell_1 + m_2 \ell_2 + \cdots + m_N \ell_N : m_j \geq 0 \text{ IN } \mathbb{Z} \}$$

WHICH IS A DISCRETE SET, BUT NOT LOCALLY UNIFORMLY BOUNDED.

$\Rightarrow \mu_X$ IS A CRYSTALLINE MEASURE SATISFYING EXOTIC PROPERTIES

(a) $\dim_{\mathbb{Q}}(\operatorname{supp} \mu_X) = \infty$, $\dim_{\mathbb{Q}}(\operatorname{supp} \hat{\mu}_X) < \infty$

(b) μ_X IS LOCALLY UNIFORMLY BOUNDED (AND POSITIVE)

NOTE THAT $\hat{\mu}_X$ CANNOT BE LOCALLY BOUNDED

<u>THEOREM</u> (LEV / OLEVSKII): IF μ IS A CRYSTALLINE MEASURE WITH BOTH μ AND $\hat{\mu}$ LOCALLY UNIFORMLY BOUNDED THEN μ CORRESPONDS TO A FINITE UNION OF ARITHMETIC PROGRESSIONS.

- ONE CAN PRODUCE SIMILAR SUCH EXOTIC CRYSTALLINE MEASURES USING ANY $P(z_1, z_2, \ldots, z_N)$ FOR WHICH $\underline{P}$ AND $\underline{P'}$ ARE D-STABLE. FOR EXAMPLE FROM THOSE ARISING IN THE LEE-YANG THEOREM AND THE THEORY OF HYPERBOLIC POLYNOMIALS WHERE THE PROOF OF STABILITY IS NOT A CONSEQUENCE OF A DETERMINANTAL FORMULA AND UNITARITY.

- IN THIS VEIN THE CRYSTALLINE MEASURES ARISING FROM THE EXPLICIT FORMULA IN THE THEORY OF PRIME NUMBERS LIES DEEPER IN A WAY THAT NEEDS EXAMINATION; AS IT IS EQUIVALENT TO "RH".