AHLFORS LECTURE II

INTEGER POINTS ON AFFINE CUBIC SURFACES

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JOINT WORK WITH A. GHOSH
AND WITH J. BOURGAIN / A. GAMBURD.

$f(x_1, x_2, \ldots, x_n)$ A POLYNOMIAL WITH INTEGRAL COEFFICIENTS AND f AND $f-k$ IRREDUCIBLE OVER $\mathbb{C}$ FOR ALL k.

$$f(x_1, x_2, \ldots x_n) = k \qquad \text{———} (*)$$

TO BE SOLVED OVER $\mathbb{Z}$ (OR $\mathcal{O}_K$ K A NUMBER FIELD)

$V_{f,k} = \{x : f(x) = k\}$, AFFINE HYPERSURFACE.

LOCAL CONGRUENCE OBSTRUCTIONS / $\mathbb{Z}$:

NEC. COND. IS THAT

$f(x) \equiv k \pmod{q}$, $q \geq 1$ HAS SOLUTION.

IF THE LOCAL CONDITION IS SUFFICIENT FOR SOLVABILITY OVER $\mathbb{Z}$; WE HAVE A LOCAL TO GLOBAL OR HASSE PRINCIPLE.

f LINEAR:

$$f(x_1, \ldots x_n) = a_1 x_1 + \cdots + a_n x_n = k$$

NEC COND IS $\gcd(a_1, \ldots a_n) \mid k$

IT IS ALSO SUFFICIENT.

f QUADRATIC (HILBERT'S ELEVENTH PROBLEM):

TO SOLVE (*) WITH

(a) $x, k \in K$

(b) $x, k \in \mathcal{O}_K$.

(a) HASSE-MINKOWSKI THEOREM:
(*) IS SOLVABLE IN K IFF IT IS SOLVABLE OVER EVERY K_v, FOR EVERY COMPLETION K_v OF K.

(b) OVER $\mathcal{O}_K$ MUCH MORE DIFFICULT; FOR EXAMPLE WHICH NUMBERS ARE SUMS OF THREE SQUARES IN $\mathcal{O}_K$?

- A STABLE LOCAL TO GLOBAL PRINCIPLE (THAT IS EXCEPT FOR FINITELY MANY EXCEPTIONS) HOLDS FOR $m \geq 3$ (SIEGEL, KNESER, DUKE/IWANIEC, COGDELL/PIATETSKI SHAPIRO/S)

KEY: $V_{f,k}$ IS A HOMOGENEOUS SPACE FOR AN ORTHOGONAL GROUP $\longrightarrow$ MODULAR FORMS.

CUBIC FORMS

- AN AFFINE CUBIC f IS A POLYNOMIAL IN $\mathbb{Z}[x_1, \ldots, x_n]$ WITH LEADING HOMOGENEOUS PART f_0 OF DEGREE 3 AND NON-DEGENERATE, WE ALSO ASSUME THAT f AND $f-k$ ARE IRREDUCIBLE.

$$(**) \qquad V_{k,f} = \{x : f(x) = k\}$$ AFFINE-HYPERSURFACE

- k IS ADMISSIBLE IF THERE ARE NO LOCAL CONGRUENCE OBSTRUCTIONS TO $(**)$ (THESE HAVE A SIMPLE DESCRIPTION)

RICHNESS OF $V_{k,f}(\mathbb{Z})$:

FOR k ADMISSIBLE IS $V_{k,f}(\mathbb{Z})$ NON-EMPTY (IE HAS A HASSE PRINCIPLE), ZARISKI-DENSE IN $V_{k,f}$, SATISFY A FORM OF STRONG-APPROXIMATION?

$\underline{n=2}$ (SUPER-CRITICAL) THUE/SIEGEL

$$|V_{k,f}(\mathbb{Z})| < \infty$$

SCHMIDT SHOWS THAT FOR VERY FEW ADMISSIBLE k's IS $V_{k,f}(\mathbb{Z}) \neq \phi$.

- $n \geq 10$ (SUBCRITICAL) BROWNING/HEATH-BROWN

f_0 NONSINGULAR THEN FOR k ADMISSIBLE $V_{k,f}(\mathbb{Z}) \neq \phi$, IT IS ZARISKI DENSE AND IT SATISFIES STRONG APPROXIMATION.

- $n \geq 4$ (SUBCRITICAL) HOOLEY

f HOMOGENEOUS NONSINGULAR AND ASSUMING THE RIEMANN HYPOTHESIS FOR CERTAIN ASSOCIATED HASSE-WEIL ZETA FUNCTIONS, $V_{k,f}(\mathbb{Z}) \neq \phi$ FOR ALMOST ALL ADMISSIBLE k'S.

- $n = 3$ (CRITICAL) AFFINE CUBIC SURFACE, VERY LITTLE IS KNOWN.

<u>EXAMPLE</u>

$$f = S(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3$$

k IS ADMISSIBLE IFF $k \not\equiv 4, 5 \pmod 9$

- IT IS POSSIBLE THAT FOR EVERY ADMISSIBLE k, $V_{k,S}(\mathbb{Z}) \neq \phi$ AND IS ZARISKI DENSE IN V_k

- A. BOOKER (2019) (USING A METHOD OF ELKIES)

$$33 = (8866128975287528)^3 + (-8778405442862239)^3 + (-2736111468807040)^3$$

$$42 = (-80538738812075974)^3 + (80435758145817515)^3 + (12602123297335631)^3$$

WITH D. SUTHERLAND

· LEHMER, BEUKERS SHOW THAT $V_{S,1}(\mathbb{Z})$ IS ZARISKI DENSE IN $V_{S,1}$.

USING CUBIC RECIPROCITY ONE CAN SHOW THAT STRONG APPROXIMATION FAILS FOR $V_{S,k}(\mathbb{Z})$.

<u>E.G.</u> $x \in V_{S,3}(\mathbb{Z}) \Rightarrow x_1 \equiv x_2 \equiv x_3 \pmod 9$

(CASSELS, HEATH-BROWN, COLLIOT-THELENE/WITTENBERG)

HOWEVER IN THE SLIGHTLY WEAKER FORM

$V_{S,k}(\mathbb{Z}) \longrightarrow V_{S,k}(\mathbb{Z}/p\mathbb{Z})$, BEING ONTO

FOR p A LARGE PRIME, MAY HOLD.

A DIOPHANTINE THEORY FOR INTEGRAL POINTS ON SOME SPECIAL CUBIC SURFACES CAN BE DEVELOPED

A. GHOSH / S, J. BOURGAIN/A. GAMBURD/S

THESE START WITH MARKOFF'S SURFACES.

MARKOFF'S CUBIC SURFACES

$$M(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - x_1 x_2 x_3$$

$$V_k = V_{k,M} = \{ x \mid M(x) = k \}.$$

$k=0$: IS MARKOFF'S SURFACE

$k=4$: IS THE CAYLEY CUBIC
(IT IS SPECIAL IN WHAT FOLLOWS)

- $V_k(\mathbb{Z})$ ARISES IN MANY CONTEXTS

DIOPHANTINE APPROXIMATION (MARKOFF)

SIMPLE CLOSE GEODESICS ON THE MODULAR SURFACE (H. COHN)

EXCEPTIONAL VECTOR BUNDLES OVER $\mathbb{P}^2$ (GORODENSTEV / RUDAKOV)

SMOOTHABLE DEL-PEZZO SURFACES (HACKING / PROKHOROV)

SYMPLECTIC 4-MANIFOLDS VIA LEFSCHETZ FIBRATIONS (AUROUX)

⋮

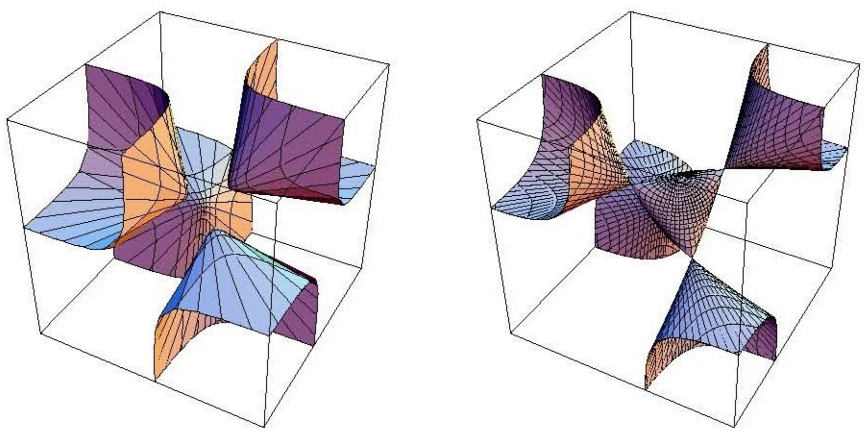
- V_k IS ALSO THE RELATIVE CHARACTER VARIETY OF REPRESENTATIONS OF $\pi_1(\Sigma_{1,1}) \to SL_2$

- IT ALSO ARISES ~~AS~~ IN THE NON-LINEAR MONODROMY GROUP OF PAINLEVÉ VI.

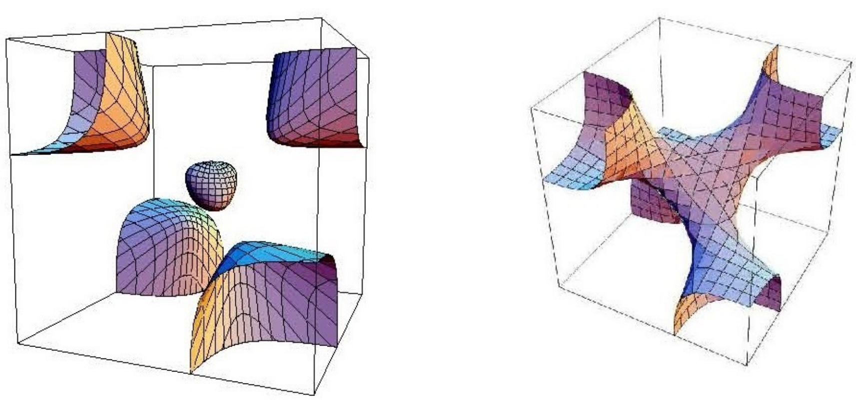
V_0 Markoff's cubic surface

V_4 Cayley's cubic surface

$V_k(\mathbb{R})$ for different k:



$k = 0$ and $k = 4$



$k = 2$ and $k = 8$

THE REASON ONE CAN STUDY $V_k(\mathbb{Z})$ IS THAT IT IS ACTED ON BY A NON-LINEAR GROUP OF MORPHISMS ALLOWING DESCENT.

Γ, THE GROUP IN $\mathrm{AUT}(\mathbb{A}^3)$ GENERATED BY PERMUTATIONS OF THE COORDINATES AND SWITCHING THE SIGNS OF TWO COORDINATES, AND THE VIETA INVOLUTIONS R_1, R_3, R_3

$$R_3(x_1, x_2, x_3) = (x_1, x_2, x_1 x_2 - x_3)$$

PRESERVES V_k AND $V_k(\mathbb{Z})$. ($\Gamma \cong PGL_2(\mathbb{Z})$)

- FOR $k \neq 4$, $V_k(\mathbb{Z})$ CONSISTS OF A FINITE NUMBER $h(k)$ OF Γ-ORBITS (MARKOFF, HURWITZ, MORDELL).

<u>CLASSICAL QUESTIONS</u>:

(i) WHEN IS $V_k(\mathbb{Z}) \neq \phi$ IE $h(k) > 0$.

(ii) IF $h(k) > 0$, IS $V_k(\mathbb{Z})$ INFINITE, ZARISKI DENSE, SATISFY A FORM OF STRONG APPROXIMATION ?

HASSE PRINCIPLE

LOCAL CONGRUENCE OBSTRUCTIONS:

$$V_k(\mathbb{Z}_p) \neq \phi \quad \text{FOR ALL } p \quad \text{IFF} \quad k \not\equiv 3(4) \text{ OR } \pm 3 \pmod 9.$$

WE RESTRICT TO k'S WHICH HAVE LOCAL INTEGRAL POINTS AND SAY THAT V_k FAILS HASSE'S PRINCIPLE IF $V_k(\mathbb{Z}) = \phi$.

FOR $|k| \geq 5$ CALL k SPECIAL IF $V_k(\mathbb{Z})$ CONTAINS A POINT x WITH $|x_j| = 0, 1, 2$. THE SPECIAL k'S ARE EASY TO DESCRIBE AND ANALYZE, THEY ARE OF ZERO DENSITY. REMAINING k'S ARE CALLED GENERIC.

- FOR $k>0$ GENERIC A POINT $x \in V_k(\mathbb{Z})$ IS GHOSH REDUCED IF IT IS OF THE FORM $(-x_1, x_2, x_3)$ WITH $3 \leq x_1 \leq x_2 \leq x_3$ AND
$$x_1^2 + x_2^2 + x_3^2 + x_1 x_2 x_3 = k$$

- (GHOSH) FOR $k>0$ GENERIC
$$\Gamma \backslash V_k(\mathbb{Z}) \cong \text{GHOSH REDUCED POINTS.}$$

COR: (a) $h(k) \ll |k|^{1/3}$

(b) $$\sum_{0<k\leq K} h(k) \sim \frac{K(\log K)^2}{36}, \quad K\to\infty$$

$$\sum_{-K\leq k<0} h(k) \sim \frac{K(\log K)^2}{48}, \quad K\to\infty.$$

THE EXPLICIT FUNDAMENTAL DOMAINS ALLOW FOR THE NUMERICAL COMPUTATIONS OF THE $h(k)$'S AND THESE INDICATE THAT

$$\left|\{ |k|\leq K : V_k \text{ FAILS HASSE}\}\right| \sim CK^{\theta}$$

WITH $C\neq 0$ AND $\theta \approx 0.887\ldots$

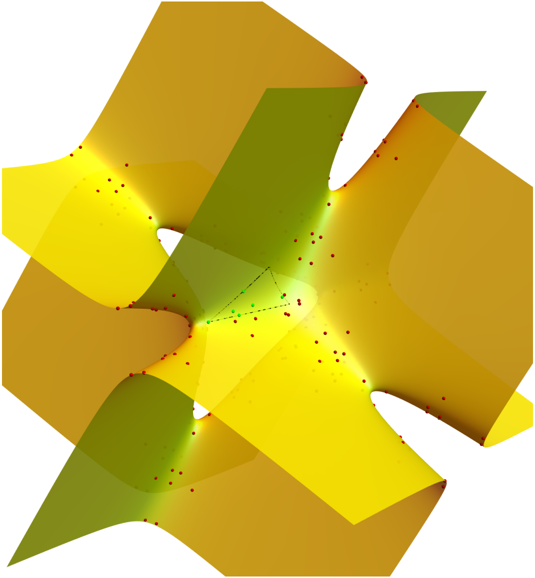


Figure: Lattice points and fundamental set (triangular) for $k = 3685$.

THEOREM 1 (GHOSH / S 2017)

(i) THERE ARE INFINITELY MANY k's WHICH ~~WHICH~~ FAIL THE HASSE PRINCIPLE. THE NUMBER OF SUCH WITH $|k| \leq K$ IS AT LEAST $K^{1/2}/\log K$.

(ii) FIX $t \geqslant 0$

$$\left| \{ |k| \leq K : k \text{ ADMISSIBLE}, h(k) = t \} \right| = o(K) \quad K \to \infty$$

$\Rightarrow$ ALMOST ALL k's SATISFY HASSE AND ALSO THESE $V_k(\mathbb{Z})$'s ARE ZARISKI DENSE.

COMMENTS:

(a) THE HASSE FAILURES ARE PRODUCED BY AN OBSTRUCTION VIA QUADRATIC RECIPROCITY. THEY COME IN TWO TYPES: ONE DIRECT USE OF RECIPROCITY AND THE SECOND WHICH ALSO INCORPORATES THE DESCENT GROUP.

RECENTLY LOUGHRAN AND MITNANKIN; AND INDEPENDENTLY COLLIOT-THELENE, DA SHENG AND F. XUI HAVE SHOWN THAT THE OBSTRUCTIONS OF THE FIRST (BUT NOT SECOND) CAN BE BE EXPAINED IN TERMS OF AN INTEGRAL BRAUER-MANIN OBSTRUCTION.

FOR EXAMPLE IF

$$k = 4 + 2\nu^2$$

WITH ν HAVING ALL OF ITS PRIME FACTORS $\equiv \pm 1 \pmod{8}$ AND $\nu \equiv 0, \pm 3, \pm 4 \pmod{9}$, THEN k IS ADMISSIBLE BUT $V_k(\mathbb{Z}) = \phi$.

(b) IS PROVED BY COMPARING THE NUMBER OF POINTS ON $V_k(\mathbb{Z})$ IN CERTAIN TENTACLED REGIONS GOTTEN BY SPECIAL PLANE SECTIONS, WITH THE EXPECTED NUMBER OF SOLUTIONS ACCORDING TO A PRODUCT OF LOCAL DENSITIES. THE KEY IS THAT THE VARIANCE OF THIS COMPARISON GOES TO ZERO ON AVERAGING $|k| \leq K$. THIS MOVING PLANE QUADRIC METHOD APPLIES TO MORE GENERAL CUBIC SURFACES INCLUDING ONES THAT DON'T CARRY MORPHISMS.

(B)(VERTICAL) ALGEBRAIC PAINLEVÉ VI

DUBROVIN / MAZZOCCO AND LISOVYY / TIKHYY CLASSIFIED THE SOLUTIONS $W(t)$ TO PAINLEVÉ VI EQUATIONS WHICH ARE ALGEBRAIC FUNCTIONS OF t.

$$\frac{d^2W}{dt^2} = \frac{1}{2}\left(\frac{1}{W} + \frac{1}{W-1} + \frac{1}{W-t}\right)\left(\frac{dW}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{W-t}\right)\frac{dW}{dt}$$

$$+ \frac{W(W-1)(W-t)}{2t^2(t-1)^2}\left((\theta_\infty - 1)^2 - \frac{\theta_x^2 t}{W^2} + \frac{\theta_y^2(t-1)}{(W-1)^2} + \frac{(1-\theta_z^2)t(t-1)}{(W-t)^2}\right)$$

HERE $\theta_\infty, \theta_x, \theta_y, \theta_z$ ARE PARAMETERS DEFINING THE P-VI. USING APPROPRIATE CO-ORDINATES THE NON-LINEAR MONODROMY OF SOLUTIONS REDUCES TO THE PROBLEM OF DETERMINING THE FINITE ORBITS OF A NONLINEAR ACTION ON A^3.

FOR EXAMPLE IF $\theta_x = \theta_y = \theta_z$ THIS GROUP G IS GENERATED BY INVOLUTIONS R_1, R_2, R_3 AND PERMUTATIONS OF THE CO-ORDS

$$R_3 : (x_1, x_2, x_3) \longrightarrow (x_1, x_2, x_1x_2 - x_3)$$

AND SIMILARLY FOR R_1 AND R_2.

THE TRANSFORATION

$$\sigma_{2,3} R_3 : (x_1, x_2, x_3) \longrightarrow (x_1, x_1 x_2 - x_3, x_2)$$

IS IN G AND IT FIXES x_1 AND INDUCES THE LINEAR TRANSF A ON (x_2, x_3).

$$A = \begin{bmatrix} x_1 & -1 \\ 1 & 0 \end{bmatrix}$$

SO IF (x_1, x_2, x_3) IS ON A FINITE ORBIT THEN

$$x_1 = \text{TRACE}(A) = 2\cos(2\pi t_1) \quad \text{WITH } t_1 \in \mathbb{Q}.$$

SIMILARLY FOR x_2 AND x_3 AND WITH A SUITABLY CHOSEN ELEMENT OF G ONE FINDS THAT IF (x_1, x_2, x_3) IS IN A FINITE ORBIT IT GIVES RISE TO A SOLUTION OF

$$\cos 2\pi\phi_1 + \cos 2\pi\phi_2 + \cos 2\pi\phi_3 + \cos 2\pi\phi_4 = 0 \qquad \text{——} (*)$$

WITH THE ϕ_j'S IN $\mathbb{Q}$ AND RELATED TO t_1, t_2, t_3.

BY LANG'S $\mathbb{G}_m$ WE CAN PARAMETRIZE ALL THE SOLUTIONS OF $(*)$ AND THEN CHECK DIRECTLY WHICH CORRESPOND TO FINITE ORBITS OF G ACTING ON A^3.

STRONG APPROXIMATION

WHEN $V_k(\mathbb{Z}) \neq \phi$ HOW RICH IS IT BEYOND ZARISKI DENSITY?

WE DISCUSS THE MARKOFF CASE $k=0$.

THE GENERAL CASE IS SIMILAR BUT FIRST REQUIRES A STUDY OF <u>ALL</u> THE FINITE Γ-ORBITS IN $\mathbb{A}^3(\overline{\mathbb{Q}})$.

• THIS IS CLOSELY RELATED TO THE DETERMINATION (DUBROVIN-MAZZACO) OF THE ALGEBRAIC PAINLEVE VI'S.

OUR TREATMENT OF THE LATTER USES THE RESOLUTION (EFFECTIVE) OF LANG'S $\mathbb{G}_m$ CONJECTURE

$$Y: \quad x_1^2 + x_2^2 + x_3^2 - 3x_1x_2x_3 = 0$$

Γ AS BEFORE EXCEPT THAT THE R_j'S ARE

$$R_3((x_1, x_2, x_3)) = (x_1, x_2, 3x_1x_2 - x_3)$$

$h = 2$, WITH ONE ORBIT BEING $\{0\}$

AND THE OTHER $Y^*(\mathbb{Z}) = \Gamma.(1,1,1)$

STRONG APPROXIMATION CONJECTURE FOR Y:

REDUCTION MOD p

$$Y^*(\mathbb{Z}) \longrightarrow Y^*(\mathbb{Z}/p\mathbb{Z})$$ IS ONTO FOR ALL PRIMES p.

- Γ ACTS ON REDUCTION mod p AS A PERMUTATION GROUP ON $Y^*(\mathbb{Z}/p\mathbb{Z})$.

- STRONG APPROXIMATION $\Longleftrightarrow$ Γ ACTING TRANSITIVELY ON $Y^*(\mathbb{Z}/p\mathbb{Z})$.

NOTE: $$|Y^*(\mathbb{Z}/p\mathbb{Z})| \sim p^2 \quad \text{AS } p \to \infty.$$

AS LONG AS $p^2 - 1$ IS NOT VERY SMOOTH (EG $= k!$) WE CAN PROVE STRONG APPROXIMATION.

THEOREM 2 (BOURGAIN-GAMBURD-S; 2014)

FOR $\varepsilon > 0$ AND p LARGE THERE IS A Γ ORBIT $\mathcal{O}(p)$ IN $Y^*(\mathbb{Z}/p\mathbb{Z})$ SUCH THAT

$$|\mathcal{O}^c(p)| = |Y^*(\mathbb{Z}/p\mathbb{Z}) \setminus \mathcal{O}(p)| \ll_\varepsilon p^\varepsilon$$

AND EVERY Γ-ORBIT $t(p)$ IN $Y^*(\mathbb{Z}/p\mathbb{Z})$ SATISFIES

$$|t(p)| \gg (\log p)^{1/3}.$$

THESE HAVE BEEN RECENTLY IMPROVED BY KONYAGIN / MAKARYCHEV / SHPARLINSKI / VYUGIN (2017) TO

$$|\mathcal{O}^c(p)| \leq \exp\left((\log p)^{\frac{2}{3}+o(1)}\right)$$

$$|t(p)| \gg (\log p)^{7/9}.$$

THEOREM 3 (B-G-S 2015)

$$|\{p \leq T : \text{STRONG APPROXIMATION FAILS FOR } p\}| \ll_\varepsilon T^\varepsilon, \quad \varepsilon > 0.$$

SO IN GENERAL THERE IS ALWAYS A GIANT COMPONENT AND THE STRONG APPROXIMATION CONJECTURE HOLDS EXCEPT PERHAPS FOR VERY FEW p'S.

THEOREM 4 (MEIRI–PUDER 2019):

- IF $p \equiv 1(4)$ AND p SATISFIES STRONG APPROXIMATION FOR $Y^*(\mathbb{Z}/p\mathbb{Z})/\pm 1$ THEN THE ACTION OF Γ ON THE LATTER IS EITHER THE FULL ALTERNATING OR SYMMETRIC GROUP.

- UNDER THE SAME ASSUMPTIONS ON p, THE ORBIT OF EVERY POINT IN $Y^*(\mathbb{Z}_p)$ IS DENSE ; HERE $\mathbb{Z}_p$ IS THE p-ADIC INTEGERS

THEOREM 4 ALLOWS ONE TO SHOW THAT

$$Y^*(\mathbb{Z}) \longrightarrow Y^*(\mathbb{Z}/q\mathbb{Z}) \text{ IS ONTO}$$

FOR $q = p_1 p_2 \cdots p_\ell$, $p_j \equiv 1\,(4)$ AND SATISFYING STRONG APPROXIMATION (GOURSAT-LEMMA).

WITH THESE WE CAN EXECUTE SOME SIMPLE SIEVING AND COUPLE IT WITH SOME TEICHMULLER DYNAMICS (MIRZAKHANI) TO ANSWER SOME OLD QUESTIONS ABOUT MARKOFF NUMBERS

$\mathcal{M}$: MARKOFF NUMBERS, THAT IS CO-ORDINATES OF A MARKOFF TRIPLE $x \in \underline{V}(\mathbb{Z})$ WITH $x_j > 0$.

$\mathcal{M}$: $1, 2, 5, 13, 29, 34, 89, 169, 194, \dots$

- FROBENIUS: $m \in \mathcal{M} \Rightarrow m \not\equiv 0, \pm 2/3 \bmod p$, IF $p \equiv 3(4)$ AND $p \neq 3$.
- STRONG APPROXIMATION $\Rightarrow$ THESE ARE THE ONLY CONGRUENCE OBSTRUCTIONS.

$\mathcal{M}$ IS LACUNARY:

$$|\{m \leq T : m \in \mathcal{M}\}| \sim c(\log T)^2,$$

ZAGIER (1982)
MIRZAKHANI (2016)

THEOREM 5: (B-G-S 2015)

ALMOST ALL $m \in \mathcal{M}$ ARE COMPOSITE

$$|\{p \leq T : p \in \mathcal{M}, p \text{ prime}\}| = o\left(|\{m \in \mathcal{M} : m \leq T\}|\right)$$

AS $T \to \infty$.

REMARKS

TOOLS ARE ELEMENTARY COMING FROM ANALYTIC NUMBER THEORY, CURVES OVER FINITE FIELDS AND COMBINATORICS. ONE INTERESTING FEATURE BEING:

$$C_\lambda : \begin{cases} y + y^{-1} = x + \lambda x^{-1}, & x, y \in \mathbb{F}_p \\ x \in H_1,\ y \in H_2 \end{cases}$$

$\lambda \neq 1$

$$H_1, H_2 \leq \mathbb{F}_p^*, \quad |H_1| \leq |H_2|.$$

NEED AN UPPER BOUND FOR $|C|$ OF THE FORM

$$|C| \leq |H_2|^\delta$$

FOR SOME $\delta < 1$ INDEPENDENT OF p.

- IF $|H_2| \geq \sqrt{p}$ ONE CAN USE THE RIEMANN HYPOTHESIS FOR CURVES OVER FINITE FIELDS TO PROVE THIS.
- FOR $|H_2|$ SMALL THIS IS OF NO USE AND WE USE STEPANOV'S ELEMENTARY PROOF OF WEIL'S THEOREM (SPECIFICALLY AUXILIARY POLYNOMIALS) TO ESTABLISH SUCH A BOUND.

CHARACTER VARIETIES

V_k IS THE RELATIVE CHARACTER VARIETY OF REPRESENTATIONS OF THE FUNDAMENTAL GROUP OF A SURFACE OF GENUS ONE WITH ONE PUNCTURE, TO SL_2. THE ACTION OF THE MAPPING CLASS GROUP IS THAT OF Γ.

MORE GENERALLY THE (AFFINE) RELATIVE CHARACTER VARIETY X_k OF REPRESENTATIONS OF $\pi_1(\Sigma_{g,n})$ INTO SL_2 IS DEFINED OVER $\mathbb{Z}$.

$\Sigma_{g,n}$ IS A SURFACE OF GENUS g WITH n PUNCTURES.

ONE CAN STUDY THE DIOPHANTINE PROPERTIES OF $X_k(\mathbb{Z})$.

J.H. WHANG (PRINCETON THESIS 2018) HAS MADE BIG STEPS IN THIS DIRECTION.

(i) X_k HAS A PROJECTIVE COMPACTIFICATION RELATIVE TO WHICH X_k IS "LOG CALABI YAU". ACCORDING TO CONJECTURES OF VOJTA THIS PLACES X_k AS BEING IN THE SAME THRESHOLD SETTING AS AFFINE CUBIC SURFACES.

(ii) $X_k(\mathbb{Z})$ HAS A FULL DESCENT IN THAT THE MAPPING CLASS GROUP ACTS VIA NON-LINEAR MORPHISMS ON $X_k(\mathbb{Z})$ WITH FINITELY MANY ORBITS.

• THESE AND MORE GENERAL CHARACTER VARIETIES CONNECTED WITH HIGHER TEICHMULLER THEORY OFFER A RICH FAMILY OF THRESHOLD AFFINE VARIETIES FOR WHICH ONE CAN APPROACH THE STUDY OF INTEGRAL POINTS.

THERE IS A RICH STUDY OF THE ACTION OF Γ ON $X_k(\mathbb{R})$ (THURSTON,.. GOLDMAN..) AND ON $X_k(\mathbb{C})$ (CANTAT, MCMULLEN... ENTROPY)

- SINCE X_k IS DEFINED OVER $\mathbb{Z}$ ONE CAN EXAMINE THE ORBIT CLOSURES OF Γ IN $X_k(\mathbb{Q}_p)$. THE BASIC COMPACT INVARIANT SET IS $X_k(\mathbb{Z}_p)$ AND WE EXPECT AS WITH $g=1, n=1$ THE ACTION OF Γ $X_k(\mathbb{Z}_p)$ IS MINIMAL (CLOSURE OF ORBITS ARE AS LARGE AS POSSIBLE).

- BISWAS | GUPTA | MJ | WHANG (2019): CLASSIFY FINITE (EVEN BOUNDED) ORBITS OF Γ ON $X_k(\mathbb{C})$ FOR $g \geq 1, n \geq 0$ IN TERMS OF THEIR LIFT ρ TO A REPRESETATION OF $\Sigma_{g,n}$ INTO $SL_2(\mathbb{C})$, BASICALLY IT SHOULD BE FINITE.

IF ONE REPLACES $\mathbb{Z}$ BY A LARGER RING D SUCH AS $\mathbb{Z}[\frac{1}{5}]$ OR $\mathbb{Z}[\sqrt{2}]$ WHICH CONTAIN INFINITELY MANY UNITS, THEN THE ACTION OF Γ ON $V_{M,k}(D)$ NEED NO LONGER CONSIST OF FINITELY MANY ORBITS (FIRST OBSERVED BY SILVERMAN).

- THIS MAKES OUR DIOPHANTINE ANALYSIS MUCH MORE CHALLENGING.

- IT APPEARS THAT THEIR ARE FEWER (IF ANY) EXCEPTIONS TO THE HASSE PRINCIPLE

- FOR THE MARKOFF SURFACES THIS HAS BEARING THROUGH FRICKE'S TRACE IDENTITIES ON COMMUTATORS IN $SL_2(D)$.

ANER SHALEV QUESTION

IS EVERY ELEMENT IN $SL_3(\mathbb{Z})$ A ONE COMMUTATOR, IE CAN ONE SOLVE FOR EVERY B IN $SL_3(\mathbb{Z})$

$$XYX^{-1}Y^{-1} = B \quad ?$$

AVNI-GELANDER-KASSABOV-SHALEV SHOW THAT ESSENTIALLY THERE IS NO LOCAL OBSTRUCTION TO THIS IN THAT IT IS TRUE IN EVERY FINITE QUOTIENT OF $SL_3(\mathbb{Z})$.

REFERENCES CAN BE FOUND IN

"INTEGRAL POINTS ON MARKOFF TYPE CUBIC SURFACES"

A. GHOSH + P. SARNAK

arXiv: 1706.06712

AND

"NONLINEAR DESCENT ON MODULI OF LOCAL SYSTEMS"

J. H. WHANG arXiv: 1710.01848.

- THE RICHNESS RESULTS FOR MARKOFF SURFACES EXTEND TO OTHER CUBIC SURFACES, THOUGH STILL SPECIAL

f_0 THE HOMOGENEOUS CUBIC PART SHOULD BE REDUCIBLE.

UNIVERSAL PERFECT FORMS

A CUBIC FORM IN THREE VARIABLES IS UNIVERSAL AND PERFECT IF IT REPRESENTS EVERY k AND RICHLY ($V_k(\mathbb{Z})$ IS ZARISKI DENSE AND A FORM OF STRONG APPROXIMATION HOLDS)

A POSSIBLE EXAMPLE ?:

EVERY k IS ADMISSIBLE FOR

$$x_1^3 + x_2^3 + 2x_3^3$$

AND PERHAPS IT IS UNIVERSAL AND PERFECT.

GHOSH / S (2017):

$$U(x_1, x_2, x_3) = x_2(x_3 - x_1) + x_1^2 + x_2^2 + x_3^2 - x_1 x_2 x_3$$

IS UNIVERSAL AND PERFECT.