AN OBSERVATION ON NORMAL FUNCTIONS (*) (**)

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1. – Normal functions were introduced by Poincaré [3] and were used by Severi [4] and Lefschetz [2] to study the (virtual) algebraic curves D lying on a smooth algebraic surface S. Very briefly, if we choose a projective embedding $S \subset \mathbf{P}^N$ and denote by C a generic hyperplane section, then after replacing D by a suitable integral linear combination $aD + bC$, we may assume that the intersection number $D \cdot C = 0$. If $J(C)$ denotes the Jacobian variety of the curve C, then the point

$$\nu_D(C) \in J(C)$$

is defined and depends only on the rational equivalence class of D. In this way the algebraic 1-cycle D gives a cross-section ν_D of the family of Jacobians of the smooth hyperplane sections of the surface. By definition, the group of *normal functions* are all of the holomorphic cross-sections of this family of Jacobians that have a certain behavior at the singular hyperplane sections—cf. [1] for further discussion and precise definitions.

In higher dimensions one may introduce normal functions using intermediate Jacobians. Specifically, suppose that M is a smooth projective variety of complex dimension $2n$. For each projective embedding $M \subset \mathbf{P}^N$ the intermediate Jacobian of a smooth hyperplane section V is the torus

$$J(V) = H^{2n-1}(V, \mathbf{R})/H^{2n-1}(V, \mathbf{Z})$$

endowed with a suitable complex structure. These fit together to give a fiber space of complex tori, whose sections satisfying a suitable

(*) Comunicazione inviata all'Istituto Nazionale di Alta Matematica Francesco Severi.

(**) Research partially supported by National Science Foundation Grant MCS 78-07348.

differential condition and growth property when V becomes singular are the normal functions (cf. [1] for details).

To each primitive algebraic n-cycle Z on M there is associated a normal function ν_Z, and in [1] we have given a theorem showing that, in a certain sense, the knowledge of ν_Z allows us to specify the polynomial equations that define Z or a cycle homologous to it. This procedure was somewhat indirect, and in this paper we shall give a more precise result along similar lines.

2. – We shall first discuss our result for the case of an algebraic 1-cycle D on a surface S. Recall that the hyperplane sections of S are parametrized by the dual projective space $\mathbf{P}^{N*}$. The singular sections correspond to the dual hypersurface $S^* \subset \mathbf{P}^{N*}$ of tangent hyperplanes, and if we denote by S^*_{sing} the singular set of this dual variety then the hyperplanes in $S^* - S^*_{\text{sing}}$ cut out curves having an isolated ordinary double point. Clearly the point $\nu_D(C) \in J(C)$ is defined for smooth sections C. Since we are allowed to vary D in its linear equivalence class, we may assume that D does not pass through any of a finite number of preassigned points on S. In this way, for $C \in S^* - S^*_{\text{sing}}$ we may still define $\nu_D(C) \in J(C)$ where $J(C)$ is the generalized Jacobian of C. We observe that this possibility was clearly understood by Severi in his great work [5] on quasi-abelian functions.

Having established that the normal functions extend to $\mathbf{P}^N - S^*_{\text{sing}}$, we may ask the question: *Over how much of $\mathbf{P}^{N*}$ can the normal function be extended without encountering an essential singularity?* More precisely, any given hyperplane section $\bar{C}_0$ may be embedded in a 1-parameter family $\{C_t\}_{t\in\Delta}$ of sections parametrized by the disc $\Delta = \{|t| < \varepsilon\}$ where C_t is smooth for $t \neq 0$ and $\bar{C}_0$ is the curve corresponding to $t = 0$ (the reason for the bar notation will appear shortly). By going to a finite covering of the punctured disc and applying stable reduction procedures we may assume that we have the following data:

i) a 2-dimensional complex manifold fibered over the disc

$$X \xrightarrow{\pi} \Delta$$

such that $C_t = \pi^{-1}(t)$ is smooth for $t \neq 0$ while C_0 is a curve with normal crossings and all of whose components have multiplicity one;

ii) a holomorphic mapping

$$X \xrightarrow{f} S$$

such that f maps the C_t for $t \neq 0$ biholomorphically onto smooth hyperplane sections of S, while in general the surjective map

$$f\colon C_0 \to \bar{C}_0$$

has several sheets and contracts curves.

If $J(C_t)$ is the Jacobian of C_t for $t \neq 0$ and generalized Jacobian of C_0, then $J = \bigcup_{t\in\Delta} J(C_t)$ forms naturally a fiber space of complex analytic groups, and the normal function $\nu_D(t)$ gives a cross-section of $J \to \Delta$ outside $t = 0$. The question now is:

When does $\nu_D(t)$ extend across $t = 0$?

We shall sketch a proof of the

THEOREM: *The normal function $\nu_D(t)$ extends across $t = 0$ if, and only if, the fundamental class $\eta_D \in H_2(S)$ of D is orthogonal under the intersection pairing to the image of*

$$H_2(C_0) \xrightarrow{f_*} H_2(S)\,.$$

PROOF: As noted above, by a linear equivalence the given 1-cycle D may be moved away from any finite set of points on S. We may thus assume that, up on X, D defines an algebraic 1-cycle that does not pass through any of the double points of C_0. If $C_{0,1}, \ldots, C_{0,n}$ are the irreducible components of C_0, then we recall that the generalized Jacobian $J(C_0)$ corresponds to divisor classes supported outside the nodes of C_0 and that have degree zero on each $C_{0,i}$. Intuitively, if we choose a basis $\omega_1(t), \ldots, \omega_r(t)$ for the space $H^0(\omega_{C_t})$ of abelian differentials on C_t, then setting $D_t = D\cdot C_t$ we want to determine a 1-chain γ_t on C_t such that $\partial\gamma_t = D_t$ and the abelian sums

$$\int_{\gamma_t} \omega_\alpha(t)$$

remain finite as $t \to 0$. The condition for this is exactly that

$$\deg(D\cdot C_{0,i}) = 0$$

for each component of C_0. This is the idea behind the theorem, and it is not difficult to supply the details for a complete argument.

3. – We shall discuss a generalization of the above theorem to higher dimensions. In a word, even though we have not tried to write out the complete proof it seems that the result carries over pretty

much verbatim. Formulating the result precisely and carrying out the proof in detail would seem to be a worthwhile problem in degeneration of Hodge structures.

Suppose that $M \subset \mathbb{P}^N$ is a smooth variety of dimension $2n$. The discussion about the dual variety $M^* \subset \mathbb{P}^{N^*}$ and the corresponding hyperplane sections of M is exactly as in the case of surfaces. If $\overline{V}_0$ is a singular section then as above we may embed $\overline{V}_0$ in a family $\{V_t\}_{t\in\Delta}$ and arrive at the situation of a smooth manifold

$$X = \bigcup_{t\in\Delta} V_t$$

where V_0 is a divisor with normal crossings all of whose components $V_{0,j}$ have multiplicity one. The intermediate Jacobian $J(V_{0,j})$ of any component is a complex torus, and we first ask under what conditions the limiting position

$$\lim_{t\to 0} \nu_Z(t) \in \bigoplus_j J(V_{0,j})$$

can be determined? We will ignore questions of torsion, which are not that important since the Jacobians will all be divisible. If

$$\eta_Z \in H^{2n}(M) \cong H_{2n}(M)$$

is the fundamental class and

$$f\colon X \to M$$

the obvious map, then the condition is that

$$f^*(\eta_Z) = 0 \quad \text{in } H^{2n}(V_{0,j}) .$$

By Poincaré duality this is equivalent to saying that $\eta_Z \in H_{2n}(M)$ is orthogonal under the intersection pairing to the image of

$$f_*\colon H_{2n}(V_{0,j}) \to H_{2n}(M) .$$

Now on the other hand the direct sum $\bigoplus_j J(V_{0,j})$ is quite different from what the generalized intermediate Jacobian should be. One expects that there should be a map

$$J(V_0) \to \bigoplus_j J(V_{0,j})$$

whose fibers are the «non-compact part» of $J(V_0)$. To construct $J(V_0)$, at least the interesting variable part of it not coming from M, we might proceed as follows: If, for a suitable positive line bundle $L \to M$, the projective embedding $M \in \mathbb{P}^N$ is given by the complete linear system $|L|$, then replacing L by a high power we may assume that the cotangent space on the variable part of $J(V)$ is given by residues of the quotient group

$$H^0\big(\Omega_M^{2n}(nV)\big)/dH^0\big(\Omega_M^{2n-1}((n-1)V)\big)$$

of meromorphic forms having a pole of order $\leq n$ along V modulo exterior derivatives of forms having a pole of order $\leq n-1$ (cf. [1]). This description still makes sense when $\overline{V}_0$ is singular. More precisely, we may lift the above quotient group to X and try to use the corresponding residues as differentials on the generalized intermediate Jacobian $J(V_0)$.

For curves this works quite well, and it also goes in higher dimensions when $\overline{V}_0$ has ordinary double points as singularities. In general, however, the monodromy around $\overline{V}_0$ will not be unipotent of index one, and the corresponding difficulty which must be understood is this: Suppose that Z is an algebraic n-cycle on M whose lift to X satisfies

$$\begin{cases} Z\cdot V_t = 0 & \text{in } H_{2n-2}(V_t) \quad \text{for } t \neq 0 \\ Z\cdot V_{0,j} = 0 & \text{in } H_{2n-2}(V_{0,j}) \quad \text{for } j = 1, \dots, n . \end{cases}$$

Then $Z\cdot V_{0,j} = \partial\gamma_j$ where γ_j is a real $(2n-1)$-chain on the smooth variety $V_{0,j}$. However, when $n \geq 2$ the γ_j will in general meet the other components of V_0. In other words, we cannot find a family of chains γ_t with $\partial\gamma_t = Z\cdot V_t$ that avoid the singularities of V_0, and so the integrals used to define $\nu_Z(t)$ are not automatically convergent as $t \to 0$. Heurestic considerations suggest that this is not an essential difficulty, but it must be understood and doing so seems to us an interesting problem. For the purposes of the remainder of this paper we will assume that above theorem has been extended to the general case.

4. – One of the potential uses of normal functions is to construct algebraic cycles. Here, the main remark is that any class

$$\eta \in H^{2n}(M, \mathbb{Z}) \cap H^{n,n}(M)$$

comes from a normal function, so that a Hodge-theoretic assumption about a given homology class η has led to the construction of a complex-analytic object whose fundamental class is η. What is missing is some way to convert a normal function ν into a cycle on M. With this in mind we will make a few heurestic remarks.

First, any given generalized intermediate Jacobian $J(V)$ has a compactification $\overline{J(V)}$. For example, when $V \in M^* - M^*_{\text{sing}}$ has one arbitrary double point there is an extension

$$0 \to \mathbf{C}^* \to J(V) \to J(\tilde{V}) \to 0$$

where $\tilde{V}$ is the desingularization of V, and we may add the cross-sections at 0 and ∞ to the corresponding $\mathbf{C}^*$-bundle over $J(\tilde{V})$. Let us imagine, then, that we have constructed a compactification $\bar{\mathbf{J}}$ of the family $\mathbf{J}$ of generalized intermediate Jacobians $J(V)$ where $V \in M^*$. Of course, now it will be necessary to blow up M^* along suitable subvarieties in order to make sense out of $\bigcup_{V \in M^*} J(V)$, but let us assume this has been done. We denote by $\bar{D} \doteqdot \bar{\mathbf{J}} - \mathbf{J}$ the divisor which compactifies $\mathbf{J}$.

Now it seems pretty clear that a given normal function extends to give a cross-section

$$\nu \colon \tilde{M}^* \to \bar{\mathbf{J}}\,.$$

According to our theorem the subvariety

$$\nu^{-1}(\bar{D})$$

corresponds to hyperplane sections $\bar{V} \subset M$ such that the fundamental class $\eta \in H_{2n}(M)$ of ν is *not* orthogonal to the image of

$$H_{2n}(\bar{V}) \to H_{2n}(M)\,.$$

If, for example, we could prove by a topological argument that

$$\nu^{-1}(\bar{D}) \neq \emptyset\,,$$

then we will have constructed a new algebraic cycle (cf. the end of [1]). More precisely, the Hodge conjecture would follow if we could show that the assumption $\nu^{-1}(\bar{D}) = \emptyset$ for all sufficiently ample embeddings implies that the fundamental class $\eta = 0$.

Of course the rub here is the last statement: Can we say in some *a priori* manner just how ample the embedding must be? Moreover,

the statement that ν extends to $J(V)$, if and only if,

$$\eta \cdot \{\text{image of } H_{2n}(V) \to H_{2n}(M)\} = 0$$

still fails to deal directly with the given normal function. It may be that what is required is one additional step dealing with duality. On the one hand, Poincaré duality appears in the statement of our theorem, while on the other hand the result itself focuses attention on the dual variety $M^* \subset \mathbf{P}^{N^*}$ as parametrizing the interesting part of the total family of Jacobians. One may suggest that the normal functions associated to M^* should also be considered. More precisely, we recall that M^* has the canonical desingularization $\mathbf{P}(N)$ where N is the normal bundle of $M \subset \mathbf{P}^N$. The « dual » of $\mathbf{P}(N)$ is just M, and it may be a worthwhile project to study the singular behavior of normal functions for $\mathbf{P}(N)$.

Testo pervenuto il 28 aprile 1979.
Bozze licenziate il 28 ottobre 1980.

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