# ARCHIMEDIAN McKAY LAW

THE UPPER BOUND (OF McKAY) FOR THE NUMBER OF SPANNING TREES $K(X)$ OF A GRAPH $X$

$$\varlimsup_{\substack{X\to\infty \\ X\in\mathcal{X}_k}} \frac{\log K(X)}{|X|} = c_k \qquad (1)$$

WHICH IS ACHIEVED FOR RANDOM $X$, CAN BE EXTENDED TO ALLOW $X$'s WHICH ARE NOT REGULAR BUT HAVE AVERAGE DEGREE AT MOST $k$ (LYONS 2003 "ASYMPTOTIC ENUMERATION OF SPANNING TREES).

THE ARCHIMEDIAN ANALOGUE, THAT IS FOR THE DETERMINANT OF THE LAPLACIAN ON RIEMANNIAN SURFACES $X$

$$\log\det{}^*(X) := \log\det{}^*(\Delta_X) \qquad (2)$$

CAN BE DERIVED USING OSGOOD-PHILLIPS-SARNAK (1988; EXTREMALS OF DETERMINANTS OF LAPLACIANS.)

IT IS SHOWN THERE THAT WITH $A(X) = \text{AREA}(X)$

(i) $$\sup_{\substack{g(X)=0 \\ A(X)=4\pi}} \det{}^*(X) = \exp\left(\tfrac{1}{2} - 4\zeta'(-1)\right)$$

WITH EQUALITY IFF X IS THE ROUND SPHERE.

(ii) $$\sup_{\substack{g(X)=1 \\ A(X)=1}} \det{}^*(X) = \frac{\sqrt{3}}{2}\left|\eta\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\right|^4$$

WITH EQUALITY IFF X IS THE HEXAGONAL FLAT TORUS.

(iii) FOR $g \geq 2$ FIXED

$$\sup_{\substack{g(X)=g \\ A(X)=4\pi(g-1)}} \frac{\log \det{}^*(X)}{A(X)} := \eta(g) < \infty$$

IS ATTAINED BY A HYPERBOLIC X (IE ONE WITH CONSTANT CURVATURE EQUAL TO $-1$)

FOR THE RANDOM HYPERBOLIC $X$ WITH $g(X) \to \infty$ NAUD (2023; "DETERMINANTS OF LAPLACIANS ON RANDOM HYPERBOLIC SURFACES) SHOWS THAT

$$\lim_{\substack{g(X)\to\infty \\ X \text{ RANDOM HYPERBOLIC}}} \frac{\log \det^{*}(X)}{4\pi(g-1)} = \left(4\zeta'(-1) - \tfrac{1}{2} + \log 2\pi\right) := E. \qquad (3)$$

HE NOTES THAT THIS LIMIT IS ALSO ACHIEVED BY TAKING A CERTAIN CONGRUENCE HYPERBOLIC SHIMURA CURVES AND USING FRACZYK (2021 "STRONG LIMIT MULTIPLICITY FOR ARITHMETIC HYPERBOLIC SURFACES AND 3-MANIFOLDS) THE SAME IS TRUE FOR ALL SUCH SEQUENCES.

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FROM NAUD ONE HAS FOR HYPERBOLIC $X$ AND WITH THE USUAL TRACE FORMULA NOTATION THAT

$$\log \det^{*}(X) = A(X)\cdot E + \gamma_0 - \int_0^1 \frac{S_X(t)}{t}\,dt - \int_1^\infty \frac{S_X(t)-1}{t}\,dt \qquad (4)$$

WHERE

$$0 \leq \delta_X(t) = \frac{e^{-t/4}}{(4\pi t)^{1/2}} \sum_{k \geq 1} \sum_{\{\gamma\}_X} \frac{l(\gamma)\, e^{-(k\, l(\gamma))^2/4t}}{2\sinh\left(\frac{k\, l(\gamma)}{2}\right)} \qquad (5)$$

FOR ALL $t > 0$;

MOREOVER

$$\sum_{j=0}^{\infty} e^{-\lambda_j t} = \delta_X(t) + \frac{A(X)\, e^{-t/4}}{(4\pi t)^{3/2}} \int_0^{\infty} \frac{r \cdot e^{-r^2/4t}}{\sinh(r/2)}\, dr \geq 1 \qquad (6)$$

FROM (6)

$$\delta_X(t) - 1 \geq -A(X) e^{-t/4}$$

FOR $t \geq 1$

$$(7)$$

HENCE FROM (4) AND $1 < \xi < \infty$

$$\log \det{}^* X \leq A(X) E + \gamma_0 + \int_1^{\xi} \frac{dt}{t} + \int_{\xi}^{\infty} \frac{A(X) e^{-t/4}}{t}\, dt$$

TAKING $\xi = \log A$ GIVES

$$\frac{\log \det{}^* X}{A(X)} \leq E + \frac{\log \log A}{A} \qquad (8)$$

AND HENCE THAT

$$\overline{\lim_{\substack{X \text{ HYPERBOLIC} \\ g(X)\to\infty}}} \frac{\log \det{}^{*} \Delta}{A(X)} \le E \qquad \text{——} (9)$$

WE CONCLUDE FROM (9) AND (3) THAT

$$\overline{\lim_{g\to\infty}} \, \eta(g) = E \qquad \text{——} (10)$$

AND THAT THE STRONG ARCHIMEDIAN MCKAY LAW HOLDS:

$$\overline{\lim_{\substack{g(X)\to\infty \\ A(X)=4\pi(g-1)}}} \frac{\log \det{}^{*} \Delta}{A(X)} = E \qquad \text{——} (11)$$

WITH RANDOM HYPERBOLIC X AS WELL AS CONGRUENCE HYPERBOLIC X'S ACHIEVING THE LIMIT IN (11).

THE EXTREME VALUE FOR $\log \det^* X$ IS RELATED TO HEIGHTS IN ARAKELOV THEORY. SPECIFICALLY THE FALTINGS HEIGHT $\delta_{FAL}(X)$ OF A (COMPLEX) CURVE $X$ OF GENUS $g$ CAN BE EXPRESSED AS (SOULÉ 1989 "GEOMETRIE D'ARAKELOV DES SURFACES ARITHMETIQUE").

$$\delta_{FAL}(X) = -6 \log\left[\det^*_{AR}(X)/A(X)\right] - 2g(X)\log\pi + 4g(X)\log 2 + (g(X)-1)(-24\zeta'(-1)+1) \qquad (12)$$

WHERE $AR(X)$ IS THE ARAKELOV METRIC ON $X$.

THE DIFFERENCE

$$\log\left[\frac{\det^*_{AR}(X)}{A(X)}\right] - \log\left[\det^*_{AH}[X]/A(X)\right] \qquad (13)$$

WITH THE LAST BEING THE HYPERBOLIC METRIC ON $X$

CAN BE EXPRESSED USING POLYAKOV'S FORMULA (SEE JORGENSON-KRAMER 2009 "BOUNDS ON FALTING'S DELTA FUNCTION THROUGH COVERS")

IN FACT J-K SHOW THAT ALL OF THE QUANTITIES FOR $X_0(N)$

$$\log\det^{*}_{HYP}(X_0(N)),\ \log\det^{*}_{AR}(X_0(N)),$$
$$\delta_{FAL}(X_0(N))$$

ARE ALL $O\left(g(X_0(N))\right)$ AS $N \to \infty$.

AN INTERESTING PROBLEM IS TO SHOW THAT EACH OF THEM IS ASYMPTOTIC TO A CONSTANT TIMES $g(X_0(N))$ WHICH WOULD YIELD A SECOND TERM IN THE MUCH STUDIED ASYMPTOTICS OF $\cancel{h_{FAL}}$ THE ARITHMETICAL HEIGHT $h_{FAL}(J_0(N))$.