0. Classical Modular group Γ:
$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$, a, b, c, d integers (rational).
generated by
$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
Acts on upper complex half plane.

Hilbert modular group Γ:
K totally real number field of degree n over the rationals, elements:
$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $\alpha\delta - \beta\gamma = 1$, $\alpha, \beta, \gamma, \delta$ integers in K. Parabolic subgroup Γ_∞, elements $\begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}$, where ω runs over integers in K.
Acts on product of n upper complex half planes.

1. 1935 Hecke asked:
When does $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -\frac{1}{c} \\ c & 0 \end{pmatrix}$ generate a discrete group?
Showed this was case if either $c = 2\cos\frac{\pi}{q}$, q integer ≥ 3, or $c \geq 2$.

More generally we may ask:
<u>When does $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ generate a discrete group?</u>

Or even more general question:

For $n \geq 1$, given a discrete lattice of translations in euclidean n space whose elements ω (each with n components $\omega^{(i)}$) are generated by n linearly independent elements $\xi_1, \ldots, \xi_n$ (with nonsingular matrix $\Omega = (\xi_j^{(i)})$. Assume the lattice is irreducible in the sense that if an element ω has one component $\omega^{(i)}$ equal to zero, then all components of ω are zero: $\omega = 0$. Denote by Γ_∞ the group with elements $\begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}$, generated by $\begin{pmatrix} 1 & \xi_j \\ 0 & 1 \end{pmatrix}$ for $j = 1, \ldots, n$.

If we now adjoin to Γ_∞ an element

$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with n components $\begin{pmatrix} \alpha^{(i)} & \beta^{(i)} \\ \gamma^{(i)} & \delta^{(i)} \end{pmatrix}$

and $\alpha\delta - \beta\gamma = 1$ ($\alpha^{(i)}, \beta^{(i)}, \gamma^{(i)}, \delta^{(i)}$ real)

<u>When is the resulting group discrete?</u> **Case $n=1$** **case $n>1$**

2. Put $D = \| \xi_j^{(i)} \| = \| \Omega \|$.

From a theorem of Minkowski:

For $t_i > 0$ and $\prod_{i=1}^{n} t_i \geq D$ $\exists\, \omega \neq 0$

with $|\omega^{(i)}| \leq t_i$ for $i = 1, \ldots, n$.

Will be used repeatedly.

We can first show that either all $\gamma^{(i)} = 0$, or all $\gamma^{(i)} \neq 0$. (Done by forming repeated commutators between elements $S^{\omega} = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}$ and M.) Since case $\gamma = 0$ is fairly trivial to discuss, we shall turn to case that all $\gamma^{(i)} \neq 0$.

Can write

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = S^{\frac{\alpha}{\gamma}} T_\gamma S^{\frac{\delta}{\gamma}}$$

where

$$S^{\frac{\alpha}{\gamma}} = \begin{pmatrix} 1 & \frac{\alpha}{\gamma} \\ 0 & 1 \end{pmatrix}, \quad T_\gamma = \begin{pmatrix} 0 & -\frac{1}{\gamma} \\ \gamma & 0 \end{pmatrix}.$$

Consider strings of elements

$$S^{\omega_0} M S^{\omega_1} M^{-1} S^{\omega_2} M \cdots S^{\omega_k} M^{(-1)^k} S^{\omega_{k+1}} =$$

$$= \pm S^{\frac{\alpha}{\gamma}} \left(S^{\omega_0} T_\gamma S^{\omega_1} T_\gamma \cdots S^{\omega_k} T_\gamma S^{\omega_{k+1}} \right) \begin{matrix} S^{\frac{\delta}{\gamma}}, k \text{ even} \\ S^{-\frac{\alpha}{\gamma}}, k \text{ odd} \end{matrix}$$

From this easily conclude: We can only get a discrete group by adjoining M to Γ_∞ if we also get a discrete group by adjoining T_γ.

Write

$$M_k = T_\gamma S^{\omega_1} T_\gamma^{\omega_2} \cdots S^{\omega_{k-1}} T_\gamma = \begin{pmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{pmatrix}.$$

Recursion formulas:

$$\alpha_{k+1} = \gamma \omega_k \alpha_k - \alpha_{k-1}; \quad \gamma_{k+1} = \gamma \omega_k \gamma_k - \gamma_{k-1}$$

$$\alpha_0 = 1, \alpha_1 = 0, \alpha_2 = -1 \quad ; \quad \gamma_0 = 0, \gamma_1 = \gamma, \gamma_2 = \omega_1 \gamma^2$$

$$\alpha_3 = -\gamma \omega_2, \alpha_4 = -\omega_2 \omega_3 \gamma^2 + 1 \quad ; \quad \gamma_3 = \omega_1 \omega_2 \gamma^3 - \gamma,$$

$\gamma_4 = \omega_1\omega_2\omega_3\gamma^4 - (\omega_1+\omega_3)\gamma^2$. (needed later)

We define:

$$N(\alpha) = \prod_{i=1}^{n} |\alpha^{(i)}|,$$

and can then show: for all $\omega \neq 0$ we have: $N(\omega\gamma) \geq 1$.

In particular $N(\gamma) \geq \frac{1}{D}$, and for all $\omega \neq 0$, $N(\omega) \geq \frac{1}{N(\gamma)}$.

We assume $N(\omega\gamma) < 1$ for some $\omega \neq 0$, and construct a sequence of $\gamma_k \neq 0$, such that all $\gamma_k^{(i)} \to 0$ as $k \to \infty$, and noting that we can always find S^{ω} and $S^{\omega'}$ in Γ so that

$$S^{\omega}\begin{pmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{pmatrix} S^{\omega'} = \begin{pmatrix} 1+\mathcal{O}(\gamma_k) & , \mathcal{O}(1+|\gamma_k|) \\ \gamma_k & , 1+\mathcal{O}(\gamma_k) \end{pmatrix},$$

where the constants implied by the $\mathcal{O}$ symbols depend on the lattice. This contradicts the discreteness.

We next turn to the structure of the lattice, using again the Minkowski theorem and the principle that a compact region can only contain finitely many different elements of a discrete group.

We pick pairs ω_1, ω_2 so that for T large and $1 \le k < \ell \le n$, $\omega_1 \neq 0$ and

$|\omega_1^{(k)}| \le T D^{\frac{1}{n}}$, $|\omega_1^{(\ell)}| \le \frac{1}{T} D^{\frac{1}{n}}$,

and $|\omega_1^{(i)}| \le D^{\frac{1}{n}}$ for $i \neq k$ and ℓ.

For $\omega_2 \neq 0$ we interchange the ~~upper~~ right hand sides of the two first inequalities.

Looking at M_3 with $\gamma_3 = \omega_1 \omega_2 \gamma^3 - \gamma$ we see since $|\omega_1^{(i)} \omega_2^{(i)}| \le D^{\frac{2}{n}}$ for $1 \le i \le n$ that the components of γ_3 remain bounded as $T \to \infty$ (and also bounded away from zero since $N(\gamma_3) \ge \frac{1}{D}$). And so we can choose ω and ω' so that

$S^{\omega} M_3 S^{\omega'}$ lies in a compact region as $T \to \infty$. We must clearly get the same element infinitely often as $T \to \infty$.

Thus we get relations of form

$\omega_1 \omega_2 - \omega_1' \omega_2' = 0$, where say, ω_1, ω_2 correspond to a T we make very large, while ω_1', ω_2' correspond to a T' kept fixed. This gives relations of the form:

$$\sum_{1\le r,s\le n} c_{r,s}^{(k,\ell)} \xi_r \xi_s = 0 ,$$

with integral rational coefficients $c_{r,s}^{(k,\ell)}$. As $T \to \infty$ we can find asymptotic expressions for the $c_{r,s}^{(k,\ell)}$ and deduce that the $\frac{n(n-1)}{2}$ relations we get by all choices of k and ℓ are linearly independent, so we can solve with respect to $\xi_r \cdot \xi_s$ with $1 \le r \le s \le n-1$, and get

$$\xi_r \xi_s = \sum_{j=1}^{n} d_j^{(r,s)} \xi_j \xi_n ,$$

with rational coefficients $d_j^{(r,s)}$.

Writing $\xi_j = \xi_j \xi_n$ (so $\xi_n = 1$)

we get

$$\xi_r \xi_s = \sum_{j=1}^{n} d_j^{(r,s)} \xi_j .$$

From this it is easy to conclude that each $\frac{\omega}{\xi_n}$ satisfies a Polynomial equation with rational coefficients of degree n at most, since any power can be expressed as

$$\frac{\omega^{(m)}}{\xi_m^{(m)}} = \sum_{j=1}^{n} c_{j,m}^{(\omega)} \xi_j$$

with rational coefficients $c_{j,m}^{(\omega)}$, and the $(n+1)$ expressions for $0 \le m \le n$ cannot be linearly independent. If we pick ω such that $\frac{\omega}{\xi_m^{(m)}}$ generates the field containing all the $\xi_j^{(m)}$, $\frac{\omega}{\xi_m^{(m)}}$ must satisfy an irreducible equation of degree n.

Thus there exists a totally real field K of degree n such that the ratios of any two ω's lie in this field and the components $\left(\frac{\omega_i}{\omega_2}\right)^{(i)}$ are simply the various n conjugates.

It is now easily seen that by a group conjugacy with a suitable $\begin{pmatrix} \rho & 0 \\ 0 & 1/\rho \end{pmatrix}$ we can bring our lattice on such a form that all ω lie in K and the lattice contains all integers in K.

We next wish to show that $\gamma\omega$ is an algebraic integer and that $(\gamma\omega)^2$ is in the field K.

We may assume our lattice contains all integers in K, and wish to show that γ^2 lies in K. Again we look at elements of the form:

$$M_3 = T_\gamma S^{\omega_1'} T_\gamma S^{\omega_2} T_\gamma ,$$

We saw earlier that we could choose ω_1, ω_2 and ω_1', ω_2' such that $\omega_1\omega_2 = \omega_1'\omega_2'$ and so the same $\gamma_3 = \omega_1\omega_2\gamma^3 - \gamma$ as we let $T \to \infty$ in our construction, and so we must have an identity

$$M_3 = S^{\omega_0} M_3' S^{\omega_3}$$

which gives

$$\alpha_3 = \alpha_3' + \omega_0\gamma_3 ,$$

or inserting the values of α_3, α_3' and γ_3,

$$-\gamma\omega_2 = -\gamma\omega_2' + \omega_0\omega_1\omega_2\gamma^3 - \gamma\omega_0 ,$$

or

$$\gamma^2 = \frac{\omega_0 + \omega_2' - \omega_2}{\omega_0\omega_1\omega_2} .$$

Thus $\underline{\gamma^2 \text{ is in } K.}$

We now assume that $\underline{\gamma^2 \text{ is not an integer in } K}$ and wish to get a contradiction.

If our γ^2 can not be written in the form $\gamma^2 = \frac{\rho}{\varkappa}$ where ρ and $\varkappa$ are integers in K with $(\rho, \varkappa) = 1$, and $\varkappa$ not a unit,

we can pass to a new γ' of the form $\gamma' = \omega \gamma^{2^k}$, where 2^k is larger than the class number of K, and choose ω such that γ' can be written as $\frac{\varrho}{\varkappa}$ with $(\varrho, \varkappa) = 1$ and $\varkappa$ not a unit. We may now write γ instead of γ'.

Forming again elements M_4 we have

$$\gamma_4 = \omega_1 \omega_2 \omega_3 \gamma^4 - (\omega_1 + \omega_3)\gamma^2$$

and

$$\alpha_4 = -\omega_2 \omega_3 \gamma^2 + 1.$$

If the ω's can be chosen as integers such that

$$(*) \qquad \omega_1 \omega_2 \omega_3 \gamma^2 = \omega_1 + \omega_3,$$

we see that

$$\gamma_4 = 0 \quad \text{and} \quad \alpha_4 = -\frac{\omega_3}{\omega_1}.$$

This leads to a contradiction if $\frac{\omega_3}{\omega_1}$ is <u>not</u> a unit, since in upper triangular elements the diagonal can only contain units, as conjugation with such an element

must carry the lattice (or Γ_∞) into itself.

(*) can be rewritten

$$\omega_1 \omega_2 \omega_3 \rho^2 = (\omega_1 + \omega_3) \varkappa^2 .$$

We may choose ω_1 as a unit η in K and ω_3 as a divisor δ of $\varkappa^2$ which is not a unit, and see then that if

$$\eta + \delta \equiv 0 \pmod{\rho^2},$$

ω_2 can also be chosen as an integer in K and we would get our contradiction.

It would clearly make it easier if $\varkappa^2$ has many divisors, so we try to pass from our $\gamma = \frac{\rho}{\varkappa}$ to a γ_k for which a solution to our congruence can be assured.

We write $$\gamma_k = \frac{\rho_k}{\varkappa^k},$$

and find

$$\rho_{k+1} = \rho \omega_k \rho_k - \varkappa^2 \rho_{k-1},$$

$\rho_1 = \rho$, $\rho_2 = \rho^2 \omega_1$, $\rho_3 = \rho^3 \omega_1 \omega_2 - \varkappa^2 \rho$, ...

In general we now write

$$\rho_{2k} = \rho^2 \Delta_{2k}, \quad \rho_{2k+1} = \rho \Delta_{2k+1},$$

and get:

$$\Delta_{2k+1} = \omega_{2k}\rho^2\Delta_{2k} - \varkappa^2\Delta_{2k-1},$$

$$\Delta_{2k+2} = \omega_{2k+1}\Delta_{2k+1} - \varkappa^2\Delta_{2k}.$$

Or,

$$\Delta_{2k+1} \equiv -\varkappa^2\Delta_{2k-1} \pmod{\rho^2\Delta_{2k}},$$

and

$$\Delta_{2k+2} \equiv -\varkappa^2\Delta_{2k} \pmod{\Delta_{2k+1}}.$$

We try to select a sequence of Δ'_s satisfying these congruences, and such that

$$\Delta_k = \eta_k \pi_k^{m_k},$$

where η_k is a unit, π_k a representative of a principal prime ideal of degree 1 (with norm a prime p_k), and such that:

π_{2k+2} is always a primitive root modulo π_{2k+1}^2,

$$\pi_{2k+1}^{p_{2k}-1} \not\equiv 1 \pmod{\pi_{2k}^2}, \quad p_{2k+1} > N(\rho^2);$$

and

$$\eta_{2k+1}\pi_{2k+1} \equiv -\varkappa^2\eta_{2k-1}\pi_{2k-1}^{m_{2k-1}} \pmod{\rho^2\pi_{2k}^2},$$

This can always be done, but we can also always select our π_k such that any L-function of the field K with a character mod π_k^2 or $\rho^2\pi_k^2$ which is not the restriction of a character mod ρ^2, has no zero in the region $|t|<p_k^{10}$, $\sigma>1-\delta$, where δ is a small positive constant depending on K.

This follows from a density result which shows that the π for which this does not hold, form a very thin set.

We can now show that we can select our π_k such that always

$$p_{2k+1} \leq \max\left(p_{2k}^{A}, C(\rho,\delta)\right),$$

where A is a constant depending on K, and always

$$p_{2k+2} \leq \max\left(p_{2k+1}^{\varepsilon}, C(\rho,\delta,\varepsilon)\right),$$

where $\varepsilon>0$ is an is an arbitrary chosen small number.

This gives

$$p_{k+2} \leq \max\left(p_k^{A\varepsilon}, C'(\rho, \delta, \varepsilon) \right),$$

so by choosing $\varepsilon < \frac{1}{A}$, we see that the p_k stay bounded.

We may now look at

$$\gamma_{2k+1} = \zeta \frac{\pi_{2k+1}^{m_{2k+1}}}{x^{2k+1}} \eta_{2k+1}.$$

Taking an arbitrary unit of infinite order, η, we see that η^s represents at least

$$p_{2k+1}^{2m_{2k+1} - r}$$

different residueclasses modulo

$$\zeta^2 \pi_{2k+1}^{2m_{2k+1}},$$

where r is the highest power of π_{2k+1} that divides $\eta^{p_{2k+1}-1} - 1$; this r is bounded since p_{2k+1} is. Now consider the numbers $\eta^s x^t$ with $0 \leq t \leq 4k+2$, they represent at least $(4k+2) p^{2m_{2k+1} - r}$ residue classes.

For $4k+2 > p_{2k+1}^r N(\zeta^2)$, this is larger than the total number of

residues modulo $\wp^2 \pi_{2k+1}^{2m_{2k+1}}$. Thus there must be duplication, and since two with the same t are not congruent we get with $t \neq t'$

$$\eta^s x^t \equiv \eta^{s'} x^{t'} \left(\text{mod. } \wp^2 \pi_{2k+1}^{2m_{2k+1}}\right),$$

and so

$$x^{t-t'} \equiv \eta^{s''} \left(\text{mod. } \wp^2 \pi_{2k+1}^{2m_{2k+1}}\right),$$

with $1 \leq |t-t'| < 4k+2$.

This finally gives our contradiction.

Thus <u>our original γ must be an algebraic integer such that γ^2 is in K.</u>

Going back to our original $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and forming higher powers M^k and looking at the lower left-hand entry, we see that the eigenvalues of M are units (not in K in general), and <u>the trace of M, $\alpha + \delta$ must be an integer whose square is in K.</u>

Considering also the trace of $S^{\omega} M$ we get that $(\gamma\omega)^2$ is an integer in K, and that $(\alpha+\delta)\gamma\omega$ also is an integer in K, as well as $(\alpha+\delta)^2$.

This last statement is in a form that is invariant under conjugacies by upper triangular matrices.

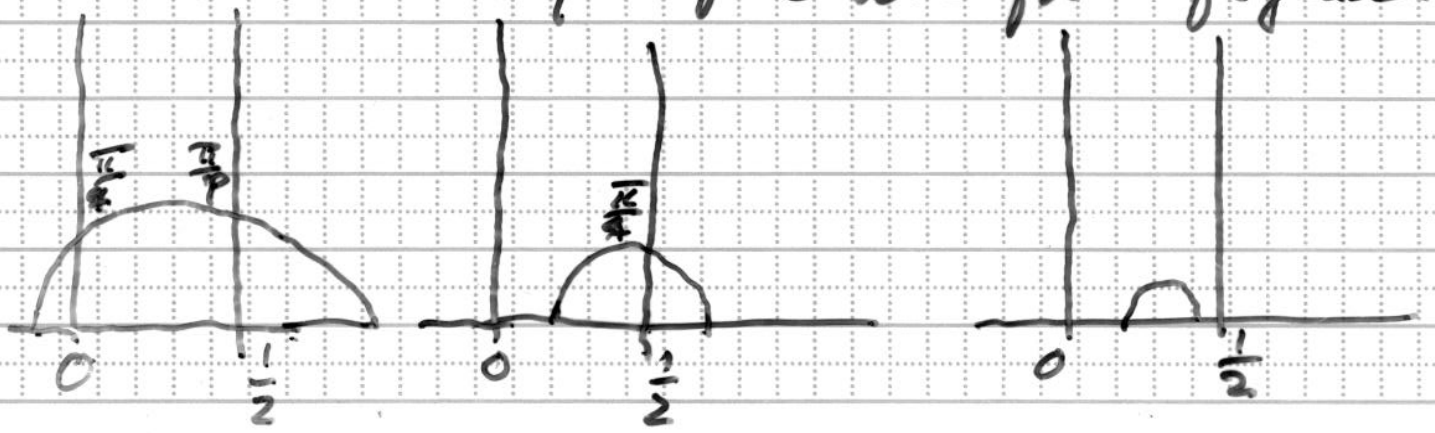
It is not hard to show that the above conditions are also sufficient to ensure that the adjunction of M to the parabolic group Γ_∞ produces a discrete group.

Finally, if we replace some or all copies of $SL(2,\mathbb{R})$ by $SL(2,\mathbb{C})$, things are similar for $n \geq 2$. The case $n=1$ with just one copy of $SL(2,\mathbb{C})$ seems quite intractable however.

Case $n=1$

For $n=1$, question best resolved by first focusing on possible elliptic elements of form $S^n M$, n integer. The answer is that (up to conjugacy by a matrix of form $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$), for $0<c<1$ there are no such M, for $1 \leq c < 2$ just the M that produce the Hecke groups for $c = 2\cos\frac{\pi}{q}$, $q \geq 3$. For $c=2$ there are two different M and for $2<c<4$, there is for each c a finite number of different M. For $c=4$ a denumerable infinity of M, and for $c>4$ a continuum of M's. (For $c>4$, the probability that an M with a and d chosen at random modulo c, produces a discrete group is actually $1-\frac{4}{c}$)

The discrete groups that can arise in this fashion, are precisely those that can arise by reflection from figures:



Some further explanation of details.

1.

Top of p. 3. Assume $\gamma^{(i)} \neq 0$ for $1 \leq i \leq r < n$, and $\gamma^{(i)} = 0$ for $r < i \leq n$. Write

$$S^{\omega} = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}$$

and form the commutators $[S^{\omega}, M]$ and $[[S^{\omega}, M], M]$. Observe that the components with $r < i \leq n$ are the identity element $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Thus $[[S^{\omega}, M], M]$ ~~remains the same~~ depends only on the first r components of ω, but since these form a dense set, the set $[[S^{\omega}, M], M]$ is easily seen not to be discrete.

p. 4. The recursion formulas at bottom of p. 3 show that we can construct an M_2 with $\gamma_2 = \gamma^2 \omega$, repeating this process using the same ω but the new M_2 we get a sequence whose $(k-1)$th term has the lower left-hand corner entry

$$\gamma^{2^{k-1}} \omega^{2^{k-1}-1} .$$

Repeating the process once with a ω_1 to be suitably chosen later, we get

$$(\gamma\omega)^{2^k - 2} \gamma^2 \omega_1 .$$

Using Minkowski we now chose ω_1 so that $N(\omega_1) \leq D$, and so that it

makes all components of $(\gamma\omega)^{2^k-2}\gamma^2\omega_1$ satisfy the inequality

$$\left|\left((\gamma\omega)^{2^k-2}\gamma^2\omega_1\right)^{(i)}\right| \leq N(\gamma\omega)^{\frac{2^k-2}{m}} N(\gamma)^{\frac{2}{m}} D^{\frac{1}{m}}.$$

Since by assumption $N(\gamma\omega)<1$, so the right-hand side of the above inequality tends to 0 as $k\to\infty$, this gives us the desired sequence.

p.6. Let $\Omega^{-1} = (\rho_{r,s})$ and write

$$\omega_1^{(k)} = T D^{\frac{1}{m}} \theta_1^{(k)}, \quad \omega_2^{(\ell)} = T D^{\frac{1}{m}} \theta_2^{(\ell)},$$

where we have

$$\frac{1}{D N(\gamma)} \leq |\theta_1^{(k)}| \leq 1, \quad \frac{1}{D N(\gamma)} \leq |\theta_2^{(\ell)}| \leq 1.$$

It is then easy to see that

$$C_{r,s}^{(k,\ell)} = T^2 D^{\frac{2}{m}} \theta_1^{(k)} \theta_2^{(\ell)} \frac{(\rho_{r,k}\rho_{s,\ell} + \rho_{r,\ell}\rho_{s,k})}{1+\delta_{k,\ell}} + \\ + Q(T),$$

where $\delta_{r,s}$ is the Kronecker symbol ($\begin{smallmatrix}1 \text{ for } r=s\\ 0 \text{ for } r\neq s\end{smallmatrix}$).

For large enough T, the determinant

$$\left| C_{r,s}^{(k,\ell)} \right| \text{ with } 1\leq k<\ell\leq n \text{ and } 1\leq r\leq s\leq n-1,$$

is then easily seen to be different from zero if we we have that the

determinant

$$\left| \frac{\rho_{r,k}\rho_{s,\ell} + \rho_{r,\ell}\rho_{s,k}}{1+\delta_{r,s}} \right| \neq 0 .$$

We cannot guarantee that this is the case with the basis $\Omega = (\xi_j^{(i)})$ we originally picked for our lattice, however, we can of course replace Ω by ΩA where A is an n by n integral matrix with determinant ± 1 (which gives the same lattice), or even ΩA where A is a nonsingular integral n by n matrix (which gives rise to a sublattice of finite index). But it is then seen that A can be chosen so that

$\|A\|^{-\frac{1}{n}} \Omega A$ approximates any given real matrix with the determinant $\pm D$ arbitrarily well. If we now use this Ω' instead of Ω it is now obvious that we can chose A such that

$$\left| \frac{\rho'_{r,k}\rho'_{s,\ell} + \rho'_{r,\ell}\rho'_{s,k}}{1+\delta_{r,s}} \right| \neq 0 .$$

Thus for $\Omega' = \Omega A$ the statement on the middle of p. 7 is proved, but it then clearly also holds for the original lattice Ω.

This really ends the part where geometry of numbers plays a significant part in the arguments. There are of course also several things in the later part that are oversimplified or left out in the exposition. That was necessary to make a 1-hour lecture out of it. To fill in all details in the later part would require many more pages. The later part is actually by far the older, dating from the early sixties if I assumed the general Riemann Hypothesis for the zeta function and L functions of the underlying field, in 1970-71 (around new year actually) I was able to replace that assumption by a density-theorem I could prove. I lectured on this right after the new year here. Huxley, who was here (there was a "number-theory year" at the time) later published the density result with somewhat more explicit constants depending on the underlying number-field.