First real case 1. k. real ~~totally~~ degree n

greek letters in k, latin not. $\begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}$; $Sl(2,R)^{n_1}\, Sl(2,C)^{n_2}$

$\underline{n_1 + n_2 > 1.}$

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let $k(\theta)$ alge no. field. θ n_1 real ; $2n_2$ complex

ω integer in k. $\omega^{(i)}$

$1 \le i \le n_1$ real. next n_2 complex.

adjoint to group $\mathfrak{P}$ of S_ω: $\begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}$ where ω runs over all integers an element M with components $\begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix}$

The resulting group cannot be discrete unless the $c^{(i)\,2}$ are the ith conjugates of an integer in k.

A) First show that if one $c^{(i)} \neq 0$ then all are. look at $[[M, S_{\omega_1}] S_{\omega_2}] = M'$ if $c^{(j)}_j = 0$ then J comp of M' is identity. since projection of $\mathfrak{P}$ on other comp. not discrete. commutator of M' with P is seen to contain elements arbitrarily close to the identity $\neq e$.

B) Next all $c^{(i)} \neq 0$, replace M by T_c with components $\begin{pmatrix} 0 & -\frac{1}{c^{(i)}} \\ c^{(i)} & 0 \end{pmatrix}$

The group resulting from adjoining M to P is discrete only if that obtained by adjoining T to P is discrete.

Write $M = S_{\frac{a}{c}}\, T_c\, S_{\frac{d}{c}}$; where notation

$S_{\frac{a}{c}}$ means parabolic element with components $S_{\frac{a^{(i)}}{c^{(i)}}}$ and similarly $S_{\frac{d}{c}}$;

We have $T_c^{\,2} = -E$;

and that an expression

$$M_k = S_{\omega_0} M S_{\omega_1} M^{-1} S_{\omega_2} \cdots M^{(-1)^{k-1}} S_{\omega_k}$$

is (apart from a simple S factor at each end) essentially the same as

$$M_k' = S_{\omega_0} T_c S_{\omega_1} T_c \cdots S_{\omega_{k-1}} T_c S_{\omega_k}$$

since the form M_k' contains <u>all</u> elements obtainable by adjunction of T_c to P the result easily follows.

c) Look at

$$T_c S_{\omega_1} T_c S_{\omega_2} T_c S_{\omega_3} T_c S_{\omega_4}$$

$\begin{pmatrix} a_4 & b_4 \\ c_4 & d_4 \end{pmatrix}$; trace $= c^4\omega_1\omega_2\omega_3\omega_4 - c^2(\omega_1+\omega_3)(\omega_2+\omega_4) + 2$

$$c_4 = \omega_1\omega_2\omega_3 c^4 - (\omega_3+\omega_1)c^2$$

or taking $\omega_3 = \eta^h$, $\omega_1 = -\eta^k$ (η unit

$\omega_2 = -\eta^{-k-h}\omega$; $c_4 = \omega c^4 - (\eta^h - \eta^k)c^2$

can form a discrete set only if c^2 in <u>k</u> (but not necessarily integral).

D) ~~Show~~ that c^2 <u>integral no</u> in k

preparations.

~~write~~

$$M_k' = \begin{pmatrix} x & x \\ c_k & x \end{pmatrix} = T_c S_{\omega_1} \cdots \quad T_c S_{\omega_k}$$

easily seen that $\quad$ $c_0 = 0,\ c_1 = c$

$$c_{k+1} = \omega_k c \cdot c_k - c_{k-1}$$

in particular $c_2 = \omega_1 c^2$, thus we may consider c^2 instead of c; denote c^2 by γ in k (have to show γ integral).

$$\gamma_{k+1} = \omega_k \gamma \gamma_k - \gamma_{k-1}$$

Assume γ not integral; we try to produce an M_k' with $\gamma_k = 0$ and such that the two elements in main diagonal are not units in k. Then easily get contradiction, since if we ~~otherwise~~ ~~cannot~~ ~~have that get that $(M_k')^\nu P (M_k')^{-\nu}$ for ν integer~~ consider the maximal ~~subgroup~~ nilpotent subgroup of M_k' & P

If we make $\gamma_4 = 0$ this requires that ω_1 and ω_3 can be chosen so that

$$\frac{\omega_1 + \omega_3}{\omega_1 \omega_3 \gamma^2} \quad \text{is an integer}$$

since $\gamma_4 = \omega_1 \omega_2 \omega_3 \gamma^4 - (\omega_1 + \omega_3)\gamma^2$

also the trace of M_4' in that case is

$$+2 + \frac{(\omega_1+\omega_3)^2}{\omega_1\omega_3} = \frac{\omega_1}{\omega_3} + \frac{\omega_3}{\omega_1} \quad \text{so that}$$

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the eigenvalues are not integral if $\frac{\omega_1}{\omega_3}$ is not a unit.

Thus we need to be able to choose ω_1 and ω_3 such that

$\frac{\omega_1}{\omega_3}$ is not a unit and

$$\frac{\omega_1 + \omega_3}{\omega_1 \omega_3} \frac{1}{\gamma^2} \text{ is an integer.}$$

This may not be possible with our original γ, but we try then to replace it with a γ_u which again is not integral and for which the corresponding conditions can be fulfilled.

First simplify: we can (by eventually replacing γ by a suitable $\omega\gamma^{2^h}$ arrange that γ can be written as $\frac{\rho}{\varkappa}$ where $(\rho, \varkappa) = 1$ and both are integers. $|N(\varkappa)| > 1$

the conditions

$$\frac{\omega_1 + \omega_3}{\omega_1 \omega_3} \frac{1}{\gamma^2} \text{ is an integer}$$

~~means~~ are then fulfilled if

$\frac{\omega_1 + \omega_3}{\omega_1 \omega_3}$ if $\omega_3 = -\varkappa_1$ is a divisor of $\varkappa^2$ not a unit. $\omega_1 = \eta$ (unit.)

and $\rho^2 / \eta - \varkappa_1$ (or)

$$\eta - \varkappa_1 \equiv 0 \ (\rho^2).$$

Thus if we can fulfill such a congruence we get a contradiction.

We now try to produce a $\gamma_k = \frac{\rho_k}{x^k}$ such that the congruence

(✻) $\eta - x^t \equiv 0 \pmod{\rho_k^2}$

holds with some unit η. some pos integer t with $0 < t < 2k$

Writing

$\rho_{2k} = \rho^2 \Delta_{2k}$; $\rho_{2k+1} = \rho \Delta_{2k+1}$

we have $\Delta_0 = 0$, $\Delta_1 = 1$

(1) $\Delta_{2k+1} = \rho^2 \omega_{2k} \Delta_{2k} - x^2 \Delta_{2k-1}$

(2) $\Delta_{2k+2} = \omega_{2k+1} \Delta_{2k+1} - x^2 \Delta_{2k}$

or

(1') $\Delta_{2k+1} \equiv - x^2 \Delta_{2k-1} \pmod{\rho^2 \Delta_{2k}}$

(2') $\Delta_{2k+2} \equiv - x^2 \Delta_{2k} \pmod{\Delta_{2k+1}}$

and all Δ' rel prime to x (<u>ω's rel prime to x</u>).

We try to ~~construct~~ choose a sequence ω_k such that always $\Delta_k = \eta_k \pi_k^{m_k}$ where π_k is a repr. of a principal prime ideal of degree 1 ($N(\pi) = \pm p$), such the norms ^{p} do not grow over all bounds as $k \to \infty$.

Construction. prime . prime ideals of degree 1.

set P of all π in k divided into two classes $P = P_N + P_E$

(1) $\pi \in P_N$ has following property: $|N(\pi)| = p \geq (N(\rho))^m$ (2) there is a field const $\delta > 0$ such that if $\pi \in P_N$ and χ is a Hecke congruence character of the field mod π^2 or $\rho^2 \pi^2$ which is such that it is not just the restriction mod π^2 or $\rho^2\pi^2$ of a character mod (1) or ρ^2,

then

$$(1) \qquad L(s, \chi) \neq 0 \quad \text{for} \quad \begin{array}{l} \sigma \geq 1-\delta \\ |t| \leq p^{80} \end{array}$$

For P_E holds

$$(2) \qquad \sum_{\substack{|N(\pi)| \leq x \\ \pi \in P_E}} 1 \leq \sqrt{x}$$

for $x > x_0$ (where x_0 depends on k and ρ and $\varkappa$). On the other hand

$$(3) \qquad \sum_{\substack{N(\pi) \leq x \\ \pi \in P}} 1 \sim \frac{x}{h \log x} \quad (h \text{ class no})$$

Thus P_E is quite thin.

Selection lemmas

(A) For any fixed pos ε and for π in P_N we can always find π' in P_N with following properties:

(a) A representative of π' is a primitive root modulo π^2, ~~(b) Either there $\exists$ an η with $\eta \equiv 1 \pmod{\pi}$ and $\eta \not\equiv 1 \pmod{\pi^2}$, or~~

(b) π' has the property that $\pi'^{\,p-1} \not\equiv 1 \pmod{\pi^2}$

(c) $p' = |N(\pi')| \leq \max(p^{\varepsilon}, x_1(\varepsilon))$

where $p = |N(\pi)|$ and x_1 depends on $\varepsilon, k, \rho, \varkappa$.

$(\varphi(\rho^2), p) = 1$ ~~$(\rho^2, p) = 1$~~;

(B) For a π in P_N and any ~~number~~ integer α such that $(\alpha, \rho^2 \pi) = 1$, we can always find a π' in P_N such that

(a') $\pi' \equiv \alpha \pmod{\rho^2 \pi}$ $/ \; p = N(\pi') > |N(\rho^2)|$

(b') $\pi'^{\,p-1} \not\equiv 1 \pmod{\pi^2}$

and finally

(c') $|N(\rho^2)| < p' = |N(\pi')| \leq \max(p^{c}, x_2)$

where c is a field constant and x_2 depends on $k, \rho, \varkappa$ only.

Conclusion.

Use B to select π_{2k+1} and suitable power of π_{2k+1} will satisfy congruence mod the power of π_{2k}.

Use A to select π_{2k+2}

Taking two steps we see that

$$p_{k+2} \leq \max\left(p_k^{C\varepsilon}, X_3(\varepsilon) \right)$$

where $X_3(\varepsilon)$ depends on ε, k, ρ, x only). Taking $\varepsilon < \frac{1}{C}$ we get that the p remain bounded as k tends to ∞.

For k large enough always solution of congruence $(*)$ for γ_{2k+1}

$$= \frac{\rho\, \pi_{2k+1}^{m_{2k+1}}}{x^{2k+1}}$$

$$\eta - x^t \equiv 0 \left(\rho^2\, \pi_{2k+1}^{2m_{2k+1}} \right)$$

with $0 < t \leq 4k+2$

enough if fixed unit η; powers represent. at least $P_{2k+1}^{2m_{k+1}} e^{-C\rho}$ diff. residues mod. $\pi_{..}^{2m_{k+1}}$