1.

Problem of Hecke. $\begin{pmatrix}1&\lambda\\0&1\end{pmatrix}, \overset{T_1}{\begin{pmatrix}0&-1\\1&0\end{pmatrix}}. \begin{pmatrix}1&1\\0&1\end{pmatrix} \overset{T_\lambda}{\begin{pmatrix}0&-\frac{1}{\lambda}\\\lambda&0\end{pmatrix}}$

$M=\begin{pmatrix}a&b\\c&d\end{pmatrix}$; $S=\begin{pmatrix}1&1\\0&1\end{pmatrix}$, $P=\begin{pmatrix}1&n\\0&1\end{pmatrix}=S^n$ $\quad \lambda = 2\cos\frac{\pi}{q}$, $\lambda \geq 1$

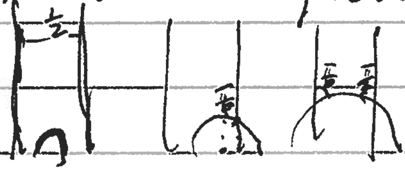
$M = S^{\frac{a}{c}}\, T_c\, S^{\frac{d}{c}}$; $T_c=\begin{pmatrix}0&-\frac{1}{c}\\c&0\end{pmatrix}$; $T_c^2 = -E$

$$S^{n_0} M S^{n_1} M^{-1} S^{n_2} M \cdots S^{n_k} M^{(-1)^k} S^{n_{k+1}}$$

$$\pm S^{\frac{a}{c}} \left(S^{n_0} T_c S^{n_1} T_c \cdots S^{n_k} T_c S^{n_{k+1}} \right) S^{-\frac{a}{c}} \text{ k odd}, \text{ or } S^{\frac{d}{c}} \text{ k even}$$

Thus. (M, P) discreate only if (T_c, P) discrete.

can be Completely solved. all discrete groups generated in this way are generalized triangle groups.



$1 - \frac{4}{|c|}$ $\quad \frac{1}{|c|}$ radius

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case n copies of Hyperbolic planes $z = (z^{(1)} \cdots z^{(n)})$

$\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}$; $\begin{pmatrix}\alpha^{(i)} & \beta^{(i)}\\ \gamma^{(i)} & \delta^{(i)}\end{pmatrix}$ $\quad i = 1 \cdots n$; $N(\alpha) = \prod \alpha^{(i)}$

a, b ... c ... integers.

$\begin{pmatrix}1&\omega\\0&1\end{pmatrix} = S^{\omega}$; P generated by $S^{\xi_1} \cdots S^{\xi_n}$

ξ_i linearly independent over rationals. P discrete. but projection of P on any subspace (by dropping some of the components. not discrete (means that closure will contain at least one one parameter group depending on a continuous variable).

adjoin $M = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$ ~~not in P~~ to P, when is resulting group discrete?

1. Either all $\gamma^{(i)} = 0$ or all $\gamma^{(i)} \neq 0$, seen by forming commutators $[M, S^{\omega_1}], [[M, S^{\omega_1}], S^{\omega_2}]$ for the (i) for which $\gamma^{(i)} = 0$, the (i) component is the identity, whereas ~~for~~ the (i) for which $\gamma^{(i)}$ ~~is not~~ $\neq 0$; we see it cannot be discrete, its closure contains elements that depend on at least one continuous parameter.

Case when all $\gamma^{(i)} = 0$ fairly trivial, so continue with case all $\gamma^{(i)} \neq 0$.

2. Then we shall show $N(\gamma\omega) \geq 1$ for all $\omega \neq 0$ in P. Conclusions $|N(\omega)|$ for $\omega \neq 0$ bounded from below by positive bound. and given P, permissible γ have $N(\gamma)$ bounded from below.

Sketch of proof.

consider M and $T_\gamma = \begin{pmatrix}0 & -\frac{1}{\gamma} \\ \gamma & 0\end{pmatrix}$; $M = S^{\frac{\alpha}{\gamma}} T_\gamma S^{\frac{\delta}{\gamma}}$

If M is permissible then compare

$$M_k = S^{\omega_0} M S^{\omega_1} M^{-1} S^{\omega_2} \cdots S^{\omega_k} M^{(-1)^k} S^{\omega_{k+1}}$$

with

$$M_k^* = S^{\omega_0} T_\gamma S^{\omega_1} \cdots S^{\omega_k} T_\gamma S^{\omega_{k+1}}$$

with $\omega_0 = 0$; $\omega_1 = \omega$, $\omega_2 = 0$

M_2.

M_2 becomes $\begin{pmatrix} * & * \\ -\omega\gamma^2 & x \end{pmatrix}$

repeating this process after k steps
we get

$$\begin{pmatrix} * & * \\ -\omega^{2^k-1}\gamma^{2^k} & * \end{pmatrix}$$

and once more with a different ω^*.
we get

$$M_k^* = \begin{pmatrix} * & * \\ -\omega^*\,\omega^{2^{k+1}-2}\gamma^{2^{k+1}} & * \end{pmatrix}$$

let $\Omega = \left(\xi_j^{(i)} \right)$ if $|N(\omega\gamma)| < 1$.

$D = \|\Omega\|$

choose k so large that

$$N(\gamma^2)\, N(\omega\gamma)^{2^{k+1}-2} \le \frac{\varepsilon_k^{\,n}}{\|\Omega\| D}$$

according to a theorem of Minkowski

$\exists\ \omega^* \not\equiv 0$ such that

$|\omega^{*(i)}| \le t_i$ if $\prod t_i \ge \|\Omega\| D$

thus $\exists\ \omega^*$ such that all components
of $\left| \left(-\omega^*\,\omega^{2^{k+1}-2}\gamma^{2^{k+1}}\right)^{(i)} \right| \le \varepsilon_k$

By suitable multiplication on left and right with suitable S^{ω} we can bring

$$\left(S^{\omega_1} M_k^{*} S^{\omega_2}\right)^{(i)} \text{ on } \begin{pmatrix} 1+O(\varepsilon_k^{(i)}) & \beta_k^{(i)} \\ \varepsilon_k^{(i)} & 1+O(\varepsilon_k^{(i)}) \end{pmatrix} ; |\varepsilon_k^{(i)}| \le \varepsilon_k$$

we see that also $\beta_k^{(i)}$ remains bounded so these elements all lie in a compact region and are different, which contradicts discreteness. This proves $|N(\gamma\omega)| \ge 1$.

Next form M_3 with ω_0 ~~=0~~; ω_3 ~~=0~~, we get

$$\gamma_3 = \omega_1 \omega_2 \gamma^3 - \gamma$$

Let $1 \le i \ne j \le n$; and let T be chosen large

By Minkowsky $\exists\ \omega_1 \ne 0$ such that

$$|\omega_1^{(i)}| \le D^{\frac{1}{n}} T$$

$$|\omega_1^{(j)}| \le D^{\frac{1}{n}} \frac{1}{T} \quad \text{for } i = j$$

$$|\omega_1^{(\nu)}| \le D^{\frac{1}{n}} \quad \text{for } \nu \ne i, j$$

and for ω_2 same inequalities with i and j interchanged. Thus $\exists\ \omega_1, \omega_2 \ne 0$ with

$$|\omega_1^{(\nu)} \omega_2^{(\nu)}| \le D^{\frac{2}{n}} \quad \text{for all } (\nu); \text{ since}$$

also $|N(\gamma^2\omega_1\omega_2)| \geq 1$ we also get the lower bound

$$\left|\omega_1^{(n)}\omega_2^{(n)}\right| \geq \frac{1}{N(\gamma^2)} D^{-\frac{2(n-1)}{n}}$$

Can in M_3 bring $\begin{pmatrix} \alpha_3 & \beta_3 \\ \gamma_3 & \delta_3 \end{pmatrix}$ into a compact region by suitable choice of ω_0 and ω_3 ~~and~~ (The possibility that $\gamma_3 \equiv 0$ can be avoided by in that case putting in a factor 2 in front of ω_1, So we can get a lower bound for components of $|\gamma_3|$) thus we get as $T \to \infty$ an infinite number of elements in a compact region, only a finite number of them can be different without violating discreteness. Thus equating $\omega_1\omega_2$ for two T of different order of magnitude we get a quadratic relation between the $\xi_k \xi_l$ with integral coefficients which can be assymptotically computed; let $\omega_1 = \sum a_k \xi_k$; $\omega_2 = \sum b_l \xi_l$, let a, and b denote column vectors then

$\Omega a = \omega_1$; $\Omega b = \omega_2$ or

$a = \Omega^{-1}\omega_1$; $b = \Omega^{-1}\omega_2$

let $\omega_1^{(i)} = T\theta^{(i,j)}$, $\omega_2^{(j)} = T\theta^{(j,i)}$; $\Omega^{-1} = (\rho_k^{(i)})$

then $a_k = \rho_k^{(i)} T \theta^{(i,j)} + O(1)$

and $b_\ell = \rho_\ell^{(j)} T \theta^{(j,i)} + O(1)$

and we have a relations

$$\sum c_{k,\ell}\, \xi_k \xi_\ell = 0$$

with
$$\varepsilon_{k,\ell} = 1;\ k \neq \ell;\quad \varepsilon_{k,\ell} = \tfrac{1}{2};\ k = \ell)$$
$$c_{k,\ell}^{(i,j)} = \varepsilon_{k,\ell} T^2 \left(\rho_k^{(i)} \rho_\ell^{(j)} + \rho_k^{(j)} \rho_\ell^{(i)}\right) \theta^{(i,j)} \theta^{(j,i)} + O(T)$$

If the determinant $1 \le i < j \le n;\ 1 \le k \le \ell \le n-1$

$$\left| c_{k,\ell}^{(i,j)} \right| \neq 0$$

we get for $1 \le k, \ell \le n-1$
$$\xi_k \xi_\ell = \sum d_r^{(k,\ell)} \xi_r \xi_m$$

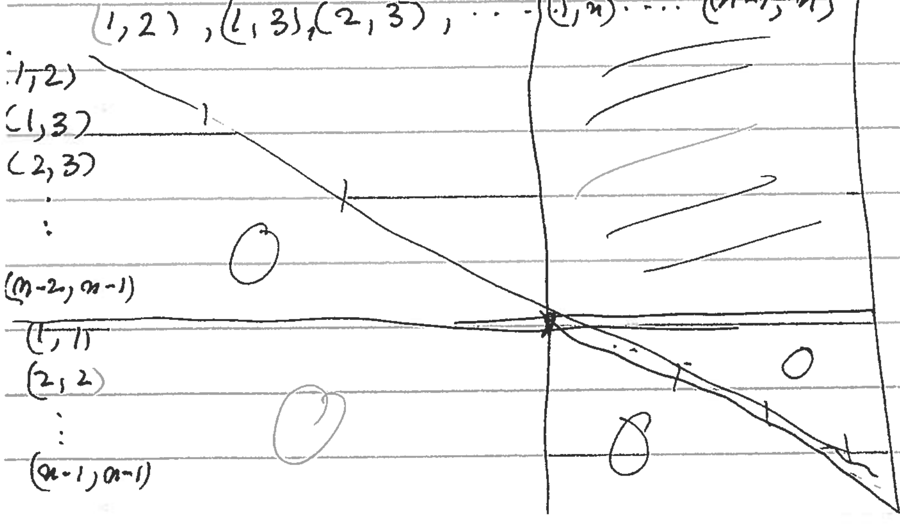
with rational $d_r^{(k,\ell)}$. Clearly this will happen for T large enough if $\frac{n(n-1)}{2} \times \frac{n(n-1)}{2}$ matrix.

$*$ $\left(\left(\rho_k^{(i)} \rho_\ell^{(j)} + \rho_k^{(j)} \rho_\ell^{(i)}\right) \varepsilon_{k,\ell} \right)$ is nonsingular.

Here we can remark that we could replace Ω with any ΩA which is A integral unimodular and still would have the same lattice P; and if we only say A integral nonsingular we would get a sublattice of finite index in P. Since everything is homogeneous we see that if we form Ω' where $\Omega' = \Omega R$ where R is any rational nonsingular n by n matrix it is enough if for the resulting Ω'^{-1} the matrix $(*)$ is nonsingular. But ~~by~~ since ~~multipl~~ the nonsingular matrices R^{-1} are everywhere dense in the set of all real n by n matrices it follows that it is enough for us to display one real matrix for ~~whi~~ $(\rho_j^{(i)})$ for which $(*)$ is nonsingular. We choose

$$\rho_k^{(i)} = \begin{cases} 1 & \text{for } k = i \le n-1 \\ 0 & \text{for } k \ne i \le n-1 \end{cases}$$

$$\rho_k^{(n)} = 1 \text{ for all } k.$$



Now easy to see that every ω satisfies an equation of degree $\frac{}{\xi_n}$ at most n with integral coefficients
can form a linear comb with integral a_k such that

$$\frac{\omega}{\xi_n} = \sum_{k=1}^{n-1} a_k \frac{\xi_k}{\xi_n}$$

~~satisfies~~ generates the field $\left(\frac{\xi_1}{\xi_n}, \ldots, \frac{\xi_{n-1}}{\xi_n}\right)$

this must be of degree n since it contains n linearly independent over the rationals $\left(\frac{\xi_1}{\xi_n}, \ldots \frac{\xi_{n-1}}{\xi_n}, 1\right)$; thus if we

do a conjugacy so that $\xi_n = 1$, we find that the $\omega^{(i)}$ are the n conjugates of numbers in a totally real field of degree n. By a suitable additional conjugacy we can obtain that the ω contain the set of all integers in the field.

Next we wish to show that γ^2 is an integer in this field. We must first show that γ^2 lies in the field.

We go back to the M_3 we constructed.

with $\gamma_3 = \omega_1 \omega_2 \gamma^3 - \gamma$; if we compute α_3 we find $\alpha_3 = \omega_0 \omega_1 \omega_2 \gamma^3 - \gamma(\omega_0 + \omega_2)$

can ~~adjust~~ only finitely many of these were different, we get

$$\omega_1 \omega_2 = \omega_1^* \omega_2^*$$

and $\gamma^2 \omega_0 \omega_1 \omega_2 \mp \omega_0 - \omega_2 = \gamma^2 \omega_0^* \omega_1^* \omega_2^* - \omega_0^* - \omega_2^*$

Here ω_1^* and ω_2^* are different from ω_1 and ω_2 but we get

$$\gamma^2 (\omega_0 - \omega_0^*) \omega_1 \omega_2 = \omega_0 + \omega_2 - \omega_0^* - \omega_2^*$$

If $\omega_0 - \omega_0^* = 0$ then we would get $\omega_2 = \omega_2^*$, so $\omega_0 - \omega_0^* \neq 0$ and we get

$$\gamma^2 = \frac{\omega_4}{\omega_5 \omega_1 \omega_2}$$ so γ^2 is in the field.

The more complicated ~~step~~ is to show that γ^2 is necessarily an integer. This proof is complicated but involves considering

$M_4 = T_\gamma S^{\omega_1} T_\gamma S^{\omega_2} T_\gamma S^{\omega_3} T_\gamma$

we find $\gamma_4 = \omega_1 \omega_2 \omega_3 \gamma^4 - (\omega_1 + \omega_3) \gamma^2$

If we can make

$$* \quad \omega_1 \omega_2 \omega_3 \gamma^2 = \omega_1 + \omega_3$$

we get $\gamma_4 = 0$. If we compute the trace of M_4 assuming $*$ we find it to be

$\pm \left(\frac{\omega_1}{\omega_3} + \frac{\omega_3}{\omega_1} \right)$ so the diagonal elements of M_4 are $\frac{\omega_1}{\omega_3}$ and $\frac{\omega_3}{\omega_1}$. If γ^2 is not an integer can show that we can at least go to another admissible γ such that $*$ is solvable with $\frac{\omega_1}{\omega_3}$ not a unit. This leads to a simple contradiction- by now considering $M^k =$ $\sigma(M) = \rho + \frac{1}{\rho}$

$\gamma_k = \frac{\rho^{k+1} - \rho^{-k-1}}{\rho - \frac{1}{\rho}} \gamma$ we see this must be

integer this means that ρ and ρ^{-1} must be algebraic integers. so the trace of M is an integer, whose square is seen to lie in the field. by suitable conjugacy can show that $\alpha, \beta, \gamma, \delta$ products of any two and the squares are in k. Can conclude that any group obtai