Problem: Parabolic group. discrete. n generators.
assumptions about projections on subspaces.
assume adjoining $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ to P leads to discrete group.

(1) either all $\gamma^{(i)} \equiv 0$ or all $\gamma^{(i)} \not\equiv 0$.
First case trivial ... second case $\gamma^{(i)} \not\equiv 0$.

(2) $|N(\gamma\omega)| \geq 1$. : Minkowski. (3) in a compact region only finitely many diff. γ. relations:

$\omega_1 \omega_2 - \omega_1' \omega_2' = 0$; .

prove for $j, k \leq n$

$$\xi_j \xi_k = \sum_{\ell=1}^{n} a_{j,k}^{(\ell)} \xi_\ell \xi_n \quad ; \quad a_{j,k}^{(\ell)} \text{ rational}$$

or $\frac{\xi_j}{\xi_n} = \xi_j$; $\xi_n = 1$

$$\xi_j \xi_k = \sum_{\ell=1}^{n} a_{j,k}^{(\ell)} \xi_\ell$$

form $1, \omega, \omega^2, \ldots \omega^n$; must be linear relation with integral coefficients so

$\mathfrak{P}(\omega) = 0$, where p is polynomial with integral coeff of degree at most n. form linear combination with rational coeff. of the ξ_j which generates field containing all ξ_j; must be root of irreducible equation of degree n, so all the ω lie in a field of degree n ... totally real field next conjugacy so $P(\omega)$ contains all integers in field. . turn attention to γ.

γ^2 in field: prove relations

$$\gamma^2 \omega_1 \omega_2 \omega_3 - \omega_1 - \omega_3 = \gamma^2 \omega_1' \omega_2' \omega_3' - \omega_1' - \omega_3'$$

with $\omega_1 \omega_2 = \omega_1' \omega_2'$; $\omega_3 \neq \omega_1'$; $\omega_3 \neq \omega_3'$

Next prove that γ^2 is an integer in field.

try to make $\gamma^2 \omega_1 \omega_2 \omega_3 - \omega_1 - \omega_3 = 0$ without $-\frac{\omega_1}{\omega_3}$, $-\frac{\omega_3}{\omega_1}$ being units.

pass from original γ to one γ' that is number in field, of form $\gamma' = \frac{\sigma}{x}$ where x, σ are integers with $(x, \sigma) = 1$ and $x\gamma^2$ is an integer. We shall show that x must be a unit. This will be shown by getting a contradiction, by making

(*) $\gamma'^2 \omega_1 \omega_2 \omega_3 - \omega_1 - \omega_3 = 0$ without $\frac{\omega_1}{\omega_3}$ being a unit. Choosing ω_1 and ω_3 as η and $-\delta$ resp where $\delta / x^2\delta$ not a unit, we see that if

$$\eta - \delta \equiv 0 \pmod{\sigma^2}.$$

ω_2 would be an integer by (*) so we get a upper triangular element where diagonal elements are not units.

If the first choice of γ' does not guarantee the solvability of (*) we go to try to construct a sequence of γ_k such that γ_k ...

$2'?$

$\underline{\gamma^3 \omega_1\omega_2 - \gamma}$ $\quad T_\gamma S_{\omega_1} T_\gamma S_{\omega_2} 1_\gamma$ $\quad | \quad \gamma_1 = \gamma$

$$\gamma_{k+1} = \gamma\omega_k \gamma_k - \gamma_{k-1}$$

$$\gamma_2 = \gamma^2\omega_1 \; ; \; \gamma_3 = \gamma^3\omega_1\omega_2 - \gamma$$

$$\gamma_4 = \gamma^4\omega_1\omega_2\omega_3 - \gamma^2(\omega_1+\omega_3)$$

$$\gamma_5 = \gamma^5\omega_1\omega_2\omega_3\omega_4 - \gamma^3(\omega_1\omega_4 + \omega_3\omega_4 + \omega_1\omega_2) + \gamma$$

$$\gamma_{2k} = \gamma^2 \cdot \rho_k$$

$$\gamma_{2k+1} = \gamma\, \rho'_k$$

$$\gamma\rho'_k = \gamma^3\omega_{2k}\rho_k - \gamma\rho'_{k-1}$$

$$\gamma^2\rho_{k+1} = \gamma^2\omega_{2k+1}\rho'_k - \gamma^2\rho_k$$

$\xi_j^{(i)}$ $\quad \psi \quad \vartheta \quad \vartheta_j^i$ $\quad \theta \quad \mu$

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\begin{aligned} \rho'_k &= \gamma^2\omega_{2k}\rho_k - \rho'_{k-1} \\ \rho_{k+1} &= \omega_{2k+1}\rho'_k - \rho_k \end{aligned}}$$

let $\dfrac{\pi^N}{x^t}$ $\quad$ ~~$t \leq c\, N$~~ $\quad N \geq c\, t.$

$$0 \leq r \leq 2t$$

$$\omega_1\omega_2\omega_3 \frac{\pi^{2N}}{x^{2t}} = \omega_1 + \omega_3$$

$$\omega_1 = \underline{\eta\, x^r \pm 1} \cdot\cdot x^{2t-r}$$

$$\frac{\pi^{2N}}{x^{2t}} \Big| \frac{\omega_1+\omega_3}{\omega_1\omega_3}.$$

$$\pi^{2N} \Big| \frac{\omega_1+\omega_3}{\omega_1\omega_3} x^{2t}$$

$$\gamma_{k+1} = \omega_k \gamma \gamma_k - \gamma_{k-1}$$

$$\gamma_4 = \omega_1 \omega_2 \omega_3 \gamma^4 - (\omega_1 + \omega_3)\gamma^2$$

$$\gamma_3 = \omega_1 \omega_2 \gamma^3 - \gamma$$

$$\gamma_2 = \omega_1 \gamma^2$$

$$\gamma_1 = \gamma$$

$$\alpha_{3} = \omega_0 \omega_1 \omega_2 \gamma^4 - (\omega_0 + \omega_2)\gamma^2$$

$$\alpha_4 = \omega_0 \omega_1 \omega_2 \omega_3 \gamma^5 - (\omega_1 \omega_0 + \omega_3 \omega_0 + \omega_2 \omega_3 + \omega_1 \omega_3)\gamma^3 + \gamma$$

if $\gamma_4 = 0$ trace of $\alpha_4 + \delta_4 = -\frac{\omega_1}{\omega_3} - \frac{\omega_3}{\omega_1}$

$\omega_1 \omega_2 \omega_3 \gamma^4 - (\omega_1 + \omega_3)\gamma^2 = 0$; $\gamma^2 \omega_2 \omega_3 = \frac{\omega_1 + \omega_3}{\omega_1}$

$\gamma^2 \omega_1 \omega_3 =$

$\alpha_4 = -(\omega_2 \omega_3 + \omega_1 \omega_3)\gamma^3 + \gamma =$

③ 3′?

call $\omega^* = \frac{\omega}{\xi_n}$; we find recursively that

$$\omega^{*r} = \sum_{k=1}^{n} e^{(r)}_k \quad \frac{\xi_r}{\xi_n}$$; where the $e^{(r)}_k$ are rational.

~~there~~ fore it follows that each ω^* is the root of a polynomial of degree at most n, it is also clear that the maximal degree for the ω^* must ~~be equal to n, and that they must all be the (n) components si~~ be equal to n, since if we consider several ω, say $\omega_1, \dots, \omega_e$ and consider ~~any~~ linear combinations with ~~rational~~ integral coefficients, these can be chosen so their field contains those generated by $\omega_1, \dots, \omega_e$, thus for can find an ω^* of maximal degree, this can not be more than n, nor can it be less than n, since the field must contain n linearly independent over rationals numbers. Conclusion $\omega^{(i)}$ are all numbers in totally real field of degree n and the components are ~~the various realizations of~~ .

may now assume that the ξ_k ; $k = 1, \dots n$, lie in