1.

modular group S, T ;

n dimensions. $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ n components.

Discrete$^{(i)}$ Parabolic group P $\quad \begin{pmatrix}1 & \omega\\ 0 & 1\end{pmatrix} = S^{\omega}$

generated by n $\begin{pmatrix}1 & \xi_j\\ 0 & 1\end{pmatrix}$; $\Omega = \left(\xi_j^{(i)}\right)$; $\|\Omega\| = D$.

assume that projection of P on any product of less than n not discrete.

Question if we adjoin an element not in P

$M = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ under what circumstances can the resulting group be discrete. Info on ω's and info on M.

(1) Either all $\gamma^{(i)} = 0$ or all $\gamma^{(i)} \neq 0$

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(2) Case all $\gamma^{(i)} = 0$, rather simple skip,

then can write

$$M = S^{\frac{\alpha}{\gamma}}\, T_\gamma\, S^{\frac{\delta}{\gamma}} \ ; \quad T_\gamma = \begin{pmatrix}0 & -\frac{1}{\gamma}\\ \gamma & 0\end{pmatrix}; \quad T_\gamma^2 = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$$

Compare

$$S^{\omega_0} M S^{\omega_1} M^{-1} \cdots\cdots \quad S^{\omega_k} M^{(-1)^k} S^{\omega_{k+1}}$$

with

$$S^{\omega_0} T_\gamma S^{\omega_1} \cdots\cdots \quad S^{\omega_k} T_\gamma S^{\omega_{k+1}}$$

See if adjoining M gives discrete group then a fortiori adjoining T_γ give discrete grp.

$$M_k = \begin{pmatrix}\alpha_k & \beta_k \\ \gamma_k & \delta_k\end{pmatrix} = T_\gamma S^{\omega_1} T_\gamma S^{\omega_2} \cdots S^{\omega_{k-1}} T_\gamma$$

Find $\alpha_0 = 1, \gamma_0 = 0 ; \alpha_1 = 0, \gamma_1 = \gamma$;

$$\alpha_{k+1} = \omega_k \gamma \alpha_k - \alpha_{k-1} ; \gamma_{k+1} = \omega_k \gamma \gamma_k - \gamma_{k-1}$$

Prove $|N(\gamma\omega)| \geq 1$ for $\omega \neq 0$.

Principles (1) $\begin{pmatrix}\alpha_k & \beta_k \\ \gamma_k & \delta_k\end{pmatrix}$ with $\gamma_k \neq 0$ can produce

$S^{\omega} \begin{pmatrix}\alpha_k & \beta_k \\ \gamma_k & \delta_k\end{pmatrix} S^{\omega^*}$ such that $= \begin{pmatrix} 1+O(\gamma_k) & O(1+|\gamma_k|) \\ \gamma_k & 1+O(\gamma_k)\end{pmatrix}$

Thus only finitely many γ_k in a compact region.

(2) Minkowski. for $\prod t_i \geq D$.

$\exists\ \omega \neq 0$ with $|\omega^{(i)}| \leq t_i$

Now form with some ω ; $T_\gamma S^{\omega_1} T_\gamma = M_2$

$\gamma_2 = -\omega_1 \gamma^2$, repeating forming $M_2 S^{\omega_1} M_2^{-1}$

$-\omega_1^3 \gamma^4 \ldots$; $-\omega^{2^{k-1}-1} \gamma^{2^{k-1}}$ once more ω_1

$-\omega_1 \omega^{2^k-2} \gamma^{2^k} = -\omega_1 \gamma^2 (\omega\gamma)^{2^k-2}$

choose ω_1 so as to equalize size of components.
Norm decreases get sequence with $\gamma_k \to 0$
contradiction if $|N(\omega\gamma)| < 1$. Q.E.D.

Next observe that $\gamma_3 = \omega_1 \omega_2 \gamma^3 - \gamma$, can arange in inf. many ways to have $|\omega_1^{(i)} \omega_2^{(i)}| \leq D^{\frac{2}{n}}$ for all i ; det $i \neq j$, T large and $|\omega_1^{(i)}| \leq T D^{\frac{1}{n}}$; $|\omega_1^{(j)}| \leq \frac{1}{T} D^{\frac{1}{n}}$ and $|\omega_1^{(k)}| \leq D^{\frac{1}{n}}$ for $k \neq i, j$. $\exists\, \omega_1 \neq 0$ satisfying. similarly ω_2 with i and j interchanged. for any specific choice of i, j will inf often have ~~same~~ γ_3 as $T \to \infty$, thus identities $\omega_1 \omega_2 - \omega_1^* \omega_2^* = 0$ or identities of form

$$\sum_{k,e} c_{k,e}^{(i,j)} \xi_k \xi_e = 0$$

where the $c_{k,e}^{(i,j)}$ are integers. different choices of $1 \leq i < j \leq n$ gives us $\frac{n(n-1)}{2}$ such which can be shown to be linearly independent and allows us to express by solving for the $\xi_k \xi_e$ with $k, e < n$,

$$\xi_k \xi_e = \sum_{r=1}^{n} d_r^{(k,e)} \xi_r \xi_n \quad , \quad d_r^{(k,e)} \text{ rational}$$

dividing by ξ_n, writing $\frac{\xi_k}{\xi_n} = \zeta_k$; $\zeta_n = 1$

$$\zeta_k \zeta_e = \sum_{r=1}^{n} d_r^{(k,e)} \zeta_r$$

thus see that for any ω we can express the integral powers

$$\omega^{m} = \sum_{1}^{n} e_r^{(m)} \xi_r$$

with rational $e_r^{(m)}$; thus any ω satisfies a polynomial equation of degree at most ~~n~~, and in particular the ξ_r are algebraic numbers, choosing a linear comb $\omega^* = \sum a_r \xi_r$ such that it generates the field $k(\xi_1, \dots, \xi_{n-1})$ we see that this ω^* is ~~the~~ must satisfy an equation of degree exactly n, and that all ω lie in this field. Thus by a simple conjugacy we can make our ω all lie in a totally real numberfield of degree n. Next we may look at α_3