

# **Automorphic forms on $GL(2)$**

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## Introduction

Two of the best known of Hecke's achievements are his theory of  $L$ -functions with Grössencharakter, which are Dirichlet series which can be represented by Euler products, and his theory of the Euler products, associated to automorphic forms on  $GL(2)$ . Since a Grössencharakter is an automorphic form on  $GL(1)$  one is tempted to ask if the Euler products associated to automorphic forms on  $GL(2)$  play a role in the theory of numbers similar to that played by the  $L$ -functions with Grössencharakter. In particular do they bear the same relation to the Artin  $L$ -functions associated to two-dimensional representations of a Galois group as the Hecke  $L$ -functions bear to the Artin  $L$ -functions associated to one-dimensional representations? Although we cannot answer the question definitively one of the principal purposes of these notes is to provide some evidence that the answer is affirmative.

The evidence is presented in §12. It comes from reexamining, along lines suggested by a recent paper of Weil, the original work of Hecke. Anything novel in our reexamination comes from our point of view which is the theory of group representations. Unfortunately the facts which we need from the representation theory of  $GL(2)$  do not seem to be in the literature so we have to review, in Chapter I, the representation theory of  $GL(2, F)$  when  $F$  is a local field. §7 is an exceptional paragraph. It is not used in the Hecke theory but in the chapter on automorphic forms and quaternion algebras.

Chapter I is long and tedious but there is nothing hard in it. None the less it is necessary and anyone who really wants to understand  $L$ -functions should take at least the results seriously for they are very suggestive.

§9 and §10 are preparatory to the Hecke theory which is finally taken up in §11. We would like to stress, since it may not be apparent, that our method is that of Hecke. In particular the principal tool is the Mellin transform. The success of this method for  $GL(2)$  is related to the equality of the dimensions of a Cartan subgroup and the unipotent radical of a Borel subgroup of  $PGL(2)$ . The implication is that our methods do not generalize. The results, with the exception of the converse theorem in the Hecke theory, may.

The right way to establish the functional equation for the Dirichlet series associated to the automorphic forms is probably that of Tate. In §13 we verify, essentially, that this method leads to the same local factors as that of Hecke and in §14 we use the method of Tate to prove the functional equation for the  $L$ -functions associated to automorphic forms on the multiplicative group of a quaternion algebra. The results of §13 suggest a relation between the characters of representations of  $GL(2)$  and the characters of representations of the multiplicative group of a quaternion algebra which is verified, using the results of §13, in §15. This relation was well-known for archimedean fields but its significance had not been stressed. Although our proof leaves something to be desired the result itself seems to us to be one of the more striking facts brought out in these notes.

Both §15 and §16 are afterthoughts; we did not discover the results in them until the rest of the notes were almost complete. The arguments of §16 are only sketched and we ourselves have not verified all the details. However the theorem of §16 is important and its proof is such a beautiful illustration of the power and ultimate simplicity of the Selberg trace formula and the theory of harmonic analysis on semi-simple groups that we could not resist adding it. Although we are very dissatisfied with the methods of the first fifteen paragraphs we see no way to improve on those of §16. They are perhaps the methods with which to attack the question left unsettled in §12.

We hope to publish a sequel to these notes which will include, among other things, a detailed proof of the theorem of §16 as well as a discussion of its implications for number theory. The theorem has, as these things go, a fairly long history. As far as we know the first forms of it were assertions about the representability of automorphic forms by theta series associated to quaternary quadratic forms.

As we said before nothing in these notes is really new. We have, in the list of references at the end of each chapter, tried to indicate our indebtedness to other authors. We could not however acknowledge completely our indebtedness to R. Godement since many of his ideas were communicated orally to one of us as a student. We hope that he does not object to the company they are forced to keep.

The notes<sup>1</sup> were typed by the secretaries of Leet Oliver Hall. The bulk of the work was done by Miss Mary Ellen Peters and to her we would like to extend our special thanks. Only time can tell if the mathematics justifies her great efforts.

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<sup>1</sup>that appeared in the SLM volume

## CHAPTER I

### Local Theory

#### §1. Weil representations

Before beginning the study of automorphic forms, we must review the representation theory of the general linear group in two variables over a local field. In particular we have to prove the existence of various series of representations. One of the quickest methods of doing this is to make use of the representations constructed by Weil in [1]. We begin by reviewing his construction adding, at appropriate places, some remarks which will be needed later.

In this paragraph  $F$  will be a local field and  $K$  will be an algebra over  $F$  of one of the following types:

- (i) The direct sum  $F \oplus F$ .
- (ii) A separable quadratic extension of  $F$ .
- (iii) The unique quaternion algebra over  $F$ .  $K$  is then a division algebra with centre  $F$ .
- (iv) The algebra  $M(2, F)$  of  $2 \times 2$  matrices over  $F$ .

In all cases we identify  $F$  with the subfield of  $K$  consisting of scalar multiples of the identity. In particular if  $K = F \oplus F$  we identify  $F$  with the set of elements of the form  $(x, x)$ . We can introduce an involution  $\iota$  of  $K$ , which will send  $x$  to  $x^\iota$ , with the following properties:

- (i) It satisfies the identities  $(x + y)^\iota = x^\iota + y^\iota$  and  $(xy)^\iota = y^\iota x^\iota$ .
- (ii) If  $x$  belongs to  $F$  then  $x = x^\iota$ .
- (iii) For any  $x$  in  $K$  both  $\tau(x) = x + x^\iota$  and  $\nu(x) = xx^\iota = x^\iota x$  belong to  $F$ .

If  $K = F \oplus F$  and  $x = (a, b)$  we set  $x^\iota = (b, a)$ . If  $K$  is a separable quadratic extension of  $F$  the involution  $\iota$  is the unique non-trivial automorphism of  $K$  over  $F$ . In this case  $\tau(x)$  is the trace of  $x$  and  $\nu(x)$  is the norm of  $x$ . If  $K$  is a quaternion algebra, a unique  $\iota$  with the required properties is known to exist.  $\tau$  and  $\nu$  are the reduced trace and reduced norm respectively. If  $K$  is  $M(2, F)$  we take  $\iota$  to be the involution sending

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

to

$$x = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Then  $\tau(x)$  and  $\nu(x)$  are the trace and determinant of  $x$ .

If  $\psi = \psi_F$  is a given non-trivial additive character of  $F$  then  $\psi_K = \psi_F \circ \tau$  is a non-trivial additive character of  $K$ . By means of the pairing

$$\langle x, y \rangle = \psi_K(xy)$$

we can identify  $K$  with its Pontrjagin dual. The function  $\nu$  is of course a quadratic form on  $K$  which is a vector space over  $F$  and  $f = \psi_F \circ \nu$  is a character of second order in the sense

of [1]. Since

$$\nu(x + y) - \nu(x) - \nu(y) = \tau(xy^\iota)$$

and

$$f(x + y)f^{-1}(x)f^{-1}(y) = \langle x, y^\iota \rangle$$

the isomorphism of  $K$  with itself associated to  $f$  is just  $\iota$ . In particular  $\nu$  and  $f$  are nondegenerate.

Let  $\mathcal{S}(K)$  be the space of Schwartz-Bruhat functions on  $K$ . There is a unique Haar measure  $dx$  on  $K$  such that if  $\Phi$  belongs to  $\mathcal{S}(K)$  and

$$\Phi'(x) = \int_K \Phi(y)\psi_K(xy) dy$$

then

$$\Phi(0) = \int_K \Phi'(x) dx.$$

The measure  $dx$ , which is the measure on  $K$  that we shall use, is said to be self-dual with respect to  $\psi_K$ .

Since the involution  $\iota$  is measure preserving the corollary to Weil's Theorem 2 can in the present case be formulated as follows.

**Lemma 1.1.** *There is a constant  $\gamma$  which depends on the  $\psi_F$  and  $K$ , such that for every function  $\Phi$  in  $\mathcal{S}(K)$*

$$\int_K (\Phi * f)(y)\psi_K(yx) dy = \gamma f^{-1}(x^\iota)\Phi'(x)$$

$\Phi * f$  is the convolution of  $\Phi$  and  $f$ . The values of  $\gamma$  are listed in the next lemma.

**Lemma 1.2.**

- (i) *If  $K = F \oplus F$  or  $M(2, F)$  then  $\gamma = 1$ .*
- (ii) *If  $K$  is the quaternion algebra over  $F$  then  $\gamma = -1$ .*
- (iii) *If  $F = \mathbf{R}$ ,  $K = \mathbf{C}$ , and*

$$\psi_F(x) = e^{2\pi i a x},$$

*then*

$$\gamma = \frac{a}{|a|}i$$

- (iv) *If  $F$  is non-archimedean and  $K$  is a separable quadratic extension of  $F$  let  $\omega$  be the quadratic character of  $F^*$  associated to  $K$  by local class field theory. If  $U_F$  is the group of units of  $F^*$  let  $m = m(\omega)$  be the smallest non-negative integer such that  $\omega$  is trivial on*

$$U_F^m = \{ a \in U_F \mid a \equiv 1 \pmod{\mathfrak{p}_F^m} \}$$

*and let  $n = n(\psi_F)$  be the largest integer such that  $\psi_F$  is trivial on the ideal  $\mathfrak{p}_F^{-n}$ . If  $a$  is any generator on the ideal  $\mathfrak{p}_F^{m+n}$  then*

$$\gamma = \omega(a) \frac{\int_{U_F} \omega^{-1}(\alpha)\psi_F(\alpha a^{-1}) d\alpha}{\left| \int_{U_F} \omega^{-1}(\alpha)\psi_F(\alpha a^{-1}) d\alpha \right|}.$$



The first two assertions are proved by Weil. To obtain the third apply the previous lemma to the function

$$\Phi(z) = e^{-2\pi z z^\ell}.$$

We prove the last. It is shown by Weil that  $|\gamma| = 1$  and that if  $\ell$  is sufficiently large  $\gamma$  differs from

$$\int_{\mathfrak{p}_K^{-\ell}} \psi_F(xx^\ell) dx$$

by a positive factor. This equals

$$\int_{\mathfrak{p}_K^{-\ell}} \psi_F(xx^\ell) |x|_K d^\times x = \int_{\mathfrak{p}_K^{-\ell}} \psi_F(xx^\ell) |xx^\ell|_F d^\times x$$

if  $d^\times x$  is a suitable multiplicative Haar measure. Since the kernel of the homomorphism  $\nu$  is compact the integral on the right is a positive multiple of

$$\int_{\nu(\mathfrak{p}_K^{-\ell})} \psi_F(x) |x|_F d^\times x.$$

Set  $k = 2\ell$  if  $K/F$  is unramified and set  $k = \ell$  if  $K/F$  is ramified. Then  $\nu(\mathfrak{p}_K^{-\ell}) = \mathfrak{p}_F^{-k} \cap \nu(K)$ . Since  $1 + \omega$  is twice the characteristic function of  $\nu(K^\times)$  the factor  $\gamma$  is a positive multiple of

$$\int_{\mathfrak{p}_F^{-k}} \psi_F(x) dx + \int_{\mathfrak{p}_F^{-k}} \psi_F(x) \omega(x) dx.$$

For  $\ell$  and therefore  $k$  sufficiently large the first integral is 0. If  $K/F$  is ramified well-known properties of Gaussian sums allow us to infer that the second integral is equal to

$$\int_{U_F} \psi_F\left(\frac{\alpha}{a}\right) \omega\left(\frac{\alpha}{a}\right) d\alpha.$$

Since  $\omega = \omega^{-1}$  we obtain the desired expression for  $\gamma$  by dividing this integral by its absolute value. If  $K/F$  is unramified we write the second integral as

$$\sum_{j=0}^{\infty} (-1)^{j-k} \left\{ \int_{\mathfrak{p}_F^{-k+j}} \psi_F(x) dx - \int_{\mathfrak{p}_F^{-k+j+1}} \psi_F(x) dx \right\}$$

In this case  $m = 0$  and

$$\int_{\mathfrak{p}_F^{-k+j}} \psi_F(x) dx$$

is 0 if  $k - j > n$  but equals  $q^{k-j}$  if  $k - j \leq n$ , where  $q$  is the number of elements in the residue class field. Since  $\omega(a) = (-1)^n$  the sum equals

$$\omega(a) \left\{ q^m + \sum_{j=0}^{\infty} (-1)^j q^{m-j} \left( 1 - \frac{1}{q} \right) \right\}$$

A little algebra shows that this equals  $\frac{2\omega(a)q^{m+1}}{q+1}$  so that  $\gamma = \omega(a)$ , which upon careful inspection is seen to equal the expression given in the lemma.

In the notation of [19] the third and fourth assertions could be formulated as an equality

$$\gamma = \lambda(K/F, \psi_F).$$

It is probably best at the moment to take this as the definition of  $\lambda(K/F, \psi_F)$ .

If  $K$  is not a separable quadratic extension of  $F$  we take  $\omega$  to be the trivial character.

**Proposition 1.3.** *There is a unique representation  $r$  of  $\mathrm{SL}(2, F)$  on  $\mathcal{S}(K)$  such that*

$$\begin{aligned} (i) \quad & r\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\right)\Phi(x) = \omega(\alpha)|\alpha|_K^{1/2}\Phi(\alpha x) \\ (ii) \quad & r\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right)\Phi(x) = \psi_F(z\nu(x))\Phi(x) \\ (iii) \quad & r\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\Phi(x) = \gamma\Phi'(x^t). \end{aligned}$$

If  $\mathcal{S}(K)$  is given its usual topology,  $r$  is continuous. It can be extended to a unitary representation of  $\mathrm{SL}(2, F)$  on  $L^2(K)$ , the space of square integrable functions on  $K$ . If  $F$  is archimedean and  $\Phi$  belongs to  $\mathcal{S}(K)$  then the function  $r(g)\Phi$  is an indefinitely differentiable function on  $\mathrm{SL}(2, F)$  with values in  $\mathcal{S}(K)$ .

This may be deduced from the results of Weil. We sketch a proof.  $\mathrm{SL}(2, F)$  is the group generated by the elements  $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ ,  $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ , and  $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  with  $\alpha$  in  $F^\times$  and  $z$  in  $F$  subject to the relations

$$\begin{aligned} (a) \quad & w\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix}w \\ (b) \quad & w^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ (c) \quad & w\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}w = \begin{pmatrix} -a^{-1} & 0 \\ 0 & -a \end{pmatrix}\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}w\begin{pmatrix} 1 & -a^{-1} \\ 0 & 1 \end{pmatrix} \end{aligned}$$

together with the obvious relations among the elements of the form  $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$  and  $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ . Thus the uniqueness of  $r$  is clear. To prove the existence one has to verify that the mapping specified by (i), (ii), (iii) preserves all relations between the generators. For all relations except (a), (b), and (c) this can be seen by inspection. (a) translates into an easily verifiable property of the Fourier transform. (b) translates into the equality  $\gamma^2 = \omega(-1)$  which follows readily from Lemma 1.2.

If  $a = 1$  the relation (c) becomes

$$(1.3.1) \quad \int_K \Phi'(y^t)\psi_F(\nu(y))\langle y, x^t \rangle dy = \gamma\psi_F(-\nu(x)) \int_K \Phi(y)\psi_F(-\nu(y))\langle y, -x^t \rangle dy$$

which can be obtained from the formula of Lemma 1.1 by replacing  $\Phi(y)$  by  $\Phi'(-y^t)$  and taking the inverse Fourier transform of the right side. If  $a$  is not 1 the relation (c) can again be reduced to (1.3.1) provided  $\psi_F$  is replaced by the character  $x \rightarrow \psi_F(ax)$  and  $\gamma$  and  $dx$  are modified accordingly. We refer to Weil's paper for the proof that  $r$  is continuous and may be extended to a unitary representation of  $\mathrm{SL}(2, F)$  in  $L^2(K)$ .

Now take  $F$  archimedean. It is enough to show that all of the functions  $r(g)\Phi$  are indefinitely differentiable in some neighbourhood of the identity. Let

$$N_F = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in F \right\}$$

and let

$$A_F = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \mid \alpha \in F^\times \right\}$$

Then  $N_F w A_F N_F$  is a neighbourhood of the identity which is diffeomorphic to  $N_F \times A_F \times N_F$ . It is enough to show that

$$\phi(n, a, n_1) = r(n w a n_1) \Phi$$

is infinitely differentiable as a function of  $n$ , as a function of  $a$ , and as a function of  $n_1$  and that the derivations are continuous on the product space. For this it is enough to show that for all  $\Phi$  all derivatives of  $r(n)\Phi$  and  $r(a)\Phi$  are continuous as functions of  $n$  and  $\Phi$  or  $a$  and  $\Phi$ . This is easily done.

The representation  $r$  depends on the choice of  $\psi_F$ . If  $a$  belongs to  $F^\times$  and  $\psi'_F(x) = \psi_F(ax)$  let  $r'$  be the corresponding representation. The constant  $\gamma' = \omega(a)\gamma$ .

**Lemma 1.4.**

(i) *The representation  $r'$  is given by*

$$r'(g) = r \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right)$$

(ii) *If  $b$  belongs to  $K^*$  let  $\lambda(b)\Phi(x) = \Phi(b^{-1}x)$  and let  $\rho(b)\Phi(x) = \Phi(xb)$ . If  $a = \nu(b)$  then*

$$r'(g)\lambda(b^{-1}) = \lambda(b^{-1})r(g)$$

*and*

$$r'(g)\rho(b) = \rho(b)r(g).$$

*In particular if  $\nu(b) = 1$  both  $\lambda(b)$  and  $\rho(b)$  commute with  $r$ .*

We leave the verification of this lemma to the reader. Take  $K$  to be a separable quadratic extension of  $F$  or a quaternion algebra of centre  $F$ . In the first case  $\nu(K^\times)$  is of index 2 in  $F^\times$ . In the second case  $\nu(K^\times)$  is  $F^\times$  if  $F$  is non-archimedean and  $\nu(K^\times)$  has index 2 in  $F^\times$  if  $F$  is  $\mathbf{R}$ .

Let  $K'$  be the compact subgroup of  $K^\times$  consisting of all  $x$  with  $\nu(x) = xx^t = 1$  and let  $G_+$  be the subgroup of  $\text{GL}(2, F)$  consisting of all  $g$  with determinant in  $\nu(K^\times)$ .  $G_+$  has index 2 or 1 in  $\text{GL}(2, F)$ . Using the lemma we shall decompose  $r$  with respect to  $K'$  and extend  $r$  to a representation of  $G_+$ .

Let  $\Omega$  be a finite-dimensional irreducible representation of  $K^\times$  in a vector space  $U$  over  $\mathbf{C}$ . Taking the tensor product of  $r$  with the trivial representation of  $\text{SL}(2, F)$  on  $U$  we obtain a representation on

$$\mathcal{S}(K) \otimes_{\mathbf{C}} U = \mathcal{S}(K, U)$$

which we still call  $r$  and which will now be the centre of attention.

**Proposition 1.5.**

(i) *If  $\mathcal{S}(K, \Omega)$  is the space of functions  $\Phi$  in  $\mathcal{S}(K, U)$  satisfying*

$$\Phi(xh) = \Omega^{-1}(h)\Phi(x)$$

*for all  $h$  in  $K'$  then  $\mathcal{S}(K, \Omega)$  is invariant under  $r(g)$  for all  $g$  in  $\text{SL}(2, F)$ .*

- (ii) The representation  $r$  of  $\mathrm{SL}(2, F)$  on  $\mathcal{S}(K, \Omega)$  can be extended to a representation  $r_\Omega$  of  $G_+$  satisfying

$$r_\Omega \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) \Phi(x) = |h|_K^{1/2} \Omega(h) \Phi(xh)$$

if  $a = \nu(h)$  belongs to  $\nu(K^\times)$ .

- (iii) If  $\eta$  is the quasi-character of  $F^\times$  such that

$$\Omega(a) = \eta(a)I$$

for  $a$  in  $F^\times$  then

$$r_\Omega \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \omega(a) \eta(a) I$$

- (iv) The representation  $r_\Omega$  is continuous and if  $F$  is archimedean all factors in  $\mathcal{S}(K, \Omega)$  are infinitely differentiable.  
(v) If  $U$  is a Hilbert space and  $\Omega$  is unitary let  $L^2(K, U)$  be the space of square integrable functions from  $K$  to  $U$  with the norm

$$\|\Phi\|^2 = \int_K \|\Phi(x)\|^2 dx$$

If  $L^2(K, \Omega)$  is the closure of  $\mathcal{S}(K, \Omega)$  in  $L^2(K, U)$  then  $r_\Omega$  can be extended to a unitary representation of  $G_+$  in  $L^2(K, \Omega)$ .

The first part of the proposition is a consequence of the previous lemma. Let  $H$  be the group of matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

with  $a$  in  $\nu(K^\times)$ . It is clear that the formula of part (ii) defines a continuous representation of  $H$  on  $\mathcal{S}(K, \Omega)$ . Moreover  $G_+$  is the semi-direct product of  $H$  and  $\mathrm{SL}(2, F)$  so that to prove (ii) we have only to show that

$$r_\Omega \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right) = r_\Omega \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) r_\Omega(g) r_\Omega \left( \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right)$$

Let  $a = \nu(h)$  and let  $r'$  be the representation associated  $\psi'_F(x) = \psi_F(ax)$ . By the first part of the previous lemma this relation reduces to

$$r'_\Omega(g) = \rho(h) r_\Omega(g) \rho^{-1}(h),$$

which is a consequence of the last part of the previous lemma.

To prove (iii) observe that

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}$$

and that  $a^2 = \nu(a)$  belongs to  $\nu(K^\times)$ . The last two assertions are easily proved.

We now insert some remarks whose significance will not be clear until we begin to discuss the local functional equations. We associate to every  $\Phi$  in  $\mathcal{S}(K, \Omega)$  a function

$$(1.5.1) \quad W_\Phi(g) = r_\Omega(g) \Phi(1)$$

on  $G_+$  and a function

$$(1.5.2) \quad \varphi_\Phi(a) = W_\Phi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right)$$

on  $\nu(K^\times)$ . The both take values in  $U$ .

It is easily verified that

$$W_\Phi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) = \psi_F(x) W_\Phi(g)$$

If  $g \in G_+$  and  $F$  is a function on  $G_+$  let  $\rho(g)F$  be the function  $h \rightarrow F(hg)$ . Then

$$\rho(g)W_\Phi = W_{r_\Omega}(g)\Phi$$

Let  $B_+$  be the group of matrices of the form

$$\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}$$

with  $a$  in  $\nu(K^\times)$ . Let  $\xi$  be the representation of  $B_+$  on the space of functions on  $\nu(K^\times)$  with values in  $U$  defined by

$$\xi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) \varphi(b) = \varphi(ba)$$

and

$$\xi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \varphi(b) = \psi_F(bx) \varphi(b).$$

Then for all  $b$  in  $B_+$

$$(1.5.3) \quad \xi(b)\varphi_\Phi = \varphi_{r_\Omega}(b)\Phi.$$

The application  $\Phi \rightarrow \varphi_\Phi$ , and therefore the application  $\Phi \rightarrow W_\Phi$ , is injective because

$$(1.5.4) \quad \varphi_\Phi(\nu(h)) = |h|_K^{1/2} \Omega(h) \Phi(h).$$

Thus we may regard  $r_\Omega$  as acting on the space  $V$  of functions  $\varphi_\Phi$ ,  $\Phi \in \mathcal{S}(K, \Omega)$ . The effect of a matrix in  $B_+$  is given by (1.5.3). The matrix  $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$  corresponds to the operator  $\omega(a)\eta(a)I$ . Since  $G_+$  is generated by  $B_+$ , the set of scalar matrices, and  $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  the representation  $r_\Omega$  on  $V$  is determined by the action of  $w$ . To specify this we introduce, formally at first, the Mellin transform of  $\varphi = \varphi_\Phi$ .

If  $\mu$  is a quasi-character of  $F^\times$  let

$$(1.5.5) \quad \widehat{\varphi}(\mu) = \int_{\nu(K^\times)} \varphi(\alpha) \mu(\alpha) d^\times \alpha.$$

Appealing to (1.5.4) we may write this as

$$(1.5.6) \quad \widehat{\varphi}_\Phi(\mu) = \widehat{\varphi}(\mu) = \int_{K^\times} |h|_K^{1/2} \mu(\nu(h)) \Omega(h) \Phi(h) d^\times h.$$

If  $\lambda$  is a quasi-character of  $F^\times$  we sometimes write  $\lambda$  for the associated quasi-character  $\lambda \circ \nu$  of  $K^\times$ . The tensor product  $\lambda \otimes \Omega$  of  $\lambda$  and  $\Omega$  is defined by

$$(\lambda \otimes \Omega)(h) = \lambda(h) \Omega(h).$$

If  $\alpha_K : h \rightarrow |h|_K$  is the module of  $K$  then

$$\alpha_K^{1/2} \mu \otimes \Omega(h) = |h|_K^{1/2} \mu(\nu(h)) \Omega(h).$$

We also introduce, again in a purely formal manner, the integrals

$$Z(\Omega, \Phi) = \int_{K^\times} \Omega(h) \Phi(h) d^\times h$$

and

$$Z(\Omega^{-1}, \Phi) = \int_{K^\times} \Omega^{-1}(h) \Phi(h) d^\times h$$

so that

$$(1.5.7) \quad \widehat{\varphi}(\mu) = Z(\mu \alpha_K^{1/2} \otimes \Omega, \Phi).$$

Now let  $\varphi' = \varphi_{r_\Omega(w)\Phi}$  and let  $\Phi'$  be the Fourier transform of  $\Phi$  so that  $r_\Omega(w)\Phi(x) = \gamma\Phi'(x')$ . If  $\mu_0 = \omega\eta$

$$\widehat{\varphi}'(\mu^{-1}\mu_0^{-1}) = Z(\mu^{-1}\mu_0^{-1}\alpha_K^{1/2} \otimes \Omega, r_\Omega(w)\Phi)$$

which equals

$$\gamma \int_K \mu^{-1}\mu_0^{-1}(\nu(h)) \Omega(h) \Phi'(h) d^\times h.$$

Since  $\mu_0(\nu(h)) = \eta(\nu(h)) = \Omega(h^\iota h) = \Omega(h^\iota) \Omega(h)$  this expression equals

$$\gamma \int_K \mu^{-1}(\nu(h)) \Omega^{-1}(h^\iota) \Phi'(h^\iota) d^\times h = \gamma \int_K \mu^{-1}(\nu(h)) \Omega^{-1}(h) \Phi'(h) d^\times h$$

so that

$$(1.5.8) \quad \widehat{\varphi}'(\mu^{-1}\mu_0^{-1}) = \gamma Z(\mu^{-1}\alpha_K^{1/2} \otimes \Omega^{-1}, \Phi').$$

Take  $\mu = \mu_1 \alpha_F^s$  where  $\mu_1$  is a fixed quasi-character and  $s$  is complex number. If  $K$  is a separable quadratic extension of  $F$  the representation  $\Omega$  is one-dimensional and therefore a quasi-character. The integral defining the function

$$Z(\mu \alpha_K^{1/2} \otimes \Omega, \Phi)$$

is known to converge for  $\text{Re } s$  sufficiently large and the function itself is essentially a local zeta-function in the sense of Tate. The integral defining

$$Z(\mu^{-1}\alpha_K^{1/2} \otimes \Omega^{-1}, \Phi')$$

converges for  $\text{Re } s$  sufficiently small, that is, large and negative. Both functions can be analytically continued to the whole  $s$ -plane as meromorphic functions. There is a scalar  $C(\mu)$  which depends analytically on  $s$  such that

$$Z(\mu \alpha_K^{1/2} \otimes \Omega, \Phi) = C(\mu) Z(\mu^{-1}\alpha_K^{1/2} \otimes \Omega^{-1}, \Phi').$$

All these assertions are also known to be valid for quaternion algebras. We shall return to the verification later. The relation

$$\widehat{\varphi}(\mu) = \gamma^{-1} C(\mu) \widehat{\varphi}'(\mu^{-1}\mu_0^{-1})$$

determines  $\varphi'$  in terms of  $\varphi$ .

If  $\lambda$  is a quasi-character of  $F^\times$  and  $\Omega_1 = \lambda \otimes \Omega$  then  $\mathcal{S}(K, \Omega_1) = \mathcal{S}(K, \Omega)$  and

$$r_{\Omega_1}(g) = \lambda(\det g) r_\Omega(g)$$

so that we may write

$$r_{\Omega_1} = \lambda \otimes r_{\Omega}$$

However the space  $V_1$  of functions on  $\nu(K^\times)$  associated to  $r_{\Omega_1}$  is not necessarily  $V$ . In fact

$$V_1 = \{ \lambda_\varphi \mid \varphi \in V \}$$

and  $r_{\Omega_1}(g)$  applied to  $\lambda_\varphi$  is the product of  $\lambda(\det g)$  with the function  $\lambda \cdot r_{\Omega}(g)_\varphi$ . Given  $\Omega$  one can always find a  $\lambda$  such that  $\lambda \otimes \Omega$  is equivalent to a unitary representation.

If  $\Omega$  is unitary the map  $\Phi \rightarrow \varphi_\Phi$  is an isometry because

$$\int_{\nu(K^\times)} \|\varphi_\Phi(a)\|^2 d^\times a = \int_{K^\times} \|\Omega(h)\Phi(h)\|^2 |h|_K d^\times h = \int_K \|\Phi(h)\|^2 dh$$

if the measures are suitably normalized.

We want to extend some of these results to the case  $K = F \oplus F$ . We regard the element of  $K$  as defining a row vector so that  $K$  becomes a right module for  $M(2, F)$ . If  $\Phi$  belongs to  $\mathcal{S}(K)$  and  $g$  belongs to  $\mathrm{GL}(2, F)$ , we set

$$\rho(g)\Phi(x) = \Phi(xg).$$

**Proposition 1.6.**

(i) If  $K = F \oplus F$  then  $r$  can be extended to a representation  $r$  of  $\mathrm{GL}(2, F)$  such that

$$r\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\Phi = \rho\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\Phi$$

for  $a$  in  $F^\times$ .

(ii) If  $\tilde{\Phi}$  is the partial Fourier transform

$$\tilde{\Phi}(a, b) = \int_F \Phi(a, y) \psi_F(by) dy$$

and the Haar measure  $dy$  is self-dual with respect to  $\psi_F$  then

$$[r(g)\Phi]^\sim = \rho(g)\tilde{\Phi}$$

for all  $\Phi$  in  $\mathcal{S}(K)$  and all  $g$  in  $G_F$ .

It is easy to prove part (ii) for  $g$  in  $\mathrm{SL}(2, F)$ . In fact one has just to check it for the standard generators and for these it is a consequence of the definitions of Proposition 1.3. The formula of part (ii) therefore defines an extension of  $r$  to  $\mathrm{GL}(2, F)$  which is easily seen to satisfy the condition of part (i).

Let  $\Omega$  be a quasi-character of  $K^\times$ . Since  $K^\times = F^\times \times F^\times$  we may identify  $\Omega$  with a pair  $(\omega_1, \omega_2)$  of quasi-characters of  $F^\times$ . Then  $r_\Omega$  will be the representation defined by

$$r_\Omega(g) = |\det g|_F^{1/2} \omega_1(\det g) r(g).$$

If  $x$  belongs to  $K^\times$  and  $\nu(x) = 1$  then  $x$  is of the form  $(t, t^{-1})$  with  $t$  in  $F^\times$ . If  $\Phi$  belongs to  $\mathcal{S}(K)$  set

$$\theta(\Omega, \Phi) = \int_{F^\times} \Omega((t, t^{-1})) \Phi((t, t^{-1})) d^\times t.$$

Since the integrand has compact support on  $F^\times$  the integral converges. We now associate to  $\Phi$  the function

$$(1.6.1) \quad W_\Phi(g) = \theta(\Omega, r_\Omega(g)\Phi)$$

on  $\mathrm{GL}(2, F)$  and the function

$$(1.6.2) \quad \varphi_\Phi(a) = W_\Phi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right)$$

on  $F^\times$ . We still have

$$\rho(g)W_\Phi = W_{r_\Omega}(g)\Phi.$$

If

$$B_F = \left\{ \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} \mid a \in F^\times, x \in F \right\}$$

and if the representation  $\xi$  of  $B_F$  on the space of functions on  $F^\times$  is defined in the same manner as the representation  $\xi$  of  $B_+$  then

$$\xi(b)\varphi_\Phi = \varphi_{r_\Omega(b)\Phi}$$

for  $b$  in  $B_F$ . The applications  $\Phi \rightarrow W_\Phi$  and  $\Phi \rightarrow \varphi_\Phi$  are no longer injective.

If  $\mu_0$  is the quasi-character defined by

$$\mu_0(a) = \Omega((a, a)) = \omega_1(a)\omega_2(a)$$

then

$$W_\Phi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g \right) = \mu_0(a)W_\Phi(g).$$

It is enough to verify this for  $g = e$ .

$$W_\Phi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \theta \left( \Omega, r_\Omega \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) \Phi \right)$$

and

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}$$

so that

$$r_\Omega \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) \Phi(x, y) = |a^2|_F^{1/2} \omega_1(a^2) |a|_K^{-1/2} \Phi(ax, a^{-1}y).$$

Consequently

$$\begin{aligned} W_\Phi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) &= \int_{F^\times} \omega_1(a^2) \omega_1(x) \omega_2^{-1}(x) \Phi(ax, a^{-1}x^{-1}) d^\times x \\ &= \omega_1(a) \omega_2(a) \int_{F^\times} \omega_1(x) \omega_2^{-1}(x) \Phi(x, x^{-1}) d^\times x \end{aligned}$$

which is the required result.

Again we introduce in a purely formal manner the distribution

$$Z(\Omega, \Phi) = Z(\omega_1, \omega_2, \Phi) = \int \Phi(x_1, x_2) \omega_1(x_2) \omega_2(x_2) d^\times x_2 d^\times x_1.$$

If  $\mu$  is a quasi-character of  $F^\times$  and  $\varphi = \varphi_\Phi$  we set

$$\widehat{\varphi}(\mu) = \int_{F^\times} \varphi(\alpha) \mu(\alpha) d^\times \alpha.$$



The integral is

$$\begin{aligned} & \int_{F^\times} \mu(\alpha) \theta \left( \Omega, r_\Omega \left( \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \right) \Phi \right) d^\times \alpha \\ &= \int_{F^\times} \mu(\alpha) \left\{ \int_{F^\times} r_\Omega \left( \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \right) \Phi(x, x^{-1}) \omega_1(x) \omega_2^{-1}(x) d^\times x \right\} d^\times \alpha \end{aligned}$$

which in turn equals

$$\int_{F^\times} \mu(\alpha) \omega_1(\alpha) |\alpha|_F^{1/2} \left\{ \int_{F^\times} \Phi(\alpha x, x^{-1}) \omega_1(x) \omega_2^{-1}(x) d^\times x \right\} d^\times \alpha.$$

Writing this as a double integral and then changing variables we obtain

$$\int_{F^\times} \int_{F^\times} \Phi(\alpha, x) \mu \omega_1(\alpha) \mu \omega_2(x) |\alpha x|_F^{1/2} d^\times \alpha d^\times x$$

so that

$$(1.6.3) \quad \widehat{\varphi}(\mu) = Z(\mu \omega_1 \alpha_F^{1/2}, \mu \omega_2 \alpha_F^{1/2}, \Phi).$$

Let  $\varphi' = \varphi_{r_\Omega(w)\Phi}$ . Then

$$\widehat{\varphi}'(\mu^{-1} \mu_0^{-1}) = Z(\mu^{-1} \omega_2^{-1} \alpha_F^{1/2}, \mu^{-1} \omega_1^{-1} \alpha_F^{1/2}, r_\Omega(w)\Phi)$$

which equals

$$\iint \Phi'(y, x) \mu^{-1} \omega_2^{-1}(x) \mu^{-1} \omega_1^{-1}(y) |xy|_F^{1/2} d^\times x d^\times y$$

so that

$$(1.6.4) \quad \widehat{\varphi}'(\mu^{-1} \mu_0^{-1}) = Z(\mu^{-1} \omega_1^{-1} \alpha_F^{1/2}, \mu^{-1} \omega_2^{-1} \alpha_F^{1/2}, \Phi').$$

Suppose  $\mu = \mu_1 \alpha_F^s$  where  $\mu_1$  is a fixed quasi-character and  $s$  is a complex number. We shall see that the integral defining the right side of (1.6.3) converges for  $\operatorname{Re} s$  sufficiently large and that the integral defining the right side of (1.6.4) converges for  $\operatorname{Re} s$  sufficiently small. Both can be analytically continued to the whole complex plane as meromorphic functions and there is a meromorphic function  $C(\mu)$  which is independent of  $\Phi$  such that

$$Z(\mu \omega_1 \alpha_F^{1/2}, \mu \omega_2 \alpha_F^{1/2}, \Phi) = C(\mu) Z(\mu^{-1} \omega_1^{-1} \alpha_F^{1/2}, \mu^{-1} \omega_2^{-1} \alpha_F^{1/2}, \Phi').$$

Thus

$$\widehat{\varphi}(\mu) = C(\mu) \widehat{\varphi}'(\mu^{-1} \mu_0^{-1})$$

The analogy with the earlier results is quite clear.

## §2. Representations of $\mathrm{GL}(2, F)$ in the non-archimedean case

In this and the next two paragraphs the ground field  $F$  is a non-archimedean local field. We shall be interested in representations  $\pi$  of  $G_F = \mathrm{GL}(2, F)$  on a vector space  $V$  over  $\mathbf{C}$  which satisfy the following condition.

**(2.1).** *For every vector  $v$  in  $V$  the stabilizer of  $v$  in  $G_F$  is an open subgroup of  $G_F$ .*

Those who are familiar with such things can verify that this is tantamount to demanding that the map  $(g, v) \rightarrow \pi(g)v$  of  $G_F \times V$  into  $V$  is continuous if  $V$  is given the trivial locally convex topology in which every semi-norm is continuous. A representation of  $G_F$  satisfying (2.1) will be called admissible if it also satisfies the following condition.

**(2.2).** *For every open subgroup  $G'$  of  $\mathrm{GL}(2, O_F)$  the space of vectors  $v$  in  $V$  stabilized by  $G'$  is finite-dimensional.  $O_F$  is the ring of integers of  $F$ .*

Let  $\mathcal{H}_F$  be the space of functions on  $G_F$  which are locally constant and compactly supported. Let  $dg$  be that Haar measure on  $G_F$  which assigns the measure 1 to  $\mathrm{GL}(2, O_F)$ . Every  $f$  in  $\mathcal{H}_F$  may be identified with the measure  $f(g) dg$ . The convolution product

$$f_1 * f_2(h) = \int_{G_F} f_1(g) f_2(g^{-1}h) dg$$

turns  $\mathcal{H}_F$  into an algebra which we refer to as the Hecke algebra. Any locally constant function on  $\mathrm{GL}(2, O_F)$  may be extended to  $G_F$  by being set equal to 0 outside of  $\mathrm{GL}(2, O_F)$  and therefore may be regarded as an element of  $\mathcal{H}_F$ . In particular if  $\pi_i$ ,  $1 \leq i \leq r$ , is a family of inequivalent finite-dimensional irreducible representations of  $\mathrm{GL}(2, O_F)$  and

$$\xi_i(g) = \dim(\pi_i) \mathrm{tr} \pi_i(g^{-1})$$

for  $g$  in  $\mathrm{GL}(2, O_F)$  we regard  $\xi_i$  as an element of  $\mathcal{H}_F$ . The function

$$\xi = \sum_{i=1}^r \xi_i$$

is an idempotent of  $\mathcal{H}_F$ . Such an idempotent will be called elementary.

Let  $\pi$  be a representation satisfying (2.1). If  $f$  belongs to  $\mathcal{H}_F$  and  $v$  belongs to  $V$  then  $f(g)\pi(g)v$  takes on only finitely many values and the integral

$$\int_{G_F} f(g)\pi(g)v dg = \pi(f)v$$

may be defined as a finite sum. Alternatively we may give  $V$  the trivial locally convex topology and use some abstract definition of the integral. The result will be the same and  $f \rightarrow \pi(f)$  is the representation of  $\mathcal{H}_F$  on  $V$ . If  $g$  belongs to  $G_F$  then  $\lambda(g)f$  is the function whose value at  $h$  is  $f(g^{-1}h)$ . It is clear that

$$\pi(\lambda(g)f) = \pi(g)\pi(f).$$

Moreover

**(2.3).** *For every  $v$  in  $V$  there is an  $f$  in  $\mathcal{H}_F$  such that  $\pi f(v) = v$ .*

In fact  $f$  can be taken to be a multiple of the characteristic function of some open and closed neighbourhood of the identity. If  $\pi$  is admissible the associated representation of  $\mathcal{H}_F$  satisfies

(2.4). For every elementary idempotent  $\xi$  of  $\mathcal{H}_F$  the operator  $\pi(\xi)$  has a finite-dimensional range.

We now verify that from a representation  $\pi$  of  $\mathcal{H}_F$  satisfying (2.3) we can construct a representation  $\pi$  of  $G_F$  satisfying (2.1) such that

$$\pi(f) = \int_{G_F} f(g)\pi(g) dg.$$

By (2.3) every vector  $v$  in  $V$  is of the form

$$v = \sum_{i=1}^r \pi(f_i)v_i$$

with  $v_i$  in  $V$  and  $f_i$  in  $\mathcal{H}_F$ . If we can show that

$$(2.3.1) \quad \sum_{i=1}^r \pi(f_i)v_i = 0$$

implies that

$$w = \sum_{i=1}^r \pi(\lambda(g)f_i)v_i$$

is 0 we can define  $\pi(g)v$  to be

$$\sum_{i=1}^r \pi(\lambda(g)f_i)v_i$$

$\pi$  will clearly be a representation of  $G_F$  satisfying (2.1).

Suppose that (2.3.1) is satisfied and choose  $f$  in  $\mathcal{H}_F$  so that  $\pi(f)w = w$ . Then

$$w = \sum_{i=1}^r \pi(f * \lambda(g)f_i)v_i.$$

If  $\rho(g)f(h) = f(hg)$  then

$$f * \lambda(g)f_i = \rho(g^{-1})f * f_i$$

so that

$$w = \sum_{i=1}^r \pi(\rho(g^{-1})f * f_i)v_i = \pi(\rho(g^{-1})f) \left\{ \sum_{i=1}^r \pi(f_i)v_i \right\} = 0.$$

It is easy to see that the representation of  $G_F$  satisfies (2.2) if the representation of  $\mathcal{H}_F$  satisfies (2.4). A representation of  $\mathcal{H}_F$  satisfying (2.3) and (2.4) will be called admissible. There is a complete correspondence between admissible representations of  $G_F$  and of  $\mathcal{H}_F$ . For example a subspace is invariant under  $G_F$  if and only if it is invariant under  $\mathcal{H}_F$  and an operator commutes with the action of  $G_F$  if and only if it commutes with the action of  $\mathcal{H}_F$ .

From now on, unless the contrary is explicitly stated, an irreducible representation of  $G_F$  or  $\mathcal{H}_F$  is to be assumed admissible. If  $\pi$  is irreducible and acts on the space  $V$  then any linear transformation  $A$  of  $V$  commuting with  $\mathcal{H}_F$  is a scalar. In fact if  $V$  is assumed, as it always will be, to be different from 0 there is an elementary idempotent  $\xi$  such that  $\pi(\xi) \neq 0$ . Its range is a finite-dimensional space invariant under  $A$ . Thus  $A$  has at least

one eigenvector and is consequently a scalar. In particular there is a homomorphism  $\omega$  of  $F^\times$  into  $\mathbf{C}^\times$  such that

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I$$

for all  $a$  in  $F^\times$ . By (2.1) the function  $\omega$  is 1 near the identity and is therefore continuous. We shall refer to a continuous homomorphism of a topological group into the multiplicative group of complex numbers as a quasi-character.

If  $\chi$  is a quasi-character of  $F^\times$  then  $g \rightarrow \chi(\det g)$  is a quasi-character of  $G_F$ . It determines a one-dimensional representation of  $G_F$  which is admissible. It will be convenient to use the letter  $\chi$  to denote this associated representation. If  $\pi$  is an admissible representation of  $G_F$  on  $V$  then  $\chi \otimes \pi$  will be the representation of  $G_F$  on  $V$  defined by

$$(\chi \otimes \pi)(g) = \chi(\det g)\pi(g).$$

It is admissible and irreducible if  $\pi$  is.

Let  $\pi$  be an admissible representation of  $G_F$  on  $V$  and let  $V^*$  be the space of all linear forms on  $V$ . We define a representation  $\pi^*$  of  $\mathcal{H}_F$  on  $V^*$  by the relation

$$\langle v, \pi^*(f)v^* \rangle = \langle \pi(\check{f})v, v^* \rangle$$

where  $\check{f}(g) = f(g^{-1})$ . Since  $\pi^*$  will not usually be admissible, we replace  $V^*$  by  $\tilde{V} = \pi^*(\mathcal{H}_F)V^*$ . The space  $\tilde{V}$  is invariant under  $\mathcal{H}_F$ . For each  $f$  in  $\mathcal{H}_F$  there is an elementary idempotent  $\xi$  such that  $\xi * f = f$  and therefore the restriction  $\tilde{\pi}$  of  $\pi^*$  to  $\tilde{V}$  satisfies (2.3). It is easily seen that if  $\xi$  is an elementary idempotent so is  $\check{\xi}$ . To show that  $\tilde{\pi}$  is admissible we have to verify that

$$\tilde{V}(\xi) = \tilde{\pi}(\xi)\tilde{V} = \pi^*(\xi)V^*$$

is finite-dimensional. Let  $V(\check{\xi}) = \pi(\check{\xi})V$  and let  $V_c = (1 - \pi(\check{\xi}))V$ .  $V$  is clearly the direct sum of  $V(\check{\xi})$ , which is finite-dimensional, and  $V_c$ . Moreover  $\tilde{V}(\xi)$  is orthogonal to  $V_c$  because

$$\langle v - \pi(\check{\xi})v, \tilde{\pi}(\xi)\tilde{v} \rangle = \langle \pi(\check{\xi})v - \pi(\check{\xi})v, \tilde{v} \rangle = 0.$$

It follows immediately that  $\tilde{V}(\xi)$  is isomorphic to a subspace of the dual of  $V(\check{\xi})$  and is therefore finite-dimensional. It is in fact isomorphic to the dual of  $V(\check{\xi})$  because if  $v^*$  annihilates  $V_c$  then, for all  $v$  in  $V$ ,

$$\langle v, \pi^*(\xi)v^* \rangle - \langle v, v^* \rangle = -\langle v - \pi(\check{\xi})v, v^* \rangle = 0$$

so that  $\pi^*(\xi)v^* = v^*$ .

$\tilde{\pi}$  will be called the representation contragredient to  $\pi$ . It is easily seen that the natural map of  $V$  into  $\tilde{V}^*$  is an isomorphism and that the image of this map is  $\tilde{\pi}^*(\mathcal{H}_F)\tilde{V}^*$  so that  $\pi$  may be identified with the contragredient of  $\tilde{\pi}$ .

If  $V_1$  is an invariant subspace of  $V$  and  $V_2 = V_1 \setminus V$  we may associate to  $\pi$  representations  $\pi_1$  and  $\pi_2$  on  $V_1$  and  $V_2$ . They are easily seen to be admissible. It is also clear that there is a natural embedding of  $\tilde{V}_2$  in  $\tilde{V}$ . Moreover any element  $\tilde{v}_1$  of  $\tilde{V}_1$  lies in  $\tilde{V}_1(\xi)$  for some  $\xi$  and therefore is determined by its effect on  $V_1(\check{\xi})$ . It annihilates  $(I - \pi(\check{\xi}))V_1$ . There is certainly a linear function  $\tilde{v}$  on  $V$  which annihilates  $(I - \pi(\check{\xi}))V$  and agrees with  $\tilde{V}_1$  on  $V_1(\check{\xi})$ .  $\tilde{v}$  is

necessarily in  $\tilde{V}$  so that  $\tilde{V}_1$  may be identified with  $\tilde{V}_2 \setminus \tilde{V}$ . Since every representation is the contragredient of its contragredient we easily deduce the following lemma.

**Lemma 2.5.**

- (a) Suppose  $V_1$  is an invariant subspace of  $V$ . If  $\tilde{V}_2$  is the annihilator of  $V_1$  in  $\tilde{V}$  then  $V_1$  is the annihilator of  $\tilde{V}_2$  in  $V$ .
- (b)  $\pi$  is irreducible if and only if  $\tilde{\pi}$  is.

Observe that for all  $g$  in  $G_F$

$$\langle \pi(g)v, \tilde{v} \rangle = \langle v, \tilde{\pi}(g^{-1})\tilde{v} \rangle.$$

If  $\pi$  is the one-dimensional representation associated to the quasi-character  $\chi$  then  $\tilde{\pi} = \chi^{-1}$ . Moreover if  $\chi$  is a quasi-character and  $\pi$  any admissible representation then the contragredient of  $\chi \otimes \pi$  is  $\chi^{-1} \otimes \tilde{\pi}$ .

Let  $V$  be a separable complete locally convex space and  $\pi$  a continuous representation of  $G_F$  on  $V$ . The space  $V_0 = \pi(\mathcal{H}_F)V$  is invariant under  $G_F$  and the restriction  $\pi_0$  of  $\pi$  to  $V_0$  satisfies (2.1). Suppose that it also satisfies (2.2). Then if  $\pi$  is irreducible in the topological sense  $\pi_0$  is algebraically irreducible. To see this take any two vectors  $v$  and  $w$  in  $V_0$  and choose an elementary idempotent  $\xi$  so that  $\pi(\xi)v = v$ .  $v$  is in the closure of  $\pi(\mathcal{H}_F)w$  and therefore in the closure of  $\pi(\mathcal{H}_F)w \cap \pi(\xi)V$ . Since, by assumption,  $\pi(\xi)V$  is finite-dimensional,  $v$  must actually lie in  $\pi(\mathcal{H}_F)w$ .

The equivalence class of  $\pi$  is not in general determined by that of  $\pi_0$ . It is, however, when  $\pi$  is unitary. To see this one has only to show that, up to a scalar factor, an irreducible admissible representation admits at most one invariant hermitian form.

**Lemma 2.6.** Suppose  $\pi_1$  and  $\pi_2$  are irreducible admissible representations of  $G_F$  on  $V_1$  and  $V_2$  respectively. Suppose  $A(v_1, v_2)$  and  $B(v_1, v_2)$  are non-degenerate forms on  $V_1 \times V_2$  which are linear in the first variable and either both linear or both conjugate linear in the second variable. Suppose moreover that, for all  $g$  in  $G_F$

$$A(\pi_1(g)v_1, \pi_2(g)v_2) = A(v_1, v_2)$$

and

$$B(\pi_1(g)v_1, \pi_2(g)v_2) = B(v_1, v_2)$$

Then there is a complex scalar  $\lambda$  such that

$$B(v_1, v_2) = \lambda A(v_1, v_2)$$

Define two mappings  $S$  and  $T$  of  $V_2$  into  $\tilde{V}_1$  by the relations

$$A(v_1, v_2) = \langle v_1, Sv_2 \rangle$$

and

$$B(v_1, v_2) = \langle v_1, Tv_2 \rangle,$$

Since  $S$  and  $T$  are both linear or conjugate linear with kernel 0 they are both embeddings. Both take  $V_2$  onto an invariant subspace of  $\tilde{V}_1$ . Since  $\tilde{V}_1$  has no non-trivial invariant subspaces they are both isomorphisms. Thus  $S^{-1}T$  is a linear map of  $V_2$  which commutes with  $G_F$  and is therefore a scalar  $\lambda I$ . The lemma follows.

An admissible representation will be called unitary if it admits an invariant positive definite hermitian form.

We now begin in earnest the study of irreducible admissible representations of  $G_F$ . The basic ideas are due to Kirillov.

**Proposition 2.7.** *Let  $\pi$  be an irreducible admissible representation of  $G_F$  on the vector space  $V$ .*

(a) *If  $V$  is finite-dimensional then  $V$  is one-dimensional and there is a quasi-character  $\chi$  of  $F^\times$  such that*

$$\pi(g) = \chi(\det g)$$

(b) *If  $V$  is infinite-dimensional there is no nonzero vector invariant by all the matrices  $\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$ ,  $x \in F$ .*

If  $\pi$  is finite-dimensional its kernel  $H$  is an open subgroup. In particular there is a positive number  $\epsilon$  such that

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

belongs to  $H$  if  $|x| < \epsilon$ . If  $x$  is any element of  $F$  there is an  $a$  in  $F^\times$  such that  $|ax| < \epsilon$ . Since

$$\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

the matrix

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

belongs to  $H$  for all  $x$  in  $F$ . For similar reasons the matrices

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$$

do also. Since the matrices generate  $\mathrm{SL}(2, F)$  the group  $H$  contains  $\mathrm{SL}(2, F)$ . Thus  $\pi(g_1)\pi(g_2) = \pi(g_2)\pi(g_1)$  for all  $g_1$  and  $g_2$  in  $G_F$ . Consequently each  $\pi(g)$  is a scalar matrix and  $\pi(g)$  is one-dimensional. In fact

$$\pi(g) = \chi(\det g)I$$

where  $\chi$  is a homomorphism of  $F^\times$  into  $\mathbf{C}^\times$ . To see that  $\chi$  is continuous we need only observe that

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = \chi(a)I.$$

Suppose  $V$  contains a nonzero vector  $v$  fixed by all the operators

$$\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right).$$

Let  $H$  be the stabilizer of the space  $\mathbf{C}v$ . To prove the second part of the proposition we need only verify that  $H$  is of finite index in  $G_F$ . Since it contains the scalar matrices and an open subgroup of  $G_F$  it will be enough to show that it contains  $\mathrm{SL}(2, F)$ . In fact we shall show that  $H_0$ , the stabilizer of  $v$ , contains  $\mathrm{SL}(2, F)$ .  $H_0$  is open and therefore contains a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with  $c \neq 0$ . It also contains

$$\begin{pmatrix} 1 & -ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -dc^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & b_0 \\ c & 0 \end{pmatrix} = w_0.$$

If  $x = \frac{b_0}{c}y$  then

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} w_0^{-1}$$

also belongs to  $H_0$ . As before we see that  $H_0$  contains  $SL(2, F)$ .

Because of this lemma we can confine our attention to infinite-dimensional representations. Let  $\psi = \psi_F$  be a non-trivial additive character of  $F$ . Let  $B_F$  be the group of matrices of the form

$$b = \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}$$

with  $a$  in  $F^\times$  and  $x$  in  $F$ . If  $X$  is a complex vector space we define a representation  $\xi_\psi$  of  $B_F$  on the space of all functions of  $F^\times$  with values in  $X$  by setting

$$(\xi_\psi(b)\varphi)(\alpha) = \psi(\alpha x)\varphi(\alpha a).$$

$\xi_\psi$  leaves invariant the space  $\mathcal{S}(F^\times, X)$  of locally constant compactly supported functions. The function  $\xi_\psi$  is continuous with respect to the trivial topology on  $\mathcal{S}(F^\times, X)$ .

**Proposition 2.8.** *Let  $\pi$  be an infinite-dimensional irreducible representation of  $G_F$  on the space  $V$ . Let  $\mathfrak{p} = \mathfrak{p}_F$  be the maximal ideal in the ring of integers of  $F$ , and let  $V'$  be the set of all vectors  $v$  in  $V$  such that*

$$\int_{\mathfrak{p}^{-n}} \psi_F(-x) \pi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) v \, dx = 0$$

for some integer  $n$ . Then

- (i) *The set  $V'$  is a subspace of  $V$ .*
- (ii) *Let  $X = V' \setminus V$  and let  $A$  be the natural map of  $V$  onto  $X$ . If  $v$  belongs to  $V$  let  $\varphi_v$  be the function defined by*

$$\varphi_v(a) = A \left( \pi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) v \right).$$

*The map  $v \rightarrow \varphi_v$  is an injection of  $V$  into the space of locally constant functions on  $F^\times$  with values in  $X$ .*

- (iii) *If  $b$  belongs to  $B_F$  and  $v$  belongs to  $V$  then*

$$\varphi_{\pi(b)v} = \xi_\psi(b)\varphi_v.$$

If  $m \geq n$  so that  $\mathfrak{p}^{-m}$  contains  $\mathfrak{p}^{-n}$  then

$$\int_{\mathfrak{p}^{-m}} \psi(-x) \pi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) v \, dx$$

is equal to

$$\sum_{y \in \mathfrak{p}^{-m}/\mathfrak{p}^{-n}} \psi(-y) \pi \left( \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \right) \int_{\mathfrak{p}^{-n}} \psi(-x) \pi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) v \, dx.$$

Thus if the integral of the lemma vanishes for some integer  $n$  it vanishes for all larger integers. The first assertion of the proposition follows immediately.

To prove the second we shall use the following lemma.

**Lemma 2.8.1.** *Let  $\mathfrak{p}^{-m}$  be the largest ideal on which  $\psi$  is trivial and let  $f$  be a locally constant function on  $\mathfrak{p}^{-\ell}$  with values in some finite-dimensional complex vector space. For any integer  $n \leq \ell$  the following two conditions are equivalent*

- (i)  *$f$  is constant on the cosets of  $\mathfrak{p}^{-n}$  in  $\mathfrak{p}^{-\ell}$*
- (ii) *The integral*

$$\int_{\mathfrak{p}^{-\ell}} \psi(-ax) f(x) dx$$

*is zero for all  $a$  outside of  $\mathfrak{p}^{-m+n}$ .*

Assume (i) and let  $a$  be an element of  $F^\times$  which is not in  $\mathfrak{p}^{-m+n}$ . Then  $x \rightarrow \psi(-ax)$  is a non-trivial character of  $\mathfrak{p}^{-n}$  and

$$\int_{\mathfrak{p}^{-\ell}} \psi(-ax) f(x) dx = \sum_{y \in \mathfrak{p}^{-\ell}/\mathfrak{p}^{-n}} \psi(-ay) \left\{ \int_{\mathfrak{p}^{-n}} \psi(-ax) dx \right\} f(y) = 0.$$

$f$  may be regarded as a locally constant function on  $F$  with support in  $\mathfrak{p}^{-\ell}$ . Assuming (ii) is tantamount to assuming that the Fourier transform  $f'$  of  $f$  has its support in  $\mathfrak{p}^{-m+n}$ . By the Fourier inversion formula

$$f(x) = \int_{\mathfrak{p}^{-m+n}} \psi(-xy) f'(y) dy.$$

If  $y$  belongs to  $\mathfrak{p}^{-m+n}$  the function  $x \rightarrow \psi(-xy)$  is constant on cosets of  $\mathfrak{p}^{-n}$ . It follows immediately that the second condition of the lemma implies the first.

To prove the second assertion of the proposition we show that if  $\varphi_v$  vanishes identically then  $v$  is fixed by the operator  $\pi\left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right)$  for all  $x$  in  $F$  and then appeal to Proposition 2.7.

Take

$$f(x) = \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)v.$$

The restriction of  $f$  to an ideal in  $F$  takes values in a finite-dimensional subspace of  $V$ . To show that  $f$  is constant on the cosets of some ideal  $\mathfrak{p}^{-n}$  it is enough to show that its restriction to some ideal  $\mathfrak{p}^{-\ell}$  containing  $\mathfrak{p}^{-n}$  has this property.

By assumption there exists an  $n_0$  such that  $f$  is constant on the cosets of  $\mathfrak{p}^{-n_0}$ . We shall now show that if  $f$  is constant on the cosets of  $\mathfrak{p}^{-n+1}$  it is also constant on the cosets of  $\mathfrak{p}^{-n}$ . Take any ideal  $\mathfrak{p}^{-\ell}$  containing  $\mathfrak{p}^{-n}$ . By the previous lemma

$$\int_{\mathfrak{p}^{-\ell}} \psi(-ax) f(x) dx = 0$$

if  $a$  is not in  $\mathfrak{p}^{-m+n-1}$ . We have to show that the integral on the left vanishes if  $a$  is a generator of  $\mathfrak{p}^{-m+n-1}$ .

If  $U_F$  is the group of units of  $O_F$  the ring of integers of  $F$  there is an open subgroup  $U_1$  of  $U_F$  such that

$$\pi\left(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}\right)v = v$$



for  $b$  in  $U_1$ . For such  $b$

$$\pi\left(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}\right) \int_{\mathfrak{p}^{-\ell}} \psi(-ax) f(x) dx = \int_{\mathfrak{p}^{-\ell}} \psi(-ax) \pi\left(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}\right) \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) v dx$$

is equal to

$$\int_{\mathfrak{p}^{-\ell}} \psi(-ax) \pi\left(\begin{pmatrix} 1 & bx \\ 0 & 1 \end{pmatrix}\right) \pi\left(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}\right) v dx = \int_{\mathfrak{p}^{-\ell}} \psi\left(-\frac{a}{b}x\right) f(x) dx.$$

Thus it will be enough to show that for some sufficiently large  $\ell$  the integral vanishes when  $a$  is taken to be one of a fixed set of representatives of the cosets of  $U_1$  in the set of generators of  $\mathfrak{p}^{-m+n-1}$ . Since there are only finitely many such cosets it is enough to show that for each  $a$  there is at least one  $\ell$  for which the integral vanishes.

By assumption there is an ideal  $\mathfrak{a}(a)$  such that

$$\int_{\mathfrak{a}(a)} \psi(-x) \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) v dx = 0$$

But this integral equals

$$|a| \pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \int_{a^{-1}\mathfrak{a}(a)} \psi(-ax) \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) v dx$$

so that  $\ell = \ell(a)$  could be chosen to make

$$\mathfrak{p}^{-\ell} = a^{-1}\mathfrak{a}(a).$$

To prove the third assertion we verify that

$$(2.8.2) \quad A\left(\pi\left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right)v\right) = \psi(y)A(v)$$

for all  $v$  in  $V$  and all  $y$  in  $F$ . The third assertion follows from this by inspection. We have to show that

$$\pi\left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right)v - \psi(y)v$$

is in  $V'$  or that, for some  $n$ ,

$$\int_{\mathfrak{p}^{-n}} \psi(-x) \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \pi\left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right) v dx - \int_{\mathfrak{p}^{-n}} \psi(-x) \psi(y) \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) v dx$$

is zero. The expression equals

$$\int_{\mathfrak{p}^{-n}} \psi(-x) \pi\left(\begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix}\right) v dx - \int_{\mathfrak{p}^{-n}} \psi(-x+y) \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) v dx.$$

If  $\mathfrak{p}^{-n}$  contains  $y$  we may change the variables in the first integral to see that it equals the second.

It will be convenient now to identify  $v$  with  $\varphi_v$  so that  $V$  becomes a space of functions on  $F^\times$  with values in  $X$ . The map  $A$  is replaced by the map  $\varphi \rightarrow \varphi(1)$ . The representation  $\pi$  now satisfies

$$\pi(b)\varphi = \xi_\psi(b)\varphi$$

if  $b$  is in  $B_F$ . There is a quasi-character  $\omega_0$  of  $F^\times$  such that

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega_0(a)I.$$

If

$$w = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$$

the representation is determined by  $\omega_0$  and  $\pi(w)$ .

**Proposition 2.9.**

(i) *The space  $V$  contains*

$$V_0 = \mathcal{S}(F^\times, X)$$

(ii) *The space  $V$  is spanned by  $V_0$  and  $\pi(w)V_0$ .*

For every  $\varphi$  in  $V$  there is a positive integer  $n$  such that

$$\pi\left(\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}\right)\varphi = \varphi$$

if  $x$  and  $a - 1$  belong to  $\mathfrak{p}^n$ . In particular  $\varphi(\alpha a) = \varphi(a)$  if  $\alpha$  belongs to  $F^\times$  and  $a - 1$  belongs to  $\mathfrak{p}^n$ . The relation

$$\psi(\alpha x)\varphi(\alpha) = \varphi(\alpha)$$

for all  $x$  in  $\mathfrak{p}^n$  implies that  $\varphi(\alpha) = 0$  if the restriction of  $\psi$  to  $\alpha\mathfrak{p}^n$  is not trivial. Let  $\mathfrak{p}^{-m}$  be the largest ideal on which  $\psi$  is trivial. Then  $\varphi(\alpha) = 0$  unless  $|\alpha| \leq |\varpi|^{-m-n}$  if  $\varpi$  is a generator of  $\mathfrak{p}$ .

Let  $V_0$  be the space of all  $\varphi$  in  $V$  such that, for some integer  $\ell$  depending on  $\varphi$ ,  $\varphi(\alpha) = 0$  unless  $|\alpha| > |\varpi|^\ell$ . To prove (i) we have to show that  $V_0 = \mathcal{S}(F^\times, X)$ . It is at least clear that  $\mathcal{S}(F^\times, X)$  contains  $V_0$ . Moreover for every  $\varphi$  in  $V$  and every  $x$  in  $F$  the difference

$$\varphi' = \varphi - \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi$$

is in  $V_0$ . To see this observe that

$$\varphi'(\alpha) = (1 - \psi(\alpha x))\varphi(\alpha)$$

is identically zero for  $x = 0$  and otherwise vanishes at least on  $x^{-1}\mathfrak{p}^{-m}$ . Since there is no function in  $V$  invariant under all the operators

$$\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)$$

the space  $V_0$  is not 0.

Before continuing with the proof of the proposition we verify a lemma we shall need.

**Lemma 2.9.1.** *The representation  $\xi_\psi$  of  $B_F$  in the space  $\mathcal{S}(F^\times)$  of locally constant, compactly supported, complex-valued functions on  $F^\times$  is irreducible.*

For every character  $\mu$  of  $U_F$  let  $\varphi_\mu$  be the function on  $F^\times$  which equals  $\mu$  on  $U_F$  and vanishes off  $U_F$ . Since these functions and their translates span  $\mathcal{S}(F^\times)$  it will be enough to show that any non-trivial invariant subspace contains all of them. Such a space must certainly contain some non-zero function  $\varphi$  which satisfies, for some character  $\nu$  of  $U_F$ , the relation

$$\varphi(a\epsilon) = \nu(\epsilon)\varphi(a)$$

for all  $a$  in  $F^\times$  and all  $\epsilon$  in  $U_F$ . Replacing  $\varphi$  by a translate if necessary we may assume that  $\varphi(1) \neq 0$ . We are going to show that the space contains  $\varphi_\mu$  if  $\mu$  is different from  $\nu$ . Since  $U_F$  has at least two characters we can then replace  $\varphi$  by some  $\varphi_\mu$  with  $\mu$  different from  $\nu$ , and replace  $\nu$  by  $\mu$  and  $\mu$  by  $\nu$  to see it also contains  $\varphi_\nu$ .

Set

$$\varphi' = \int_{U_F} \mu^{-1}(\epsilon) \xi_\psi \left( \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \right) \xi_\psi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \varphi d\epsilon$$

where  $x$  is still to be determined.  $\mu$  is to be different from  $\nu$ .  $\varphi'$  belongs to the invariant subspace and

$$\varphi'(a\epsilon) = \mu(\epsilon)\varphi'(a)$$

for all  $a$  in  $F^\times$  and all  $\epsilon$  in  $U_F$ . We have

$$\varphi'(a) = \varphi(a) \int_{U_F} \mu^{-1}(\epsilon) \nu(\epsilon) \psi(ax\epsilon) d\epsilon$$

The character  $\mu^{-1}\nu$  has a conductor  $\mathfrak{p}^n$  with  $n$  positive. Take  $x$  to be of order  $-n-m$ . The integral, which can be rewritten as a Gaussian sum, is then, as is well-known, zero if  $a$  is not in  $U_F$  but different from zero if  $a$  is in  $U_F$ . Since  $\varphi(1)$  is not zero  $\varphi'$  must be a nonzero multiple of  $\varphi_\mu$ .

To prove the first assertion of the proposition we need only verify that if  $u$  belongs to  $X$  then  $V_0$  contains all functions of the form  $\alpha \rightarrow \eta(\alpha)u$  with  $\eta$  in  $\mathcal{S}(F^\times)$ . There is a  $\varphi$  in  $V$  such that  $\varphi(1) = u$ . Take  $x$  such that  $\psi(x) \neq 1$ . Then

$$\varphi' = \varphi - \pi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \varphi$$

is in  $V_0$  and  $\varphi'(1) = (1 - \psi(x))u$ . Consequently every  $u$  is of the form  $\varphi(1)$  for some  $\varphi$  in  $V_0$ .

If  $\mu$  is a character of  $U_F$  let  $V_0(\mu)$  be the space of functions  $\varphi$  in  $V_0$  satisfying

$$\varphi(a\epsilon) = \mu(\epsilon)\varphi(a)$$

for all  $a$  in  $F^\times$  and all  $\epsilon$  in  $U_F$ .  $V_0$  is clearly the direct sum of the space  $V_0(\mu)$ . In particular every vector  $u$  in  $X$  can be written as a finite sum

$$u = \sum \varphi_i(1)$$

where  $\varphi_i$  belongs to some  $V_0(\mu_i)$ .

If we make use of the lemma we need only show that if  $u$  can be written as  $u = \varphi(1)$  where  $\varphi$  is in  $V_0(\nu)$  for some  $\nu$  then there is at least one function in  $V_0$  of the form  $\alpha \rightarrow \eta(\alpha)u$

where  $\eta$  is a nonzero function in  $\mathcal{S}(F^\times)$ . Choose  $\mu$  different from  $\nu$  and let  $\mathfrak{p}^n$  be the conductor of  $\mu^{-1}\nu$ . We again consider

$$\varphi' = \int_{U_F} \mu^{-1}(\epsilon) \xi_\psi \left( \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \varphi d\epsilon$$

where  $x$  is of order  $-n - m$ . Then

$$\varphi'(a) = \varphi(a) \int_{U_F} \mu^{-1}(\epsilon) \nu(\epsilon) \psi_F(ax\epsilon) d\epsilon$$

The properties of Gaussian sums used before show that  $\varphi'$  is a function of the required kind.

The second part of the proposition is easier to verify. Let  $P_F$  be the group of upper triangular matrices in  $G_F$ . Since  $V_0$  is invariant under  $P_F$  and  $V$  is irreducible under  $G_F$  the space  $V$  is spanned by  $V_0$  and the vectors

$$\varphi' = \pi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \pi(w)\varphi$$

with  $\varphi$  in  $V_0$ . But

$$\varphi' = \{\varphi' - \pi(w)\varphi\} + \pi(w)\varphi$$

and as we saw,  $\varphi' - \pi(w)\varphi$  is in  $V_0$ . The proposition is proved.

To study the effect of  $w$  we introduce a formal Mellin transform. Let  $\varpi$  be a generator of  $\mathfrak{p}$ . If  $\varphi$  is a locally constant function on  $F^\times$  with values in  $X$  then for every integer  $n$  the function  $\epsilon \rightarrow \varphi(\epsilon\varpi^n)$  on  $U_F$  takes its values in a finite-dimensional subspace of  $X$  so that the integral

$$\int_{U_F} \varphi(\epsilon\varpi^n) \nu(\epsilon) d\epsilon = \widehat{\varphi}_n(\nu)$$

is defined. In this integral we take the total measure of  $U_F$  to be 1. It is a vector in  $X$ . The expression  $\widehat{\varphi}(\nu, t)$  will be the Formal Laurent series

$$\sum_n t^n \widehat{\varphi}_n(\nu)$$

If  $\varphi$  is in  $V$  the series has only a finite number of terms with negative exponent. Moreover the series  $\widehat{\varphi}(\nu, t)$  is different from zero for only finitely many  $\nu$ . If  $\varphi$  belongs to  $V_0$  these series have only finitely many terms. It is clear that if  $\varphi$  is locally constant and all the formal series  $\widehat{\varphi}(\nu, t)$  vanish then  $\varphi = 0$ .

Suppose  $\varphi$  takes values in a finite-dimensional subspace of  $X$ ,  $\omega$  is a quasi-character of  $F^\times$ , and the integral

$$(2.10.1) \quad \int_{F^\times} \omega(a) \varphi(a) d^\times a$$

is absolutely convergent. If  $\omega'$  is the restriction of  $\omega$  to  $U_F$  this integral equals

$$\sum_n z^n \int_{U_F} \varphi(\varpi^n \epsilon) \omega'(\epsilon) d\epsilon = \sum_n z^n \widehat{\varphi}_n(\omega')$$

if  $z = \omega(\varpi)$ . Consequently the formal series  $\widehat{\varphi}(\omega', t)$  converges absolutely for  $t = z$  and the sum is equal to (2.10.1). We shall see that  $X$  is one-dimensional and that there is a

constant  $c_0 = c_0(\varphi)$  such that if  $|\omega(\varpi)| = |\varpi|^c$  with  $c > c_0$  then the integral (2.10.1) is absolutely convergent. Consequently all the series  $\widehat{\varphi}(\nu, t)$  have positive radii of convergence.

If  $\psi = \psi_F$  is a given non-trivial additive character of  $F$ ,  $\mu$  any character of  $U_F$ , and  $x$  any element of  $F$  we set

$$\eta(\mu, x) = \int_{U_F} \mu(\epsilon) \psi(\epsilon x) d\epsilon$$

The integral is taken with respect to the normalized Haar measure on  $U_F$ . If  $g$  belongs to  $G_F$ ,  $\varphi$  belongs to  $V$ , and  $\varphi' = \pi(g)\varphi$  we shall set

$$\pi(g)\widehat{\varphi}(\nu, t) = \widehat{\varphi}'(\nu, t).$$

**Proposition 2.10.**

(i) If  $\delta$  belongs to  $U_F$  and  $\ell$  belongs to  $\mathbf{Z}$  then

$$\pi\left(\begin{pmatrix} \delta\varpi^\ell & 0 \\ 0 & 1 \end{pmatrix}\right)\widehat{\varphi}(\nu, t) = t^{-\ell}\nu^{-1}(\delta)\widehat{\varphi}(\nu, t)$$

(ii) If  $x$  belongs to  $F$  then

$$\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\widehat{\varphi}(\nu, t) = \sum_n t^n \left\{ \sum_\mu \eta(\mu^{-1}\nu, \varpi^n x) \widehat{\varphi}_n(\mu) \right\}$$

where the inner sum is taken over all characters of  $U_F$

(iii) Let  $\omega_0$  be the quasi-character defined by

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega_0(a)I$$

for  $a$  in  $F^\times$ . Let  $\nu_0$  be its restriction to  $U_F$  and let  $z_0 = \omega_0(\varpi)$ . For each character  $\nu$  of  $U_F$  there is a formal series  $C(\nu, t)$  with coefficients in the space of linear operators on  $X$  such that for every  $\varphi$  in  $V_0$

$$\pi\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\widehat{\varphi}(\nu, t) = C(\nu, t)\widehat{\varphi}(\nu^{-1}\nu_0^{-1}, t^{-1}z_0^{-1}).$$

Set

$$\varphi' = \pi\left(\begin{pmatrix} \delta\varpi^\ell & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi.$$

Then

$$\widehat{\varphi}'(\nu, t) = \sum_n t^n \int_{U_F} \nu(\epsilon) \varphi(\varpi^{n+\ell} \delta \epsilon) d\epsilon.$$

Changing variables in the integration and in the summation we obtain the first formula of the proposition.

Now set

$$\varphi' = \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi.$$

Then

$$\widehat{\varphi}'(\nu, t) = \sum_n t^n \int_{U_F} \psi(\varpi^n \epsilon x) \nu(\epsilon) \varphi(\varpi^n \epsilon) d\epsilon.$$

By Fourier inversion

$$\varphi(\varpi^n \epsilon) = \sum_{\mu} \widehat{\varphi}_n(\mu) \mu^{-1}(\epsilon).$$

The sum on the right is in reality finite. Substituting we obtain

$$\widehat{\varphi}'(\nu, t) = \sum_n t^n \left\{ \sum_{\mu} \int_{U_F} \mu^{-1} \nu(\epsilon) \psi(\epsilon \varpi^n x) d\epsilon \widehat{\varphi}_n(\mu) \right\}$$

as asserted.

Suppose  $\nu$  is a character of  $U_F$  and  $\varphi$  in  $V_0$  is such that  $\widehat{\varphi}(\mu, t) = 0$  unless  $\mu = \nu^{-1} \nu_0^{-1}$ . This means that

$$\varphi(a\epsilon) \equiv \nu \nu_0(\epsilon) \varphi(a)$$

or that

$$\pi \left( \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \right) \varphi = \nu \nu_0(\epsilon) \varphi$$

for all  $\epsilon$  in  $U_F$ . If  $\varphi' = \pi(w) \varphi$  then

$$\pi \left( \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \right) \varphi' = \pi \left( \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \right) \pi(w) \varphi = \pi(w) \pi \left( \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \right) \varphi.$$

Since

$$\left( \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \right) = \left( \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \right) \left( \begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right)$$

the expression on the right is equal to

$$\nu^{-1}(\epsilon) \pi(w) \varphi = \nu^{-1}(\epsilon) \varphi',$$

so that  $\widehat{\varphi}'(\mu, t) = 0$  unless  $\mu = \nu$ .

Now take a vector  $u$  in  $X$  and a character  $\nu$  of  $U_F$  and let  $\varphi$  be the function in  $V_0$  which is zero outside of  $U_F$  and on  $U_F$  is given by

$$(2.10.2) \quad \varphi(\epsilon) = \nu(\epsilon) \nu_0(\epsilon) u.$$

If  $\varphi' = \pi(w) \varphi$  then  $\widehat{\varphi}'_n$  is a function of  $n$ ,  $\nu$ , and  $u$  which depends linearly on  $u$  and we may write

$$\widehat{\varphi}'_n(\nu) = C_n(\nu) u$$

where  $C_n(\nu)$  is a linear operator on  $X$ .

We introduce the formal series

$$C(\nu, t) = \sum t^n C_n(\nu).$$

We have now to verify the third formula of the proposition. Since  $\varphi$  is in  $V_0$  the product on the right is defined. Since both sides are linear in  $\varphi$  we need only verify it for a set of generators of  $V_0$ . This set can be taken to be the functions defined by (2.10.2) together with their translates by powers of  $\varpi$ . For functions of the form (2.10.2) the formula is valid

because of the way the various series  $C(\nu, t)$  were defined. Thus all we have to do is show that if the formula is valid for a given function  $\varphi$  it remains valid when  $\varphi$  is replaced by

$$\pi \left( \begin{pmatrix} \varpi^\ell & 0 \\ 0 & 1 \end{pmatrix} \right) \varphi.$$

By part (i) the right side is replaced by

$$z_0^\ell t^\ell C(\nu, t) \widehat{\varphi}(\nu^{-1} \nu_0^{-1}, t^{-1} z_0^{-1}).$$

Since

$$\pi(w) \pi \left( \begin{pmatrix} \varpi^\ell & 0 \\ 0 & 1 \end{pmatrix} \right) \varphi = \pi \left( \begin{pmatrix} 1 & 0 \\ 0 & \varpi^\ell \end{pmatrix} \right) \pi(w) \varphi$$

and  $\pi(w) \widehat{\varphi}(\nu, t)$  is known we can use part (i) and the relation

$$\begin{pmatrix} 1 & 0 \\ 0 & \varpi^\ell \end{pmatrix} = \begin{pmatrix} \varpi^\ell & 0 \\ 0 & \varpi^\ell \end{pmatrix} \begin{pmatrix} \varpi^{-\ell} & 0 \\ 0 & 1 \end{pmatrix}$$

to see that the left side is replaced by

$$z_0^\ell t^\ell \pi(w) \widehat{\varphi}(\nu, t) = z_0^\ell t^\ell C(\nu, t) \widehat{\varphi}(\nu^{-1} \nu_0^{-1}, t^{-1} z_0^{-1}).$$

For a given  $u$  in  $X$  and a given character  $\nu$  of  $U_F$  there must exist a  $\varphi$  in  $V$  such that

$$\widehat{\varphi}(\nu, t) = \sum t^n C_n(\nu) u$$

Consequently there is an  $n_0$  such that  $C_n(\nu) u = 0$  for  $n < n_0$ . Of course  $n_0$  may depend on  $u$  and  $\nu$ . This observation together with standard properties of Gaussian sums shows that the infinite sums occurring in the following proposition are meaningful, for when each term is multiplied on the right by a fixed vector in  $X$  all but finitely many disappear.

**Proposition 2.11.** *Let  $\mathfrak{p}^{-\ell}$  be the largest ideal on which  $\psi$  is trivial.*

- (i) *Let  $\nu$  and  $\rho$  be two characters of  $U_F$  such that  $\nu \rho \nu_0$  is not 1. Let  $\mathfrak{p}^m$  be its conductor. Then*

$$\sum_{\sigma} \eta(\sigma^{-1} \nu, \varpi^n) \eta(\sigma^{-1} \rho, \varpi^p) C_{p+n}(\sigma)$$

*is equal to*

$$\eta(\nu^{-1} \rho^{-1} \nu_0^{-1}, \varpi^{-m-\ell}) z_0^{m+\ell} \nu \rho \nu_0(-1) C_{n-m-\ell}(\nu) C_{p-m-\ell}(\rho)$$

*for all integers  $n$  and  $p$ .*

- (ii) *Let  $\nu$  be any character of  $U_F$  and let  $\tilde{\nu} = \nu^{-1} \nu_0^{-1}$ . Then*

$$\sum_{\sigma} \eta(\sigma^{-1} \nu, \varpi^n) \eta(\sigma^{-1} \tilde{\nu}, \varpi p) C_{p+n}(\sigma)$$

*is equal to*

$$z_0^p \nu_0(-1) \delta_{n,p} + (|\varpi| - 1)^{-1} z_0^{\ell+1} C_{n-1-\ell}(\nu) C_{p-1-\ell}(\tilde{\nu}) - \sum_{-2-\ell}^{-\infty} z^{-r} C_{n+r}(\nu) C_{p+r}(\tilde{\nu})$$

*for all integers  $n$  and  $p$ .*

The left hand sums are taken over all characters  $\sigma$  of  $U_F$  and  $\delta_{n,p}$  is Kronecker's delta. The relation

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = - \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

implies that

$$\pi(w)\pi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)\pi(w)\varphi = \nu_0(-1)\pi\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right)\pi(w)\pi\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right)\varphi$$

for all  $\varphi$  in  $V_0$ . Since  $\pi(w)\varphi$  is not necessarily in  $V_0$  we write this relation as

$$\begin{aligned} \pi(w)\left\{\pi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)\pi(w)\varphi - \pi(w)\varphi\right\} + \pi^2(w)\varphi \\ = \nu_0(-1)\pi\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right)\pi(w)\pi\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right)\varphi. \end{aligned}$$

The term  $\pi^2(w)\varphi$  is equal to  $\nu_0(-1)\varphi$ .

We compute the Mellin transforms of both sides

$$\pi\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right)\widehat{\varphi}(\nu, t) = \sum_n t^n \left\{ \sum_\rho \eta(\rho^{-1}\nu, -\varpi^n) \widehat{\varphi}_n(\rho) \right\}$$

and

$$\pi(w)\pi\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right)\widehat{\varphi}(\nu, t) = \sum_n t^n \sum_{p,\rho} \eta(\rho^{-1}\nu^{-1}\nu_0^{-1}, -\varpi^p) z_0^{-p} C_{p+n}(\nu) \widehat{\varphi}_p(\rho)$$

so that the Mellin transform of the right side is

$$(2.11.1) \quad \nu_0(-1) \sum_n t^n \sum_{p,\rho,\sigma} \eta(\sigma^{-1}\nu, -\varpi^n) \eta(\rho^{-1}\sigma^{-1}\nu_0^{-1}, -\varpi^p) z_0^{-p} C_{p+n}(\sigma) \widehat{\varphi}_p(\rho).$$

On the other hand

$$\pi(w)\widehat{\varphi}(\nu, t) = \sum_n t^n \sum_p z_0^{-p} C_{p+n}(\nu) \widehat{\varphi}_p(\nu^{-1}\nu_0^{-1})$$

and

$$\pi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)\pi(w)\widehat{\varphi}(\nu, t) = \sum_n t^n \sum_{p,\rho} z_0^{-p} \eta(\rho^{-1}\nu, \varpi^n) C_{p+n}(\rho) \widehat{\varphi}_p(\rho^{-1}\nu_0^{-1})$$

so that

$$\pi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)\pi(w)\widehat{\varphi}(\nu, t) - \pi(w)\widehat{\varphi}(\nu, t)$$

is equal to

$$\sum_n t^n \sum_{p,\rho} z_0^{-p} [\eta(\rho\nu\nu_0, \varpi^n) - \delta(\rho\nu\nu_0)] C_{p+n}(\rho^{-1}\nu_0^{-1}) \widehat{\varphi}_p(\rho).$$



Here  $\delta(\rho\nu\nu_0)$  is 1 if  $\rho\nu\nu_0$  is the trivial character and 0 otherwise. The Mellin transform of the left hand side is therefore

$$(2.11.2) \quad \sum_{p,r,\rho} t^n \sum z_0^{-p-r} [\eta(\rho\nu^{-1}, \varpi^r) - \delta(\rho\nu^{-1})] C_{n+r}(\nu) C_{p+r}(\rho^{-1}\nu_0^{-1}) \widehat{\varphi}_p(\rho) + \nu_0(-1) \sum t^n \widehat{\varphi}_n(\nu).$$

The coefficient of  $t^n \widehat{\varphi}_p(\rho)$  in (2.11.1) is

$$(2.11.3) \quad \nu_0(-1) \sum_{\sigma} \eta(\sigma^{-1}\nu, -\varpi^n) \eta(\rho^{-1}\sigma^{-1}\nu_0^{-1}, -\varpi^p) z_0^{-1} C_{p+n}(\sigma)$$

and in (2.11.2) it is

$$(2.11.4) \quad \sum_r [\eta(\rho\nu^{-1}, \varpi^r) - \delta(\rho\nu^{-1})] z_0^{-p-r} C_{n+r}(\nu) C_{p+r}(\rho^{-1}\nu_0^{-1}) + \nu_0(-1) \delta_{n,\rho} \delta(\rho\nu^{-1}) I$$

These two expressions are equal for all choice of  $n, p, \rho$ , and  $\nu$ .

If  $\rho \neq \nu$  and the conductor of  $\nu\rho^{-1}$  is  $\mathfrak{p}^m$  the gaussian sum  $\eta(\rho\nu^{-1}, \varpi^r)$  is zero unless  $r = -m - \ell$ . Thus (2.11.4) reduces to

$$\eta(\rho\nu^{-1}, \varpi^{-m-\ell}) z_0^{-p-m-\ell} C_{n-m-\ell}(\nu) C_{p-m-\ell}(\rho^{-1}\nu_0^{-1}).$$

Since

$$\eta(\mu, -x) = \mu(-1) \eta(\mu, x)$$

the expression (2.11.3) is equal to

$$\rho^{-1}\nu(-1) \sum_{\sigma} \eta(\sigma^{-1}\nu, \varpi^n) \eta(\rho^{-1}\sigma^{-1}\nu_0^{-1}\varpi^p) z_0^{-p} C_{p+n}(\sigma).$$

Replacing  $\rho$  by  $\rho^{-1}\nu_0^{-1}$  we obtain the first part of the proposition.

If  $\rho = \nu$  then  $\delta(\rho\nu^{-1}) = 1$ . Moreover, as is well-known and easily verified,  $\eta(\rho\nu^{-1}, \varpi^r) = 1$  if  $r \geq -\ell$ ,

$$\eta(\rho\nu^{-1}, \varpi^{-\ell-1}) = |\varpi|(|\varpi| - 1)^{-1}$$

and  $\eta(\rho\nu^{-1}, \varpi^r) = 0$  if  $r \leq -\ell - 2$ . Thus (2.11.4) is equal to

$$\begin{aligned} \nu_0(-1) \delta_{n,p} I + (|\varpi| - 1)^{-1} z_0^{-p+\ell+1} C_{n-\ell-1}(\nu) C_{n-\ell-1}(\nu^{-1}\nu_0^{-1}) \\ - \sum_{r=-\ell-2}^{-\infty} z_0^{-p-r} C_{n+r}(\nu) C_{n+r}(\nu^{-1}\nu_0^{-1}). \end{aligned}$$

The second part of the proposition follows.

**Proposition 2.12.**

(i) For every  $n, p, \nu$  and  $\rho$

$$C_n(\nu) C_p(\rho) = C_p(\rho) C_n(\nu)$$

(ii) There is no non-trivial subspace of  $X$  invariant under all the operators  $C_n(\nu)$ .

(iii) The space  $X$  is one-dimensional.

Suppose  $\rho\nu\nu_0 \neq 1$ . The left side of the first identity in the previous proposition is symmetric in the two pairs  $(n, \nu)$  and  $(p, \rho)$ . Since  $\eta(\nu^{-1}\rho^{-1}\nu_0^{-1}, \varpi^{-m-\ell})$  is not zero we conclude that

$$C_{n-m-\ell}(\nu) C_{p-m-\ell}(\rho) = C_{p-m-\ell}(\rho) C_{n-m-\ell}(\nu)$$

for all choices of  $n$  and  $p$ . The first part of the proposition is therefore valid in  $\rho \neq \tilde{\nu}$ .

Now suppose  $\rho = \tilde{\nu}$ . We are going to prove that if  $(p, n)$  is a given pair of integers and  $u$  belongs to  $X$  then

$$C_{n+r}(\nu)C_{p+r}(\tilde{\nu})u = C_{p+r}(\tilde{\nu})C_{n+r}(\nu)u$$

for all  $r$  in  $\mathbf{Z}$ . If  $r \ll 0$  both sides are 0 and the relation is valid so the proof can proceed by induction on  $r$ . For the induction one uses the second relation of Proposition 2.11 in the same way as the first was used above.

Suppose  $X_1$  is a non-trivial subspace of  $X$  invariant under all the operators  $C_n(\nu)$ . Let  $V_1$  be the space of all functions in  $V_0$  which take values in  $X_1$  and let  $V'_1$  be the invariant subspace generated by  $V_1$ . We shall show that all functions in  $V'_1$  take values in  $X_1$  so that  $V'_1$  is a non-trivial invariant subspace of  $V$ . This will be a contradiction. If  $\varphi$  in  $V$  takes value in  $X_1$  and  $g$  belongs to  $P_F$  then  $\pi(g)\varphi$  also takes values in  $X_1$ . Therefore all we need to do is show that if  $\varphi$  is in  $V_1$  then  $\pi(w)\varphi$  takes values in  $X_1$ . This follows immediately from the assumption and Proposition 2.10.

To prove (iii) we show that the operators  $C_n(\nu)$  are all scalar multiples of the identity. Because of (i) we need only show that every linear transformation of  $X$  which commutes with all the operators  $C_n(\nu)$  is a scalar. Suppose  $T$  is such an operator. If  $\varphi$  belongs to  $V$  let  $T_\varphi$  be the function from  $F^\times$  to  $X$  defined by

$$T_\varphi(a) = T(\varphi(a)).$$

Observe that  $T\varphi$  is still in  $V$ . This is clear if  $\varphi$  belongs to  $V_0$  and if  $\varphi = \pi(w)\varphi_0$  we see on examining the Mellin transforms of both sides that

$$T_\varphi = \pi(w)T_{\varphi_0}.$$

Since  $V = V_0 + \pi(w)V_0$  the observation follows.  $T$  therefore defines a linear transformation of  $V$  which clearly commutes with the action of any  $g$  in  $P_F$ . If we can show that it commutes with the action of  $w$  it will follow that it and, therefore, the original operator on  $X$  are scalars. We have to verify that

$$\pi(w)T_\varphi = T\pi(w)\varphi$$

at least for  $\varphi$  on  $V_0$  and for  $\varphi = \pi(w)\varphi_0$  with  $\varphi_0$  in  $V_0$ . We have already seen that the identity holds for  $\varphi$  in  $V_0$ . Thus if  $\varphi = \pi(w)\varphi_0$  the left side is

$$\pi(w)T\pi(w)\varphi_0 = \pi^2(w)T_{\varphi_0} = \nu_0(-1)T_{\varphi_0}$$

and the right side is

$$T\pi^2(w)\varphi_0 = \nu_0(-1)T_{\varphi_0}.$$

Because of this proposition we can identify  $X$  with  $\mathbf{C}$  and regard the operators  $C_n(\nu)$  as complex numbers. For each  $r$  the formal Laurent series  $C(\nu, t)$  has only finitely many negative terms. We now want to show that the realization of  $\pi$  on a space of functions on  $F^\times$  is, when certain simple conditions are imposed, unique so that the series  $C(\nu, t)$  are determined by the class of  $\pi$  and that conversely the series  $C(\nu, t)$  determine the class of  $\pi$ .

**Theorem 2.13.** *Suppose an equivalence class of infinite-dimensional irreducible admissible representations of  $G_F$  is given. Then there exists exactly one space  $V$  of complex-valued functions on  $F^\times$  and exactly one representation  $\pi$  of  $G_F$  on  $V$  which is in this class and which is such that*

$$\pi(b)\varphi = \xi_\psi(b)\varphi$$

*if  $b$  is in  $B_F$  and  $\varphi$  is in  $V$ .*

We have proved the existence of one such  $V$  and  $\pi$ . Suppose  $V'$  is another such space of functions and  $\pi'$  a representation of  $G_F$  on  $V'$  which is equivalent to  $\pi$ . We suppose of course that

$$\pi'(b)\varphi = \xi_\psi(b)\varphi$$

if  $b$  is in  $B_F$  and  $\varphi$  is in  $V'$ . Let  $A$  be an isomorphism of  $V$  with  $V'$  such that  $A\pi(g) = \pi'(g)A$  for all  $g$ . Let  $L$  be the linear functional

$$L(\varphi) = A\varphi(1)$$

on  $V$ . Then

$$L\left(\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi\right) = A\varphi(a)$$

so that  $A$  is determined by  $L$ . If we could prove the existence of a scalar  $\lambda$  such that  $L(\varphi) = \lambda\varphi(1)$  it would follow that

$$A\varphi(a) = \lambda\varphi(a)$$

for all  $a$  such that  $A\varphi = \lambda\varphi$ . This equality of course implies the theorem.

Observe that

$$(2.13.1) \quad L\left(\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi\right) = \pi'\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)A\varphi(1) = \psi(x)L(\varphi).$$

Thus we need the following lemma.

**Lemma 2.13.2.** *If  $L$  is a linear functional on  $V$  satisfying (2.13.1) there is a scalar  $\lambda$  such that*

$$L(\varphi) = \lambda\varphi(1).$$

This is a consequence of a slightly different lemma.

**Lemma 2.13.3.** *Suppose  $L$  is a linear functional on the space  $\mathcal{S}(F^\times)$  of locally constant compactly supported functions on  $F^\times$  such that*

$$L\left(\xi_\psi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi\right) = \psi(x)L(\varphi)$$

*for all  $\varphi$  in  $\mathcal{S}(F^\times)$  and all  $x$  in  $F$ . Then there is a scalar  $\lambda$  such that  $L(\varphi) = \lambda\varphi(1)$ .*

Suppose for a moment that the second lemma is true. Then given a linear functional  $L$  on  $V$  satisfying (2.13.1) there is a  $\lambda$  such that  $L(\varphi) = \lambda\varphi(1)$  for all  $\varphi$  in  $V_0 = \mathcal{S}(F^\times)$ . Take  $x$  in  $F$  such that  $\psi(x) \neq 1$  and  $\varphi$  in  $V$ . Then

$$L(\varphi) = L\left(\varphi - \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi\right) + L\left(\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi\right).$$

Since

$$\varphi - \pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi$$

is in  $V_0$  the right side is equal to

$$\lambda\varphi(1) - \lambda\psi(x)\varphi(1) + \psi(x)L(\varphi)$$

so that

$$(1 - \psi(x))L(\varphi) = \lambda(1 - \psi(x))\varphi(1)$$

which implies that  $L(\varphi) = \lambda\varphi(1)$ .

To prove the second lemma we have only to show that  $\varphi(1) = 0$  implies  $L(\varphi) = 0$ . If we set  $\varphi(0) = 0$  then  $\varphi$  becomes a locally constant function with compact support in  $F$ . Let  $\varphi'$  be its Fourier transform so that

$$\varphi(a) = \int_F \psi(ba)\varphi'(-b) db.$$

Let  $\Omega$  be an open compact subset of  $F^\times$  containing 1 and the support of  $\varphi$ . There is an ideal  $\mathfrak{a}$  in  $F$  so that for all  $a$  in  $\Omega$  the function  $\varphi'(-b)\psi(ba)$  is constant on the cosets of  $\mathfrak{a}$  in  $F$ . Choose an ideal  $\mathfrak{b}$  containing  $\mathfrak{a}$  and the support of  $\varphi'$ . If  $S$  is a set of representatives of  $\mathfrak{b}/\mathfrak{a}$  and if  $c$  is the measure of  $\mathfrak{a}$  then

$$\varphi(a) = c \sum_{b \in S} \psi(ba)\varphi'(-b).$$

If  $\varphi_0$  is the characteristic function of  $\Omega$  this relation may be written

$$\varphi = \sum_{b \in S} \lambda_b \xi_\psi \left( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) \varphi_0$$

with  $\lambda_b = c\varphi'(-b)$ . If  $\varphi(1) = 0$  then

$$\sum_{b \in S} \lambda_b \psi(b) = 0$$

so that

$$\varphi = \sum \lambda_b \left\{ \xi_\psi \left( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) \varphi_0 - \psi(b)\varphi_0 \right\}$$

It is clear that  $L(\varphi) = 0$ .

The representation of the theorem will be called the Kirillov model. There is another model which will be used extensively. It is called the Whittaker model. Its properties are described in the next theorem.

**Theorem 2.14.**

(i) For any  $\varphi$  in  $V$  set

$$W_\varphi(g) = (\pi(g)\varphi)(1)$$

so that  $W_\varphi$  is a function in  $G_F$ . Let  $W(\pi, \psi)$  be the space of such functions. The map  $\varphi \rightarrow W_\varphi$  is an isomorphism of  $V$  with  $W(\pi, \psi)$ . Moreover

$$W_{\pi(g)\varphi} = \rho(g)W_\varphi$$

(ii) Let  $W(\psi)$  be the space of all functions  $W$  on  $G_F$  such that

$$W \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) g = \psi(x)W(g)$$

for all  $x$  in  $F$  and  $g$  in  $G$ . Then  $W(\pi, \psi)$  is contained in  $W(\psi)$  and is the only invariant subspace which transforms according to  $\pi$  under right translations.

Since

$$W_\varphi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = \left(\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi\right)(1) = \varphi(a)$$

the function  $W_\varphi$  is 0 only if  $\varphi$  is. Since

$$\rho(g)W(h) = W(hg)$$

the relation

$$W_{\pi(g)\varphi} = \rho(g)W_\varphi$$

is clear. Then  $W(\pi, \psi)$  is invariant under right translation and transforms according to  $\pi$ .

Since

$$W_\varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}g\right) = \left(\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\pi(g)\varphi\right)(1) = \psi(x)\{\pi(g)\varphi(1)\}$$

the space  $W(\pi, \psi)$  is contained in  $W(\psi)$ . Suppose  $W$  is an invariant subspace of  $W(\psi)$  which transforms according to  $\pi$ . There is an isomorphism  $A$  of  $V$  with  $W$  such that

$$A(\pi(g)\varphi) = \rho(g)(A\varphi).$$

Let

$$L(\varphi) = A\varphi(1).$$

Since

$$L(\pi(g)\varphi) = A\pi(g)\varphi(1) = \rho(g)A\varphi(1) = A\varphi(g)$$

the map  $A$  is determined by  $L$ . Also

$$L\left(\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi\right) = A\varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = \psi(x)A\varphi(1) = \psi(x)L(\varphi)$$

so that by Lemma 2.13.2 there is a scalar  $\lambda$  such that

$$L(\varphi) = \lambda\varphi(1).$$

Consequently  $A\varphi = \lambda W_\varphi$  and  $W = W(\pi, \psi)$ .

The realization of  $\pi$  on  $W(\pi, \psi)$  will be called the Whittaker model. Observe that the representation of  $G_F$  on  $W(\psi)$  contains no irreducible finite-dimensional representations. In fact any such representation is of the form

$$\pi(g) = \chi(\det g).$$

If  $\pi$  were contained in the representation on  $W(\psi)$  there would be a nonzero function  $W$  on  $G_F$  such that

$$W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}g\right) = \psi(x)\chi(\det g)W(e)$$

In particular taking  $g = e$  we find that

$$W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = \psi(x)W(e)$$

However it is also clear that

$$W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = \chi\left(\det\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)W(e) = W(e)$$

so that  $\psi(x) = 1$  for all  $x$ . This is a contradiction. We shall see however that  $\pi$  is a constituent of the representation on  $W(\psi)$ . That is, there are two invariant subspaces  $W_1$  and  $W_2$  of  $W(\psi)$  such that  $W_1$  contains  $W_2$  and the representation of the quotient space  $W_1/W_2$  is equivalent to  $\pi$ .

**Proposition 2.15.** *Let  $\pi$  and  $\pi'$  be two infinite-dimensional irreducible representations of  $G_F$  realized in the Kirillov form on spaces  $V$  and  $V'$ . Assume that the two quasi-characters defined by*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I \quad \pi'\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega'(a)I$$

*are the same. Let  $\{C(\nu, t)\}$  and  $\{C'(\nu, t)\}$  be the families of formal series associated to the two representations. If*

$$C(\nu, t) = C'(\nu, t)$$

*for all  $\nu$  then  $\pi = \pi'$ .*

If  $\varphi$  belongs to  $\mathcal{S}(F^\times)$  then, by hypothesis,

$$\pi(w)\widehat{\varphi}(\nu, t) = \pi'(w)\widehat{\varphi}(\nu, t)$$

so that  $\pi(w)\varphi = \pi'(w)\varphi$ . Since  $V$  is spanned by  $\mathcal{S}(F^\times)$  and  $\pi(w)\mathcal{S}(F^\times)$  and  $V'$  is spanned by  $\mathcal{S}(F^\times)$  and  $\pi'(w)\mathcal{S}(F^\times)$  the spaces  $V$  and  $V'$  are the same. We have to show that

$$\pi(g)\varphi = \pi'(g)\varphi$$

for all  $\varphi$  in  $V$  and all  $g$  in  $G_F$ . This is clear if  $g$  is in  $P_F$  so it is enough to verify it for  $g = w$ . We have already observed that  $\pi(w)\varphi_0 = \pi'(w)\varphi_0$  if  $\varphi_0$  is in  $\mathcal{S}(F^\times)$  so we need only show that  $\pi(w)\varphi = \pi'(w)\varphi$  if  $\varphi$  is of the form  $\pi(w)\varphi_0$  with  $\varphi_0$  in  $\mathcal{S}(F^\times)$ . But  $\pi(w)\varphi = \pi^2(w)\varphi_0 = \omega(-1)\varphi_0$  and, since  $\pi(w)\varphi_0 = \pi'(w)\varphi_0$ ,  $\pi'(w)\varphi = \omega'(-1)\varphi_0$ .

Let  $N_F$  be the group of matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

with  $x$  in  $F$  and let  $\mathcal{B}$  be the space of functions on  $G_F$  invariant under left translations by elements of  $N_F$ .  $\mathcal{B}$  is invariant under right translations and the question of whether or not a given irreducible representation  $\pi$  is contained in  $\mathcal{B}$  arises. The answer is obviously positive when  $\pi = \chi$  is one-dimensional for then the function  $g \rightarrow \chi(\det g)$  is itself contained in  $\mathcal{B}$ .

Assume that the representation  $\pi$  which is given in the Kirillov form acts on  $\mathcal{B}$ . Then there is a map  $A$  of  $V$  into  $\mathcal{B}$  such that

$$A\pi(g)\varphi = \rho(g)A\varphi$$

If  $L(\varphi) = A\varphi(1)$  then

$$(2.15.1) \quad L\left(\xi_\psi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi\right) = L(\varphi)$$

for all  $\varphi$  in  $V$  and all  $x$  in  $F$ . Conversely given such a linear form the map  $\varphi \rightarrow A\varphi$  defined by

$$A\varphi(g) = L(\pi(g)\varphi)$$

satisfies the relation  $A\pi(g) = \rho(g)A$  and takes  $V$  into  $\mathcal{B}$ . Thus  $\pi$  is contained in  $\mathcal{B}$  if and only if there is a non-trivial linear form  $L$  on  $V$  which satisfies (2.15.1).

**Lemma 2.15.2.** *If  $L$  is a linear form on  $\mathcal{S}(F^\times)$  which satisfies (2.15.1) for all  $x$  in  $F$  and for all  $\varphi$  in  $\mathcal{S}(F^\times)$  then  $L$  is zero.*

We are assuming that  $L$  annihilates all functions of the form

$$\xi_\psi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \varphi - \varphi$$

so it will be enough to show that they span  $\mathcal{S}(F^\times)$ . If  $\varphi$  belongs to  $\mathcal{S}(F^\times)$  we may set  $\varphi(0) = 0$  and regard  $\varphi$  as an element of  $\mathcal{S}(F)$ . Let  $\varphi'$  be its Fourier transform so that

$$\varphi(x) = \int_F \varphi'(-b)\psi(bx) db.$$

Let  $\Omega$  be an open compact subset of  $F^\times$  containing the support of  $\varphi$  and let  $\mathfrak{p}^{-n}$  be an ideal containing  $\Omega$ . There is an ideal  $\mathfrak{a}$  of  $F$  such that  $\varphi'(-b)\psi(bx)$  is, as a function of  $b$ , constant on cosets of  $\mathfrak{a}$  for all  $x$  in  $\mathfrak{p}^{-n}$ . Let  $\mathfrak{b}$  be an ideal containing both  $\mathfrak{a}$  and the support of  $\varphi'$ . If  $S$  is a set of representatives for the cosets of  $\mathfrak{a}$  in  $\mathfrak{b}$ , if  $c$  is the measure of  $\mathfrak{a}$ , and if  $\varphi_0$  is the characteristic function of  $\Omega$  then

$$\varphi(x) = \sum_{b \in S} \lambda_b \psi(bx) \varphi_0(x)$$

if  $\lambda_b = c\varphi'(-b)$ . Thus

$$\varphi = \sum_b \lambda_b \xi_\psi \left( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) \varphi_0.$$

Since  $\varphi(0) = 0$  we have

$$\sum_b \lambda_b = 0$$

so that

$$\varphi = \sum_b \lambda_b \left\{ \xi_\psi \left( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) \varphi_0 - \varphi_0 \right\}$$

as required.

Thus any linear form on  $V$  verifying (2.15.1) annihilates  $\mathcal{S}(F^\times)$ . Conversely any linear form on  $V$  annihilating  $\mathcal{S}(F^\times)$  satisfies (2.15.1) because

$$\xi_\psi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \varphi - \varphi$$

is in  $\mathcal{S}(F^\times)$  if  $\varphi$  is in  $V$ . We have therefore proved

**Proposition 2.16.** *For any infinite-dimensional irreducible representation  $\pi$  the following two properties are equivalent:*

(i)  $\pi$  is not contained in  $\mathcal{B}$ .

(ii) The Kirillov model of  $\pi$  is realized in the space  $\mathcal{S}(F^\times)$ .

A representation satisfying these two conditions will be called absolutely cuspidal.

**Lemma 2.16.1.** *Let  $\pi$  be an infinite-dimensional irreducible representation realized in the Kirillov form on the space  $V$ . Then  $V_0 = \mathcal{S}(F^\times)$  is of finite codimension in  $V$ .*

We recall that  $V = V_0 + \pi(w)V_0$ . Let  $V_1$  be the space of all  $\varphi$  in  $V_0$  with support in  $U_F$ . An element of  $\pi(w)V_0$  may always be written as a linear combination of functions of the form

$$\pi\left(\begin{pmatrix} \varpi^p & 0 \\ 0 & 1 \end{pmatrix}\right)\pi(w)\varphi$$

with  $\varphi$  in  $V_1$  and  $p$  in  $\mathbf{Z}$ . For each character  $\mu$  of  $U_F$  let  $\varphi_\mu$  be the function in  $V_1$  such that  $\varphi_\mu(\epsilon) = \mu(\epsilon)\nu_0(\epsilon)$  for  $\epsilon$  in  $U_F$ . Then

$$\widehat{\varphi}_\mu(\nu, t) = \delta(\nu\mu\nu_0)$$

and

$$\pi(w)\widehat{\varphi}_\mu(\nu, t) = \delta(\nu\mu^{-1})C(\nu, t).$$

Let  $\eta_\mu = \pi(w)\varphi_\mu$ . The space  $V$  is spanned by  $V_0$  and the functions

$$\pi\left(\begin{pmatrix} \varpi^p & 0 \\ 0 & 1 \end{pmatrix}\right)\eta_\mu$$

For the moment we take the following two lemmas for granted.

**Lemma 2.16.2.** *For any character  $\mu$  of  $\widehat{U}_F$  there is an integer  $n_0$  and a family of constants  $\lambda_i$ ,  $1 \leq i \leq p$ , such that*

$$C_n(\mu) = \sum_{i=1}^p \lambda_i C_{n-i}(\mu)$$

for  $n \geq n_0$ .

**Lemma 2.16.3.** *There is a finite set  $S$  of characters of  $U_F$  such that for  $\nu$  not in  $S$  the numbers  $C_n(\nu)$  are 0 for all but finitely many  $n$ .*

If  $\mu$  is not in  $S$  the function  $\eta_\mu$  is in  $V_0$ . Choose  $\mu$  in  $S$  and let  $V_\mu$  be the space spanned by the functions

$$\pi\left(\begin{pmatrix} \varpi^p & 0 \\ 0 & 1 \end{pmatrix}\right)\eta_\mu$$

and the functions  $\varphi$  in  $V_0$  satisfying  $\varphi(a\epsilon) = \varphi(a)\mu^{-1}(\epsilon)$  for all  $a$  in  $F^\times$  and all  $\epsilon$  in  $U_F$ . It will be enough to show that  $V_\mu/(V_\mu \cap V_0)$  is finite-dimensional.

If  $\varphi$  is in  $V_\mu$  then  $\widehat{\varphi}(\nu, t) = 0$  unless  $\nu = \mu$  and we may identify  $\varphi$  with the sequence  $\{\widehat{\varphi}_n(\mu)\}$ . The elements of  $V_\mu \cap V_0$  are the elements corresponding to sequences with only finitely many nonzero terms. Referring to Proposition 2.10 we see that all of the sequences satisfy the recursion relation

$$\widehat{\varphi}_n(\mu) = \sum_{i=1}^p \lambda_i \widehat{\varphi}_{n-i}(\mu)$$

for  $n \geq n_1$ . The integer  $n_1$  depends on  $\varphi$ .



Lemma 2.16.1 is therefore a consequence of the following elementary lemma whose proof we postpone to Paragraph 8.

**Lemma 2.16.4.** *Let  $\lambda_i$ ,  $1 \leq i \leq p$ , be  $p$  complex numbers. Let  $A$  be the space of all sequences  $\{a_n\}$ ,  $n \in \mathbf{Z}$  for which there exist two integers  $n_1$  and  $n_2$  such that*

$$a_n = \sum_{1 \leq i \leq p} \lambda_i a_{n-i}$$

*for  $n \geq n_1$  and such that  $a_n = 0$  for  $n \leq n_2$ . Let  $A_0$  be the space of all sequences with only a finite number of nonzero terms. Then  $A/A_0$  is finite-dimensional.*

We now prove Lemma 2.16.2. According to Proposition 2.11

$$\sum_{\sigma} \eta(\sigma^{-1}\nu, \varpi^n) \eta(\sigma^{-1}\tilde{\nu}, \varpi^p) C_{p+n}(\sigma)$$

is equal to

$$z_0^p \nu_0(-1) \delta_{n,p} + (|\varpi| - 1)^{-1} z_0^{\ell+1} C_{n-1-\ell}(\nu) C_{p-1-\ell}(\tilde{\nu}) - \sum_{-2-\ell}^{-\infty} z_0^{-r} C_{n+r}(\nu) C_{p+r}(\tilde{\nu}).$$

Remember that  $\mathfrak{p}^{-\ell}$  is the largest ideal on which  $\psi$  is trivial. Suppose first that  $\tilde{\nu} = \nu$ .

Take  $p = -\ell$  and  $n > -\ell$ . Then  $\delta(n-p) = 0$  and

$$\eta(\sigma^{-1}\nu, \varpi^n) \eta(\sigma^{-1}\nu, \varpi^p) = 0$$

unless  $\sigma = \nu$ . Hence

$$C_{n-\ell}(\nu) = (|\varpi| - 1)^{-1} z_0^{\ell+1} C_{n-1-\ell}(\nu) C_{-2\ell-1}(\nu) - \sum_{-2-\ell}^{-\infty} z_0^{-r} C_{n+r}(\nu) C_{-\ell+r}(\nu)$$

which, since almost all of the coefficients  $C_{-\ell+r}(\nu)$  in the sum are zero, is the relation required.

If  $\nu \neq \tilde{\nu}$  take  $p \geq -\ell$  and  $n > p$ . Then  $\eta(\sigma^{-1}\nu, \varpi^n) = 0$  unless  $\sigma = \nu$  and  $\eta(\sigma^{-1}\nu, \varpi^p) = 0$  unless  $\sigma = \tilde{\nu}$ . Thus

$$(2.16.5) \quad (|\varpi| - 1)^{-1} z_0^{\ell+1} C_{n-1-\ell}(\nu) C_{p-1-\ell}(\tilde{\nu}) - \sum_{-2-\ell}^{-\infty} z_0^{-r} C_{n+r}(\nu) C_{p+r}(\tilde{\nu}) = 0.$$

There is certainly at least one  $i$  for which  $C_i(\tilde{\nu}) \neq 0$ . Take  $p-1-\ell \geq i$ . Then from (2.16.5) we deduce a relation of the form

$$C_{n+r}(\nu) = \sum_{i=1}^q \lambda_i C_{n+r-i}(\nu)$$

where  $r$  is a fixed integer and  $n$  is any integer greater than  $p$ .

Lemma 2.16.3 is a consequence of the following more precise lemma. If  $\mathfrak{p}^m$  is the conductor of a character  $\rho$  we refer to  $m$  as the order of  $\rho$ .

**Lemma 2.16.6.** *Let  $m_0$  be of the order  $\nu_0$  and let  $m_1$  be an integer greater than  $m_0$ . Write  $\nu_0$  in any manner in the form  $\nu_0 = \nu_1^{-1} \nu_2^{-1}$  where the orders of  $\nu_1$  and  $\nu_2$  are strictly less than  $m_1$ . If the order  $m$  of  $\rho$  is large enough*

$$C_{-2m-2\ell}(\rho) = \nu_2^{-1} \rho(-1) z_0^{-m-\ell} \frac{\eta(\nu_1^{-1} \rho, \varpi^{-m-\ell})}{\eta(\nu_2 \rho^{-1}, \varpi^{-m-\ell})}$$

and  $C_p(\rho) = 0$  if  $p \neq -2m - 2\ell$ .

Suppose the order of  $\rho$  is at least  $m_1$ . Then  $\rho\nu_1\nu_0 = \rho\nu_2^{-1}$  is still of order  $m$ . Applying Proposition 2.11 we see that

$$\sum_{\sigma} \eta(\sigma^{-1}\nu_1, \varpi^{n+m+\ell}) \eta(\sigma^{-1}\rho, \varpi^{p+m+\ell}) C_{p+n+2m+2\ell}(\sigma)$$

is equal to

$$\eta(\nu_1^{-1}\rho^{-1}\nu_0^{-1}, \varpi^{-m-\ell}) z_0^{m+\ell} \nu_1 \rho \nu_0(-1) C_{n-m-\ell}(\nu) C_{p-m-\ell}(\rho)$$

for all integers  $n$  and  $p$ . Choose  $n$  such that  $C_n(\nu_1) \neq 0$ . Assume also that  $m + n + \ell \geq -\ell$  or that  $m \geq -2\ell - n$ . Then  $\eta(\sigma^{-1}\nu_1, \varpi^{n+m+\ell}) = 0$  unless  $\sigma = \nu_1$  so that

$$\eta(\nu_1^{-1}\rho, \varpi^{p+m+\ell}) C_{p+n+2m+2\ell}(\nu_1) = \eta(\nu_2\rho^{-1}, \varpi^{-m-\ell}) z_0^{m+\ell} \nu_1 \rho \nu_0(-1) C_n(\nu_1) C_p(\rho).$$

Since  $\nu_1^{-1}\rho$  is still of order  $m$  the left side is zero unless  $p = -2m - 2\ell$ . The only term on the right side that can vanish is  $C_p(\rho)$ . On the other hand if  $p = -2m - 2\ell$  we can cancel the terms  $C_n(\nu_1)$  from both sides to obtain the relation of the lemma.

Apart from Lemma 2.16.4 the proof of Lemma 2.16.1 is complete. We have now to discuss its consequences. If  $\omega_1$  and  $\omega_2$  are two quasi-characters of  $F^\times$  let  $\mathcal{B}(\omega_1, \omega_2)$  be the space of all functions  $\varphi$  on  $G_F$  which satisfy

(i) For all  $g$  in  $G_F$ ,  $a_1, a_2$  in  $F^\times$ , and  $x$  in  $F$

$$\varphi\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right) = \omega_1(a_1)\omega_2(a_2) \left|\frac{a_1}{a_2}\right|^{1/2} \varphi(g).$$

(ii) There is an open subgroup  $U$  of  $\mathrm{GL}(2, O_F)$  such that  $\varphi(gu) \equiv \varphi(g)$  for all  $u$  in  $U$ .

Since

$$G_F = N_F A_F \mathrm{GL}(2, O_F)$$

where  $A_F$  is the group of diagonal matrices the elements of  $\mathcal{B}(\omega_1, \omega_2)$  are determined by their restrictions to  $\mathrm{GL}(2, O_F)$  and the second condition is tantamount to the condition that  $\varphi$  be locally constant.  $\mathcal{B}(\omega_1, \omega_2)$  is invariant under right translations by elements of  $G_F$  so that we have a representation  $\rho(\omega_1, \omega_2)$  of  $G_F$  on  $\mathcal{B}(\omega_1, \omega_2)$ . It is admissible.

**Proposition 2.17.** *If  $\pi$  is an infinite-dimensional irreducible representation of  $G_F$  which is not absolutely cuspidal then for some choice of  $\mu_1$  and  $\mu_2$  it is contained in  $\rho(\mu_1, \mu_2)$ .*

We take  $\pi$  in the Kirillov form. Since  $V_0$  is invariant under the group  $P_F$  the representation  $\pi$  defines a representation  $\sigma$  of  $P_F$  on the finite-dimensional space  $V/V_0$ . It is clear that  $\sigma$  is trivial on  $N_F$  and that the kernel of  $\sigma$  is open. The contragredient representation has the same properties. Since  $P_F/N_F$  is abelian there is a nonzero linear form  $L$  on  $V/V_0$  such that

$$\tilde{\sigma}\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix}\right) L = \mu_1^{-1}(a_1) \mu_1^{-1}(a_2) L$$

for all  $a_1, a_2$ , and  $x$ .  $\mu_1$  and  $\mu_2$  are homomorphisms of  $F^\times$  into  $\mathbf{C}^\times$  which are necessarily continuous.  $L$  may be regarded as a linear form on  $V$ . Then

$$L\left(\pi\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix}\right)\varphi\right) = \mu_1(a_1)\mu_2(a_2)L(\varphi).$$

If  $\varphi$  is in  $V$  let  $A\varphi$  be the function

$$A\varphi(g) = L(\pi(g)\varphi)$$

on  $G_F$ .  $A$  is clearly an injection of  $V$  into  $\mathcal{B}(\mu_1, \mu_2)$  which commutes with the action of  $G_F$ .

Before passing to the next theorem we make a few simple remarks. Suppose  $\pi$  is an infinite-dimensional irreducible representation of  $G_F$  and  $\omega$  is a quasi-character of  $F^\times$ . It is clear that  $W(\omega \otimes \pi, \psi)$  consists of the functions

$$g \rightarrow W(g)\omega(\det g)$$

with  $W$  in  $W(\pi, \psi)$ . If  $V$  is the space of the Kirillov model of  $\pi$  the space of the Kirillov model of  $\omega \otimes \pi$  consists of the functions  $a \rightarrow \varphi(a)\omega(a)$  with  $\varphi$  in  $V$ . To see this take  $\pi$  in the Kirillov form and observe first of all that the map  $A : \varphi \rightarrow \varphi\omega$  is an isomorphism of  $V$  with another space  $V'$  on which  $G_F$  acts by means of the representation  $\pi' = A(\omega \otimes \pi)A^{-1}$ . If

$$b \begin{pmatrix} \alpha & x \\ 0 & 1 \end{pmatrix}$$

belongs to  $B_F$  and  $\varphi' = \varphi\omega$  then

$$\pi'(b)\varphi'(a) = \omega(a)\{\omega(\alpha)\psi(ax)\varphi(\alpha a)\} = \psi(ax)\varphi'(\alpha a)$$

so that  $\pi'(b)\varphi' = \xi_\psi(b)\varphi'$ . By definition then  $\pi'$  is the Kirillov model of  $\omega \otimes \pi$ . Let  $\omega_1$  be the restriction of  $\omega$  to  $U_F$  and let  $z_1 = \omega(\varpi)$ . If  $\varphi' = \varphi\omega$  then

$$\widehat{\varphi}'(\nu, t) = \widehat{\varphi}(\nu\omega_1, z_1 t).$$

Thus

$$\pi'(w)\varphi'(\nu, t) = \pi(w)\widehat{\varphi}(\nu\omega_1, z_1 t) = C(\nu\omega_1, z_1 t)\widehat{\varphi}(\nu^{-1}\omega_1^{-1}\nu_0^{-1}, z_0^{-1}z_1^{-1}t^{-1}).$$

The right side is equal to

$$C(\nu\omega_1, z_1 t)\widehat{\varphi}'(\nu^{-1}\nu_0^{-1}\omega_1^{-2}, z_0^{-1}z_1^{-2}t^{-1})$$

so that when we replace  $\pi$  by  $\omega \otimes \pi$  we replace  $C(\nu, t)$  by  $C(\nu\omega_1, z_1 t)$ .

Suppose  $\psi'(x) = \psi(bx)$  with  $b$  in  $F^\times$  is another non-trivial additive character. Then  $W(\pi, \psi')$  consists of the functions

$$W'(g) = W\left(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}g\right)$$

with  $W$  in  $W(\pi, \psi)$ .

The last identity of the following theorem is referred to as the local functional equation. It is the starting point of our approach to the Hecke theory.

**Theorem 2.18.** *Let  $\pi$  be an irreducible infinite-dimensional admissible representation of  $G_F$ .*

(i) *If  $\omega$  is the quasi-character of  $G_F$  defined by*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I$$

*then the contragredient representation  $\widetilde{\pi}$  is equivalent to  $\omega^{-1} \otimes \pi$ .*

(ii) *There is a real number  $s_0$  such that for all  $g$  in  $G_F$  and all  $W$  in  $W(\pi, \psi)$  the integrals*

$$\int_{F^\times} W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) |a|^{s-1/2} d^\times a = \Psi(g, s, W)$$

$$\int_{F^\times} W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) |a|^{s-1/2} \omega^{-1}(a) d^\times a = \tilde{\Psi}(g, s, W)$$

*converge absolutely for  $\operatorname{Re} s > s_0$ .*

(iii) *There is a unique Euler factor  $L(s, \pi)$  with the following property: if*

$$\Psi(g, s, W) = L(s, \pi) \Phi(g, s, W)$$

*then  $\Phi(g, s, W)$  is a holomorphic function of  $s$  for all  $g$  and all  $W$  and there is at least one  $W$  in  $W(\pi, \psi)$  such that*

$$\Phi(e, s, W) = a^s$$

*where  $a$  is a positive constant.*

(iv) *If*

$$\tilde{\Psi}(g, s, W) = L(s, \pi) \tilde{\Phi}(g, s, W)$$

*there is a unique factor  $\epsilon(s, \pi, \psi)$  which, as a function of  $s$ , is an exponential such that*

$$\tilde{\Phi}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g, 1-s, W\right) = \epsilon(s, \pi, \psi) \Phi(g, s, W)$$

*for all  $g$  in  $G_F$  and all  $W$  in  $W(\pi, \psi)$ .*

To say that  $L(s, \pi)$  is an Euler factor is to say that  $L(s, \pi) = P^{-1}(q^{-s})$  where  $P$  is a polynomial with constant term 1 and  $q = |\varpi|^{-1}$  is the number of elements in the residue field. If  $L(s, \pi)$  and  $L'(s, \pi)$  were two Euler factors satisfying the conditions of the lemma their quotient would be an entire function with no zero. This clearly implies uniqueness.

If  $\psi$  is replaced by  $\psi'$  where  $\psi'(x) = \psi(bx)$  the functions  $W$  are replaced by the functions  $W'$  with

$$W'(g) = W\left(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

and

$$\Psi(g, s, W') = |b|^{1/2-s} \Psi(g, s, W)$$

while

$$\tilde{\Psi}(g, s, W') = |b|^{1/2-s} \omega(b) \tilde{\Psi}(g, s, W).$$

Thus  $L(s, \pi)$  will not depend on  $\psi$ . However

$$\epsilon(s, \pi, \psi') = \omega(b) |b|^{2s-1} \epsilon(s, \pi, \psi).$$

According to the first part of the theorem if  $W$  belongs to  $W(\pi, \psi)$  the function

$$\widetilde{W}(g) = W(g) \omega^{-1}(\det g)$$

is in  $W(\pi, \psi)$ . It is clear that

$$\tilde{\Psi}(g, s, W) = \omega(\det g) \Psi(g, s, \widetilde{W})$$

so that if the third part of the theorem is valid when  $\pi$  is replaced by  $\tilde{\pi}$  the function  $\tilde{\Phi}(g, s, W)$  is a holomorphic function of  $s$ . Combining the functional equation for  $\pi$  and for  $\tilde{\pi}$  one sees that

$$\epsilon(s, \pi, \psi)\epsilon(1-s, \tilde{\pi}, \psi) = \omega(-1).$$

Let  $V$  be the space on which the Kirillov model of  $\pi$  acts. For every  $W$  in  $W(\pi, \psi)$  there is a unique  $\varphi$  in  $V$  such that

$$W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = \varphi(a).$$

If  $\pi$  is itself the canonical model

$$\pi(w)\varphi(a) = W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}w\right)$$

where

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

If  $\chi$  is any quasi-character of  $F^\times$  we set

$$\widehat{\varphi}(\chi) = \int_{F^\times} \varphi(a)\chi(a) d^\times a$$

if the integral converges. If  $\chi_0$  is the restriction of  $\chi$  to  $U_F$  then

$$\widehat{\varphi}(\chi) = \widehat{\varphi}(\chi_0, \chi(\varpi)).$$

Thus if  $\alpha_F$  is the quasi-character  $\alpha_F(x) = |x|$  and the appropriate integrals converge

$$\Psi(e, s, W) = \widehat{\varphi}(\alpha_F^{s-1/2}) = \widehat{\varphi}(1, q^{1/2-s})$$

$$\tilde{\Psi}(e, s, W) = \widehat{\varphi}(\alpha_F^{s-1/2}\omega^{-1}) = \widehat{\varphi}(\nu_0^{-1}, z_0^{-1}q^{1/2-s})$$

if  $\nu_0$  is the restriction of  $\omega$  to  $U_F$  and  $z_0 = \omega(\varpi)$ . Thus if the theorem is valid the series  $\widehat{\varphi}(1, t)$  and  $\widehat{\varphi}(\nu_0^{-1}, t)$  have positive radii of convergence and define functions which are meromorphic in the whole  $t$ -plane.

It is also clear that

$$\tilde{\Psi}(w, 1-s, W) = \pi(w)\widehat{\varphi}(\nu_0^{-1}, z_0^{-1}q^{s-1/2}).$$

If  $\varphi$  belongs to  $V_0$  then

$$\pi(w)\widehat{\varphi}(\nu_0^{-1}, z_0^{-1}q^{-1/2}t) = C(\nu_0^{-1}, z_0^{-1/2}q^{-1/2}t)\widehat{\varphi}(1, q^{1/2}t^{-1}).$$

Choosing  $\varphi$  in  $V_0$  such that  $\widehat{\varphi}(1, t) \equiv 1$  we see that  $C(\nu_0^{-1}, t)$  is convergent in some disc and has an analytic continuation to a function meromorphic in the whole plane.

Comparing the relation

$$\pi(w)\widehat{\varphi}(\nu_0^{-1}z_0^{-1}q^{-1/2}q^s) = C(\nu_0^{-1}, z_0^{-1/2}q^{-1/2}q^s)\widehat{\varphi}(1, q^{1/2}q^{-s})$$

with the functional equation we see that

$$(2.18.1) \quad C(\nu_0^{-1}, z_0^{-1}q^{-1/2}q^s) = \frac{L(1-s, \tilde{\pi})\epsilon(s, \pi, \psi)}{L(s, \pi)}.$$

Replacing  $\pi$  by  $\chi \otimes \pi$  we obtain the formula

$$C(\nu_0^{-1}\chi_0^{-1}, z_0^{-1}z_1^{-1}q^{-1/2}q^s) = \frac{L(1-s, \chi^{-1} \otimes \tilde{\pi})\epsilon(s, \chi \otimes \pi, \psi)}{L(s, \chi \otimes \pi)}.$$

Appealing to Proposition 2.15 we obtain the following corollary.

**Corollary 2.19.** *Let  $\pi$  and  $\pi'$  be two irreducible infinite-dimensional representations of  $G_F$ . Assume that the quasi-characters  $\omega$  and  $\omega'$  defined by*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I \quad \pi'\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega'(a)I$$

*are equal. Then  $\pi$  and  $\pi'$  are equivalent if and only if*

$$\frac{L(1-s, \chi^{-1} \otimes \tilde{\pi})\epsilon(s, \chi \otimes \pi, \psi)}{L(s, \chi \otimes \pi)} = \frac{L(1-s, \chi^{-1} \otimes \tilde{\pi}')\epsilon(s, \chi \otimes \pi', \psi)}{L(s, \chi \otimes \pi')}$$

*for all quasi-characters.*

We begin the proof of the first part of the theorem. If  $\varphi_1$  and  $\varphi_2$  are numerical functions on  $F^\times$  we set

$$\langle \varphi_1, \varphi_2 \rangle = \int \varphi_1(a)\varphi_2(-a) d^\times a.$$

The Haar measure is the one which assigns the measure 1 to  $U_F$ . If one of the functions is in  $\mathcal{S}(F^\times)$  and the other is locally constant the integral is certainly defined. By the Plancherel theorem for  $U_F$

$$\langle \varphi, \varphi' \rangle = \sum_n \sum_\nu \nu(-1) \widehat{\varphi}_n(\nu) \widehat{\varphi}'_n(\nu^{-1}).$$

The sum is in reality finite. It is easy to see that if  $b$  belongs to  $B$

$$\langle \xi_\psi(b)\varphi, \xi_\psi(b)\varphi' \rangle = \langle \varphi, \varphi' \rangle.$$

Suppose  $\pi$  is given in the Kirillov form and acts on  $V$ . Let  $\pi'$ , the Kirillov model of  $\omega^{-1} \otimes \pi$ , act on  $V'$ . To prove part (i) we have only to construct an invariant non-degenerate bilinear form  $\beta$  on  $V \times V'$ . If  $\varphi$  belongs to  $V_0$  and  $\varphi'$  belongs to  $V'$  or if  $\varphi$  belongs to  $V$  and  $\varphi'$  belongs to  $V'_0$  we set

$$\beta(\varphi, \varphi') = \langle \varphi, \varphi' \rangle.$$

If  $\varphi$  and  $\varphi'$  are arbitrary vectors in  $V$  and  $V'$  we may write  $\varphi = \varphi_1 + \pi(w)\varphi_2$  and  $\varphi' = \varphi'_1 + \pi'(w)\varphi'_2$  with  $\varphi, \varphi_2$  in  $V_0$  and  $\varphi'_1, \varphi'_2$  in  $V'_0$ . We want to set

$$\beta(\varphi, \varphi') = \langle \varphi_1, \varphi'_1 \rangle + \langle \varphi_1, \pi'(w)\varphi'_2 \rangle + \langle \pi(w)\varphi_2, \varphi'_1 \rangle + \langle \varphi_2, \varphi'_2 \rangle.$$

The second part of the next lemma shows that  $\beta$  is well-defined.

**Lemma 2.19.1.** *Let  $\varphi$  and  $\varphi'$  belong to  $V_0$  and  $V'_0$  respectively. Then*

(i)

$$\langle \pi(w)\varphi, \varphi' \rangle = \nu_0(-1) \langle \varphi, \pi'(w)\varphi' \rangle$$

(ii) *If either  $\pi(w)\varphi$  belongs to  $V_0$  or  $\pi'(w)\varphi'$  belongs to  $V'_0$  then*

$$\langle \pi(w)\varphi, \pi'(w)\varphi' \rangle = \langle \varphi, \varphi' \rangle.$$

The relation

$$\pi(w)\widehat{\varphi}(\nu, t) = \sum_n t^n \sum_p C_{n+p}(\nu) \widehat{\varphi}_p(\nu^{-1}\nu_0^{-1}) z_0^{-p}$$

implies that

$$(2.19.2) \quad \langle \pi(w)\varphi, \varphi' \rangle = \sum_{n,p,\nu} \nu(-1) C_{n+p}(\nu) \widehat{\varphi}_p(\nu^{-1}\nu_0^{-1}) z_0^{-p} \widehat{\varphi}'_n(\nu^{-1}).$$

Replacing  $\pi$  by  $\pi'$  replaces  $\omega$  by  $\omega^{-1}$ ,  $\nu_0$  by  $\nu_0^{-1}$ ,  $z_0$  by  $z_0^{-1}$ , and  $C(\nu, t)$  by  $C(\nu\nu_0^{-1}, z_0^{-1}t)$ . Thus

$$(2.19.3) \quad \langle \varphi, \pi'(w)\varphi' \rangle = \sum_{n,p,\nu} \nu(-1) C_{n+p}(\nu\nu_0^{-1}) z_0^{-n} \widehat{\varphi}'_p(\nu^{-1}\nu_0) \widehat{\varphi}_n(\nu^{-1}).$$

Replacing  $\nu$  by  $\nu\nu_0$  in (2.19.3) and comparing with (2.19.2) we obtain the first part of the lemma.

Because of the symmetry it will be enough to consider the second part when  $\pi(w)\varphi$  belongs to  $V_0$ . By the first part

$$\langle \pi(w)\varphi, \pi'(w)\varphi' \rangle = \nu_0(-1) \langle \pi^2(w)\varphi, \varphi' \rangle = \langle \varphi, \varphi' \rangle.$$

It follows immediately from the lemma that

$$\beta(\pi(w)\varphi, \pi'(w)\varphi') = \beta(\varphi, \varphi')$$

so that to establish the invariance of  $\beta$  we need only show that

$$\beta(\pi(p)\varphi, \pi'(p)\varphi') = \beta(\varphi, \varphi')$$

for all triangular matrices  $p$ . If  $\varphi$  is in  $V_0$  or  $\varphi'$  is in  $V'_0$  this is clear. We need only verify it for  $\varphi$  in  $\pi(w)V_0$  and  $\varphi'$  in  $\pi'(w)V'_0$ .

If  $\varphi$  is in  $V_0$ ,  $\varphi'$  is in  $V'_0$  and  $p$  is diagonal then

$$\beta(\pi(p)\pi(w)\varphi, \pi'(p)\pi'(w)\varphi') = \beta(\pi(w)\pi(p_1)\varphi, \pi'(w)\pi'(p_1)\varphi')$$

where  $p_1 = w^{-1}pw$  is also diagonal. The right side is equal to

$$\beta(\pi(p_1)\varphi, \pi'(p_1)\varphi') = \beta(\varphi, \varphi') = \beta(\pi(w)\varphi, \pi'(w)\varphi').$$

Finally we have to show that<sup>1</sup>

$$(2.19.2) \quad \beta\left(\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi, \pi'\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi'\right) = \beta(\varphi, \varphi')$$

for all  $x$  in  $F$  and all  $\varphi$  and  $\varphi'$ . Let  $\varphi_i$ ,  $1 < i < r$ , generate  $V$  modulo  $V_0$  and let  $\varphi'_j$ ,  $1 \leq j \leq r'$ , generate  $V'$  modulo  $V'_0$ . There certainly is an ideal  $\mathfrak{a}$  of  $F$  such that

$$\pi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi_i = \varphi_i$$

and

$$\pi'\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi'_j = \varphi'_j$$

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<sup>1</sup>The tags on Equations 2.19.2 and 2.19.3 have inadvertently been repeated.

for all  $i$  and  $j$  if  $x$  belongs to  $\mathfrak{a}$ . Then

$$\beta \left( \pi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \varphi_i, \pi' \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \varphi'_j \right) = \beta(\varphi_i, \varphi_j).$$

Since (2.19.2) is valid  $\varphi$  is in  $V_0$  or  $\varphi'$  is in  $V'_0$  it is valid for all  $\varphi$  and  $\varphi'$  provided that  $x$  is in  $\mathfrak{a}$ . Any  $y$  in  $F$  may be written as  $ax$  with  $a$  in  $F^\times$  and  $x$  in  $\mathfrak{a}$ . Then

$$\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

and it follows readily that

$$\beta \left( \pi \left( \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \right) \varphi, \pi' \left( \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \right) \varphi' \right) = \beta(\varphi, \varphi').$$

Since  $\beta$  is invariant and not identically zero it is non-degenerate. The rest of the theorem will now be proved for absolutely cuspidal representations. The remaining representations will be considered in the next chapter. We observe that since  $W(\pi, \psi)$  is invariant under right translations the assertions need only be established when  $g$  is the identity matrix  $e$ .

If  $\pi$  is absolutely cuspidal then  $V = V_0 = \mathcal{S}(F^\times)$  and  $W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = \varphi(a)$  is locally constant with compact support. Therefore the integrals defining  $\Psi(e, s, W)$  and  $\tilde{\Psi}(e, s, W)$  are absolutely convergent for all values of  $s$  and the two functions are entire. We may take  $L(s, \pi) = 1$ . If  $\varphi$  is taken to be the characteristic function of  $U_F$  then  $\Phi(e, s, W) = 1$ .

Referring to the discussion preceding Corollary 2.19 we see that if we take

$$\epsilon(s, \pi, \psi) = C(\nu_0^{-1}, z_0^{-1} q^{-1/2} q^s)$$

the local functional equation of part (iv) will be satisfied. It remains to show that  $\epsilon(s, \pi, \psi)$  is an exponential function or, what is at least as strong, to show that, for all  $\nu$ ,  $C(\nu, t)$  is a multiple of a power of  $t$ . It is a finite linear combination of powers of  $t$  and if it is not of the form indicated it has a zero at some point different from 0.  $C(\nu \nu_0^{-1}, z_0^{-1} t^{-1})$  is also a linear combination of powers of  $t$  and so cannot have a pole except at zero. To show that  $C(\nu, t)$  has the required form we have only to show that

$$(2.19.3) \quad C(\nu, t) C(\nu^{-1} \nu_0^{-1}, z_0^{-1} t^{-1}) = \nu_0(-1).$$

Choose  $\varphi$  in  $V_0$  and set  $\varphi' = \pi(w)\varphi$ . We may suppose that  $\varphi'(\nu, t) \neq 0$ . The identity is obtained by combining the two relations

$$\hat{\varphi}'(\nu, t) = C(\nu, t) \hat{\varphi}(\nu^{-1} \nu_0^{-1}, z_0^{-1} t^{-1})$$

and

$$\nu_0(-1) \hat{\varphi}(\nu^{-1} \nu_0^{-1}, t) = C(\nu^{-1} \nu_0^{-1}, t) \hat{\varphi}'(\nu, z_0^{-1} t^{-1}).$$

We close this paragraph with a number of facts about absolutely cuspidal representations which will be useful later.

**Proposition 2.20.** *Let  $\pi$  be an absolutely cuspidal representation of  $G_F$ . If the quasi-character  $\omega$  defined by*

$$\pi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \omega(a) I$$

*is actually a character then  $\pi$  is unitary.*



As usual we take  $\pi$  and  $\tilde{\pi}$  in the Kirillov form. We have to establish the existence of a positive-definite invariant hermitian form on  $V$ . We show first that if  $\varphi$  belongs to  $V$  and  $\tilde{\varphi}$  belongs to  $\tilde{V}$  then there is a compact set  $\Omega$  in  $G_F$  such that if

$$Z_F = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in F \right\}$$

the support of  $\langle \pi(g)\varphi, \tilde{\varphi} \rangle$ , a function of  $g$ , is contained in  $Z_F\Omega$ . If  $A_F$  is the group of diagonal matrices  $G_F = \mathrm{GL}(2, O_F)A_F\mathrm{GL}(2, O_F)$ . Since  $\varphi$  and  $\tilde{\varphi}$  are both invariant under subgroups of finite index in  $\mathrm{GL}(2, O_F)$  it is enough to show that the function  $\langle \pi(b)\varphi, \tilde{\varphi} \rangle$  on  $A_F$  has support in a set  $Z_F\Omega$  with  $\Omega$  compact. Since

$$\left\langle \pi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} b \right) \varphi, \tilde{\varphi} \right\rangle = \omega(a) \langle \pi(b)\varphi, \tilde{\varphi} \rangle$$

it is enough to show that the function

$$\left\langle \pi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) \varphi, \tilde{\varphi} \right\rangle$$

has compact support in  $F^\times$ . This matrix element is equal to

$$\int_{F^\times} \varphi(ax) \tilde{\varphi}(-x) d^\times x.$$

Since  $\varphi$  and  $\tilde{\varphi}$  are functions with compact support the result is clear.

Choose  $\tilde{\varphi}_0 \neq 0$  in  $\tilde{V}$  and set

$$(\varphi_1, \varphi_2) = \int_{Z_F \backslash G_F} \langle \pi(g)\varphi_1, \tilde{\varphi}_0 \rangle \overline{\langle \pi(g)\varphi_2, \tilde{\varphi}_0 \rangle} dg.$$

This is a positive invariant hermitian form on  $V$ .

We have incidentally shown that  $\pi$  is square-integrable. Observe that even if the absolutely cuspidal representation  $\pi$  is not unitary one can choose a quasi-character  $\chi$  such that  $\chi \otimes \pi$  is unitary.

If  $\pi$  is unitary there is a conjugate linear map  $A : V \rightarrow \tilde{V}$  defined by

$$(\varphi_1, \varphi_2) = \langle \varphi_1, A\varphi_2 \rangle.$$

Clearly  $A\xi_\psi(b) = \xi_\psi(b)A$  for all  $b$  in  $B_F$ . The map  $A_0$  defined by

$$A_0\varphi(a) = \overline{\varphi}(-a)$$

has the same properties. We claim that

$$A = \lambda A_0$$

with  $\lambda$  in  $\mathbf{C}^\times$ . To see this we have only to apply the following lemma to  $A_0^{-1}A$ .

**Lemma 2.21.1.** *Let  $T$  be a linear operator on  $\mathcal{S}(F^\times)$  which commutes with  $\xi_\psi(b)$  for all  $b$  in  $B_F$ . Then  $T$  is a scalar.*

Since  $\xi_\psi$  is irreducible it will be enough to show that  $T$  has an eigenvector. Let  $\mathfrak{p}^{-\ell}$  be the largest ideal on which  $\psi$  is trivial. Let  $\mu$  be a non-trivial character of  $U_F$  and let  $\mathfrak{p}^n$  be its conductor.  $T$  commutes with the operator

$$S = \int_{U_F} \mu^{-1}(\epsilon) \xi_\psi \left( \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \varpi^{-\ell-n} \\ 0 & 1 \end{pmatrix} \right) d\epsilon$$

and it leaves the range of the restriction of  $S$  to the functions invariant under  $U_F$  invariant. If  $\varphi$  is such a function

$$S\varphi(a) = \varphi(a) \int_{U_F} \mu^{-1}(\epsilon) \psi(a\epsilon\varpi^{-\ell-n}) d\epsilon.$$

The Gaussian sum is 0 unless  $a$  lies in  $U_F$ . Therefore  $S\varphi$  is equal to  $\varphi(1)$  times the function which is zero outside of  $U_F$  and equals  $\mu$  on  $U_F$ . Since  $T$  leaves a one-dimensional space invariant it has an eigenvector.

Since  $A = \lambda A_0$  the hermitian form  $(\varphi_1, \varphi_2)$  is equal to

$$\lambda \int_{F^\times} \varphi_1(a) \overline{\varphi_2}(a) d^\times a.$$

**Proposition 2.21.2.** *Let  $\pi$  be an absolutely cuspidal representation of  $G_F$  for which the quasi-character  $\omega$  defined by*

$$\pi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \omega(a) I$$

*is a character.*

(i) *If  $\pi$  is in the Kirillov form the hermitian form*

$$\int_{F^\times} \varphi_1(a) \overline{\varphi_2}(a) d^\times a$$

*is invariant.*

(ii) *If  $|z| = 1$  then  $|C(\nu, z)| = 1$  and if  $\operatorname{Re} s = 1/2$*

$$|\epsilon(s, \pi, \psi)| = 1.$$

Since  $|z_0| = 1$  the second relation of part (ii) follows from the first and the relation

$$\epsilon(s, \pi, \psi) = C(\nu_0^{-1}, q^{s-1/2} z_0^{-1}).$$

If  $n$  is in  $\mathbf{Z}$  and  $\nu$  is a character of  $U_F$  let

$$\varphi(\epsilon\varpi^m) = \delta_{n,m} \nu(\epsilon) \nu_0(\epsilon)$$

for  $m$  in  $\mathbf{Z}$  and  $\epsilon$  in  $U_F$ . Then

$$\int_{F^\times} |\varphi(a)|^2 da = 1.$$

If  $\varphi' = \pi(w)\varphi$  and  $C(\nu, t) = C_\ell(\nu)t^\ell$  then

$$\varphi'(\epsilon\varpi^m) = \delta_{\ell-n,m} C_\ell(\nu) z_0^{-n} \nu^{-1}(\epsilon).$$

Since  $|z_0| = 1$

$$\int_{F^\times} |\varphi'(a)|^2 da = |C_\ell(\nu)|^2.$$

Applying the first part of the lemma we see that, if  $|z| = 1$ , both  $|C_\ell(\nu)|^2$  and  $|C(\nu, z)|^2 = |C_\ell(\nu)|^2 |z|^{2\ell}$  are 1.

**Proposition 2.22.** *Let  $\pi$  be an irreducible representation of  $G_F$ . It is absolutely cuspidal if and only if for every vector  $v$  there is an ideal  $\mathfrak{a}$  in  $F$  such that*

$$\int_{\mathfrak{a}} \pi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) v \, dx = 0.$$

It is clear that the condition cannot be satisfied by a finite-dimensional representation. Suppose that  $\pi$  is infinite-dimensional and in the Kirillov form. If  $\varphi$  is in  $V$  then

$$\int_{\mathfrak{a}} \pi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \varphi \, dx = 0$$

if and only if

$$\varphi(a) \int_{\mathfrak{a}} \psi(ax) \, dx = 0$$

for all  $a$ . If this is so the character  $x \rightarrow \psi(ax)$  must be non-trivial on  $\mathfrak{a}$  for all  $a$  in the support of  $\varphi$ . This happens if and only if  $\varphi$  is in  $\mathcal{S}(F^\times)$ . The proposition follows.

**Proposition 2.23.** *Let  $\pi$  be an absolutely cuspidal representation and assume the largest ideal on which  $\psi$  is trivial is  $O_F$ . Then, for all characters  $\nu$ ,  $C_n(\nu) = 0$  if  $n \geq -1$ .*

Take a character  $\nu$  and choose  $n_1$  such that  $C_{n_1}(\nu) \neq 0$ . Then  $C_n(\nu) = 0$  for  $n \neq n_1$ . If  $\tilde{\nu} = \nu^{-1}\nu_0^{-1}$  then, as we have seen,

$$C(\nu, t)C(\tilde{\nu}, t^{-1}z_0^{-1}) = \nu_0(-1)$$

so that

$$C_n(\tilde{\nu}) = 0$$

for  $n \neq n_1$  and

$$C_{n_1}(\nu)C_{n_1}(\tilde{\nu}) = \nu_0(-1)z_0^{n_1}.$$

In the second part of Proposition 2.11 take  $n = p = n_1 + 1$  to obtain

$$\sum_{\sigma} \eta(\sigma^{-1}\nu, \varpi^{n_1+1})\eta(\sigma^{-1}\tilde{\nu}, \varpi^{n_1+1})C_{2n_1+2}(\sigma) = z_0^{n_1+1}\nu_0(-1) + (|\varpi| - 1)^{-1}z_0C_{n_1}(\nu)C_{n_1}(\tilde{\nu}).$$

The right side is equal to

$$z_0^{n_1+1}\nu_0(-1) \cdot \frac{|\varpi|}{|\varpi| - 1}.$$

Assume  $n_1 \geq -1$ . Then  $\eta(\sigma^{-1}\nu, \varpi^{n_1+1})$  is 0 unless  $\sigma = \nu$  and  $\eta(\sigma^{-1}\tilde{\nu}, \varpi^{n_1+1})$  is 0 unless  $\sigma = \tilde{\nu}$ . Thus the left side is 0 unless  $\nu = \tilde{\nu}$ . However if  $\nu = \tilde{\nu}$  the left side equals  $C_{2n_1+2}(\nu)$ . Since this cannot be zero  $2n_1 + 2$  must equal  $n_1$  so that  $n_1 = -2$ . This is a contradiction.

### §3. The principal series for non-archimedean fields

In order to complete the discussion of the previous paragraph we have to consider representations which are not absolutely cuspidal. This we shall now do. We recall that if  $\mu_1, \mu_2$  is a pair of quasi-characters of  $F^\times$  the space  $\mathcal{B}(\mu_1, \mu_2)$  consists of all locally constant functions  $f$  on  $G_F$  which satisfy

$$(3.1) \quad f\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right) = \mu_1(a_1)\mu_2(a_2) \left|\frac{a_1}{a_2}\right|^{1/2} f(g)$$

for all  $g$  in  $G_F$ ,  $a_1, a_2$ , in  $F^\times$ , and  $x$  in  $F$ .  $\rho(\mu_1, \mu_2)$  is the representation of  $G_F$  on  $\mathcal{B}(\mu_1, \mu_2)$ .

Because of the Iwasawa decomposition  $G_F = P_F \text{GL}(2, O_F)$  the functions in  $\mathcal{B}(\mu_1, \mu_2)$  are determined by their restrictions to  $\text{GL}(2, O_F)$ . The restriction can be any locally constant function on  $\text{GL}(2, O_F)$  satisfying

$$f\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right) = \mu_1(a_1)\mu_2(a_2) f(g)$$

for all  $g$  in  $\text{GL}(2, O_F)$ ,  $a_1, a_2$  in  $U_F$ , and  $x$  in  $O_F$ . If  $U$  is an open subgroup of  $\text{GL}(2, O_F)$  the restriction of any function invariant under  $U$  is a function on  $\text{GL}(2, O_F)/U$  which is a finite set. Thus the space of all such functions is finite-dimensional and as observed before  $\rho(\mu_1, \mu_2)$  is admissible.

Let  $\mathcal{F}$  be the space of continuous functions  $f$  on  $G_F$  which satisfy

$$f\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right) = \left|\frac{a_1}{a_2}\right| f(g)$$

for all  $g$  in  $G_F$ ,  $a_1, a_2$  in  $F^\times$ , and  $x$  in  $F$ . We observe that  $\mathcal{F}$  contains  $\mathcal{B}(\alpha_F^{1/2}, \alpha_F^{-1/2})$ .  $G_F$  acts on  $\mathcal{F}$ . The Haar measure on  $G_F$  if suitably normalized satisfies

$$\int_{G_F} f(g) dg = \int_{N_F} \int_{A_F} \int_{\text{GL}(2, O_F)} \left|\frac{a_1}{a_2}\right|^{-1} f(nak) dn da dk$$

if

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

It follows easily from this that

$$\int_{\text{GL}(2, O_F)} f(k) dk$$

is a  $G_F$ -invariant linear form on  $\mathcal{F}$ . There is also a positive constant  $c$  such that

$$\int_{G_F} f(g) dg = c \int_{N_F} \int_{A_F} \int_{N_F} \left|\frac{a_1}{a_2}\right|^{-1} f\left(na \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} n_1\right) dn da dn_1.$$

Consequently

$$\int_{\text{GL}(2, O_F)} f(k) dk = c \int_F f\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) dx.$$

If  $\varphi_1$  belongs to  $\mathcal{B}(\mu_1, \mu_2)$  and  $\varphi_2$  belongs to  $\mathcal{B}(\mu_1^{-1}, \mu_2^{-1})$  then  $\varphi_1\varphi_2$  belongs to  $\mathcal{F}$  and we set

$$\langle \varphi_1, \varphi_2 \rangle = \int_{\mathrm{GL}(2, O_F)} \varphi_1(k) \varphi_2(k) dk.$$

Clearly

$$\langle \rho(g)\varphi_1, \rho(g)\varphi_2 \rangle = \langle \varphi_1, \varphi_2 \rangle$$

so that this bilinear form is invariant. Since both  $\varphi_1$  and  $\varphi_2$  are determined by their restrictions to  $\mathrm{GL}(2, O_F)$  it is also non-degenerate. Thus  $\rho(\mu_1^{-1}, \mu_2^{-1})$  is equivalent to the contragredient of  $\rho(\mu_1, \mu_2)$ .

In Proposition 1.6 we introduced a representation  $r$  of  $G_F$  and then we introduced a representation  $r_\Omega = r_{\mu_1, \mu_2}$ . Both representations acted on  $\mathcal{S}(F^2)$ . If

$$\tilde{\Phi}(a, b) = \int_F \Phi(a, y) \psi(by) dy$$

is the partial Fourier transform

$$(3.1.1) \quad [r(g)\Phi]^\sim = \rho(g)\tilde{\Phi}$$

and

$$(3.1.2) \quad r_{\mu_1, \mu_2}(g) = \mu_1(\det g) |\det g|^{1/2} r(g).$$

We also introduced the integral

$$\theta(\mu_1, \mu_2; \Phi) = \int_{F^\times} \mu_1(t) \mu_2^{-1}(t) \Phi(t, t^{-1}) d^\times t$$

and we set

$$(3.1.3) \quad W_\Phi(g) = \theta(\mu_1, \mu_2; r_{\mu_1, \mu_2}(g)\Phi).$$

The space of functions  $W_\Phi$  is denoted  $W(\mu_1, \mu_2; \psi)$ .

If  $\omega$  is a quasi-character of  $F^\times$  and if  $|\omega(\varpi)| = |\varpi|^s$  with  $s > 0$  the integral

$$z(\omega, \Phi) = \int_{F^\times} \Phi(0, t) \omega(t) d^\times t$$

is defined for all  $\Phi$  in  $\mathcal{S}(F^2)$ . In particular if  $|\mu_1(\varpi) \mu_2^{-1}(\varpi)| = |\varpi|^s$  with  $s > -1$  we can consider the function

$$f_\Phi(g) = \mu_1(\det g) |\det g|^{1/2} z(\alpha_F \mu_1 \mu_2^{-1}, \rho(g)\Phi)$$

on  $G_F$ . Recall that  $\alpha_F(a) = |a|$ . Clearly

$$(3.1.4) \quad \rho(h) f_\Phi = f_\Psi$$

if

$$\Psi = \mu_1(\det h) |\det h|^{1/2} \rho(h)\Phi.$$

We claim that  $f_\Phi$  belongs to  $\mathcal{B}(\mu_1, \mu_2)$ . Since the stabilizer of every  $\Phi$  under the representation  $g \rightarrow \mu_1(\det g) |\det g|^{1/2} \rho(g)$  is an open subgroup of  $G_F$  the functions  $f_\Phi$  are locally constant. Since the space of functions  $f_\Phi$  is invariant under right translations we need verify (3.1) only for  $g = e$ .

$$f_\Phi \left( \begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} \right) = z \left( \mu_1 \mu_2^{-1} \alpha_F, \rho \left( \begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} \right) \Phi \mu_1(a_1 a_2) |a_1 a_2|^{1/2} \right).$$

By definition the right side is equal to

$$\mu_1(a_1 a_2) |a_1 a_2|^{1/2} \int \mu_1(t) \mu_2^{-1}(t) |t| \Phi(0, a_2 t) d^\times t.$$

Changing variables we obtain

$$\mu_1(a_1) \mu_2(a_2) \left| \frac{a_1}{a_2} \right|^{1/2} \int \mu_1(t) \mu_2^{-1}(t) |t| \Phi(0, t) d^\times t.$$

The integral is equal to  $f_\Phi(e)$ . Hence our assertion.

**Proposition 3.2.** *Assume  $|\mu_1(\varpi) \mu_2^{-1}(\varpi)| = |\varpi|^s$  with  $s > -1$ .*

- (i) *There is a linear transformation  $A$  of  $W(\mu_1, \mu_2; \psi)$  into  $\mathcal{B}(\mu_1, \mu_2)$  which, for all  $\Phi$  in  $\mathcal{S}(F^2)$ , sends  $W_\Phi$  to  $f_{\tilde{\Phi}}$ .*
- (ii)  *$A$  is bijective and commutes with right translations.*

To establish the first part of the proposition we have to show that  $f_{\tilde{\Phi}}$  is 0 if  $W_\Phi$  is. Since  $N_F A_F \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} N_F$  is a dense subset of  $G_F$  this will be a consequence of the following lemma.

**Lemma 3.2.1.** *If the assumptions of the proposition are satisfied then, for all  $\Phi$  in  $\mathcal{S}(F^2)$ , the function*

$$a \longrightarrow \mu_2^{-1}(a) |a|^{-1/2} W_\Phi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right)$$

*is integrable with respect to the additive Haar measure on  $F$  and*

$$\int W_\Phi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) \mu_2^{-1}(a) |a|^{-1/2} \psi(ax) da = f_{\tilde{\Phi}} \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right).$$

By definition

$$f_{\tilde{\Phi}} \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) = \int \tilde{\Phi}(t, tx) \mu_1(t) \mu_2^{-1}(t) |t| d^\times t$$

while

$$(3.2.2) \quad W_\Phi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) \mu_2^{-1}(a) |a|^{-1/2} = \mu_1(a) \mu_2^{-1}(a) \int \Phi(at, t^{-1}) \mu_1(t) \mu_2^{-1}(t) d^\times t.$$

After a change of variable the right side becomes

$$\int \Phi(t, at^{-1}) \mu_1(t) \mu_2^{-1}(t) d^\times t.$$

Computing formally we see that

$$\int W_\Phi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) \mu_2^{-1}(a) |a|^{-1/2} \psi(ax) da$$

is equal to

$$\begin{aligned} \int_F \psi(ax) \left\{ \int_{F^\times} \Phi(t, at^{-1}) \mu_1(t) \mu_2^{-1}(t) d^\times t \right\} da \\ = \int_{F^\times} \mu_1(t) \mu_2^{-1}(t) \left\{ \int_F \Phi(t, at^{-1}) \psi(ax) da \right\} d^\times t \end{aligned}$$

which in turn equals

$$\int_{F^\times} \mu_1(t) \mu_2^{-1}(t) |t| \left\{ \int_F \Phi(t, a) \psi(amt) da \right\} d^\times t = \int_{F^\times} \tilde{\Phi}(t, xt) \mu_1(t) \mu_2^{-1}(t) |t| d^\times t.$$

Our computation will be justified and the lemma proved if we show that the integral

$$\int_{F^\times} \int_F |\Phi(t, at^{-1}) \mu_1(t) \mu_2^{-1}(t)| d^\times t da$$

is convergent. It equals

$$\int_{F^\times} \int_F |\Phi(t, a)| |t|^{s+1} d^\times t da$$

which is finite because  $s$  is greater than  $-1$ .

To show that  $A$  is surjective we show that every function  $f$  in  $\mathcal{B}(\mu_1, \mu_2)$  is of the form  $f_\Phi$  for some  $\Phi$  in  $\mathcal{S}(F^2)$ . Given  $f$  let  $\Phi(x, y)$  be 0 if  $(x, y)$  is not of the form  $(0, 1)g$  for some  $g$  in  $\text{GL}(2, O_F)$  but if  $(x, y)$  is of this form let  $\Phi(x, y) = \mu_1^{-1}(\det g) f(g)$ . It is easy to see that  $\Phi$  is well-defined and belongs to  $\mathcal{S}(F^2)$ . To show that  $f = f_\Phi$  we need only show that  $f(g) = f_\Phi(g)$  for all  $g$  in  $\text{GL}(2, O_F)$ . If  $g$  belongs to  $\text{GL}(2, O_F)$  then  $\Phi((0, t)g) = 0$  unless  $t$  belongs to  $U_F$  so that

$$f_\Phi(g) = \mu_1(\det g) \int_{U_F} \Phi((0, t)g) \mu_1(t) \mu_2^{-1}(t) dt.$$

Since

$$\Phi((0, t)g) = \mu_1^{-1}(t) \mu_1^{-1}(\det g) f\left(\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} g\right) = \mu_1^{-1}(t) \mu_2(t) \mu_1^{-1}(\det g) f(g)$$

the required equality follows.

Formulae (3.1.2) to (3.1.4) show that  $A$  commutes with right translations. Thus to show that  $A$  is injective we have to show that  $W_\Phi(e) = 0$  if  $f_{\tilde{\Phi}}$  is 0. It follows from the previous lemma that

$$W_\Phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)$$

is zero for almost all  $a$  if  $f_{\tilde{\Phi}}$  is 0. Since  $W_\Phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)$  is a locally constant function on  $F^\times$  it must vanish everywhere.

We have incidentally proved the following lemma.

**Lemma 3.2.3.** *Suppose  $|\mu_1(\varpi) \mu_2^{-1}(\varpi)| = |\varpi|^s$  with  $s > -1$  and  $W$  belongs to  $W(\mu_1, \mu_2; \psi)$ . If*

$$W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

*for all  $a$  in  $F^\times$  then  $W$  is 0.*

**Theorem 3.3.** *Let  $\mu_1$  and  $\mu_2$  be two quasi-characters of  $F^\times$ .*

- (i) *If neither  $\mu_1 \mu_2^{-1}$  nor  $\mu_1^{-1} \mu_2$  is  $\alpha_F$  the representations  $\rho(\mu_1, \mu_2)$  and  $\rho(\mu_2, \mu_1)$  are equivalent and irreducible.*

- (ii) If  $\mu_1\mu_2^{-1} = \alpha_F$  write  $\mu_1 = \chi\alpha_F^{1/2}$ ,  $\mu_2 = \chi\alpha_F^{-1/2}$ . Then  $\mathcal{B}(\mu_1, \mu_2)$  contains a unique proper invariant subspace  $\mathcal{B}_s(\mu_1, \mu_2)$  which is irreducible.  $\mathcal{B}(\mu_2, \mu_1)$  also contains a unique proper invariant subspace  $\mathcal{B}_f(\mu_2, \mu_1)$ . It is one-dimensional and contains the function  $\chi(\det g)$ . Moreover the  $G_F$ -modules  $\mathcal{B}_s(\mu_1, \mu_2)$  and  $\mathcal{B}(\mu_2, \mu_1)/\mathcal{B}_f(\mu_2, \mu_1)$  are equivalent as are the modules  $\mathcal{B}(\mu_1, \mu_2)/\mathcal{B}_s(\mu_1, \mu_2)$  and  $\mathcal{B}_f(\mu_2, \mu_1)$ .

We start with a simple lemma.

**Lemma 3.3.1.** *Suppose there is a non-zero function  $f$  in  $\mathcal{B}(\mu_1, \mu_2)$  invariant under right translations by elements of  $N_F$ . Then there is a quasi-character  $\chi$  such that  $\mu_1 = \chi\alpha_F^{1/2}$  and  $\mu_2 = \chi\alpha_F^{-1/2}$  and  $f$  is a multiple of  $\chi$ .*

Since  $N_F A_F \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} N_F$  is an open subset of  $G_F$  the function  $f$  is determined by its value at  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . Thus if  $\mu_1$  and  $\mu_2$  have the indicated form it must be a multiple of  $\chi$ .

If  $c$  belongs to  $F^\times$  then

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \begin{pmatrix} c^{-1} & 1 \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & c^{-1} \\ 0 & 1 \end{pmatrix}.$$

Thus if  $f$  exists and  $\omega = \mu_2\mu_1^{-1}\alpha_F^{-1}$

$$f\left(\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\right) = \omega(c)f\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right).$$

Since  $f$  is locally constant there is an ideal  $\mathfrak{a}$  in  $F$  such that  $\omega$  is constant on  $\mathfrak{a} - \{0\}$ . It follows immediately that  $\omega$  is identically 1 and that  $\mu_1$  and  $\mu_2$  have the desired form.

The next lemma is the key to the theorem.

**Lemma 3.3.2.** *If  $|\mu_1\mu_2(\varpi)| = |\varpi|^s$  with  $s > -1$  there is a minimal non-zero invariant subspace  $X$  of  $\mathcal{B}(\mu_1, \mu_2)$ . For all  $f$  in  $\mathcal{B}(\mu_1, \mu_2)$  and all  $n$  in  $N_F$  the difference  $f - \rho(n)f$  belongs to  $X$ .*

By Proposition 3.2 it is enough to prove the lemma when  $\mathcal{B}(\mu_1, \mu_2)$  is replaced by  $W(\mu_1, \mu_2; \psi)$ . Associate to each function  $W$  in  $W(\mu_1, \mu_2; \psi)$  a function

$$\varphi(a) = W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)$$

on  $F^\times$ . If  $\varphi$  is 0 so is  $W$ . We may regard  $\pi = \rho(\mu_1, \mu_2)$  as acting on the space  $V$  of such functions. If  $b$  is in  $B_F$

$$\pi(b)\varphi = \xi_\psi(b)\varphi.$$

Appealing to (3.2.2) we see that every function  $\varphi$  in  $V$  has its support in a set of the form

$$\{a \in F^\times \mid |a| \leq c\}$$

where  $c = c(\varphi)$  is a constant. As in the second paragraph the difference  $\varphi - \pi(n)\varphi = \varphi - \xi_\psi(n)\varphi$  is in  $\mathcal{S}(F^\times)$  for all  $n$  in  $N_F$ . Thus  $V \cap \mathcal{S}(F^\times)$  is not 0. Since the representation  $\xi_\psi$  of  $B_F$  on  $\mathcal{S}(F^\times)$  is irreducible,  $V$  and every non-trivial invariant subspace of  $V$  contains  $\mathcal{S}(F^\times)$ . Taking the intersection of all such spaces we obtain the subspace of the lemma.

We first prove the theorem assuming that  $|\mu_1(\varpi)\mu_2^{-1}(\varpi)| = |\varpi|^s$  with  $s > -1$ . We have defined a non-degenerate pairing between  $\mathcal{B}(\mu_1, \mu_2)$  and  $\mathcal{B}(\mu_1^{-1}, \mu_2^{-1})$ . All elements of the orthogonal complement of  $X$  are invariant under  $N_F$ . Thus if  $\mu_1\mu_2^{-1}$  is not  $\alpha_F$  the



orthogonal complement is 0 and  $X$  is  $\mathcal{B}(\mu_1, \mu_2)$  so that the representation is irreducible. The contragredient representation  $\rho(\mu_1^{-1}, \mu_2^{-1})$  is also irreducible.

If  $\mu_1 \mu_2^{-1} = \alpha_F$  we write  $\mu_1 = \chi \alpha_F^{1/2}$ ,  $\mu_2 = \chi \alpha_F^{-1/2}$ . In this case  $X$  is the space of the functions orthogonal to the function  $\chi^{-1}$  in  $\mathcal{B}(\mu_1^{-1}, \mu_2^{-1})$ . We set  $\mathcal{B}_s(\mu_1, \mu_2) = X$  and we let  $\mathcal{B}_f(\mu_1^{-1}, \mu_2^{-1})$  be the space of scalar multiples of  $\chi^{-1}$ . The representation of  $G_F$  on  $\mathcal{B}_s(\mu_1, \mu_2)$  is irreducible. Since  $\mathcal{B}_s(\mu_1, \mu_2)$  is of codimension one it is the only proper invariant subspace of  $\mathcal{B}(\mu_1, \mu_2)$ . Therefore  $\mathcal{B}_f(\mu_1^{-1}, \mu_2^{-1})$  is the only proper invariant subspace of  $\mathcal{B}(\mu_1^{-1}, \mu_2^{-1})$ .

If  $|\mu_1(\varpi)\mu_2^{-1}(\varpi)| = |\varpi|^s$  then  $|\mu_1^{-1}(\varpi)\mu_2(\varpi)| = |\varpi|^{-s}$  and either  $s > -1$  or  $-s > -1$ . Thus if  $\mu_1^{-1}\mu_2$  is neither  $\alpha_F$  nor  $\alpha_F^{-1}$  the representation  $\pi = \rho(\mu_1, \mu_2)$  is irreducible. If  $\omega = \mu_1\mu_2$  then

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I$$

so that  $\pi$  is equivalent to  $\omega \otimes \tilde{\pi}$  or to  $\omega \otimes \rho(\mu_1^{-1}, \mu_2^{-1})$ . It is easily seen that  $\omega \otimes \rho(\mu_1^{-1}, \mu_2^{-1})$  is equivalent to  $\rho(\omega\mu_1^{-1}, \omega\mu_2^{-1}) = \rho(\mu_2, \mu_1)$ .

If  $\mu_1\mu_2^{-1} = \alpha_F$  and  $\pi$  is the restriction of  $\rho$  to  $\mathcal{B}_s(\mu_1, \mu_2)$  then  $\tilde{\pi}$  is the representation on  $\mathcal{B}(\mu_1^{-1}, \mu_2^{-1})/\mathcal{B}_f(\mu_1^{-1}, \mu_2^{-1})$  defined by  $\rho(\mu_1^{-1}, \mu_2^{-1})$ . Thus  $\pi$  is equivalent to the tensor product of  $\omega = \mu_1\mu_2$  and this representation. The tensor product is of course equivalent to the representation on  $\mathcal{B}(\mu_2, \mu_1)/\mathcal{B}_f(\mu_2, \mu_1)$ . If  $\mu_1 = \chi \alpha_F^{1/2}$  and  $\mu_2 = \chi \alpha_F^{-1/2}$  the representations on  $\mathcal{B}(\mu_1, \mu_2)/\mathcal{B}_s(\mu_1, \mu_2)$  and  $\mathcal{B}_f(\mu_2, \mu_1)$  are both equivalent to the representation  $g \rightarrow \chi(\det g)$ .

The space  $W(\mu_1, \mu_2; \psi)$  has been defined for all pairs  $\mu_1, \mu_2$ .

**Proposition 3.4.**

(i) For all pairs  $\mu_1, \mu_2$

$$W(\mu_1, \mu_2; \psi) = W(\mu_2, \mu_1; \psi)$$

(ii) In particular if  $\mu_1\mu_2^{-1} \neq \alpha_F^{-1}$  the representation of  $G_F$  on  $W(\mu_1, \mu_2; \psi)$  is equivalent to  $\rho(\mu_1, \mu_2)$ .

If  $\Phi$  is a function on  $F^2$  define  $\Phi^\iota$  by

$$\Phi^\iota(x, y) = \Phi(y, x).$$

To prove the proposition we show that, if  $\Phi$  is in  $\mathcal{S}(F^2)$ ,

$$\mu_1(\det g)|\det g|^{1/2}\theta(\mu_1, \mu_2; r(g)\Phi^\iota) = \mu_2(\det g)|\det g|^{1/2}\theta(\mu_2, \mu_1; r(g)\Phi).$$

If  $g$  is the identity this relation follows upon inspection of the definition of  $\theta(\mu_1, \mu_2; \Phi^\iota)$ . It is also easily seen that

$$r(g)\Phi^\iota = [r(g)\Phi]^\iota$$

if  $g$  is in  $\text{SL}(2, F)$  so that it is enough to prove the identity for

$$g = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

It reduces to

$$\mu_1(a) \int \Phi^\iota(at, t^{-1})\mu_1(t)\mu_2^{-1}(t) d^\times t = \mu_2(a) \int \Phi(at, t^{-1})\mu_2(t)\mu_2^{-1}(t) d^\times t.$$

The left side equals

$$\mu_1(a) \int \Phi(t^{-1}, at)\mu_1(t)\mu_2^{-1}(t) d^\times t$$

which, after changing the variable of integration, one sees is equal to the right side.

If  $\mu_1\mu_2^{-1}$  is not  $\alpha_F$  or  $\alpha_F^{-1}$  so that  $\rho(\mu_1, \mu_2)$  is irreducible we let  $\pi(\mu_1, \mu_2)$  be any representation in the class of  $\rho(\mu_1, \mu_2)$ . If  $\rho(\mu_1, \mu_2)$  is reducible it has two constituents one finite-dimensional and one infinite-dimensional. A representation in the class of the first will be called  $\pi(\mu_1, \mu_2)$ . A representation in the class of the second will be called  $\sigma(\mu_1, \mu_2)$ . Any irreducible representation which is not absolutely cuspidal is either a  $\pi(\mu_1, \mu_2)$  or a  $\sigma(\mu_1, \mu_2)$ . The representations  $\sigma(\mu_1, \mu_2)$  which are defined only for certain values of  $\mu_1$  and  $\mu_2$  are called special representations.

Before proceeding to the proof of Theorem 2.18 for representations which are not absolutely cuspidal we introduce some notation. If  $\omega$  is an unramified quasi-character of  $F^\times$  the associated  $L$ -function is

$$L(s, \omega) = \frac{1}{1 - \omega(\varpi)|\varpi|^s}.$$

It is independent of the choice of the generator  $\varpi$  of  $\mathfrak{p}$ . If  $\omega$  is ramified  $L(s, \omega) = 1$ . If  $\varphi$  belongs to  $\mathcal{S}(F)$  the integral

$$Z(\omega\alpha_F^s, \varphi) = \int_{F^\times} \varphi(\alpha)\omega(\alpha)|\alpha|^s d^\times\alpha$$

is absolutely convergent in some half-plane  $\text{Re } s > s_0$  and the quotient

$$\frac{Z(\omega\alpha_F^s, \varphi)}{L(s, \omega)}$$

can be analytically continued to a function holomorphic in the whole complex plane. Moreover for a suitable choice of  $\varphi$  the quotient is 1. If  $\omega$  is unramified and

$$\int_{U_F} d^\times\alpha = 1$$

one could take the characteristic function of  $O_F$ . There is a factor  $\epsilon(s, \omega, \psi)$  which, for a given  $\omega$  and  $\psi$ , is of the form  $ab^s$  so that if  $\hat{\varphi}$  is the Fourier transform of  $\varphi$

$$\frac{Z(\omega^{-1}\alpha_F^{1-s}, \hat{\varphi})}{L(1-s, \omega^{-1})} = \epsilon(s, \omega, \psi) \frac{Z(\omega\alpha_F^s, \varphi)}{L(s, \omega)}.$$

If  $\omega$  is unramified and  $O_F$  is the largest ideal on which  $\psi$  is trivial  $\epsilon(s, \omega, \psi) = 1$ .

**Proposition 3.5.** *Suppose  $\mu_1$  and  $\mu_2$  are two quasi-characters of  $F^\times$  such that neither  $\mu_1^{-1}\mu_2$  nor  $\mu_1\mu_2^{-1}$  is  $\alpha_F$  and  $\pi$  is  $\pi(\mu_1, \mu_2)$ . Then*

$$W(\pi, \psi) = W(\mu_1, \mu_2; \psi)$$

and if

$$\begin{aligned} L(s, \pi) &= L(s, \mu_1)L(s, \mu_2) \\ L(s, \tilde{\pi}) &= L(s, \mu_1^{-1})L(s, \mu_2^{-1}) \\ \epsilon(s, \pi, \psi) &= \epsilon(s, \mu_1, \psi)\epsilon(s, \mu_2, \psi) \end{aligned}$$

all assertions of Theorem 2.18 are valid. In particular if  $|\mu_1(\varpi)| = |\varpi|^{-s_1}$  and  $|\mu_2(\varpi)| = |\varpi|^{-s_2}$  the integrals defining  $\Psi(g, s, W)$  are absolutely convergent if  $\text{Re } s > \max\{s_1, s_2\}$ . If  $\mu_1$  and  $\mu_2$  are unramified and  $O_F$  is the largest ideal of  $F$  on which  $\psi$  is trivial there is a

unique function  $W_0$  in  $W(\pi, \psi)$  which is invariant under  $\mathrm{GL}(2, O_F)$  and assumes the value 1 at the identity. If

$$\int_{U_F} d^\times \alpha = 1$$

then  $\Phi(e, s, W_0) = 1$ .

That  $W(\pi, \psi) = W(\mu_1, \mu_2; \psi)$  is of course a consequence of the previous proposition. As we observed the various assertions need be established only for  $g = e$ . Take  $\Phi$  in  $\mathcal{S}(F^2)$  and let  $W = W_\Phi$  be the corresponding element of  $W(\pi, \psi)$ . Then

$$\varphi(a) = W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)$$

belongs to the space of the Kirillov model of  $\pi$ . As we saw in the closing pages of the first paragraph

$$\Psi(e, s, W) = \int_{F^\times} W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) |a|^{s-1/2} d^\times a = \widehat{\varphi}(\alpha_F^{s-1/2})$$

is equal to

$$Z(\mu_1 \alpha_F^s, \mu_2 \alpha_F^s, \Phi)$$

if the last and therefore all of the integrals are defined.

Also

$$\widetilde{\Psi}(e, s, W) = Z(\mu_2^{-1} \alpha_F^s, \mu_1^{-1} \alpha_F^s, \Phi).$$

Any function in  $\mathcal{S}(F^2)$  is a linear combination of functions of the form

$$\Phi(x, y) = \varphi_1(x) \varphi_2(y).$$

Since the assertions to be proved are all linear we need only consider the functions  $\Phi$  which are given as products. Then

$$Z(\mu_1 \alpha_F^s, \mu_2 \alpha_F^s, \Phi) = Z(\mu_1 \alpha_F^s, \varphi_1) Z(\mu_2 \alpha_F^s, \varphi_2)$$

so that the integral does converge in the indicated region. Moreover

$$Z(\mu_2^{-1} \alpha_F^s, \mu_1^{-1} \alpha_F^s, \Phi) = Z(\mu_2^{-1} \alpha_F^s, \varphi_1) Z(\mu_1^{-1} \alpha_F^s, \varphi_2)$$

also converges for  $\mathrm{Re} s$  sufficiently large.  $\Phi(e, s, W)$  is equal to

$$\frac{Z(\mu_1 \alpha_F^s, \varphi_1)}{L(s, \mu_1)} \frac{Z(\mu_2 \alpha_F^s, \varphi_2)}{L(s, \mu_2)}$$

and is holomorphic in the whole complex plane. We can choose  $\varphi_1$  and  $\varphi_2$  so that both factors are 1.

It follows from the Iwasawa decomposition  $G_F = P_F \mathrm{GL}(2, O_F)$  that if both  $\mu_1$  and  $\mu_2$  are unramified there is a non-zero function on  $\mathcal{B}(\mu_1, \mu_2)$  which is invariant under  $\mathrm{GL}(2, O_F)$  and that it is unique up to a scalar factor. If the largest ideal on which  $\psi$  is trivial is  $O_F$ , if  $\Phi_0$  is the characteristic function of  $O_F^2$ , and if  $\widetilde{\Phi}_0$  is the partial Fourier transform introduced in Proposition 1.6 then  $\widetilde{\Phi}_0 = \Phi_0$ . Consequently

$$r_{\mu_1, \mu_2}(g) \Phi_0 = \Phi_0$$

for all  $g$  in  $\mathrm{GL}(2, O_F)$ . If  $W_0 = W_{\Phi_0}$  then, since  $\Phi_0$  is the product of the characteristic function of  $O_F$  with itself,  $\Phi(e, s, W_0) = 1$  if

$$\int_{U_F} d^\times \alpha = 1.$$

The only thing left to prove is the local functional equation. Observe that

$$\tilde{\Phi}(w, s, W) = \tilde{\Phi}(e, s, \rho(w)W),$$

that if  $W = W_\Phi$  then  $\rho(w)W = W_{r(w)\Phi}$ , and that  $r(w)\Phi(x, y) = \Phi'(y, x)$  if  $\Phi'$  is the Fourier transform of  $\Phi$ . Thus if  $\Phi(x, y)$  is a product  $\varphi_1(x)\varphi_2(y)$

$$\tilde{\Phi}(w, s, W) = \frac{Z(\mu_1^{-1}\alpha_F^s, \varphi'_1)}{L(s, \mu_1^{-1})} \frac{Z(\mu_2^{-1}\alpha_F^s, \varphi'_2)}{L(s, \mu_2^{-1})}.$$

The functional equation follows immediately.

If  $\mu_1\mu_2^{-1}$  is  $\alpha_F$  or  $\alpha_F^{-1}$  and  $\pi = \pi(\mu_1, \mu_2)$  we still set

$$L(s, \pi) = L(s, \mu_1)L(s, \mu_2)$$

and

$$\epsilon(s, \pi, \psi) = \epsilon(s, \mu_1, \psi)\epsilon(s, \mu_2, \psi).$$

Since  $\tilde{\pi}$  is equivalent to  $\pi(\mu_1^{-1}, \mu_2^{-1})$

$$L(s, \tilde{\pi}) = L(s, \mu_1^{-1})L(s, \mu_2^{-1}).$$

Theorem 2.18 is not applicable in this case. It has however yet to be proved for the special representations. Any special representation  $\sigma$  is of the form  $\sigma(\mu_1, \mu_2)$  with  $\mu_1 = \chi\alpha_F^{1/2}$  and  $\mu_2 = \chi\alpha_F^{-1/2}$ . The contragredient representation of  $\tilde{\sigma}$  is  $\sigma(\mu_2^{-1}, \mu_1^{-1})$ . This choice of  $\mu_1$  and  $\mu_2$  is implicit in the following proposition.

**Proposition 3.6.**  *$W(\sigma, \psi)$  is the space of functions  $W = W_\Phi$  in  $W(\mu_1, \mu_2; \psi)$  for which*

$$\int_F \Phi(x, 0) dx = 0.$$

*Theorem 2.18 will be valid if we set  $L(s, \sigma) = L(s, \tilde{\sigma}) = 1$  and  $\epsilon(s, \sigma, \psi) = \epsilon(s, \mu_1, \psi)\epsilon(s, \mu_2, \psi)$  when  $\chi$  is ramified and we set  $L(s, \sigma) = L(s, \mu_1)$ ,  $L(s, \tilde{\sigma}) = L(s, \mu_2^{-1})$ , and*

$$\epsilon(s, \sigma, \psi) = \epsilon(s, \mu_1, \psi)\epsilon(s, \mu_2, \psi) \frac{L(1-s, \mu_1^{-1})}{L(s, \mu_2)}$$

*when  $\chi$  is unramified.*

$W(\sigma, \psi)$  is of course the subspace of  $W(\mu_1, \mu_2; \psi)$  corresponding to the space  $\mathcal{B}_s(\mu_1, \mu_2)$  under the transformation  $A$  of Proposition 3.2. If  $W = W_\Phi$  then  $A$  takes  $W$  to the function  $f = f_{\tilde{\Phi}}$  defined by

$$f(g) = z \left( \mu_1 \mu_2^{-1} \alpha_F, \rho(g) \tilde{\Phi} \right) \mu_1(\det g) |\det g|^{1/2}.$$

$f$  belongs to  $\mathcal{B}_s(\mu_1, \mu_2)$  if and only if

$$\int_{\mathrm{GL}(2, O_F)} \chi^{-1}(g) f(g) dg = 0.$$

As we observed this integral is equal to a constant times

$$\int_F \chi^{-1} \left( w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) f \left( w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) dx = \int_F f \left( w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) dx$$

which equals

$$\int z \left( \alpha_F^2, \rho(w) \rho \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \tilde{\Phi} \right) dx = \int \left\{ \int \tilde{\Phi}(-t, -tx) |t|^2 d^\times t \right\} dx.$$

The double integral does converge and equals, apart from a constant factor,

$$\iint \tilde{\Phi}(t, tx) |t| dt dx = \iint \tilde{\Phi}(t, x) dt dx$$

which in turn equals

$$\int \Phi(t, 0) dt.$$

We now verify not only the remainder of the theorem but also the following corollary.

**Corollary 3.7.**

(i) If  $\pi = \pi(\mu_1, \mu_2)$  then

$$\epsilon(s, \sigma, \psi) \frac{L(1-s, \tilde{\sigma})}{L(s, \sigma)} = \epsilon(s, \pi, \psi) \frac{L(1-s, \tilde{\pi})}{L(s, \pi)}.$$

(ii) The quotient

$$\frac{L(s, \pi)}{L(s, \sigma)}$$

is holomorphic.

(iii) For all  $\Phi$  such that

$$\int \Phi(x, 0) dx = 0,$$

the quotient

$$\frac{Z(\mu_1 \alpha_F^s, \mu_2 \alpha_F^s, \Phi)}{L(s, \sigma)}$$

is holomorphic and there exists such a  $\Phi$  for which the quotient is one.

The first and second assertions of the corollary are little more than matters of definition. Although  $W(\mu_1, \mu_2 \psi)$  is not irreducible we may still, for all  $W$  in this space, define the integrals

$$\begin{aligned} \Psi(g, s, W) &= \int W \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right) |a|^{s-1/2} d^\times a \\ \tilde{\Psi}(g, s, W) &= \int W \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right) |a|^{s-1/2} \omega^{-1}(a) d^\times a. \end{aligned}$$

They may be treated in the same way as the integrals appearing in the proof of Proposition 3.5. In particular they converge to the right of some vertical line and if  $W = W_\Phi$

$$\begin{aligned}\Psi(e, s, W) &= Z(\mu_1 \alpha_F^s, \mu_2 \alpha_F^s, \Phi) \\ \tilde{\Psi}(e, s, W) &= Z(\mu_2^{-1} \alpha_F^s, \mu_1^{-1} \alpha_F^s, \Phi).\end{aligned}$$

Moreover

$$\frac{\Psi(g, s, W)}{L(s, \pi)}$$

is a holomorphic function of  $s$  and

$$\frac{\tilde{\Psi}(g, 1-s, W)}{L(1-s, \tilde{\pi})} = \epsilon(s, \pi, \psi) \frac{\Psi(g, s, W)}{L(s, \pi)}.$$

Therefore

$$\Phi(g, s, W) = \frac{\Psi(g, s, W)}{L(s, \sigma)}$$

and

$$\tilde{\Phi}(g, s, W) = \frac{\tilde{\Psi}(g, s, W)}{L(s, \tilde{\sigma})}$$

are meromorphic functions of  $s$  and satisfy the local functional equation

$$\tilde{\Phi}(wg, 1-s, W) = \epsilon(s, \sigma, \psi) \Phi(g, s, W).$$

To complete the proof of the theorem we have to show that  $\epsilon(s, \sigma, \psi)$  is an exponential function of  $s$  and we have to verify the third part of the corollary. The first point is taken care of by the observation that  $\mu_1^{-1}(\varpi)|\varpi| = \mu_2^{-1}(\varpi)$  so that

$$\frac{L(1-s, \mu_1^{-1})}{L(s, \mu_2)} = \frac{1 - \mu_2(\varpi)|\varpi|^s}{1 - \mu_1^{-1}(\varpi)|\varpi|^{1-s}} = -\mu_1(\varpi)|\varpi|^{s-1}.$$

If  $\chi$  is ramified so that  $L(s, \sigma) = L(s, \pi)$  the quotient of part (iii) of the corollary is holomorphic. Moreover a  $\Phi$  in  $\mathcal{S}(F^2)$  for which

$$Z(\mu_1 \alpha_F^s, \mu_2 \alpha_F^s, \Phi) = L(s, \sigma) = 1$$

can be so chosen that

$$\Phi(\epsilon x, \eta y) = \chi(\epsilon \eta) \Phi(x, y)$$

for  $\epsilon$  and  $\eta$  in  $U_F$ . Then

$$\int_F \Phi(x, 0) dx = 0.$$

Now take  $\chi$  unramified so that  $\chi(a) = |a|^r$  for some complex number  $r$ . We have to show that if

$$\int_F \Phi(x, 0) dx = 0$$

then

$$\frac{Z(\mu_1 \alpha_F^s, \mu_2 \alpha_F^s, \Phi)}{L(s, \mu_1)}$$

is a holomorphic function of  $s$ . Replacing  $s$  by  $s - r + 1/2$  we see that it is enough to show that

$$(1 - |\varpi|^{s+1}) \iint \Phi(x, y) |x|^{s+1} |y|^s d^\times x d^\times y$$

is a holomorphic function of  $s$ . Without any hypothesis on  $\Phi$  the integral converges for  $\operatorname{Re} s > 0$  and the product has an analytic continuation whose only poles are at the roots of  $|\varpi|^s = 1$ . To see that these poles do not occur we have only to check that there is no pole at  $s = 0$ . For a given  $\Phi$  in  $\mathcal{S}(F^2)$  there is an ideal  $\mathfrak{a}$  such that

$$\Phi(x, y) = \Phi(x, 0)$$

for  $y$  in  $\mathfrak{a}$ . If  $\mathfrak{a}'$  is the complement of  $\mathfrak{a}$

$$\iint \Phi(x, y) |x|^{s+1} |y|^s d^\times x d^\times y$$

is equal to the sum of

$$\int_F \int_{\mathfrak{a}'} \Phi(x, y) |x|^{s+1} |y|^s d^\times x d^\times y$$

which has no pole at  $s = 0$  and a constant times

$$\left\{ \int_F \Phi(x, 0) |x|^s dx \right\} \left\{ \int_{\mathfrak{a}} |y|^s d^\times y \right\}$$

If  $\mathfrak{a} = \mathfrak{p}^n$  the second integral is equal to

$$|\varpi|^{ns} (1 - |\varpi|^s)^{-1}$$

If

$$\int_F \Phi(x, 0) dx = 0$$

the first term, which defines a holomorphic function of  $s$ , vanishes at  $s = 0$  and the product has no pole there.

If  $\varphi_0$  is the characteristic function of  $O_F$  set

$$\Phi(x, y) = \{\varphi_0(x) - |\varpi|^{-1} \varphi_0(\varpi^{-1}x)\} \varphi_0(y).$$

Then

$$\int_F \Phi(x, 0) dx = 0$$

and

$$Z(\mu_1 \alpha_F^s, \mu_2 \alpha_F^s, \Phi)$$

is equal to

$$\left\{ \int (\varphi_0(x) - |\varpi|^{-1} \varphi_0(\varpi^{-1}x)) \mu_1(x) |x|^s d^\times x \right\} \left\{ \int \varphi_0(y) \mu_2(y) |y|^s d^\times y \right\}$$

The second integral equals  $L(s, \mu_2)$  and the first equals

$$(1 - \mu_1(\varpi) |\varpi|^{s-1}) L(s, \mu_1)$$

so their product is  $L(s, \mu_1) = L(s, \sigma)$ .

Theorem 2.18 is now completely proved. The properties of the local  $L$ -functions  $L(s, \pi)$  and the factors  $\epsilon(s, \pi, \psi)$  described in the next proposition will not be used until the paragraph on extraordinary representations.

**Proposition 3.8.**

- (i) If  $\pi$  is an irreducible representation there is an integer  $m$  such that if the order of  $\chi$  is greater than  $m$  both  $L(s, \chi \otimes \pi)$  and  $L(s, \chi \otimes \tilde{\pi})$  are 1.

(ii) Suppose  $\pi_1$  and  $\pi_2$  are two irreducible representations of  $G_F$  and that there is a quasi-character  $\omega$  such that

$$\pi_1 \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \omega(a)I \quad \pi_2 \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \omega(a)I$$

Then there is an integer  $m$  such that if the order of  $\chi$  is greater than  $m$

$$\epsilon(s, \chi \otimes \pi_1, \psi) = \epsilon(s, \chi \otimes \pi_2, \psi)$$

(iii) Let  $\pi$  be an irreducible representation and let  $\omega$  be the quasi-character defined by

$$\pi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \omega(a)I$$

Write  $\omega$  in any manner as  $\omega = \mu_1 \mu_2$ . Then if the order of  $\chi$  is sufficiently large in comparison to the orders of  $\mu_1$  and  $\mu_2$

$$\epsilon(s, \chi \otimes \pi, \psi) = \epsilon(s, \chi \mu_1, \psi) \epsilon(s, \chi \mu_2, \psi).$$

It is enough to treat infinite-dimensional representations because if  $\sigma = \sigma(\mu_1, \mu_2)$  and  $\pi = \pi(\mu_1, \mu_2)$  are both defined  $L(s, \chi \otimes \sigma) = L(s, \chi \otimes \pi)$ ,  $L(s, \chi \otimes \tilde{\sigma}) = L(s, \chi \otimes \tilde{\pi})$ , and  $\epsilon(s, \chi \otimes \sigma, \psi) = \epsilon(s, \chi \otimes \pi, \psi)$  if the order of  $\chi$  is sufficiently large.

If  $\pi$  is not absolutely cuspidal the first part of the proposition is a matter of definition. If  $\pi$  is absolutely cuspidal we have shown that  $L(s, \chi \otimes \pi) = L(s, \chi \otimes \tilde{\pi}) = 1$  for all  $\pi$ .

According to the relation (2.18.1)

$$\epsilon(s, \chi \otimes \pi, \psi) = C(\nu_0^{-1} \nu_1^{-1}, z_0^{-1} z_1^{-1} q^{-1/2} z^{-1})$$

if the order of  $\chi$  is so large that  $L(s, \chi \otimes \pi) = L(s, \chi^{-1} \otimes \tilde{\pi}) = 1$ . Thus to prove the second part we have only to show that if  $\{C_1(\nu, t)\}$  and  $\{C_2(\nu, t)\}$  are the series associated to  $\pi_1$  and  $\pi_2$  then

$$C_1(\nu, t) = C_2(\nu, t)$$

if the order of  $\nu$  is sufficiently large. This was proved in Lemma 2.16.6. The third part is also a consequence of that lemma but we can obtain it by applying the second part to  $\pi_1 = \pi$  and to  $\pi_2 = \pi(\mu_1, \mu_2)$ .

We finish up this paragraph with some results which will be used in the Hecke theory to be developed in the second chapter.

**Lemma 3.9.** *The restriction of the irreducible representation  $\pi$  to  $\text{GL}(2, O_F)$  contains the trivial representation if and only if there are two unramified characters  $\mu_1$  and  $\mu_2$  such that  $\pi = \pi(\mu_1, \mu_2)$ .*

This is clear if  $\pi$  is one-dimensional so we may as well suppose that  $\pi$  is infinite-dimensional. If  $\pi = \pi(\mu_1, \mu_2)$  we may let  $\pi = \rho(\mu_1, \mu_2)$ . It is clear that there is a non-zero vector in  $\mathcal{B}(\mu_1, \mu_2)$  invariant under  $\text{GL}(2, O_F)$  if and only if  $\mu_1$  and  $\mu_2$  are unramified and that if there is such a vector it is determined up to a scalar factor. If  $\pi = \sigma(\mu_1, \mu_2)$  and  $\mu_1 \mu_2^{-1} = \alpha_F$  we can suppose that  $\pi$  is the restriction of  $\rho(\mu_1, \mu_2)$  to  $\mathcal{B}_s(\mu_1, \mu_2)$ . The vectors in  $\mathcal{B}(\mu_1, \mu_2)$  invariant under  $\text{GL}(2, O_F)$  clearly do not lie in  $\mathcal{B}_s(\mu_1, \mu_2)$  so that the restriction of  $\pi$  to  $\text{GL}(2, O_F)$  does not contain the trivial representation. All that we have left to do is to show that the restriction of an absolutely cuspidal representation to  $\text{GL}(2, O_F)$  does not contain the trivial representation.



Suppose the infinite-dimensional irreducible representation  $\pi$  is given in the Kirillov form with respect to an additive character  $\psi$  such that  $O_F$  is the largest ideal on which  $\psi$  is trivial. Suppose the non-zero vector  $\varphi$  is invariant under  $\mathrm{GL}(2, O_F)$ . It is clear that if

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I$$

then  $\omega$  is unramified, that  $\varphi(\nu, t) = 0$  unless  $\nu = 1$  is the trivial character, and that  $\varphi(\nu, t)$  has no pole at  $t = 0$ . Suppose  $\pi$  is absolutely cuspidal so that  $\varphi$  belongs to  $\mathcal{S}(F^\times)$ . Since  $\pi(w)\varphi = \varphi$  and the restriction of  $\omega$  to  $U_F$  is trivial

$$\widehat{\varphi}(1, t) = C(1, t)\widehat{\varphi}(1, z_0^{-1}t^{-1})$$

if  $z_0 = \omega(\varpi)$ . Since  $C(1, t)$  is a constant times a negative power of  $t$  the series on the left involves no negative powers of  $t$  and that on the right involves only negative powers. This is a contradiction.

Let  $\mathcal{H}_0$  be the subalgebra of the Hecke algebra formed by the functions which are invariant under left and right translations by elements of  $\mathrm{GL}(2, O_F)$ . Suppose the irreducible representation  $\pi$  acts on the space  $X$  and there is a non-zero vector  $x$  in  $X$  invariant under  $\mathrm{GL}(2, O_F)$ . If  $f$  is in  $\mathcal{H}_0$  the vector  $\pi(f)x$  has the same property and is therefore a multiple  $\lambda(f)x$  of  $x$ . The map  $f \rightarrow \lambda(f)$  is a non-trivial homomorphism of  $\mathcal{H}_0$  into the complex numbers.

**Lemma 3.10.** *Suppose  $\pi = \pi(\mu_1, \mu_2)$  where  $\mu_1$  and  $\mu_2$  are unramified and  $\lambda$  is the associated homomorphism of  $\mathcal{H}_0$  into  $\mathbf{C}$ . There is a constant  $c$  such that*

$$(3.10.1) \quad |\lambda(f)| \leq c \int_{G_F} |f(g)| dg$$

for all  $f$  in  $\mathcal{H}_0$  if and only if  $\mu_1\mu_2$  is a character and  $|\mu_1(\varpi)\mu_2^{-1}(\varpi)| = |\varpi|^s$  with  $-1 \leq s \leq 1$ .

Let  $\tilde{\pi}$  act on  $\tilde{X}$  and let  $\tilde{x}$  in  $\tilde{X}$  be such that  $\langle x, \tilde{x} \rangle \neq 0$ . Replacing  $\tilde{x}$  by

$$\int_{\mathrm{GL}(2, O_F)} \tilde{\pi}(g)\tilde{x} dg$$

if necessary we may suppose that  $\tilde{x}$  is invariant under  $\mathrm{GL}(2, O_F)$ . We may also assume that  $\langle x, \tilde{x} \rangle = 1$ . If  $\eta(g) = \langle \pi(g)x, \tilde{x} \rangle$  then

$$\lambda(f)\eta(g) = \int_{G_F} \eta(gh)f(h) dh$$

for all  $f$  in  $\mathcal{H}_0$ . In particular

$$\lambda(f) = \int_{G_F} \eta(h)f(h) dh.$$

If  $|\eta(h)| \leq c$  for all  $h$  the inequality (3.10.1) is certainly valid. Conversely, since  $\eta$  is invariant under left and right translations by  $\mathrm{GL}(2, O_F)$  we can, if the inequality holds, apply it to the characteristic functions of double cosets of this group to see that  $|\eta(h)| \leq c$  for all  $h$ . Since

$$\eta\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}h\right) = \mu_1(a)\mu_2(a)\eta(h)$$

the function  $\eta$  is bounded only if  $\mu_1\mu_2$  is a character as we now assume it to be. The finite-dimensional representations take care of themselves so we now assume  $\pi$  is infinite-dimensional.

Since  $\pi$  and  $\tilde{\pi}$  are irreducible the function  $\langle \pi(g)x, \tilde{x} \rangle$  is bounded for a given pair of non-zero vectors if and only if it is bounded for all pairs. Since  $G_F = \mathrm{GL}(2, O_F)A_F\mathrm{GL}(2, O_F)$  and  $\mu_1\mu_2$  is a character these functions are bounded if and only if the functions

$$\left\langle \pi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) x, \tilde{x} \right\rangle$$

are bounded on  $F^\times$ . Take  $\pi$  and  $\tilde{\pi}$  in the Kirillov form. If  $\varphi$  is in  $V$  and  $\tilde{\varphi}$  is in  $V$  then

$$\left\langle \pi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) \varphi, \tilde{\pi}(w)\tilde{\varphi} \right\rangle$$

is equal to

$$\left\langle \pi^{-1}(w)\pi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) \varphi, \tilde{\varphi} \right\rangle = \mu_1(a)\mu_2(a) \left\langle \pi \left( \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right) \pi^{-1}(w)\varphi, \tilde{\varphi} \right\rangle$$

Thus  $\eta(g)$  is bounded if and only if the functions

$$\left\langle \pi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) \varphi, \tilde{\varphi} \right\rangle$$

are bounded for all  $\varphi$  in  $V$  and all  $\tilde{\varphi}$  in  $\mathcal{S}(F^\times)$ .

It is not necessary to consider all  $\tilde{\varphi}$  in  $\mathcal{S}(F^\times)$  but only a set which together with its translates by the diagonal matrices spans  $\mathcal{S}(F^\times)$ . If  $\mu$  is a character of  $U_F$  let  $\varphi_\mu$  be the function on  $F^\times$  which is 0 outside of  $U_F$  and equals  $\mu$  on  $U_F$ . It will be sufficient to consider the functions  $\tilde{\varphi} = \varphi_\mu$  and all we need show is that

$$(3.10.2) \quad \left\langle \pi \left( \begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix} \right) \varphi, \varphi_\mu \right\rangle$$

is a bounded function of  $n$  for all  $\mu$  and all  $\varphi$ . The expression (3.10.2) is equal to  $\hat{\varphi}_n(\mu)$ . If  $\varphi$  belongs to  $\mathcal{S}(F^\times)$  the sequence  $\{\hat{\varphi}_n(\mu)\}$  has only finitely many non-zero terms and there is no problem. If  $\varphi = \pi(w)\varphi_0$  then

$$\sum_n \hat{\varphi}_n(\mu)t^n = C(\mu, t)\eta(t)$$

where  $\eta(t)$  depends on  $\varphi_0$  and is an arbitrary finite Laurent series. We conclude that (3.10.1) is valid if and only if  $\mu_1\mu_2$  is a character and the coefficients of the Laurent series  $C(\mu, t)$  are bounded for every choice of  $\mu$ .

It follows from Proposition 3.5 and formula (2.18.1) that, in the present case, the series has only one term if  $\mu$  is ramified but that if  $\mu$  is trivial

$$C\left(\mu, |\varpi|^{1/2}\mu_1^{-1}(\varpi)\mu_2^{-1}(\varpi)t\right) = \frac{(1 - \mu_1(\varpi)t^{-1})(1 - \mu_2(\varpi)t^{-1})}{(1 - \mu_1^{-1}(\varpi)|\varpi|t)(1 - \mu_2^{-1}(\varpi)|\varpi|t)}.$$

The function on the right has zeros at  $t = \mu_1(\varpi)$  and  $t = \mu_2(\varpi)$  and poles at  $t = 0$ ,  $t = |\varpi|^{-1}\mu_1(\varpi)$ , and  $t = |\varpi|^{-1}\mu_2(\varpi)$ . A zero can cancel a pole only if  $\mu_2(\varpi) = |\varpi|^{-1}\mu_1(\varpi)$

or  $\mu_1(\varpi) = |\varpi|^{-1}\mu_2(\varpi)$ . Since  $\mu_1$  and  $\mu_2$  are unramified this would mean that  $\mu_1^{-1}\mu_2$  equals  $\alpha_F$  or  $\alpha_F^{-1}$  which is impossible when  $\pi = \pi(\mu_1, \mu_2)$  is infinite-dimensional.

If  $C(\mu, t)$  has bounded coefficients and  $\mu_1\mu_2$  is a character the function on the right has no poles for  $|t| < |\varpi|^{-1/2}$  and therefore  $|\mu_1(\varpi)| \geq |\varpi|^{1/2}$  and  $|\mu_2(\varpi)| \geq |\varpi|^{1/2}$ . Since

$$|\mu_1(\varpi)\mu_2^{-1}(\varpi)| = |\mu_1(\varpi)|^2 = |\mu_2^{-1}(\varpi)|^2$$

where  $\mu_1\mu_2$  is a character these two inequalities are equivalent to that of the lemma. Conversely if these two inequalities are satisfied the rational function on the right has no pole except that at 0 inside the circle  $|t| = |\varpi|^{-1/2}$  and at most simple poles on the circle itself. Applying, for example, partial fractions to find its Laurent series expansion about 0 one finds that the coefficients of  $C(\mu, t)$  are bounded.

**Lemma 3.11.** *Suppose  $\mu_1$  and  $\mu_2$  are unramified,  $\mu_1\mu_2$  is a character, and  $\pi = \pi(\mu_1, \mu_2)$  is infinite-dimensional. Let  $|\mu_1(\varpi)| = |\varpi|^r$  where  $r$  is real so that  $|\mu_2(\varpi)| = |\varpi|^{-r}$ . Assume  $O_F$  is the largest ideal on which  $\psi$  is trivial and let  $W_0$  be that element of  $W(\pi, \psi)$  which is invariant under  $\text{GL}(2, O_F)$  and takes the value 1 at the identity. If  $s > |r|$  then*

$$\int_{F^\times} \left| W_0 \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) \right| |a|^{s-1/2} d^\times a \leq \frac{1}{(1 - |\varpi|^{s+r})(1 - |\varpi|^{s-r})}$$

if the Haar measure is so normalized that the measure of  $U_F$  is one.

If  $\Phi$  is the characteristic function of  $O_F^2$  then

$$W_0 \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) = \mu_1(a) |a|^{1/2} \int_{F^\times} \Phi(at, t^{-1}) \mu_1(t) \mu_2^{-1} d^\times t$$

and

$$\int_{F^\times} \left| W_0 \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) \right| |a|^{s-1/2} d^\times a \leq \iint \Phi(at, t^{-1}) |a|^{s+r} |t|^{2r} d^\times a d^\times t.$$

Changing variables in the left-hand side we obtain

$$\int_{O_F} \int_{O_F} |a|^{s+r} |b|^{s-r} d^\times a d^\times b = \frac{1}{(1 - |\varpi|^{s+r})(1 - |\varpi|^{s-r})}.$$

#### §4. Examples of absolutely cuspidal representations

In this paragraph we will use the results of the first paragraph to construct some examples of absolutely cuspidal representations.

First of all let  $K$  be a quaternion algebra over  $F$ .  $K$  is of course unique up to isomorphism. As in the first paragraph  $\Omega$  will denote a continuous finite-dimensional representation of  $K^\times$  the multiplicative group of  $K$ . If  $\chi$  is a quasi-character of  $F^\times$  and  $\nu$  is the reduced norm on  $K$  we denote the one-dimensional representation  $g \rightarrow \chi(\nu(g))$  of  $K^\times$  by  $\chi$  also. If  $\Omega$  is any representation  $\chi \otimes \Omega$  is the representation  $g \rightarrow \chi(g)\Omega(g)$ . If  $\Omega$  is irreducible all operators commuting with the action of  $K^\times$  are scalars. In particular there is a quasi-character  $\omega$  of  $F^\times$  such that

$$\Omega(a) = \omega(a)I$$

for all  $a$  in  $F^\times$  which is of course a subgroup of  $K^\times$ . If  $\Omega$  is replaced by  $\chi \otimes \Omega$  then  $\omega$  is replaced by  $\chi^2\omega$ .  $\tilde{\Omega}$  will denote the representation contragredient to  $\Omega$ .

Suppose  $\Omega$  is irreducible, acts on  $V$ , and the quasi-character  $\omega$  is a character. Since  $K^\times/F^\times$  is compact there is a positive definite hermitian form on  $V$  invariant under  $K^\times$ . When this is so we call  $\Omega$  unitary.

It is a consequence of the following lemma that any one-dimensional representation of  $K^\times$  is the representation associated to a quasi-character of  $F^\times$ .

**Lemma 4.1.** *Let  $K_1$  be the subgroup of  $K^\times$  consisting of those  $x$  for which  $\nu(x) = 1$ . Then  $K_1$  is the commutator subgroup, in the sense of group theory, of  $K^\times$ .*

$K_1$  certainly contains the commutator subgroup. Suppose  $x$  belongs to  $K_1$ . If  $x = x'$  then  $x^2 = xx' = 1$  so that  $x = \pm 1$ . Otherwise  $x$  determines a separable quadratic extension of  $F$ . Thus, in all cases, if  $xx' = 1$  there is a subfield  $L$  of  $K$  which contains  $x$  and is quadratic and separable over  $L$ . By Hilbert's Theorem 90 there is a  $y$  in  $L$  such that  $x = yy^{-t}$ . Moreover there is an element  $\sigma$  in  $K$  such that  $\sigma z \sigma^{-1} = z^t$  for all  $z$  in  $L$ . Thus  $x = y\sigma y^{-1}\sigma^{-1}$  is in the commutator subgroup.

In the first paragraph we associated to  $\Omega$  a representation  $r_\Omega$  of a group  $G_+$  on the space  $\mathcal{S}(K, \Omega)$ . Since  $F$  is now non-archimedean the group  $G_+$  is now  $G_F = \text{GL}(2, F)$ .

**Theorem 4.2.**

- (i) *The representation  $r_\Omega$  is admissible.*
- (ii) *Let  $d = \text{degree } \Omega$ . Then  $r_\Omega$  is equivalent to the direct sum of  $d$  copies of an irreducible representation  $\pi(\Omega)$ .*
- (iii) *If  $\Omega$  is the representation associated to a quasi-character  $\chi$  of  $F^\times$  then*

$$\pi(\Omega) = \sigma(\chi\alpha_F^{1/2}, \chi\alpha_F^{-1/2}).$$

- (iv) *If  $d > 1$  the representation  $\pi(\Omega)$  is absolutely cuspidal.*

If  $n$  is a natural number we set

$$G_n = \{ g \in \text{GL}(2, O_F) \mid g = I \pmod{\mathfrak{p}^n} \}$$

We have first to show that if  $\Phi$  is in  $\mathcal{S}(K, \Omega)$  there is an  $n$  such that  $r_\Omega(g)\Phi = \Phi$  if  $g$  is in  $G_n$  and that for a given  $n$  the space of  $\Phi$  in  $\mathcal{S}(K, \Omega)$  for which  $r_\Omega(g)\Phi = \Phi$  for all  $g$  in  $G_n$  is finite-dimensional.

Any

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in  $G_n$  may be written as

$$g = \begin{pmatrix} 1 & 0 \\ ca^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & b' \\ 0 & d' \end{pmatrix}$$

and both the matrices on the right are in  $G_n$ . Thus  $G_n$  is generated by the matrices of the forms

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad w \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} w^{-1} \quad w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} w^{-1}$$

with  $a \equiv 1 \pmod{\mathfrak{p}^n}$  and  $x \equiv 0 \pmod{\mathfrak{p}^n}$ . It will therefore be enough to verify the following three assertions.

(4.2.1) Given  $\Phi$  there is an  $n > 0$  such that

$$r_\Omega \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) \Phi = \Phi$$

if  $a \equiv 1 \pmod{\mathfrak{p}^n}$ .

(4.2.2) Given  $\Phi$  there is an  $n > 0$  such that

$$r_\Omega \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \Phi = \Phi$$

if  $x \equiv 0 \pmod{\mathfrak{p}^n}$ .

(4.2.3) Given  $n > 0$  the space of  $\Phi$  in  $\mathcal{S}(K, \Omega)$  such that

$$r_\Omega \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \Phi = \Phi$$

and

$$r_\Omega(w^{-1}) r_\Omega \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) r_\Omega(w) \Phi = \Phi$$

for all  $x$  in  $\mathfrak{p}^n$  is finite-dimensional.

If  $a = \nu(h)$  then

$$r_\Omega \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) \Phi = |h|_K^{1/2} \Omega(h) \Phi(xh).$$

Since  $\Phi$  has compact support in  $K$  and is locally constant there is a neighbourhood  $U$  of 1 in  $K^\times$  such that

$$\Omega(h) \Phi(xh) |h|_K^{1/2} = \Phi(x)$$

for all  $h$  in  $U$  and all  $x$  in  $K$ . The assertion (4.2.1) now follows from the observation that  $\nu$  is an open mapping of  $K^\times$  onto  $F^\times$ .

We recall that

$$r_\Omega \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \Phi(z) = \psi(x\nu(z)) \Phi(z)$$

Let  $\mathfrak{p}^{-\ell}$  be the largest ideal on which  $\psi$  is trivial and let  $\mathfrak{p}_K$  be the prime ideal of  $K$ . Since  $\nu(\mathfrak{p}_K^m) = \mathfrak{p}_F^m$

$$r_\Omega \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \Phi = \Phi$$

for all  $x$  in  $\mathfrak{p}^n$  if and only if the support of  $\Phi$  is contained in  $\mathfrak{p}_K^{-n-\ell}$ . With this (4.2.2) is established.

$\Phi$  satisfies the two conditions of (4.2.3) if and only if both  $\Phi$  and  $r(w)\Phi$  have support in  $\mathfrak{p}_K^{-n-\ell}$  or, since  $r(w)\Phi = -\Phi'$ , if and only if  $\Phi$  and  $\Phi'$ , its Fourier transform, have support in this set. There is certainly a natural number  $k$  such that  $\psi(\tau(y)) = 1$  for all  $y$  in  $\mathfrak{p}_K^k$ . Assertion (4.2.3) is therefore a consequence of the following simple lemma.

**Lemma 4.2.4.** *If the support of  $\Phi$  is contained in  $\mathfrak{p}_K^{-n}$  and  $\psi(\tau(y)) = 1$  for all  $y$  in  $\mathfrak{p}_K^k$  the Fourier transform of  $\Phi$  is constant on cosets of  $\mathfrak{p}_K^{k+n}$ .*

Since

$$\Phi'(x) = \int_{\mathfrak{p}_K^{-n}} \Phi(y) \psi(\tau(x, y)) dy$$

the lemma is clear.

We prove the second part of the theorem for one-dimensional  $\Omega$  first. Let  $\Omega$  be the representation associated to  $\chi$ .  $\mathcal{S}(K, \Omega)$  is the space of  $\Phi$  in  $\mathcal{S}(K)$  such that  $\Phi(xh) = \Phi(x)$  for all  $h$  in  $K_1$ . Thus to every  $\Phi$  in  $\mathcal{S}(K, \Omega)$  we may associate the function  $\varphi_\Phi$  on  $F^\times$  defined by

$$\varphi_\Phi(a) = |h|_K^{1/2} \Omega(h) \Phi(h)$$

if  $a = \nu(h)$ . The map  $\Phi \rightarrow \varphi_\Phi$  is clearly injective. If  $\varphi$  belongs to  $\mathcal{S}(F^\times)$  the function  $\Phi$  defined by

$$\Phi(h) = |h|_K^{-1/2} \Omega^{-1}(h) \varphi(\nu(h))$$

if  $h \neq 0$  and by

$$\Phi(0) = 0$$

belongs to  $\mathcal{S}(K, \Omega)$  and  $\varphi = \varphi_\Phi$ . Let  $\mathcal{S}_0(K, \Omega)$  be the space of functions obtained in this way. It is the space of functions in  $\mathcal{S}(K, \Omega)$  which vanish at 0 and therefore is of codimension one. If  $\Phi$  belongs to  $\mathcal{S}_0(K, \Omega)$ , is non-negative, does not vanish identically and  $\Phi'$  is its Fourier transform then

$$\Phi'(0) = \int \Phi(x) dx \neq 0.$$

Thus  $r_\Omega(w)\Phi$  does not belong to  $\mathcal{S}_0(K, \Omega)$  and  $\mathcal{S}_0(K, \Omega)$  is not invariant. Since it is of codimension one there is no proper invariant subspace containing it.

Let  $V$  be the image of  $\mathcal{S}(K, \omega)$  under the map  $\Phi \rightarrow \varphi_\Phi$ . We may regard  $r_\Omega$  as acting in  $V$ . From the original definitions we see that

$$r_\Omega(b)\varphi = \xi_\psi(b)\varphi$$

if  $b$  is in  $B_F$ . If  $V_1$  is a non-trivial invariant subspace of  $V$  the difference

$$\varphi - r_\Omega \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \varphi$$

is in  $V_0 \cap V_1$  for all  $\varphi$  in  $V_1$  and all  $x$  in  $F$ . If  $\varphi$  is not zero we can certainly find an  $x$  for which the difference is not zero. Consequently  $V_0 \cap V_1$  is not 0 so that  $V_1$  contains  $V_0$  and hence all of  $V$ .

The representation  $r_\Omega$  is therefore irreducible and when considered as acting on  $V$  it is in the Kirillov form. Since  $V_0$  is not  $V$  it is not absolutely cuspidal. It is thus a  $\pi(\mu_1, \mu_2)$  or a  $\sigma(\mu_1, \mu_2)$ . To see which we have to find a linear form on  $V$  which is trivial on  $V_0$ . The obvious choice is

$$L(\varphi) = \Phi(0)$$

if  $\varphi = \varphi_\Phi$ . Then

$$L\left(r_\Omega\left(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\varphi\right)\right) = \chi(a_1 a_2) \left|\frac{a_1}{a_2}\right| L(\varphi).$$

To see this we have only to recall that

$$r_\Omega\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \Omega(a)I = \chi^2(a)I$$

and that

$$r_\Omega\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\Phi(0) = |h|_K^{1/2}\Omega(h)\Phi(0)$$

where  $a = \nu(h)$  so that  $|h|_K^{1/2} = |a|_F$  and  $\Omega(h) = \chi(a)I$ . Thus if

$$A\varphi(g) = L(r_\Omega(g)\varphi)$$

$A$  is an injection of  $V$  into an irreducible invariant subspace of  $\mathcal{B}(\chi\alpha_F^{1/2}, \chi\alpha_F^{-1/2})$ . The only such subspace is  $\mathcal{B}_s(\chi\alpha_F^{1/2}, \chi\alpha_F^{-1/2})$  and  $r_\Omega$  is therefore  $\sigma(\chi\alpha_F^{1/2}, \chi\alpha_F^{-1/2})$ .

Suppose now that  $\Omega$  is not one-dimensional. Let  $\Omega$  act on  $U$ . Since  $K_1$  is normal and  $K/K_1$  is abelian there is no non-zero vector in  $U$  fixed by every element of  $K_1$ . If  $\Phi$  is in  $\mathcal{S}(K, \Omega)$  then

$$\Phi(xh) = \Omega^{-1}(h)\Phi(x)$$

for all  $h$  in  $K_1$ . In particular  $\Phi(0)$  is fixed by every element in  $K_1$  and is therefore 0. Thus all functions in  $\mathcal{S}(K, \Omega)$  have compact supports in  $K^\times$  and if we associate to every  $\Phi$  in  $\mathcal{S}(K, \Omega)$  the function

$$\varphi_\Phi(a) = |h|_K^{1/2}\Omega(h)\Phi(h)$$

where  $a = \nu(h)$  we obtain a bijection from  $\mathcal{S}(K, \Omega)$  to  $\mathcal{S}(F^\times, U)$ . It is again clear that

$$\varphi_{\Phi_1} = \xi_\psi(b)\varphi_\Phi$$

if  $b$  is in  $B_F$  and  $\Phi_1 = r_\Omega(b)\Phi$ .

**Lemma 4.2.5.** *Let  $\Omega$  be an irreducible representation of  $K^\times$  in the complex vector space  $U$ . Assume that  $U$  has dimension greater than one.*

(i) *For any  $\Phi$  in  $\mathcal{S}(K, U)$  the integrals*

$$\begin{aligned} Z(\alpha_F^s \otimes \Omega, \Phi) &= \int_{K^\times} |a|_K^{s/2} \Omega(a) \Phi(a) d^\times a \\ Z(\alpha_F^s \otimes \Omega^{-1}, \Phi) &= \int_{K^\times} |a|_K^{s/2} \Omega^{-1}(a) \Phi(a) d^\times a \end{aligned}$$

are absolutely convergent in some half-plane  $\operatorname{Re} s > s_0$ .

- (ii) The functions  $Z(\alpha_F^s \otimes \Omega, \Phi)$  and  $Z(\alpha_F^s \otimes \Omega^{-1}, \Phi)$  can be analytically continued to functions meromorphic in the whole complex plane.
- (iii) Given  $u$  in  $U$  there is a  $\Phi$  in  $\mathcal{S}(K, U)$  such that

$$Z(\alpha_F^s \otimes \Omega, \Phi) \equiv u.$$

- (iv) There is a scalar function  $\epsilon(s, \Omega, \psi)$  such that for all  $\Phi$  in  $\mathcal{S}(K, U)$

$$Z(\alpha_F^{3/2-s} \otimes \Omega^{-1}, \Phi') = -\epsilon(s, \Omega, \psi) Z(\alpha_F^{s+1/2} \otimes \Omega, \Phi)$$

if  $\Phi'$  is the Fourier transform of  $\Phi$ . Moreover, as a function of  $s$ ,  $\epsilon(s, \Omega, \psi)$  is a constant times an exponential.

There is no need to verify the first part of the lemma. Observe that  $\alpha_F(\nu(x)) = |\nu(x)|_F = |x|_K^{1/2}$  so that

$$(\alpha_F^s \otimes \Omega)(x) = |x|_K^{s/2} \Omega(x).$$

If  $\Phi$  belongs to  $\mathcal{S}(K, U)$  set

$$\Phi_1(x) = \int_{K_1} \Omega(h) \Phi(xh).$$

The integration is taken with respect to the normalized Haar measure on the compact group  $K_1$ .  $\Phi_1$  clearly belongs to  $\mathcal{S}(K, U)$  and

$$(4.2.6) \quad Z(\alpha_F^s \otimes \Omega, \Phi) = Z(\alpha_F^s \otimes \Omega, \Phi_1)$$

and the Fourier transform  $\Phi'_1$  of  $\Phi_1$  is given by

$$\Phi'_1(x) = \int_{K_1} \Omega(h^{-1}) \Phi'(hx)$$

The function  $\Phi'_1(x')$  belongs to  $\mathcal{S}(K, \Omega)$  and

$$(4.2.7) \quad Z(\alpha_F^s \otimes \Omega^{-1}, \Phi') = Z(\alpha_F^s \otimes \Omega^{-1}, \Phi'_1).$$

Since  $\Phi_1$  and  $\Phi'_1$  both have compact support in  $K^\times$  the second assertion is clear.

If  $u$  is in  $U$  and we let  $\Phi_u$  be the function which is  $O$  outside of  $U_K$ , the group of units of  $O_K$ , and on  $U_K$  is given by  $\Phi_u(x) = \Omega^{-1}(x)u$  then

$$Z(\alpha_F^s \otimes \Omega, \Phi_u) = cu$$

if

$$c = \int_{U_K} d^\times a.$$

If  $\varphi$  belongs to  $\mathcal{S}(K^\times)$  let  $A(\varphi)$  and  $B(\varphi)$  be the linear transformations of  $U$  defined by

$$A(\varphi)U = Z(\alpha_F^{s+1/2} \otimes \Omega, \varphi^u)$$

$$B(\varphi)u = Z(\alpha_F^{-s+3/2} \otimes \Omega^{-1}, \varphi'u)$$

where  $\varphi'$  is the Fourier transform of  $\varphi$ . If  $\lambda(h)\varphi(x) = \varphi(h^{-1}x)$  and  $\rho(h)\varphi(x) = \varphi(xh)$  then

$$A(\lambda(h)\varphi) = |h|_K^{s/2+1/4} \Omega(h) A(\varphi)$$

and

$$A(\rho(h)\varphi) = |h|_K^{-s/2-1/4} A(\varphi) \Omega^{-1}(h).$$



Since the Fourier transform of  $\lambda(h)\varphi$  is  $|h|_K\rho(h)\varphi'$  and the Fourier transform of  $\rho(h)\varphi$  is  $|h|_K^{-1}\lambda(h)\varphi'$ , the map  $\varphi \rightarrow B(\varphi)$  has the same two properties. Since the kernel of  $\Omega$  is open it is easily seen that  $A(\varphi)$  and  $B(\varphi)$  are obtained by integrating  $\varphi$  against locally constant functions  $\alpha$  and  $\beta$ . They will of course take values in the space of linear transformations of  $U$ . We will have

$$\alpha(ha) = |h|_K^{s/2+1/4}\Omega(h)\alpha(a)$$

and

$$\alpha(ah^{-1}) = |h|_K^{-s/2-1/4}\alpha(a)\Omega^{-1}(h)$$

$\beta$  will satisfy similar identities. Thus

$$\alpha(h) = |h|_K^{s/2+1/4}\Omega(h)\alpha(1),$$

$$\beta(h) = |h|_K^{s/2+1/4}\Omega(h)\beta(1),$$

where  $\alpha(1)$  is of course the identity. However  $\beta(1)$  must commute with  $\Omega(h)$  for all  $h$  in  $K^\times$  and therefore it is a scalar multiple of the identity. Take this scalar to be  $-\epsilon(s, \Omega, \psi)$ .

The identity of part (iv) is therefore valid for  $\Phi$  in  $\mathcal{S}(K^\times, U)$  and in particular for  $\Phi$  in  $\mathcal{S}(K, \Omega)$ . The general case follows from (4.2.6) and (4.2.7). Since

$$\epsilon(s, \Omega, \psi) = -\frac{1}{c}Z(\alpha_F^{3/2-s} \otimes \Omega^{-1}, \Phi'_u)$$

the function  $\epsilon(s, \Omega, \psi)$  is a finite linear combination of powers  $|\varpi|^s$  if  $\varpi$  is a generator of  $\mathfrak{p}_F$ . Exchanging the roles of  $\Phi_u$  and  $\Phi'_u$  we see that  $\epsilon^{-1}(s, \Omega, \psi)$  has the same property. The factor  $\epsilon(s, \Omega, \psi)$  is therefore a multiple of some power of  $|\varpi|^s$ .

We have yet to complete the proof of the theorem. Suppose  $\varphi = \varphi_\Phi$  belongs to  $\mathcal{S}(F^\times, U)$  and  $\varphi' = \varphi_{r_\Omega(w)\Phi}$ . We saw in the first paragraph that if  $\chi$  is a quasi-character of  $F^\times$  then

$$(4.2.8) \quad \widehat{\varphi}(\chi) = Z(\alpha_F\chi \otimes \Omega, \Phi)$$

and, if  $\Omega(a) = \omega(a)I$  for  $a$  in  $F^\times$ ,

$$(4.2.9) \quad \widehat{\varphi}'(\chi^{-1}\omega^{-1}) = -Z(\alpha_F\chi^{-1} \otimes \Omega^{-1}, \Phi').$$

Suppose  $U_0$  is a subspace of  $U$  and  $\varphi$  takes its values in  $U_0$ . Then, by the previous lemma,  $\widehat{\varphi}(\chi)$  and  $\widehat{\varphi}'(\chi^{-1}\omega^{-1})$  also lie in  $U_0$  for all choices of  $\chi$ . Since  $\varphi'$  lies in  $\mathcal{S}(F^\times, U)$  we may apply Fourier inversion to the multiplicative group to see that  $\varphi'$  takes values in  $U_0$ .

We may regard  $r_\Omega$  as acting on  $\mathcal{S}(F^\times, U)$ . Then  $\mathcal{S}(F^\times, U_0)$  is invariant under  $r_\Omega(w)$ . Since  $r_\Omega(b)\varphi = \xi_\psi(b)\varphi$  for  $b$  in  $B_F$  it is also invariant under the action of  $B_F$ . Finally  $r_\Omega\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right)\varphi = \omega(a)\varphi$  so that  $\mathcal{S}(F^\times, U_0)$  is invariant under the action of  $G_F$  itself. If we take  $U_0$  to have dimension one then  $\mathcal{S}(F^\times, U_0)$  may be identified with  $\mathcal{S}(F^\times)$  and the representation  $r_\Omega$  restricted to  $\mathcal{S}(F^\times, U_0)$  is irreducible. From (4.2.8) and (4.2.9) we obtain

$$\begin{aligned} \widehat{\varphi}(\alpha_F^{s-1/2}\chi) &= Z(\alpha_F^{s+1/2}\chi \otimes \Omega, \Phi) \\ \widehat{\varphi}'(\alpha_F^{-s+1/2}\chi^{-1}\omega^{-1}) &= -Z(\alpha_F^{-s+3/2}\chi^{-1} \otimes \Omega^{-1}, \Phi') \end{aligned}$$

so that

$$\widehat{\varphi}'(\alpha_F^{-s+1/2}\chi^{-1}\omega^{-1}) = \epsilon(s, \chi \otimes \Omega, \psi)\widehat{\varphi}(\alpha_F^{s-1/2}\chi).$$

Thus if  $\pi_0$  is the restriction of  $r_\Omega$  to  $\mathcal{S}(F^\times, U_0)$

$$\epsilon(s, \chi \otimes \pi_0, \psi) = \epsilon(s, \chi \otimes \Omega, \psi)$$

so that  $\pi_0 = \pi(\Omega)$  is, apart from equivalence, independent of  $U_0$ . The theorem follows.

Let  $\Omega$  be any irreducible finite-dimensional representation of  $K^\times$  and let  $\Omega$  act on  $U$ . The contragredient representation  $\tilde{\Omega}$  acts on the dual space  $\tilde{U}$  of  $U$ . If  $u$  belongs to  $U$  and  $\tilde{u}$  belongs to  $\tilde{U}$

$$\langle u, \tilde{\Omega}(h)\tilde{u} \rangle = \langle \Omega^{-1}(h)u, \tilde{u} \rangle.$$

If  $\Phi$  belongs to  $\mathcal{S}(K)$  set

$$Z(\alpha_F^s \otimes \Omega, \Phi; u, \tilde{u}) = \int_{K^\times} |\nu(h)|^s \Phi(h) \langle \Omega(h)u, \tilde{u} \rangle d^\times h$$

and set

$$Z(\alpha_F^s \otimes \tilde{\Omega}, \Phi; u, \tilde{u}) = \int_{K^\times} |\nu(h)|^s \Phi(h) \langle u, \tilde{\Omega}(h)\tilde{u} \rangle d^\times h.$$

**Theorem 4.3.** *Let  $\Omega$  be an irreducible representation of  $K^\times$  in the space  $U$ .*

(i) *For any quasi-character  $\chi$  of  $F^\times$*

$$\pi(\chi \otimes \Omega) = \chi \otimes \pi(\Omega).$$

(ii) *There is a real number  $s_0$  such that for all  $u, \tilde{u}$  and  $\Phi$  and all  $s$  with  $\operatorname{Re} s > s_0$  the integral defining  $Z(\alpha_F^s \otimes \Omega, \Phi; u, \tilde{u})$  is absolutely convergent.*

(iii) *There is a unique Euler factor  $L(s, \Omega)$  such that the quotient*

$$\frac{Z(\alpha_F^{s+1/2} \otimes \Omega, \Phi, u, \tilde{u})}{L(s, \Omega)}$$

*is holomorphic for all  $u, \tilde{u}, \Phi$  and for some choice of these variables is a non-zero constant.*

(iv) *There is a functional equation*

$$\frac{Z(\alpha_F^{3/2-s} \otimes \tilde{\Omega}, \Phi', u, \tilde{u})}{L(1-s, \tilde{\Omega})} = -\epsilon(s, \Omega, \psi) \frac{Z(\alpha_F^{s+1/2} \otimes \Omega, \Phi, u, \tilde{u})}{L(s, \Omega)}$$

*where  $\epsilon(s, \Omega, \psi)$  is, as a function of  $s$ , an exponential.*

(v) *If  $\Omega(a) = \omega(a)I$  for  $a$  in  $F^\times$  and if  $\pi = \pi(\Omega)$  then*

$$\pi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \omega(a)I.$$

*Moreover  $L(s, \pi) = L(s, \Omega)$ ,  $L(s, \tilde{\pi}) = L(s, \tilde{\Omega})$  and  $\epsilon(s, \pi, \psi) = \epsilon(s, \Omega, \psi)$ .*

The first assertion is a consequence of the definitions. We have just proved all the others when  $\Omega$  has a degree greater than one. Suppose then that  $\Omega(h) = \chi(\nu(h))$  where  $\chi$  is a quasi-character of  $F^\times$ . Then  $\pi(\Omega) = \pi(\chi\alpha_F^{1/2}, \chi\alpha_F^{-1/2})$  and if the last part of the theorem is to hold  $L(s, \Omega)$ , which is of course uniquely determined by the conditions of part (iii), must equal  $L(s, \pi) = L(s, \chi\alpha_F^{1/2})$ . Also  $L(s, \tilde{\Omega})$  must equal  $L(s, \tilde{\pi}) = L(s, \chi^{-1}\alpha_F^{1/2})$ .

In the case under consideration  $U = \mathbf{C}$  and we need only consider

$$Z(\alpha_F^s \otimes \Omega, \Phi; 1, 1) = Z(\alpha_F^s \otimes \Omega, \Phi).$$

As before the second part is trivial and

$$Z(\alpha_F^s \otimes \Omega, \Phi) = Z(\alpha_F^s \otimes \Omega, \Phi_1)$$

if

$$\Phi_1(x) = \int_{K_1} \Phi(xh).$$

The Fourier transform of  $\Phi_1$  is

$$\Phi'_1(x) = \int_{K_1} \Phi'(hx) = \int_{K_1} \Phi'(xh)$$

and

$$Z(\alpha_F^s \otimes \tilde{\Omega}, \Phi') = Z(\alpha_F^s \otimes \tilde{\Omega}, \Phi'_1).$$

It is therefore enough to consider the functions in  $\mathcal{S}(K, \Omega)$ .

If  $\varphi = \varphi_\Phi$  is defined as before then  $\varphi$  lies in the space on which the Kirillov model of  $\pi$  acts and

$$\widehat{\varphi}(\alpha_F^{s-1/2}) = Z(\alpha_F^{s+1/2} \otimes \Omega, \Phi).$$

The third assertion follows from the properties of  $L(s, \pi)$ . The fourth follows from the relation

$$\widehat{\varphi}'(\alpha_F^{1/2-s} \omega^{-1}) = -Z(\alpha_F^{3/2-s} \otimes \Omega^{-1}, \Phi'),$$

which was proved in the first paragraph, and the relation

$$\frac{\widehat{\varphi}'(\alpha_F^{1/2-s} \omega^{-1})}{L(1-s, \tilde{\pi})} = \epsilon(s, \pi, \psi) \frac{\widehat{\varphi}(\alpha_F^{s-1/2})}{L(s, \pi)},$$

which was proved in the second, if we observe that  $\tilde{\Omega}(h) = \Omega^{-1}(h)$ . Here  $\varphi'$  is of course  $\pi(w)\varphi$ .

**Corollary 4.4.** *If  $\pi = \pi(\Omega)$  then  $\tilde{\pi} = \pi(\tilde{\Omega})$ .*

This is clear if  $\Omega$  is of degree one so suppose it is of degree greater than one. Combining the identity of part (iv) with that obtained upon interchanging the roles of  $\Omega$  and  $\tilde{\Omega}$  and of  $\Phi$  and  $\Phi'$  we find that

$$\epsilon(s, \Omega, \psi) \epsilon(1-s, \tilde{\Omega}, \psi) = \omega(-1).$$

The same considerations show that

$$\epsilon(s, \pi, \psi) \epsilon(1-s, \tilde{\pi}, \psi) = \omega(-1).$$

Consequently

$$\epsilon(s, \tilde{\pi}, \psi) = \epsilon(s, \tilde{\Omega}, \psi).$$

Replacing  $\Omega$  by  $\chi \otimes \Omega$  we see that

$$\epsilon(s, \chi^{-1} \otimes \tilde{\pi}, \psi) = \epsilon(s, \chi^{-1} \otimes \tilde{\Omega}, \psi) = \epsilon(s, \chi^{-1} \otimes \pi(\tilde{\Omega}), \psi)$$

for all quasi-characters  $\chi$ . Since  $\tilde{\pi}$  and  $\pi(\tilde{\Omega})$  are both absolutely cuspidal they are equivalent.

There is a consequence of the theorem whose significance we do not completely understand.

**Proposition 4.5.** *Let  $\Omega$  be an irreducible representation of  $K^\times$  on the space  $U$  and suppose that the dimension of  $U$  is greater than one. Let  $\tilde{U}$  be the dual space of  $U$ . Let  $\pi$  be the Kirillov model of  $\pi(\Omega)$ , let  $\varphi$  lie in  $\mathcal{S}(F^\times)$ , and let  $\varphi' = \pi(w)\varphi$ . If  $u$  belongs to  $U$  and  $\tilde{u}$  belong to  $\tilde{U}$  the function  $\Phi$  on  $K$  which vanishes at 0 and on  $K^\times$  is defined by*

$$\Phi(x) = \varphi(\nu(x)) |\nu(x)|^{-1} \langle u, \tilde{\Omega}(x) \tilde{u} \rangle$$

is in  $\mathcal{S}(K)$  and its Fourier transform  $\Phi'$  vanishes at 0 and on  $K^\times$  is given by

$$\Phi'(x) = -\varphi'(\nu(x))|\nu(x)|^{-1}\omega^{-1}(\nu(x))\langle\Omega(x)u, \tilde{u}\rangle$$

if  $\Omega(a) = \omega(a)I$  for  $a$  in  $F^\times$ .

It is clear that  $\Phi$  belongs not merely to  $\mathcal{S}(K)$  but in fact to  $\mathcal{S}(K^\times)$ . So does the function  $\Phi_1$  which we are claiming is equal to  $\Phi'$ . The Schur orthogonality relations for the group  $K_1$  show that  $\Phi'(0) = 0$  so that  $\Phi'$  also belongs to  $\mathcal{S}(K^\times)$ .

We are going to show that for every irreducible representation of  $\Omega'$  of  $K^\times$

$$\int \frac{\Phi_1(x), \langle u', \tilde{\Omega}'(x)\tilde{u}' \rangle |\nu(x)|^{3/2-s} d^\times x}{L(1-s, \tilde{\Omega}')} = - \int \frac{\epsilon(s, \Omega', \psi)\Phi(x) \langle \Omega'(x)u', \tilde{u}' \rangle |\nu(x)|^{s+1/2} d^\times x}{L(s, \Omega')}$$

for all choices of  $u'$  and  $\tilde{u}'$ . Applying the theorem we see that

$$\int \{\Phi_1(x) - \Phi'(x)\} \langle u', \tilde{\Omega}'(x)\tilde{u}' \rangle |\nu(x)|^{3/2-s} d^\times x = 0$$

for all choices of  $\Omega'$ ,  $u'$ ,  $\tilde{u}'$ , and all  $s$ . An obvious and easy generalization of the Peter-Weyl theorem, which we do not even bother to state, shows that  $\Phi_1 = \Phi'$ .

If

$$\Psi(x) = \int_{K_1} \langle u, \tilde{\Omega}(hx)\tilde{u} \rangle \langle \Omega'(hx)u', \tilde{u}' \rangle dh$$

then

$$\int_{K^\times} \Phi(x) \langle \Omega'(x)u', \tilde{u}' \rangle |\nu(x)|^{s+1/2} d^\times x = \int_{K^\times/K_1} \varphi(\nu(x)) |\nu(x)|^{s-1/2} \Psi(x) d^\times x$$

while

$$\begin{aligned} \int_{K^\times} \Phi_1(x) \langle u' \tilde{\Omega}'(x), \tilde{u}' \rangle |\nu(x)|^{3/2-s} d^\times x \\ = - \int_{K^\times/K_1} \varphi'(\nu(x)) \omega^{-1}(\nu(x)) |\nu(x)|^{1/2-s} \Psi(x^{-1}) d^\times x \end{aligned}$$

If  $\Psi$  is 0 for all choice of  $u'$  and  $\tilde{u}'$  the required identity is certainly true. Suppose then  $\Psi$  is different from 0 for some choice  $u'$  and  $\tilde{u}'$ .

Let  $U$  be the intersection of the kernels of  $\Omega'$  and  $\Omega$ . It is an open normal subgroup of  $K^\times$  and  $H = UK_1F^\times$  is open, normal, and of finite index in  $K^\times$ . Suppose that  $\Omega'(a) = \omega'(a)I$  for  $a$  in  $F^\times$ . If  $h$  belongs to  $H$

$$\Psi(xh) = \chi_0(h)\Psi(x)$$

where  $\chi_0$  is a quasi-character of  $H$  trivial on  $U$  and  $K_1$  and equal to  $\omega'\omega^{-1}$  on  $F^\times$ . Moreover  $\chi_0$  extends to a quasi-character  $\chi$  of  $K^\times$  so that

$$\int_{K^\times/H} \Psi(x) \chi^{-1}(x) = \int_{K^\times/F^\times} \psi(x) \chi^{-1}(x) \neq 0$$

$\chi$  may of course be identified with a quasi-character of  $F^\times$ .

**Lemma 4.5.1.** *If*

$$\int_{K^\times/F^\times} \Psi(x) \chi^{-1}(x) \neq 0$$

*then  $\Omega'$  is equivalent to  $\chi \otimes \Omega$ .*

The representation  $\Omega'$  and  $\chi \otimes \Omega$  agree on  $F^\times$  and

$$\int_{K^\times/F^\times} \left\langle u, \widetilde{\chi \otimes \Omega(x)\tilde{u}} \right\rangle \langle \Omega'(x)u', \tilde{u}' \rangle \neq 0.$$

The lemma follows from the Schur orthogonality relations.

We have therefore only to prove the identity for  $\Omega' = \chi \otimes \Omega$ . Set

$$F(x) = \int_{K_1} \left\langle u, \tilde{\Omega}(hx)\tilde{u} \right\rangle \langle \Omega(hx)u', \tilde{u}' \rangle dh.$$

The vectors  $u'$  and  $\tilde{u}'$  now belong to the spaces  $U$  and  $\tilde{U}$ . There is a function  $f$  on  $F^\times$  such that

$$F(x) = f(\nu(x))$$

The identity we are trying to prove may be written as

$$(4.5.2) \quad \frac{\int_{F^\times} \varphi'(a) \chi^{-1}(a) \omega^{-1}(a) f(a^{-1}) |a|^{1/2-s} d^\times a}{L(1-s, \chi^{-1} \otimes \tilde{\pi})} = \epsilon(s, \chi \otimes \pi, \psi) \frac{\int_{F^\times} \varphi(a) \chi(a) f(a) |a|^{s-1/2} d^\times a}{L(s, \chi \otimes \pi)}.$$

Let  $H$  be the group constructed as before with  $U$  taken as the kernel of  $\Omega$ . The image  $F'$  of  $H$  under  $\nu$  is a subgroup of finite index in  $F^\times$  and  $f$ , which is a function on  $F^\times/F'$ , may be written as a sum

$$f(a) = \sum_{i=1}^p \lambda_i \chi_i(a)$$

where  $\{\chi_1, \dots, \chi_p\}$  are the characters of  $F^\times/F'$  which are not orthogonal to  $f$ . By the lemma  $\Omega$  is equivalent to  $\chi_i \otimes \Omega$  for  $1 \leq i \leq p$  and therefore  $\pi$  is equivalent to  $\chi_i \otimes \pi$ . Consequently

$$\epsilon(s, \chi \otimes \pi, \psi) = \epsilon(s, \chi \chi_i \otimes \pi, \psi)$$

and

$$\frac{\int_{F^\times} \varphi'(a) \chi^{-1}(a) \chi_1^{-1}(a) \omega^{-1}(a) |a|^{1/2-s} d^\times a}{L(1-s, \chi^{-1} \otimes \tilde{\pi})} = \epsilon(s, \chi \otimes \pi, \psi) \frac{\int_{F^\times} \varphi(a) \chi(a) \chi_i(a) |a|^{s-1/2} d^\times a}{L(s, \chi \otimes \pi)}.$$

The identity (4.5.2) follows.

Now let  $K$  be a separable quadratic extension of  $F$ . We are going to associate to each quasi-character  $\omega$  of  $K^\times$  an irreducible representation  $\pi(\omega)$  of  $G_F$ . If  $G_+$  is the set of all  $g$  in  $G_F$  whose determinants belong to  $\nu(K^\times)$  we have already, in the first paragraph, associated to  $\omega$  a representation  $r_\omega$  of  $G_+$ . To emphasize the possible dependence of  $r_\omega$  on  $\psi$  we now denote it by  $\pi(\omega, \psi)$ . The group  $G_+$  is of index 2 in  $G_F$ . Let  $\pi(\omega)$  be the representation of  $G_F$  induced from  $\pi(\omega, \psi)$ .

**Theorem 4.6.**

- (i) *The representation  $\pi(\omega, \psi)$  is irreducible.*
- (ii) *The representation  $\pi(\omega)$  is admissible and irreducible and its class does not depend on the choice of  $\psi$ .*
- (iii) *If there is no quasi-character  $\chi$  of  $F^\times$  such that  $\omega = \chi_0 \nu$  the representation  $\pi(\omega)$  is absolutely cuspidal.*
- (iv) *If  $\omega = \chi_0 \nu$  and  $\eta$  is the character of  $F^\times$  associated to  $K$  by local class field theory then  $\pi(\omega)$  is  $\pi(\chi, \chi_\eta)$ .*

It is clear what the notion of admissibility for a representation of  $G_+$  should be. The proof that  $\pi(\omega, \psi)$  is admissible proceeds like the proof of the first part of Theorem 4.2 and there is little point in presenting it.

To every  $\Phi$  in  $\mathcal{S}(K, \omega)$  we associate the function  $\varphi_\Phi$  on  $F_+ = \nu(K^\times)$  defined by

$$\varphi_\Phi(a) = \omega(h)|h|_K^{1/2}\Phi(h)$$

if  $a = \nu(h)$ . Clearly  $\varphi_\Phi = 0$  if and only if  $\Phi = 0$ . Let  $V_+$  be the space of functions on  $F_+$  obtained in this manner. Then  $V_+$  clearly contains the space  $\mathcal{S}(F_+)$  of locally constant compactly supported functions on  $F_+$ . In fact if  $\varphi$  belongs to  $\mathcal{S}(F_+)$  and

$$\Phi(h) = \omega^{-1}(h)|h|_K^{-1/2}\varphi(\nu(h))$$

then  $\varphi = \varphi_\Phi$ . If the restriction of  $\omega$  to the group  $K_1$  of elements of norm 1 in  $K^\times$  is not trivial so that every element of  $\mathcal{S}(K, \omega)$  vanishes at 0 then  $V_+ = \mathcal{S}(F_+)$ . Otherwise  $\mathcal{S}(F_+)$  is of codimension one in  $V_+$ .

Let  $B_+$  be the group of matrices of the form

$$\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}$$

with  $a$  in  $F_+$  and  $x$  in  $F$ . In the first paragraph we introduced a representation  $\xi = \xi_\psi$  of  $B_+$  on the space of functions on  $F_+$ . It was defined by

$$\xi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi(b) = \varphi(ba)$$

and

$$\xi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi(b) = \psi(bx)\varphi(b).$$

We may regard  $\pi(\omega, \psi)$  as acting on  $V_+$  and if we do the restriction of  $\pi(\omega, \psi)$  to  $B_+$  is  $\xi_\psi$ .

**Lemma 4.6.1.** *The representation of  $B_F$  induced from the representation  $\xi_\psi$  of  $B_+$  on  $\mathcal{S}(F_+)$  is the representation  $\xi_\psi$  of  $B_F$ . In particular the representation  $\xi_\psi$  of  $B_+$  is irreducible.*

The induced representation is of course obtained by letting  $B_F$  act by right translations on the space of all functions  $\tilde{\varphi}$  on  $B_F$  with values in  $\mathcal{S}(F_+)$  which satisfy

$$\tilde{\varphi}(b_1b) = \xi_\psi(b_1)\tilde{\varphi}(b)$$

for all  $b_1$  in  $B_+$ . Let  $L$  be the linear functional in  $\mathcal{S}(F_+)$  which associates to a function its value at 1. Associate to  $\tilde{\varphi}$  the function

$$\varphi(a) = L\left(\tilde{\varphi}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\right) = L\left(\rho\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\tilde{\varphi}(e)\right)$$

The value of  $\tilde{\varphi}\left(\begin{pmatrix} a & 0 \\ x & 1 \end{pmatrix}\right)$  at  $\alpha$  in  $F_+$  is

$$\begin{aligned} L\left(\tilde{\varphi}\left(\begin{pmatrix} \alpha a & \alpha x \\ 0 & 1 \end{pmatrix}\right)\right) &= L\left(\xi_\psi\left(\begin{pmatrix} 1 & \alpha x \\ 0 & 1 \end{pmatrix}\right)\tilde{\varphi}\left(\begin{pmatrix} \alpha a & 0 \\ 0 & 1 \end{pmatrix}\right)\right) \\ &= \psi(ax)L\left(\tilde{\varphi}\left(\begin{pmatrix} \alpha a & 0 \\ 0 & 1 \end{pmatrix}\right)\right) = \psi(ax)\varphi(\alpha a). \end{aligned}$$

Since  $F^\times/F_+$  is finite it follows immediately that  $\varphi$  is in  $\mathcal{S}(F^\times)$  and that  $\tilde{\varphi}$  is 0 if  $\varphi$  is. It also shows that  $\varphi$  can be any function in  $\mathcal{S}(F^\times)$  and that if  $\tilde{\varphi}' = \rho(b)\tilde{\varphi}$  then  $\varphi' = \xi_\psi(b)\varphi$  for all  $b$  in  $B_F$ . Since a representation obtained by induction cannot be irreducible unless the original representation is, the second assertion follows from Lemma 2.9.1.

If the restriction of  $\omega$  to  $K_1$  is not trivial the first assertion of the theorem follows immediately. If it is then, by an argument used a number of times previously, any non-zero invariant subspace of  $V_+$  contains  $\mathcal{S}(F_+)$  so that to prove the assertion we have only to show that  $\mathcal{S}(F_+)$  is not invariant.

As before we observe that if  $\Phi$  in  $\mathcal{S}(K, \omega) = \mathcal{S}(K)$  is taken to vanish at 0 but to be non-negative and not identically 0 then

$$r_\omega(w)\Phi(0) = \gamma \int_K \Phi(x) dx \neq 0$$

so that  $\varphi_\Phi$  is in  $\mathcal{S}(F_+)$  but  $\varphi_{r_\omega(w)\Phi}$  is not.

The representation  $\pi(\omega)$  is the representation obtained by letting  $G_+$  act to the right on the space of functions  $\tilde{\varphi}$  on  $G_+$  with values in  $V_+$  which satisfy

$$\tilde{\varphi}(hg) = \pi(\omega, \psi)(h)\tilde{\varphi}(g)$$

for  $h$  in  $G_+$ . Replacing the functions  $\tilde{\varphi}$  by the functions

$$\tilde{\varphi}'(g) = \tilde{\varphi}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}g\right)$$

we obtain an equivalent representation, that induced from the representation

$$g \rightarrow \pi(\omega, \psi)\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}g\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)$$

of  $G_+$ . It follows from Lemma 1.4 that this representation is equivalent to  $\pi(\omega, \psi')$  if  $\psi'(x) = \psi(ax)$ . Thus  $\pi(\omega)$  is, apart from equivalence, independent of  $\psi$ .

Since

$$G_F = \left\{ g \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \mid g \in G_+, a \in F^\times \right\}$$

$\tilde{\varphi}$  is determined by its restrictions to  $B_F$ . This restriction, which we again call  $\tilde{\varphi}$ , is any one of the functions considered in Lemma 4.6.1. Thus, by the construction used in the proof of that lemma, we can associate to any  $\tilde{\varphi}$  a function  $\varphi$  on  $F^\times$ . Let  $V$  be the space of functions so obtained. We can regard  $\pi = \pi(\omega)$  as acting on  $V$ . It is clear that, for all  $\varphi$  in  $V$ ,

$$\pi(b)\varphi = \xi_\psi(b)\varphi$$

if  $b$  is in  $B_F$ . Every function on  $F_+$  can, by setting it equal to 0 outside of  $F_+$ , be regarded as a function  $F^\times$ . Since

$$\tilde{\varphi}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)(\alpha) = \varphi(\alpha a)$$

$V$  is the space generated by the translates of the functions in  $V_+$ . Thus if  $V_+ = \mathcal{S}(F_+)$  then  $V = \mathcal{S}(F^\times)$  and if  $\mathcal{S}(F_+)$  is of codimension one in  $V_+$  then  $\mathcal{S}(F^\times)$  is of codimension two in  $V$ .

It follows immediately that  $\pi(\omega)$  is irreducible and absolutely cuspidal if the restriction of  $\omega$  to  $K_1$  is not trivial.

The function  $\varphi$  in  $V_+$  corresponds to the function  $\tilde{\varphi}$  which is 0 outside of  $G_+$  and on  $G_+$  is given by

$$\tilde{\varphi}(g) = \pi(\omega, \psi)(g)\varphi.$$

It is clear that

$$\pi(\omega)(g)\varphi = \pi(\omega, \psi)(g)\varphi$$

if  $g$  is in  $G_+$ . Any non-trivial invariant subspace of  $V$  will have to contain  $\mathcal{S}(F^\times)$  and therefore  $\mathcal{S}(F_+)$ . Since  $\pi(\omega, \psi)$  is irreducible it will have to contain  $V_+$  and therefore will be  $V$  itself. Thus  $\pi(\omega)$  is irreducible for all  $\omega$ .

If the restriction of  $\omega$  to  $K_1$  is trivial there is a quasi-character  $\chi$  of  $F^\times$  such that  $\omega = \chi \circ \nu$ . To establish the last assertion of the lemma all we have to do is construct a non-zero linear form  $L$  on  $V$  which annihilates  $\mathcal{S}(F^\times)$  and satisfies

$$L\left(\pi\left(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\right)\varphi\right) = \chi(a_1 a_2) \eta(a_2) \left|\frac{a_1}{a_2}\right|^{1/2} L(\varphi)$$

if  $\pi = \pi(\omega)$ . We saw in Proposition 1.5 that

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right)\varphi = \chi^2(a) \eta(a) \varphi$$

so will only have to verify that

$$L\left(\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi\right) = \chi(a) |a|^{1/2} L(\varphi)$$

If  $\varphi = \varphi_\Phi$  is in  $V_+$  we set

$$L(\varphi) = \Phi(0)$$

so that if  $a$  is in  $F_+$

$$L\left(\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi\right) = r_\omega\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\Phi(0) = \chi(a) |a|^{1/2} L(\varphi).$$

If  $\epsilon$  is in  $F^\times$  but not in  $F_+$  any function  $\varphi$  in  $V$  can be written uniquely as

$$\varphi = \varphi_1 + \pi\left(\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi_2$$



with  $\varphi_1$  and  $\varphi_2$  in  $V_+$ . We set

$$L(\varphi) = L(\varphi_1) + \chi(\epsilon)L(\varphi_2).$$

**Theorem 4.7.**

- (i) If  $\pi = \pi(\omega)$  then  $\pi = \pi(\omega')$  if  $\omega'(a) = \omega(a')$ ,  $\tilde{\pi} = \pi(\omega^{-1})$  and  $\chi \otimes \pi = \pi(\omega\chi')$  if  $\chi$  is a quasi-character of  $F^\times$  and  $\chi' = \chi \circ \nu$ .
- (ii) If  $a$  is in  $F^\times$  then

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)\eta(a)I.$$

- (iii)  $L(s, \pi) = L(s, \omega)$  and  $L(s, \tilde{\pi}) = L(s, \omega^{-1})$ . Moreover if  $\psi_K(x) = \psi_F(\xi(x))$  for  $x$  in  $K$  and if  $\lambda(K/F, \psi_F)$  is the factor introduced in the first paragraph then

$$\epsilon(s, \pi, \psi_F) = \epsilon(s, \omega, \psi_K)\lambda(K/F, \psi_F)$$

It is clear that  $\chi \otimes \pi$  is the representation of  $G_F$  induced from the representation  $\chi \otimes \pi(\omega, \psi)$  of  $G_+$ . However by its very construction  $\chi \otimes \pi(\omega, \psi) = \pi(\omega\chi', \psi)$ . The relation

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)\eta(a)I$$

is a consequence of part (iii) of Proposition 1.5 and has been used before. Since  $\eta' = \eta \circ \nu$  is trivial and  $\omega(\nu(a)) = \omega(a)\omega'(a)$

$$\tilde{\pi} = \omega^{-1}\eta^{-1} \otimes \pi = \pi(\omega^{-\iota})$$

To complete the proof of the first part of the theorem we have to show that  $\pi(\omega) = \pi(\omega')$ . It is enough to verify that  $\pi(\omega, \psi) = \pi(\omega', \psi)$ . If  $\Phi$  belongs to  $\mathcal{S}(K)$  let  $\Phi'(x) = \Phi(x')$ . The mapping  $\Phi \rightarrow \Phi'$  is a bijection of  $\mathcal{S}(K, \omega)$  with  $\mathcal{S}(K, \omega')$  which changes  $\pi(\omega, \psi)$  into  $\pi(\omega', \psi)$ . Observe that here as elsewhere we have written an equality when we really mean an equivalence.

We saw in the first paragraph that if  $\varphi = \varphi_\Phi$  is in  $V_+$  then

$$\widehat{\varphi}(\alpha_F^{1/2-s}) = Z(\alpha_K^s \omega, \Phi)$$

and that if  $\varphi' = \pi(w)\varphi$  and  $\Phi'$  is the Fourier transform of  $\Phi$  then, if  $\omega_0(a) = \omega(a)\eta(a)$  for  $a$  in  $F^\times$ ,

$$\widehat{\varphi}'(\omega_0^{-1}\alpha_F^{s-1/2}) = \gamma Z(\alpha_K^{1-s}\omega^{-1}, \Phi')$$

if  $\gamma = \lambda(K/F, \psi_F)$ . Thus for all  $\varphi$  in  $V_+$  the quotient

$$\frac{\widehat{\varphi}(\alpha_F^{s-1/2})}{L(s, \omega)}$$

has an analytic continuation as a holomorphic function of  $s$  and for some  $\varphi$  it is a non-zero constant. Also

$$\frac{\widehat{\varphi}'(w_0^{-1}\alpha_F^{1/2-s})}{L(1-s, \omega^{-1})} = \lambda(K/F, \psi_F)\epsilon(s, \omega, \psi_K)\frac{\widehat{\varphi}(\alpha_F^{s-1/2})}{L(s, \omega)}.$$

To prove the theorem we have merely to check that these assertions remain valid when  $\varphi$  is allowed to vary in  $V$ . In fact we need only consider functions of the form

$$\varphi = \pi\left(\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi_0$$

where  $\varphi_0$  is in  $V_+$  and  $\epsilon$  is not in  $F_+$ . Since

$$\widehat{\varphi}(\alpha_F^{s-1/2}) = |\epsilon|^{1/2-s} \widehat{\varphi}_0(\alpha_F^{s-1/2})$$

the quotient

$$\frac{\widehat{\varphi}(\alpha_F^{s-1/2})}{L(s, \omega)}$$

is certainly holomorphic in the whole plane. Since

$$\widehat{\varphi}'(\omega_0^{-1} \alpha_F^{1/2-s}) = \omega_0(\epsilon) \omega_0^{-1}(\epsilon) |\epsilon|^{1/2-s} \widehat{\varphi}'_0(\omega_0^{-1} \alpha_F^{1/2-s}) = |\epsilon|^{1/2-s} \widehat{\varphi}'_0(\omega_0^{-1} \alpha_F^{1/2-s})$$

the functional equation is also satisfied.

Observe that if  $\omega = \chi \circ \nu$  then  $\pi(\omega) = \pi(\chi, \chi_\eta)$  so that

$$L(s, \omega) = L(s, \chi) L(s, \chi_\eta)$$

and

$$\epsilon(s, \omega, \psi_K) \lambda(K/F, \psi_F) = \epsilon(s, \chi, \psi_F) \epsilon(s, \chi_\eta, \psi_F)$$

These are special cases of the identities of [19].

### §5. Representations of $GL(2, \mathbf{R})$

We must also prove a local functional equation for the real and complex fields. In this paragraph we consider the field  $\mathbf{R}$  of real numbers. The standard maximal compact subgroup of  $GL(2, \mathbf{R})$  is the orthogonal group  $O(2, \mathbf{R})$ . Neither  $GL(2, \mathbf{R})$  nor  $O(2, \mathbf{R})$  is connected.

Let  $\mathcal{H}_1$  be the space of infinitely differentiable compactly supported functions on  $GL(2, \mathbf{R})$  which are  $O(2, \mathbf{R})$  finite on both sides. Once a Haar measure on  $G_{\mathbf{R}} = GL(2, \mathbf{R})$  has been chosen we may regard the elements of  $\mathcal{H}_1$  as measures and it is then an algebra under convolution.

$$f_1 \times f_2(g) = \int_{G_{\mathbf{R}}} f_1(gh^{-1})f_2(h) dh.$$

On  $O(2, \mathbf{R})$  we choose the normalized Haar measure. Then every function  $\xi$  on  $O(2, \mathbf{R})$  which is a finite sum of matrix elements of irreducible representations of  $O(2, \mathbf{R})$  may be identified with a measure on  $O(2, \mathbf{R})$  and therefore on  $GL(2, \mathbf{R})$ . Under convolution these measures form an algebra  $\mathcal{H}_2$ . Let  $\mathcal{H}_{\mathbf{R}}$  be the sum of  $\mathcal{H}_1$  and  $\mathcal{H}_2$ . It is also an algebra under convolution of measures. In particular if  $\xi$  belongs to  $\mathcal{H}_2$  and  $f$  belongs to  $\mathcal{H}_1$

$$\xi * f(g) = \int_{O(2, \mathbf{R})} \xi(u)f(u^{-1}g) du$$

and

$$f * \xi(g) = \int_{O(2, \mathbf{R})} f(gu^{-1})\xi(u) du.$$

If  $\sigma_i$ ,  $1 \leq i \leq p$ , is a family of inequivalent irreducible representations of  $O(2, \mathbf{R})$  and

$$\xi_i(u) = \dim \sigma_i \text{ trace } \sigma_i(u^{-1})$$

then

$$\xi = \sum_{i=1}^p \xi_i$$

is an idempotent of  $\mathcal{H}_{\mathbf{R}}$ . Such an idempotent is called elementary.

It is a consequence of the definitions that for any  $f$  in  $\mathcal{H}_1$  there is an elementary idempotent  $\xi$  such that

$$\xi * f = f * \xi = f.$$

Moreover for any elementary idempotent  $\xi$

$$\xi * \mathcal{H}_1 * \xi = \xi * C_c^\infty(G_{\mathbf{R}}) * \xi$$

is a closed subspace of  $C_c^\infty(G_{\mathbf{R}})$ , in the Schwartz topology. We give it the induced topology.

A representation  $\pi$  of the algebra  $\mathcal{H}_{\mathbf{R}}$  on the complex vector space  $V$  is said to be admissible if the following conditions are satisfied.

(5.1) Every vector  $v$  in  $V$  is of the form

$$v = \sum_{i=1}^r \pi(f_i)v_i$$

with  $f_i$  in  $\mathcal{H}_1$  and  $v_i$  in  $V$ .

(5.2) For every elementary idempotent  $\xi$  the range of  $\pi(\xi)$  is finite-dimensional.

(5.3) For every elementary idempotent  $\xi$  and every vector  $v$  in  $\pi(\xi)V$  the map  $f \rightarrow \pi(f)v$  of  $\xi\mathcal{H}_1\xi$  into the finite-dimensional space  $\pi(\xi)V$  is continuous.

If  $v = \sum_{i=1}^r \pi(f_i)v_i$  we can choose an elementary idempotent  $\xi$  so that  $\xi f_i = f_i \xi = f_i$  for  $1 \leq i \leq r$ . Then  $\pi(\xi)v = v$ . Let  $\{\varphi_n\}$  be a sequence in  $C_c^\infty(G_{\mathbf{R}})$  which converges, in the space of distributions, towards the Dirac distribution at the origin. Set  $\varphi'_n = \xi * \varphi_n * \xi$ . For each  $i$  the sequence  $\{\varphi'_n * f_i\}$  converges to  $f_i$  in the space  $\xi \mathcal{H}_1 \xi$ . Thus by (5.3) the sequence  $\{\pi(\varphi'_n)v\}$  converges to  $v$  in the finite-dimensional space  $\pi(\xi)v$ . Thus  $v$  is in the closure of the subspace  $\pi(\xi \mathcal{H}_1 \xi)v$  and therefore belongs to it.

As in the second paragraph the conditions (5.1) and (5.2) enable us to define the representation  $\tilde{\pi}$  contragredient to  $\pi$ . Up to equivalence it is characterized by demanding that it satisfy (5.1) and (5.2) and that there be a non-degenerate bilinear form on  $V \times \tilde{V}$  satisfying

$$\langle \pi(f)v, \tilde{v} \rangle = \langle v, \pi(\check{f})\tilde{v} \rangle$$

for all  $f$  in  $\mathcal{H}_{\mathbf{R}}$ . Here  $\tilde{V}$  is the space on which  $\tilde{\pi}$  acts and  $\check{f}$  is the image of the measure  $f$  under the map  $g \rightarrow g^{-1}$ . Notice that we allow ourselves to use the symbol  $f$  for all elements of  $\mathcal{H}_{\mathbf{R}}$ . The condition (5.3) means that for every  $v$  in  $V$  and every  $\tilde{v}$  in  $\tilde{V}$  the linear form

$$f \rightarrow \langle \pi(f)v, \tilde{v} \rangle$$

is continuous on each of the spaces  $\xi \mathcal{H}_1 \xi$ . Therefore  $\tilde{\pi}$  is also admissible.

Choose  $\xi$  so that  $\pi(\xi)v = v$  and  $\tilde{\pi}(\check{\xi})\tilde{v} = \tilde{v}$ . Then for any  $f$  in  $\mathcal{H}_1$

$$\langle \pi(f)v, \tilde{v} \rangle = \langle \pi(\xi f \xi)v, \tilde{v} \rangle.$$

There is therefore a unique distribution  $\mu$  on  $G_{\mathbf{R}}$  such that

$$\mu(f) = \langle \pi(f)v, \tilde{v} \rangle$$

for  $f$  in  $\mathcal{H}_1$ . Choose  $\varphi$  in  $\xi \mathcal{H}_1 \xi$  so that  $\pi(\varphi)v = v$ . Then

$$\mu(f\varphi) = \mu(\xi f \varphi \xi) = \mu(\xi f \xi \varphi) = \langle \pi(\xi f \xi \varphi)v, \tilde{v} \rangle = \langle \pi(\xi f \xi)v, \tilde{v} \rangle$$

so that  $\mu(f\varphi) = \mu(f)$ . Consequently the distribution  $\mu$  is actually a function and it is not unreasonable to write it as  $g \rightarrow \langle \pi(g)v, \tilde{v} \rangle$  even though  $\pi$  is not a representation of  $G_{\mathbf{R}}$ . For a fixed  $g$ ,  $\langle \pi(g)v, \tilde{v} \rangle$  depends linearly on  $v$  and  $\tilde{v}$ . If the roles of  $\pi$  and  $\tilde{\pi}$  are reversed we obtain a function  $\langle v, \tilde{\pi}(g)\tilde{v} \rangle$ . It is clear from the definition that

$$\langle \pi(g)v, \tilde{v} \rangle = \langle v, \tilde{\pi}(g^{-1})\tilde{v} \rangle.$$

Let  $\mathfrak{g}$  be the Lie algebra of  $G_{\mathbf{R}}$  and let  $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$ . Let  $\mathfrak{A}$  be the universal enveloping algebra of  $\mathfrak{g}_{\mathbf{C}}$ . If we regard the elements of  $\mathfrak{A}$  as distributions on  $G_{\mathbf{R}}$  with support at the identity we can take their convolution product with the elements of  $C_c^\infty(G_{\mathbf{R}})$ . More precisely if  $X$  belongs to  $\mathfrak{g}$

$$X * f(g) = \left. \frac{d}{dt} f(\exp(-tX)) \right|_{t=0}$$

and

$$f * X(g) = \left. \frac{d}{dt} f(g \exp(-tX)) \right|_{t=0}$$

If  $f$  belongs to  $\mathcal{H}_1$  so do  $f * X$  and  $X * f$ .

We want to associate to the representation  $\pi$  of  $\mathcal{H}_{\mathbf{R}}$  on  $V$  a representation  $\pi$  of  $\mathfrak{A}$  on  $V$  such that

$$\pi(X)\pi(f) = \pi(X * f)$$

and

$$\pi(f)\pi(X) = \pi(f * X)$$

for all  $X$  in  $\mathfrak{A}$  and all  $f$  in  $\mathcal{H}_1$ . If  $v = \sum \pi(f_i)v_i$  we will set

$$\pi(X)v = \sum_i \pi(X * f_i)v_i$$

and the first condition will be satisfied. However we must first verify that if

$$\sum_i \pi(f_i)v_i = 0$$

then

$$w = \sum_i \pi(X * f_i)v_i$$

is also 0. Choose  $f$  so that  $w = \pi(f)w$ . Then

$$w = \sum_i \pi(f)\pi(X * f_i)v_i = \sum_i \pi(f * X * f_i)v_i = \pi(f * X) \left\{ \sum_i \pi(f_i)v_i \right\} = 0.$$

From the same calculation we extract the relation

$$\pi(f) \left\{ \sum_i \pi(X * f_i)v_i \right\} = \pi(f * X) \left\{ \sum_i \pi(f_i)v_i \right\}$$

for all  $f$  so that  $\pi(f)\pi(X) = \pi(f * X)$ .

If  $g$  is in  $G_{\mathbf{R}}$  then  $\lambda(g)f = \delta_g * f$  if  $\delta_g$  is the Dirac function at  $g$ . If  $g$  is in  $O(2, \mathbf{R})$  or in  $Z_{\mathbf{R}}$ , the groups of scalar matrices,  $\delta_g * f$  is in  $\mathcal{H}_1$  if  $f$  is, so that the same considerations allow us to associate to  $\pi$  a representation  $\pi$  of  $O(2, \mathbf{R})$  and a representation  $\pi$  of  $Z_{\mathbf{R}}$ . It is easy to see that if  $h$  is in either of these groups then

$$\pi(\text{Ad } hX) = \pi(h)\pi(X)\pi(h^{-1}).$$

To dispel any doubts about possible ambiguities of notation there is a remark we should make. For any  $f$  in  $\mathcal{H}_1$

$$\langle \pi(f)v, \tilde{v} \rangle = \int_{G_{\mathbf{R}}} f(g) \langle \pi(g)v, \tilde{v} \rangle dg.$$

Thus if  $h$  is in  $O(2, \mathbf{R})$  or  $Z_{\mathbf{R}}$

$$\langle \pi(f * \delta_h)v, \tilde{v} \rangle = \int_{G_{\mathbf{R}}} f(g) \langle \pi(gh)v, \tilde{v} \rangle dg$$

and

$$\langle \pi(f)\pi(h)v, \tilde{v} \rangle = \int_{G_{\mathbf{R}}} f(g) \langle \pi(g)\pi(h)v, \tilde{v} \rangle dg$$

so that

$$\langle \pi(gh)v, \tilde{v} \rangle = \langle \pi(g)\pi(h)v, \tilde{v} \rangle.$$

A similar argument shows that

$$\langle \pi(hg)v, \tilde{v} \rangle = \langle \pi(g)v, \tilde{\pi}(h^{-1})\tilde{v} \rangle.$$

It is easily seen that the function  $\langle \pi(g)v, \tilde{v} \rangle$  takes the value  $\langle v, \tilde{v} \rangle$  at  $g = e$ . Thus if  $h$  belongs to  $O(2, \mathbf{R})$  or  $Z_{\mathbf{R}}$  the two possible interpretations of  $\langle \pi(h)v, \tilde{v} \rangle$  give the same result.

It is not possible to construct a representation of  $G_{\mathbf{R}}$  on  $V$  and the representation of  $\mathfrak{A}$  is supposed to be a substitute. Since  $G_{\mathbf{R}}$  is not connected, it is not adequate and we introduce instead the notion of a representation  $\pi_1$  of the system  $\{\mathfrak{A}, \epsilon\}$  where

$$\epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is a representation  $\pi_1$  of  $\mathfrak{A}$  and an operator  $\pi_1(\epsilon)$  which satisfy the relations

$$\pi_1^2(\epsilon) = I$$

and

$$\pi_1(\text{Ad } \epsilon X) = \pi_1(\epsilon)\pi_1(X)\pi_1(\epsilon^{-1}).$$

Combining the representation  $\pi$  with  $\mathfrak{A}$  with the operator  $\pi(\epsilon)$  we obtain a representation of the system  $\{\mathfrak{A}, \epsilon\}$ .

There is also a representation  $\tilde{\pi}$  of  $\mathfrak{A}$  associated to  $\tilde{\pi}$  and it is not difficult to see that

$$\langle \pi(X)v, \tilde{v} \rangle = \langle v, \tilde{\pi}(\check{X})\tilde{v} \rangle$$

if  $X \rightarrow \check{X}$  is the automorphism of  $\mathfrak{A}$  which sends  $X$  in  $\mathfrak{g}$  to  $-X$ .

Let

$$\varphi(g) = \langle \pi(g)v, \tilde{v} \rangle.$$

The function  $\varphi$  is certainly infinitely differentiable. Integrating by parts we see that

$$\int_{G_{\mathbf{R}}} f(g)\varphi * X(g) dg = \int_{G_{\mathbf{R}}} f * \check{X}(g)\varphi(g) dg$$

The right side is

$$\langle \pi(f)\pi(\check{X})v, \tilde{v} \rangle = \int_{G_{\mathbf{R}}} f(g) \langle \pi(g)\pi(\check{X})v, \tilde{v} \rangle$$

so that

$$\varphi * \check{X}(g) = \langle \pi(g)\pi(\check{X})v, \tilde{v} \rangle.$$

Assume now that the operators  $\pi(X)$  are scalar if  $X$  is in the centre  $\mathfrak{Z}$  of  $\mathfrak{A}$ . Then the standard proof, which uses the theory of elliptic operators, shows that the functions  $\varphi$  are analytic on  $G_{\mathbf{R}}$ . Since

$$\begin{aligned} \varphi * \check{X}(e) &= \langle \pi(\check{X})v, \tilde{v} \rangle \\ \varphi * \check{X}(\epsilon) &= \langle \pi(\epsilon)\pi(\check{X})v, \tilde{v} \rangle \end{aligned}$$

and  $G_{\mathbf{R}}$  has only two components, one containing  $e$  and the other containing  $\epsilon$ . The function  $\varphi$  vanishes identically if  $\langle \pi(\check{X})v, \tilde{v} \rangle$  and  $\langle \pi(\epsilon)\pi(\check{X})v, \tilde{v} \rangle$  are 0 for all  $X$  in  $\mathfrak{A}$ . Any subspace  $V_1$  of  $V$  invariant under  $\mathfrak{A}$  and  $\epsilon$  is certainly invariant under  $O(2, \mathbf{R})$  and therefore is determined by its annihilator in  $\tilde{V}$ . If  $v$  is in  $V_1$  and  $\tilde{v}$  annihilates  $V_1$  the function  $\langle \pi(g)v, \tilde{v} \rangle$  is 0 so that

$$\langle \pi(f)v, \tilde{v} \rangle = 0$$

for all  $f$  in  $\mathcal{H}_1$ . Thus  $\pi(f)v$  is also in  $V_1$ . Since  $\mathcal{H}_2$  clearly leaves  $V_1$  invariant this space is left invariant by all of  $\mathcal{H}_{\mathbf{R}}$ .

By the very construction any subspace of  $V$  invariant under  $\mathcal{H}_{\mathbf{R}}$  is invariant under  $\mathfrak{A}$  and  $\epsilon$  so that we have almost proved the following lemma.

**Lemma 5.4.** *The representation  $\pi$  of  $\mathcal{H}_{\mathbf{R}}$  is irreducible if and only if the associated representation  $\pi$  of  $\{\mathfrak{A}, \epsilon\}$  is.*

To prove it completely we have to show that if the representation of  $\{\mathfrak{A}, \epsilon\}$  is irreducible the operator  $\pi(X)$  is a scalar for all  $X$  in  $\mathfrak{Z}$ . As  $\pi(X)$  has to have a non-zero eigenfunction we have only to check that  $\pi(X)$  commutes with  $\pi(Y)$  for  $Y$  in  $\mathfrak{A}$  with  $\pi(\epsilon)$ . It certainly commutes with  $\pi(Y)$ .  $X$  is invariant under the adjoint action not only of the connected component of  $G_{\mathbf{R}}$  but also of the connected component of  $GL(2, \mathbf{C})$ . Since  $GL(2, \mathbf{C})$  is connected and contains  $\epsilon$

$$\pi(\epsilon)\pi(X)\pi^{-1}(\epsilon) = \pi(\text{Ad } \epsilon(X)) = \pi(X).$$

Slight modifications, which we do not describe, of the proof of Lemma 5.4 lead to the following lemma.

**Lemma 5.5.** *Suppose  $\pi$  and  $\pi'$  are two irreducible admissible representations of  $\mathcal{H}_{\mathbf{R}}$ . Then  $\pi$  and  $\pi'$  are equivalent if and only if the associated representations of  $\{\mathfrak{A}, \epsilon\}$  are.*

We comment briefly on the relation between representations of  $G_{\mathbf{R}}$  and representations of  $\mathcal{H}_{\mathbf{R}}$ . Let  $V$  be a complete separable locally convex topological space and  $\pi$  a continuous representation of  $G_{\mathbf{R}}$  on  $V$ . Thus the map  $(g, v) \rightarrow \pi(g)v$  of  $G_{\mathbf{R}} \times V$  to  $V$  is continuous and for  $f$  in  $C_c^\infty(G_{\mathbf{R}})$  the operator

$$\pi(f) = \int_{G_{\mathbf{R}}} f(x)\pi(x) dx$$

is defined. So is  $\pi(f)$  for  $f$  in  $\mathcal{H}_2$ . Thus we have a representation of  $\mathcal{H}_{\mathbf{R}}$  on  $V$ . Let  $V_0$  be the space of  $O(2, \mathbf{R})$ -finite vectors in  $V$ . It is the union of the space  $\pi(\xi)V$  as  $\xi$  ranges over the elementary idempotents and is invariant under  $\mathcal{H}_{\mathbf{R}}$ . Assume, as is often the case, that the representation  $\pi_0$  of  $\mathcal{H}_{\mathbf{R}}$  on  $V_0$  is admissible. Then  $\pi_0$  is irreducible if and only if  $\pi$  is irreducible in the topological sense.

Suppose  $\pi'$  is another continuous representation of  $G_{\mathbf{R}}$  in a space  $V'$  and there is a continuous non-degenerate bilinear form on  $V \times V'$  such that

$$\langle \pi(g)v, v' \rangle = \langle v, \pi'(g^{-1})v' \rangle.$$

Then the restriction of this form to  $V_0 \times V'_0$  is non-degenerate and

$$\langle \pi(f)v, v' \rangle = \langle v, \pi'(\check{f})v' \rangle$$

for all  $f$  in  $\mathcal{H}_{\mathbf{R}}$ ,  $v$  in  $V_0$ , and  $v'$  in  $V'_0$ . Thus  $\pi'_0$  is the contragredient of  $\pi_0$ . Since

$$\langle \pi_0(f)v, v' \rangle = \int_{G_{\mathbf{R}}} f(g) \langle \pi(g)v, v' \rangle$$

we have

$$\langle \pi_0(g)v, v' \rangle = \langle \pi(g)v, v' \rangle.$$

The special orthogonal group  $SO(2, \mathbf{R})$  is abelian and so is its Lie algebra. The one-dimensional representation

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \rightarrow e^{in\theta}$$

of  $SO(2, \mathbf{R})$  and the associated representation of its Lie algebra will be both denoted by  $\kappa_n$ . A representation  $\pi$  of  $\mathfrak{A}$  or of  $\{\mathfrak{A}, \epsilon\}$  will be called admissible if its restrictions to the Lie

algebra of  $\mathrm{SO}(2, \mathbf{R})$  decomposes into a direct sum of the representations  $\kappa_n$  each occurring with finite multiplicity. If  $\pi$  is an admissible representation of  $\mathcal{H}_{\mathbf{R}}$  the corresponding representation of  $\{\mathfrak{A}, \epsilon\}$  is also admissible. We begin the classification of the irreducible admissible representations of  $\mathcal{H}_{\mathbf{R}}$  and of  $\{\mathfrak{A}, \epsilon\}$  with the introduction of some particular representations.

Let  $\mu_1$  and  $\mu_2$  be two quasi-characters of  $F^\times$ . Let  $\mathcal{B}(\mu_1, \mu_2)$  be the space of functions  $f$  on  $G_{\mathbf{R}}$  which satisfy the following two conditions.

(i)

$$f\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right) = \mu_1(a_1)\mu_2(a_2) \left|\frac{a_1}{a_2}\right|^{1/2} f(g)$$

for all  $g$  in  $G_{\mathbf{R}}$ ,  $a_1, a_2$  in  $\mathbf{R}^\times$ , and  $x$  in  $\mathbf{R}$ .

(ii)  $f$  is  $\mathrm{SO}(2, \mathbf{R})$  finite on the right.

Because of the Iwasawa decomposition

$$G_{\mathbf{R}} = N_{\mathbf{R}} A_{\mathbf{R}} \mathrm{SO}(2, \mathbf{R})$$

these functions are complete determined by their restrictions to  $\mathrm{SO}(2, \mathbf{R})$  and in particular are infinitely differentiable. Write

$$\mu_i(t) = |t|^{s_i} \left(\frac{t}{|t|}\right)^{m_i}$$

where  $s_i$  is a complex number and  $m_i$  is 0 or 1. Set  $s = s_1 - s_2$  and  $m = |m_1 - m_2|$  so that  $\mu_1\mu_2^{-1}(t) = |t|^s \left(\frac{t}{|t|}\right)^m$ . If  $n$  has the same parity as  $m$  let  $\varphi_n$  be the function in  $\mathcal{B}(\mu_1, \mu_2)$  defined by

$$\varphi_n\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}\right) = \mu_1(a_1)\mu_2(a_2) \left|\frac{a_1}{a_2}\right|^{1/2} e^{in\theta}.$$

The collection  $\{\varphi_n\}$  is a basis of  $\mathcal{B}(\mu_1, \mu_2)$ .

For any infinitely differentiable function  $f$  on  $G_{\mathbf{R}}$  and any compactly supported distribution  $\mu$  we defined  $\lambda(\mu)f$  by

$$\lambda(\mu)f(g) = \check{\mu}(\rho(g)f)$$

and  $\rho(\mu)f$  by

$$\rho(\mu)f(g) = \mu(\lambda(g^{-1})f).$$

If, for example,  $\mu$  is a measure

$$\lambda(\mu)f(g) = \int_{G_{\mathbf{R}}} f(h^{-1}g) d\mu(h)$$

and

$$\rho(\mu)f(g) = \int_{G_{\mathbf{R}}} f(gh) d\mu(h).$$

In all cases  $\lambda(\mu)f$  and  $\rho(\mu)f$  are again infinitely differentiable. For all  $f$  in  $\mathcal{H}_{\mathbf{R}}$  the space  $\mathcal{B}(\mu_1, \mu_2)$  is invariant under  $\rho(f)$  so that we have a representation  $\rho(\mu_1, \mu_2)$  of  $\mathcal{H}_{\mathbf{R}}$  on  $\mathcal{B}(\mu_1, \mu_2)$ . It is clearly admissible and the associated representation  $\rho(\mu_1, \mu_2)$  of  $\{\mathfrak{A}, \epsilon\}$  is also defined by right convolution.



We introduce the following elements of  $\mathfrak{g}$  which is identified with the Lie algebra of  $2 \times 2$  matrices.

$$U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad V_+ = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad V_- = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix},$$

$$X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

as well as

$$D = X_+X_- + X_-X_+ + \frac{Z^2}{2},$$

which belongs to  $\mathfrak{A}$ .

**Lemma 5.6.** *The following relations are valid*

$$\begin{array}{ll} (i) & \rho(U)\varphi_n = in\varphi_n \\ (iii) & \rho(V_+)\varphi_n = (s+1+n)\varphi_{n+2} \\ (v) & \rho(D)\varphi_n = \frac{s^2-1}{2}\varphi_n \end{array} \quad \begin{array}{ll} (ii) & \rho(\epsilon)\varphi_n = (-1)^{m_1}\varphi_{-n} \\ (iv) & \rho(V_-)\varphi_n = (s+1-n)\varphi_{n-2} \\ (vi) & \rho(J)\varphi_n = (s_1+s_2)\varphi_n \end{array}$$

The relations (i), (ii), and (vi) are easily proved. It is also clear that for all  $\varphi$  in  $\mathcal{B}(\mu_1, \mu_2)$

$$\rho(Z)\varphi(e) = (s+1)\varphi(e)$$

and

$$\rho(X_+)\varphi(e) = 0.$$

The relations

$$\text{Ad} \left( \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \right) V_+ = e^{2i\theta} V_+$$

and

$$\text{Ad} \left( \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \right) V_- = e^{-2i\theta} V_-$$

show that  $\rho(V_+)\varphi_n$  is a multiple of  $\varphi_{n+2}$  and that  $\rho(V_-)\varphi_n$  is a multiple of  $\varphi_{n-2}$ . Since

$$V_+ = Z - iU + 2iX_+$$

and

$$V_- = Z + iU - 2iX_+$$

the value of  $\rho(V_+)\varphi_n$  at the identity  $e$  is  $s+1+n$  and that of  $\rho(V_-)\varphi_n$  is  $s+1-n$ . Relations (iii) and (iv) follow.

It is not difficult to see that  $D$  belongs to  $\mathfrak{Z}$  the centre of  $\mathfrak{A}$ . Therefore  $\rho(D)\varphi = \lambda(\check{D})\varphi = \lambda(D)\varphi$  since  $D = \check{D}$ . If we write  $D$  as

$$2X_-X_+ + Z + \frac{Z^2}{2}$$

and observe that  $\lambda(X_+)\varphi = 0$  and  $\lambda(Z)\varphi = -(s+1)\varphi$  if  $\varphi$  is in  $\mathcal{B}(\mu_1, \mu_2)$  we see that

$$\rho(D)\varphi_n = \left\{ -(s+1) + \frac{(s+1)^2}{2} \right\} \varphi_n = \frac{s^2-1}{2} \varphi_n.$$

**Lemma 5.7.**

(i) *If  $s-m$  is not an odd integer  $\mathcal{B}(\mu_1, \mu_2)$  is irreducible under the action of  $\mathfrak{g}$ .*

(ii) If  $s - m$  is an odd integer and  $s \geq 0$  the only proper subspaces of  $\mathcal{B}(\mu_1, \mu_2)$  invariant under  $\mathfrak{g}$  are

$$\begin{aligned}\mathcal{B}_1(\mu_1, \mu_2) &= \sum_{\substack{n \geq s+1 \\ n \equiv s+1 \pmod{2}}} \mathbf{C}\varphi_n \\ \mathcal{B}_2(\mu_1, \mu_2) &= \sum_{\substack{n \leq -s-1 \\ n \equiv s+1 \pmod{2}}} \mathbf{C}\varphi_n\end{aligned}$$

and, when it is different from  $\mathcal{B}(\mu_1, \mu_2)$ ,

$$\mathcal{B}_s(\mu_1, \mu_2) = \mathcal{B}_1(\mu_1, \mu_2) + \mathcal{B}_2(\mu_1, \mu_2).$$

(iii) If  $s - m$  is an odd integer and  $s < 0$  the only proper subspaces of  $\mathcal{B}(\mu_1, \mu_2)$  invariant under  $\mathfrak{g}$  are

$$\begin{aligned}\mathcal{B}_1(\mu_1, \mu_2) &= \sum_{\substack{n \geq s+1 \\ n \equiv s+1 \pmod{2}}} \mathbf{C}\varphi_n \\ \mathcal{B}_2(\mu_1, \mu_2) &= \sum_{\substack{n \leq -s-1 \\ n \equiv s+1 \pmod{2}}} \mathbf{C}\varphi_n\end{aligned}$$

and

$$\mathcal{B}_f(\mu_1, \mu_2) = \mathcal{B}_1(\mu_1, \mu_2) \cap \mathcal{B}_2(\mu_1, \mu_2).$$

Since a subspace of  $\mathcal{B}(\mu_1, \mu_2)$  invariant under  $\mathfrak{g}$  is spanned by those of the vectors  $\varphi_n$  that it contains, this lemma is an easy consequence of the relations of Lemma 5.6. Before stating the corresponding results for  $\{\mathfrak{A}, \epsilon\}$  we state some simple lemmas.

**Lemma 5.8.** *If  $\pi$  is an irreducible admissible representation of  $\{\mathfrak{A}, \epsilon\}$  there are two possibilities:*

- (i) *The restriction of  $\pi$  to  $\mathfrak{A}$  is irreducible and the representations  $X \rightarrow \pi(X)$  and  $X \rightarrow \pi(\text{Ad } \epsilon(X))$  are equivalent.*
- (ii) *The space  $V$  on which  $\pi$  acts decomposes into a direct sum  $V_1 \oplus V_2$  where  $V_1$  and  $V_2$  are both invariant and irreducible under  $\mathfrak{A}$ . The representations  $\pi_1$  and  $\pi_2$  of  $\mathfrak{A}$  on  $V_1$  and  $V_2$  are not equivalent but  $\pi_2$  is equivalent to the representation  $X \rightarrow \pi(\text{Ad } \epsilon(X))$ .*

If the restriction of  $\pi$  to  $\mathfrak{A}$  is irreducible the representations  $X \rightarrow \pi(X)$  and  $X \rightarrow \pi(\text{Ad } \epsilon(X))$  are certainly equivalent. If it is not irreducible let  $V_1$  be a proper subspace invariant under  $\mathfrak{A}$ . If  $V_2 = \pi(\epsilon)V_1$  then  $V_1 \cap V_2$  and  $V_1 + V_2$  are all invariant under  $\{\mathfrak{A}, \epsilon\}$ . Thus  $V_1 \cap V_2 = \{0\}$  and  $V = V_1 \oplus V_2$ . If  $V_1$  had a proper subspace  $V'_1$  invariant under  $\mathfrak{A}$  the same considerations would show that  $V = V'_1 \oplus V'_2$  with  $V'_2 = \pi(\epsilon)V'_1$ . Since this is impossible  $V_1$  and  $V_2$  are irreducible under  $\mathfrak{A}$ .

If  $v_1$  is in  $V_1$

$$\pi_2(X)\pi(\epsilon)v_1 = \pi(\epsilon)\pi_1(\text{ad } \epsilon(X))v_1$$

so that the representations  $X \rightarrow \pi_2(X)$  and  $X \rightarrow \pi_1(\text{Ad } \epsilon(X))$  are equivalent. If  $\pi_1$  and  $\pi_2$  were equivalent there would be an invertible linear transformation  $A$  from  $V_1$  to  $V_2$  so that  $A\pi_1(X) = \pi_2(X)A$ . If  $v_1$  is in  $V_1$

$$A^{-1}\pi(\epsilon)\pi_1(X)v_1 = A^{-1}\pi_2(\text{ad } \epsilon(X))\pi(\epsilon)v_1 = \pi_1(\text{Ad } \epsilon(X))A^{-1}\pi(\epsilon)v_1$$

Consequently  $\{A^{-1}\pi(\epsilon)\}^2$  regarded as a linear transformation of  $V_1$  commutes with  $\mathfrak{A}$  and is therefore a scalar. There is no harm in supposing that it is the identity. The linear transformation

$$v_1 + v_2 \rightarrow A^{-1}v_2 + Av_1$$

then commutes with the action of  $\{\mathfrak{A}, \epsilon\}$ . This is a contradiction.

Let  $\chi$  be a quasi-character of  $\mathbf{R}^\times$  and let  $\chi(t) = t^c$  for  $t$  positive. For any admissible representation  $\pi$  of  $\mathfrak{A}$  and therefore of  $\mathfrak{g}$  we define a representation  $\chi \otimes \pi$  of  $\mathfrak{g}$  and therefore  $\mathfrak{A}$  by setting

$$\chi \otimes \pi(X) = \frac{c}{2} \text{trace } X + \pi(X)$$

if  $X$  is in  $\mathfrak{g}$ . If  $\pi$  is a representation of  $\{\mathfrak{A}, \epsilon\}$  we extend  $\chi \otimes \pi$  to  $\{\mathfrak{A}, \epsilon\}$  by setting

$$\chi \otimes \pi(\epsilon) = \chi(-1)\pi(\epsilon)$$

If  $\pi$  is associated to a representation  $\pi$  of  $\mathcal{H}_{\mathbf{R}}$  then  $\chi \otimes \pi$  is associated to the representation of  $\mathcal{H}_{\mathbf{R}}$  defined by

$$\chi \otimes \pi(f) = \pi(\chi f)$$

if  $\chi f$  is the product of the functions  $\chi$  and  $f$ .

**Lemma 5.9.** *Let  $\pi_0$  be an irreducible admissible representation of  $\mathfrak{A}$ . Assume that  $\pi_0$  is equivalent to the representation  $X \rightarrow \pi_0(\text{Ad } \epsilon(X))$ . Then there is an irreducible representation  $\pi$  of  $\{\mathfrak{A}, \epsilon\}$  whose restriction to  $\mathfrak{A}$  is  $\pi_0$ . If  $\eta$  is the non-trivial quadratic character of  $\mathbf{R}^\times$  the representations  $\pi$  and  $\eta \otimes \pi$  are not equivalent but any representation of  $\{\mathfrak{A}, \epsilon\}$  whose restriction to  $\mathfrak{A}$  is equivalent to  $\pi_0$  is equivalent to one of them.*

Let  $\pi_0$  act on  $V$ . There is an invertible linear transformation  $A$  of  $V$  such that  $A\pi_0(X) = \pi_0(\text{Ad } \epsilon(X))A$  for all  $X$  in  $\mathfrak{A}$ . Then  $A^2$  commutes with all  $\pi_0(X)$  and is therefore a scalar. We may suppose that  $A^2 = I$ . If we set  $\pi(\epsilon) = A$  and  $\pi(X) = \pi_0(X)$  for  $X$  in  $\mathfrak{A}$  we obtain the required representation. If we replace  $A$  by  $-A$  we obtain the representation  $\eta \otimes \pi$ .  $\pi$  and  $\eta \otimes \pi$  are not equivalent because any operator giving the equivalence would have to commute with all of the  $\pi(X)$  and would therefore be a scalar. Any representation  $\pi'$  of  $\{\mathfrak{A}, \epsilon\}$  whose restriction to  $\mathfrak{A}$  is equivalent to  $\pi_0$  can be realized on  $V_0$  in such a way that  $\pi'(X) = \pi_0(X)$  for all  $X$ . Then  $\pi'(\epsilon) = \pm A$ .

**Lemma 5.10.** *Let  $\pi_1$  be an irreducible admissible representation of  $\mathfrak{A}$ . Assume that  $\pi_1$  and  $\pi_2$ , with  $\pi_2(X) = \pi_1(\text{Ad } \epsilon(X))$ , are not equivalent. Then there is an irreducible representation  $\pi$  of  $\{\mathfrak{A}, \epsilon\}$  whose restriction to  $\mathfrak{A}$  is the direct sum of  $\pi_1$  and  $\pi_2$ . Every irreducible admissible representation of  $\{\mathfrak{A}, \epsilon\}$  whose restriction to  $\mathfrak{A}$  contains  $\pi_1$  is equivalent to  $\pi$ . In particular  $\eta \otimes \pi$  is equivalent to  $\pi$ .*

Let  $\pi_1$  act on  $V_1$ . To construct  $\pi$  we set  $V = V_1 \oplus V_2$  and we set

$$\pi(X)(v_1 \oplus v_2) = \pi_1(X)v_1 \oplus \pi_2(X)v_2$$

and

$$\pi(\epsilon)(v_1 \oplus v_2) = v_2 \oplus v_1.$$

The last assertion of the lemma is little more than a restatement of the second half of Lemma 5.8.

**Theorem 5.11.** *Let  $\mu_1$  and  $\mu_2$  be two quasi-characters of  $F^\times$ .*

- (i) If  $\mu_1\mu_2^{-1}$  is not of the form  $t \rightarrow t^p \operatorname{sgn} t$  with  $p$  a non-zero integer the space  $\mathcal{B}(\mu_1, \mu_2)$  is irreducible under the action of  $\{\mathfrak{A}, \epsilon\}$  or  $\mathcal{H}_{\mathbf{R}}$ .  $\pi(\mu_1, \mu_2)$  is any representation equivalent to  $\rho(\mu_1, \mu_2)$ .
- (ii) If  $\mu_1\mu_2^{-1}(t) = t^p \operatorname{sgn} t$ , where  $p$  is a positive integer, the space  $\mathcal{B}(\mu_1, \mu_2)$  contains exactly one proper subspace  $\mathcal{B}_s(\mu_1, \mu_2)$  invariant under  $\{\mathfrak{A}, \epsilon\}$ . It is infinite-dimensional and any representation of  $\{\mathfrak{A}, \epsilon\}$  equivalent to the restriction of  $\rho(\mu_1, \mu_2)$  to  $\mathcal{B}_s(\mu_1, \mu_2)$  will be denoted by  $\sigma(\mu_1, \mu_2)$ . The quotient space

$$\mathcal{B}_f(\mu_1, \mu_2) = \mathcal{B}(\mu_1, \mu_2) / \mathcal{B}_s(\mu_1, \mu_2)$$

is finite-dimensional and  $\pi(\mu_1, \mu_2)$  will be any representation equivalent to the representation of  $\{\mathfrak{A}, \epsilon\}$  on this quotient space.

- (iii) If  $\mu_1\mu_2^{-1}(t) = t^p \operatorname{sgn} t$ , where  $p$  is a negative integer, the space  $\mathcal{B}(\mu_1, \mu_2)$  contains exactly one proper subspace  $\mathcal{B}_f(\mu_1, \mu_2)$  invariant under  $\{\mathfrak{A}, \epsilon\}$ . It is finite-dimensional and  $\pi(\mu_1, \mu_2)$  will be any representation equivalent to the restriction of  $\rho(\mu_1, \mu_2)$  to  $\mathcal{B}_f(\mu_1, \mu_2)$ . Moreover  $\sigma(\mu_1, \mu_2)$  will be any representation equivalent to the representation on the quotient space

$$\mathcal{B}_s(\mu_1, \mu_2) = \mathcal{B}(\mu_1, \mu_2) / \mathcal{B}_f(\mu_1, \mu_2).$$

- (iv) A representation  $\pi(\mu_1, \mu_2)$  is never equivalent to a representation  $\sigma(\mu'_1, \mu'_2)$ .
- (v) The representations  $\pi(\mu_1, \mu_2)$  and  $\pi(\mu'_1, \mu'_2)$  are equivalent if and only if either  $(\mu_1, \mu_2) = (\mu'_1, \mu'_2)$  or  $(\mu_1, \mu_2) = (\mu'_2, \mu'_1)$ .
- (vi) The representations  $\sigma(\mu_1, \mu_2)$  and  $\sigma(\mu'_1, \mu'_2)$  are equivalent if and only if  $(\mu_1, \mu_2)$  is one of the four pairs  $(\mu'_1, \mu'_2)$ ,  $(\mu'_2, \mu'_1)$ ,  $(\mu'_1\eta, \mu'_2\eta)$ , or  $(\mu'_2\eta, \mu'_1\eta)$ .
- (vii) Every irreducible admissible representation of  $\{\mathfrak{A}, \epsilon\}$  is either a  $\pi(\mu_1, \mu_2)$  or a  $\sigma(\mu_1, \mu_2)$ .

Let  $\mu_1\mu_2^{-1}(t) = |t|^s \left(\frac{t}{|t|}\right)^m$ .  $s - m$  is an odd integer if and only if  $s$  is an integer  $p$  and  $\mu_1\mu_2^{-1}(t) = t^p \operatorname{sgn} t$ . Thus the first three parts of the lemma are consequences of Lemma 5.6 and 5.7. The fourth follows from the observation that  $\pi(\mu_1, \mu_2)$  and  $\sigma(\mu'_1, \mu'_2)$  cannot contain the same representations of the Lie algebra of  $\mathrm{SO}(2, \mathbf{R})$ .

We suppose first that  $s - m$  is not an odd integer and construct an invertible transformation  $T$  from  $\mathcal{B}(\mu_1, \mu_2)$  to  $\mathcal{B}(\mu_2, \mu_1)$  which commutes with the action of  $\{\mathfrak{A}, \epsilon\}$ . We have introduced a basis  $\{\varphi_n\}$  of  $\mathcal{B}(\mu_1, \mu_2)$ . Let  $\{\varphi'_n\}$  be the analogous basis of  $\mathcal{B}(\mu_2, \mu_1)$ . The transformation  $T$  will have to take  $\varphi_n$  to a multiple  $a_n\varphi'_n$  of  $\varphi'_n$ . Appealing to Lemma 5.6 we see that it commutes with the action of  $\{\mathfrak{A}, \epsilon\}$  if and only if

$$\begin{aligned} (s + 1 + n)a_{n+2} &= (-s + 1 + n)a_n \\ (s + 1 - n)a_{n-2} &= (-s + 1 - n)a_n \end{aligned}$$

and

$$a_n = (-1)^m a_{-n}.$$

These relations will be satisfied if we set

$$a_n = a_n(s) = \frac{\Gamma\left(\frac{-s+1+n}{2}\right)}{\Gamma\left(\frac{s+1+n}{2}\right)}$$

Since  $n \equiv m \pmod{2}$  and  $s - m - 1$  is not an even integer all these numbers are defined and different from 0.

If  $s \leq 0$  and  $s - m$  is an odd integer we let

$$a_n(s) = \lim_{z \rightarrow s} a_n(z)$$

The numbers  $a_n(s)$  are still defined although some of them may be 0. The associated operator  $T$  maps  $\mathcal{B}(\mu_1, \mu_2)$  into  $\mathcal{B}(\mu_2, \mu_1)$  and commutes with the action of  $\{\mathfrak{A}, \epsilon\}$ . If  $s = 0$  the operator  $T$  is non-singular. If  $s < 0$  its kernel is  $\mathcal{B}_f(\mu_1, \mu_2)$  and it defines an invertible linear transformation from  $\mathcal{B}_s(\mu_1, \mu_2)$  to  $\mathcal{B}_s(\mu_2, \mu_1)$ . If  $s > 0$  and  $s - m$  is an odd integer the functions  $a_n(z)$  have at most simple poles at  $s$ . Let

$$b_n(s) = \lim_{z \rightarrow s} (z - s) a_n(z)$$

The operator  $T$  associated to the family  $\{b_n(s)\}$  maps  $\mathcal{B}(\mu_1, \mu_2)$  into  $\mathcal{B}(\mu_2, \mu_1)$  and commutes with the action of  $\{\mathfrak{A}, \epsilon\}$ . Its kernel is  $\mathcal{B}_s(\mu_1, \mu_2)$  so that it defines an invertible linear transformation from  $\mathcal{B}_f(\mu_1, \mu_2)$  to  $\mathcal{B}_f(\mu_2, \mu_1)$ . These considerations together with Lemma 5.10 give us the equivalences of parts (v) and (vi).

Now we assume that  $\pi = \pi(\mu_1, \mu_2)$  and  $\pi' = \pi(\mu'_1, \mu'_2)$  or  $\pi = \sigma(\mu_1, \mu_2)$  and  $\pi' = \sigma(\mu'_1, \mu'_2)$  are equivalent. Let  $\mu_i(T) = |t|^{s_i} \left(\frac{t}{|t|}\right)^{m_i}$  and let  $\mu'_i(t) = |t|^{s'_i} \left(\frac{t}{|t|}\right)^{m'_i}$ . Let  $s = s_1 - s_2$ ,  $m = |m_1 - m_2|$ ,  $s' = s'_1 - s'_2$ ,  $m' = |m'_1 - m'_2|$ . Since the two representations must contain the same representations of the Lie algebra of  $SO(2, \mathbf{R})$  the numbers  $m$  and  $m'$  are equal. Since  $\pi(D)$  and  $\pi'(D)$  must be the same scalar Lemma 5.6 shows that  $s' = \pm s$ .  $\pi(J)$  and  $\pi'(J)$  must also be the same scalar so  $s'_1 + s'_2 = s_1 + s_2$ . Thus if  $\eta(t) = \text{sgn } t$  the pair  $(\mu_1, \mu_2)$  must be one of the four pairs  $(\mu'_1, \mu'_2)$ ,  $(\mu'_2, \mu'_1)$ ,  $(\eta\mu'_1, \eta\mu'_2)$ ,  $(\eta\mu'_2, \eta\mu'_1)$ . Lemma 5.9 shows that  $\pi(\mu'_1\mu'_2)$  and  $\pi(\eta\mu'_1, \eta\mu'_2)$  are not equivalent. Parts (v) and (vi) of the theorem follow immediately.

Lemmas 5.8, 5.9, and 5.10 show that to prove the last part of the theorem we need only show that any irreducible admissible representation  $\pi$  of  $\mathfrak{A}$  is, for a suitable choice of  $\mu_1$  and  $\mu_2$ , a constituent of  $\rho(\mu_1, \mu_2)$ . That is there should be two subspaces  $\mathcal{B}_1$  and  $\mathcal{B}_2$  of  $\mathcal{B}(\mu_1, \mu_2)$  invariant under  $\mathfrak{A}$  so that  $\mathcal{B}_1$  contains  $\mathcal{B}_2$  and  $\pi$  is equivalent to the representation of  $\mathfrak{A}$  on the quotient  $\mathcal{B}_1/\mathcal{B}_2$ . If  $\chi$  is a quasi-character of  $F^\times$  then  $\pi$  is a constituent of  $\rho(\mu_1, \mu_2)$  if and only if  $\chi \otimes \pi$  is a constituent of  $\rho(\chi\mu_1, \chi\mu_2)$ . Thus we may suppose that  $\pi(J)$  is 0 so that  $\pi$  is actually a representation of  $\mathfrak{A}_0$ , the universal enveloping algebra of the Lie algebra of  $Z_{\mathbf{R}} \backslash G_{\mathbf{R}}$ . Since this group is semi-simple the desired result is a consequence of the general theorem of Harish-Chandra [6].

It is an immediate consequence of the last part of the theorem that every irreducible admissible representation of  $\{\mathfrak{A}, \epsilon\}$  is the representation associated to an irreducible admissible representation of  $\mathcal{H}_{\mathbf{R}}$ . Thus we have classified the irreducible admissible representations of  $\{\mathfrak{A}, \epsilon\}$  and of  $\mathcal{H}_{\mathbf{R}}$ . We can write such a representation of  $\mathcal{H}_{\mathbf{R}}$  as  $\pi(\mu_1, \mu_2)$  or  $\sigma(\mu_1, \mu_2)$ .

In the first paragraph we associated to every quasi-character  $\omega$  of  $\mathbf{C}^\times$  a representation of  $r_\omega$  of  $G_+$  the group of matrices with positive determinant. The representation  $r_\omega$  acts on the space of functions  $\Phi$  in  $\mathcal{S}(\mathbf{C})$  which satisfy

$$\Phi(xh) = \omega^{-1}(h)\Phi(x)$$

for all  $h$  such that  $h\bar{h} = 1$ . All elements of  $\mathcal{S}(\mathbf{C}, \omega)$  are infinitely differentiable vectors for  $r_\omega$  so that  $r_\omega$  also determines a representation, again called  $r_\omega$ , of  $\mathfrak{A}$ .  $r_\omega$  depended on the choice of a character of  $\mathbf{R}$ . If that character is

$$\psi(x) = e^{2\pi u x i}$$

then

$$r_\omega(X_+)\Phi(z) = (2\pi u z \bar{z} i)\Phi(z).$$

**Lemma 5.12.** *Let  $\mathcal{S}_0(\mathbf{C}, \omega)$  be the space of functions  $\Phi$  in  $\mathcal{S}(C, \omega)$  of the form*

$$\Phi(z) = e^{-2\pi|u|z\bar{z}}P(z, \bar{z})$$

*where  $P(z, \bar{z})$  is a polynomial in  $z$  and  $\bar{z}$ . Then  $\mathcal{S}_0(\mathbf{C}, \omega)$  is invariant under  $\mathfrak{A}$  and the restriction of  $r_\omega$  to  $\mathcal{S}_0(\mathbf{C}, \omega)$  is admissible and irreducible.*

It is well known and easily verified that the function  $e^{-2\pi|u|z\bar{z}}$  is its own Fourier transform provided of course that the transform is taken with respect to the character

$$\psi_{\mathbf{C}}(z) = \psi(z + \bar{z})$$

and the self-dual measure for that character. From the elementary properties of the Fourier transform one deduces that the Fourier transform of a function

$$\Phi(z) = e^{-2\pi|u|z\bar{z}}P(z, \bar{z})$$

where  $P$  is a polynomial in  $z$  and  $\bar{z}$  is of the same form. Thus  $r_\omega(w)$  leaves  $\mathcal{S}_0(\mathbf{C}, \omega)$  invariant. Recall that

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

$\mathcal{S}_0(\mathbf{C}, \omega)$  is clearly invariant under  $r_\omega(X_+)$ . Since  $X_- = \text{Ad } w(X_+)$  it is also invariant under  $X_-$ . But  $X_+X_- - X_-X_+ = Z$ , so that it is also invariant under  $Z$ . We saw in the first paragraph that if  $\omega_0$  is the restriction of  $\omega$  to  $\mathbf{R}^\times$  then

$$r_\omega\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = (\text{sgn } a)\omega_0(a)I$$

thus  $r_\omega(J) = cI$  if  $\omega_0(a) = a^c$  for a positive  $a$ . In conclusion  $\mathcal{S}_0(\mathbf{C}, \omega)$  is invariant under  $\mathfrak{g}$  and therefore under  $\mathfrak{A}$ .

If

$$\omega(z) = (z\bar{z})^r \frac{z^m \bar{z}^n}{(z\bar{z})^{\frac{m+n}{2}}}$$

where  $r$  is a complex number and  $m$  and  $n$  are two integers, one 0 and the other non-negative, the functions

$$\Phi_p(z) = e^{-2\pi|u|z\bar{z}} z^{n+p} \bar{z}^{m+p},$$

with  $p$  a non-negative integer, form a basis of  $\mathcal{S}_0(\mathbf{C}, \omega)$ . Suppose as usual that  $\frac{\partial}{\partial z} = \frac{1}{2} \frac{\partial}{\partial x} + \frac{1}{2i} \frac{\partial}{\partial y}$  and that  $\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \frac{\partial}{\partial x} - \frac{1}{2i} \frac{\partial}{\partial y}$ . Then the Fourier transform  $\Phi'_p$  of  $\Phi_p$  is given by

$$\Phi'_p(z) = \frac{1}{(2\pi i u)^{m+n+2p}} \frac{\partial^{n+p}}{\partial z^{n+p}} \frac{\partial^{m+p}}{\partial \bar{z}^{m+p}} e^{-2\pi|u|z\bar{z}}$$

which is a function of the form

$$(i \text{sgn } u)^{m+n+2p} e^{-2\pi|u|z\bar{z}} \bar{z}^{n+p} z^{m+p} + \sum_{q=0}^{p-1} a_q e^{-2\pi|u|z\bar{z}} \bar{z}^{n+q} z^{m+q}.$$

Only the coefficient  $a_{p-1}$  interests us. It equals

$$\frac{(i \text{sgn } u)^{m+n+2p-1}}{2\pi i u} \{p(n+m+1+p-1)\}.$$

Since

$$r_\omega(w)\Phi(z) = (i \operatorname{sgn} u)\Phi'(\bar{z})$$

and

$$r_\omega(X_-) = (-1)^{m+n}r_\omega(w)r_\omega(X_+)r(w)$$

while

$$r_\omega(X_+)\Phi_p = (2\pi ui)\Phi_{p+1}$$

we see that

$$r_\omega(X_-)\Phi_p = (2\pi ui)\Phi_{p+1} - (i \operatorname{sgn} u)(n + m + 2p + 1)\Phi_p + \sum_{q=0}^{p-1} b_q \Phi_q.$$

Since  $U = X_+ - X_-$  we have

$$r_\omega(U)\Phi_p = (i \operatorname{sgn} u)(n + m + 2p + 1)\Phi_p - \sum_{q=0}^{p-1} b_q \Phi_q$$

and we can find the functions  $\Psi_p$ ,  $p = 0, 1, \dots$ , such that

$$\Psi_p = \Phi_p + \sum_{q=0}^{p-1} a_{pq} \Phi_q$$

while

$$r_\omega(U)\Psi_p = (i \operatorname{sgn} u)(n + m + 2p + 1)\Psi_p.$$

These functions form a basis of  $\mathcal{S}_0(\mathbf{C}, \omega)$ . Consequently  $r_\omega$  is admissible.

If it were not irreducible there would be a proper invariant subspace which may or not contain  $\Phi_0$ . In any case if  $\mathcal{S}_1$  is the intersection of all invariant subspaces containing  $\Phi_0$  and  $\mathcal{S}_2$  is the sum of all invariant subspaces which do not contain  $\Phi_0$  both  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are invariant and the representation  $\pi_1$  of  $\mathfrak{A}$  on  $\mathcal{S}_1/\mathcal{S}_2 \cap \mathcal{S}_1$  is irreducible. If the restriction of  $\pi_1$  to the Lie algebra of  $SO(2, \mathbf{R})$  contains  $\kappa_p$  it does not contain  $\kappa_{-p}$ . Thus  $\pi_1$  is not equivalent to the representation  $X \rightarrow \pi_1(\operatorname{Ad} \epsilon(X))$ . Consequently the irreducible representation  $\pi$  of  $\{\mathfrak{A}, \epsilon\}$  whose restriction to  $\mathfrak{A}$  is  $\pi_1$  must be one of the special representations  $\sigma(\mu_1, \mu_2)$  or a representation  $\pi(\mu_1, \mu_2 \eta)$ . Examining these we see that since  $\pi$  contains  $\kappa_q$  with  $q = \operatorname{sgn} u(n + m + 1)$  it contains all the representations  $\kappa_q$  with  $q = \operatorname{sgn} u(n + m + 2p + 1)$ ,  $p = 0, 1, 2, \dots$ . Thus  $\mathcal{S}_1$  contains all the functions  $\Psi_p$  and  $\mathcal{S}_2$  contains none of them. Since this contradicts the assumption that  $\mathcal{S}_0(\mathbf{C}, \omega)$  contains a proper invariant subspace the representation  $r_\omega$  is irreducible.

For the reasons just given the representation  $\pi$  of  $\{\mathfrak{A}, \epsilon\}$  whose restriction to  $\mathfrak{A}$  contains  $r_\omega$  is either a  $\sigma(\mu_1, \mu_2)$  or a  $\pi(\mu_1, \mu_1 \eta)$ . It is a  $\pi(\mu_1, \mu_1 \eta)$  if and only if  $n + m = 0$ . Since

$$\pi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \omega(a) \operatorname{sgn} a I = \omega(a) \eta(a) I,$$

we must have  $\mu_1 \mu_2 = \omega_0 \eta$  in the first case and  $\mu_1^2 = \omega_0$  in the second.  $\omega_0$  is the restriction of  $\omega$  to  $\mathbf{R}^\times$ . Since the two solutions  $\mu_1^2 = \omega_0$  differ by  $\eta$  they lead to the same representation. If  $n + m = 0$  then  $\mu_1^2 = \omega_0$  if and only if  $\omega(z) = \mu_1(\nu(z))$  for all  $z$  in  $\mathbf{C}^\times$ . Of course  $\nu(z) = z\bar{z}$ .

Suppose  $n + m > 0$  so that  $\pi$  is a  $\sigma(\mu_1, \mu_2)$ . Let  $\mu_i(t) = |t|^{s_i} \left( \frac{t}{|t|} \right)^{m_i}$ . Because of Theorem 5.11 we can suppose that  $m_1 = 0$ . Let  $s = s_1 - s_2$ . We can also suppose that  $s$  is non-negative. If  $m = |m_1 - m_2|$  then  $s - m$  is an odd integer so  $m$  and  $m_2$  are determined

by  $s$ . We know what representations of the Lie algebra of  $\mathrm{SO}(2, \mathbf{R})$  are contained in  $\pi$ . Appealing to Lemma 5.7 we see that  $s = n + m$ . Since  $\mu_1 \mu_2 = \eta \omega_0$  we have  $s_1 + s_2 = 2r$ . Thus  $s_1 = r + \frac{m+n}{2}$  and  $s_2 = r - \frac{n+m}{2}$ . In all cases the representation  $\pi$  is determined by  $\omega$  alone and does not depend on  $\psi$ . We refer to it as  $\pi(\omega)$ . Every special representation  $\sigma(\mu_1, \mu_2)$  is a  $\pi(\omega)$  and  $\pi(\omega)$  is equivalent to  $\pi(\omega')$  if and only if  $\omega = \omega'$  or  $\omega'(z) = \omega(\bar{z})$ .

We can now take the first step in the proof of the local functional equation.

**Theorem 5.13.** *Let  $\pi$  be an infinite-dimensional irreducible admissible representation of  $\mathcal{H}_{\mathbf{R}}$ . If  $\psi$  is a non-trivial additive character of  $\mathbf{R}$  there exists exactly one space  $W(\pi, \psi)$  of functions  $W$  on  $G_{\mathbf{R}}$  with the following properties*

(i) *If  $W$  is in  $W(\pi, \psi)$  then*

$$W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi(x)W(g)$$

*for all  $x$  in  $F$ .*

(ii) *The functions  $W$  are continuous and  $W(\pi, \psi)$  is invariant under  $\rho(f)$  for all  $f$  in  $\mathcal{H}_{\mathbf{R}}$ . Moreover the representation of  $\mathcal{H}_{\mathbf{R}}$  on  $W(\pi, \psi)$  is equivalent to  $\pi$ .*

(iii) *If  $W$  is in  $W(\pi, \psi)$  there is a positive number  $N$  such that*

$$W\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right) = O(|t|^N)$$

*as  $|t| \rightarrow \infty$ .*

We prove first the existence of such a space. Suppose  $\pi = \pi(\omega)$  is the representation associated to some quasi-character  $\omega$  of  $\mathbf{C}^\times$ . An additive character  $\psi$  being given the restriction of  $\pi$  to  $\mathfrak{A}$  contains the representation  $r_\omega$  determined by  $\omega$  and  $\psi$ . For any  $\Phi$  in  $\mathcal{S}(\mathbf{C}, \omega)$  define a function  $W_\Phi$  on  $G_+$  by

$$W_\Phi(g) = r_\omega(g)\Phi(1)$$

Since  $\rho(g)W_\Phi = W_{r_\omega(g)\Phi}$  the space of such functions is invariant under right translations. Moreover

$$W_\Phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi(x)W_\Phi(g)$$

Every vector in  $\mathcal{S}(\mathbf{C}, \omega)$  is infinitely differentiable for the representation  $r_\omega$ . Therefore the functions  $W_\Phi$  are all infinitely differentiable and, if  $X$  is in  $\mathfrak{A}$ ,

$$\rho(X)W_\Phi = W_{r_\omega(X)\Phi}.$$

In particular the space  $W_1(\pi, \psi)$  of those  $W_\Phi$  for which  $\Phi$  is in  $\mathcal{S}_0(\mathbf{C}, \omega)$  is invariant under  $\mathfrak{A}$ . We set  $W_\Phi$  equal to 0 outside of  $G_+$  and regard it as a function on  $G_{\mathbf{R}}$ .

We want to take  $W(\pi, \psi)$  to be the sum of  $W_1(\pi, \psi)$  and its right translate by  $\epsilon$ . If we do it will be invariant under  $\{\mathfrak{A}, \epsilon\}$  and transform according to the representation  $\pi$  of  $\{\mathfrak{A}, \epsilon\}$ . To verify the second condition we have to show that it is invariant under  $\mathcal{H}_{\mathbf{R}}$ . For this it is enough to show that  $\mathcal{S}_0(\mathbf{C}, \omega)$  is invariant under the elements of  $\mathcal{H}_{\mathbf{R}}$  with support in  $G_+$ . The elements certainly leave the space of functions in  $\mathcal{S}(\mathbf{C}, \omega)$  spanned by the functions transforming according to a one-dimensional representation of  $\mathrm{SO}(2, \mathbf{R})$  invariant. Any function in  $\mathcal{S}(\mathbf{C}, \omega)$  can be approximated uniformly on compact sets by a function in  $\mathcal{S}_0(\mathbf{C}, \omega)$ . If in addition it transforms according to the representation  $\kappa_n$



of  $SO(2, \mathbf{R})$  it can be approximated by functions in  $\mathcal{S}_0(\mathbf{C}, \omega)$  transforming according to the same representation. In other words it can be approximated by multiples of a single function in  $\mathcal{S}_0(\mathbf{C}, \omega)$  and therefore is already in  $\mathcal{S}_0(\mathbf{C}, \omega)$ .

The growth condition need only be checked for the functions  $W_\Phi$  in  $W_1(\pi, \psi)$ . If  $a$  is negative

$$W_\Phi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) = 0$$

but if  $a$  is positive and

$$\Phi(z) = e^{-2\pi|u|z\bar{z}} P(z, \bar{z})$$

it is equal to

$$e^{-2\pi|u|a} P(a^{1/2}, a^{1/2}) \omega(a) |a|^{1/2},$$

and certainly satisfies the required condition.

We have still to prove the existence of  $W(\pi, \psi)$  when  $\pi = \pi(\mu_1, \mu_2)$  and is infinite-dimensional. As in the first paragraph we set

$$\theta(\mu_1, \mu_2, \Phi) = \int_{\mathbf{R}^\times} \mu_1(t) \mu_2^{-1}(t) \Phi(t, t^{-1}) d^\times t$$

for  $\Phi$  in  $\mathcal{S}(\mathbf{R}^s)$  and we set

$$\begin{aligned} W_\Phi(g) &= \mu_1(\det g) |\det g|^{1/2} \theta(\mu_1, \mu_2, r(g)\Phi) \\ &= \theta(\mu_1, \mu_2, r_{\mu_1, \mu_2}(g)\Phi). \end{aligned}$$

$r_{\mu_1, \mu_2}$  is the representation associated to the quasi-character  $(a, b) \rightarrow \mu_1(a)\mu_2(b)$  of  $\mathbf{R}^\times \times \mathbf{R}^\times$ . If  $X$  is in  $\mathfrak{A}$

$$\rho(X)W_\Phi(g) = W_{r_{\mu_1, \mu_2}(X)\Phi}(g)$$

Let  $W(\mu_1, \mu_2; \psi)$  be the space of those  $W_\Phi$  which are associated to  $O(2, \mathbf{R})$ -finite functions  $\Phi$ .  $W(\mu_1, \mu_2; \psi)$  is invariant under  $\{\mathfrak{A}, \epsilon\}$  and under  $\mathcal{H}_{\mathbf{R}}$ .

**Lemma 5.13.1.** *Assume  $\mu_1(x)\mu_2^{-1}(x) = |x|^s \left(\frac{x}{|x|}\right)^m$  with  $\operatorname{Re} s > -1$  and  $m$  equal to 0 or 1. Then there exists a bijection  $A$  of  $W(\mu_1, \mu_2; \psi)$  with  $\mathcal{B}(\mu_1, \mu_2)$  which commutes with the action of  $\{\mathfrak{A}, \epsilon\}$ .*

We have already proved a lemma like this in the non-archimedean case. If  $\Phi$  is in  $\mathcal{S}(\mathbf{R}^2)$  and  $\omega$  is a quasi-character of  $\mathbf{R}^\times$  set

$$z(\omega, \Phi) = \int \Phi(0, t) \omega(t) d^\times(t)$$

The integral converges if  $\omega(t) = |t|^r (\operatorname{sgn} t)^n$  with  $r > 0$ . In particular under the circumstances of the lemma

$$f_\Phi(g) = \mu_1(\det g) |\det g|^{1/2} z(\mu_1 \mu_2^{-1} \alpha_{\mathbf{R}}, \rho(g)\Phi)$$

is defined. As usual  $\alpha_{\mathbf{R}}(x) = |x|$ . A simple calculation shows that

$$f_\Phi \left( \begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g \right) = \mu_1(a_1) \mu_2(a_2) \left| \frac{a_1}{a_2} \right|^{1/2} f_\Phi(g).$$

If  $\tilde{\Phi}$  is the partial Fourier transform of  $\Phi$  introduced in the first paragraph then

$$\rho(g) f_{\tilde{\Phi}} = f_{\tilde{\Phi}_1}$$

if  $\Phi_1 = r_{\mu_1, \mu_2}(f)\Phi$ . A similar relation will be valid for a function  $f$  in  $\mathcal{H}_{\mathbf{R}}$ , that is

$$\rho(f)f_{\tilde{\Phi}} = f_{\tilde{\Phi}_1}$$

if  $\Phi_1 = r_{\mu_1, \mu_2}(f)\Phi$ . In particular if  $f_{\tilde{\Phi}}$  is  $O(2, \mathbf{R})$ -finite there is an elementary idempotent  $\xi$  such that  $\rho(\xi)f_{\tilde{\Phi}} = f_{\tilde{\Phi}}$ . Thus, if  $\Phi_1 = r_{\mu_1, \mu_2}(\xi)\Phi$ ,  $f_{\tilde{\Phi}} = f_{\tilde{\Phi}_1}$  and  $\tilde{\Phi}_1$  is  $O(2, \mathbf{R})$  finite. Of course  $f_{\tilde{\Phi}}$  is  $O(2, \mathbf{R})$ -finite if and only if it belongs to  $\mathcal{B}(\mu_1, \mu_2)$ .

We next show that given any  $f$  in  $\mathcal{B}(\mu_1, \mu_2)$  there is an  $O(2, \mathbf{R})$ -finite function  $\Phi$  in  $\mathcal{S}(\mathbf{R}^2)$  such that  $f = f_{\tilde{\Phi}}$ . According to the preceding observation together with the self-duality of  $\mathcal{S}(\mathbf{R}^2)$  under Fourier transforms it will be enough to show that for some  $\Phi$  in  $\mathcal{S}(\mathbf{R}^2)$ ,  $f = f_{\Phi}$ . In fact, by linearity, it is sufficient to consider the functions  $\varphi_n$  in  $\mathcal{B}(\mu_1, \mu_2)$  defined earlier by demanding that

$$\varphi_n \left( \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \right) = e^{in\theta}$$

$n$  must be of the same parity as  $m$ . If  $\delta = \text{sgn } n$  set

$$\Phi(x, y) = e^{-\pi(x^2+y^2)}(x + i\delta y)^{|n|}$$

Then

$$\rho \left( \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \right) \Phi = e^{in\theta} \Phi$$

Since  $\rho(g)f_{\Phi} = f_{\rho(g)\Phi}$  when  $\det g = 1$  the function  $f_{\Phi}$  is a multiple of  $\varphi_n$ . Since

$$\begin{aligned} f_{\Phi}(e) &= (i)^{|n|} \int_{-\infty}^{\infty} e^{-\pi t^2} t^{|n|+s+1} d^{\times} t \\ &= (i)^n \frac{\pi^{\frac{-(|n|+s+1)}{2}}}{2} \Gamma \left( \frac{|n|+s+1}{2} \right) \end{aligned}$$

which is not 0, the function  $f_{\Phi}$  is not 0.

The map  $A$  will transform the function  $W_{\Phi}$  to  $f_{\tilde{\Phi}}$ . It will certainly commute with the action of  $\{\mathfrak{A}, \epsilon\}$ . That  $A$  exists and is injective follows from a lemma which, together with its proof, is almost identical to the statement and proof of Lemma 3.2.1.

The same proof as that used in the non-archimedean case also shows that  $W(\mu_1, \mu_2; \psi) = W(\mu_2, \mu_1; \psi)$  for all  $\psi$ . To prove the existence of  $W(\pi, \psi)$  when  $\pi = \pi(\mu_1, \mu_2)$  and is infinite-dimensional we need only show that when  $\mu_1$  and  $\mu_2$  satisfy the condition the previous lemma the functions  $W$  in  $W(\mu_1, \mu_2; \psi)$  satisfy the growth condition of the theorem. We have seen that we can take  $W = W_{\Phi}$  with

$$\tilde{\Phi}(x, y) = e^{-\pi(x^2+y^2)} P(x, y)$$

where  $P(x, y)$  is a polynomial in  $x$  and  $y$ . Then

$$\Phi(x, y) = e^{-\pi(x^2+u^2y^2)} Q(x, y)$$

where  $Q(x, y)$  is another polynomial. Recall that  $\psi(x) = e^{2\pi i u x}$ . Then

$$W_{\Phi} \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) = \mu_1(a) |a|^{1/2} \int_{-\infty}^{\infty} e^{-\pi(a^2 t^2 + u^2 t^{-2})} Q(at, ut^{-1}) |t|^s (\text{sgn } t)^m d^{\times} t$$

The factor in front certainly causes no harm. If  $\delta > 0$  the integrals from  $-\infty$  to  $-\delta$  and from  $\delta$  to  $\infty$  decrease rapidly as  $|a| \rightarrow \infty$  and we need only consider integrals of the form

$$\int_0^\delta e^{-\pi(a^2 t^2 + u^2 t^{-2})} t^r dt$$

where  $r$  is any real number and  $u$  is fixed and positive. If  $v = \frac{u}{2}$  then  $u^2 = v^2 + \frac{3u^2}{4}$  and  $e^{-\frac{3}{4}\pi u^2 t^{-2}} t^r$  is bounded in the interval  $[0, \delta]$  so we can replace  $u$  by  $v$  and suppose  $r$  is 0. We may also suppose that  $a$  and  $v$  are positive and write the integral as

$$e^{-2\pi av} \int_0^\delta e^{-\pi(at + vt^{-1})^2} dt.$$

The integrand is bounded by 1 so that the integral is  $O(1)$ . In any case the growth condition is more than satisfied.

We have still to prove uniqueness. Suppose  $W_1(\pi, \psi)$  is a space of functions satisfying the first two conditions of the lemma. Let  $\kappa_n$  be a representation of the Lie algebra of  $SO(2, \mathbf{R})$  occurring in  $\pi$  and let  $W_1$  be a function in  $W_1(\pi, \psi)$  satisfying

$$W_1\left(g\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}\right) = e^{in\theta} W_1(g).$$

If

$$\varphi_1(t) = W_1\left(\begin{pmatrix} \frac{t}{|t|^{1/2}} & 0 \\ 0 & \frac{1}{|t|^{1/2}} \end{pmatrix}\right)$$

the function  $W_1$  is completely determined by  $\varphi_1$ . It is easily seen that

$$\begin{aligned} \rho(U)W_1\left(\begin{pmatrix} \frac{t}{|t|^{1/2}} & 0 \\ 0 & \frac{1}{|t|^{1/2}} \end{pmatrix}\right) &= in\varphi_1(t) \\ \rho(Z)W_1\left(\begin{pmatrix} \frac{t}{|t|^{1/2}} & 0 \\ 0 & \frac{1}{|t|^{1/2}} \end{pmatrix}\right) &= 2t \frac{d\varphi_1}{dt} \\ \rho(X_+)W_1\left(\begin{pmatrix} \frac{t}{|t|^{1/2}} & 0 \\ 0 & \frac{1}{|t|^{1/2}} \end{pmatrix}\right) &= iut\varphi_1(t). \end{aligned}$$

Thus if  $\varphi_1^+$  and  $\varphi_1^-$  correspond to  $\rho(V_+)W_1$  and  $\rho(V_-)W_1$

$$\varphi_1^+(t) = 2t \frac{d\varphi_1}{dt} - (2ut - n)\varphi_1(t)$$

and

$$\varphi_1^-(t) = 2t \frac{d\varphi_1}{dt} + (2ut - n)\varphi_1(t).$$

Since

$$D = \frac{1}{2}V_-V_+ - iU - \frac{U^2}{2}$$

$\rho(D)W_1$  corresponds to

$$2t \frac{d}{dt} \left( t \frac{d\varphi_1}{dt} - 2t \frac{d\varphi_2}{dt} \right) + (2nut - 2u^2t^2)\varphi_1.$$

Finally  $\rho(\epsilon)W_1$  corresponds to  $\varphi_1(-t)$ .

Suppose that  $\pi$  is either  $\pi(\mu_1, \mu_2)$  or  $\sigma(\mu_1, \mu_2)$ . Let  $\mu_1\mu_2^{-1}(t) = |t|^s(\text{sgn } t)^m$ . If  $s - m$  is an odd integer we can take  $n = |s| + 1$ . From Lemma 5.6 we have  $\rho(V_-)W_1 = 0$  so that  $\varphi_1$  satisfies the equation

$$2t \frac{d\varphi_1}{dt} + (2ut - n)\varphi_1 = 0.$$

If the growth condition is to be satisfied  $\varphi_1$  must be 0 for  $ut < 0$  and a multiple of  $|t|^{n/2}e^{-ut}$  for  $ut > 0$ . Thus  $W_1$  is determined up to a scalar factor and the space  $W(\pi, \psi)$  is unique.

Suppose  $s - m$  is not an odd integer. Since  $\rho(D)W_1 = \frac{s^2-1}{2}W_1$  the function  $\varphi_1$  satisfies the equation

$$\frac{d^2\varphi_1}{dt^2} + \left\{ -u^2 + \frac{nu}{t} + \frac{(1-s^2)}{4t^2} \right\} \varphi_1 = 0$$

We have already constructed a candidate for the space  $W(\pi, \psi)$ . Let's call this candidate  $W_2(\pi, \psi)$ . There will be a non-zero function  $\varphi_2$  in it satisfying the same equation as  $\varphi_1$ . Now  $\varphi_1$  and all of its derivatives go to infinity no faster than some power of  $|t|$  as  $t \rightarrow \infty$  while as we saw  $\varphi_2$  and its derivations go to 0 at least exponentially as  $|t| \rightarrow \infty$ . Thus the Wronskian

$$\varphi_1 \frac{d\varphi_2}{dt} - \varphi_2 \frac{d\varphi_1}{dt}$$

goes to 0 as  $|t| \rightarrow \infty$ . By the form of the equation the Wronskian is constant. Therefore it is identically 0 and  $\varphi_1(t) = \alpha\varphi_2(t)$  for  $t > 0$  and  $\varphi_1(t) = \beta\varphi_2(t)$  for  $t < 0$  where  $\alpha$  and  $\beta$  are two constants. The uniqueness will follow if we can show that for suitable choice of  $n$  we have  $\alpha = \beta$ . If  $m = 0$  we can take  $n = 0$ . If  $\mu_1(t) = |t|^{s_1}(\text{sgn } t)^{m_1}$  then  $\pi(\epsilon)W_1 = (-1)^{m_2}W_1$  so that  $\varphi_1(-t) = (-1)^{m_1}\varphi_1(t)$  and  $\varphi_2(-t) = (-1)^{m_2}\varphi_2(t)$ . Thus  $\alpha = \beta$ . If  $m = 1$  we can take  $n = 1$ . From Lemma 5.6

$$\pi(V_{-1})W_1 = (-1)^{m_1}s\pi(\epsilon)W_1$$

so that

$$2t \frac{d\varphi_1}{dt} + (2ut - 1)\varphi_1(t) = (-1)^{m_1}s\varphi_1(-t).$$

Since  $\varphi_2$  satisfies the same equation  $\alpha = \beta$ .

If  $\mu$  is a quasi-character of  $\mathbf{R}^\times$  and  $\omega$  is the character of  $\mathbf{C}^\times$  defined by  $\omega(z) = \mu(z\bar{z})$  then  $\pi(\omega) = \pi(\mu, \mu\eta)$ . We have defined  $W(\pi(\omega), \psi)$  in terms of  $\omega$  and also as  $W(\mu_1, \mu_2; \psi)$ . Because of the uniqueness the two resulting spaces must be equal.

**Corollary 5.14.** *Let  $m$  and  $n$  be two integers, one positive and the other 0. Let  $\omega$  be a quasi-character of  $\mathbf{C}^\times$  of the form*

$$\omega(z) = (z\bar{z})^{r-\frac{m+n}{2}} z^m \bar{z}^n$$

*and let  $\mu_1$  and  $\mu_2$  be two quasi-characters of  $\mathbf{R}^\times$  satisfying  $\mu_1\mu_2(x) = |x|^{2r}(\text{sgn } x)^{m+n+1}$  and  $\mu_1\mu_2^{-1}(x) = x^{m+n}\text{sgn } x$  so that  $\pi(\omega) = \sigma(\mu_1, \mu_2)$ . Then the subspace  $\mathcal{B}_s(\mu_1, \mu_2)$  of  $\mathcal{B}(\mu_1, \mu_2)$  is defined and there is an isomorphism of  $\mathcal{B}(\mu_1, \mu_2)$  with  $W(\mu_1, \mu_2; \psi)$  which commutes with*

the action of  $\{\mathfrak{A}, \epsilon\}$ . The image  $W_s(\mu_1, \mu_2; \psi)$  of  $\mathcal{B}_s(\mu_1, \mu_2)$  is  $W(\pi(\omega), \psi)$ . If  $\Phi$  belongs to  $\mathcal{S}(\mathbf{R}^2)$  and  $W_\Phi$  belongs to  $W(\mu_1, \mu_2; \psi)$  then  $W_\Phi$  belongs to  $W_s(\mu_1, \mu_2; \psi)$  if and only if

$$\int_{-\infty}^{\infty} x^i \frac{\partial^j}{\partial y^j} \Phi(x, 0) dx = 0$$

for any two non-negative integers  $i$  and  $j$  with  $i + j = m + n - 1$ .

Only the last assertion is not a restatement of previously verified facts. To prove it we have to show that  $f_\Phi$  belongs to  $\mathcal{B}_s(\mu_1, \mu_2)$  if and only if  $\Phi$  satisfies the given relations. Let  $f = f_\Phi$ . It is in  $\mathcal{B}_s(\mu_1, \mu_2)$  if and only if it is orthogonal to the functions in  $\mathcal{B}_f(\mu_1^{-1}, \mu_2^{-1})$ . Since  $\mathcal{B}_f(\mu_1^{-1}, \mu_2^{-1})$  is finite-dimensional there is a non-zero vector  $f_0$  in it such that  $\rho(X_+)f_0 = 0$ . Then

$$f_0 \left( w \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \right) = f_0(w)$$

and  $f$  is orthogonal to  $f_0$  if and only if

$$(5.14.1) \quad \int_{\mathbf{R}} f \left( w \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \right) dy = 0.$$

The dimension of  $\mathcal{B}_f(\mu_1^{-1}, \mu_2^{-1})$  is  $m + n$ . It follows easily from Lemmas 5.6 and 5.7 that the vectors  $\rho(X_+^p)\rho(w)f_0$ ,  $0 \leq p \leq m + n - 1$  span it. Thus  $f$  is in  $\mathcal{B}_s(\mu_1, \mu_2)$  if and only if each of the functions  $\rho(X_+^p)\rho(w)f$  satisfy (5.14.1). For  $f$  itself the left side of (5.14.1) is equal to

$$\int \left\{ \int \tilde{\Phi} \left( (0, t) w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \mu_1(t) \mu_2^{-1}(t) |t| d^\times t \right\} dx.$$

Apart from a positive constant which relates the additive and multiplicative Haar measure this equals

$$\iint \tilde{\Phi}(-t, -tx) t^{m+n} \operatorname{sgn} t dt dx$$

which is

$$(-1)^{m+n-1} \iint \tilde{\Phi}(t, x) t^{m+n-1} dt dx$$

or, in terms of  $\Phi$ ,

$$(5.14.2) \quad (-1)^{m+n-1} \int \Phi(t, 0) t^{m+n-1} dt.$$

By definition

$$r_{\mu_1, \mu_2}(w) \Phi(x, y) = \Phi'(y, x)$$

and an easy calculation based on the definition shows that

$$r_{\mu_1, \mu_2}(X_+^p) \Phi(x, y) = (2i\pi uxy)^p \Phi(x, y).$$

Thus  $r_{\mu_1, \mu_2}(X_+^p)r_{\mu_1, \mu_2}(w)\Phi$  is a non-zero scalar times

$$\frac{\partial^{2p}}{\partial x^p \partial y^p} \Phi'(y, x)$$

For this function (5.14.2) is the product of a non-zero scalar and

$$\iint \frac{\partial^{2p}}{\partial x^p \partial y^p} \Phi'(0, x) x^{m+n-1} dx.$$

Integrating by parts we obtain

$$\int \frac{\partial^p}{\partial y^p} \Phi'(0, x) x^{m+n-p-1} dx$$

except perhaps for sign. If we again ignore a non-zero scalar this can be expressed in terms of  $\Phi$  as

$$\int \frac{\partial^{m+n-p-1}}{\partial y^{m+n-p-1}} \Phi(x, 0) x^p dx.$$

The proof of the corollary is now complete.

Before stating the local functional equation we recall a few facts from the theory of local zeta-functions. If  $F$  is  $\mathbf{R}$  or  $\mathbf{C}$  and if  $\Phi$  belongs to  $\mathcal{S}(F)$  we set

$$Z(\omega \alpha_F^s, \Phi) = \int \Phi(a) \omega(a) |a|_F^s d^\times a.$$

$\omega$  is a quasi-character. The integral converges in a right half-plane. One defines functions  $L(s, \omega)$  and  $\epsilon(s, \omega, \psi_F)$  with the following properties:

(i) For every  $\Phi$  the quotient

$$\frac{Z(\omega \alpha_F^s, \Phi)}{L(s, \omega)}$$

has an analytic continuation to the whole complex plane as a holomorphic function. Moreover for a suitable choice of  $\Phi$  it is an exponential function and in fact a constant.

(ii) If  $\Phi'$  is the Fourier transform of  $\Phi$  with respect to the character  $\psi_F$  then

$$\frac{Z(\omega^{-1} \alpha_F^{1-s}, \Phi')}{L(1-s, \omega^{-1})} = \epsilon(s, \omega, \psi_F) \frac{Z(\omega \alpha_F^s, \Phi)}{L(s, \omega)}.$$

If  $F = \mathbf{R}$  and  $\omega(x) = |x|_{\mathbf{R}}^r (\text{sgn } x)^m$  with  $m$  equal to 0 or 1 then

$$L(s, \omega) = \pi^{-\frac{1}{2}(s+r+m)} \Gamma\left(\frac{s+r+m}{2}\right)$$

and if  $\psi_F(x) = e^{2\pi i u x}$

$$\epsilon(s, \omega, \psi_F) = (i \text{sgn } u)^m |u|_{\mathbf{R}}^{s+r-\frac{1}{2}}.$$

If  $F = \mathbf{C}$  and

$$\omega(x) = |x|_{\mathbf{C}}^r x^m \bar{x}^n$$

where  $m$  and  $n$  are non-negative integers, one of which is 0, then

$$L(s, \omega) = 2(2\pi)^{-(s+r+m+n)} \Gamma(s+r+m+n).$$

Recall that  $|x|_{\mathbf{C}} = x\bar{x}$ . If  $\psi_F(x) = e^{4\pi i \text{Re}(wz)}$

$$\epsilon(s, \omega, \psi_F) = i^{m+n} \omega(w) |w|_{\mathbf{C}}^{s-1/2}.$$

These facts recalled, let  $\pi$  be an irreducible admissible representation of  $\mathcal{H}_{\mathbf{R}}$ . If  $\pi = \pi(\mu_1, \mu_2)$  we set

$$L(s, \pi) = L(s, \mu_1) L(s, \mu_2)$$

and

$$\epsilon(s, \pi, \psi_{\mathbf{R}}) = \epsilon(s, \mu_1, \psi_{\mathbf{R}}) \epsilon(s, \mu_2, \psi_{\mathbf{R}})$$

and if  $\pi = \pi(\omega)$  where  $\omega$  is a character of  $\mathbf{C}^*$  we set

$$L(s, \pi) = L(s, \omega)$$

and

$$\epsilon(s, \pi, \psi_{\mathbf{R}}) = \lambda(\mathbf{C}/\mathbf{R}, \psi_{\mathbf{R}}) \epsilon(s, \omega, \psi_{\mathbf{C}/\mathbf{R}})$$

if  $\psi_{\mathbf{C}/\mathbf{R}}(z) = \psi_{\mathbf{R}}(z + \bar{z})$ . The factor  $\lambda(\mathbf{C}/\mathbf{R}, \psi_{\mathbf{R}})$  was defined in the first paragraph. It is of course necessary to check that the two definitions coincide if  $\pi(\omega) = \pi(\mu_1, \mu_2)$ . This is an immediate consequence of the duplication formula.

**Theorem 5.15.** *Let  $\pi$  be an infinite-dimensional irreducible admissible representation of  $\mathcal{H}_{\mathbf{R}}$ . Let  $\omega$  be the quasi-character of  $\mathbf{R}^\times$  defined by*

$$\pi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \omega(a)I$$

*If  $W$  is in  $W(\pi, \psi)$  set*

$$\begin{aligned} \Psi(g, s, W) &= \int_{\mathbf{R}^\times} W \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right) |a|^{s-1/2} d^\times a \\ \tilde{\Psi}(g, s, W) &= \int_{\mathbf{R}^\times} W \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right) \omega^{-1}(a) |a|^{s-1/2} d^\times a \end{aligned}$$

*and let*

$$\Psi(g, s, W) = L(s, \pi) \Phi(g, s, W)$$

$$\tilde{\Psi}(g, s, W) = L(s, \tilde{\pi}) \tilde{\Phi}(g, s, W).$$

- (i) *The integrals defined  $\Psi(g, s, W)$  and  $\tilde{\Psi}(g, s, W)$  are absolutely convergent in some right half-plane.*
- (ii) *The functions  $\Phi(g, s, W)$  and  $\tilde{\Phi}(g, s, W)$  can be analytically continued to the whole complex plane as meromorphic functions. Moreover there exists a  $W$  for which  $\Phi(e, s, W)$  is an exponential function of  $s$ .*
- (iii) *The functional equation*

$$\tilde{\Phi}(wg, 1-s, W) = \epsilon(s, \pi, \psi) \Phi(g, s, W)$$

*is satisfied.*

- (iv) *If  $W$  is fixed  $\Psi(g, s, W)$  remains bounded as  $g$  varies in a compact set and  $s$  varies in the region obtained by removing discs centred at the poles of  $L(s, \pi)$  from a vertical strip of finite width.*

We suppose first that  $\pi = \pi(\mu_1, \mu_2)$ . Then  $W(\pi, \psi) = W(\mu_1, \mu_2; \psi)$ . Each  $W$  in  $W(\mu_1, \mu_2; \psi)$  is of the form  $W = W_\Phi$  where

$$\Phi(x, y) = e^{-\pi(x^2 + u^2 y^2)} P(x, y)$$

with  $P(x, y)$  a polynomial. However we shall verify the assertions of the theorem not merely for  $W$  in  $W(\pi, \psi)$  but for any function  $W = W_\Phi$  with  $\Phi$  in  $\mathcal{S}(\mathbf{R}^2)$ . Since this class of

functions is invariant under right translations most of the assertions need then be verified only for  $g = e$ .

A computation already performed in the non-archimedean case shows that

$$\Psi(e, s, W) = Z(\mu_1 \alpha_{\mathbf{R}}^s, \mu_2 \alpha_{\mathbf{R}}^s, \Phi)$$

the integrals defining these functions both being absolutely convergent in a right half-plane. Also for  $s$  in some left half-plane

$$\tilde{\Psi}(w, 1-s, W) = Z(\mu_1^{-1} \alpha_{\mathbf{R}}^{1-s}, \mu_2^{-1} \alpha_{\mathbf{R}}^{1-s}, \Phi')$$

if  $\Phi'$  is the Fourier transform of  $\Phi$ .

Since  $\Phi$  can always be taken to be a function of the form  $\Phi(x, y) = \Phi_1(x)\Phi_2(y)$  the last assertion of part (ii) is clear. All other assertions of the theorem except the last are consequence of the following lemma.

**Lemma 5.15.1.** *For every  $\Phi$  in  $\mathcal{S}(\mathbf{R}^2)$  the quotient*

$$\frac{Z(\mu_1 \alpha_{\mathbf{R}}^{s_1}, \mu_2 \alpha_{\mathbf{R}}^{s_2}, \Phi)}{L(s, \mu_1)L(s, \mu_2)}$$

*is a holomorphic function of  $(s_1, s_2)$  and*

$$\frac{Z(\mu_1^{-1} \alpha_{\mathbf{R}}^{1-s_1}, \mu_2^{-1} \alpha_{\mathbf{R}}^{1-s_2}, \Phi')}{L(1-s_1, \mu_1^{-1})L(1-s_2, \mu_2^{-1})}$$

*is equal to*

$$\epsilon(s_1, \mu_1, \psi)\epsilon(s_2, \mu_2, \psi) \frac{Z(\mu_1 \alpha_{\mathbf{R}}^{s_1}, \mu_2 \alpha_{\mathbf{R}}^{s_2}, \Phi)}{L(s_1, \mu_1)L(s_2, \mu_2)}.$$

We may as well assume that  $\mu_1$  and  $\mu_2$  are characters so that the integrals converge for  $\text{Re } s_1 > 0$  and  $\text{Re } s_2 > 0$ . We shall show that when  $0 < \text{Re } s_1 < 1$  and  $0 < \text{Re } s_2 < 1$

$$Z(\mu_1 \alpha_{\mathbf{R}}^{s_1}, \mu_2 \alpha_{\mathbf{R}}^{s_2}, \Phi) Z(\mu_1^{-1} \alpha_{\mathbf{R}}^{1-s_1}, \mu_2^{-1} \alpha_{\mathbf{R}}^{1-s_2}, \Psi')$$

is equal to

$$Z(\mu_1^{-1} \alpha_{\mathbf{R}}^{1-s_1}, \mu_2^{-1} \alpha_{\mathbf{R}}^{1-s_2}, \Phi') Z(\mu_1 \alpha_{\mathbf{R}}^{s_1}, \mu_2 \alpha_{\mathbf{R}}^{s_2}, \Psi)$$

if  $\Phi$  and  $\Psi$  belong to  $\mathcal{S}(\mathbf{R}^2)$ .

The first of these expressions is equal to

$$\int \Phi(x, y) \Psi'(u, v) \mu_1\left(\frac{x}{u}\right) \mu_2\left(\frac{y}{v}\right) \left|\frac{x}{u}\right|^{s_1} \left|\frac{y}{v}\right|^{s_2} d^\times x d^\times y du dv$$

if we assume, as we may, that  $d^\times x = |x|^{-1} dx$ . Changing variables we obtain

$$\int \mu_1(x) \mu_2(y) |x|^{s_1} |y|^{s_2} \left\{ \int \Phi(xu, yv) \Psi'(u, v) du dv \right\} d^\times x d^\times y$$

The second expression is equal to

$$\int \mu_1^{-1}(x) \mu_2^{-1}(y) |x|^{1-s_1} |y|^{1-s_2} \left\{ \int \Phi'(xu, yv) \Psi(u, v) du dv \right\} d^\times x d^\times y$$

which equals

$$\int \mu_1(x) \mu_2(y) |x|^{s_1} |y|^{s_2} \left\{ \int |xy|^{-1} \Phi'(x^{-1}u, y^{-1}v) \Psi(u, v) du dv \right\} d^\times x d^\times y.$$



Since the Fourier transform of the function  $(u, v) \rightarrow \Phi(xu, yv)$  is the function

$$|xy|^{-1} \Phi'(x^{-1}u, y^{-1}v),$$

the Plancherel theorem implies that

$$\int \Phi(xu, yv) \Psi'(u, v) du dv = \int |xy|^{-1} \Phi'(x^{-1}u, y^{-1}v) \Psi(u, v) du dv.$$

The desired equality follows.

Choose  $\Phi_1$  and  $\Phi_2$  in  $\mathcal{S}(\mathbf{R})$  such that

$$L(s, \mu_i) = Z(\mu_i \alpha_{\mathbf{R}}^s, \Phi_i)$$

and take  $\Psi(x, y) = \Phi_1(x) \Phi_2(y)$ . The functional equation of the lemma follows immediately if  $0 < s_1 < 1$  and  $0 < s_2 < 1$ . The expression on one side of the equation is holomorphic for  $0 < \operatorname{Re} s_1$  and  $0 < \operatorname{Re} s_2$ . The expression on the other side is holomorphic for  $\operatorname{Re} s_1 < 1$  and  $\operatorname{Re} s_2 < 1$ . Standard and easily proved theorems in the theory of functions of several complex variables show that the function they define is actually an entire function of  $s_1$  and  $s_2$ . The lemma is completely proved.

For  $\pi = \pi(\mu_1, \mu_2)$  the final assertion of the theorem is a consequence of the following lemma.

**Lemma 5.15.2.** *Let  $\Omega$  be a compact subset of  $\mathcal{S}(\mathbf{R}^2)$  and  $C$  a domain in  $\mathbf{C}^2$  obtained by removing balls about the poles of  $L(s_1, \mu_1)L(s_2, \mu_2)$  from a tube  $a_1 \leq \operatorname{Re} s_1 \leq b_1$ ,  $a_2 \leq \operatorname{Re} s_2 \leq b_2$ . Then*

$$Z(\mu_1 \alpha_{\mathbf{R}}^{s_1}, \mu_2 \alpha_{\mathbf{R}}^{s_2}, \Phi)$$

*remains bounded as  $\Phi$  varies in  $\Omega$  and  $(s_1, s_2)$  varies in  $C$ .*

The theorems in the theory of functions alluded to earlier show that it is enough to prove this when either both  $a_1$  and  $a_2$  are greater than 0 or both  $b_1$  and  $b_2$  are less than 1. On a region of the first type the function  $Z(\mu_1 \alpha_{\mathbf{R}}^s, \mu_2 \alpha_{\mathbf{R}}^s, \Phi)$  is defined by a definite integral. Integrating by parts as in the theory of Fourier transforms one finds that

$$Z(\mu_1 \alpha_{\mathbf{R}}^{\sigma_1 + i\tau_1}, \mu_2 \alpha_{\mathbf{R}}^{\sigma_2 + i\tau_2}, \Phi) = O(\tau_1^2 + \tau_2^2)^{-n}$$

as  $\tau_1^2 + \tau_2^2 \rightarrow \infty$  uniformly for  $\Phi$  in  $\Omega$  and  $a_1 \leq \sigma_1 \leq b_1$ ,  $a_2 \leq \sigma_2 \leq b_2$  which is a much stronger estimate than required. For a region of the second type one combines the estimates just obtained with the functional equation and the known asymptotic behaviour of the  $\Gamma$ -function.

Now let  $\omega$  be a quasi-character of  $\mathbf{C}^\times$  which is not of the form  $\omega(z) = \chi(z\bar{z})$  with  $\chi$  a quasi-character of  $\mathbf{R}^\times$  and let  $\pi = \pi(\omega)$ .  $W(\pi, \psi)$  is the sum of  $W_1(\pi, \psi)$  and its right translate by  $\epsilon$ . It is easily seen that

$$\Phi(g, s, \rho(\epsilon)W) = \omega(-1)\Phi(\epsilon^{-1}g\epsilon, s, W)$$

and that

$$\tilde{\Phi}(wg, s, \rho(\epsilon)W) = \omega(-1)\tilde{\Phi}(w\epsilon^{-1}g\epsilon, s, W)$$

Thus it will be enough to prove the theorem for  $W$  in  $W_1(\pi, \psi)$ . Since

$$\Phi(\epsilon g, s, W) = \Phi(g, s, W)$$

and

$$\tilde{\Phi}(w\epsilon g, s, W) = \tilde{\Phi}(wg, s, W)$$

we can also take  $g$  in  $G_+$ .  $W_1(\pi, \psi)$  consists of the functions  $W_\Phi$  with  $\Phi$  in  $\mathcal{S}_0(\mathbf{C}, \omega)$ . We prove the assertions for functions  $W_\Phi$  with  $\Phi$  in  $\mathcal{S}(\mathbf{C}, \omega)$ . Since this class of functions is invariant under right translations by elements of  $G_+$  we may take  $g = e$ .

As we observed in the first paragraph we will have

$$\begin{aligned}\Psi(e, s, W) &= Z(\omega \alpha_{\mathbf{C}}^s, \Phi) \\ \tilde{\Psi}(w, 1-s, W) &= \lambda(\mathbf{C}/\mathbf{R}, \psi) Z(\omega^{-1} \alpha_{\mathbf{C}}^{1-s}, \Phi')\end{aligned}$$

in some right half plane and the proof proceeds as before. If  $\omega(z) = (z\bar{z})^r z^m \bar{z}^n$  and  $p - q = n - m$  the function

$$\Phi(z) = e^{-2\pi|u|z\bar{z}} z^p \bar{z}^q$$

belongs to  $\mathcal{S}_0(\mathbf{C}, \omega)$  and

$$\begin{aligned}Z(\omega \alpha_{\mathbf{C}}^s, \Phi) &= 2\pi \int_0^\infty e^{-2\pi|u|t^2} t^{2(s+r+p+m)} dt \\ &= \pi(2\pi|u|)^{-(s+r+p+m)} \Gamma(s+r+p+m)\end{aligned}$$

Taking  $p = n$  we obtain an exponential times  $L(s, \omega)$ . The last part of the theorem follows from an analogue of Lemma 5.15.2.

The local functional equation which we have just proved is central to the Hecke theory. We complete the paragraph with some results which will be used in the paragraph on extraordinary representations and the chapter on quaternion algebras.

**Lemma 5.16.** *Suppose  $\mu_1$  and  $\mu_2$  are two quasi-characters for which both  $\pi = \pi(\mu_1, \mu_2)$  and  $\sigma = \sigma(\mu_1, \mu_2)$  are defined. Then*

$$\frac{L(1-s, \tilde{\sigma})\epsilon(s, \sigma, \psi)}{L(s, \sigma)} = \frac{L(1-s, \tilde{\pi})\epsilon(s, \pi, \psi)}{L(s, \pi)}$$

and the quotient

$$\frac{L(s, \sigma)}{L(s, \pi)}$$

is an exponential times a polynomial.

Interchanging  $\mu_1$  and  $\mu_2$  if necessary we may suppose that  $\mu_1 \mu_2^{-1}(x) = |x|^s (\text{sgn } x)^m$  with  $s > 0$ . According to Corollary 5.14,  $W(\sigma, \psi)$  is a subspace of  $W(\mu_1, \mu_2, \psi)$ . Although  $W(\mu_1, \mu_2, \psi)$  is not irreducible it is still possible to define  $\Psi(g, s, W)$  and  $\tilde{\Psi}(g, s, W)$  when  $W$  lies in  $W(\mu_1, \mu_2, \psi)$  and to use the method used to prove Theorem 5.15 to show that

$$\frac{\tilde{\Psi}(wg, 1-s, W)}{L(1-s, \tilde{\pi})}$$

is equal to

$$\epsilon(s, \pi, \psi) \frac{\Psi(g, s, W)}{L(s, \pi)}$$

Applying the equality to an element of  $W(\sigma, \psi)$  we obtain the first assertion of the lemma. The second is most easily obtained by calculation. Replacing  $\mu_1$  and  $\mu_2$  by  $\mu_1 \alpha_{\mathbf{R}}^t$  and  $\mu_2 \alpha_{\mathbf{R}}^t$  is equivalent to a translation in  $s$  so we may assume that  $\mu_2$  is of the form  $\mu_2(x) = (\text{sgn } x)^{m_2}$ . There is a quasi-character  $\omega$  of  $\mathbf{C}^\times$  such that  $\sigma = \pi(\omega)$ . If  $\omega(z) = (z\bar{z})^r z^m \bar{z}^n$  then  $\mu_1(x) = |x|^{2r+m+n} (\text{sgn } x)^{m+n+m_2+1}$ ,  $\mu_1(x) = x^{m+n} (\text{sgn } x)^{m_2+1}$  so that  $r = 0$ . Apart from an

exponential factor  $L(s, \sigma)$  is equal to  $\Gamma(s + m + n)$  while  $L(s, \pi)$  is, again apart from an exponential factor,

$$(5.16.1) \quad \Gamma\left(\frac{s + m + n + m_2}{2}\right) \Gamma\left(\frac{s + m_2}{2}\right)$$

where  $m_1 = m + n + m_2 + 1 \pmod{2}$ . Since  $m + n > 0$  the number

$$k = \frac{1}{2}(m + n + 1 + m_1 - m_2) - 1$$

is a non-negative integer and  $m_2 + 2k = m + n + m_1 - 1$ . Thus

$$\Gamma\left(\frac{s + m_2}{2}\right) = \left\{ \frac{1}{2^{k+1}} \prod_{j=0}^k (s + m_2 + 2j) \right\}^{-1} \Gamma\left(\frac{s + m + n + m_1 + 1}{2}\right).$$

By the duplication formula the product (5.16.1) is a constant times an exponential times

$$\frac{\Gamma(s + m + n + m_1)}{\prod_{j=0}^k (s + m_2 + 2j)}.$$

If  $m_1 = 0$  the lemma follows immediately. If  $m_1 = 1$

$$\Gamma(s + m + n + m_2) = (s + m + n) \Gamma(s + m + n)$$

and  $m_2 + 2k = m + n$ . The lemma again follows.

**Lemma 5.17.** *Suppose  $\omega(z) = (z\bar{z})^r z^m \bar{z}^n$  is a quasi-character of  $\mathbf{C}^\times$  with  $mn = 0$  and  $m + n > 0$ . Suppose  $\mu_1$  and  $\mu_2$  are two quasi-characters of  $F^\times$  with  $\mu_1 \mu_2(x) = |x|^{2r} x^{m+n} \operatorname{sgn} x$  and  $\mu_1 \mu_2^{-1}(x) = x^{m+n} \operatorname{sgn} x$ . Then for every  $\Phi$  in  $\mathcal{S}(\mathbf{R}^2)$  such that*

$$\int x^i \frac{\partial^j \Phi}{\partial y^j}(x, 0) dx = 0$$

for  $i > 0$ ,  $j \geq 0$ , and  $i + j + 1 = m + n$  the quotient

$$\frac{Z(\mu_1 \alpha_{\mathbf{R}}^s, \mu_2 \alpha_{\mathbf{R}}^s, \Phi)}{L(s, \pi(w))}$$

is a holomorphic function of  $s$  and for some  $\Phi$  it is an exponential.

If  $W_\Phi$  belongs to  $W(\mu_1, \mu_2, \psi)$  this is a consequence of Corollary 5.14 and Theorem 5.15. Unfortunately we need the result for all  $\Phi$ . The observations made during the proof of Lemma 5.16 show that if  $\pi = \pi(\mu_1, \mu_2)$  the quotient

$$\frac{Z(\mu_1 \alpha_{\mathbf{R}}^s, \mu_2 \alpha_{\mathbf{R}}^s, \Phi)}{L(s, \pi)}$$

is holomorphic. Since  $L(s, \pi)$  and  $L(s, \sigma)$  have no zeros we have only to show that the extra poles of  $L(s, \pi)$  are not really needed to cancel poles of  $Z(\mu_1 \alpha_{\mathbf{R}}^s, \mu_2 \alpha_{\mathbf{R}}^s, \Phi)$ . As in the proof of Lemma 5.16 we may take  $r = 0$ . We have to show that  $Z(\mu_1 \alpha_{\mathbf{R}}^s, \mu_2 \alpha_{\mathbf{R}}^s, \Phi)$  is holomorphic at  $s = -m_2 - 2j$ ,  $0 \leq j \leq k$  if  $m_1 = 0$  and at  $s = -m_2 - 2j$ ,  $0 \leq j \leq k$  if  $m_1 = 1$ . We remark first that if  $\mu_1$  and  $\mu_2$  are two quasi-characters of  $\mathbf{R}^\times$ ,  $\Phi$  belongs to  $\mathcal{S}(\mathbf{R}^2)$ , and  $\operatorname{Re} s$  is sufficiently large then, by a partial integration,

$$\int \mu_1(x) \mu_2(y) |x|^s |y|^s \Phi(x, y) d^\times x d^\times y = -\frac{1}{s} \int \mu_1(x) \mu_2(y) \eta(y) |x|^s |y|^{s+1} \frac{\partial \Phi}{\partial y}(x, y) d^\times x d^\times y$$

if  $\eta(y) = \text{sgn } y$ . Integrating by parts again we obtain

$$\int \mu_1(x)\mu_2(y)|x|^s|y|^s\Phi(x,y) d^\times x d^\times y = \frac{1}{s(s+1)} \int \mu_1(x)\mu_2(y)|x|^s|y|^{s+2}\frac{\partial^2\Phi}{\partial y^2}(x,y) d^\times x d^\times y.$$

If  $\Phi$  belongs to  $\mathcal{S}(\mathbf{R}^2)$  the function defined by

$$(5.17.1) \quad \int \Phi(x,y)|x|^{s+1}|y|^s d^\times x d^\times y$$

is certainly holomorphic for  $\text{Re } s > 0$ . We have to show that if

$$\int \Phi(x,0) dx = 0$$

it is holomorphic for  $\text{Re } s > -1$ . Suppose first that  $\Phi(x,0) \equiv 0$ . Since

$$\Phi(x,y) = y \frac{\partial \Phi}{\partial y}(x,0) + \int_0^y (y-u) \frac{\partial^2 \Phi}{\partial y^2}(x,u) du$$

the function

$$\Psi(x,y) = \frac{1}{y} \Phi(x,y)$$

is dominated by the inverse of any polynomial. Thus (5.17.1) which equals

$$\int \Psi(x,y)|x|^{s+1}|y|^{s+1}\eta(y) d^\times x d^\times y$$

is absolutely convergent for  $\text{Re } s > -1$ . In the general case we set

$$\begin{aligned} \Phi(x,y) &= \left\{ \Phi(x,y) - \Phi(x,0)e^{-y^2} \right\} + \Phi(x,0)e^{-y^2} \\ &= \Phi_1(x,y) + \Phi_2(x,y). \end{aligned}$$

Since  $\Phi_1(x,0) = 0$  we need only consider

$$\int \Phi_2(x,0)e^{-y^2}|x|^{s+1}|y|^s d^\times x d^\times y$$

which is the product of a constant and

$$\Gamma\left(\frac{s}{2}\right) \int \Phi_2(x,0)|x|^s dx.$$

The integral defines a function which is holomorphic for  $\text{Re } s > -1$  and, when the assumptions are satisfied, vanishes at  $s = 0$ .

We have to show that if  $0 \leq j \leq m+n-1$  and  $j-m_2$  is even then  $Z(\mu_1\alpha_{\mathbf{R}}^s, \mu_2\alpha_{\mathbf{R}}^s, \Phi)$  is holomorphic at  $-j$ . Under these circumstances the function  $Z(\mu_1\alpha_{\mathbf{R}}^s, \mu_2\alpha_{\mathbf{R}}^s, \Phi)$  is equal to

$$\int \eta(x)^{m_1}\eta(y)^{m_2}|x|^{m+n}|x|^s|y|^s\Phi(x,y) d^\times x d^\times y$$

which equals

$$\frac{(-1)^j}{\prod_{i=0}^{j-1}(s+i)} \int \eta(x)^{m_1}|x|^{s+m+n}|y|^{s+j}\frac{\partial^j\Phi}{\partial y^j}(x,y) d^\times x d^\times y.$$

The factor in front is holomorphic at  $s = -j$ . If

$$\Psi(x,y) = x^{m+n-j-1}\frac{\partial^j\Phi}{\partial y^j}(x,y)$$

the integral itself is equal to

$$\int |x|^{s+j+1} |y|^{s+j} \Psi(x, y) d^\times x d^\times y.$$

Since, by assumption,

$$\int \Psi(x, 0) dx = 0,$$

it is holomorphic at  $s = -j$ .

We observe that if  $m + n$  is even

$$\Phi(x, y) = e^{-\pi(x^2+y^2)} xy^{m+n}$$

satisfies the conditions of the lemma and, if  $r = 0$  and  $m_2 = 0$ ,  $Z(\mu_1 \alpha_{\mathbf{R}}^s, \mu_2 \alpha_{\mathbf{R}}^s, \Phi)$  is equal to

$$\int e^{-\pi(x^2+y^2)} |x|^{m+n+s+1} |y|^{m+n+s} d^\times x d^\times y$$

which differs by an exponential from  $\Gamma(s + m + n)$  and  $L(s, \pi(\omega))$ . If  $m_2 = 1$  we take  $\Phi(x, y) = e^{-\pi(x^2+y^2)} y^{m+n+1}$  to obtain the same result. If  $m + n$  is odd and  $m_2 = 0$  the polynomial factor will be  $y^{m+n+1}$  but if  $m + n$  is odd and  $m_2 = 1$  it will again be  $xy^{m+n}$ .

**Proposition 5.18.** *Suppose  $\pi$  and  $\pi'$  are two infinite-dimensional irreducible admissible representations of  $\mathcal{H}_{\mathbf{R}}$  such that, for some quasi-character  $\omega$  of  $F^\times$ ,*

$$\pi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \omega(a)I \quad \pi' \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \omega(a)I.$$

If

$$\frac{L(1-s, \chi^{-1} \otimes \tilde{\pi})}{L(s, \chi \otimes \pi)} \epsilon(s, \chi \otimes \pi, \psi) = \frac{L(1-s, \chi^{-1} \otimes \tilde{\pi}')}{L(s, \chi \otimes \pi')} \epsilon(s, \chi \otimes \pi', \psi)$$

for all quasi-characters  $\chi$  and  $\pi$  and  $\pi'$  are equivalent.

Suppose  $\pi = \pi(\mu_1, \mu_2)$  or  $\sigma(\mu_1, \mu_2)$ . From Lemma 5.16 and the definitions the expression on the left is equal to

$$(i \operatorname{sgn} u)^{m_1+m_2} |u|^{2s+s_1+s_2-1} \pi^{2s+s_1+s_2-1} \frac{\Gamma\left(\frac{1-s-r_1+m_1}{2}\right) \Gamma\left(\frac{1-s-r_2+m_2}{2}\right)}{\Gamma\left(\frac{s+r_1+m_1}{2}\right) \Gamma\left(\frac{s+r_2+m_2}{2}\right)}$$

if  $\chi$  is trivial and  $\mu_i(x) = |x|^{r_i} (\operatorname{sgn} x)^{m_i}$ . If  $\chi(x) = \operatorname{sgn} x$  and  $n_i$  is 0 or 1 while  $m_i + n_i = 1 \pmod{2}$  the quotient is

$$(i \operatorname{sgn} u)^{m_1+m_2} |u|^{2s+s_1+s_2-1} \pi^{2s+s_1+s_2-1} \frac{\Gamma\left(\frac{1-s-r_1-n_1}{2}\right) \Gamma\left(\frac{1-s-r_2+n_2}{2}\right)}{\Gamma\left(\frac{s+r_1+n_1}{2}\right) \Gamma\left(\frac{s+r_2+n_2}{2}\right)}.$$

If we let  $\pi'$  be  $\pi(\mu'_1, \mu'_2)$  or  $\sigma(\mu'_1, \mu'_2)$  we obtain similar formulae with  $r_i$  replaced by  $r'_i$  and  $m_i$  by  $m'_i$ .

Consider first the quotients for  $\pi$ . The first has an infinite number of zeros of the form  $-r_1 - m_1 - 2p$  where  $p$  is a non-negative integer and an infinite number of the form  $-r_2 - m_2 - 2p$  where  $p$  is a non-negative integer, but no other zeros. Similarly the zeros of the second are at points  $-r_1 - n_1 - 2p$  or  $-r_2 - n_2 - 2p$ . Thus if the quotients are equal  $r_1 + m_1 \equiv r_2 + n_2 \equiv r_2 + m_2 + 1 \pmod{2}$ . Moreover if  $r_1 + m_1 = r_2 + m_2 + 1 \pmod{2}$  then  $\pi = \sigma(\mu_1, \mu_2)$  and, as we saw in Theorem 5.11,  $\sigma(\mu_1 \eta, \mu_2 \eta) = \sigma(\mu_1, \mu_2)$  so

that the two quotients are equal. As a result either  $r_1 + m_1 = r_2 + m_2 + 1 \pmod{2}$  and  $r'_1 + m'_1 = r'_2 + m'_2 + 1 \pmod{2}$  or neither of these congruences hold.

Suppose first that  $\pi = \pi(\mu_1, \mu_2)$  and  $\pi' = \pi'(\mu'_1, \mu'_2)$ . Then the first quotient for  $\pi$  has zeros at the points  $-r_1 - m_1, -r_1 - m_1 - 2, \dots$  and  $-r_2 - m_2, -r_2 - m_2 - 2, \dots$  while that for  $\pi'$  has zeros at  $-r'_1 - m'_1, -r'_1 - m'_1 - 2, \dots$  and  $-r'_2 - m'_2, -r'_2 - m'_2 - 2, \dots$ . Thus either  $r_1 + m_1 = r'_1 + m'_1$  or  $r_1 + m_1 = r'_2 + m'_2$ . Interchanging  $\mu'_1$  and  $\mu'_2$  if necessary we may assume that the first of these two alternatives hold. Then  $r_2 + m_2 = r'_2 + m'_2$ . Moreover  $r_1 + r_2 = r'_1 + r'_2$  and  $|m_1 - m_2| = |m'_1 - m'_2|$ . If  $m_1 = m'_1$  it follows immediately that  $\mu_1 = \mu'_1$  and  $\mu_2 = \mu'_2$ . Suppose that  $m_1 \neq m'_1$ . Examining the second quotient we see that either  $r_1 + n_1 = r'_1 + n'_1$  or  $r_1 + n_1 = r'_2 + n'_2$ . The first equality is incompatible with the relations  $r_1 + m_1 = r'_1 + m'_1$  and  $m_1 \neq m'_1$ . Thus  $r_1 + n_1 = r'_2 + n'_2$ . For the same reason  $r_2 + n_2 = r'_1 + n'_1$ . Interchanging the roles of  $\mu_1, \mu_2$  and  $\mu'_1, \mu'_2$  if necessary we may suppose that  $m_1 = 0$  and  $m'_1 = 1$ . Then  $r_1 = r'_1 + 1$ . Since  $r_1 + r_2 = r'_1 + r'_2$  we have  $r_2 = r'_2 - 1$  so that  $m_2 = 1, m'_2 = 0$ . Thus  $n_1 = n'_2 = 1$  and  $r_1 = r'_2$  so that  $r_2 = r'_1$ . It follows that  $\mu_1 = \mu'_2$  and  $\mu_2 = \mu'_1$ .

Finally we suppose that  $\pi = \sigma(\mu_1, \mu_2)$  and  $\pi' = \sigma(\mu'_1, \mu'_2)$ . Then there are quasi-characters  $\omega_1$  and  $\omega'_1$  of  $\mathbf{C}^\times$  such that  $\pi = \pi(\omega_1)$  and  $\pi' = \pi(\omega'_1)$ . Replacing  $\omega_1$  by the quasi-character  $z \rightarrow \omega_1(\bar{z})$  does not change  $\pi(\omega_1)$  so we may suppose that  $\omega_1(z) = (z\bar{z})^r z^m$  while  $\omega'_1(z) = (z\bar{z})^{r'} z^{m'}$ . Since  $\omega_1$  and  $\omega'_1$  must have the same restriction to  $\mathbf{R}^\times$  the numbers  $2r + m$  and  $2r' + m'$  are equal while  $m \equiv m' \pmod{2}$ . Apart from a constant and an exponential factor the quotient

$$\epsilon(s, \pi, \psi) \frac{L(1-s, \tilde{\pi})}{L(s, \pi)}$$

is given by

$$\frac{\Gamma(1-s-r)}{\Gamma(s+r+m)}$$

whose pole furthest to the left is at  $1-r$ . Consequently  $r = r'$  and  $m = m'$ .

**Corollary 5.19.** *Suppose  $\pi$  and  $\pi'$  are two irreducible admissible representations of  $\mathcal{H}_{\mathbf{R}}$ . Suppose there is a quasi-character  $\omega$  of  $\mathbf{R}^\times$  such that*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I \quad \pi'\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I$$

*If for all quasi-characters  $\chi$ ,  $L(s, \chi \otimes \pi) = L(s, \chi \otimes \pi')$ ,  $L(s, \chi^{-1} \otimes \tilde{\pi}) = L(s, \chi^{-1} \otimes \tilde{\pi}')$ , and  $\epsilon(s, \chi \otimes \pi, \psi) = \epsilon(s, \chi \otimes \pi', \psi)$  then  $\pi$  and  $\pi'$  are equivalent.*

Combining Lemma 5.16 with the previous proposition we infer that there is a pair of quasi-characters  $\mu_1$  and  $\mu_2$  such that both  $\pi$  and  $\pi'$  are one of the representations  $\pi(\mu_1, \mu_2)$  or  $\sigma(\mu_1, \mu_2)$ . However the computations made during the proof of Lemma 5.16 show that  $L(s, \chi \otimes \pi(\mu_1, \mu_2))$  differs from  $L(s, \chi \otimes \sigma(\mu_1, \mu_2))$  for a suitable choice of  $\chi$ .

Let  $K$  be the quaternion algebra over  $\mathbf{R}$ . We could proceed along the lines of the fourth paragraph and associate to every finite-dimensional irreducible representation  $\Omega$  of  $K^\times$  a representation  $\pi(\Omega)$  of  $G_{\mathbf{R}}$ . Since we have just classified the representations of  $G_{\mathbf{R}}$  we can actually proceed in a more direct manner.

We identify  $K$  with the algebra of  $2 \times 2$  complex matrices of the form

$$z = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

Then

$$z^\iota = \begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix}$$

and  $\nu(z) = zz^\iota$  is the scalar matrix  $(|a|^2 + |b|^2)I$  while  $\tau(z)$  is the scalar matrix  $(a + \bar{a} + b + \bar{b})I$ . Let  $\rho_1$  be the two-dimensional representation of  $K^\times$  associated to this identification and let  $\rho_n$  be the  $n$ th symmetric power of  $\rho_1$ . Any irreducible representation is equivalent to a representation of the form  $\chi \otimes \rho_n$  where  $\chi$  is a quasi-character of  $\mathbf{R}^\times$ . Thus

$$(\chi \otimes \rho_n)(h) = \chi(\nu(h))\rho_n(h)$$

Since  $\nu(h)$  is always positive we may suppose that  $\chi$  is of the form  $\chi(x) = |x|^r$ .

Let  $\Omega$  be a finite-dimensional representation and let  $\Omega$  act on  $U$ . In the first paragraph we introduced the space  $\mathcal{S}(K, U)$ . It is clear that if  $\Phi$  is in  $\mathcal{S}(K, U)$  the integrals

$$Z(\alpha_{\mathbf{R}}^s \otimes \Omega, \Phi) = \int_{K^\times} \Omega(h) |\nu(h)|^s \Phi(h) d^\times h$$

and

$$Z(\alpha_{\mathbf{R}}^s \otimes \Omega^{-1}, \Phi) = \int_{K^\times} \Omega^{-1}(h) |\nu(h)|^s \Phi(h) d^\times h$$

converge absolutely in some right half-plane.

**Proposition 5.20.** *Suppose  $\chi(x) = |x|^r$  and  $\Omega = \chi \otimes \rho_n$ . Let  $\omega$  be the character of  $\mathbf{C}^\times$  defined by  $\omega(z) = (z\bar{z})^{r-1/2} z^{n+1}$ . Set  $L(s, \Omega) = L(s, \omega)$  and*

$$\epsilon(s, \Omega, \psi_{\mathbf{R}}) = \lambda(\mathbf{C}/\mathbf{R}, \psi_{\mathbf{R}}) \epsilon(s, \omega, \psi_{\mathbf{C}/\mathbf{R}})$$

*The quotient*

$$\frac{Z(\alpha_{\mathbf{R}}^{s+1/2} \otimes \Omega, \Phi)}{L(s, \Omega)}$$

*can be analytically continued to the whole complex plane as a holomorphic function. Given  $u$  in  $U$  there exists a  $\Phi$  in  $\mathcal{S}(K, U)$  such that*

$$\frac{Z(\alpha_{\mathbf{R}}^{s+1/2}, \Phi)}{L(s, \Omega)} = a^s u.$$

*For all  $\Phi$  the two functions*

$$\frac{Z(\alpha_{\mathbf{R}}^{3/2-s} \otimes \Omega^{-1}, \Phi')}{L(1-s, \tilde{\Omega})}$$

*and*

$$-\epsilon(s, \Omega, \psi_{\mathbf{R}}) \frac{Z(\alpha_{\mathbf{R}}^{s+1/2} \otimes \Omega, \Phi)}{L(s, \Omega)}$$

*are equal. Finally  $Z(\alpha_{\mathbf{R}}^{s+1/2} \otimes \Omega, \Phi)$  is bounded in any region obtained by removing discs about the poles  $L(s, \Omega)$  from a vertical strip of finite width.*

Suppose  $K_1$  is the subgroup of  $K^\times$  formed by the elements of reduced norm one. Let  $\Phi_1$  be the function on  $\mathbf{R}$  defined by

$$\Phi_1(t) = \int_{K_1} \Omega(h) \Phi(th) dh$$

$\Phi_1$  belongs to  $\mathcal{S}(\mathbf{R})$  and if  $\omega_0$  is the quasi-character of  $\mathbf{R}^\times$  defined by  $\Omega(t) = \omega_0(t)I$  the function  $\omega_0(t)\Phi_1(t)$  is even. Moreover if the multiplicative Haar measures are suitably normalized

$$Z(\alpha_{\mathbf{R}}^{s+1/2} \otimes \Omega, \Phi) = Z(\alpha_{\mathbf{R}}^{2s+1} \omega_0, \Phi_1).$$

Since  $\omega_0(t) = |t|^{2r} t^n$  we can integrate by parts as in the proof of Lemma 5.17 to see that for any non-negative integer  $m$

$$Z(\alpha_{\mathbf{R}}^{2s+1} \omega_0, \Phi_1) = \frac{(-1)^m}{\prod_{j=0}^{m-1} (2s + 2r + n + j + 1)} \int \eta(t)^{m+n} |t|^{2s+2r+m+n+1} \frac{\partial^m \Phi_1}{\partial t^m} d^\times t.$$

The integral is holomorphic for  $\operatorname{Re}(2s + 2r + m + n) > -1$  and, if  $\frac{\partial^m \Phi_1}{\partial t^m}$  vanishes at  $t = 0$ , for  $\operatorname{Re}(2s + 2r + m + n) > -2$ . Thus the function on the left has an analytic continuation to the whole complex plane as a meromorphic function with simple poles. Since

$$L(s, \Omega) = 2(2\pi)^{-(s+r+n+1/2)} \Gamma\left(s + r + n + \frac{1}{2}\right)$$

we have to show that its poles occur at the points  $s + r + n + \frac{1}{2} + j = 0$  with  $j = 0, 1, 2, \dots$ . Since  $\frac{\partial^m \Phi_1}{\partial t^m}$  vanishes at 0 if  $m+n$  is odd its only poles are at the points  $2s + 2r + 2n + 2j + 1 = 0$  with  $n + 2j \geq 0$ . To exclude the remaining unwanted poles we have to show that  $\frac{\partial^m \Phi_1}{\partial t^m} = 0$  at 0 if  $m < n$ . If we expand  $\Phi$  in a Taylor's series about 0 we see that  $\frac{\partial^m \Phi_1}{\partial t^m} = 0$  at 0 unless the restriction of  $\rho_n$  to  $K_1$  is contained in the representation on the polynomials of degree  $m$  on  $K$ . This can happen only if  $m \geq n$ .

Since  $\tilde{\Omega}$  is equivalent to the representation  $h \rightarrow \Omega^{-1}(h^t)$  the quotient

$$\frac{Z(\alpha_{\mathbf{R}}^{3/2-s} \otimes \Omega^{-1}, \Phi')}{L(1-s, \tilde{\Omega})}$$

is also holomorphic. The argument used to prove Lemma 5.15.1 shows that there is a scalar  $\lambda(s)$  such that, for all  $\Phi$ ,

$$\frac{Z(\alpha_{\mathbf{R}}^{3/2-s} \otimes \Omega^{-1}, \Phi')}{L(1-s, \tilde{\Omega})} = \lambda(s) \frac{Z(\alpha_{\mathbf{R}}^{s+1/2} \otimes \Omega, \Phi)}{L(s, \Omega)}.$$

We shall use the following lemma to evaluate  $\lambda(s)$ .

**Lemma 5.20.1.** *Let  $\varphi$  be a function in  $\mathcal{S}(\mathbf{C})$  of the form*

$$\varphi(x) = e^{-2\pi x \bar{x}} P(x, \bar{x})$$

*where  $P$  is a polynomial in  $x$  and  $\bar{x}$ . Suppose  $\varphi(xu) = \varphi(x)\omega^{-1}(u)$  if  $u\bar{u} = 1$ . Define the function  $\Phi$  in  $K^\times$  by*

$$\Phi(z) = \varphi(\alpha)\omega(\alpha)(\alpha\bar{\alpha})^{-1/2} \left\langle u, \tilde{\Omega}(z)\tilde{u} \right\rangle$$

*if  $\nu(z) = \alpha\bar{\alpha}$ . Then  $\Phi$  extends to a function in  $\mathcal{S}(K)$  and its Fourier transform is given by*

$$\Phi'(z) = -\lambda(\mathbf{C}/\mathbf{R}, \psi_{\mathbf{R}}) \varphi'(\alpha) \omega^{-1}(\alpha) (\alpha\bar{\alpha})^{-1/2} \left\langle \Omega(z)u, \tilde{u} \right\rangle$$



if  $\varphi'$  is the Fourier transform of  $\varphi$ .

By linearity we may assume that  $\varphi$  is of the form

$$\varphi(x) = e^{-2\pi x \bar{x}} (x \bar{x})^p \bar{x}^{n+1}$$

where  $p$  is a non-negative integer. We may suppose that the restriction of  $\rho_n$  to the elements of norm one is orthogonal and identify the space  $U$  on which it acts with its dual  $\tilde{U}$ . Then  $\tilde{\Omega} = \alpha_{\mathbf{R}}^{-r-n} \otimes \rho_n$ . Thus if

$$z = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

the value of  $\Phi$  at  $z$  is

$$e^{-2\pi(a\bar{a}+b\bar{b})} (a\bar{a} + b\bar{b})^{r+n+p} \langle u, \tilde{\Omega}(z)\tilde{u} \rangle = e^{-2\pi(a\bar{a}+b\bar{b})} (a\bar{a} + b\bar{b})^p \langle u, \rho_n(z)\tilde{u} \rangle$$

The expression on the right certainly defines a function in  $\mathcal{S}(K)$ .

We are trying to show that if

$$F(z) = \varphi(\alpha)\omega(\alpha)(\alpha\bar{\alpha})^{-1/2}\Omega^{-1}(z)$$

when  $z = \alpha\bar{\alpha}$  then the Fourier transform of  $F$  is given by

$$(5.20.2) \quad F'(z) = -\lambda(\mathbf{C}/\mathbf{R}, \psi_{\mathbf{R}})\varphi'(\alpha)\omega^{-1}(\alpha)(\alpha\bar{\alpha})^{-1/2}\Omega(z).$$

If  $h_1$  and  $h_2$  have norm one

$$F(h_1 z h_2) = \Omega(h_2^{-1})F(z)\Omega(h_1^{-1})$$

and therefore

$$F'(h_1 z h_2) = \Omega(h_1)F'(z)\Omega(h_2)$$

In particular if  $z$  is a scalar in  $K$  the operator  $F'(z)$  commutes with the elements of norm one and is therefore a scalar operator. The expression  $F_1(z)$  on the right of (5.20.2) has the same properties so that all we need do is show that for some pair of vectors  $u$  and  $\tilde{u}$  which are not orthogonal

$$\langle F'(z)u, \tilde{u} \rangle = \langle F_1(z)u, \tilde{u} \rangle$$

for all positive scalars  $z$ .

If we only wanted to show that  $F'(z) = cF_1(z)$  where  $c$  is a positive constant it would be enough to show that

$$(5.20.3) \quad \langle F'(z)u, \tilde{u} \rangle = c \langle F_1(z)u, \tilde{u} \rangle.$$

Once this was done we could interchange the roles of  $\varphi$  and  $\varphi'$  and  $\Phi$  and  $\Phi'$  to show that  $c^2 = 1$ . To obviate any fuss with Haar measures we prove (5.20.3).

Recall that if

$$a(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ \theta & e^{-i\theta} \end{pmatrix}$$

then, apart from a positive constant,

$$\int_{K_1} \langle u, \tilde{\Omega}(k)\tilde{u} \rangle f(k) dk$$

is equal to

$$\langle u, \tilde{u} \rangle \int_0^\pi \sin(n+1)\theta \sin \theta f(a(\theta)) d\theta$$

if  $f$  is a class function on  $K_1$ , the group of elements of norm one. The equality is of course a consequence of the Weyl character formula and the Schur orthogonality relations.

If  $x$  is a positive scalar in  $K$  then, apart from a positive constant,  $\Phi'(x)$  is given by

$$\int_{K^\times} \Phi(z) \psi_{\mathbf{R}}(\tau(xz)) |\nu(z)|^2 d^\times z$$

which is a positive multiple of

$$\int_0^\infty t^3 \varphi(t) \left\{ \int_{K_1} \langle u, \tilde{\Omega}(k) \tilde{u} \rangle \psi_{\mathbf{R}}(xt\tau(k)) dk \right\} d^\times t.$$

Since  $\tau(k)$  is a class function this expression is a positive multiple of

$$\langle u, \tilde{u} \rangle \int_0^\infty t^3 \varphi(t) \left\{ \int_0^\pi \sin(n+1)\theta \sin \theta \psi_{\mathbf{R}}(2xt \cos \theta) d\theta \right\} d^\times t$$

Integrating the inner integral by parts we obtain

$$\langle u, \tilde{u} \rangle \frac{n+1}{4\pi i u x} \int_0^\infty t^2 \varphi(t) \left\{ \int_0^\pi \cos(n+1)\theta \psi_{\mathbf{R}}(2xt \cos \theta) d\theta \right\} d^\times t.$$

On the other hand if  $x$ , which is a positive real number, is regarded as an element of  $\mathbf{C}$  then  $\varphi'(x)$  is a positive multiple of

$$\int_{\mathbf{C}^\times} \varphi(z) \psi_{\mathbf{R}}(\tau(xz)) z \bar{z} d^\times z$$

or of

$$\int_0^\infty t^2 \varphi(t) \left\{ \int_0^{2\pi} e^{-i(n+1)\theta} \psi_{\mathbf{R}}(xt \cos \theta) d\theta \right\} d^\times t.$$

Since

$$\int_0^{2\pi} e^{-i(n+1)\theta} \psi_{\mathbf{R}}(xt \cos \theta) d\theta = 2 \int_0^\pi \cos(n+1)\theta \psi_{\mathbf{R}}(xt \cos \theta) d\theta$$

and  $\lambda(\mathbf{C}/\mathbf{R}) = i \operatorname{sgn} u$  the identity (5.20.3) follows for any choice of  $u$  and  $\tilde{u}$ .

To evaluate  $\lambda(s)$  we choose  $\Phi$  as in the lemma and compute

$$\left\langle Z \left( \alpha_{\mathbf{R}}^{s+\frac{1}{2}} \otimes \Omega, \Phi v \right), \tilde{v} \right\rangle = \int \Phi(z) |\nu(z)|^{s+\frac{1}{2}} \langle \Omega(z)v, \tilde{v} \rangle d^\times z$$

and

$$\left\langle Z \left( \alpha_{\mathbf{R}}^{\frac{3}{2}-s} \otimes \Omega^{-1}, \Phi v \right), \tilde{v} \right\rangle = \int \Phi(z) |\nu(z)|^{\frac{3}{2}-s} \langle v, \tilde{\Omega}(z) \tilde{v} \rangle d^\times z.$$

The first is equal to

$$\int_{K^\times/K_1} |\nu(z)|^{s+\frac{1}{2}} \left\{ \int_{K_1} \Phi(zk) \langle \Omega(zk)v, \tilde{v} \rangle dk \right\} d^\times z.$$

Since

$$\int_{K_1} \langle \Omega(zk)v, \tilde{v} \rangle \langle u, \tilde{\Omega}(zk) \tilde{u} \rangle dk$$

is, by the Schur orthogonality relations, equal to

$$\frac{1}{\deg \Omega} \langle v, \tilde{u} \rangle \langle u, \tilde{v} \rangle$$

the double integral is equal to

$$\frac{1}{\deg \Omega} \langle v, \tilde{u} \rangle \langle u, \tilde{v} \rangle \int_{K^\times} \varphi(\alpha) \omega(\alpha) (\alpha \bar{\alpha})^s d^\times z$$

where  $\alpha \bar{\alpha} = \nu(z)$ . If the Haar measure on  $\mathbf{C}^\times$  is suitably chosen the integral here is equal to  $Z(\omega \alpha_{\mathbf{C}}^s, \varphi)$ . The same choice of Haar measures lead to the relation

$$\left\langle Z\left(\alpha_{\mathbf{R}}^{\frac{3}{2}-s} \otimes \Omega^{-1}, \Phi v\right), \tilde{v}\right\rangle = \frac{-\lambda(\mathbf{C}/\mathbf{R}, \psi_{\mathbf{R}})}{\deg \Omega} \langle v, \tilde{u} \rangle \langle u, \tilde{v} \rangle Z(\omega^{-1} \alpha_{\mathbf{C}}^{1-s}, \varphi').$$

Since  $L(s, \Omega) = L(s, \omega)$  and  $L(s, \tilde{\Omega}) = L(s, \omega^{-1})$  we can compare the functional equation for  $Z(\omega \alpha_{\mathbf{C}}^s, \varphi)$  with that for  $Z(\alpha_{\mathbf{R}}^{s+1/2} \otimes \Omega, \Phi v)$  to see that

$$\lambda(s) = -\lambda(\mathbf{C}/\mathbf{R}, \psi_{\mathbf{R}}) \epsilon(s, \omega, \psi_{\mathbf{C}/\mathbf{R}})$$

as asserted.

If

$$\varphi(x) = e^{-2\pi x \bar{x}}$$

then  $Z(\alpha_{\mathbf{C}}^s \omega, \varphi)$  is an exponential times  $L(s, \omega)$  so that  $Z(\alpha_{\mathbf{R}}^{s+1/2} \otimes \Omega, \Phi v)$  is, with a suitable choice of  $v$  and  $\tilde{u}$ , a non-zero scalar times an exponential times  $L(s, \omega)u$ . The last assertion of the proposition is proved in the same way as Lemma 5.15.2.

We end this paragraph with the observation that the space  $W(\pi, \psi)$  of Theorem 5.13 cannot exist when  $\pi$  is finite-dimensional. If  $W = W(\pi, \psi)$  did exist the contragredient representation  $\tilde{\pi}$  on the dual space  $\tilde{W}$  would also be finite-dimensional and  $\tilde{\pi}(X_+)$  would be nilpotent. However if  $\lambda$  is the linear functional  $\varphi \rightarrow \varphi(e)$  then  $\tilde{\pi}(X_+) \lambda = -2i\pi a$  if  $\psi(x) = e^{2i\pi ax}$ .

### §6. Representations of $GL(2, \mathbf{C})$

In this paragraph we have to review the representation theory of  $G_{\mathbf{C}} = GL(2, \mathbf{C})$  and prove the local functional equation for the complex field. Many of the definitions and results of the previous paragraph are applicable, after simple modifications which we do not always make explicit, to the present situation.

The standard maximal compact subgroup of  $GL(2, \mathbf{C})$  is the group  $U(2, \mathbf{C})$  of unitary matrices.  $\mathcal{H}_1$  will be the space of infinitely differentiable compactly supported functions on  $G_{\mathbf{C}}$ .  $\mathcal{H}_2$  will be the space of functions on  $U(2, \mathbf{C})$  which are finite linear combinations of the matrix elements of finite-dimensional representations.  $\mathcal{H}_{\mathbf{C}} = \mathcal{H}_1 \oplus \mathcal{H}_2$  can be regarded as a space of measures. Under convolution it forms an algebra called the Hecke algebra. The notion of an elementary idempotent and the notion of an admissible representation of  $\mathcal{H}_{\mathbf{C}}$  are defined more or less as before.

Let  $\mathfrak{g}$  be the Lie algebra of the real Lie group of  $GL(2, \mathbf{C})$  and let  $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$ .  $\mathfrak{A}$  will be the universal enveloping algebra of  $\mathfrak{g}_{\mathbf{C}}$ . A representation of  $\mathfrak{A}$  will be said to be admissible if its restriction to the Lie algebra of  $U(2, \mathbf{C})$  decomposes into a direct sum of irreducible finite-dimensional representations each occurring with finite multiplicity. There is a one-to-one correspondence between classes of irreducible admissible representations of  $\mathcal{H}_{\mathbf{C}}$  and those of  $\mathfrak{A}$ . We do not usually distinguish between the two. The representation  $\tilde{\pi}$  contragredient to  $\pi$  and the tensor product of  $\pi$  with a quasi-character of  $\mathbf{C}^{\times}$  are defined as before.

If  $\mu_1$  and  $\mu_2$  are two quasi-characters of  $\mathbf{C}^{\times}$  we can introduce the space  $\mathcal{B}(\mu_1, \mu_2)$  and the representation  $\rho(\mu_1, \mu_2)$  of  $\mathcal{H}_{\mathbf{C}}$  or of  $\mathfrak{A}$  on  $\mathcal{B}(\mu_1, \mu_2)$ . In order to study this representation we identify  $\mathfrak{g}_{\mathbf{C}}$  with  $\mathfrak{g}_{\ell}(2, \mathbf{C}) \oplus \mathfrak{g}_{\ell}(2, \mathbf{C})$  in such a way that  $\mathfrak{g}$  corresponds to the elements of  $X \oplus \overline{X}$ . If  $\mathfrak{A}_1$  is the universal enveloping algebra of  $\mathfrak{g}_{\ell}(2, \mathbf{C})$  we may then identify  $\mathfrak{A}$  with  $\mathfrak{A}_1 \otimes \mathfrak{A}_1$ .

In the previous paragraph we introduced the elements  $D$  and  $J$  of  $\mathfrak{A}_1$ . Set  $D_1 = D \otimes 1$ ,  $D_2 = 1 \otimes D$ ,  $J_1 = J \otimes 1$ , and  $J_2 = 1 \otimes J$ . These four elements lie in the centre of  $\mathfrak{A}$ . A representation of  $\mathfrak{A}$  is admissible if its restriction to the Lie algebra of the group  $SU(2, \mathbf{C})$  of unitary matrices of determinant one decomposes into the direct sum of irreducible finite-dimensional representations each occurring with finite multiplicity.

The first part of the next lemma is verified by calculations like those used in the proof of Lemma 5.6. The second is a consequence of the Frobenius reciprocity law applied to the pair  $SU(2, \mathbf{C})$  and its subgroup of diagonal matrices.

**Lemma 6.1.** *Let*

$$\mu_i(z) = (z\bar{z})^{s_i - \frac{1}{2}(a_i + b_i)} z^{a_i} \bar{z}^{b_i}$$

and

$$\mu_1 \mu_2^{-1}(z) = \mu(z) = (z\bar{z})^{s - \frac{1}{2}(a+b)} z^a \bar{z}^b$$

where  $a_i$ ,  $b_i$ ,  $a$ , and  $b$  are non-negative integers and  $a_i b_i = ab = 0$ .

(i) On  $\mathcal{B}(\mu_1, \mu_2)$  we have the following four relations

$$\begin{aligned}\rho(D_1) &= \frac{1}{2} \left\{ \left( s + \frac{a-b}{2} \right)^2 - 1 \right\} I \\ \rho(D_2) &= \frac{1}{2} \left\{ \left( s + \frac{b-a}{2} \right)^2 - 1 \right\} I \\ \rho(J_1) &= \left\{ (s_1 + s_2) + \frac{a_1 - b_1 + a_2 - b_2}{2} \right\} I \\ \rho(J_2) &= \left\{ (s_1 + s_2) + \frac{b_1 - a_1 + b_2 - a_2}{2} \right\} I\end{aligned}$$

(ii)  $\rho(\mu_1, \mu_2)$  is admissible and contains the representation  $\rho_n$  of the Lie algebra of  $SU(2, \mathbf{C})$  if and only if  $n \geq a + b$  and  $n \equiv a + b \pmod{2}$  and then it contains it just once.

$\rho_n$  is the unique irreducible representation of  $SU(2, \mathbf{C})$  of degree  $n + 1$ . Let  $\mathcal{B}(\mu_1, \mu_2, \rho_n)$  be the space of functions in  $\mathcal{B}(\mu_1, \mu_2)$  transforming according to  $\rho_n$ .

**Theorem 6.2.**

- (i) If  $\mu$  is not of the form  $z \rightarrow z^p \bar{z}^q$  or  $z \rightarrow z^{-p} \bar{z}^{-q}$  with  $p \geq 1$  and  $q \geq 1$  then  $\rho(\mu_1, \mu_2)$  is irreducible. A representation equivalent to  $\rho(\mu_1, \mu_2)$  will be denoted by  $\pi(\mu_1, \mu_2)$ ,  
(ii) If  $\mu(z) = z^p \bar{z}^q$  with  $p \geq 1, q \geq 1$  then

$$\mathcal{B}_s(\mu_1, \mu_2) = \sum_{\substack{n \geq p+q \\ n \equiv p+q \pmod{2}}} \mathcal{B}(\mu_1, \mu_2, \rho_n)$$

is the only proper invariant subspace of  $\mathcal{B}(\mu_1, \mu_2)$ .  $\sigma(\mu_1, \mu_2)$  will be any representation equivalent to the representation on  $\mathcal{B}_s(\mu_1, \mu_2)$  and  $\pi(\mu_1, \mu_2)$  will be any representation equivalent to the representation on the quotient space

$$\mathcal{B}_f(\mu_1, \mu_2) = \mathcal{B}(\mu_1, \mu_2) / \mathcal{B}_s(\mu_1, \mu_2)$$

- (iii) If  $\mu(z) = z^{-p} \bar{z}^{-q}$  with  $p \geq 1, q \geq 1$  then

$$\mathcal{B}_f(\mu_1, \mu_2) = \sum_{\substack{|p-q| \leq n < p+q \\ n \equiv p+q \pmod{2}}} \mathcal{B}(\mu_1, \mu_2, \rho_n)$$

is the only proper invariant subspace of  $\mathcal{B}(\mu_1, \mu_2)$ .  $\pi(\mu_1, \mu_2)$  will be any representation equivalent to the representation on  $\mathcal{B}_f(\mu_1, \mu_2)$  and  $\sigma(\mu_1, \mu_2)$  will be any representation equivalent to the representation on the quotient space

$$\mathcal{B}_s(\mu_1, \mu_2) = \mathcal{B}(\mu_1, \mu_2) / \mathcal{B}_f(\mu_1, \mu_2).$$

- (iv)  $\pi(\mu_1, \mu_2)$  is equivalent to  $\pi(\mu'_1, \mu'_2)$  if and only if  $(\mu_1, \mu_2) = (\mu'_1, \mu'_2)$  or  $(\mu_1, \mu_2) = (\mu'_2, \mu'_1)$ .  
(v) If  $\sigma(\mu_1, \mu_2)$  and  $\sigma(\mu'_1, \mu'_2)$  are defined they are equivalent if and only if  $(\mu_1, \mu_2) = (\mu'_1, \mu'_2)$  or  $(\mu_1, \mu_2) = (\mu'_2, \mu'_1)$ .  
(vi) If  $\mu(z) = z^p \bar{z}^q$  with  $p \geq 1, q \geq 1$  there is a pair of characters  $\nu_1, \nu_2$  such that  $\mu_1 \mu_2 = \nu_1 \nu_2$  and  $\nu_1 \nu_2^{-1} = z^p \bar{z}^{-q}$  and  $\sigma(\mu_1, \mu_2)$  is equivalent to  $\pi(\nu_1, \nu_2)$ .

(vii) Every irreducible admissible representation of  $\mathcal{H}_{\mathbf{C}}$  or  $\mathfrak{A}$  is a  $\pi(\mu_1, \mu_2)$  for some choice of  $\mu_1$  and  $\mu_2$ .

The proofs of the first three assertions will be based on two lemmas.

**Lemma 6.2.1.** *If there exists a proper invariant subspace  $V$  of  $\mathcal{B}(\mu_1, \mu_2)$  which is finite-dimensional then  $\mu_1\mu_2^{-1}(z) = z^{-p}\bar{z}^{-q}$  with  $p \geq 1$ ,  $q \geq 1$  and  $V = \mathcal{B}_f(\mu_1, \mu_2)$ .*

**Lemma 6.2.2.** *Let  $V$  be a proper invariant subspace of  $\mathcal{B}(\mu_1, \mu_2)$  and let  $n_0$  be the smallest integer such that some subspace of  $V$  transforms according to the representation  $\rho_{n_0}$  of the Lie algebra of  $\mathrm{SU}(2, \mathbf{C})$ . Either*

$$V = \sum_{n \geq n_0} \mathcal{B}(\mu_1, \mu_2, \rho_n)$$

*or  $V$  contains a finite-dimensional invariant subspace.*

Grant these lemmas for a moment and let  $V$  be a proper invariant subspace of  $\mathcal{B}(\mu_1, \mu_2)$ . As in the case of the non-archimedean and real fields there is an invariant non-degenerate bilinear form on  $\mathcal{B}(\mu_1, \mu_2) \times \mathcal{B}(\mu_1^{-1}, \mu_2^{-1})$ . The orthogonal complement  $V^\perp$  of  $V$  in  $\mathcal{B}(\mu_1^{-1}, \mu_2^{-1})$  is a proper invariant subspace. By Lemma 6.2.1 they cannot both contain an invariant finite-dimensional subspace. Therefore by Lemma 6.2.2 one of them is of finite codimension. The other must be of finite dimension. If  $V$  is finite-dimensional then  $\mu_1\mu_2^{-1}(z) = z^{-p}\bar{z}^{-q}$  and  $V = \mathcal{B}_f(\mu_1, \mu_2)$ . If  $V^\perp$  is finite-dimensional then  $\mu_1\mu_2^{-1}(z) = z^p\bar{z}^q$ . Since the orthogonal complement of  $\mathcal{B}_f(\mu_1, \mu_2)$  is  $\mathcal{B}_s(\mu_1, \mu_2)$  we must have  $V = \mathcal{B}_s(\mu_1, \mu_2)$ .

We shall now show that  $\mathcal{B}_f(\mu_1, \mu_2)$  is invariant when  $\mu_1\mu_2^{-1}(z) = z^{-p}\bar{z}^{-1}$ . It will follow from duality that  $\mathcal{B}_s(\mu_1, \mu_2)$  is invariant when  $\mu_1\mu_2^{-1}(z) = z^p\bar{z}^q$ . Every irreducible finite-dimensional representation  $\pi$  of  $\mathfrak{A}$  determines a representation  $\pi$  of  $G_{\mathbf{C}}$ . If  $\pi$  acts on  $X$  there is a nonzero vector  $v_0$  in  $X$  such that

$$\pi \left( \begin{pmatrix} z & x \\ 0 & z^{-1} \end{pmatrix} \right) v_0 = z^m \bar{z}^n v_0$$

for all  $z$  in  $\mathbf{C}^\times$  and all  $x$  in  $\mathbf{C}$ .  $v_0$  is determined up to a scalar factor and  $m$  and  $n$  are non-negative integers. Moreover there is a quasi-character  $\omega_0$  of  $\mathbf{C}^\times$  such that

$$\pi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \omega_0(a) I$$

Thus

$$\pi \left( \begin{pmatrix} z_1 & x \\ 0 & z_2 \end{pmatrix} \right) v_0 = \omega_1(z_1) \omega_2(z_2) v_0$$

where  $\omega_1\omega_2^{-1}(z) = z^m\bar{z}^n$ .  $\pi$  is determined up to equivalence by  $\omega_1$  and  $\omega_2$  so we write  $\pi = \kappa(\omega_1, \omega_2)$ . As long as  $\omega_1\omega_2^{-1}(z) = z^m\bar{z}^n$  with non-negative integers  $m$  and  $n$  the representation  $\kappa(\omega_1, \omega_2)$  exists. By the Clebsch-Gordan formula the restriction of  $\kappa(\omega_1, \omega_2)$  or its contragredient to  $\mathrm{SU}(2, \mathbf{C})$  breaks up into the direct sum of the representations  $\rho_i$  with  $|m - n| \leq i \leq m + n$  and  $1 \equiv m + n \pmod{2}$ . Let  $\pi$  be  $\kappa(\omega_1, \omega_2)$  and let  $\tilde{\pi}$ , the contragredient representation, act on  $\tilde{X}$ . To each vector  $\tilde{v}$  in  $\tilde{X}$  we associate the function

$$\varphi(g) = \langle v_0, \tilde{\pi}(g)\tilde{v} \rangle$$

on  $G_{\mathbf{C}}$ . The map  $\tilde{v} \rightarrow \varphi$  is linear and injective. Moreover  $\tilde{\pi}(g)\tilde{v} \rightarrow \rho(g)\varphi$  while

$$\varphi\left(\begin{pmatrix} z_1 & x \\ 0 & z_2 \end{pmatrix}g\right) = \omega_1^{-1}(z_1)\omega_2^{-1}(z_2)\varphi(g)$$

so that if  $\mu_1 = \omega_1^{-1}\alpha_{\mathbf{C}}^{-1/2}$  and  $\mu_2 = \omega_2^{-1}\alpha_{\mathbf{C}}^{1/2}$  the function  $\varphi$  belongs to  $\mathcal{B}(\mu_1, \mu_2)$ . As we vary  $\omega_1$  and  $\omega_2$  the quasi-characters  $\mu_1$  and  $\mu_2$  vary over all pairs such that  $\mu_1\mu_2^{-1}(z) = z^{-p}\bar{z}^{-q}$  with  $p \geq 1$  and  $q \geq 1$ .

We have still to prove the two lemmas. Suppose  $V$  is a proper finite-dimensional subspace of  $\mathcal{B}(\mu_1, \mu_2)$ . The representation of  $\mathfrak{A}$  on  $V$  is certainly a direct sum of irreducible representations each occurring with multiplicity one. Let  $V'$  be an irreducible subspace of  $V$  and let  $\tilde{V}'$  be the dual space of  $V'$ . Let  $\lambda$  be the linear functional  $\lambda: \varphi \rightarrow \varphi(e)$  on  $V'$ . If  $\pi$  is the representation of  $\mathfrak{A}$  or of  $G_{\mathbf{C}}$  on  $V'$  then

$$\tilde{\pi}\left(\begin{pmatrix} z_1 & x \\ 0 & z_2 \end{pmatrix}\right)\lambda = \mu_1^{-1}(z_1)\mu_2^{-1}(z_2)(z_1\bar{z}_1z_2^{-1}(\bar{z}_2)^{-1})^{-1/2}\lambda$$

Thus if  $\omega_1 = \mu_1^{-1}\alpha_{\mathbf{C}}^{-1/2}$  and  $\omega_2 = \mu_2^{-1}\alpha_{\mathbf{C}}^{1/2}$  the representation  $\tilde{\pi}$  is  $\kappa(\omega_1, \omega_2)$ . It follows immediately that  $\mu_1\mu_2^{-1}$  is of the form  $\mu_1\mu_2^{-1}(z) = z^{-p}\bar{z}^{-q}$  with  $p \geq 1$  and  $q \geq 1$  and that  $V'$  and therefore  $V$  is  $\mathcal{B}_f(\mu_1, \mu_2)$ .

To prove the second lemma we regard  $\mathfrak{g}$  as the real Lie algebra of  $2 \times 2$  complex matrices. Then

$$\mathfrak{a} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbf{C} \right\}$$

is the centre of  $\mathfrak{g}$  and

$$\mathfrak{u} = \left\{ \begin{pmatrix} ia & b \\ -\bar{b} & -ia \end{pmatrix} \mid a \in \mathbf{R}, b \in \mathbf{C} \right\}$$

is the Lie algebra of  $SU(2, \mathbf{C})$ . If

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & b \\ \bar{b} & -a \end{pmatrix} \mid a \in \mathbf{R}, b \in \mathbf{C} \right\}$$

then  $\mathfrak{u} \oplus \mathfrak{g}$  is the Cartan decomposition of the Lie algebra of the special linear group. The space  $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$  is invariant under the adjoint action of  $\mathfrak{u}$  on  $\mathfrak{g}_{\mathbf{C}}$ . Moreover  $\mathfrak{u}$  acts on  $\mathfrak{g}_{\mathbf{C}}$  according to the representation  $\rho_2$ . One knows that  $\rho_2 \otimes \rho_n$  is equivalent to  $\rho_{n+2} \oplus \rho_n \oplus \rho_{n-2}$  if  $n \geq 2$ , that  $\rho_2 \otimes \rho_1$  is equivalent to  $\rho_3 \oplus \rho_1$  and, of course, that  $\rho_2 \otimes \rho_0$  is equivalent to  $\rho_2$ . The map of  $\mathfrak{g}_{\mathbf{C}} \otimes \mathcal{B}(\mu_1, \mu_2, \rho_n)$  into  $\mathcal{B}(\mu_1, \mu_2)$  which sends  $X \otimes f$  to  $\rho(X)f$  commutes with the action of  $\mathfrak{u}$ . Thus  $\rho(X)f$  is contained in

$$\mathcal{B}(\mu_1, \mu_2, \rho_{n+2}) \oplus \mathcal{B}(\mu_1, \mu_2, \rho_n) \oplus \mathcal{B}(\mu_1, \mu_2, \rho_{n-2}).$$

It is understood that  $\mathcal{B}(\mu_1, \mu_2, \rho_\ell) = 0$  if  $\ell < 0$ .

Now let  $V$  be a proper invariant subspace of  $\mathcal{B}(\mu_1, \mu_2)$ . Let  $n_0$  be the smallest non-negative integer  $n$  for which  $V$  contains  $\mathcal{B}(\mu_1, \mu_2, \rho_n)$ . If  $n \geq n_0$  set

$$V(n) = \sum_{\substack{n \geq k \geq n_0 \\ k \equiv n_0 \pmod{2}}} \mathcal{B}(\mu_1, \mu_2, \rho_k)$$

If  $V$  contains every  $V(n)$  there is nothing to prove so assume that there is a largest integer  $n_1$  for which  $V$  contains  $V(n_1)$ . All we need do is show that  $V(n_1)$  is invariant under  $\mathfrak{g}$ . It is invariant under  $\mathfrak{a}$  and  $\mathfrak{u}$  by construction so we need only verify that if  $X$  lies in  $\mathfrak{g}_{\mathbf{C}}$  then  $\rho(X)$  takes  $V(n_1)$  into itself. It is clear that  $\rho(X)$  takes  $V(n_1 - 2)$  into  $V(n_1)$  so we have only to show that it takes  $\mathcal{B}(\mu_1, \mu_2, \rho_{n_1})$  into  $V(n_1)$ . Take  $f$  in  $\mathcal{B}(\mu_1, \mu_2, \rho_{n_1})$  and let  $\rho(X)f = f_1 + f_2$  with  $f_1$  in  $V(n_1)$  and  $f_2$  in  $\mathcal{B}(\mu_1, \mu_2, \rho_{n_1+2})$ . Certainly  $f_2$  lies in  $V$ . Since

$$V \cap \mathcal{B}(\mu_1, \mu_2, \rho_{n_1+2})$$

is either 0 or  $\mathcal{B}(\mu_1, \mu_2, \rho_{n_1+2})$  and since, by construction, it is not  $\mathcal{B}(\mu_1, \mu_2, \rho_{n_1+2})$  the function  $f_2$  is 0.

The first three assertions of the theorem are now proved and we consider the remaining ones. We make use of the fact that  $D_1, D_2, J_1$  and  $J_2$  generate the centre of  $\mathfrak{A}$  as well as a result of Harish-Chandra to be quoted later. Suppose  $\pi$  and  $\pi'$  are two irreducible representations of  $\mathfrak{A}$  which are constituents of  $\rho(\mu_1, \mu_2)$  and  $\rho(\mu'_1, \mu'_2)$  respectively. Assume  $\pi$  and  $\pi'$  contain the same representations of the Lie algebra of  $\mathrm{SU}(2, \mathbf{C})$  and are associated to the same homomorphism of the centre of  $\mathfrak{A}$  into  $\mathbf{C}$ . Comparing the scalars  $\pi(J_1)$  and  $\pi(J_2)$  with  $\pi'(J_1)$  and  $\pi'(J_2)$  we find that  $\mu_1\mu_2 = \mu'_1\mu'_2$ . Let  $\mu_1\mu_2^{-1}(z) = (z\bar{z})^{s-\frac{a+b}{2}}z^a\bar{z}^b$  and let  $\mu'_1\mu'_2^{-1}(z) = (z\bar{z})^{s'-\frac{a'+b'}{2}}z^{a'}\bar{z}^{b'}$ . Comparing  $\pi(D_1)$  and  $\pi(\nu_2)$  with  $\pi'(D_1)$  and  $\pi'(D_2)$  we see that

$$\left(s + \frac{a-b}{2}\right)^2 = \left(s' + \frac{a'-b'}{2}\right)^2$$

and

$$\left(s + \frac{b-a}{2}\right)^2 = \left(s' + \frac{b'-a'}{2}\right)^2.$$

These relations will hold if  $\mu_1\mu_2^{-1} = \mu'_1\mu'_2^{-1}$  or  $\mu_1^{-1}\mu_2 = \mu'_1\mu'_2^{-1}$  and therefore, when  $\mu_1\mu_2 = \mu'_1\mu'_2$ ,  $(\mu_1, \mu_2) = (\mu'_1, \mu'_2)$  or  $(\mu_1, \mu_2) = (\mu'_2, \mu'_1)$ . If neither of these alternatives hold we must have

$$s = \frac{a'-b'}{2}, \quad s' = \frac{a-b}{2},$$

or

$$s = \frac{b'-a'}{2}, \quad s' = \frac{b-a}{2}.$$

Since  $\mu_1\mu_2 = \mu'_1\mu'_2$  the integers  $a+b$  and  $a'+b'$  must have the same parity. Let  $\mu = \mu_1\mu_2^{-1}$  and  $\mu' = \mu'_1\mu'_2^{-1}$ . In the first case  $\mu\mu'$  is of the form  $\mu\mu'(z) = z^{2p}$  and  $\mu\mu'^{-1}$  is of the form  $\bar{z}^{2q}$  and in the second  $\mu\mu'(z) = \bar{z}^{2p}$  while  $\mu\mu'^{-1}(z) = z^{2q}$ . Since  $\{\mu_1, \mu_2\}$  is not  $\{\mu'_1, \mu'_2\}$  neither  $p$  nor  $q$  is 0. In the first case  $\mu = z^p\bar{z}^q$  and  $\mu' = z^p\bar{z}^{-q}$  and in the second  $\mu = z^q\bar{z}^p$  while  $\mu' = z^{-q}\bar{z}^p$ .

In conclusion we see that  $\pi$  and  $\pi'$  contain the same representations of the Lie algebra of  $\mathrm{SU}(2, \mathbf{C})$  and are associated to the same homomorphism of the centre of  $\mathfrak{A}$  into  $\mathbf{C}$  if and only if one of the following alternatives holds.

- (i) For some pair of quasi-characters  $\nu_1$  and  $\nu_2$  we have  $\{\pi, \pi'\} = \{\pi(\nu_1, \nu_2), \pi(\nu_1, \nu_2)\}$  or  $\{\pi, \pi'\} = \{\pi(\nu_1, \nu_2), \pi(\nu_2, \nu_1)\}$ .
- (ii) For some pair of quasi-characters  $\nu_1$  and  $\nu_2$  we have  $\{\pi, \pi'\} = \{\sigma(\nu_1, \nu_2), \sigma(\nu_1, \nu_2)\}$  or  $\{\pi, \pi'\} = \{\sigma(\nu_1, \nu_2), \sigma(\nu_2, \nu_1)\}$ .



- (iii) For some pair of quasi-characters  $\nu_1$  and  $\nu_2$  with  $\nu_1\nu_2^{-1}(z) = z^p\bar{z}^q$  where  $p \geq 1$ ,  $q \geq 1$  we have  $\{\pi, \pi'\} = \{\sigma(\nu_1, \nu_2), \pi(\nu'_1, \nu'_2)\}$  where  $\nu_1\nu_2 = \nu'_1\nu'_2$  and  $\nu'_1\nu'_2^{-1}(z)$  is either  $z^p\bar{z}^{-1}$  or  $z^{-p}\bar{z}^q$ .
- (iv) For some pair of quasi-characters  $\nu_1$  and  $\nu_2$  with  $\nu_1\nu_2^{-1}(z) = z^{-p}\bar{z}^{-q}$  where  $p \geq 1$ ,  $q \geq 1$  we have  $\{\pi, \pi'\} = \{\sigma(\nu_1, \nu_2), \pi(\nu'_1, \nu'_2)\}$  where  $\nu_1\nu_2 = \nu'_1\nu'_2$  and  $\nu'_1\nu'_2^{-1}(z)$  is either  $z^p\bar{z}^{-q}$  or  $z^{-p}\bar{z}^q$ .

The remaining assertions are now all consequences of a theorem of Harish-Chandra which, in the special case of interest to us, we may state in the following manner.

**Lemma 6.2.3.** *If  $\pi$  is an irreducible admissible representation of  $\mathfrak{A}$  there exists a pair of quasi-characters  $\mu_1$  and  $\mu_2$  such that  $\rho(\mu_1, \mu_2)$  and  $\pi$  contain at least one irreducible representation of the Lie algebra of  $SU(2, \mathbf{C})$  in common and are associated to the same homomorphism of the centre of  $\mathfrak{A}$  into  $\mathbf{C}$ . When this is so  $\pi$  is a constituent of  $\rho(\mu_1, \mu_2)$ .*

As before  $\chi \otimes \pi(\mu_1, \mu_2)$  is  $\pi(\chi\mu_1, \chi\mu_2)$  and  $\chi \otimes \sigma(\mu_1, \mu_2)$  is  $\sigma(\chi\mu_1, \chi\mu_2)$ . If

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega_0(a)I$$

then  $\tilde{\pi} = \omega_0^{-1} \otimes \pi$ .

**Theorem 6.3.** *Let  $\pi$  be an infinite-dimensional irreducible admissible representation of  $\mathcal{H}_{\mathbf{C}}$  and let  $\psi$  be a non-trivial additive character of  $\mathbf{C}$ . There is exactly one space  $W(\pi, \psi)$  of functions on  $G_{\mathbf{C}}$  which satisfies the following three conditions.*

(i) *Every function  $W$  in  $W(\pi, \psi)$  satisfies*

$$W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}g\right) = \psi(x)W(g).$$

(ii) *The functions in  $W(\pi, \psi)$  are continuous and  $W(\pi, \psi)$  is invariant under the operators  $\rho(f)$  for  $f$  in  $\mathcal{H}_{\mathbf{C}}$ . Moreover the representation of  $\mathcal{H}_{\mathbf{C}}$  on  $W(\pi, \psi)$  is equivalent to  $\pi$ .*

(iii) *If  $W$  is in  $W(\pi, \psi)$  there is a positive number  $N$  such that*

$$W\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right) = O(|t|^N)$$

*as  $|t| \rightarrow \infty$ .*

Since every  $\pi$  is of the form  $\pi = \pi(\mu_1, \mu_2)$  the existence is rather easy to prove. If  $\Phi$  is in  $\mathcal{S}(\mathbf{C}^2)$  let

$$\theta(\mu_1, \mu_2, \Phi) = \int_{\mathbf{C}^\times} \Phi(t, t^{-1})\mu_1(t)\mu_2^{-1}(t) d^\times t$$

We let  $W(\mu_1, \mu_2, \psi)$  be the space of functions on  $G_{\mathbf{C}}$  of the form

$$W(g) = W_\Phi(g) = \mu_1(\det g)|\det g|_{\mathbf{C}}^{1/2}\theta(\mu_1, \mu_2, r(g)\Phi)$$

where  $\Phi$  in  $\mathcal{S}(\mathbf{C}^2)$  is  $SU(2, \mathbf{C})$ -finite under the action defined by  $r$ . It is clear that

$$W(\mu_1, \mu_2, \psi) = W(\mu_2, \mu_1, \psi)$$

and that  $W(\mu_1, \mu_2, \psi)$  is invariant under right translations by elements of  $\mathcal{H}_{\mathbf{C}}$  and of  $\mathfrak{A}$ .

The existence of  $W(\pi, \psi)$  will, as before, be a consequence of the following analogue of Lemma 5.13.1.

**Lemma 6.3.1.** *Suppose  $\mu_1\mu_2^{-1}(t) = (t\bar{t})^{s-\frac{a+b}{2}}t^a\bar{t}^b$  with  $\operatorname{Re} s > -1$ . Then there is a bijection  $A$  of  $W(\mu_1, \mu_2, \psi)$  with  $\mathcal{B}(\mu_1, \mu_2)$  which commutes with the action of  $\mathcal{H}_{\mathbf{C}}$ .*

As before  $A$  associates to  $W_{\Phi}$  the function

$$f_{\Phi}(g) = \mu_1(\det g)|\det g|_{\mathbf{C}}^{1/2}z\left(\mu_1\mu_2^{-1}\alpha_{\mathbf{C}}, \rho(g)\tilde{\Phi}\right)$$

The proof of course proceeds as before. However we should check that  $A$  is surjective. Theorem 6.2 shows that, under the present circumstances, there is no proper invariant subspace of  $\mathcal{B}(\mu_1, \mu_2)$  containing  $\mathcal{B}(\mu_1, \mu_2, \rho_{a+b})$  so that we need only show that at least one nonzero function in  $\mathcal{B}(\mu_1, \mu_2, \rho_{a+b})$  is of the form  $f_{\Phi}$  where  $\Phi$  is in  $\mathcal{S}(\mathbf{C}^2)$  and  $\mathrm{SU}(2, \mathbf{C})$ -finite under right translations.

If

$$\Phi(x, y) = e^{-2\pi(x\bar{x}+y\bar{y})}\bar{y}^ay^b$$

then, since  $a + b = 0$ ,  $\Phi$  transforms under right translations by  $\mathrm{SU}(2, \mathbf{C})$  according to  $\rho_{a+b}$  so we need only check that  $f_{\Phi}$  is not 0. Proceeding according to the definition we see that

$$\begin{aligned} f_{\Phi}(e) &= \int_{\mathbf{C}^{\times}} \Phi(0, t)(t\bar{t})^{s-\frac{a+b}{2}}t^a\bar{t}^b d^{\times}t \\ &= \int_{\mathbf{C}^{\times}} e^{-2\pi t\bar{t}}(t\bar{t})^{1+s+\frac{a+b}{2}} d^{\times}t. \end{aligned}$$

Apart from a constant which depends on the choice of Haar measure this is

$$(2\pi)^{-s-\frac{a+b}{2}}\Gamma\left(1+s+\frac{a+b}{2}\right)$$

and is thus not 0.

Just as in the previous paragraph  $W(\mu_1, \mu_2, \psi)$  is spanned by functions  $W_{\Phi}$  where  $\Phi$  is of the form

$$\Phi(x, y) = e^{-2\pi(x\bar{x}+u\bar{u}y\bar{y})}x^p\bar{x}^qy^m\bar{y}^n$$

where  $p, q, m$ , and  $n$  are integers. The complex number  $u$  is determined by the relation  $\psi(z) = e^{4\pi i \operatorname{Re} uz}$ . We can show that

$$W_{\Phi}\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right)$$

decreases exponentially as  $|t| \rightarrow \infty$ .

To prove the uniqueness we will use a differential equation as in the previous chapter. This time the equations are a little more complicated. Suppose  $W_1(\pi, \psi)$  is a space of functions satisfying the first two conditions of the theorem. We regard  $\rho_n$  as acting on the space  $V_n$  of binary forms of degree  $n$  according to the rule

$$\rho_n\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\varphi(x, y) = \varphi(ax + cy, bx + dy)$$

If

$$\varphi(x, y) = \sum_{\substack{|k| \leq n \\ \frac{n}{2} - k \in \mathbf{Z}}} \varphi^k x^{\frac{n}{2}+k} y^{\frac{n}{2}-k}$$

then  $\varphi^k$  is called the  $k$ th coordinate of  $\varphi$ . On the dual space  $\tilde{V}_n$  we introduce the dual coordinates.

If  $\rho_n$  is contained in  $\pi$  there is an injection  $A$  of  $V_n$  into  $W_1(\pi, \psi)$  which commutes with the action of  $SU(2, \mathbf{C})$ . Let  $\Phi(g)$  be the function on  $G_{\mathbf{C}}$  with values in  $\tilde{V}_n$  defined by

$$\langle \varphi, \Phi(g) \rangle = A\varphi(g).$$

It is clear that  $W_1(\pi, \psi)$  is determined by  $\Phi$  which is in turn determined by  $W_1(\pi, \psi)$  up to a scalar factor. The function  $\Phi(g)$  is determined by the function

$$\varphi(t) = \Phi \left( \begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2} \end{pmatrix} \right)$$

on the positive real numbers. If  $\varphi^k(t)$  is the  $k$ th coordinate of  $\varphi(t)$  and if  $\pi$  is a constituent of  $\rho(\mu_1, \mu_2)$  the differential equations

$$\begin{aligned} \rho(D_1)\Phi &= \frac{1}{2} \left\{ \left( s + \frac{a-b}{2} \right)^2 - 1 \right\} \Phi \\ \rho(D_2)\Phi &= \frac{1}{2} \left\{ \left( s + \frac{b-a}{2} \right)^2 - 1 \right\} \Phi \end{aligned}$$

may, if our calculations are correct, be written as

$$\begin{aligned} \frac{1}{2} \left[ t \frac{d}{dt} + k - 1 \right]^2 \varphi^k - t^2 \frac{|u|^2}{2} \varphi^k + \left( \frac{n}{2} + k \right) t i u \varphi^{k-1} &= \frac{1}{2} \left( s + \frac{a-b}{2} \right)^2 \varphi^k \\ \frac{1}{2} \left[ t \frac{d}{dt} - k - 1 \right]^2 \varphi^k - t^2 \frac{|u|^2}{2} \varphi^k - \left( \frac{n}{2} - k \right) t i \bar{u} \varphi^{k+1} &= \frac{1}{2} \left( s + \frac{b-a}{2} \right)^2 \varphi^k. \end{aligned}$$

We have set  $\varphi^k = 0$  if  $|k| \geq n/2$ . Recall that  $\psi(z) = e^{4\pi i \operatorname{Re} uz}$ . These equations allow one to solve for all  $\varphi^k$  in terms of  $\varphi^{n/2}$  or  $\varphi^{-n/2}$ .

For  $k = \frac{n}{2}$  the second equation may be written as

$$(*) \quad \frac{1}{2} \frac{d^2 \varphi^{n/2}}{dt^2} + \left( -\frac{1}{2} - \frac{n}{2} \right) \frac{1}{t} \frac{d \varphi^{n/2}}{dt} + \left\{ -\frac{|u|^2}{2} + \frac{\left( \frac{n}{2} + 1 \right)^2}{2t^2} \right\} \varphi^{n/2} = \frac{1}{2t^2} \left( s + \frac{b-a}{2} \right)^2 \varphi^{n/2}.$$

If we have two independent solutions of this equation their Wronskian  $W(t)$  is a non-trivial solution of the equation

$$\frac{dW}{dt} = \frac{(n+1)}{t} W$$

and therefore a non-zero multiple of  $t^{n+1}$ . Since we already have shown the existence of a solution of  $(*)$  which decreases exponentially we see that there cannot be another solution which is bounded by a power of  $t$  as  $t \rightarrow \infty$ . The uniqueness of the space  $W(\pi, \psi)$  follows.

Every irreducible admissible representation of  $\mathcal{H}_{\mathbf{C}}$  is of the form  $\pi = \pi(\mu_1, \mu_2)$ . Moreover  $\pi(\mu_1, \mu_2) = \pi(\mu'_1, \mu'_2)$  if and only if  $\{\mu_1, \mu_2\} = \{\mu'_1, \mu'_2\}$ . Thus we may set

$$L(s, \pi) = L(s, \mu_1)L(s, \mu_2)$$

and

$$\epsilon(s, \pi, \psi) = \epsilon(s, \mu_1, \psi)\epsilon(s, \mu_2, \psi).$$

Then

$$L(s, \tilde{\pi}) = L(s, \mu_1^{-1})L(s, \mu_2^{-1}).$$

The local functional equation which is proved just as in the real case reads as follows.

**Theorem 6.4.** *Let  $\pi$  be an infinite-dimensional irreducible admissible representation of  $\mathcal{H}_{\mathbf{C}}$ . Let  $\omega$  be the quasi-character of  $\mathbf{C}^\times$  defined by*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I$$

for  $a$  in  $\mathbf{C}^\times$ . If  $W$  is in  $W(\pi, \psi)$  the integrals

$$\begin{aligned} \Psi(g, s, W) &= \int_{\mathbf{C}^\times} W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}g\right) |a|_{\mathbf{C}}^{s-1/2} d^\times a, \\ \tilde{\Psi}(g, s, W) &= \int_{\mathbf{C}^\times} W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}g\right) |a|_{\mathbf{C}}^{s-1/2} \omega^{-1}(a) d^\times a \end{aligned}$$

converge absolutely in some right half-plane. Set

$$\begin{aligned} \Psi(g, s, W) &= L(s, \pi)\Phi(g, s, W), \\ \tilde{\Psi}(g, s, W) &= L(s, \tilde{\pi})\tilde{\Phi}(g, s, W). \end{aligned}$$

The functions  $\Phi(g, s, W)$  and  $\tilde{\Phi}(g, s, W)$  can be analytically continued to the whole complex plane as holomorphic functions of  $s$ . For a suitable choice of  $W$  the function  $\Phi(e, s, W)$  is an exponential function of  $s$ . The functional equation

$$\tilde{\Phi}(wg, 1-s, W) = \epsilon(s, \pi, \psi)\Phi(g, s, W)$$

is satisfied. Moreover, if  $W$  is fixed  $|\Psi(g, s, W)|$  remains bounded as  $g$  varies over a compact subset of  $G_{\mathbf{C}}$  and  $s$  varies in a vertical strip of finite width from which discs about the poles of  $L(s, \pi)$  have been removed.

The following lemma can be verified by an explicit computation. The first assertion may also be proved by the method of Lemma 5.16.

**Lemma 6.5.** *If  $\sigma = \sigma(\mu_1, \mu_2)$  and  $\pi = \pi(\mu_1, \mu_2)$  are defined then*

$$\frac{L(1-s, \tilde{\sigma})\epsilon(s, \sigma, \psi)}{L(s, \sigma)} = \frac{L(1-s, \tilde{\pi})\epsilon(s, \pi, \psi)}{L(s, \pi)}$$

and the quotient

$$\frac{L(s, \chi \otimes \sigma)}{L(s, \chi \otimes \pi)}$$

is the product of a constant, a polynomial, and an exponential. Moreover the polynomial is of positive degree for some choice of the quasi-character  $\chi$ .

We verify the last assertion. There is no harm in supposing that  $\sigma = \pi(\nu_1, \nu_2)$  and that  $\chi\mu_1(z) = z^{a+p}\bar{z}^{b+q}$ ,  $\chi\mu_2(z) = z^a\bar{z}^b$ ,  $\chi\nu_1(z) = z^{a+p}\bar{z}^b$ , and  $\chi\nu_2(z) = z^a\bar{z}^{b+q}$ , where  $p \geq 1$  and  $q \geq 1$  are integers. Varying  $\chi$  is equivalent to varying  $a$  and  $b$  through all the integers. If  $m_1$  is the largest of  $a + p$  and  $b + q$  and  $m_2$  is the largest of  $a$  and  $b$  while  $n_1$  is the largest of  $a + p$  and  $b$  and  $n_2$  is the largest of  $a$  and  $b + q$  the quotient

$$\frac{L(s, \chi \otimes \sigma)}{L(s, \chi \otimes \pi)}$$

differs from

$$\frac{\Gamma(s + n_1)\Gamma(s + n_2)}{\Gamma(s + m_1)\Gamma(s + m_2)}$$

by a constant times an exponential. It is clear that  $n_1$  and  $n_2$  are both greater than or equal to  $m_2$  and that either  $n_1$  or  $n_2$  is greater than or equal to  $m_1$ . Thus the quotient is a polynomial. If  $p \geq q$  choose  $a$  and  $b$  so that  $b + q > a \geq b$ . Then  $n_1 = m_1$  and  $n_2 > m_2$  so that the quotient is of positive degree. If  $q \geq p$  choose  $a$  and  $b$  so that  $a + p > b \geq a$ . Then  $n_2 = m_1$  and  $n_1 > m_2$ .

**Lemma 6.6.** *Let  $\pi$  and  $\pi'$  be two infinite-dimensional irreducible representations of  $\mathcal{H}_{\mathbf{C}}$ . Suppose there is a quasi-character  $\omega$  of  $\mathbf{C}^\times$  such that*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I$$

and

$$\pi'\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)I$$

for all  $a$  in  $\mathbf{C}^\times$ . If

$$\epsilon(s, \chi \otimes \pi', \psi) \frac{L(1-s, \chi^{-1} \otimes \tilde{\pi}')}{L(s, \chi \otimes \pi')} = \epsilon(s, \chi \otimes \pi, \psi) \frac{L(1-s, \chi^{-1} \otimes \tilde{\pi})}{L(s, \chi \otimes \pi)}$$

for all quasi-characters  $\chi$  then  $\pi$  and  $\pi'$  are equivalent.

Let  $\pi = \pi(\mu_1, \mu_2)$  and let  $\pi' = \pi(\mu'_1, \mu'_2)$ . We let

$$\mu_i(z) = (z\bar{z})^{s_i} \left\{ \frac{z}{(z\bar{z})^{1/2}} \right\}^{a_i}$$

and

$$\mu'_i(z) = (z\bar{z})^{s'_i} \left\{ \frac{z}{(z\bar{z})^{1/2}} \right\}^{a'_i}$$

with  $a_i$  and  $a'_i$  in  $\mathbf{Z}$ . By assumption,  $s_1 + s_2 = s'_1 + s'_2$  and  $a_1 + a_2 = a'_1 + a'_2$ . Choose

$$\chi(z) = \left\{ \frac{z}{(z\bar{z})^{1/2}} \right\}^n$$

with  $n$  in  $\mathbf{Z}$ . The quotient on the right has the same zeros and poles as

$$\frac{\Gamma\left(1-s-s_1+\left|\frac{n+a_1}{2}\right|\right)}{\Gamma\left(s+s_1+\left|\frac{n+a_1}{2}\right|\right)} \cdot \frac{\Gamma\left(1-s-s_2+\left|\frac{n+a_2}{2}\right|\right)}{\Gamma\left(s+s_2+\left|\frac{n+a_2}{2}\right|\right)}.$$

A pole of the numerator can cancel a pole of the denominator if and only if there are two non-negative integers  $\ell$  and  $m$  such that

$$s_1 - s_2 = 1 + \ell + m + \left\lfloor \frac{n + a_1}{2} \right\rfloor + \left\lfloor \frac{n + a_2}{2} \right\rfloor$$

or

$$s_2 - s_1 = 1 + \ell + m + \left\lfloor \frac{n + a_1}{2} \right\rfloor + \left\lfloor \frac{n + a_2}{2} \right\rfloor.$$

This can happen only if  $\mu_1\mu_2^{-1}$  is of the form  $\mu_1\mu_2^{-1}(z) = z^p\bar{z}^q$  or  $\mu_1\mu_2^{-1}(z) = z^{-p}\bar{z}^{-q}$  where  $p \geq 1$  and  $q \geq 1$  are integers. Since  $\pi(\mu_1, \mu_2)$  is infinite-dimensional it cannot be of either these forms and no poles cancel.

Consequently for every integer  $n$ ,

$$\left\{ s_1 + \left\lfloor \frac{n + a_1}{2} \right\rfloor, s_2 + \left\lfloor \frac{n + a_2}{2} \right\rfloor \right\} = \left\{ s'_1 + \left\lfloor \frac{n + a'_1}{2} \right\rfloor, s'_2 + \left\lfloor \frac{n + a'_2}{2} \right\rfloor \right\}.$$

This can happen only if  $s_1 = s'_1$ ,  $a_1 = a'_1$ ,  $s_2 = s'_2$ , and  $a_2 = a'_2$  or  $s_1 = s'_2$ ,  $a_1 = a'_2$ ,  $s_2 = s'_1$ , and  $a_2 = a'_1$ . Thus  $\pi$  and  $\pi'$  are equivalent.

The following proposition is an easy consequence of these two lemmas.

**Proposition 6.7.** *Suppose  $\pi$  and  $\pi'$  are two irreducible admissible representations of  $\mathcal{H}_{\mathbf{C}}$ . Suppose there is a quasi-character  $\omega$  of  $\mathbf{C}^*$  such that*

$$\pi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \omega(a)I$$

and

$$\pi' \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \omega(a)I.$$

If  $L(s, \chi \otimes \pi) = L(s, \chi \otimes \pi')$ ,  $L(s, \chi^{-1} \otimes \tilde{\pi}) = L(s, \chi^{-1} \otimes \tilde{\pi}')$  and

$$\epsilon(s, \chi \otimes \pi, \psi) = \epsilon(s, \chi \otimes \pi', \psi)$$

for all quasi-characters  $\chi$  the representations  $\pi$  and  $\pi'$  are equivalent.

### §7. Characters

If  $F$  is a non-archimedean local field and  $\pi$  is an admissible representation of  $G_F$  the operator  $\pi(f)$  is of finite rank for every  $f$  in  $\mathcal{H}_F$  and therefore has a trace  $\text{Tr } \pi(f)$ . In this paragraph we prove that if  $\pi$  is irreducible there is a locally integrable function  $\chi_\pi$  on  $G_F$  such that

$$\text{Tr } \pi(f) = \int_{G_F} f(g) \chi_\pi(g) dg.$$

Although  $\text{Tr } \pi(f)$  depends on the choice of the Haar measure the function  $\chi_\pi$  does not.

The following simple lemma shows that  $\chi_\pi$  determines the class of  $\pi$ .

**Lemma 7.1.** *If  $\{\pi_1, \dots, \pi_p\}$  is a set of inequivalent irreducible admissible representations of  $\mathcal{H}_F$  the set of linear forms  $\text{Tr } \pi_1, \text{Tr } \pi_2, \dots, \text{Tr } \pi_p$  is linearly independent.*

Let  $\pi_i$  act on  $V_i$  and let  $\xi$  be an elementary idempotent such that none of the spaces  $\pi_i(\xi)V_i$ ,  $1 \leq i \leq p$ , are 0. Let  $\bar{\pi}_i$  be the representation of  $\xi\mathcal{H}_F\xi$  on the finite-dimensional space  $\pi_i(\xi)V_i = V_i(\xi)$ . Suppose  $\bar{\pi}_i$  and  $\bar{\pi}_j$  are equivalent. Then there is an invertible linear map  $A$  from  $V_i(\xi)$  to  $V_j(\xi)$  which commutes with the action of  $\xi\mathcal{H}_F\xi$ . Choose a non-zero vector  $v_i$  in  $V_i(\xi)$  and let  $v_j = Av_i$ . We are going to show that  $\pi_i$  and  $\pi_j$  are equivalent. It is enough to show that, for any  $f$  in  $\mathcal{H}_F$ ,  $\pi_i(f)v_i = 0$  if and only if  $\pi_j(f)v_j = 0$ . But  $\pi_i(f)v_i = 0$  if and only if  $\pi_i(\xi * h)\pi_i(f)v_i = 0$  for all  $h$  in  $\mathcal{H}_F$ . Since  $\pi_i(\xi * h)\pi_i(f)v_i = \pi_i(\xi * h * f * \xi)v_i$  and  $\xi * h * f * \xi$  is in  $\xi\mathcal{H}_F\xi$  the assertion follows.

Thus the representations  $\bar{\pi}_1, \dots, \bar{\pi}_p$  are inequivalent. Using this we shall show that the linear forms  $\text{Tr } \bar{\pi}_1, \dots, \text{Tr } \bar{\pi}_p$  on  $\xi\mathcal{H}_F\xi$  are linearly independent. The lemma will then be proved. Take  $h$  in  $\xi\mathcal{H}_F\xi$ . Since  $\bar{\pi}_i$  is irreducible and finite-dimensional  $\text{Tr } \bar{\pi}_i(hf) = 0$  for all  $f$  in  $\xi\mathcal{H}_F\xi$  if and only if  $\bar{\pi}_i(h) = 0$ . Suppose we had  $h_1, \dots, h_p$  in  $\xi\mathcal{H}_F\xi$  so that for at least one  $i$  the operator  $\bar{\pi}_i(h_i)$  was not 0 while

$$\sum_{i=1}^p \text{Tr } \bar{\pi}_i(h_i f) = 0$$

for all  $f$  in  $\xi\mathcal{H}_F\xi$ . There must then be at least two integers  $j$  and  $k$  such that  $\bar{\pi}_j(h_j) \neq 0$  and  $\bar{\pi}_k(h_k) \neq 0$ . Since  $\bar{\pi}_j$  and  $\bar{\pi}_k$  are not equivalent we can find an  $h$  in  $\xi\mathcal{H}_F\xi$  such that  $\bar{\pi}_j(h) = 0$  while  $\bar{\pi}_k(h)$  is invertible. Replacing  $h_i$  by  $h_i h$  we obtain a relation of the same type in which the number of  $i$  for which  $\bar{\pi}_i(h_i) = 0$  has been increased. By induction we see that no such relation is possible. Since  $\xi\mathcal{H}_F\xi$  contains a unit the required independence follows.

For most of these notes the existence of  $\chi_\pi$  is irrelevant. It is used only toward the end. The reader who is more interested in automorphic forms than in group representations will probably want to take the existence of  $\chi_\pi$  for granted and, for the moment at least, skip this paragraph. To do so will cause no harm. However he will eventually have to turn back to read the first few pages in order to review the definition of the Tamagawa measure.

Choose a non-trivial additive character  $\psi$  of  $F$ . If  $X$  is an analytic manifold over  $F$  and  $\omega$  is a differential form of highest degree on  $X$  we can associate to  $\omega$  a measure on  $X$  which is denoted by  $|\omega|_F$  or sometimes simply by  $\omega$ . If  $X = F$  and  $\omega = dx$  is the differential of the identity application the measure  $|\omega|_F = dx$  is by definition the Haar measure on  $F$  which is self-dual with respect to  $\psi$ . In general if  $p$  belongs to  $X$  and  $x^1, \dots, x^n$  are local coordinates near  $p$  so that

$$\omega = a(x^1, \dots, x^n) dx^1 \wedge \dots \wedge dx^n$$

then, if  $f$  is a continuous real-valued functions with support in a small neighbourhood of  $p$ ,

$$\int_X f|\omega|_F = \int f(x^1, \dots, x^n) |a(x^1, \dots, x^n)| dx^1 \cdots dx^n.$$

The absolute value  $|a(x^1, \dots, x^n)|$  is the normalized absolute value in the field  $F$ . To prove the existence of the measure  $\omega$  one has to establish the usual formula for a change of variable in a multiple integral. For this and other facts about these measures we refer to the notes of Weil [12].

If  $G$  is an algebraic group over  $F$  then  $G_F$  is an analytic space. If  $\omega$  is a left-invariant form of highest degree on  $G_F$  the measure  $|\omega|_F$  is a Haar measure on  $G_F$ . It is called the Tamagawa measure. It depends on  $\omega$  and  $\psi$ .

If  $M$  is the algebra of  $2 \times 2$  matrices over  $F$  the additive group of  $M$  is an algebraic group. If a typical element of  $M$  is

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then

$$\mu = da \wedge db \wedge dc \wedge dd$$

is an invariant form of highest degree and  $|\mu| = dx$  is the additive Haar measure which is self-dual with respect to the character  $\psi_M(x) = \psi_F(\tau(x))$  if  $\tau$  is the trace of  $x$ .

On the multiplicative group  $G$  of  $M$  we take the form  $\omega(x) = (\det x)^{-2} \mu(x)$ . The associated Haar measure is

$$|\omega(x)| = |\det x|_F^{-2} dx = |x|_M^{-1} dx.$$

An element of  $G$  is said to be regular if its eigenvalues are distinct. The centralizer in  $G_F$  of a regular element in  $G_F$  is a Cartan subgroup of  $G_F$ . Such a Cartan subgroup  $B_F$  is of course abelian. There seems to be no canonical choice for the invariant form on  $B_F$ . However the centralizer of  $B_F$  in  $M_F$  is an algebra  $E$  of degree two over  $F$ . It is either isomorphic to the direct sum of  $F$  with itself or it is a separable quadratic of  $F$ . The subgroup  $B_F$  is the multiplicative group of  $E$ . In the first paragraph we introduced a map  $\nu$  from  $E$  to  $F$ . Once a form  $\mu_E$  on  $E$  which is invariant for the additive group has been chosen we can set  $\mu_B(x) = \nu(x)^{-1} \mu_E(x)$ , and  $\mu_B$  is then an invariant form on  $B_F$ . The associated measure is invariant under all automorphisms of  $E$  over  $F$ . We should also recall at this point that two Cartan subgroups  $B_F$  and  $B'_F$  are conjugate in  $G_F$  if and only if the corresponding algebras are isomorphic.

Once  $\mu_E$  and therefore  $\mu_B$  has been chosen we can introduce on  $B_F \backslash G_F$  which is also an analytic manifold the form  $\omega_B$  which is the quotient of  $\omega$  by  $\mu_B$ . Then

$$\int_{G_F} f(g) \omega(g) = \int_{B_F \backslash G_F} \left\{ \int_{B_F} f(bg) \mu_B(b) \right\} \omega_B(g).$$

The centre of the algebra of  $M_F$  is isomorphic to  $F$  and the centre  $Z_F$  of  $G_F$  is isomorphic to  $F^\times$ . On  $F^\times$  we have the form  $x^{-1} dx$ . We take  $\mu_Z$  to be the corresponding form on  $Z_F$ .  $\mu_B^0$  will be the quotient of  $\mu_B$  by  $\mu_Z$  and  $\omega^0$  will be the quotient of  $\omega$  by  $\mu_Z$ . The corresponding integration formulae are

$$\int_{B_F} f(b) \mu_B(b) = \int_{Z_F \backslash B_F} \left\{ \int_{Z_F} f(zb) \mu_Z(z) \right\} \mu_B^0(b)$$



and

$$\int_{G_F} f(g)\omega(g) = \int_{Z_F \backslash G_F} \left\{ \int_{Z_F} f(zg)\mu_Z(z) \right\} \omega^0(g).$$

If  $g$  belongs to  $G_F$  its eigenvalues  $\alpha_1$  and  $\alpha_2$  are the roots of the equation

$$X^2 - \tau(g)X + \nu(g) = 0$$

and

$$\frac{(\alpha_1 - \alpha_2)^2}{\alpha_1 \alpha_2} = \frac{\{\tau(g)\}^2 - 4\nu(g)}{\nu(g)}$$

belongs to  $F$ . Set

$$\delta(g) = \left| \frac{(\alpha_1 - \alpha_2)^2}{\alpha_1 \alpha_2} \right|_F.$$

Since  $g$  is regular if and only if  $\delta(g) \neq 0$  the set  $\widehat{G}_F$  of regular elements is open in  $G_F$  and its complement has measure zero.

There are two more integration formulae that we shall need. Their proof proceeds as for archimedean fields. Choose a system  $S$  of representatives of the conjugacy classes of Cartan subgroups of  $G_F$ . Then

$$(7.2.1) \quad \int_{G_F} f(g)\omega(g) = \sum_{B_F \in S} \frac{1}{2} \int_{B_F} \delta(b) \left\{ \int_{B_F \backslash G_F} f(g^{-1}bg)\omega_B(g) \right\} \mu_B(b)$$

$$(7.2.2) \quad \int_{Z_F \backslash G_F} f(g)\omega^0(g) = \sum_{B_F \in S} \frac{1}{2} \int_{Z_F \backslash B_F} \delta(b) \left\{ \int_{B_F \backslash G_F} f(g^{-1}bg)\omega_B(g) \right\} \mu_B^0(b)$$

if  $f$  is an integrable function on  $G_F$  or  $Z_F \backslash G_F$ . Notice that the sum on the right is not necessarily finite. Let  $\widehat{B}_F = B_F \cap \widehat{G}_F$  and let

$$\widehat{B}_F^G = \left\{ g^{-1}bg \mid b \in \widehat{B}_F, g \in G_F \right\}.$$

Then  $\widehat{G}_F$  is the disjoint union

$$\bigcup_{B_F \in S} \widehat{B}_F^G.$$

There is a simple lemma to be verified.

**Lemma 7.2.**

- (i) For any Cartan subgroup  $B_F$  the set  $\widehat{B}_F^G$  is open.
- (ii) The set  $\widehat{G}_F$  is open.
- (iii) The set  $\widetilde{G}_F$  of  $g$  in  $G_F$  whose eigenvalues do not belong to  $F$  is open.

The second statement is a consequence of the first. If  $B_F$  corresponds to the separable quadratic extension  $E$  then  $\widehat{B}_F^G$  is the set of matrices with distinct eigenvalues in  $E$  and if  $B_F$  splits and therefore corresponds to the direct sum of  $F$  with itself,  $\widehat{B}_F^G$  is the set of matrices with distinct eigenvalues in  $F$ . Thus the first assertion is a consequence of the following lemma which is a form of Hensel's lemma or of the implicit function theorem.

**Lemma 7.2.1.** *Let  $E$  be a separable extension of  $F$ . Assume the equation*

$$X^p + a_1 X^{p-1} + \cdots + a_p = 0$$

*with coefficients in  $F$  has a simple root  $\lambda$  in  $E$ . Given  $\epsilon > 0$  there is a  $\delta > 0$  such that whenever  $b_1, \dots, b_p$  are in  $F$  and  $|b_i - a_i|_F < \delta$  for  $1 \leq i \leq p$  the equation*

$$X^p + b_1 X^{p-1} + \cdots + b_p = 0$$

*has a root  $\mu$  in  $E$  for which  $|\lambda - \mu|_E < \epsilon$ .*

There is no need to prove this lemma. To prove the third assertion we have to show that the set of matrices with eigenvalues in  $F$  is closed. Suppose  $g_n \rightarrow g$  and  $g_n$  has eigenvalues  $\lambda_n$  and  $\mu_n$  in  $F$ . Then  $\lambda_n + \mu_n \rightarrow \tau(g)$  and  $\lambda_n \mu_n \rightarrow \nu(g)$ . If  $\lambda_n$  and  $\mu_n$  did not remain in a compact subset of  $F^\times$  then, since their product does, we would have, after passing to a subsequence,  $|\lambda_n| \rightarrow 0$ ,  $|\mu_n| \rightarrow \infty$  or  $|\lambda_n| \rightarrow \infty$ ,  $|\mu_n| \rightarrow 0$ . In either case  $\lambda_n + \mu_n$  could not converge. Thus, again passing to a subsequence, we have  $\lambda_n \rightarrow \lambda$  and  $\mu_n \rightarrow \mu$ .  $\lambda$  and  $\mu$  are the eigenvalues of  $g$ .

If the characteristic of  $F$  is not two the sets  $\widehat{G}_F$  and  $\widetilde{G}_F$  are the same. We now introduce a function on  $G_F$  which plays an important role in the discussion of characters. If  $B_F$  is a split Cartan subgroup we set  $c(B_F) = 1$  but if  $B_F$  is not split and corresponds to the quadratic extension  $E$  we set

$$c(B_F) = |\varpi|_F^{\frac{t+1}{2}}$$

where  $\varpi$  is a generator of  $\mathfrak{p}_F$  and  $\mathfrak{p}_F^{t+1}$  is the discriminant of  $E$  over  $F$ . If  $g$  in  $\widehat{G}_F$  belongs to the Cartan subgroup  $B_F$  we set

$$\xi(g) = c(B_F) \delta^{-1/2}(g).$$

If  $g$  is singular we set  $\xi(g) = \infty$ . The factor  $c(B_F)$  is important only in characteristic two when there are an infinite number of conjugacy classes of Cartan subgroups.

**Lemma 7.3.** *The function  $\xi$  is locally constant on  $\widehat{G}_F$  and bounded away from zero on any compact subset of  $G_F$ . It is locally integrable on  $Z_F \backslash G_F$  and on  $G_F$ .*

It is of course implicit in the statement of the lemma that  $\xi$  is constant on cosets of  $Z_F$ . The two previous lemmas show that  $\xi$  is locally constant on  $\widehat{G}_F$ . To prove the remaining assertions we recall some facts about orders and modules in separable quadratic extensions of non-archimedean fields.

If  $E$  is a separable quadratic extension of  $F$  an order  $R$  of  $E$  is a subring of  $O_E$  which contains  $O_F$  and a basis of  $E$ . A module  $I$  in  $E$  is a finitely generated  $O_F$  submodule of  $E$  which contains a basis of  $E$ . If  $I$  is a module the set

$$\{\alpha \in E \mid \alpha I \subseteq I\}$$

is an order  $R_I$ . It is clear that an order is a module and that  $R_R = R$ . Two modules  $I$  and  $J$  are said to be equivalent if there is an  $\alpha$  in  $E^\times$  so that  $J = \alpha I$ . Then  $R_I = R_J$ .

Suppose the module  $I$  is contained on  $O_E$  and contains 1. Since  $I/O_F$  is a torsion-free  $O_F$  module the module  $I$  has a basis of the form  $\{1, \delta\}$ . Since  $\delta$  is integral  $\delta^2$  belongs to  $I$ . Therefore  $I$  is an order and  $R_I = I$ . Since any module is equivalent to a module which contains 1 and lies in  $O_E$  the collection of modules  $I$  for which  $R_I = R$  forms, for a given order  $R$ , a single equivalence class.

As observed any order has a basis, over  $O_F$ , of the form  $\{1, \delta\}$ . The absolute values of the numbers  $\delta$  occurring in such bases are bounded below. A basis  $\{1, \delta\}$  is said to be normal if  $\delta$  has the smallest possible absolute value. It is easily seen, by considering the ramified and unramified extensions separately, that if  $\{1, \delta\}$  is normal

$$R = O_F + \delta O_E.$$

Thus  $R$  determines and is determined by  $|\delta|_E$ . It is easily seen that if  $E/F$  is unramified  $|\delta|_E$  is any number of the form  $|\varpi_E|_E^n$  with  $n \geq 0$ , where  $\varpi_E$  is a generator of  $\mathfrak{p}_E$ . We set  $n = \omega(R)$ . If  $E/F$  is ramified  $|\delta|_E$  is any number of the form  $|\varpi_E|_E^{2n+1}$  with  $n \geq 0$ . We set  $\omega(R) = n$ . In the ramified case

$$[E^\times : F^\times(U_E \cap R)] = 2|\varpi_F|_F^{-\omega(R)}.$$

In the unramified case

$$[E^\times : F^\times(U_E \cap R)] = |\varpi_F|_F^{-\omega(R)}(1 + |\varpi_F|_F)$$

unless  $\omega(R) = 0$  and then

$$[E^\times : F^\times(U_E \cap R)] = 1.$$

It is clear that  $R'$  contains  $R$  if and only if  $\omega(R') \leq \omega(R)$ . Thus  $\omega(R) + 1$  is the number of orders which contain  $R$ . If  $\gamma$  belongs to  $O_E$  but not to  $O_F$  let  $R(\gamma)$  be the order with basis  $\{1, \gamma\}$  and let  $\omega(\gamma) = \omega(R(\gamma))$ .

**Lemma 7.3.1.** *Let  $\bar{\gamma}$  be the conjugate of  $\gamma$  in  $E$  and let*

$$|(\gamma - \bar{\gamma})^2|_F^{1/2} = |\varpi_F|_F^{m(\gamma)}.$$

*If  $\mathfrak{p}_E^{t+1}$  is the discriminant of  $E$  and  $\gamma$  belongs to  $O_K$  but not to  $O_F$  then*

$$m(\gamma) = \omega(\gamma) + \frac{t+1}{2}.$$

Let  $\{1, \delta\}$  be a normal basis of  $R(\gamma)$ . Then  $\gamma = a + b\delta$  with  $a$  and  $b$  in  $O_F$ . Moreover  $\delta = c + d\gamma$  with  $c$  and  $d$  in  $O_F$ . Thus  $\gamma = (a + bc) + bd\gamma$  so that  $a + bc = 0$  and  $bd = 1$ . Therefore  $b$  is a unit and  $|\gamma - \bar{\gamma}| = |\delta - \bar{\delta}|$ . We can thus replace  $\gamma$  by  $\delta$ . Suppose first that  $E/F$  is unramified so that  $t+1 = 0$ . We take  $\delta = \epsilon\varpi_F^n$  where  $n = \omega(R(\gamma))$  and  $\epsilon$  is a unit of  $O_E$ . Since

$$\delta - \bar{\delta} = (\epsilon - \bar{\epsilon})\varpi_F^n$$

we have only to show that  $\epsilon - \bar{\epsilon}$  is a unit.  $\epsilon$  is not congruent to an element of  $O_F$  modulo  $\mathfrak{p}_E$  and therefore  $\{1, \epsilon\}$  determines a basis of  $O_E/\mathfrak{p}_E$ . Since the Galois group acts faithfully on  $O_E/\mathfrak{p}_E$  the number  $\epsilon - \bar{\epsilon}$  is not in  $\mathfrak{p}_E$ .

If  $E/F$  is ramified we may take  $\delta = \varpi_F^n \varpi_E$  with  $n = \omega(\delta)$ . It is well-known that

$$|\varpi_E - \bar{\varpi}_E| = |\varpi_E|_E^{t+1}$$

Thus

$$|(\delta - \bar{\delta})^2|_F^{1/2} = |\delta - \bar{\delta}|_E^{1/2} = |\varpi_F|_F^n |\varpi_K|_K^{\frac{t+1}{2}} = |\varpi_F|_F^{n + \frac{t+1}{2}}$$

The lemma follows.

There are two more lemmas to be proved before we return to the proof of Lemma 7.3.

**Lemma 7.3.2.** *Let  $C$  be a compact subset of  $Z_F \backslash G_F$  and let  $\chi_C$  be the characteristic function of  $C$  and of its inverse image in  $G_F$ . There is a constant  $c$  such that for every  $b$  in  $G_F$  which is contained in an anisotropic Cartan subgroup*

$$\int_{Z_F \backslash G_F} \chi_C(g^{-1}bg) \omega^0(g) \leq c\xi(b).$$

The assertion is trivial unless  $b$  is regular. Then the assumption is that its eigenvalues are distinct and do not lie in  $F$ . Any  $h$  in  $G_F$  can be written as

$$g_1 \begin{pmatrix} \varpi_F^p & \\ & \varpi_F^q \end{pmatrix} g_2$$

where  $g_1$  and  $g_2$  belongs to  $\mathrm{GL}(2, O_F)$  and  $p \leq q$ . The numbers  $\varpi_F^p$  and  $\varpi_F^q$  are the elementary divisors of  $h$ . Let  $T_r$  be the set of all those  $h$  for which  $q - p \leq r$ . This set is the inverse image of a compact subset  $T'_r$  of  $Z_F \backslash G_F$ . If  $r$  is sufficiently large  $C$  is contained in  $T'_r$ . Thus we may replace  $\chi_C$  and  $\chi_r$  the characteristic function of  $T'_r$ .

If  $h$  belongs to  $\mathrm{GL}(2, O_F)$  then  $h^{-1}g^{-1}bgh$  belongs to  $T_r$  if and only if  $g^{-1}bg$  belongs to  $T_r$ . Thus the integral is the product of the measure of  $\mathrm{GL}(2, O_F) \cap Z_F \backslash \mathrm{GL}(2, O_F)$  by the number of right cosets of  $Z_F \mathrm{GL}(2, O_F)$  whose elements  $g$  are such that  $g^{-1}bg$  belong to  $T_r$ . If  $H$  is such a coset and  $B_F$  is the Cartan subgroup containing  $b$  then for any  $b'$  in  $B_F$  the coset  $b'H$  has the same property. Thus the integral equals

$$\text{measure}(\mathrm{GL}(2, O_F) \cap Z_F \backslash \mathrm{GL}(2, O_F)) \sum [B_F g \mathrm{GL}(2, O_F) : Z_F \mathrm{GL}(2, O_F)].$$

The sum is over a set of representatives of the cosets in  $B_F \backslash G_F / \mathrm{GL}(2, O_F)$ .

Let  $B_F$  correspond to the separable quadratic extension  $E$ . Choose a basis of  $O_E$  over  $O_F$ . It will also be a basis of  $E$  over  $F$ . By means of this basis we identify  $G_F$  with the group of invertible linear transformations of  $E$  over  $F$ .  $\mathrm{GL}(2, O_F)$  is the stabilizer of  $O_E$ . Every  $\gamma$  in  $E^\times$  determines a linear transformation  $b_\gamma : x \rightarrow \gamma x$  of  $E$ . The set of all such linear transformations is a Cartan subgroup conjugate to  $B_F$  and with no loss of generality we may assume that it is  $B_F$ . Choose  $\gamma$  so that  $b = b_\gamma$ .

Every module is of the form  $gO_E$  with  $g$  in  $G_F$ . Moreover  $g_1O_E$  and  $g_2O_E$  are equivalent if and only if  $g_1$  and  $g_2$  belong to the same double coset in  $B_F \backslash G_F / \mathrm{GL}(2, O_F)$ . Thus there is a one-to-one correspondence between the collection of double cosets and the collection of orders of  $E$ . Let  $B_F g \mathrm{GL}(2, O_F)$  correspond to the order  $R$ . The index

$$[B_F g \mathrm{GL}(2, O_F) : Z_F \mathrm{GL}(2, O_F)]$$

is equal to

$$[B_F : B_F \cap Z_F g \mathrm{GL}(2, O_F) g^{-1}]$$

Two elements  $b_1$  and  $b_2$  in  $B_F$  belong to the same coset of  $B_F \cap Z_F g \mathrm{GL}(2, O_F) g^{-1}$  if and only if there is a  $z$  in  $Z_F$  and an  $h$  in  $\mathrm{GL}(2, O_F)$  such that

$$b_1 g = b_2 z g h$$

This can happen if and only if

$$b_1 g O_E = b_2 z g O_E.$$

Let  $I = gO_E$  and let  $b_i = b_{\gamma_i}$ . If we identify  $Z_F$  and  $F^\times$  so that  $z$  may be regarded as an element of  $F^\times$  the last relation is equivalent to

$$\gamma_1 I = \gamma_2 z I$$

or  $\gamma_1^{-1}\gamma_2 z \in R \cap U_E$ . Thus

$$[B_F g \text{GL}(2, O_F) : Z_F \text{GL}(2, O_F)] = [E^\times : F^\times(R \cap U_E)].$$

Let  $|\det b|_F = |\gamma|_K = |\varpi_F|_F^m$ . Let  $\varpi_F^p$  and  $\varpi_F^q$  with  $p \leq q$  be the elementary divisors of  $g^{-1}bg$ . Certainly  $p+q = m$ . The matrix  $g^{-1}bg$  belongs to  $T_r$  if and only if  $q-p = m-2p \leq r$ . If  $s$  is the integral part of  $\frac{r-m}{2}$  this is so if and only if  $\varpi_F^s g^{-1}bg$  has integral coefficients, that is if and only if

$$\varpi_F^s g^{-1}bg O_E \subseteq O_E$$

or  $\varpi_F^s \gamma \in R$ .

Our integral is therefore equal to

$$(*) \quad \text{measure}(\text{GL}(2, O_F) \cap Z_F \backslash \text{GL}(2, O_F)) \sum_{\varpi_F^s \gamma \in R} [E^\times : F^\times(R \cap U_E)].$$

The sum is over all orders of  $E$  which contains  $\varpi_F^s \gamma$ . The element  $\varpi_F^s \gamma$  does not lie in  $F$ . If it does not lie in  $O_K$  the sum is zero. If it lies in  $O_K$  then  $\varpi_F^s \gamma$  belongs to  $R$  if and only if  $\omega(R) \leq \omega(\varpi_F^s \gamma)$ . In this case the expression  $(*)$  is bounded by

$$2 \text{ measure}(\text{GL}(2, O_F) \cap Z_F \backslash \text{GL}(2, O_F)) \sum_{0 \leq k \leq \omega(\varpi_F^s \gamma)} |\varpi_F|_F^{-k}.$$

This in turn is bounded by a constant, which is independent of  $B_F$  and  $r$ , times

$$|\varpi_F|_F^{-\omega(\varpi_F^s \gamma)}$$

We have  $c(B_F) = |\varpi_F|_F^{\frac{t+1}{2}}$ ,  $m(\varpi_F^s \gamma) = s + m(\gamma) \leq \frac{r-m}{2} + m(\gamma)$ , and

$$\delta(b)^{1/2} = \frac{|(\gamma - \bar{\gamma})^2|_F^{1/2}}{|\gamma \bar{\gamma}|_F^{1/2}} = |\varpi_F|_F^{-m/2} |\varpi_F|_F^{m(\gamma)}.$$

To prove the lemma we have only to show that

$$-m(\gamma) + \frac{m}{2} + \frac{t+1}{2} + \omega(\varpi_F^s \gamma)$$

is bounded above by a constant which depends only on  $r$ . By the previous lemma

$$\omega(\varpi_F^s \gamma) = m(\varpi_F^s \gamma) - \frac{t+1}{2}$$

so that

$$-m(\gamma) + \frac{m}{2} + \frac{t+1}{2} + \omega(\varpi_F^s \gamma) \leq \frac{r-m}{2} + \frac{m}{2} = \frac{r}{2}.$$

Suppose the Cartan subalgebra  $B_F$  corresponds to the algebra  $E$ . Once the measure  $\mu_E$  on  $E$  has been chosen we can form the measure  $\mu_B$  on  $B_F$  and the measure  $\omega_B$  on  $B_F \backslash G_F$ . Once  $\mu_E$  and therefore  $\mu_B$  and  $\omega_B$  are chosen we let  $n(B_F)$  be that constant which makes  $n(B_F)\mu_E$  self-dual with respect to the character  $x \rightarrow \psi(\tau(x))$  on  $E$ .

**Lemma 7.3.3.** *If  $r$  is a non-negative integer there is a constant  $d_r$  such that for any Cartan subgroup  $B_F$  and any  $b$  in  $B_F$*

$$\int_{B_F \backslash G_F} \chi_r(g^{-1}bg) \omega_B(g) \leq d_r n(B_F) \delta(b)^{-1/2}.$$

We may again suppose that  $b$  belongs to  $\widehat{B}_F$ . If  $B_F$  is anisotropic the left side is equal to

$$\frac{1}{\text{measure}(Z_F \backslash B_F)} \int_{Z_F \backslash G_F} \chi_r(g^{-1}bg) \omega^0(g).$$

Suppose  $B_F$  corresponds to the quadratic extension  $E$ . If  $E/F$  is unramified

$$\text{measure}(Z_F \backslash B_F) = \frac{1}{n(B_F)} (1 + |\varpi_F|)$$

because  $n(B_F)\mu_E$  assigns the measure 1 to  $O_E$ . If  $E/F$  is ramified  $n(B_F)\mu_E$  assigns the measure  $|\varpi_F|^{\frac{t+1}{2}}$  to  $O_E$  and

$$\text{measure}(Z_F \backslash B_F) = \frac{2}{n(B_F)} |\varpi_F|^{\frac{t+1}{2}} = \frac{2}{n(B_F)} c(B_F)$$

In these cases the assertion is therefore a consequence of the previous lemma.

If the inequality of the lemma is true for one Cartan subgroup it is true for all conjugate subgroups. To complete the proof we have to verify it when  $B_F$  is the group  $A_F$  of diagonal matrices. Since we are now dealing with a fixed Cartan subgroup the choice of Haar measure on  $B_F \backslash G_F$  is not important. Moreover  $\text{GL}(2, O_F) T_r \text{GL}(2, O_F) = T_r$  so that, using the Iwasawa decomposition and the associated decomposition of measures, we may take the integral to be

$$\int_F \chi_r \left( \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) dx$$

if

$$b = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

The argument in the integrand is

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & \left(1 - \frac{\beta}{\alpha}\right)x \\ 0 & 1 \end{pmatrix}$$

Changing the variables in the integral we obtain

$$\frac{1}{\left|1 - \frac{\beta}{\alpha}\right|} \int_F \chi_r \left( \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) dx.$$

Let  $|\alpha| = |\varpi_F|^\ell$ ,  $|\beta| = |\varpi_F|^m$ , and  $|x| = |\varpi_F|^n$ . With no loss of generality we may suppose  $|\alpha| \geq |\beta|$ . If  $n \geq 0$  the elementary divisors of

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

are  $\varpi_F^\ell$  and  $\varpi_F^m$  so that it is in  $T_r$  if and only if  $m - \ell \leq r$ . If  $n < 0$  its elementary divisors are  $\varpi_F^{\ell+n}$  and  $\varpi_F^{m-n}$  so that it is in  $T_r$  if and only if  $m - \ell - 2n \leq r$ . Thus the integral is at most

$$\text{measure} \left\{ x \mid |x| \leq |\varpi_F|^{\frac{m-\ell-r}{2}} \right\}$$

which is, apart from a factor depending on the choice of the Haar measure,  $|\varpi_F|^{\frac{m-\ell-r}{2}}$ . Since

$$|\varpi_F|^{\frac{m-\ell-r}{2}} = \left| \frac{\beta}{\alpha} \right|^{1/2} |\varpi_F|^{-r/2}$$

and

$$\frac{\left| \frac{\beta}{\alpha} \right|^{1/2}}{\left| 1 - \frac{\beta}{\alpha} \right|} = \delta(b)^{-1/2}$$

the lemma follows.

We return to Lemma 7.3 and prove first that  $\xi$  is bounded away from zero on each compact subset  $C$ . In other words we show that there is a positive constant  $c$  such that  $\xi(h) \geq c$  on  $C$ . There is a  $z$  in  $Z_F$  such that every matrix in  $zC$  has integral entries. Since  $\xi(zh) = \xi(h)$  we may as well assume that every matrix in  $C$  itself has integral entries. There is a constant  $c_1 > 0$  such that

$$|\det h|_F^{1/2} \geq c_1$$

on  $C$  and a constant  $c_2$  such that

$$|\tau(h)^2 - 4\nu(h)|^{1/2} \leq c_2$$

on  $C$ .  $\tau$  and  $\nu$  are the trace and determinant of  $h$ . Thus

$$\delta^{-1/2}(h) \geq \frac{c_1}{c_2}$$

on  $C$ . Here  $\xi(h)$  is certainly bounded away from 0 on the singular elements and the preceding inequality shows that it is bounded away from 0 on the regular elements in  $C$  which lie in a split Cartan subalgebra. Suppose  $h$  is regular and lies in the anisotropic Cartan subgroup  $B_F$ . Let  $B_F$  correspond to the field  $E$  and let  $h$  have eigenvalues  $\gamma$  and  $\bar{\gamma}$  in  $E$ . Then

$$|(\gamma - \bar{\gamma})^2|_F^{-1/2} c(B_F) = |\varpi_F|^{-m(\gamma)} |\varpi_F|^{\frac{t+1}{2}} = |\varpi_F|^{-\omega(\gamma)}$$

Since  $\omega(\gamma) \geq 0$  we have  $\xi(h) \geq c_1$ .

The function  $\xi$  is certainly measurable. It is locally integrable in  $G_F$  if and only if it is locally integrable on  $Z_F \backslash G_F$ . Let  $C$  be a compact set in  $Z_F \backslash G_F$ . We have to show

$$\int_{Z_F \backslash G_F} \chi_C(g) \xi(g) \omega^0(g)$$

is finite. As usual it will be enough to show that

$$\int_{Z_F \backslash G_F} \chi_r(g) \xi(g) \omega^0(g)$$

is finite for every non-negative integer  $r$ . According to formula (7.2.2) this integral is the sum of

$$\frac{1}{2} \int_{Z_F \backslash A_F} \xi(a) \delta(a) \left\{ \int_{A_F \backslash G_F} \chi_r(g^{-1} a g) \omega_A(g) \right\} \mu_A^0(a)$$

and

$$\frac{1}{2} \sum_{B_F \in S'} \int_{Z_F \backslash B_F} \xi(b) \delta(b) \left\{ \int_{B_F \backslash G_F} \chi_r(g^{-1} b g) \omega_B(g) \right\} \mu_B^0(b).$$

It is easy to see that there is a compact set  $C_0$  in  $Z_F \backslash A_F$  such that  $\chi_r(g^{-1}ag) = 0$  for all  $g$  unless the projection of  $a$  lies in  $C_0$ . Thus the first integral need only be taken over  $C_0$ . The inner integral is at most  $d_r n(A_F) \delta(a)^{-1/2}$ . Since  $\xi(a) \delta(a) \delta(a)^{-1/2} = 1$  on  $A_F$  the first integral causes no trouble. We can also use Lemma 7.3.3 to see that the sum over  $S'$ , which is by the way a set of representatives for the conjugacy classes of anisotropic Cartan subgroups, is less than or equal to

$$\frac{1}{2} \sum_{B_F \in S'} d_r n(B_F) c(B_F) \int_{Z_F \backslash B_F} \mu_B^0(b).$$

If the characteristic is not two this sum is finite and there is no problem.

In general if  $B_F$  corresponds to the field  $E$  and  $\mathfrak{p}_F^{t_E+1}$  is the discriminant of  $E$  we have  $c(B_F) = |\varpi_F|^{\frac{t_E+1}{2}}$  and

$$n(B_F) \int_{Z_F \backslash B_F} \mu_B^0(b) \leq 2 |\varpi_F|^{(t_E+1)/2}$$

To complete the proof we have to show that

$$\sum_E |\varpi_F|^{t_E+1}$$

is finite if  $F$  has characteristic 2. The sum is over all separable quadratic extensions of  $F$ . Let  $M(t)$  be the number of extensions  $E$  for which  $t_E = t$ . Associated to any such  $E$  is a quadratic character of  $F^\times$  with conductor  $\mathfrak{p}_F^{t+1}$ . Thus

$$M(t) \leq [F^\times : (F^\times)^2 (1 + \mathfrak{p}_F^{t+1})] = 2[U_F : U_F^2 (1 + \mathfrak{p}_F^{t+1})]$$

if  $t \geq 0$ . Of course  $M(-1) = 1$ . Any element of  $U_F$  is congruent modulo  $1 + \mathfrak{p}_F^{t+1}$  to an element of the form

$$a_0 + a_1 \varpi_F + \cdots + a_t \varpi_F^t.$$

Such a number is a square if  $a_i = 0$  for  $i$  odd. Thus

$$M(t) = O\left(|\varpi_F|^{-\frac{t+1}{2}}\right)$$

and the series converges.

We can now begin the study of characters.

**Proposition 7.4.** *The character of an absolutely cuspidal representation exists as a locally integrable function whose absolute value is bounded by a multiple of  $\xi$ . It is continuous on  $\widehat{G}_F \cup \widetilde{G}_F$ .*

If the character  $\chi_\pi$  of  $\pi$  exists and  $\chi$  is a quasi-character of  $F^\times$  then the character of  $\pi' = \chi \otimes \pi$  also exists and  $\chi_{\pi'}(g) = \chi(\det g) \chi_\pi(g)$ . Thus the proposition has only to be proved for unitary representations  $\pi$ . Then  $\pi$  is square integrable and we can make use of the following lemma for which, although it is well-known, we provide a proof.

**Lemma 7.4.1.** *Let  $f$  belong to  $\mathcal{H}_F$  and let  $u$  be a vector of length one in the space on which the absolutely cuspidal unitary representation  $\pi$  acts. Then*

$$\mathrm{Tr} \pi(f) = d(\pi) \int_{Z_F \backslash G_F} \left\{ \int_{G_F} f(h) (\pi(g^{-1}hg)u, u) dh \right\} dg$$

if  $d(\pi)$  is the formal degree of  $\pi$ .



Let  $Q$  be the operator

$$\pi(f) = \int_{G_F} f(h)\pi(h) dh.$$

Let  $\{v_i\}$  be an orthonormal basis of the space on which  $\pi$  acts. All but a finite number of the coefficients

$$Q_{ij} = (Qv_i, v_j)$$

are zero. We have

$$(\pi(g^{-1})Q\pi(g)u, u) = (Q\pi(g)u, \pi(g)u)$$

The right side equals

$$\sum_i (Q\pi(g)u, v_i)(v_i, \pi(g)u)$$

which in turn equals

$$\sum_i \sum_j (\pi(g)u, v_j)Q_{ji}(v_i, \pi(g)u)$$

In both series there are only a finite number of non-zero terms. Thus

$$\int_{Z_F \backslash G_F} (\pi(g^{-1})Q\pi(g)u, u) dg = \sum_{i,j} Q_{ji} \int_{Z_F \backslash G_F} (\pi(g)u, v_j)(v_i, \pi(g)u) dg$$

The integrals on the right exist because the representation is square-integrable. Applying the Schur orthogonality relations we see that the right side is equal to

$$\frac{1}{d(\pi)} \sum_{i,j} Q_{ij}(v_i, v_j) = \frac{1}{d(\pi)} \sum_i Q_{ii} = \frac{1}{d(\pi)} \text{Tr } \pi(f).$$

Since

$$(\pi(g^{-1})Q\pi(g)u, u) = \int_{G_F} f(h)(\pi(g^{-1})\pi(h)\pi(g)u, u) dh$$

the lemma follows.

Observe that the integral of the lemma is an iterated and not a double integral. It is the limit as  $r$  approaches infinity of

$$\int_{T'_r} \left\{ \int_{G_F} f(h)(\pi(g^{-1}hg)u, u) dh \right\} dg$$

Since  $T'_r$  is compact this integral is absolutely convergent and equals

$$\int_{G_F} f(h) \left\{ \int_{T'_r} (\pi(g^{-1}hg)u, u) dg \right\} dh.$$

To prove the first part of the proposition we show that the sequence of functions

$$\varphi_r(h) = \int_{T'_r} (\pi(g^{-1}hg)u, u) dg$$

is dominated locally by a multiple of  $\xi$  and converges almost everywhere on  $G_F$ . We shall set

$$\chi_\pi(h) = d(\pi) \lim_{r \rightarrow \infty} \varphi_r(h)$$

whenever the limit exists.

When proving the second part of the proposition we shall make use of the following lemma.

**Lemma 7.4.2.** *Let  $C_1$  be a compact subset of  $\tilde{G}_F$  and let  $C_2$  be a compact set in  $G_F$ . The image in  $Z_F \backslash G_F$  of*

$$\{ g \in G_F \mid g^{-1}C_1g \cap Z_FC_2 \neq \emptyset \}$$

*is compact.*

The set is clearly closed so we have only to show that it is contained in some compact set. We may suppose that  $\mathrm{GL}(2, O_F)C_2\mathrm{GL}(2, O_F) = C_2$ . Let

$$g = \begin{pmatrix} \alpha & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix} h$$

with  $h$  in  $\mathrm{GL}(2, O_F)$ . Then

$$g^{-1}C_1g \cap Z_FC_2 \neq \emptyset$$

if and only if

$$\begin{pmatrix} \alpha & x \\ 0 & 1 \end{pmatrix}^{-1} C_1 \begin{pmatrix} \alpha & x \\ 0 & 1 \end{pmatrix} \cap Z_FC_2 \neq \emptyset.$$

We have to show that this condition forces  $\alpha$  to lie in a compact subset of  $F^\times$  and  $x$  to lie in a compact subset of  $F$ . Since

$$\det(g^{-1}cg) = \det c$$

we may replace  $Z_FC_2$  by the compact set

$$C_3 = \{ h \in Z_FC_2 \mid \det h \in \det C_1 \}.$$

Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a typical element of  $C_1$ . The entry  $c$  is never 0 on  $C_1$  and therefore its absolute value is bounded below,

$$\begin{pmatrix} \alpha & x \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a - xc & y \\ c\alpha & cx + d \end{pmatrix}.$$

The number  $y$  is of no interest. The matrix on the right cannot lie in  $C_3$  unless  $|cx + d|$  is bounded above by some number depending on  $C_3$ . Since  $|d|$  is bounded above and  $|c|$  is bounded below  $x$  is forced to lie in some compact set  $\Omega$  of  $F$ . If  $C_4$  is the compact set

$$\left\{ \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} h \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \Omega, h \in C_1 \right\}$$

we have finally to show that if

$$\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix} C_4 \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \cap C_3 \neq \emptyset$$

then  $\alpha$  is forced to lie in a compact subset of  $F^\times$ . We now let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a typical element of  $C_4$ . On  $C_4$  both  $|b|$  and  $|c|$  are bounded below. Since

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b/\alpha \\ c\alpha & d \end{pmatrix}$$

and all matrix entries are bounded above in absolute value on  $C_3$  the absolute value  $|\alpha|$  must indeed be bounded above and below.

If  $\pi$  acts on  $V$  then for any  $u$  in  $V$  the support of the function  $(\pi(g)u, u)$  has been shown, in the second paragraph during the proof of proposition 2.20, to be compact modulo  $Z_F$ . Let  $C$  be its compact image in  $Z_F \backslash G_F$ . Let  $C_1$  be a compact subset of  $G_F$ . By the previous lemma the set of  $g$  in  $G_F$  such that

$$(\pi(g^{-1}hg)u, u) \neq 0$$

for some  $h$  in  $C_1$  has an image in  $Z_F \backslash G_F$  which is contained in a compact set  $C_2$ . Therefore the integral

$$\int_{Z_F \backslash G_F} (\pi(g^{-1}hg)u, u) dg = \int_{C_2} (\pi(g^{-1}hg)u, u) dg$$

is convergent for  $h$  in  $C_1$ . Moreover if  $r$  is large enough  $T'_r$  contains  $C_2$  and

$$\varphi_r(h) = \int_{Z_F \backslash G_F} (\pi(g^{-1}hg)u, u) dg.$$

Therefore the sequence  $\{\varphi_r\}$  converges uniformly on any compact subset of  $\tilde{G}_F$  and its limit  $d^{-1}(\pi)\chi_\pi(h)$  is continuous on  $\tilde{G}_F$ . We may state the following proposition.

**Proposition 7.5.** *If  $h$  belongs to  $\tilde{G}_F$  then*

$$\int_{Z_F \backslash G_F} (\pi(g^{-1}hg)u, u) dg$$

*exists and is equal to  $d^{-1}(\pi)\chi_\pi(h)$ .*

Since

$$\left| (\pi(g)u, u) \right| \leq \chi_C(g)$$

it follows from Lemma 7.3.2 that, for some constant  $c$ ,

$$|\varphi_r(h)| \leq c\xi(h)$$

on  $\tilde{G}_F$ . The set  $\hat{G}_F - \tilde{G}_F$  is  $\hat{A}_F^G$  which is open. To complete the proof of Proposition 7.4 we show that on the intersection of  $\hat{A}_F^G$  with a compact subset of  $G_F$  the sequence  $\{\varphi_r\}$  is dominated by a multiple of  $\xi$  and that it converges uniformly in a compact subset of  $\hat{A}_F^G$ .

Let  $C_3$  be a compact subset of  $G_F$ . Any  $h$  in  $\hat{A}_F^G$  may be written in the form

$$h = h_1^{-1} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} h_1$$

where  $h_1$  belongs to  $\text{GL}(2, O_F)$  and  $\alpha \neq \beta$ . In  $C_3 \cap \hat{A}_F^G$  the absolute values of  $\alpha$  and  $\beta$  are bounded above and below. If  $C_3$  is contained in  $\hat{A}_F^G$  the absolute value of  $1 - \frac{\beta}{\alpha}$  is also bounded above and below on  $C_3$ . Since

$$\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & \left(1 - \frac{\beta}{\alpha}\right)x \\ 0 & 1 \end{pmatrix}$$

the absolute value of  $x$  will be bounded above.

Since  $\mathrm{GL}(2, O_F)T_r\mathrm{GL}(2, O_F) = T_r$  the integral which defines  $\varphi_r(h)$  is equal to

$$\int_{T'_r} (\pi(g^{-1}h'g)u, u) dg$$

if

$$h' = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

and we may as well assume that  $h$  itself is of this form. We are going to show that there is a constant  $c$  such that

$$|\varphi_r(h)| \leq c \left| 1 - \frac{\beta}{\alpha} \right|^{-1}$$

for all  $r$  and all such  $h$  and that the sequence  $\{\varphi_r\}$  converges uniformly if  $x$  remains in a compact subset of  $F$  and  $\alpha, \beta$  and  $1 - \frac{\beta}{\alpha}$  remain in a compact subset of  $F^\times$ . Then the proof of the proposition will be complete.

The stabilizer of  $u$  is some open subgroup  $U$  of  $\mathrm{GL}(2, O_F)$ . Let  $h_1, \dots, h_p$  be a set of coset representatives for  $\mathrm{GL}(2, O_F)/U$  and let  $u_i = \pi(h_i)u$ . Apart from an unimportant factor coming from the Haar measure  $\varphi_r(h)$  is given by

$$\sum_{i=1}^p \varphi_r^i(h)$$

with

$$\varphi_r^i(h) = \int \left( \pi \left( \begin{pmatrix} \gamma & \gamma x_1 \\ 0 & 1 \end{pmatrix} h \begin{pmatrix} \gamma & \gamma x_1 \\ 0 & 1 \end{pmatrix} \right) u_i, u_i \right) dx_1 d^\times \gamma.$$

The integral is taken over the set of all those  $\gamma$  and  $x_1$  for which

$$\begin{pmatrix} \gamma & \gamma x_1 \\ 0 & 1 \end{pmatrix}$$

belongs to  $T_r$ . Since

$$\begin{pmatrix} \gamma & \gamma x_1 \\ 0 & 1 \end{pmatrix}^{-1} h \begin{pmatrix} \gamma & \gamma x_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & (1 - \frac{\beta}{\alpha})(\gamma^{-1}x + x_1) \\ 0 & 1 \end{pmatrix}$$

we can change variables in the integral to obtain

$$(7.4.3) \quad \left| 1 - \frac{\beta}{\alpha} \right|^{-1} \int \left( \pi \left( \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & \gamma^{-1}(1 - \frac{\beta}{\alpha})x + x_1 \\ 0 & 1 \end{pmatrix} \right) u_i, u_i \right) dx_1 d^\times \gamma.$$

The integration is now taken over all those  $x_1$  and  $\gamma$  for which

$$(7.4.4) \quad \begin{pmatrix} \gamma & \gamma \left(1 - \frac{\beta}{\alpha}\right)^{-1} x_1 \\ 0 & 1 \end{pmatrix}$$

is in  $T_r$ .

Let  $\left|1 - \frac{\beta}{\alpha}\right| = |\varpi_F|^t$ ,  $|\gamma| = |\varpi_F|^m$ , and  $|x| = |\varpi_F|^n$ . Let  $\varpi_F^p$  and  $\varpi_F^q$  be the elementary divisors of the matrix (7.4.4). We now list the possibilities for  $p$  and  $q$  together with the condition that the matrix belong to  $T_r$ , that is that  $q - p$  be at most  $r$ .

- (i)  $m \geq 0$ ,  $-t + m + n \geq 0$ ,  $p = 0$ ,  $q = m : 0 \leq m \leq r$
- (ii)  $m \geq 0$ ,  $-t + m + n \leq 0$ ,  $p = -t + m + n$ ,  $q = n - t : -r \leq m + 2n - 2t$
- (iii)  $m \leq 0$ ,  $-t + m + n \leq m$ ,  $p = -t + m + n$ ,  $q = n - t : -r \leq m + 2n - 2t$
- (iv)  $m \leq 0$ ,  $-t + m + n \geq m$ ,  $p = m$ ,  $q = 0 : -r \leq m \leq 0$ .

These conditions amount to the demand that  $-r \leq m \leq r$  and that  $2n \geq 2t - r - m$ . On the other hand we know that there is an integer  $s$  such that

$$\int_{|x| \leq |\varpi_F|^j} \pi \left( \begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} \right) u_i dx = 0$$

for  $1 \leq i \leq p$  if  $j \leq s$ .

Thus if  $|\gamma| = |\varpi_F|^m$  the integral

$$\int \left( \pi \left( \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & \gamma^{-1} \left(1 - \frac{\beta}{\alpha}\right)^{x+x_1} \\ 0 & 1 \end{pmatrix} \right) u_i, u_i \right) dx_1$$

taken over all  $x_1$  for which

$$\begin{pmatrix} \gamma & \gamma \left(1 - \frac{\beta}{\alpha}\right) x_1 \\ 0 & 1 \end{pmatrix}$$

is in  $T_r$  is zero if  $2t - r - m \leq 2s$ . Therefore in (7.4.3) we need only take the integral over those  $\gamma$  and  $x$  for which  $|\gamma| = |\varpi_F|^m$  with  $0 \leq m + r \leq 2(t - s)$  and  $|x| \leq |\varpi_F|^{t - \frac{m+r}{2}}$ . We should also have  $m \leq r$  but since we are about to replace the integrand by its absolute value that does not matter. For each such  $\gamma$  the integration with respect to  $x$  gives a result which is bounded in absolute value by a constant times  $|\varpi_F|^{t - \frac{m+r}{2}}$ . Integrating with respect to  $\gamma$  we obtain a result which is bounded in absolute value by a constant times

$$|\varpi_F|^t \sum_{k=0}^{2(t-s)-1} |\varpi_F|^{-k/2} \leq |\varpi_F|^s \sum_{k=0}^{\infty} |\varpi_F|^{k/2}$$

The right side depends on neither  $r$  nor  $t$ .

The value of  $\left|1 - \frac{\beta}{\alpha}\right| \varphi_r^i(h)$  is

$$\int \left( \pi \left( \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \right) \pi \left( \begin{pmatrix} 1 & \varpi_F^r \gamma^{-1} \left(1 - \frac{\beta}{\alpha}\right) x + x_1 \\ 0 & 1 \end{pmatrix} \right) u_i, u_i \right) dx_1 d^\times \gamma.$$

The integration is taken over those  $\gamma$  and  $x_1$  for which  $|\gamma| = |\varpi_F|^m$  with  $0 \leq m < 2(t - s)$  and  $|x| \leq |\varpi_F|^{t - \frac{m}{2}}$ . Of course  $\left|1 - \frac{\beta}{\alpha}\right| = |\varpi_F|^t$ . Since we are now interested in a set of  $\alpha$  and  $\beta$  on which  $t$  takes only a finite number of values we may as well assume it is constant. Then the integral is taken over a fixed compact subset of  $F \times F^\times$ . The integrand converges

uniformly on this set uniformly in the  $\alpha$ ,  $\beta$  and  $x$  under consideration as  $r$  approaches infinity.

We have still to prove the existence of the character of a representation which is not absolutely cuspidal. Most of them are taken care of by the next proposition.

**Proposition 7.6.** *Let  $\mu_1$  and  $\mu_2$  be a pair of quasi-characters of  $F^\times$ . Let  $\chi_{\mu_1, \mu_2}$  be the function which is 0 on  $\widehat{G}_F \cap \widetilde{G}_F$ , undefined on the singular elements, and equal to*

$$\{\mu_1(\alpha)\mu_2(\beta) + \mu_2(\beta)\mu_1(\alpha)\} \left| \frac{\alpha\beta}{(\alpha - \beta)^2} \right|^{1/2}$$

*at an element of  $g$  of  $\widehat{A}_F^G$  with eigenvalues  $\alpha$  and  $\beta$ . Then  $\chi_{\mu_1, \mu_2}$  is continuous on  $\widehat{G}_F$  and is dominated in absolute value by some multiple of  $\xi$ . Moreover if  $\pi = \rho(\mu_1, \mu_2)$*

$$\text{Tr } \pi(f) = \int_{G_F} \chi_{\mu_1, \mu_2}(g) f(g) dg$$

*for all  $f$  in  $\mathcal{H}_F$ .*

Only the last assertion requires verification. Since the absolute value of  $\chi_{\mu_1, \mu_2}$  is bounded by a multiple of  $\xi$  the function  $\chi_{\mu_1, \mu_2}$  is locally integrable. Suppose  $f$  belongs to  $\mathcal{H}_F$ . When applied to the function  $\chi_{\mu_1, \mu_2} f$  the relation (7.2.1) shows that

$$(7.6.1) \quad \int_{G_F} \chi_{\mu_1, \mu_2}(g) f(g) dg$$

is equal to

$$\frac{1}{2} \int_{A_F} \delta(a) \left\{ \int_{A_F \setminus G_F} \chi_{\mu_1, \mu_2}(g^{-1}ag) f(g^{-1}ag) dg \right\} da.$$

Since  $\chi_{\mu_1, \mu_2}$  is a class function this may be written as

$$\frac{1}{2} \int_{A_F} \{\mu_1(\alpha)\mu_2(\beta) + \mu_2(\alpha)\mu_1(\beta)\} \left| \frac{(\alpha - \beta)^2}{\alpha\beta} \right|^{1/2} \left\{ \int_{A_F \setminus G_F} f\left(g^{-1} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} g\right) dg \right\} da$$

if

$$a = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Since  $a$  is conjugate to

$$\begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}$$

we have

$$\int_{A_F \setminus G_F} f\left(g^{-1} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} g\right) dg = \int_{A_F \setminus G_F} f\left(g^{-1} \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix} g\right) dg.$$

Thus (7.6.1) is equal to

$$(7.6.2) \quad \int_{A_F} \mu_1(\alpha)\mu_2(\beta) \left| \frac{(\alpha - \beta)^2}{\alpha\beta} \right|^{1/2} \left\{ \int_{A_F \setminus G_F} f\left(g^{-1} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} g\right) dg \right\} da.$$

As long as the measure on  $A_F \backslash G_F$  is the quotient of the measure on  $G_F$  by that on  $A_F$  the choice of Haar measure on  $A_F$  and  $G_F$  is not relevant. Thus we may write (7.6.2) as

$$\int_{A_F} \mu_1(\alpha) \mu_2(\beta) \left| \frac{(\alpha - \beta)^2}{\alpha\beta} \right|^{1/2} \left\{ f \left( k^{-1} n^{-1} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} n k \right) dk dn \right\} da.$$

The inner integral is taken over  $\mathrm{GL}(2, O_F) \times N_F$ . If

$$n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

then

$$n^{-1} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} n = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & (1 - \frac{\beta}{\alpha})x \\ 0 & 1 \end{pmatrix}$$

Changing variables in the last integral we obtain

$$(7.6.3) \quad \int_{A_F} \mu_1(\alpha) \mu_2(\beta) \left| \frac{\alpha}{\beta} \right|^{1/2} \left\{ \int f \left( k^{-1} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} n k \right) dk dn \right\} da.$$

To evaluate  $\mathrm{Tr} \pi(f)$  we observe that if  $\varphi$  belongs to  $\mathcal{B}(\mu_1, \mu_2)$  then, if  $k_1$  is in  $\mathrm{GL}(2, O_F)$

$$\pi(f) \varphi(k_1) = \int_{G_F} \varphi(k_1 g) f(g) dg.$$

Replacing  $g$  by  $k_1^{-1} g$  and writing the integral out in terms of the Haar measure we have chosen we obtain

$$\int_{\mathrm{GL}(2, O_F)} \varphi(k_2) \left\{ \int f \left( k_1^{-1} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} n k_2 \right) \mu_1(\alpha) \mu_2(\beta) \left| \frac{\alpha}{\beta} \right|^{1/2} da dn \right\} dk_2.$$

The inner integral is taken over  $A_F \times N_F$ . We have of course used the relation

$$\varphi \left( \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} n k_2 \right) = \mu_1(\alpha) \mu_2(\beta) \left| \frac{\alpha}{\beta} \right|^{1/2} \varphi(k_2).$$

If

$$K(k_1, k_2) = \int f \left( k_1^{-1} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} n k_2 \right) \mu_1(\alpha) \mu_2(\beta) \left| \frac{\alpha}{\beta} \right|^{1/2} da dn$$

then

$$\pi(f) \varphi(k_1) = \int_{\mathrm{GL}(2, O_F)} K(k_1, k_2) \varphi(k_2) dk_2.$$

$\mathcal{B}(\mu_1, \mu_2)$  may be regarded as a space of functions on  $\mathrm{GL}(2, O_F)$ . Then  $\pi(f)$  is the integral operator with kernel  $K(k_1, k_2)$ . It is easily seen that this operator, when allowed to act on the space of all  $\mathrm{GL}(2, O_F)$ -finite functions on  $\mathrm{GL}(2, O_F)$ , has range in  $\mathcal{B}(\mu_1, \mu_2)$ . Thus the trace of  $\pi(f)$  is the same as the trace of the integral operator which is of course

$$\int_{\mathrm{GL}(2, O_F)} K(k, k) dk.$$

When written out in full this integral becomes (7.6.3).

**Theorem 7.7.** *Let  $\pi$  be an irreducible admissible representation of  $\mathcal{H}_F$ . There is a function  $\chi_\pi$  which is continuous on  $G_F$  and locally bounded in absolute value of  $\widehat{G}_F$  by a multiple of  $\xi$  such that*

$$\mathrm{Tr} \pi(f) = \int_{G_F} \chi_\pi(g) f(g) dg$$

for all  $f$  in  $\mathcal{H}_F$ .

The theorem has only to be verified for the one-dimensional and the special representations. If  $\pi$  is a one-dimensional representation associated to the quasi-character  $\chi$  we may take  $\chi_\pi(g) = \chi(\det g)$ . The character  $\chi_\pi$  is locally bounded and therefore, by Lemma 7.3, locally bounded by a multiple of  $\xi$ .

Suppose  $\pi_1$ ,  $\pi_2$  and  $\pi_3$  are three admissible representations of  $F$  on the spaces  $V_1$ ,  $V_2$ , and  $V_3$  respectively. Suppose also that there is an exact sequence

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$$

of  $\mathcal{H}_F$ -modules. If  $f$  is in  $\mathcal{H}_F$  all the operators  $\pi_1(f)$ ,  $\pi_2(f)$  and  $\pi_3(f)$  are of finite rank so that

$$\mathrm{Tr} \pi_2(f) = \mathrm{Tr} \pi_1(f) + \mathrm{Tr} \pi_3(f).$$

Thus if  $\chi_{\pi_1}$  and  $\chi_{\pi_2}$  exist so does  $\chi_{\pi_3}$ . Applying this observation to  $\pi_3 = \sigma(\mu_1, \mu_2)$ ,  $\pi_2 = \rho(\mu_1, \mu_2)$ , and  $\pi_1 = \pi(\mu_1, \mu_2)$  we obtain the theorem.

If  $F$  is taken to be the real or complex field Theorem 7.7 is a special case of a general and difficult theorem of Harish-Chandra. The special case is proved rather easily however. In fact Proposition 7.6 is clearly valid for archimedean fields and Theorem 7.7 is clearly valid for archimedean fields if  $\pi$  is finite-dimensional. There remains only the special representations and these are taken care of as before.



### §8. Odds and ends

In this paragraph various facts which will be used in the discussion of the constant term in the Fourier expansion of an automorphic form are collected together. If  $H$  is a locally compact abelian group a continuous complex-valued function  $f$  on  $H$  will be called  $H$ -finite or simply finite if the space spanned by the translates of  $f$  is finite-dimensional.

Let  $H$  be a group of the form

$$H = H_0 \times \mathbf{Z}^m \times \mathbf{R}^n$$

where  $H_0$  is compact. We regard  $\mathbf{Z}^m \times \mathbf{R}^n$  as a subgroup of  $\mathbf{R}^{m+n}$ . The projection

$$\xi_i : h = (h_0, x_1, \dots, x_{m+n}) \rightarrow x_i$$

may be regarded as a function on  $H$  with values in  $\mathbf{R}$ . If  $p_1, \dots, p_{m+n}$  is a sequence of non-negative integers and  $\chi$  is a quasi-character we may introduce the function

$$\chi \prod_{i=1}^{m+n} \xi_i^{p_i}$$

on  $H$ .

**Lemma 8.1.** *For any sequence  $p_1, \dots, p_{m+n}$  and any quasi-character  $\chi$  the function*

$$\chi \prod_{i=1}^{m+n} \xi_i^{p_i}$$

*is continuous and finite. These functions form a basis of the space of continuous finite functions on  $H$ .*

If  $\chi$  is a fixed quasi-character of  $H$  and  $p$  is a non-negative integer let  $V(\chi, p)$  be the space spanned by the functions  $\chi \prod_{i=1}^{m+n} \xi_i^{p_i}$  with  $0 \leq p_i \leq p$ . Since it is finite-dimensional and invariant under translations the first assertion of the lemma is clear.

To show that these functions are linearly independent we shall use the following simple lemma.

**Lemma 8.1.1.** *Suppose  $E_1, \dots, E_r$  are  $r$  sets and  $\mathcal{F}_1, \dots, \mathcal{F}_r$  are linearly independent sets of complex-valued functions on  $E_1, \dots, E_r$  respectively. Let  $\mathcal{F}$  be the set of functions*

$$(x_1, \dots, x_r) \rightarrow f_1(x_1)f_2(x_2) \cdots f_r(x_r)$$

*on  $E_1 \times \cdots \times E_r$ . Here  $f_i$  belongs to  $\mathcal{F}_i$ . Then  $\mathcal{F}$  is also linearly independent.*

Any relation

$$\sum_{f_1, \dots, f_r} a(f_1, \dots, f_r) f_1(x_1) \cdots f_r(x_r) \equiv 0$$

leads to

$$\sum_{f_r} \left\{ \sum_{f_1, \dots, f_{r-1}} a(f_1, \dots, f_r) f_1(x_1) \cdots f_{r-1}(x_{r-1}) \right\} f_r(x_r) \equiv 0$$

As  $\mathcal{F}_r$  is linearly independent this implies that

$$\sum_{f_1, \dots, f_{r-1}} a(f_1, \dots, f_r) f_1(x_1) \cdots f_{r-1}(x_{r-1}) \equiv 0$$

and the lemma follows by induction.

To show that the functions  $\chi \prod_{i=1}^{m+n} \xi_i^{p_i}$  span the space of continuous finite functions we use another simple lemma.

**Lemma 8.1.2.** *Let  $H_1$  and  $H_2$  be two locally compact abelian groups and let  $H = H_1 \times H_2$ . Then every continuous finite function  $f$  on  $H$  is a finite linear combination of the form*

$$f(x, y) = \sum_i \lambda_i \varphi_i(x) \psi_i(y)$$

where the  $\varphi_i$  and  $\psi_i$  are continuous finite functions on  $H_1$  and  $H_2$  respectively.

Let  $V$  be any finite-dimensional space of continuous functions on  $H$ . We associate to any point  $\xi$  in  $H$  the linear functional  $f \rightarrow f(\xi)$  on  $V$ . Since no function but zero is annihilated by all these functionals we can choose  $\xi_1, \dots, \xi_p$  so that the corresponding functionals form a basis of the dual of  $V$ . Then we can choose a basis  $f_1, \dots, f_p$  of  $V$  so that  $f_i(\xi_j) = \delta_{ij}$ .

Now suppose  $V$  is invariant under translations. It could for example be the space spanned by the translates of a single finite continuous function. The space  $V_1$  of functions  $\varphi$  on  $H_1$  defined by  $\varphi(x) = f(x, 0)$  with  $f$  in  $V$  is finite-dimensional and translation invariant. Therefore the functions in it are finite and of course continuous. We define  $V_2$  in a similar manner. If  $f$  is in  $V$  the function  $h \rightarrow f(g + h)$  is, for any  $g$  in  $H$ , also in  $V$ . Thus

$$f(g + h) = \sum_i \lambda_i(g) f_i(h).$$

Since

$$\lambda_i(g) = f(g + \xi_i)$$

the function  $\lambda_i$  belongs to  $V$ . If  $\varphi_i(x) = \lambda_i(x, 0)$  and  $\psi_i(y) = f_i(0, y)$  then

$$f(x, y) = \sum_i \varphi_i(x) \psi_i(y)$$

as required.

These two lemmas show that we need prove the final assertions of Lemma 8.1 only for  $H$  compact,  $H = \mathbf{Z}$ , or  $H = \mathbf{R}$ .

Suppose  $H$  is compact. If we have a non-trivial relation

$$\sum_{i=1}^r a_i \chi_i(h) \equiv 0$$

we may replace  $h$  by  $g + h$  to obtain

$$\sum_{i=1}^r a_i \chi_i(g) \chi_i(h) \equiv 0.$$

If such a relation holds we must have  $r \geq 2$  and at least two coefficients say  $a_1$  and  $a_2$  must be different from zero. Choose  $g$  so that  $\chi_1(g) \neq \chi_2(g)$ . Multiplying the first relation by  $\chi_1(g)$  and subtracting the second relation from the result we obtain a relation

$$\sum_{i=2}^r b_i \chi_i(h) \equiv 0.$$

Since  $b_2 = \{\chi_1(g) - \chi_2(g)\} a_2$  the new relation is non-trivial. The independence of the quasi-characters can therefore be proved by induction on  $r$ .

To prove that when  $H$  is compact the quasi-characters span the space of finite continuous functions we have just to show that any finite-dimensional space  $V$  of continuous functions which is translation invariant is spanned by the quasi-characters it contains. Choose a basis  $\{f_i\}$  of  $V$  as before and let

$$\rho(g)f_i = \sum \lambda_{ij}(g)f_j.$$

We saw that the functions  $\lambda_{ij}(g)$  are continuous. Thus the action of  $H$  on  $V$  by right translations is continuous and  $V$  is the direct sum of one-dimensional translation invariant spaces. Each such space is easily seen to contain a character.

When applied to a locally compact abelian group the argument of the previous paragraph leads to weaker conclusions. We can then find subspaces  $V_1, \dots, V_r$  of  $V$  and quasi-characters  $\chi_1, \dots, \chi_r$  of  $H$  such that

$$V = \bigoplus_{i=1}^r V_i$$

and, for every  $h$  in  $H$ ,

$$\{\rho(h) - \chi_i(h)\}^{\dim V_i}$$

annihilates  $V_i$ . Now we want to take  $H$  equal to  $\mathbf{Z}$  or  $\mathbf{R}$ . Then  $H$  is not the union of a finite number of proper closed subgroups. Suppose  $\mu_1, \dots, \mu_s$  are quasi-characters of  $H$  and for every  $h$  in  $H$  the operator

$$(8.1.3) \quad \prod_{i=1}^s \{\rho(h) - \mu_i(h)\}$$

on  $V$  is singular. Then for every  $h$  in  $H$  there is an  $i$  and a  $j$  such that  $\mu_i(h) = \chi_j(h)$ . If

$$H_{ij} = \{h \mid \mu_i(h) = \chi_j(h)\}$$

then  $H_{ij}$  is a closed subgroup of  $H$ . Since the union of these closed subgroups is  $H$  there must be an  $i$  and a  $j$  such that  $H_{ij} = H$  and  $\mu_i = \chi_j$ . If the operator (8.1.3) were zero the same argument would show that for every  $j$  there is an  $i$  such that  $\mu_i = \chi_j$ .

If  $\mu$  is a quasi-character of  $H$ , now taken to be  $\mathbf{Z}$  or  $\mathbf{R}$ , we let  $V(\mu, p)$  be the space spanned by the functions  $\mu\xi^i$ , with  $0 \leq i \leq p$ . Here  $\xi$  is the coordinate function on  $H$ . It is clear that  $V(\mu, p)$  is annihilated by  $\{\rho(h) - \mu(h)\}^{p+1}$  for all  $h$  in  $H$ . Suppose  $\mu, \mu_1, \dots, \mu_s$  are distinct and

$$V = V(\mu, p) \sum_{i=1}^s V(\mu_i, p_i)$$

is not zero. Decomposing  $V$  as above we see that  $\chi_1, \dots, \chi_r$  must all be equal to  $\mu$  on one hand and on the other that every  $\mu_i$  is a  $\chi_j$ . This is a contradiction. Thus if there is any non-trivial relation at all between the functions  $\chi\xi^i$  where  $\chi$  is any quasi-character and  $i$  is a non-negative integer there is one of the form

$$\sum_{i=0}^p a_i \mu \xi^i = 0.$$

Since the polynomial  $\sum_{i=0}^p a_i \xi^i$  would then have an infinite number of zeros this is impossible.

To prove the functions  $\chi\xi^i$  span the space of finite continuous functions we have only to show that if  $\chi$  is a given quasi-character and  $V$  is a finite-dimensional space of continuous functions which is invariant under translations and annihilated by  $\{\rho(h) - \chi(h)\}^{\dim V}$  for

all  $h$  in  $H$  then every function in  $V$  is the product of  $\chi$  and a polynomial. Since we can always multiply the functions in  $V$  by  $\chi^{-1}$  we may as well suppose that  $\chi$  is trivial. We have only to observe that any function  $f$  annihilated by the operator  $\{\rho(h) - 1\}^n$  for all  $h$  in  $H$  is a polynomial of degree at most  $n$ . This is clear if  $n = 1$  so by induction we can assume that  $\rho(h)f - f$  is a polynomial  $\sum_{i=0}^{n-1} a_i(h)\xi^i$ . We can certainly find a polynomial  $f'$  of degree  $n$  such that

$$\rho(1)f' - f' = \sum_{i=0}^{n-1} a_i(1)\xi^i$$

and we may as well replace  $f$  by  $f - f'$ . The new  $f$  satisfies  $\rho(1)f = f$ . It is therefore bounded. Moreover  $\rho(h)f - f$  is a bounded polynomial function and therefore a constant  $c(h)$ .  $c(h)$  is a bounded function of  $h$  and satisfies  $c(h_1 + h_2) = c(h_1) + c(h_2)$ . It is therefore zero and the new  $f$  is a constant.

Lemma 8.1 is now completely proved. Although it is trivial it is important to the notes and we thought it best to provide a proof. We might as well prove Lemma 2.16.4 at the same time. Let  $B$  be the space of all functions  $f$  on  $\mathbf{Z}$  such that for some  $n_0$  depending on  $f$  we have  $f(n) = 0$  for  $n < n_0$ . Let  $A_0$  be the space of functions on  $\mathbf{Z}$  which vanish outside a finite set.  $\mathbf{Z}$  acts on  $B$  and on  $A_0$  by right translations and therefore it also acts on  $\overline{B} = B/A_0$ . In particular let  $D = \rho(1)$ . We have merely to show that if  $P$  is a polynomial with leading coefficient 1 then the null space of  $P(D)$  in  $\overline{B}$  is finite-dimensional. If

$$P(X) = \prod_{i=1}^r (X - \alpha_i)^{p_i}$$

the null space of  $P(D)$  is the direct sum of the null spaces of the operators  $(D - \alpha_i)^{p_i}$ . The null space of  $(D - \alpha)^p$  is the image in  $\overline{B}$  of the functions in  $B$  which are zero to the left of 0 and of the form

$$n \rightarrow \alpha^n Q(n)$$

to the right of 0.  $Q$  is a polynomial of degree at most  $p$ .

Lemma 8.1 is certainly applicable to the direct product of a finite number of copies of the multiplicative group of a local field  $F$ . If  $H = (F^\times)^n$  any finite continuous function on  $H$  is a linear combination of functions of the form

$$f(x_1, \dots, x_n) = \prod_{i=1}^n \left\{ \chi_i(x_i) (\log |x_i|_F)^{n_i} \right\}.$$

Let  $\mathcal{B} = \mathcal{B}_F$  be the space of continuous functions  $f$  on  $G_F$  which satisfy the following three conditions.

- (i)  $f$  is finite on the right under the standard maximal compact subgroup  $K$  of  $G_F$ .
- (ii)  $f$  is invariant on the left under  $N_F$ .
- (iii)  $f$  is  $A_F$ -finite on the left.

$\mathcal{B}_F$  is invariant under left translations by elements of  $A_F$ . If  $f$  is in  $\mathcal{B}_F$  let  $V$  be the finite-dimensional space generated by these left translates. Choose  $g_1, \dots, g_p$  in  $G_F$  so that the linear functions  $\varphi \rightarrow \varphi(g_i)$  are a basis of the dual of  $V$  and let  $f_1, \dots, f_p$  be the dual basis. If  $a$  is  $A_F$  we may write

$$f(a, g) = \sum_{i=1}^p \theta_i(a) f_i(g).$$

Then

$$\theta_i(a) = f(ag_i)$$

so that

$$\theta_i(ab) = \sum_{j=1}^p \theta_j(a) f_j(bg_i).$$

Thus the functions  $\theta_i$  are continuous and finite. We may write them in the form

$$\theta_i(a) = \sum c_{m,n,\mu,\nu}^i \mu(\alpha_1) \nu(\alpha_2) (\log|\alpha_1|)^m (\log|\alpha_2|)^n$$

if

$$a = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}.$$

The sum is over all quasi-characters  $\mu$  and  $\nu$  of  $F^\times$  and all non-negative integers  $m$  and  $n$ . Of course only a finite number of the coefficients  $c_{m,n,\mu,\nu}^i$  are different from zero.

We may replace  $\mu$  by  $\alpha_F^{1/2} \mu$  and  $\nu$  by  $\alpha_F^{-1/2} \nu$  in the sum. Thus if

$$f_{m,n,\mu,\nu} = \sum_{i=1}^p c_{m,n,\mu,\nu}^i f_i$$

we have

$$(8.2) \quad f(ag) = \left| \frac{\alpha_1}{\alpha_2} \right|^{1/2} \sum \mu(\alpha_1) \nu(\alpha_2) (\log|\alpha_1|)^m (\log|\alpha_2|)^n f_{m,n,\mu,\nu}(g).$$

Let  $M$  be a non-negative integer and  $S$  a finite set of pairs of quasi-characters of  $F^\times$ . The set  $\mathcal{B}(S, M)$  will be the collection of  $f$  in  $\mathcal{B}$  for which the sum in (8.2) need only be taken over those  $m, n, \mu, \nu$  for which  $m+n \leq M$  and  $(\mu, \nu)$  belong to  $S$ . Observe that the functions  $f_{m,n,\mu,\nu}$  are determined by  $f$ .  $\mathcal{B}$  is the union of the spaces  $\mathcal{B}(S, M)$ ; if  $S$  consists of the single pair  $(\mu_1, \mu_2)$  we write  $\mathcal{B}(\mu_1, \mu_2, M)$  instead of  $\mathcal{B}(S, M)$ . If  $f$  is in  $(\mu_1, \mu_2, M)$

$$f(ag) = \left| \frac{\alpha_1}{\alpha_2} \right|^{1/2} \mu_1(\alpha_1) \mu_2(\alpha_2) \sum (\log|\alpha_1|)^m (\log|\alpha_2|)^n f_{m,n}(g).$$

The space  $\mathcal{B}(\mu_1, \mu_2, 0)$  is just  $\mathcal{B}(\mu_1, \mu_2)$ .

The functions  $f_{m,n,\mu,\nu}$  are uniquely determined and by their construction belong to the space spanned by left translates of  $f$  by elements of  $A_F$ . Thus if  $f$  belongs to  $\mathcal{B}(S, M)$  so do the functions  $f_{m,n,\mu,\nu}$ . We want to verify that  $f_{0,0,\mu,\nu}$  belongs to  $\mathcal{B}(\mu, \nu, M)$ . If

$$b = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}$$

and we replace  $a$  by  $ab$  in the relation (8.2) we find that

$$\left| \frac{\alpha_1}{\alpha_2} \right|^{1/2} \sum \mu(\alpha_1) \nu(\alpha_2) (\log|\alpha_1|)^m (\log|\alpha_2|)^n f_{m,n,\mu,\nu}$$

is equal to

$$\left| \frac{\alpha_1 \beta_1}{\alpha_2 \beta_2} \right|^{1/2} \sum \mu(\alpha_1 \beta_1) \nu(\alpha_2 \beta_2) (\log|\alpha_1| + \log|\beta_1|)^m (\log|\alpha_2| + \log|\beta_2|)^n f_{m,n,\mu,\nu}(g).$$

Fix  $b$  and  $g$  and regard this equality as an identity in the variable  $a$ . Because of Lemma 8.1 we can compare the coefficients of the basic finite functions. The coefficient of  $\mu(\alpha_1)\nu(\alpha_2)$  on one side is  $f_{0,0,\mu,\nu}(bg)$ . On the other it is

$$\left| \frac{\beta_1}{\beta_2} \right|^{1/2} \sum_{m+n \leq M} \mu(\beta_1)\nu(\beta_2)(\log|\beta_1|)^m (\log|\beta_2|)^n f_{m,n,\mu,\nu}(g).$$

The resulting identity is the one we wanted to verify.

Taking  $a = 1$  in (8.2) we see that

$$f(g) = \sum_{(\mu,\nu) \in S} f_{0,0,\mu,\nu}(g).$$

Therefore

$$\mathcal{B}(S, M) = \sum_{(\mu,\nu) \in S} \mathcal{B}(\mu, \nu, M).$$

The sum is direct.

It is fortunately possible to give a simple characterization of  $\mathcal{B}$ .

**Proposition 8.3.** *Let  $\varphi$  be a continuous function on  $G_F$ . Assume  $\varphi$  is  $K$ -finite on the right and invariant under  $N_F$  on the left. Then  $\varphi$  belongs to  $\mathcal{B}$  if and only if the space*

$$\{ \rho(\xi f)\varphi \mid f \in \mathcal{H}_F \}$$

*is finite-dimensional for every elementary idempotent in  $\mathcal{H}_F$ .*

We have first to show that if  $\varphi$  belongs to  $\mathcal{B}$

$$\{ \rho(\xi f)\varphi \mid f \in \mathcal{H}_F \}$$

is finite-dimensional. Certainly  $\varphi$  belongs to some  $\mathcal{B}(S, M)$ . Both  $\mathcal{B}$  and  $\mathcal{B}(S, M)$  are invariant under right translations by elements of  $\mathcal{H}_F$ . Thus we have only to show that the range of  $\rho(\xi)$  as an operator on  $\mathcal{B}(S, M)$  is finite-dimensional. This is tantamount to showing that any irreducible representation of  $K$  occurs with finite multiplicity in the representation of  $\mathcal{B}(S, M)$ .

Let  $\sigma$  be such a representation and let  $V$  be the space of continuous functions on  $K$  which transform according to  $\sigma$  under right translations.  $V$  is finite-dimensional. If  $f$  is in  $\mathcal{B}(S, M)$  we may write

$$f(ag) \left| \frac{\alpha_1}{\alpha_2} \right|^{1/2} \sum \mu(\alpha_1)\nu(\alpha_2)(\log|\alpha_1|)^m (\log|\alpha_2|)^n f_{m,n,\mu,\nu}(g)$$

The restriction of  $f_{m,n,\mu,\nu}$  to  $K$  lies in  $V$ . Call this restriction  $\bar{f}_{m,n,\mu,\nu}$ . Moreover  $f$  is determined by its restriction to  $A_F K$ . Thus

$$f \rightarrow \bigoplus_{\substack{(\mu,\nu) \in S \\ m+n \leq M}} \bar{f}_{m,n,\mu,\nu}$$

is an injection of the space of functions under consideration into the direct sum of a finite number of copies of  $V$ .

The converse is more complicated. Suppose  $\varphi$  is  $K$ -finite on the right, invariant under  $N_F$  on the left, and the space

$$\{ \rho(\xi f)\varphi \mid f \in \mathcal{H}_F \}$$

is finite-dimensional for every elementary idempotent  $\xi$ . Choose  $\xi$  so that  $\rho(\xi)\varphi = \varphi$ . There is actually a function  $f$  in  $\xi\mathcal{H}_F\xi$  such that  $\rho(f)\varphi = \varphi$ . If  $F$  is non-archimedean  $\xi$  is itself a function so this is clear. If  $F$  is archimedean we observe that if  $f_1$  is an approximation to the  $\delta$ -function then  $\rho(f_1)\varphi$  is close to  $\varphi$ . Then if  $f'_1 = \xi * f_1 * \xi$  the function  $f'_1$  is in  $\xi\mathcal{H}_F\xi$  and  $\rho(f'_1)\varphi$  is also close to  $\varphi$ . The existence of  $f$  then follows from the fact that  $\rho(\xi\mathcal{H}_F\xi)\varphi$  is finite-dimensional. This argument was used before in Paragraph 5.

Take  $F$  to be archimedean. Then  $\varphi$  must be an infinitely differentiable function on  $G_F$ . Let  $\mathfrak{Z}$  be the centre of the universal enveloping algebra of the Lie algebra of  $G_F$ . If  $Z$  is in  $\mathfrak{Z}$  then

$$\rho(Z)\varphi = \rho(Z)\rho(f)\varphi = \rho(Z * f)\varphi$$

and  $Z * f$  is still in  $\xi\mathcal{H}_F\xi$ . Thus  $\varphi$  is also  $\mathfrak{Z}$ -finite. For the rest of the proof in the archimedean case we refer to Chapter I of [11].

Now take  $F$  non-archimedean. We may replace  $\xi$  by any elementary idempotent  $\xi'$  for which  $\xi'\xi = \xi$ . In particular if we choose  $n$  to be a sufficiently large positive integer and let  $K'$  be the elements of  $K$  which are congruent to the identity modulo  $\mathfrak{p}^n$  we may take

$$\xi = \sum \xi_i$$

where the sum is over all elementary idempotents corresponding to irreducible representations of  $K$  whose kernel contains  $K'$ . Notice that  $n$  is at least 1. Then  $\xi\mathcal{H}_F\xi$  is the space of functions on  $G_F$  which are constant on double cosets of  $K'$ .

Let  $V$  be the space spanned by the functions  $\rho(k)\varphi$  with  $k$  in  $K$ . It is finite-dimensional and all the functions in  $V$  satisfy the same conditions as  $\varphi$ . Let  $\varphi_i$ ,  $1 \leq i \leq p$ , be a basis of  $V$ . If  $k$  belongs to  $K$  we may write

$$\varphi(gk) = \sum_{i=1}^p \theta_i(k) \varphi_i(g)$$

and  $\varphi$  is determined by the functions  $\theta_i$  and the restrictions of the functions  $\varphi_i$  to  $A_F$ . To show that  $\varphi$  is  $A_F$ -finite on the left we have merely to show that the restriction of each  $\varphi_i$  to  $A_F$  is finite. We may as well just show that the restriction of  $\varphi$  to  $A_F$  is finite.

Suppose  $f$  is in  $\xi\mathcal{H}_F\xi$  and  $\rho(f)\varphi = \varphi$ . If  $a$  is in  $Z_F$  then

$$\lambda(a)\varphi = \rho(a^{-1})\varphi = \rho(\delta_{a^{-1}} * f)\varphi$$

if  $\delta_{a^{-1}}$  is the  $\delta$ -function at  $a^{-1}$ . Since  $\delta_{a^{-1}} * f$  is still in  $\xi\mathcal{H}_F\xi$  the function  $\varphi$  is certainly  $Z_F$ -finite and so is its restriction  $\bar{\varphi}$  to  $A_F$ . If  $\alpha$  and  $\beta$  are units and  $\alpha \equiv \beta \equiv 1 \pmod{\mathfrak{p}^n}$  then

$$\lambda\left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\right)\bar{\varphi} = \bar{\varphi}.$$

Thus the translates of  $\bar{\varphi}$  by the elements of  $A_f \cap K$  span a finite-dimensional space and if  $\varpi$  is a generator of  $\mathfrak{p}$  we have only to show that the translates of  $\bar{\varphi}$  by the group

$$H = \left\{ \begin{pmatrix} \varpi p & 0 \\ 0 & 1 \end{pmatrix} \mid p \in \mathbf{Z} \right\}$$

span a finite-dimensional space. Suppose the span  $W$  of

$$\left\{ \lambda \left( \begin{pmatrix} \varpi^p & 0 \\ 0 & 1 \end{pmatrix} \right) \bar{\varphi} \mid p \leq 0 \right\}$$

is finite-dimensional. Then

$$\lambda \left( \begin{pmatrix} \varpi^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right)$$

maps  $W$  into itself and annihilates no vector but zero so that it has an inverse on  $W$  which must be

$$\lambda \left( \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix} \right).$$

Thus  $W$  is invariant under  $H$  and  $\bar{\varphi}$  is finite.

To show that  $W$  is finite-dimensional we show that if

$$a = \begin{pmatrix} \varpi^{-p} & 0 \\ 0 & 1 \end{pmatrix}$$

with  $p > 0$  there is a function  $f_a$  in  $\xi \mathcal{H}_F \xi$  such that

$$\lambda(a) \bar{\varphi} = \bar{\varphi}'$$

if  $\varphi' = \rho(f_a) \varphi$ . There is an  $f$  in  $\xi \mathcal{H}_F \xi$  such that

$$\varphi(g) = \int_{G_F} \varphi(gh) f(h) dh$$

for all  $g$  in  $G_F$ . Thus if  $b$  belongs to  $A_F$

$$\lambda(a) \bar{\varphi}(b) = \varphi(a^{-1}b) = \int_{G_F} \varphi(ba^{-1}h) f(h) dh.$$

If  $f_1(h) = f(ah)$  the integral is equal to

$$\int_{G_F} \varphi(bh) f_1(h) dh.$$

If  $f_1$  were in  $\xi \mathcal{H}_F \xi$  we would be done. Unfortunately this may not be so. However  $f_1(hk) = f_1(h)$  if  $k$  belongs to  $K'$ . If

$$k = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

then

$$f_1(kh) = f \left( \begin{pmatrix} \alpha & \varpi^{-p} \beta \\ \varpi^p \gamma & \delta \end{pmatrix} ah \right).$$

Thus  $f_1(kh) = f_1(h)$  if  $\alpha \equiv \delta \equiv 1 \pmod{\mathfrak{p}^n}$ ,  $\gamma \equiv 0 \pmod{\mathfrak{p}^n}$ , and  $\beta \equiv 0 \pmod{\mathfrak{p}^{n+p}}$ . Set

$$f_2(h) = \int_{\mathfrak{p}^n/\mathfrak{p}^{n+p}} f_1 \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} h \right) dx$$



where the Haar measure is so chosen that the measure of the underlying space  $\mathfrak{p}^n/\mathfrak{p}^{n+p}$  is 1. Since  $\varphi(bnh) = \varphi(bh)$  for all  $n$  in  $N_F$

$$\lambda(a)\overline{\varphi}(b) = \int_{G_F} \varphi(bh)f_2(h) dh.$$

We show that  $f_2$  lies in  $\xi\mathcal{H}_F\xi$ .

Certainly  $f_2(hk) = f_2(h)$  if  $k$  is in  $K'$ . Moreover, because of its construction,  $f_2(kh) = f_2(h)$  if

$$k = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$$

with  $\alpha \equiv \delta \equiv 1 \pmod{\mathfrak{p}^n}$  and  $\beta \equiv 0 \pmod{\mathfrak{p}^n}$ . Since every element of  $K'$  is a product

$$\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$$

where both terms lie in  $K'$  we have only to show that  $f_2$  is invariant under the first factor. If

$$k = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$$

with  $\gamma \equiv 0 \pmod{\mathfrak{p}^n}$  and

$$k_1(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1+x\gamma \end{pmatrix} \begin{pmatrix} \frac{1}{1+x\gamma} & 0 \\ \frac{-\gamma}{1+x\gamma} & 1 \end{pmatrix}$$

then

$$k_1(x) \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} k = \begin{pmatrix} 1 & \frac{x}{1+x\gamma} \\ 0 & 1 \end{pmatrix}.$$

Moreover if  $x$  is in  $O_F$

$$f_1(k_1(x)g) = f_1(g).$$

Thus  $f_2(kg)$  which is given by

$$\int_{\mathfrak{p}^n/\mathfrak{p}^{n+p}} f_1 \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} kh \right) dx$$

is equal to

$$\int_{\mathfrak{p}^n/\mathfrak{p}^{n+p}} f_1 \left( \begin{pmatrix} 1 & \frac{x}{1+x\gamma} \\ 0 & 1 \end{pmatrix} h \right) dx.$$

Since the map  $x \rightarrow \frac{x}{1+x\gamma}$  is a one-to-one map of the finite set  $\mathfrak{p}^n/\mathfrak{p}^{n+p}$  onto itself it is measure preserving and the above integral is equal to  $f_2(h)$ .

Analyzing the above proof one sees that in the non-archimedean case the left translates of  $\overline{\varphi}$  are contained in the space  $X$  obtained by restricting the functions in  $\rho(\xi\mathcal{H}_F\xi)\varphi$  to  $A_F$ . Thus if  $Y$  is the space of the functions on  $K/K'$  the left translates of  $\varphi$  by elements of  $A_F$  are contained in the space of functions on  $N_F \backslash G_F$  of the form

$$\varphi'(ak) = \sum \theta_i(k)\varphi_i(a)$$

with  $\theta_i$  in  $Y$  and  $\varphi_i$  in  $X$ .

In the archimedean case  $Y$  is the space of continuous functions  $\theta$  on  $K$  for which  $\theta * \xi = \xi * \theta = \theta$ . It is again finite-dimensional.  $X$  is defined in the same way. In this case there are a finite number of invariant differential operators  $D_1, \dots, D_r$  on  $A_F$  such that the left translates of  $\varphi$  by elements of  $A_F$  are contained in the space of functions  $N_F \backslash G_F$  of the form

$$\varphi'(ak) = \sum \theta_i(k) \varphi_i(a)$$

with  $\theta_i$  in  $Y$  and  $\varphi_i$  in  $\sum_{j=1}^r D_j X$ .

There is a corollary of these observations. Let  $F_1, \dots, F_n$  be a finite collection of local fields. Let  $G_i = G_{F_i}$ ,  $N_i = N_{F_i}$ ,  $A_i = A_{F_i}$ , and let  $K_i$  be the standard maximal compact subgroup of  $G_i$ . We set  $G = \prod_{i=1}^n G_i$ ,  $N = \prod_{i=1}^n N_i$  and so on. If  $\mathcal{H}_i = \mathcal{H}_{F_i}$  we let  $\mathcal{H} = \bigotimes_i \mathcal{H}_i$ . Then  $\mathcal{H}$  may be regarded as an algebra of measures on  $G$ .

**Corollary 8.4.** *Let  $\varphi$  be a continuous function on  $N \backslash G$  which is  $K$ -finite on the right. If for every elementary idempotent  $\xi$  in  $\mathcal{H}$  the space*

$$\{ \rho(\xi f) \varphi \mid f \in \mathcal{H} \}$$

*is finite-dimensional  $\varphi$  is  $A$ -finite on the left.*

If  $\varphi$  satisfies the conditions of the lemma so does any left translate by an element of  $A$ . Thus we need only show that  $\varphi$  is  $A_i$ -finite on the left for each  $i$ . If  $g$  is in  $G$  we write  $g = (g_i, \widehat{g}_i)$  where  $g_i$  is in  $G_i$  and  $\widehat{g}_i$  is in  $\widehat{G}_i = \prod_{j \neq i} G_j$ . We may suppose that there is a  $\xi'$  of the form  $\xi' = \bigotimes_i \xi'_i$  where  $\xi'_i$  is an elementary idempotent of  $\mathcal{H}_i$  such that  $\rho(\xi') \varphi = \varphi$ . By means of the imbedding  $f \rightarrow f \otimes \prod_{j \neq i} \xi'_j$  the algebra  $\mathcal{H}_i$  becomes a subalgebra of  $\mathcal{H}$ . The left translates of  $\varphi$  by  $A_i$  all lie in the space of functions of the form

$$\varphi(a_i k_i, \widehat{g}_1) = \sum_j \theta_j(k_i) \varphi_j(a_i, \widehat{g}_1)$$

where the  $\theta_j$  lie in a certain finite-dimensional space determined by  $\xi'_i$  and the  $\varphi_j$  lie in the space obtained by restricting the functions in  $\rho(\xi_i \mathcal{H}_i) \varphi$  to  $A_i \times \widehat{G}_i$  or, in the archimedean case, the space obtained from this space by applying certain invariant differential operators. Here  $\xi_i$  is a certain elementary idempotent which may be different from  $\xi'_i$ .

With the odds taken care of we come to the ends.

**Proposition 8.5.** *Let  $\mathcal{B}(\mu, \nu, \infty) = \bigcup_{M \geq 0} \mathcal{B}(\mu, \nu, M)$ . If an irreducible admissible representation  $\pi$  of  $\mathcal{H}_F$  is a constituent of the representation  $\rho(\mu, \nu, \infty)$  on  $\mathcal{B}(\mu, \nu, \infty)$  it is a constituent of  $\rho(\mu, \nu)$ .*

There are two invariant subspaces  $V_1$  and  $V_2$  of  $\mathcal{B}(\mu, \nu, \infty)$  such that  $V_1$  contains  $V_2$  and  $\pi$  is equivalent to the representation on  $\mathcal{H}_F$  on  $V_1/V_2$ . Choose  $M$  so that  $V_1 \cap \mathcal{B}(\mu, \nu, M)$  is not contained in  $V_2$ . Since  $\pi$  is irreducible

$$V_1 = V_2 + (V_1 \cap \mathcal{B}(\mu, \nu, M))$$

and

$$V_1/V_2 = \left\{ V_2 + (V_1 \cap \mathcal{B}(\mu, \nu, M)) \right\} / V_2$$

is isomorphic as an  $\mathcal{H}_F$  module to

$$V_1 \cap \mathcal{B}(\mu, \nu, M) / V_2 \cap \mathcal{B}(\mu, \nu, M)$$

so that we may as well suppose that  $V_1$  is contained in  $\mathcal{B}(\mu, \nu, M)$ .

Given  $\pi$  we choose  $M$  as small as possible. If  $M = 0$  there is nothing to prove so assume  $M$  is positive. If  $\varphi$  is in  $\mathcal{B}(\mu, \nu, M)$  we can express

$$\varphi \left( \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} g \right)$$

as

$$\left| \frac{\alpha_1}{\alpha_2} \right|^{1/2} \mu(\alpha_1) \nu(\alpha_2) \sum_{m+n \leq M} (\log |\alpha_1|)^m (\log |\alpha_2|)^n \varphi_{m,n}(g)$$

We can express

$$\varphi \left( \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix} g \right)$$

in two ways because the second factor can be absorbed into the first or the third. One way we obtain

$$\left| \frac{\alpha_1}{\alpha_2} \right|^{1/2} \mu(\alpha_1) \nu(\alpha_2) \sum_{m+n \leq M} (\log |\alpha_1|)^m (\log |\alpha_2|)^n \varphi_{m,n} \left( \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix} g \right)$$

and the other way we obtain

$$\left| \frac{\alpha_1 \beta_1}{\alpha_2 \beta_2} \right|^{1/2} \mu(\alpha_1 \beta_1) \nu(\alpha_2 \beta_2) \sum_{m+n \leq M} (\log |\alpha_1| + \log |\beta_1|)^m (\log |\alpha_2| + \log |\beta_2|)^n \varphi_{m,n}(g).$$

On comparing coefficients we see that if  $m + n = M$

$$\varphi_{m,n} \left( \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix} g \right) = \left| \frac{\beta_1}{\beta_2} \right|^{1/2} \mu(\beta_1) \nu(\beta_2) \varphi_{m,n}(g)$$

so that  $\varphi_{m,n}$  is in  $\mathcal{B}(\mu, \nu)$ . Consider the map

$$\varphi \rightarrow \bigoplus_{m+n=M} \varphi_{m,n}$$

of  $V_1$  into

$$\bigoplus_{m+n=M} \mathcal{B}(\mu, \nu).$$

Its kernel is  $V_1 \cap \mathcal{B}(\mu, \nu, M-1)$ . Since  $V_2 + (V_1 \cap \mathcal{B}(\mu, \nu, M-1))$  cannot be  $V_1$  the image of  $V_2$  is not the same as the image of  $V_1$ . Since the map clearly commutes with the action of  $\mathcal{H}_F$  the representation  $\pi$  is a constituent of  $\bigoplus_{m+n=M} \rho(\mu, \nu)$ .

Proposition 8.5 is now a consequence of the following simple lemma.

**Lemma 8.6.** *Suppose  $\pi$  is an irreducible representation of an algebra  $H$ . Suppose  $\rho$  is a representation of  $H$  of which  $\pi$  is a constituent and that  $\rho$  is the direct sum of the representations  $\rho_\lambda$ ,  $\lambda \in \Lambda$ . Then  $\pi$  is a constituent of at least one of the  $\rho_\lambda$ .*

Let  $\rho_\lambda$  act on  $X_\lambda$  and let  $\rho$  act on  $X$  the direct sum of  $X_\lambda$ . Suppose that  $Y_1$  and  $Y_2$  are invariant subspaces of  $X$  and that the representation on the quotient  $Y_1/Y_2$  is equivalent to  $\pi$ . There is a finite subset  $\Lambda_0$  of  $\Lambda$  such that

$$Y_1 \cap \left( \sum_{\lambda \in \Lambda_0} X_\lambda \right)$$

is not contained in  $Y_2$ . We may as well replace  $Y_1$  by  $Y_1 \cap \left(\sum_{\lambda \in \Lambda_0} X_\lambda\right)$  and  $Y_2$  by  $Y_2 \cap \left(\sum_{\lambda \in \Lambda_0} X_\lambda\right)$  and suppose that  $\Lambda$  is finite. If  $\Lambda = \{\lambda_1, \dots, \lambda_p\}$  we have only to show that  $\pi$  is a constituent of  $\rho_{\lambda_1}$  or of  $\rho_{\lambda_2} \oplus \dots \oplus \rho_{\lambda_p}$  for we can then use induction. Thus we may as well take  $p = 2$ . If the projections of  $Y_1$  and  $Y_2$  on  $X_{\lambda_1}$  are not equal we can replace  $Y_1$  and  $Y_2$  by these projections to see that  $\pi$  is a constituent of  $\rho_{\lambda_1}$ . If they are equal  $Y_1 = Y_2 + (Y_1 \cap X_{\lambda_2})$  and we can replace  $Y_1$  and  $Y_2$  by  $Y_1 \cap X_{\lambda_2}$  and  $Y_2 \cap X_{\lambda_2}$  to see that  $\pi$  is a constituent of  $\rho_{\lambda_2}$ .

## References for Chapter I

The Weil representation is constructed in:

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One of the first to study representations of groups over non-archimedean local fields was F. Mautner in:
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Absolutely cuspidal representations were first constructed by Gelfand and Graev. References to their work and that of Kirillov will be found in:
- [3] Gelfand, I. M., M. I. Graev, and I. I. Pyatetskii-Shapiro, *Representation Theory and Automorphic Functions*, W.B. Saunders Co., 1966.  
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- [6] Harish-Chandra, *Representations of semisimple Lie groups*, II, T.A.M.S., vol 76, 1954.  
Our discussion of characters owes much to:
- [7] Sally, P. J. and J. A. Shalika, *Characters of the discrete series of representations of  $SL(2)$  over a local field*, P.N.A.S., 1968.  
The following three books are standard references to the theory of  $L$ -functions are:
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In Paragraph 8 we have used a result from:
- [11] Harish-Chandra, *Automorphic forms on semisimple Lie groups*, Springer-Verlag, 1968.  
Tamagawa measures are discussed in:
- [12] Weil, A., *Adèles and algebraic groups*, Institute for Advanced Study, 1961.



## CHAPTER II

### Global Theory

#### §9. The global Hecke algebra

Let  $F$  be a global field, that is, an algebraic number field of finite degree over the rationals or a function field in one variable over a finite field.  $\mathbf{A}$  will be the adèle ring of  $F$ . Before studying the representations of  $\mathrm{GL}(2, \mathbf{A})$  or, more precisely, the representations of a suitable group algebra of  $\mathrm{GL}(2, \mathbf{A})$  we introduce some simple algebraic notions.

Let  $\{V_\lambda \mid \lambda \in \Lambda\}$  be a family of complex vector spaces. Suppose that for all but a finite number of  $\lambda$  we are given a non-zero vector  $e_\lambda$  in  $V_\lambda$ . Let  $V^0$  be the set of all  $x = \prod_\lambda x_\lambda$  in  $\prod_\lambda V_\lambda$  such that  $x_\lambda = e_\lambda$  for all but a finite number of  $\lambda$ . Let  $C$  be the free vector space with complex coefficients over  $V^0$  and let  $D$  be the subspace generated by vectors of the form

$$\left\{ (aY_\mu + bZ_\mu) \times \prod_{\lambda \neq \mu} x_\lambda \right\} - a \left\{ y_\mu \times \prod_{\lambda \neq \mu} x_\lambda \right\} - b \left\{ z_\mu \times \prod_{\lambda \neq \mu} x_\lambda \right\}.$$

$a$  and  $b$  belong to  $\mathbf{C}$  and  $\mu$  is any element of  $\Lambda$ . The quotient of  $C$  by  $D$  is called the tensor product of the  $V_\lambda$  with respect to the family  $e_\lambda$  and is written

$$V = \bigotimes_{e_\lambda} V_\lambda$$

or simply  $\bigotimes V_\lambda$ . It has an obvious universal property which characterizes it up to isomorphism. The image of  $\prod x_\lambda$  in  $V$  is written  $\bigotimes x_\lambda$ .

If  $\Lambda'$  is a subset of  $\Lambda$  with finite complement we may form the ordinary tensor product

$$\bigotimes_{\lambda \in \Lambda - \Lambda'} V_\lambda$$

and we may form

$$\bigotimes_{\lambda \in \Lambda'} V_\lambda$$

with respect to the family  $e_\lambda$ . Then  $\bigotimes_{\lambda \in \Lambda} V_\lambda$  is canonically isomorphic to

$$\left\{ \bigotimes_{\lambda \in \Lambda - \Lambda'} V_\lambda \right\} \otimes \left\{ \bigotimes_{\lambda \in \Lambda'} V_\lambda \right\}$$

If  $S$  is a finite subset of  $\Lambda$  let

$$V_S = \bigotimes_{\lambda \in S} V_\lambda$$

If  $S$  is so large that  $e_\lambda$  is defined for  $\lambda$  not in  $S$  let  $\varphi_S$  be the map of  $V_S$  into  $V$  which sends  $\bigotimes_{\lambda \in S} x_\lambda$  to  $\left\{ \bigotimes_{\lambda \in S} x_\lambda \right\} \otimes \left\{ \bigotimes_{\lambda \notin S} e_\lambda \right\}$ . If  $S'$  contains  $S$  there is a unique map  $\varphi_{S, S'}$  of  $V_S$  into  $V_{S'}$  which makes

$$\begin{array}{ccc}
V_S & \xrightarrow{\varphi_{S,S'}} & V'_S \\
& \searrow \varphi_S \quad \swarrow \varphi_{S'} & \\
& V &
\end{array}$$

commutative. If we use these maps to form the inductive limit of the spaces  $V_S$  we obtain a space which the layman is unable to distinguish from  $V$ .

Suppose that for every  $\lambda$  we are given a linear map  $B_\lambda$  of  $V_\lambda$  into itself. If  $B_\lambda e_\lambda = e_\lambda$  for all but a finite number of  $\lambda$  there is exactly one linear transformation  $B$  of  $\bigotimes V_\lambda$  such that

$$B : \bigotimes x_\lambda \rightarrow \bigotimes B_\lambda x_\lambda$$

$B$  is denoted by  $\bigotimes B_\lambda$ .

For example if  $A_\lambda$ ,  $\lambda \in \Lambda$  is a family of associative algebras, which may or may not have a unit, and if, for almost all  $\lambda$ ,  $\xi_\lambda$  is a given idempotent of  $A_\lambda$  one may turn

$$A = \bigotimes_{\xi_\lambda} A_\lambda$$

into an algebra in such a way that

$$\left( \bigotimes a_\lambda \right) \left( \bigotimes b_\lambda \right) = \bigotimes (a_\lambda b_\lambda).$$

Let  $V_\lambda$ ,  $\lambda \in \Lambda$ , be an  $A_\lambda$  module. If for almost all  $\lambda$  a vector  $e_\lambda$  such that  $\xi_\lambda e_\lambda = e_\lambda$  is given we may turn  $V = \bigotimes_{e_\lambda} V_\lambda$  into an  $A = \bigotimes_{\xi_\lambda} A_\lambda$  module in such a way that

$$\left( \bigotimes a_\lambda \right) \left( \bigotimes x_\lambda \right) = \bigotimes (a_\lambda x_\lambda)$$

Suppose the family  $\{e_\lambda\}$  is replaced by a family  $\{e'_\lambda\}$  but that, for all but a finite number of  $\lambda$ ,  $e'_\lambda = \alpha_\lambda e_\lambda$  where  $\alpha_\lambda$  is a non-zero scalar. Suppose for example that  $e'_\lambda = \alpha_\lambda e_\lambda$  if  $\lambda$  is not in the finite set  $S$ . There is a unique map of  $\bigotimes_{e_\lambda} V_\lambda$  to  $\bigotimes_{e'_\lambda} V_\lambda$  which sends

$$\left\{ \bigotimes_{\lambda \in S} x_\lambda \right\} \otimes \left\{ \bigotimes_{\lambda \notin S} x_\lambda \right\}$$

to

$$\left\{ \bigotimes_{\lambda \in S} x_\lambda \right\} \otimes \left\{ \bigotimes_{\lambda \notin S} \alpha_\lambda x_\lambda \right\}$$

It is invertible and commutes with the action of  $A$ . Moreover apart from a scalar factor it is independent of  $S$ .

Now suppose  $F$  is a global field. A place of  $F$  is an equivalence class of injections, with dense image, of  $F$  into a local field. If  $\lambda_1$  takes  $F$  into  $F_1$  and  $\lambda_2$  takes  $F$  into  $F_2$  they are equivalent if there is a topological isomorphism  $\varphi$  of  $F_1$  with  $F_2$  such that  $\lambda_2 = \varphi \circ \lambda_1$ . The symbol for a place will be  $v$ . If  $v$  contains the imbedding  $\lambda_1$  and  $a$  belongs to  $F$  we set  $|a|_v = |\lambda_1(a)|$ . To be definite we let  $F_v$  be the completion of  $F$  with respect to the absolute value  $a \rightarrow |a|_v$ . Where  $v$  is archimedean or non-archimedean according to the nature of  $F_v$ . Non-archimedean places will sometimes be denoted by  $\mathfrak{p}$ .

If  $G_F = \text{GL}(2, F)$  we set

$$G_v = G_{F_v} = \text{GL}(2, F_v).$$



The group  $K_v$  will be the standard maximal compact subgroup of  $G_v$ . Then  $G_{\mathbf{A}} = \mathrm{GL}(2, \mathbf{A})$  is the restricted direct product of the groups  $G_v$  with respect to the subgroups  $K_v$ .

If  $v$  is non-archimedean we set  $O_v = O_{F_v}$  and  $U_v = U_{F_v}$ .  $O_v$  is the ring of integers of  $F_v$  and  $U_v$  is the group of units of  $O_v$ . Suppose  $M'$  is a quaternion algebra over  $F$ . Let  $M'_v = M'_{F_v} = M' \otimes_F F_v$ . For almost all  $v$  the algebra  $M'_v$  is split, that is, there is an isomorphism

$$\theta_v : M'_v \rightarrow M(2, F_v)$$

where  $M(2, F_v)$  is the algebra of  $2 \times 2$  matrices over  $F_v$ . For every place  $v$  at which  $M'_v$  is split we want to fix such an isomorphism  $\theta_v$ . Let  $B$  be a basis of  $M$  over  $F$  and let  $L_v$  be the  $O_v$  module generated in  $M_v$  by  $B$ . We may and do choose  $\theta_v$  so that for almost all  $v$

$$\theta_v(L_v) = M(2, O_v).$$

If  $B'$  is another basis and  $\{\theta'_v\}$  a family of isomorphisms associated to  $B'$  then for every place  $v$  at which  $M'_v$  splits there is a  $g_v$  in  $\mathrm{GL}(2, F_v)$  such that

$$\theta'_v \theta_v^{-1} a = g_v a g_v^{-1}$$

for all  $a$  in  $M(2, F_v)$ . Moreover  $g_v$  belongs to  $K_v$  for all but a finite number of  $v$ .

Suppose the family of isomorphisms  $\theta_v$  has been chosen. If  $M'_v$  is split we define a maximal compact subgroup  $K'_v$  of  $G'_v$ , the group of invertible elements of  $M'_v$ , by the condition

$$\theta_v(K'_v) = K_v.$$

If  $M'_v$  is not split we set

$$K'_v = \left\{ x \in M'_v \mid |\nu(x)|_v = 1 \right\}.$$

This group is compact. In any case  $K'_v$  is defined for all  $v$ . Since many of the constructions to be made depend on the family  $K'_v$ , which in turn depends on the family of  $\theta_v$  it is very unfortunate that the family of  $\theta_v$  is not unique. We should really check at every stage of the discussion that the constructions are, apart from some kind of equivalence, independent of the initial choice of  $\theta_v$ . We prefer to pretend that the difficulty does not exist. As a matter of fact for anyone lucky enough not to have been indoctrinated in the functorial point of view it doesn't. We do however remark that any two choices of the family of  $K'_v$  lead to the same result for almost all  $v$ . The adelic group  $G'_{\mathbf{A}}$  is the restricted direct product of the groups  $G'_v$  with respect to the subgroups  $K'_v$ .

We have now to introduce the Hecke algebras  $\mathcal{H}$  and  $\mathcal{H}'$  of  $G_{\mathbf{A}}$  and  $G'_{\mathbf{A}}$ . Let  $\mathcal{H}_v$  be  $\mathcal{H}_{F_v}$ . If  $M'_v$  is split  $G'_v$  is isomorphic, by means of  $\theta_v$ , to  $G_v$  and we let  $\mathcal{H}'_v$  be the algebra of measures on  $G'_v$  corresponding to  $\mathcal{H}_v$ . Suppose  $M'_v$  is not split. If  $v$  is non-archimedean  $\mathcal{H}'_v$  is the algebra of measures defined by the locally constant compactly supported functions on  $G'_v$ . If  $v$  is archimedean  $\mathcal{H}'_v$  will be the sum of two subspaces, the space of measures defined by infinitely differentiable compactly supported functions on  $G'_v$  which are  $K'_v$ -finite on both sides and the space of measures on  $K'_v$  defined by the matrix coefficients of finite-dimensional representations of  $K'_v$ .

Let  $\epsilon_v$  and  $\epsilon'_v$  be the normalized Haar measures on  $K_v$  and  $K'_v$ . The measure  $\epsilon_v$  is an elementary idempotent of  $\mathcal{H}_v$  and  $\epsilon'_v$  is an elementary idempotent of  $\mathcal{H}'_v$ . We set

$$\mathcal{H} = \bigotimes_{\epsilon_v} \mathcal{H}_v$$

and

$$\mathcal{H}' = \bigotimes_{\epsilon'_v} \mathcal{H}'_v$$

If  $S$  is the finite set of places at which  $M'_v$  does not split we may write

$$\mathcal{H} = \left\{ \bigotimes_{v \in S} \mathcal{H}_v \right\} \otimes \left\{ \bigotimes_{v \notin S} \mathcal{H}_v \right\} = \mathcal{H}_S \otimes \widehat{\mathcal{H}}_S$$

and

$$\mathcal{H}' = \left\{ \bigotimes_{v \in S} \mathcal{H}'_v \right\} \otimes \left\{ \bigotimes_{v \notin S} \mathcal{H}'_v \right\} = \mathcal{H}'_S \otimes \widehat{\mathcal{H}}'_S$$

By construction, if  $M'_v$  is split,  $\mathcal{H}_v$  and  $\mathcal{H}'_v$  are isomorphic in such a way that  $\epsilon_v$  and  $\epsilon'_v$  correspond. Using these isomorphism we may construct an isomorphism of  $\widehat{\mathcal{H}}_S$  and  $\widehat{\mathcal{H}}'_S$ . We may also write

$$G_{\mathbf{A}} = \left\{ \prod_{v \in S} G_v \right\} \times \left\{ \prod_{v \notin S} G_v \right\} = G_S \times \widehat{G}_S$$

and

$$G'_{\mathbf{A}} = \left\{ \prod_{v \in S} G'_v \right\} \times \left\{ \prod_{v \notin S} G'_v \right\} = G'_S \times \widehat{G}'_S.$$

The second factor is in both cases a restricted direct product. There is an isomorphism  $\theta : \widehat{G}'_S \rightarrow \widehat{G}_S$  defined by

$$\theta \left( \prod_{v \notin S} g'_v \right) = \prod_{v \notin S} \theta_v(g'_v)$$

We will interpret  $\widehat{\mathcal{H}}_S$  and  $\widehat{\mathcal{H}}'_S$  as algebras of measures on  $\widehat{G}_S$  and  $\widehat{G}'_S$  and then the isomorphism between them will be that associated to  $\theta$ .

We can also interpret the elements of  $\mathcal{H}$  and  $\mathcal{H}'$  as measures on  $G_{\mathbf{A}}$  and  $G'_{\mathbf{A}}$ . For example any element of  $\mathcal{H}$  is a linear combination of elements of the form  $f = \bigotimes_v f_v$ . Let  $T$  be a finite set of places and suppose that  $f_v = \epsilon_v$  for  $v$  not in  $T$ . If  $T'$  contains  $T$ , on the group

$$G_{\mathbf{A}(T')} = \left\{ \prod_{v \in T'} G_v \right\} \times \left\{ \prod_{v \notin T'} K_v \right\}$$

we can introduce the product of the measures  $f_v$ . Since  $G_{\mathbf{A}}$  is the union of these groups and the measures on them are consistent we can put the measures together to form a measure  $f$  on  $G_{\mathbf{A}}$ . If each  $f_v$  is the measure associated to a function then  $f$  is also. Such measures form a subalgebra  $\mathcal{H}_1$  and  $\mathcal{H}$ .

The notion of an elementary idempotent of  $\mathcal{H}$  or  $\mathcal{H}'$  is defined in the obvious way. If  $\xi$  is an elementary idempotent of  $\mathcal{H}$  there is another elementary idempotent  $\xi_1$  of the form  $\xi_1 = \bigotimes_v \xi_v$  where  $\xi_v$  is an elementary idempotent of  $\mathcal{H}_v$  and  $\xi_v = \epsilon_v$  for almost all  $v$  so that  $\xi_1 \xi = \xi$ .

We shall now discuss the representations of  $\mathcal{H}$ . A representation  $\pi$  of  $\mathcal{H}$  on the vector space  $V$  over  $\mathbf{C}$  will be called admissible if the following conditions are satisfied

- (i) Every  $w$  in  $V$  is a linear combination of the form  $\sum \pi(f_i)w_i$  with  $f_i$  in  $\mathcal{H}_1$ .
- (ii) If  $\xi$  is an elementary idempotent the range of  $\pi(\xi)$  is finite-dimensional.
- (iii) Let  $v_0$  be an archimedean place. Suppose that for each  $v$  an elementary idempotent  $\xi_v$  is given and that  $\xi_v = \epsilon_v$  for almost all  $v$ . Let  $\xi = \bigotimes_v \xi_v$ . If  $w$  is in  $V$  the map

$$f_{v_0} \rightarrow \pi \left( f_{v_0} \otimes \left\{ \bigotimes_{v \neq v_0} \xi_v \right\} \right) w$$

of  $\xi_{v_0} \mathcal{H}_{v_0} \xi_{v_0}$  into the finite-dimensional space  $\pi(\xi)V$  is continuous.

Suppose that an admissible representation  $\pi_v$  of  $\mathcal{H}_v$  on  $V_v$  is given for each  $v$ . Assume that for almost all  $v$  the range of  $\pi_v(\epsilon_v)$  is not zero. Assume also that the range of  $\pi_v(\epsilon_v)$  has dimension one when it is not zero. As we saw in the first chapter this supplementary condition is satisfied if the representations  $\pi_v$  are irreducible. Choosing for almost all  $v$  a vector  $e_v$  such that  $\pi_v(\epsilon_v)e_v = e_v$  we may form  $V = \bigotimes_{e_v} V_v$ . Let  $\pi$  be the representation  $\bigotimes_v \pi_v$  on  $V$ . Because of the supplementary condition it is, apart from equivalence, independent of the choice of the  $e_v$ .

The representations  $\pi$  will be admissible. To see this observe first of all that condition (i) has only to be verified for vectors of the form  $w = \bigotimes_v w_v$ . Suppose  $w_v = e_v$  when  $v$  is not in the finite set  $T$  which we suppose contains all archimedean places. If  $v$  is not in  $T$  let  $f_v = \epsilon_v$  so that  $w_v = \pi(f_v)w_v$ . If  $v$  is in  $T$  let

$$w_v = \sum \pi_v(f_v^i)w_v^i.$$

Then

$$w = \left\{ \bigotimes_{v \in T} \sum \pi_v(f_v^i)w_v^i \right\} \otimes \left\{ \bigotimes_{v \notin T} \pi(f_v)w_v \right\}.$$

Expanding the right hand side we obtain the desired relation. The second condition has only to be verified for elementary idempotents of the form  $\xi = \bigotimes_v \xi_v$ . Then

$$\pi(\xi)V = \bigotimes_v \pi(\xi_v)V_v$$

Since  $\pi(\xi_v)V_v$  is finite-dimensional for all  $v$  and  $\pi(\xi_v)V_v = \pi(\epsilon_v)V_v$ , which has dimension one, for almost all  $v$  the right side is finite-dimensional. The last condition results from the admissibility of  $\pi_{v_0}$ .

Certainly  $\pi$  cannot be irreducible unless each  $\pi_v$  is. Suppose however that each  $\pi_v$  is irreducible. If  $\xi_v$  is an elementary idempotent of  $\mathcal{H}_v$  and if  $\pi_v(\xi_v) \neq 0$  we have a representation  $\pi_{\xi_v}$  of  $\xi_v \mathcal{H}_v \xi_v$  on  $\pi_v(\xi_v)V_v$ . Since it is irreducible  $\pi_{\xi_v}$  determines a surjective map

$$\pi_{\xi_v} : \xi_v \mathcal{H}_v \xi_v \rightarrow L(\xi_v)$$

if  $L(\xi_v)$  is the ring of linear transformations of  $V(\xi_v) = \pi_v(\xi_v)V_v$ . To show that  $\pi$  is irreducible we have only to show that for every elementary idempotent of the form  $\xi = \bigotimes_v \xi_v$  the representation of  $\xi \mathcal{H} \xi$  on  $V(\xi) = \pi(\xi)V$  is irreducible. Suppose that  $\xi_v = \epsilon_v$  if  $v$  is not in  $T$ .

Then

$$V(\xi) = \bigotimes_v V(\xi_v)$$

is isomorphic to  $\bigotimes_{v \in T} V(\xi_v)$ . The full ring of linear transformations of this space is

$$\bigotimes_{v \in T} L(\xi_v)$$

and therefore the full ring of linear transformations of  $V(\xi)$  is

$$\left\{ \bigotimes_{v \in T} L(\xi_v) \right\} \otimes \left\{ \bigotimes_{v \notin T} \pi_v(\epsilon_v) \right\}.$$

This is the image under  $\pi$  of

$$\left\{ \bigotimes_{v \in T} \xi_v \mathcal{H}_v \xi_v \right\} \otimes \left\{ \bigotimes_{v \notin T} \epsilon_v \right\}$$

which is contained in  $\xi \mathcal{H} \xi$ .

An admissible representation equivalent to one constructed by tensor products is said to be factorizable.

**Proposition 9.1.** *Every irreducible admissible representation of  $\mathcal{H}$  is factorizable. The factors are unique up to equivalence.*

Suppose  $\pi$  is such a representation. Let  $I$  be the set of elementary idempotents of the form  $\xi = \bigotimes \xi_v$  for which  $\pi(\xi)$  is not 0.  $I$  is certainly not empty. Let  $V(\xi) = \pi(\xi)V$  if  $V$  is the space on which  $\pi$  acts. If  $\xi$  and  $\xi'$  are elementary idempotents we write  $\xi \leq \xi'$  if  $\xi'\xi = \xi$ . Then  $\xi\xi'$  will also equal  $\xi$ . If  $\xi = \bigotimes \xi_v$  and  $\xi' = \bigotimes \xi'_v$  then  $\xi \leq \xi'$  if and only if  $\xi_v \xi'_v = \xi'_v \xi_v = \xi_v$  for all  $v$ . If  $\xi \leq \xi'$  and  $\xi$  belongs to  $I$  so does  $\xi'$ . Moreover  $\xi \mathcal{H} \xi$  is a subalgebra of  $\xi' \mathcal{H} \xi'$ . Let  $\iota(\xi', \xi)$  be the corresponding injection and let  $L(\xi)$  and  $L(\xi')$  be the spaces of linear transformations of  $V(\xi)$  and  $V(\xi')$ . There is exactly one map

$$\varphi(\xi', \xi) : L(\xi) \rightarrow L(\xi')$$

which makes

$$\begin{array}{ccc} \xi \mathcal{H} \xi & \xrightarrow{\iota(\xi', \xi)} & \iota(\xi', \xi) \\ \pi_\xi \downarrow & & \downarrow \pi_{\xi'} \\ L(\xi) & \xrightarrow{\varphi(\xi', \xi)} & L(\xi') \end{array}$$

commutative.

There is a map of  $\xi_v \mathcal{H}_v \xi_v$  into  $\xi \mathcal{H} \xi$  which sends  $f_v$  to  $f_v \otimes \left\{ \bigotimes_{w \neq v} \xi_w \right\}$ . Composing this map with  $\pi_\xi$  we obtain a map  $\pi_\xi^v$  of  $\xi_v \mathcal{H}_v \xi_v$  onto a subalgebra  $L_v(\xi)$  of  $L(\xi)$ .  $L(\xi)$  and  $L_v(\xi)$  have the same unit, namely  $\pi_\xi(\xi)$ . If  $v \neq w$  the elements of  $L_v(\xi)$  commute with those of  $L_w(\xi)$ . If we form the tensor product of the algebras  $L_v(\xi)$  with respect to the family of units there is a map from  $\bigotimes_v L_v(\xi)$  to  $L(\xi)$  which sends  $\bigotimes_v \lambda_v$  to  $\prod_v \lambda_v$ . Moreover we may identify  $\bigotimes_v \xi_v \mathcal{H}_v \xi_v$  and  $\xi \mathcal{H} \xi$ . Since the diagram

$$\begin{array}{ccc}
\bigotimes_v \xi_v \mathcal{H}_v \xi_v & \longrightarrow & \xi \mathcal{H} \xi \\
\bigotimes_v \pi_\xi^v \downarrow & & \downarrow \pi_\xi \\
\bigotimes_v L_v(\xi) & \xrightarrow{\varphi(\xi', \xi)} & L(\xi)
\end{array}$$

is commutative the bottom arrow is surjective.

**Lemma 9.1.1.** *The algebras  $L_v(\xi)$  are simple and the map  $\bigotimes_v L_v(\xi) \rightarrow L(\xi)$  is an isomorphism.*

To show that  $L_v(\xi)$  is simple we need only show that the faithful  $L_v(\xi)$ -module  $V(\xi)$  is spanned by a family of equivalent irreducible submodules. Let  $M$  be any irreducible submodule. Then the family  $\{TM\}$  where  $T$  runs over the image of  $1_v \otimes \left\{ \bigotimes_{w \neq v} L_w(\xi) \right\}$  spans  $V(\xi)$  and each  $TM$  is 0 or equivalent to  $M$  because  $T$  commutes with the elements of  $L_v(\xi)$ . The element  $1_v$  is the unit of  $L_v(\xi)$ . We have only to show that  $\bigotimes_v L_v(\xi) \hookrightarrow L(\xi)$ . Since  $\bigotimes_v L_v(\xi)$  is the inductive limit of  $\bigotimes_{v \in T} L_v(\xi)$ , where  $T$  is a finite set, we have only to show that the map is injective on these subalgebras. As they are tensor products of simple algebras they are simple and the map is certainly injective on them.

If  $\xi \leq \xi'$  there is a commutative diagram

$$\begin{array}{ccc}
\bigotimes_v \xi_v \mathcal{H}_v \xi_v & \xrightarrow{\iota(\xi', \xi)} & \bigotimes_v \xi'_v \mathcal{H}_v \xi'_v \\
\downarrow & & \downarrow \\
\bigotimes_v L_v(\xi) & & \bigotimes_v L_v(\xi') \\
\downarrow & & \downarrow \\
L(\xi) & \xrightarrow{\varphi(\xi', \xi)} & L(\xi')
\end{array}$$

Moreover if  $\iota_v(\xi', \xi)$  is the imbedding of  $\xi_v \mathcal{H}_v \xi_v$  into  $\xi'_v \mathcal{H}_v \xi'_v$  then  $\iota(\xi', \xi) = \bigotimes_v \iota_v(\xi', \xi)$ . We want to verify that a horizontal arrow  $\bigotimes_v \varphi_v(\xi', \xi)$  can be inserted in the middle without destroying the commutativity. To do this we have only to show that if  $f_v$  is in  $\xi_v \mathcal{H}_v \xi_v$  and therefore in  $\xi'_v \mathcal{H}_v \xi'_v$  then  $\pi_\xi^v(f_v) = 0$  if and only if  $\pi_{\xi'}^v(f_v) = 0$ . Let  $U = \pi_\xi^v(f_v)$  and let  $T = \pi_{\xi'}^v(f_v)$ . If

$$E = \pi_{\xi'} \left( \xi'_v \otimes \left\{ \bigotimes_{w \neq v} \xi_w \right\} \right)$$

then

$$TE = \pi_{\xi'} \left( f_v \otimes \left\{ \bigotimes_{w \neq v} \xi_w \right\} \right)$$

is determined by its restriction to  $V(\xi)$  and that restriction is  $U$ .

It is clear that if  $S$  is a sufficiently large finite set the map  $\bigotimes_{w \in S} L_w(\xi') \rightarrow L(\xi')$  is an isomorphism. We suppose that  $S$  contains  $v$ .  $E$  belongs to the image  $M$  of  $1_v \otimes \left\{ \bigotimes_{w \neq v} L_w(\xi') \right\}$ . Since  $M$  is simple and  $E$  is not 0 there are  $A_i, B_i$   $1 \leq i \leq r$  in  $M$  such

that

$$\sum_{i=1}^r A_i E B_i = 1$$

Thus

$$T = \sum_i T A_i E B_i = \sum_i A_i T E B_i$$

and  $T = 0$  if and only if  $U = 0$ .

Since the necessary compatibility conditions are satisfied we can take inductive limits, over  $I$ , to the left and right. The inductive limit of the  $\xi \mathcal{H} \xi$  is  $\mathcal{H}$  and that of the  $\xi_v \mathcal{H}_v \xi_v$  is  $\mathcal{H}_v$ . Let  $L_v$  be that of  $L_v(\xi)$  and  $L$  that of  $L(\xi)$ . There is a map  $\pi^v : \mathcal{H}_v \rightarrow L_v$  and, for almost all  $v$ ,  $\pi^v(\epsilon_v) = \mu_v$  is not zero. We have a commutative diagram

$$\begin{array}{ccc} \bigotimes \mathcal{H}_v & \longrightarrow & \mathcal{H} \\ \bigotimes \pi^v \downarrow & & \downarrow \\ \bigotimes_v L_v & \longrightarrow & L \end{array}$$

in which the rows are isomorphisms. Moreover  $L$  acts faithfully on  $V$  and the representation of  $\mathcal{H}$  on  $V$  can be factored through  $L$ .

If  $A$  is an algebra with a minimal left ideal  $J$  then any faithful irreducible representation of  $A$  on a vector space  $X$  is equivalent to the representation on  $J$ . In fact we can choose  $x_0$  in  $X$  so that  $Jx_0 \neq 0$ . The map  $j \rightarrow jx_0$  of  $J$  to  $X$  gives the equivalence. Thus to prove that  $\pi$  is factorizable it will be enough to show that  $L$  has a minimal left ideal, that the representation of  $L$  on this minimal left ideal is a tensor product of representations  $\sigma_v$  of  $L_v$ , and that  $\sigma_v \circ \pi^v$  is admissible.

Suppose  $A$  is a simple algebra and  $J$  is a left ideal in  $A$ . If  $a$  in  $A$  is not 0 and  $aJ = 0$  then  $AaAJ = AJ = 0$ . If  $J$  is not 0 this is impossible. Suppose  $e$  is an idempotent of  $A$  and  $A_1 = eAe$ . Let  $J_1$  be a minimal left ideal of  $A_1$  and let  $J = AJ_1$ . If  $J$  were not minimal it would properly contain a non-zero ideal  $J'$ . Moreover  $J' \cap A_1$  would have to be 0. Since  $Je = J$  we must have  $eJ = eJe = 0$ . Since this is a contradiction  $J$  is minimal. Suppose for example that  $A$  is the union of a family  $\{A_\lambda\}$  of matrix algebras. Suppose that for each  $\lambda$  there is an idempotent  $e_\lambda$  in  $A$  such that  $A_\lambda = e_\lambda A e_\lambda$  and that given  $\lambda_1$  and  $\lambda_2$  there is a  $\lambda_3$  such that  $A_{\lambda_3}$  contains  $A_{\lambda_1}$  and  $A_{\lambda_2}$ . Then  $A$  is certainly simple and, by the preceding discussion, contains a minimal left ideal.

The algebras  $L$  and  $L_v$  satisfy these conditions. In fact, speaking a little loosely,  $L$  is the union of the  $L(\xi)$  and  $L_v$  is the union of  $L_v(\xi)$ . Choose  $\xi$  so that  $V(\xi) \neq 0$  and let  $J_v$  be a minimal left ideal in  $L_v(\xi)$ . Since  $L_v(\xi)$  is one-dimensional for almost all  $v$  the ideal  $J_v = L_v(\xi)$  for almost all  $v$ . Thus  $J = \bigotimes J_v$  exists and is a minimal left ideal of  $L(\xi)$ . Thus  $LJ = \bigotimes L_v J_v$ .  $LJ$  is a minimal left ideal of  $L$  and  $L_v J_v$  is a minimal left ideal of  $L_v$ . The representation of  $L$  on  $LJ$  is clearly the tensor product of the representations  $\sigma_v$  of  $L_v$  on  $L_v J_v$ .

Thus  $\pi$  is equivalent to the tensor product of the representations  $\pi_v = \sigma_v \circ \pi^v$ . The representations  $\pi_v$  are irreducible. Since it is easily seen that a tensor product  $\bigotimes \pi_v$  is admissible only if each factor is admissible we may regard the first assertion of the proposition as proved.

If  $\pi$  is an admissible representation of  $\mathcal{H}$  on  $V$  and  $v$  is a place we may also introduce a representation of  $\mathcal{H}_v$  on  $V$  which we still call  $\pi$ . If  $u$  is in  $V$  we choose  $\xi = \bigotimes_w \xi_w$  so that

$\pi(\xi)u = u$ . Then if  $f$  belongs to  $\mathcal{H}_v$  we set

$$\pi(f)u = \pi \left( f\xi_v \otimes \left\{ \bigotimes_{w \neq v} \xi_w \right\} \right) u$$

The second part of the proposition is a consequence of the following lemma whose proof is immediate.

**Lemma 9.1.2.** *Suppose  $\pi = \bigotimes_w \pi_w$ . Then the representation  $\pi$  of  $\mathcal{H}_v$  is the direct sum of representations equivalent to  $\pi_v$ .*

Let  $S_a$  be the set of archimedean primes. One can also associate to an admissible representation  $\pi$  of  $\mathcal{H}$  on  $V$  a representation of  $\widehat{G}_{S_a}$ , the group formed by the elements of  $G_A$  whose components at every archimedean place are 1, on  $V$ . If  $v$  is archimedean one can associate to  $\pi$  a representation of  $\mathfrak{A}_v$ , the universal enveloping algebra of the Lie algebra of  $G_v$ , on  $V$ . Finally  $\pi$  determines a representation of the group  $Z_A$  of scalar matrices in  $\mathrm{GL}(2, A)$ . If  $\pi$  is irreducible there is a quasi-character  $\eta$  of  $I$  the group of ideles such that

$$\pi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \eta(a)I$$

for all  $a$  in  $I$ . If  $\pi_v$  is associated to  $\eta_v$  and  $\pi = \bigotimes_v \pi_v$  then  $\pi$  is associated to the quasi-character  $\eta$  defined by

$$\eta(a) = \prod_v \eta_v(a_v).$$

One may define the contragredient of  $\pi$  and the tensor product of  $\pi$  with a quasi-character of  $I$ . All the expected formal relations hold. In particular  $\tilde{\pi}$  is equivalent to  $\eta^{-1} \otimes \pi$  if  $\pi$  is irreducible.

The above discussion applies, *mutatis mutandis*, to the algebra  $\mathcal{H}'$ . The next proposition, which brings us a step closer to the theory of automorphic forms, applies to  $\mathcal{H}$  alone.

**Proposition 9.2.** *Let  $\pi = \bigotimes_v \pi_v$  be an irreducible admissible representation of  $\mathcal{H}$ . Suppose that  $\pi_v$  is infinite-dimensional for all  $v$ . Let  $\psi$  be a non-trivial character  $A/F$ . There is exactly one space  $W(\pi, \psi)$  of continuous functions on  $G_A$  with the following properties:*

(i) *If  $W$  is in  $W(\pi, \psi)$  then for all  $g$  in  $G_A$  and all  $x$  in  $A$*

$$W \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) = \psi(x)W(g)$$

(ii) *The space  $W(\pi, \psi)$  is invariant under the operators  $\rho(f)$ ,  $f \in \mathcal{H}$ , and transforms according to the representation  $\pi$  of  $\mathcal{H}$ . In particular it is irreducible under the action of  $\mathcal{H}$ .*

(iii) *If  $F$  is a number field and  $v$  an archimedean place then for each  $W$  in  $W(\pi, \psi)$  there is a real number  $N$  such that*

$$W \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) = O(|a|^N)$$

*as  $a \rightarrow \infty$  in  $F_v^\times$ .*

In the last assertion  $F_v^\times$  is regarded as a subgroup of  $I$ .  $F_v$  is a subgroup of  $\mathbf{A}$  and the restriction  $\psi_v$  of  $\psi$  to  $F_v$  is non-trivial. Thus for each place  $v$  the space  $W(\pi_v, \psi_v)$  is defined and we may suppose that  $\pi_v$  acts on it. Moreover for almost all  $v$  the largest ideal of  $F_v$  on which  $\psi_v$  is trivial is  $O_v$  and  $\pi_v$  contains the trivial representation of  $K_v$ . Thus by Proposition 3.5 there is a unique function  $\varphi_v^0$  in  $W(\pi_v, \psi_v)$  such that  $\varphi_v^0(g_v k_v) = \varphi_v^0(g_v)$  for all  $k_v$  in  $K_v$  and  $\varphi_v^0(I) = 1$ . Then  $\varphi_v^0(k_v) = 1$  for all  $k_v$  in  $K_v$ . The representation  $\pi$  acts on

$$\bigotimes_{\varphi_v^0} W(\pi_v, \psi_v)$$

If  $g$  is in  $G_{\mathbf{A}}$  and  $\bigotimes \varphi_v$  belongs to this space then  $\varphi_v(g_v) = 1$  for almost all  $v$  so that we can define a function  $\varphi$  on  $G_{\mathbf{A}}$  by

$$\varphi(g) = \prod_v \varphi_v(g_v).$$

The map  $\bigotimes \varphi_v \rightarrow \varphi$  extends to a map of  $\bigotimes W(\pi_v, \psi_v)$  into a space  $W(\pi, \psi)$  of functions on  $G_{\mathbf{A}}$ .  $W(\pi, \psi)$  certainly has the required properties. We have to show that it is characterized by these properties.

Suppose  $\mathfrak{M}$  is another space with these properties. There is an isomorphism  $T$  of  $\bigotimes W(\pi_v, \psi_v)$  and  $\mathfrak{M}$  which commutes with the action of  $\mathcal{H}$ . All we have to do is show that there is a constant  $c$  such that if  $\varphi = \bigotimes \varphi_v$  then

$$T\varphi(g) = c \prod_v \varphi_v(g_v).$$

Let  $S$  be a finite set of places and let

$$W_S = \bigotimes_{v \in S} W(\pi_v, \psi_v)$$

and

$$\widehat{W}_S = \bigotimes_{v \notin S} W(\pi_v, \psi_v).$$

Then

$$\bigotimes W(\pi_v, \psi_v) = W_S \otimes \widehat{W}_S.$$

We first show that if  $S$  is given there is a function  $c_S$  on  $\widehat{G}_S \times \widehat{W}_S$  such that if

$$f = T \left( \left\{ \bigotimes_{v \in S} \varphi_v \right\} \otimes \varphi \right)$$

with  $\varphi$  in  $\widehat{W}_S$  then

$$f(gh) = c_S(h, \varphi) \prod_{v \in S} \varphi_v(g_v)$$

if  $g$  is in  $G_S$  and  $h$  is in  $\widehat{G}_S$ .

Suppose that  $S$  consists of the single place  $v$ . If  $\varphi$  belongs to  $\widehat{W}_S$  and  $h$  belongs to  $\widehat{G}_S$  associate to every function  $\varphi_v$  in  $W(\pi_v, \psi_v)$  the function

$$\varphi'_v(g_v) = f(g_v h)$$



on  $G_v$ . The function  $f$  is  $T(\varphi_v \otimes \varphi)$ . By construction, if  $\varphi_v$  is replaced by  $\rho(f_v)\varphi_v$  with  $f_v$  in  $\mathcal{H}_v$  the function  $\varphi'_v$  is replaced by  $\rho(f_v)\varphi'_v$ . Moreover if  $x$  is in  $F_v$

$$\varphi'_v \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g_v \right) = \psi_v(x) \varphi'_v(g_v).$$

Since any conditions on rates of growth can easily be verified we see that the functions  $\varphi'_v$  are either all zero or they fill up the space  $W(\pi_v, \psi_v)$ . In both cases the map  $\varphi_v \rightarrow \varphi'_v$  is a map of  $W(\pi_v, \psi_v)$  into itself which commutes with the action of  $\mathcal{H}_v$  and therefore consists merely of multiplication by a scalar  $c_S(h, \varphi)$ .

Now suppose that  $S'$  is obtained by adjoining the place  $w$  to  $S$  and that our assertion is true for  $S$ . Take  $h$  in  $\widehat{G}_{S'}$  and  $\varphi$  in  $\widehat{W}_{S'}$ . If

$$f = T \left( \left\{ \bigotimes_{v \in S'} \varphi_v \right\} \otimes \varphi \right)$$

then, for  $g$  in  $G_S$ , and  $g_w$  in  $G_w$ ,

$$f(gg_w h) = c_S(g_w h, \varphi_w \otimes \varphi) \prod_{v \in S} \varphi_v(g_v).$$

The argument used before shows that for a given  $h$  and  $\varphi$  the function

$$g_w \rightarrow c_S(g_w h, \varphi_w \otimes \varphi)$$

is a multiple  $c_{S'}(h, \varphi)$  of  $\varphi_w$ .

To prove the existence of  $c$  we observe first that if  $S$  is the disjoint union of  $S_1$  and  $S_2$  we may write any  $h_1$  in  $\widehat{G}_{S_1}$  as  $h_1 = h \prod_{v \in S_2} h_v$  with  $h$  in  $\widehat{G}_S$ . Suppose  $\varphi_1 = \left\{ \bigotimes_{v \in S_2} \varphi_v \right\} \otimes \varphi$  with  $\varphi$  in  $\widehat{W}_S$  is in  $\widehat{W}_{S_1}$ . Then

$$(9.2.1) \quad c_{S_1}(h_1, \varphi_1) = \left\{ \prod_{v \in S_2} \varphi_v(h_v) \right\} c_S(h, \varphi)$$

because the right hand side has all the properties demanded of the left. If  $S_1$  is large enough that  $\varphi_v^0$  exists for  $v$  not in  $S_1$  then, by its definition,  $c_{S_1} \left( h, \bigotimes_{v \notin S_1} \varphi_v^0 \right)$  has a constant value  $c(S_1)$  on

$$\prod_{v \notin S_1} K_v$$

The formula (9.2.1) shows that  $c(S) = c(S_1)$  if  $S$  contains  $S_1$ . We take  $c$  to be the common value of these constants. Given  $\varphi = \bigotimes \varphi_v$  and  $g = \prod g_v$  we choose  $S$  so that  $\varphi_v = \varphi_v^0$  and  $g_v \in K_v$  for  $v$  not in  $S$ . Then

$$\begin{aligned} T\varphi(g) &= c \left( \prod_{v \notin S} g_v, \bigotimes_{v \notin S} \varphi_v \right) \prod_{v \in S} \varphi_v(g_v) \\ &= c \prod_v \varphi_v(g_v). \end{aligned}$$

We observed that if  $\pi_v$  is finite-dimensional the space  $W(\pi_v, \psi_v)$  cannot exist if  $v$  is non-archimedean or real. Although we neglected to mention it, the argument used for the real field also shows that  $W(\pi_v, \psi_v)$  cannot exist if  $v$  is complex. The proof of Proposition 9.2 can therefore be used, with minor changes, to verify the next proposition.

**Proposition 9.3.** *If  $\pi = \bigotimes \pi_v$  is given and if one of the representations  $\pi_v$  is finite-dimensional there can exist no space  $W(\pi, \psi)$  satisfying the first two conditions of the previous proposition.*

An admissible representation  $\pi$  of  $\mathcal{H}$  on the space  $V$  is said to be unitary if there is a positive definite hermitian form  $(v_1, v_2)$  on  $V$  such that, if  $f^*(g) = \bar{f}(g^{-1})$ ,

$$(\pi(f)v_1, v_2) = (v_1, \pi(f^*)v_2)$$

for all  $f$  in  $\mathcal{H}$ .

**Lemma 9.4.** *If  $\pi$  is unitary and admissible then  $V$  is the direct sum of mutually orthogonal invariant irreducible subspaces.*

The direct sum of the lemma is to be taken in the algebraic sense. We first verify that if  $V_1$  is an invariant subspace and  $V_2$  is its orthogonal complement then  $V = V_1 \oplus V_2$ . Certainly  $V_1 \cap V_2 = 0$ . Let  $\xi$  be an elementary idempotent and let  $V(\xi)$ ,  $V_1(\xi)$ ,  $V_2(\xi)$  be the ranges of  $\pi(\xi)$  in  $V$ ,  $V_1$ , and  $V_2$ . Let  $V_1^\perp(\xi)$  be the range of  $1 - \pi(\xi)$  acting on  $V_1$ . Then  $V(\xi)$  and  $V_1^\perp(\xi)$  are orthogonal and

$$V_1 = V_1(\xi) \oplus V_1^\perp(\xi).$$

Thus  $V_2(\xi)$  is just the orthogonal complement of  $V_1(\xi)$  in  $V(\xi)$ . Since  $V(\xi)$  is finite-dimensional

$$V(\xi) = V_1(\xi) \oplus V_2(\xi).$$

Since every element of  $V$  is contained in some  $V(\xi)$  we have  $V = V_1 + V_2$ .

To complete the proof we shall use the following lemma.

**Lemma 9.4.1.** *If  $\pi$  is a unitary admissible representation of  $\mathcal{H}$  on the space  $V$  then  $V$  contains a minimal non-zero invariant subspace.*

Choose an idempotent  $\xi$  so that  $V(\xi) = \pi(\xi)V \neq 0$ . Since  $V(\xi)$  is finite-dimensional amongst all the non-zero subspaces of it obtained by intersecting it with an invariant subspace of  $V$  there is a minimal one  $N$ . Let  $M$  be the intersection of all invariant subspaces containing  $N$ . If  $M$  is not irreducible it is the direct sum of two orthogonal invariant subspaces  $M_1$  and  $M_2$ . Then

$$N = M \cap V(\xi) = \pi(\xi)M = \pi(\xi)M_1 \oplus \pi(\xi)M_2$$

The right side is

$$\{M_1 \cap V(\xi)\} \oplus \{M_2 \cap V(\xi)\}$$

so that one of  $M_1 \cap V(\xi)$  and  $M_2 \cap V(\xi)$  is  $N$ . Then  $M_1$  or  $M_2$  contains  $M$ . This is a contradiction.

Let  $A$  be the set consisting of families of mutually orthogonal invariant, and irreducible subspaces of  $V$ . Each member of the family is to be non-zero. Let  $\{V_\lambda\}$  be a maximal family. Then  $V = \bigoplus_\lambda V_\lambda$ . If not let  $V_1 = \bigoplus_\lambda V_\lambda$ . The orthogonal complement of  $V_1$  would be different from zero and therefore would contain a minimal non-zero invariant subspace which when added to the family  $\{V_\lambda\}$  would make it larger.

If  $T$  is a finite set of places most of the results of this paragraph are valid for representations  $\pi$  of  $\widehat{\mathcal{H}}_T$ . For example  $\pi$  is factorizable and  $W(\pi, \psi)$  exists as a space of functions on  $\widehat{G}_T$ .

### §10. Automorphic forms

In this paragraph  $F$  is still a global field. We shall begin by recalling a simple result from reduction theory. If  $v$  is a place of  $\mathbf{A}$  and  $a$  is in  $\mathbf{A}$  then  $|a|_v$  is the absolute value of  $a_v$  the  $v$ th component of  $a$ . If  $a$  is in  $I$

$$|a| = \prod_v |a|_v$$

**Lemma 10.1.** *There is a constant  $c_0$  such that if  $g$  belongs to  $G_{\mathbf{A}}$  there is a  $\gamma$  in  $G_F$  for which*

$$\prod_v \max\{|c|_v, |d|_v\} \leq c_0 |\det g|^{1/2}$$

if

$$\gamma g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

If  $F$  is a number field let  $O_F$  be the ring of integers in  $F$  and if  $F$  is a function field take any transcendental element  $x$  of  $F$  over which  $F$  is separable and let  $O_F$  be the integral closure in  $F$  of the ring generated by 1 and  $x$ . A place  $v$  will be called finite if  $|a|_v \leq 1$  for all  $a$  in  $O_F$ ; otherwise it will be called infinite. If  $S$  is a finite set of places which contains all the infinite places let

$$\begin{aligned} \mathbf{A}(S) &= \{a \in \mathbf{A} \mid |a|_v \leq 1 \text{ if } v \notin S\} \\ I(S) &= \{a \in I \mid |a|_v = 1 \text{ if } v \notin S\} \end{aligned}$$

Then  $\mathbf{A} = F + \mathbf{A}(S)$  and if  $S$  is sufficiently large  $I = F^\times I(S)$ . We first verify that if  $I = F^\times I(S)$  then

$$G_{\mathbf{A}} = G_F G_{\mathbf{A}(S)}$$

where  $G_{\mathbf{A}(S)} = \text{GL}(2, \mathbf{A}(S))$ . If  $v$  is not in  $S$  then  $v$  is non-archimedean and we can speak of ideals of  $F_v$ . Any element of  $G_{\mathbf{A}}$  may be written as a product

$$g = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in which the second factor belongs to

$$K = \prod_v K_v$$

and therefore to  $G_{\mathbf{A}(S)}$ . It will be sufficient to show that the first factor is in  $G_F G_{\mathbf{A}(S)}$ . If  $\alpha = \alpha_1 \alpha_2$  and  $\gamma = \gamma_1 \gamma_2$  with  $\alpha_1$  and  $\gamma_1$  in  $F^\times$  and  $\alpha_2$  and  $\gamma_2$  in  $I(S)$

$$\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \gamma_1 \end{pmatrix} \begin{pmatrix} 1 & \beta/\alpha_1 \gamma_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_2 & 0 \\ 0 & \gamma_2 \end{pmatrix}$$

The first factor is in  $G_F$  and the third in  $G_{\mathbf{A}(S)}$ . Since  $\frac{\beta}{\alpha_1 \gamma_2}$  belongs to  $F + \mathbf{A}(S)$  the second factor is in  $G_F G_{\mathbf{A}(S)}$  and the assertion follows.

There is certainly a  $u$  in  $O_F$  such that  $|u|_v < 1$  at all finite places in  $S$ . Enlarging  $S$  if necessary we may assume that a finite place  $v$  belongs to  $S$  if and only if  $|u|_v < 1$ . Then

$$F \cap \mathbf{A}(S) = \left\{ \frac{x}{u^m} \mid x \in O_F, m \in \mathbf{Z} \right\}.$$

We identify the prime ideals of  $O_F$  with the places corresponding to them. By the theory of rings of quotients the proper ideals of  $F \cap \mathbf{A}(S)$  are the ideals of the form

$$(F \cap \mathbf{A}(S)) \prod_{\mathfrak{p} \notin S} \mathfrak{p}^{m_{\mathfrak{p}}}$$

Since  $I = F^\times I(S)$  every such ideal is principal. Thus  $F \cap \mathbf{A}(S)$  is a principal ideal domain.

To prove the lemma we show that there is a constant  $c_0$  such that if  $g$  belongs to  $G_{\mathbf{A}(S)}$  there is a  $\gamma$  in  $G_{F \cap \mathbf{A}(S)}$  such that

$$\prod_{v \in S} \max\{|c|_v, |d|_v\} \leq c_0 |\det g|^{1/2}$$

if

$$\gamma g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Fix a Haar measure on the additive group  $\mathbf{A}(S)$ . This determines a measure on  $\mathbf{A}(S) \oplus \mathbf{A}(S)$ . The group  $L = (F \cap \mathbf{A}(S)) \oplus (F \cap \mathbf{A}(S))$  is a discrete subgroup of  $\mathbf{A}(S) \oplus \mathbf{A}(S)$  and the quotient  $\mathbf{A}(S) \oplus \mathbf{A}(S)/L$  is compact and has finite measure  $c_1$ . If  $g$  belongs to  $G_{\mathbf{A}(S)}$  the lattice  $Lg$  is also discrete and the quotient  $\mathbf{A}(S) \oplus \mathbf{A}(S)/Lg$  has measure  $c_1 |\det g|$ .

Suppose  $(m, n) = (\mu, \nu)g$  belongs to  $Lg$ . If  $a \neq 0$  belongs to  $F \cap \mathbf{A}(S)$  then

$$\prod_{v \in S} \max\{|am|_v, |an|_v\} = \left( \prod_{v \in S} |a|_v \right) \left( \prod_{v \in S} \max\{|c|_v, |d|_v\} \right).$$

Since

$$1 = \prod_v |a|_v = \left( \prod_{v \in S} |a|_v \right) \left( \prod_{v \notin S} |a|_v \right)$$

the product  $\prod_{v \in S} |a|_v$  is at least 1 and

$$\prod_{v \in S} \max\{|am|_v, |an|_v\} \geq \prod_{v \in S} \max\{|m|_v, |n|_v\}.$$

Let  $R$  be a positive number and consider the set

$$E = \left\{ (m, n) \in Lg \mid \prod_{v \in S} \max\{|m|_v, |n|_v\} \leq R \right\}.$$

The previous inequality shows that if  $E$  contains a non-zero element of  $Lg$  it contains one  $(m, n) = (\mu, \nu)g$  for which  $\mu$  and  $\nu$  are relatively prime. Then we may choose  $\kappa$  and  $\lambda$  in  $F \cap \mathbf{A}(S)$  so that  $\kappa\nu - \lambda\mu = 1$ . If

$$\gamma = \begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix}$$

then  $\gamma$  belongs to  $G_{F \cap \mathbf{A}(S)}$  and if

$$\gamma g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then  $c = m$  and  $d = n$  so that

$$\prod_{v \in S} \max\{|c|_v, |d|_v\} \leq R.$$

To prove the lemma we have to show that there is a constant  $c_0$  such that if  $g$  is in  $G_{\mathbf{A}(S)}$  and  $R = c_0 |\det g|^{1/2}$  the set  $E$  is not reduced to  $\{0\}$ . We will show in fact that there is a constant  $c_2$  such that for all  $g$  there is a non-zero vector  $(m, n)$  in  $Lg$  with

$$\sup_{v \in S} \max\{|m|_v, |n|_v\} \leq c_2 |\det g|^{\frac{1}{2s}}$$

if  $s$  is the number of elements in  $S$ . There is certainly a positive constant  $c_3$  such that the measure of

$$\left\{ (m, n) \in \mathbf{A}(S) \oplus \mathbf{A}(S) \mid \sup_{v \in S} \max\{|m|_v, |n|_v\} \leq R \right\}$$

is, for any choice of  $R$ , at least  $c_3 R^{2s}$ . Choose  $c_2$  so that

$$c_2 > 2 \left( \frac{c_1}{c_3} \right)^{\frac{1}{2s}}.$$

If  $Lg$  contained no non-zero vector satisfying the desired inequality the set

$$\left\{ (m, n) \in \mathbf{A}_s \oplus \mathbf{A}_S \mid \sup_{v \in S} \max\{|m|_v, |n|_v\} \leq \frac{c_2}{2} |\det g|^{\frac{1}{2s}} \right\}$$

would intersect none of its translates by the elements of  $Lg$ . Therefore its measure would not be changed by projection on  $\mathbf{A}(S) \oplus \mathbf{A}(S)/Lg$  and we would have

$$c_1 \leq c_3 \left( \frac{c_2}{2} \right)^{2s}$$

which is impossible.

Choose some place  $v$  of  $F$  which is to be archimedean if  $F$  is a number field. If  $c$  is any positive constant there is a compact set  $C$  in  $I$  such that

$$\{a \in I \mid |a| \geq c\}$$

is contained in

$$\{ab \mid a \in F_v^\times, |a| \geq c, b \in C\}$$

If  $\omega_1$  is a compact subset of  $\mathbf{A}$ ,  $\omega_2$  a compact subset of  $I$ , and  $c$  a positive constant we may introduce the Siegel domain  $\mathfrak{S} = \mathfrak{S}(\omega_1, \omega_2, c, v)$  consisting of all

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} bb_1 & 0 \\ 0 & 1 \end{pmatrix} k$$

with  $x$  in  $\omega_1$ ,  $a$  in  $I$ ,  $b$  in  $\omega_2$ ,  $b_1$  in  $F_v^\times$  with  $|b_1| \geq c$ , and  $k$  in  $K$ . Then  $Z_{\mathbf{A}}\mathfrak{S} = \mathfrak{S}$ . If we use the Iwasawa decomposition of  $G_{\mathbf{A}}$  to calculate integrals we easily see that the projection of  $\mathfrak{S}$  on  $Z_{\mathbf{A}} \backslash G_{\mathbf{A}}$  has finite measure. Moreover it follows readily from the previous lemma that, for a suitable choice of  $\omega_1$ ,  $\omega_2$ , and  $c$ ,

$$G_{\mathbf{A}} = G_F \mathfrak{S}.$$

Thus  $Z_{\mathbf{A}} G_F \backslash G_{\mathbf{A}}$  has finite measure.

Let  $\varphi$  be a continuous function on  $G_F \backslash G_{\mathbf{A}}$ . If it is  $Z_{\mathbf{A}}$ -finite the space  $V$  spanned by the functions  $\rho(a)\varphi$ ,  $a \in Z_{\mathbf{A}}$ , is finite-dimensional. We may choose a finite set of points  $g_1, \dots, g_p$  and a basis  $\varphi_1, \dots, \varphi_p$  of  $V$  so that  $\varphi_i(g_j) = \delta_{ij}$ . Then

$$\rho(a)\varphi = \sum_{i=1}^p \lambda_i(a)\varphi_i.$$

Since  $\lambda_i(a) = \varphi(ag_i)$  the function  $\lambda_i$  are continuous and finite as functions on  $Z_{\mathbf{A}}$  or  $Z_F \backslash Z_{\mathbf{A}}$ . Since  $Z_F \backslash Z_{\mathbf{A}}$  is isomorphic to  $F^{\times} \backslash I$  it satisfies the hypothesis of Lemma 8.1 and  $\lambda_i$  is a finite linear combination of functions of the form

$$\lambda_i \left( \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \right) = \chi(a)(\log|\alpha|)^m$$

where  $\chi$  is a quasi-character of  $F^{\times} \backslash I$ .

A continuous function  $\varphi$  on  $G_F \backslash G_{\mathbf{A}}$  which is  $Z_{\mathbf{A}}$ -finite will be called slowly increasing if for any compact set  $\Omega$  in  $G_{\mathbf{A}}$  and any  $c > 0$  there are constants  $M_1$  and  $M_2$  such that

$$\left| \varphi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right) \right| \leq M_2 |a|^{M_1}$$

for  $g$  in  $\Omega$ ,  $a$  in  $I$ , and  $|a| \geq c$ . If such an inequality is valid, with suitable choice of  $M_2$ , for any  $M_1$  we will say, for lack of a better terminology, that  $\varphi$  is rapidly decreasing.

Suppose  $\varphi$  is a continuous function on  $G_F \backslash G_{\mathbf{A}}$ . Assume it is  $K$ -finite on the right and that for every elementary idempotent  $\xi$  in  $\mathcal{H}$  the space

$$\{ \rho(\xi f)\varphi \mid f \in \mathcal{H} \}$$

is finite-dimensional. An argument used more than once already shows that there is a  $\xi$  and an  $f$  in  $\xi \mathcal{H}_1 \xi$  such that  $\rho(f)\varphi = \varphi$ . If  $a$  belongs to  $Z_{\mathbf{A}}$

$$\rho(a)\varphi = \rho(\delta_a * f)\varphi$$

so that  $\varphi$  is  $Z_{\mathbf{A}}$ -finite. Thus we can make the following definition.

**Definition 10.2.** A continuous function  $\varphi$  on  $G_F \backslash G_{\mathbf{A}}$  is said to be an automorphic form if

- (i) It is  $K$ -finite on the right.
- (ii) For every elementary idempotent  $\xi$  in  $\mathcal{H}$  the space

$$\{ \rho(\xi f)\varphi \mid f \in \mathcal{H} \}$$

is finite-dimensional.

- (iii) If  $F$  is a number field  $\xi$  is slowly increasing.

We observe, with regret, in passing that there has been a tendency of late to confuse the terms automorphic form and automorphic function. If not the result it is certainly the cause of much misunderstanding and is to be deplored.

Let  $\mathcal{A}$  be the vector space of automorphic forms. If  $\varphi$  is in  $\mathcal{A}$  and  $f$  is in  $\mathcal{H}$  then  $\rho(f)\varphi$  is in  $\mathcal{A}$  so that  $\mathcal{H}$  operates on  $\mathcal{A}$ . A continuous function on  $\varphi$  on  $G_F \backslash G_{\mathbf{A}}$  is said to be cuspidal if

$$\int_{F \backslash \mathbf{A}} \varphi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx = 0$$

for all  $g$  in  $G_{\mathbf{A}}$ . An automorphic form which is cuspidal is called a cusp form. The space  $\mathcal{A}_0$  of cusp forms is stable under the action of  $\mathcal{H}$ .

**Proposition 10.3.** *Let  $F$  be a function field and let  $\varphi$  be a function on  $G_F \backslash G_{\mathbf{A}}$ . If  $\varphi$  satisfies the following three conditions it is a cusp form.*

- (i)  $\varphi$  is  $K$ -finite on the right.
- (ii)  $\varphi$  is cuspidal.
- (iii) There is a quasi-character  $\eta$  of  $F^\times \backslash I$  such that

$$\varphi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g \right) = \eta(a) \varphi(g)$$

for all  $a$  in  $I$ .

If  $\xi$  is an elementary idempotent of  $\mathcal{H}$  there is an open subgroup  $K'$  of  $K$  such that  $\xi$  is invariant under translations on either side by the elements of  $K'$ . Therefore the functions  $\rho(\xi f)\varphi$  are invariant under right translations. To prove the proposition we show that if  $K'$  is a given open subgroup of  $K$  and  $\eta$  is a given quasi-character of  $F^\times \backslash I$  then the space  $V$  of all continuous functions  $\varphi$  on  $G_F \backslash G_{\mathbf{A}}$  which are cuspidal and satisfy  $\varphi(gk) = \varphi(g)$  for all  $k$  in  $K'$  as well as

$$\varphi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g \right) = \eta(a) \varphi(g)$$

for all  $a$  in  $F^\times \backslash I$  is finite-dimensional.

We shall show that there is a compact set  $C$  in  $G_{\mathbf{A}}$  such that the support of every  $\varphi$  in  $V$  is contained in  $G_F Z_{\mathbf{Z}} C$ . Then the functions in  $V$  will be determined by their restrictions to  $C$ . Since  $C$  is contained in the union of a finite number of left translates of  $K'$  they will actually be determined by their values on a finite set and  $V$  will be finite-dimensional.

Choose a Siegel domain  $\mathfrak{S} = \mathfrak{S}(\omega_1, \omega_2, c, v)$  so that  $G_{\mathbf{A}} = G_F \mathfrak{S}$ . If

$$\mathfrak{S}' = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} bb_1 & 0 \\ 0 & 1 \end{pmatrix} k \mid x \in \omega_1, b \in \omega_2, b_1 \in F_v^\times, |b_1| \geq c, k \in K \right\}$$

we have just to show that the support in  $\mathfrak{S}'$  of every  $\varphi$  in  $V$  is contained in a certain compact set which is independent of  $\varphi$ . In fact we have to show the existence of a constant  $c_1$  such that  $\varphi$  vanishes on

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} bb_1 & 0 \\ 0 & 1 \end{pmatrix} k$$

as soon as  $|b_1| \geq c_1$ . Let  $k_1, \dots, k_n$  be a set of representatives of the cosets of  $K/K'$  and let  $\varphi_i(g) = \varphi(gk_i)$ . If  $k$  belongs to  $k_i K'$  then  $\varphi(gk) = \varphi_i(g)$  and it will be enough to show that there is a constant  $c_2$  such that, for  $1 \leq i \leq n$ ,

$$\varphi_i \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) = 0$$

if  $x$  belongs to  $\mathbf{A}$  and  $|a| > c_2$ . It is enough to show this for a single, but arbitrary,  $\varphi_i$ . Since  $\varphi_i$  satisfies the same hypothesis as  $\varphi$ , perhaps with a different group  $K'$ , we just prove the corresponding fact for  $\varphi$ .

We use the following lemma which is an immediate consequence of the theorem of Riemann-Roch as described in reference [10] of Chapter I.



**Lemma 10.3.1.** *Let  $X$  be an open subgroup of  $\mathbf{A}$ . There is a constant  $c_2$  such that  $\mathbf{A} = F + aX$  if  $a$  belongs to  $I$  and  $|a| > c_2$ .*

Let  $X$  be the set of all  $y$  for which

$$\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$$

belongs to  $K'$ . Since

$$\varphi\left(\begin{pmatrix} 1 & ay \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right)$$

we have

$$\varphi\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}\right)$$

if  $z$  is in  $aX$ . The equation also holds for  $z$  in  $F$  and therefore for all  $z$  in  $\mathbf{A}$  if  $|a| > c_2$ . Then

$$\varphi\left(\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}\right) = \frac{1}{\text{measure}(F \backslash \mathbf{A})} \int_{F \backslash \mathbf{A}} \varphi\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}\right) dz$$

which by assumption is zero.

There is a corollary.

**Proposition 10.4.** *Suppose  $\varphi$  is a cusp form and for some quasi-character  $\eta$  of  $F^\times \backslash I$*

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}g\right) = \eta(a)\varphi(g)$$

*for all  $a$  in  $I$ . Then  $\varphi$  is compactly supported modulo  $G_F Z_{\mathbf{A}}$ . Moreover the function*

$$a \rightarrow \varphi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)$$

*on  $F^\times \backslash I$  is compactly supported.*

The first assertion has just been verified. We know moreover that there is a constant  $c$  such that

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)$$

is 0 for  $|a| \geq c$ . If

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and  $\varphi'(g) = \varphi(gw)$  then  $\varphi'$  is also a cusp form. Since

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = \varphi\left(w^{-1}\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}w\right) = \eta(a)\varphi\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)$$

there is also a constant  $c_1$  such that it vanishes for  $|a| \leq c_1$ .

**Proposition 10.5.** *Let  $F$  be a function field and  $\eta$  a quasi-character of  $F^\times \backslash I$ . Let  $\mathcal{A}_0(\eta)$  be the space of cusp forms  $\varphi$  for which*

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}g\right) = \eta(a)\varphi(g)$$

*for all  $a$  in  $I$ . The representation of  $\mathcal{H}$  on  $\mathcal{A}_0(\eta)$  is the direct sum of irreducible admissible representations each occurring with finite multiplicity.*

The proof of Proposition 10.3 showed that the representation  $\pi$  of  $\mathcal{H}$  on  $\mathcal{A}_0(\eta)$  is admissible. Let  $\eta'(\alpha) = |\eta(\alpha)|^{-1}\eta(\alpha)$ . The map  $\varphi \rightarrow \varphi'$  is an isomorphism of  $\mathcal{A}_0(\eta)$  with  $\mathcal{A}_0(\eta')$  which replaces  $\pi$  by  $\eta_1 \otimes \pi$  if  $\eta_1(\alpha) = |\eta(\alpha)|^{-1/2}$ . Thus we may as well suppose that  $\eta$  is a character. Then if  $\varphi_1$  and  $\varphi_2$  belong to  $\mathcal{A}_0(\eta)$  the function  $\varphi_1 \overline{\varphi_2}$  is a function on  $G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}}$ . Since it has compact support we may set

$$(\varphi_1, \varphi_2) = \int_{G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}}} \varphi_1(g) \overline{\varphi_2(g)} dg.$$

It is easily seen that

$$(\rho(f)\varphi_1, \varphi_2) = (\varphi_1, \rho(f^*)\varphi_2)$$

so that, by Lemma 9.4,  $\pi$  is the direct sum of irreducible admissible representations. Since  $\pi$  is admissible the range of  $\pi(\xi)$  is finite-dimensional for all  $\xi$  so that no irreducible representation occurs an infinite number of times.

The analogue of this proposition for a number field is somewhat more complicated. If  $\varphi$  is a continuous function on  $G_{\mathbf{A}}$ , if  $v$  is a place of  $F$ , and if  $f_v$  belongs to  $\mathcal{H}_v$  we set

$$\rho(f_v)\varphi = \int_{G_v} \varphi(gh_v) f_v(h_v) dh_v.$$

Since  $f_v$  may be a measure the expression on the right is not always to be taken literally. If  $v$  is archimedean and if the function  $\varphi(hg_v)$  on  $G_v$  is infinitely differentiable for any  $h$  in  $G_{\mathbf{A}}$  then for any  $X$  in  $\mathfrak{A}_v$  the universal enveloping algebra of  $G_v$ , we can also define  $\rho(X)\varphi$ . If  $S$  is a finite set of places we can in a similar fashion let the elements of

$$\mathcal{H}_S = \bigotimes_{v \in S} \mathcal{H}_v$$

or, if every place in  $S$  is archimedean,

$$\mathfrak{A}_S = \bigotimes_{v \in S} \mathfrak{A}_v$$

act on  $\varphi$ . It is clear what an elementary idempotent in  $\mathcal{H}_S$  is to be. If  $S = S_a$  is the set of archimedean places we set  $\mathcal{H}_a = \mathcal{H}_{S_a}$ .

**Proposition 10.6.** *Suppose  $F$  is a number field. A continuous function  $\varphi$  on  $G_F \backslash G_{\mathbf{A}}$  is a cusp form if it satisfies the following five conditions.*

- (i)  $\varphi$  is  $K$ -finite on the right.
- (ii)  $\varphi$  is cuspidal.
- (iii) There is a quasi-character  $\eta$  of  $F^\times \backslash I$  such that

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}g\right) = \eta(a)\varphi(g)$$

for all  $a$  in  $I$ .

(iv) For any elementary idempotent  $\xi$  in  $\mathcal{H}_a$  the space

$$\{ \rho(\xi f)\varphi \mid f \in \mathcal{H}_a \}$$

is finite-dimensional.

(v)  $\varphi$  is slowly increasing.

There is a  $\xi$  in  $\mathcal{H}_a$  such that  $\rho(\xi)\varphi = \varphi$ . Because of the fourth condition  $\varphi$  transforms according to a finite-dimensional representation of  $\xi\mathcal{H}_a\xi$  and the usual argument shows that there is a function  $f$  in  $\mathcal{H}_a$  such that  $\rho(f)\varphi = \varphi$ .

Since  $\varphi$  is invariant under right translations by the elements of an open subgroup of  $\prod_{v \notin S_a} K_v$  this implies in turn the existence of another function  $f$  in  $\mathcal{H}$  such that  $\rho(f)\varphi = \varphi$ . From Theorem 2 of [14] one infers that  $\varphi$  is rapidly decreasing.

As before we may assume that  $\eta$  is a character. Then  $\varphi$  is bounded and therefore its absolute value is square integrable on  $G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}}$  which has finite measure. Let  $L^2(\eta)$  be the space of measurable functions  $h$  on  $G_F \backslash G_{\mathbf{A}}$  such that

$$h\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) = \eta(a)h(g)$$

for all  $g$  in  $G_{\mathbf{A}}$  and all  $a$  in  $I$  and

$$\int_{G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}}} |h(g)|^2 dg < \infty.$$

According to a theorem of Godement (see reference [11] to Chapter I) any closed subspace of  $L^2(\eta)$  which consists entirely of bounded functions is finite-dimensional.

What we show now is that if  $\xi$  is an elementary idempotent of  $\mathcal{H}$  the space

$$V = \{ \rho(\xi f)\varphi \mid f \in \mathcal{H} \}$$

is contained in such a closed subspace. The functions in  $V$  itself certainly satisfy the five conditions of the proposition and therefore are bounded and in  $L^2(\eta)$ . Replacing  $\xi$  by a larger idempotent if necessary we may suppose that  $\xi = \xi_a \otimes \widehat{\xi}_a$  where  $\xi_a$  is an elementary idempotent in  $\mathcal{H}_a$ . There is a two-sided ideal  $\mathfrak{a}$  in  $\xi_a \mathcal{H}_a \xi_a$  such that  $\rho(f)\varphi = 0$  if  $f$  belongs to  $\mathfrak{a}$ . The elements of  $\mathfrak{a}$  continue to annihilate  $V$  and its closure in  $L^2(\eta)$ . Approximating the  $\delta$ -function as usual we see that there is a function  $f_1$  in  $\mathcal{H}_a$  and a polynomial  $P$  with non-zero constant term such that  $P(f_1)$  belongs to  $\mathfrak{a}$ . Therefore there is a function  $f_2$  in  $\mathcal{H}_a$  such that  $f_2 - 1$  belongs to  $\mathfrak{a}$ . To complete the proof of the proposition we have merely to refer to Theorem 2 of [14] once again.

For a number field the analogue to Proposition 10.4 is the following.

**Proposition 10.7.** *Suppose  $\varphi$  is a cusp form and for some quasi-character  $\eta$  of  $F^\times \backslash I$*

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) = \eta(a)\varphi(g)$$

for all  $a$  in  $I$ . Then for any real number  $M_1$  there is a real number  $M_2$  such that

$$\left| \varphi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) \right| \leq M_2 |a|^{M_1}$$

for all  $a$  in  $I$ . Moreover the absolute value of  $\varphi$  is square integrable on  $G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}}$ .

We need another corollary of Proposition 10.6. To prove it one has just to explain the relation between automorphic forms on  $G_{\mathbf{A}}$  and  $G_{\mathbf{R}}$ , which is usually assumed to be universally known, and then refer to the first chapter of reference [11] to Chapter I. It is perhaps best to dispense with any pretence of a proof and to rely entirely on the reader's initiative. We do not however go so far as to leave the proposition itself unstated.

**Proposition 10.8.** *Let  $\mathfrak{Z}_v$  be the centre of  $\mathfrak{A}_v$  and let  $\mathfrak{a}$  be an ideal of finite codimension in  $\mathfrak{Z} = \bigotimes_{v \in S_a} \mathfrak{Z}_v$ . Let  $\xi$  be an elementary idempotent of  $\mathcal{H}$  and  $\eta$  a quasi-character of  $F^\times \backslash I$ . Then the space of infinitely differentiable functions  $\varphi$  on  $G_F \backslash G_{\mathbf{A}}$  which satisfy the following five conditions is finite-dimensional.*

- (i)  $\varphi$  is cuspidal.
- (ii)  $\rho(\xi)\varphi = \varphi$ .
- (iii) If  $a$  is in  $I$  then

$$\varphi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g \right) = \eta(a) \varphi(g).$$

- (iv)  $\rho(X)\varphi = 0$  for all  $X$  in  $\mathfrak{a}$
- (v)  $\varphi$  is slowly increasing.

**Proposition 10.9.** *Let  $\eta$  be a quasi-character of  $F^\times \backslash I$  and let  $\mathcal{A}_0(\eta)$  be the space of cusp forms  $\varphi$  for which*

$$\varphi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g \right) = \eta(a) \varphi(g)$$

for all  $a$  in  $I$ . The representation of  $\mathcal{H}$  on  $\mathcal{A}_0(\eta)$  is the direct sum of irreducible admissible representations each occurring with finite multiplicity.

Every element of  $\mathcal{A}_0(\eta)$  is annihilated by some ideal of finite codimension in  $\mathfrak{Z}$ . If  $\mathfrak{a}$  is such an ideal let  $\mathcal{A}_0(\eta, \mathfrak{a})$  be the space of functions in  $\mathcal{A}_0(\eta)$  annihilated by  $\mathfrak{a}$ . It is enough to prove the first part of the proposition for the space  $\mathcal{A}_0(\eta, \mathfrak{a})$ . Then one may use the previous proposition and argue as in the proof of Proposition 10.5. To show that every representation occurs with finite multiplicity one combines the previous proposition with the observation that two functions transforming under the same representation of  $\mathcal{H}$  are annihilated by the same ideal in  $\mathfrak{Z}$ .

The algebra  $\mathcal{H}$  acts on the space  $\mathcal{A}$ . An irreducible admissible representation  $\pi$  of  $\mathcal{H}$  is a constituent of the representation on  $\mathcal{A}$  or, more briefly, a constituent of  $\mathcal{A}$  if there are two invariant subspaces  $U$  and  $V$  of  $\mathcal{A}$  such that  $U$  contains  $V$  and the action on the quotient space  $U/V$  is equivalent to  $\pi$ . A constituent of  $\mathcal{A}_0$  is defined in a similar fashion. The constituents of  $\mathcal{A}_0$  are more interesting than the constituents of  $\mathcal{A}$  which are not constituents of  $\mathcal{A}_0$ .

**Theorem 10.10.** *Let  $\pi = \bigotimes \pi_v$  be an irreducible admissible representation of  $\mathcal{H}$  which is a constituent of  $\mathcal{A}$  but not of  $\mathcal{A}_0$ . Then there are two quasi-characters  $\mu$  and  $\nu$  of  $F^\times \backslash I$  such that for each place  $v$  the representation  $\pi_v$  is a constituent of  $\rho(\mu_v, \nu_v)$ .*

The character  $\mu_v$  is the restriction of  $\mu$  to  $F_v^\times$ . Let  $\mathcal{B}$  be the space of all continuous functions  $\varphi$  on  $G_{\mathbf{A}}$  satisfying the following conditions.

(i) For all  $x$  in  $\mathbf{A}$

$$\varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}g\right) = \varphi(g).$$

(ii) For all  $\alpha$  and  $\beta$  in  $F^\times$

$$\varphi\left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}g\right) = \varphi(g).$$

(iii)  $\varphi$  is  $K$ -finite on the right.

(iv) For every elementary idempotent  $\xi$  in  $\mathcal{H}$  the space

$$\{\rho(\xi f)\varphi \mid f \in \mathcal{H}\}$$

is finite-dimensional.

**Lemma 10.10.1.** *A continuous function  $\varphi$  on  $G_{\mathbf{A}}$  which satisfies the first three of these conditions satisfies the fourth if and only if it is  $A_{\mathbf{A}}$ -finite on the left.*

$A$  is the group of diagonal matrices. Since  $\varphi$  is a function on  $A_F \backslash G_{\mathbf{A}}$  it is  $A_{\mathbf{A}}$  finite if and only if it is  $A_F \backslash A_{\mathbf{A}}$  finite. If it is  $A_F \backslash A_{\mathbf{A}}$  finite there is a relation of the form

$$\varphi(ag) = \sum_i \lambda_i(a) \varphi_i(g)$$

where the  $\lambda_i$  are finite continuous functions on  $A_F \backslash A_{\mathbf{A}}$ . Since  $A_F \backslash A_{\mathbf{A}}$  is isomorphic to the direct product of  $F^\times \backslash I$  with itself it is a group to which Lemma 8.1 can be applied. Thus there is a unique family  $\varphi_{m,n,\mu,\nu}$  of functions on  $G_{\mathbf{A}}$  such that

$$\varphi\left(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}g\right) = \left|\frac{a_1}{a_2}\right| \sum \mu(a_1)\nu(a_2)(\log|a_1|)^m(\log|a_2|)^n \varphi_{m,n,\mu,\nu}(g)$$

The functions  $\varphi_{m,n,\mu,\nu}$  also satisfy the first three conditions. Moreover there is a finite set  $S$  of pairs  $(\mu, \nu)$  and a non-negative integer  $M$  such that  $\varphi_{m,n,\mu,\nu}$  is 0 if  $(\mu, \nu)$  does not belong to  $S$  or  $m + n > M$ .

Given  $S$  and  $M$  let  $\mathcal{B}(S, M)$  be the space of continuous functions  $f$  on  $G_{\mathbf{A}}$  which satisfy the first three conditions and for which

$$f\left(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}g\right)$$

can be expanded in the form

$$\left|\frac{a_1}{a_2}\right|^{1/2} \sum \mu(a_1)\nu(a_2)(\log|a_1|)^m(\log|a_2|)^n f_{m,n,\mu,\nu}(g)$$

where the sum is taken only over the pairs  $(\mu, \nu)$  in  $S$  the pairs  $(m, n)$  for which  $m + n \leq M$ .  $\mathcal{B}(S, M)$  is invariant under  $\mathcal{H}$ . To show that if  $\varphi$  is  $A_F \backslash A_{\mathbf{A}}$  finite it satisfies the fourth condition we show that the range of  $\rho(\xi)$  on  $\mathcal{B}(S, M)$  is finite-dimensional.

A function  $f$  in  $\mathcal{B}(S, M)$  is determined by the restriction of the finitely many functions  $f_{m,n,\mu,\nu}$  to  $K$ . If  $f$  is in the range of  $\rho(\xi)$  these restrictions lie in the range of  $\rho(\xi)$  acting on the continuous functions on  $K$ . That range is finite-dimensional.

We have also to show that if  $\varphi$  satisfies the fourth condition it is  $A_{\mathbf{A}}$  finite. The space  $V$  spanned by the right translates of  $\varphi$  by the elements of  $K$  is finite-dimensional and each

element in it satisfies all four conditions. Let  $\varphi_1, \dots, \varphi_p$  be a basis of  $V$ . We can express  $\varphi(gk)$  as

$$\sum_{i=1}^p \lambda_i(k) \varphi_i(g).$$

Because of the Iwasawa decomposition  $G_{\mathbf{A}} = N_{\mathbf{A}} A_{\mathbf{A}} K$  it is enough to show that the restriction of each  $\varphi_i$  to  $A_{\mathbf{A}}$  is finite. Since  $\varphi_i$  satisfies the same conditions as  $\varphi$  we need only consider the restriction of  $\varphi$ .

Since  $\varphi$  is  $K$  finite there is a finite set  $S$  of places such that  $\varphi$  is invariant under right translations by the elements of  $\prod_{v \notin S} K_v$ . Let

$$I_S = \prod_{v \in S} F_v^\times.$$

We regard  $I_S$  as a subgroup of  $I$ . If we choose  $S$  so large that  $I = F^\times I(S)$  then every element  $\alpha$  of  $I$  is a product of  $\alpha = \alpha_1 \alpha_2 \alpha_3$  with  $\alpha_1$  in  $F^\times$ ,  $\alpha_2$  in  $I_S$ , and  $\alpha_3$  in  $I(S)$  such that its component at any place in  $S$  is 1. If  $\beta$  in  $I$  is factored in a similar fashion

$$\varphi \left( \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \right) = \varphi \left( \begin{pmatrix} \alpha_2 & 0 \\ 0 & \beta_2 \end{pmatrix} \right).$$

Thus we need only show that the restriction of  $\varphi$  to

$$A_S = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \mid \alpha, \beta \in I_S \right\}$$

is finite. This is a consequence of Corollary 8.4 since the restriction of  $\varphi$  to  $G_S$  clearly satisfies the conditions of the corollary.

The next lemma explains the introduction of  $\mathcal{B}$ .

**Lemma 10.10.2.** *If  $\pi$  is a constituent of  $\mathcal{A}$  but not of  $\mathcal{A}_0$  then it is a constituent of  $\mathcal{B}$ .*

If  $\varphi$  belongs to  $\mathcal{A}$  the functions

$$\varphi_0(g) = \frac{1}{\text{measure}(F \backslash \mathbf{A})} \int_{F \backslash \mathbf{A}} \varphi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx$$

belongs to  $\mathcal{B}$ . The map  $\varphi \rightarrow \varphi_0$  commutes with the action of  $\mathcal{H}$  and its kernel is  $\mathcal{A}_0$ . Suppose  $U$  and  $V$  are two invariant subspaces of  $\mathcal{A}$  and  $\pi$  occurs on the quotient of  $U$  by  $V$ . Let  $U_0$  be the image of  $U$  and  $V_0$  be the image of  $V$  in  $\mathcal{B}$ . Since  $\pi$  is irreducible there are two possibilities. Either  $U_0 \neq V_0$  in which case  $\pi$  is equivalent to the representation on  $U_0/V_0$  and is a constituent of  $\mathcal{B}$  or  $U_0 = V_0$ . In the latter case

$$U = V + U \cap \mathcal{A}_0$$

and  $\pi$  is equivalent to the representation on

$$U \cap \mathcal{A}_0 / V \cap \mathcal{A}_0$$

which is precisely the possibility we have excluded.

**Lemma 10.10.3.** *If  $\pi$  is a constituent of  $\mathcal{B}$  then there is a pair of quasi-characters  $\mu, \nu$  and a non-negative integer  $M$  such that  $\pi$  is a constituent of  $\mathcal{B}(\mu, \nu, M)$ .*

If  $S$  consists of the single pair  $(\mu, \nu)$  then, by definition,  $\mathcal{B}(\mu, \nu, M) = \mathcal{B}(S, M)$ . Suppose  $\pi$  occurs on the quotient of  $U$  by  $V$ . Choose the finite set  $S$  of pairs of quasi-characters and the non-negative integer  $M$  so that  $U \cap \mathcal{B}(S, M)$  is different from  $V \cap \mathcal{B}(S, M)$ . Then  $\pi$  occurs on the quotient of  $U \cap \mathcal{B}(S, M)$  by  $V \cap \mathcal{B}(S, M)$  and we may as well assume that  $U$  is contained in  $\mathcal{B}(S, M)$ . The argument used in the eighth paragraph in an almost identical context shows that

$$\mathcal{B}(S, M) = \bigoplus_{(\mu, \nu) \in S} \mathcal{B}(\mu, \nu, M)$$

so that the lemma is a consequence of Lemma 8.6.

The next lemma is proved in exactly the same way as Proposition 8.5.

**Lemma 10.10.4.** *If  $\pi$  is a constituent of  $\mathcal{B}(\mu, \nu, M)$  for some  $M$  then it is a constituent of  $\mathcal{B}(\mu, \nu) = \mathcal{B}(\mu, \nu, 0)$ .*

Let  $\mu_v$  and  $\nu_v$  be the restrictions of  $\mu$  and  $\nu$  to  $F_v^\times$ . For almost all  $v$  the quasi-characters  $\mu_v$  and  $\nu_v$  are unramified and there is a unique function  $\varphi_v^0$  in  $\mathcal{B}(\mu_v, \nu_v)$  such that  $\varphi_v^0(g_v k_v) = \varphi_v^0(g_v)$  for all  $k_v$  in  $K_v$  while  $\varphi_v^0(e) = 1$ . We can form

$$\bigotimes_{\varphi_v^0} \mathcal{B}(\mu_v, \nu_v)$$

There is clearly a linear map of this space into  $\mathcal{B}(\mu, \nu)$  which sends  $\bigotimes \varphi_v$  to the function

$$\varphi(g) = \prod_v \varphi_v(g_v)$$

It is easily seen to be surjective and is in fact, although this is irrelevant to our purposes, an isomorphism. In any case an irreducible constituent of  $\mathcal{B}(\mu, \nu)$  is a constituent of  $\bigotimes_v \rho(\mu_v, \nu_v)$ .

With the following lemma the proof of Theorem 10.10 is complete.

**Lemma 10.10.5.** *If the irreducible admissible representation  $\pi = \bigotimes_v \pi_v$  is a constituent of  $\rho = \bigotimes \rho_v$ , the tensor product of admissible representations which are not necessarily irreducible, then, for each  $v$ ,  $\pi_v$  is a constituent of  $\rho_v$ .*

As in the ninth paragraph  $\pi$  and  $\rho$  determine representations  $\pi$  and  $\rho$  of  $\mathcal{H}_v$ . The new  $\pi$  will be a constituent of the new  $\rho$ . By Lemma 9.12 the representation  $\pi$  of  $\mathcal{H}_v$  is the direct sum of representations equivalent to  $\pi_v$ . Thus  $\pi_v$  is a constituent of  $\pi$  and therefore of  $\rho$ . Since  $\rho$  is the direct sum of representations equivalent to  $\rho_v$ , Lemma 8.6 shows that  $\pi_v$  is a constituent of  $\rho_v$ .

The considerations which led to Proposition 8.5 and its proof will also prove the following proposition.

**Proposition 10.11.** *If  $\pi$  is an irreducible constituent of the space  $\mathcal{A}_0$  then for some quasi-character  $\eta$  it is a constituent of  $\mathcal{A}_0(\eta)$ .*

Observe that if  $\pi$  is a constituent of  $\mathcal{A}_0(\eta)$  then

$$\pi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \eta(a)I$$

for all  $a$  in  $I$ . There are two more lemmas to be proved to complete the preparations for the Hecke theory.

**Lemma 10.12.** *Suppose there is a continuous function  $\varphi$  on  $G_{\mathbf{A}}$  with the following properties.*

- (i)  $\varphi$  is  $K$  finite on the right.
- (ii) For all  $\alpha$  and  $\beta$  in  $F^\times$  and all  $x$  in  $\mathbf{A}$

$$\varphi\left(\begin{pmatrix} \alpha & x \\ 0 & \beta \end{pmatrix}g\right) = \varphi(g).$$

- (iii) There is a quasi-character  $\eta$  of  $F^\times \backslash I$  such that

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}g\right) = \eta(a)\varphi(g)$$

for all  $a$  in  $I$ .

- (iv) There is a finite set  $S$  of non-archimedean places such that the space

$$V = \rho(\widehat{\mathcal{H}}_S)\varphi$$

transforms under  $\widehat{\mathcal{H}}_S$  according to the irreducible admissible representation  $\pi = \bigotimes_{v \notin S} \pi_v$ .

Then  $V$  is a subspace of  $\mathcal{B}$  and there are two quasi-characters  $\mu$  and  $\nu$  of  $F^\times \backslash I$  such that  $\pi_v$  is a constituent of  $\rho(\mu_v, \nu_v)$  for all  $v$  not in  $S$ .

If one observes that there is a finite set  $T$  of places which is disjoint from  $S$  such that  $I = F^\times I_T$  one can proceed as in Lemma 10.10.1 to show that  $\varphi$  is  $\mathbf{A}$ -finite on the right. Thus there is a finite set  $R$  of pairs of quasi-characters and a non-negative integer  $M$  such that  $V$  is contained in  $\mathcal{B}(R, M)$ . The same reduction as before shows that  $\pi$  is a constituent of the representation of  $\widehat{\mathcal{H}}_S$  on some  $\mathcal{B}(\mu, \nu)$  and that  $\pi_v$  is a constituent of  $\rho(\mu_v, \nu_v)$  if  $v$  is not in  $S$ .

**Lemma 10.13.** *Let  $\varphi$  be a continuous function on  $G_F \backslash G_{\mathbf{A}}$ . If  $\varphi$  satisfies the four following conditions it is an automorphic form.*

- (i)  $\varphi$  is  $K$  finite on the right.
- (ii) There is a quasi-character  $\eta$  of  $F^\times \backslash I$  such that

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}g\right) = \eta(a)\varphi(g)$$

for all  $a$  in  $I$ .

- (iii) There is a finite set  $S$  of non-archimedean places such that  $\rho(\widehat{\mathcal{H}}_S)\varphi$  transforms according to an irreducible admissible representation of  $\widehat{\mathcal{H}}_S$ .
- (iv) If  $F$  is a number field  $\varphi$  is slowly increasing.

We have to show that for every elementary idempotent  $\xi$  in  $\mathcal{H}$  the space  $\rho(\xi\mathcal{H})\varphi$  is finite-dimensional. If  $f$  is a continuous function on  $G_F \backslash G_{\mathbf{A}}$  let

$$f_0(g) = \frac{1}{\text{measure}(F \backslash \mathbf{A})} \int_{F \backslash \mathbf{A}} f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}g\right) dx.$$



The map  $f \rightarrow f_0$  commutes with the action of  $\mathcal{H}$  or of  $\widehat{\mathcal{H}}_S$ . Consequently  $\varphi_0$  satisfies the conditions of the previous lemma and belongs to a space  $\mathcal{B}(R, M)$  invariant under  $\mathcal{H}$  on which  $\rho(\xi)$  has a finite-dimensional range.

We need only show that

$$V = \{ f \in \rho(\xi\mathcal{H})\varphi \mid f_0 = 0 \}$$

is finite-dimensional. If  $F$  is a function field then, by Proposition 10.3,  $V$  is contained in  $\mathcal{A}_0(\eta)$ . More precisely it is contained in the range of  $\rho(\xi)$ , as an operator on  $\mathcal{A}_0(\eta)$ , which we know is finite-dimensional. Suppose  $F$  is a number field. Since every place of  $S$  is non-archimedean the third condition guarantees that  $\varphi$  is an eigenfunction of every element of  $\mathfrak{Z}$ . In particular there is an ideal  $\mathfrak{a}$  of finite codimension in  $\mathfrak{Z}$  which annihilates  $\varphi$  and therefore every element of  $\rho(\xi\mathcal{H})\varphi$ . By Proposition 10.6 the space  $V$  is contained in  $\mathcal{A}_0(\eta)$  and therefore in  $\mathcal{A}_0(\eta, \mathfrak{a})$ . By Proposition 10.8 the range of  $\rho(\xi)$  in  $\mathcal{A}_0(\eta, \mathfrak{a})$  is finite-dimensional.

### §11. Hecke theory

The preliminaries are now complete and we can broach the central topic of these notes. Let  $\psi$  be a non-trivial character of  $F \setminus \mathbf{A}$ . For each place  $v$  the restriction  $\psi_v$  of  $\psi$  to  $F_v$  is non-trivial. Let  $\pi = \bigotimes_v \pi_v$  be an irreducible admissible representation of  $\mathcal{H}$ . The local  $L$ -functions  $L(s, \pi_v)$  and the factors  $\epsilon(s, \pi_v, \psi_v)$  have all been defined. Since for almost all  $v$  the representation  $\pi_v$  contains the trivial representation of  $K_v$  and  $O_v$  is the largest ideal on which  $\psi_v$  is trivial, almost all of the factors  $\epsilon(s, \pi_v, \psi_v)$  are identically 1 and we can form the product

$$\epsilon(s, \pi) = \prod_v \epsilon(s, \pi_v, \psi_v).$$

In general it depends on  $\psi$ . Suppose however that

$$\pi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \eta(a)I$$

and that  $\eta$  is trivial on  $F^\times$ . If  $\psi$  is replaced by the character  $x \rightarrow \psi(\alpha x)$  with  $\alpha$  in  $F^\times$  then  $\epsilon(s, \pi_v, \psi_v)$  is multiplied by  $\eta_v(\alpha)|\alpha|_v^{2s-1}$  so that  $\epsilon(s, \pi)$  is multiplied by

$$\prod_v \eta_v(\alpha)|\alpha|_v^{2s-1} = \eta(\alpha)|\alpha|^{2s-1} = 1$$

The product

$$\prod_v L(s, \pi_v)$$

does not converge and define a function  $L(s, \pi)$  unless  $\pi$  satisfies some further conditions.

**Theorem 11.1.** *Suppose the irreducible admissible representation  $\pi = \bigotimes_v \pi_v$  is a constituent of  $\mathcal{A}$ . Then the infinite products defining  $L(s, \pi)$  and  $L(s, \tilde{\pi})$  converge absolutely in a right half-plane and the functions  $L(s, \pi)$  and  $L(s, \tilde{\pi})$  themselves can be analytically continued to the whole complex plane as meromorphic functions of  $s$ . If  $\pi$  is a constituent of  $\mathcal{A}_0$  they are entire. If  $F$  is a number field they have only a finite number of poles and are bounded at infinity in any vertical strip of finite width. If  $F$  is a function field with field of constants  $\mathbf{F}_q$  they are rational functions of  $q^{-s}$ . Finally they satisfy the functional equation*

$$L(s, \pi) = \epsilon(s, \pi) L(1-s, \tilde{\pi}).$$

Observe that if  $\pi = \bigotimes_v \pi_v$  then  $\tilde{\pi} = \bigotimes_v \tilde{\pi}_v$ . Consider first a representation  $\pi$  which is a constituent of  $\mathcal{A}$  but not of  $\mathcal{A}_0$ . There are quasi-characters  $\mu$  and  $\nu$  of  $F^\times \setminus I$  such that  $\pi_v$  is a constituent of  $\rho(\mu_v, \nu_v)$  for all  $v$ . Since  $\pi_v$  has to contain the trivial representation of  $K_v$  for all but a finite number of  $v$  it is equal to  $\pi(\mu_v, \nu_v)$  for almost all  $v$ .

Consider first the representation  $\pi' = \bigotimes_v \pi(\mu_v, \nu_v)$ . Recall that

$$\begin{aligned} L(s, \pi(\mu_v, \nu_v)) &= L(s, \mu_v) L(s, \nu_v) \\ L(s, \tilde{\pi}(\mu_v, \nu_v)) &= L(s, \mu_v^{-1}) L(s, \nu_v^{-1}) \end{aligned}$$

and

$$\epsilon(s, \pi(\mu_v, \nu_v), \psi_v) = \epsilon(s, \mu_v, \psi_v) \epsilon(s, \nu_v, \psi_v)$$

If  $\chi$  is any quasi-character of  $F^\times \setminus I$  the product

$$\prod_v L(s, \chi_v)$$

is known to converge in a right half plane and the function  $L(s, \chi)$  it defines is known to be analytically continuable to the whole plane as a meromorphic function. Moreover if

$$\epsilon(s, \chi) = \prod_v \epsilon(s, \chi_v, \psi_v)$$

the functional equation

$$L(s, \chi) = \epsilon(s, \chi) L(1-s, \chi^{-1})$$

is satisfied. Since

$$L(s, \pi') = L(s, \mu) L(s, \nu)$$

and

$$L(s, \tilde{\pi}') = L(s, \mu^{-1}) L(s, \nu^{-1})$$

they too are defined and meromorphic in the whole plane and satisfy the functional equation

$$L(s, \pi') = \epsilon(s, \pi') L(1-s, \tilde{\pi}').$$

The other properties of  $L(s, \pi')$  demanded by the lemma, at least when  $\pi'$  is a constituent of  $\mathcal{A}$ , can be inferred from the corresponding properties of  $L(s, \mu)$  and  $L(s, \nu)$  which are well known.

When  $\pi_v$  is not  $\pi(\mu_v, \nu_v)$  it is  $\sigma(\mu_v, \nu_v)$ . We saw in the first chapter that

$$\frac{L(s, \sigma(\mu_v, \nu_v))}{L(s, \pi(\mu_v, \nu_v))}$$

is the product of a polynomial and an exponential. In particular it is entire. If we replace  $\pi(\mu_v, \nu_v)$  by  $\pi_v$  we change only a finite number of the local factors and do not disturb the convergence of the infinite product. If  $S$  is the finite set of places  $v$  at which  $\pi_v = \sigma(\mu_v, \nu_v)$  then

$$L(s, \pi) = L(s, \pi') \prod_{v \in S} \frac{L(s, \sigma(\mu_v, \nu_v))}{L(s, \pi(\mu_v, \nu_v))}$$

and therefore is meromorphic with no more poles than  $L(s, \pi')$ . For  $L(s, \tilde{\pi})$  the corresponding equation is

$$L(s, \tilde{\pi}) = L(s, \tilde{\pi}') \prod_{v \in S} \frac{L(s, \sigma(\mu_v^{-1}, \nu_v^{-1}))}{L(s, \pi(\mu_v^{-1}, \nu_v^{-1}))}.$$

The functional equation of  $L(s, \pi)$  is a consequence of the relations

$$\frac{L(s, \sigma(\mu_v, \nu_v))}{L(s, \pi(\mu_v, \nu_v))} = \frac{\epsilon(s, \sigma(\mu_v, \nu_v), \psi_v) L(1-s, \sigma(\mu_v^{-1}, \nu_v^{-1}))}{\epsilon(s, \pi(\mu_v, \nu_v), \psi_v) L(1-s, \pi(\mu_v^{-1}, \nu_v^{-1}))}$$

which were verified in the first chapter. It also follows from the form of the local factors that  $L(s, \pi)$  and  $L(s, \tilde{\pi})$  are rational functions of  $q^{-s}$  when  $F$  is a function field. If  $F$  is a number field  $L(s, \pi)$  is bounded in vertical strips of finite width in a right half-plane and, because of the functional equation, in vertical strips in a left half-plane. Its expression in terms of  $L(s, \pi')$  prevents it from growing very fast at infinity in any vertical strip of finite width. The Phragmén-Lindelöf principle implies that it is bounded at infinity in any such strip.

Now suppose  $\pi$  is a constituent of  $\mathcal{A}_0$ . It is then a constituent of  $\mathcal{A}_0(\eta)$  if

$$\pi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \eta(a) I$$

for  $a$  in  $I$ . Since the representation of  $\mathcal{H}$  in  $\mathcal{A}_0(\eta)$  is the direct sum of invariant irreducible subspaces there is an invariant subspace  $U$  of  $\mathcal{A}_0(\eta)$  which transforms according to  $\pi$ . Let  $\varphi$  belong to  $U$ . If  $g$  is in  $G_{\mathbf{A}}$

$$\varphi_g(x) = \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}g\right)$$

is a function on  $F \backslash \mathbf{A}$ . Since  $\varphi_g$  is continuous it is determined by its Fourier series. The constant term is

$$\frac{1}{\text{measure } F \backslash \mathbf{A}} \int_{F \backslash \mathbf{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}g\right) dx$$

which is 0 because  $\varphi$  is a cusp form. If  $\psi$  is a given non-trivial character of  $F \backslash \mathbf{A}$  the other non-trivial characters are the functions  $x \rightarrow \psi(\alpha x)$  with  $\alpha$  in  $F^\times$ . Set

$$\varphi_1(g) = \frac{1}{\text{measure } F \backslash \mathbf{A}} \int_{F \backslash \mathbf{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}g\right) \psi(-x) dx.$$

Since  $\varphi$  is a function on  $G_F \backslash G_{\mathbf{A}}$ .

$$\varphi_1\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}g\right) = \frac{1}{\text{measure } F \backslash \mathbf{A}} \int_{F \backslash \mathbf{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}g\right) \psi(-\alpha x) dx$$

if  $\alpha$  belongs to  $F^\times$ . Thus, formally at least,

$$\varphi(g) = \varphi_g(e) = \sum_{\alpha \in F^\times} \varphi_1\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}g\right).$$

In any case it is clear that  $\varphi_1$  is not 0 unless  $\varphi$  is.

Let

$$U_1 = \{\varphi_1 \mid \varphi \in U\}.$$

Since the map  $\varphi \rightarrow \varphi_1$  commutes with the action of  $\mathcal{H}$  the space  $U_1$  is invariant and transforms according to  $\pi$  under right translation by  $\mathcal{H}$ . Moreover

$$\varphi_1\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}g\right) = \psi(x)\varphi_1(g)$$

if  $x$  is in  $\mathbf{A}$ . If  $F$  is a number field  $\varphi$  is slowly increasing. Therefore if  $\Omega$  is a compact subset of  $G_{\mathbf{A}}$  there is a real number  $M$  such that

$$\varphi_1\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}g\right) = O(|a|^M)$$

as  $|a| \rightarrow \infty$  for all  $g$  in  $\Omega$ . Propositions 9.2 and 9.3 imply that all  $\pi_v$  are infinite-dimensional and that  $U_1$  is  $W(\pi, \psi)$ . Therefore  $U_1$  is completely determined by  $\pi$  and  $\psi$  and  $U$  is completely determined by  $\pi$ . We have therefore proved the following curious proposition.

**Proposition 11.1.1.** *If an irreducible representation of  $\mathcal{H}$  is contained in  $\mathcal{A}_0(\eta)$  it is contained with multiplicity one.*

For almost all  $v$  there is in  $W(\pi_v, \psi_v)$  a function  $\varphi_v^0$  such that  $\varphi_v^0(g_v k_v) = \varphi_v^0(g_v)$  for all  $k_v$  in  $K_v$  while  $\varphi_v^0(e) = 1$ . The space  $W(\pi, \psi)$  is spanned by functions of the form

$$(11.1.2) \quad \varphi_1(g) = \prod_v \varphi_v(g_v)$$

where  $\varphi_v$  is in  $W(\pi_v, \psi_v)$  for all  $v$  and equal to  $\varphi_v^0$  for almost all  $v$ .

Suppose  $\varphi$  corresponds to a function  $\varphi_1$  of the form (11.1.2). Suppose  $\varphi_v = \varphi_v^0$  so that  $\pi_v$  contains the trivial representation of  $K_v$ . If  $\epsilon_v$  is the normalized Haar measure on  $K_v$  let  $\lambda_v$  be the homomorphism of  $\epsilon_v \mathcal{H}_v \epsilon_v$  into  $\mathbf{C}$  associated to  $\pi_v$ . If  $f_v$  is in  $\epsilon_v \mathcal{H}_v \epsilon_v$  then

$$\lambda_v(f_v) \varphi(g) = \int_{G_v} \varphi(gh) f_v(h) dh$$

and if  $\lambda'_v$  is the homomorphism associated to  $|\eta_v|^{-1/2} \otimes \pi_v$

$$\lambda'_v(f_v) |\eta(\det g)|^{-1/2} \varphi(g)$$

is equal to

$$\int_{G_v} |\eta(\det gh)|^{-1/2} \varphi(gh) f_v(h) dh.$$

Since  $\varphi$  is a cusp form the function  $|\eta(\det g)|^{-1/2} \varphi(g)$  is bounded and  $\lambda'_v$  satisfies the conditions of Lemma 3.10. Thus if  $\pi_v = \pi(\mu_v, \nu_v)$  both  $\mu_v$  and  $\nu_v$  are unramified and

$$\begin{aligned} |\eta(\varpi_v)|^{1/2} |\varpi_v|^{1/2} &\leq |\mu_v(\varpi_v)| \leq |\eta(\varpi_v)|^{1/2} |\varpi_v|^{-1/2} \\ |\eta(\varpi_v)|^{1/2} |\varpi_v|^{1/2} &\leq |\nu_v(\varpi_v)| \leq |\eta(\varpi_v)|^{1/2} |\varpi_v|^{-1/2} \end{aligned}$$

if  $\varpi_v$  is the generator of the maximal ideal of  $O_v$ . Consequently the infinite products defining  $L(s, \pi)$  and  $L(s, \tilde{\pi})$  converge absolutely for  $\text{Re } s$  sufficiently large.

We know that for any  $v$  and any  $\varphi_v$  in  $W(\pi_v, \psi_v)$  the integral

$$\int_{F_v^\times} \varphi_v \left( \begin{pmatrix} a_v & 0 \\ 0 & 1 \end{pmatrix} g_v \right) |a_v|^{s-\frac{1}{2}} d^\times a_v$$

converges absolutely for  $\text{Re } s$  large enough. Suppose that, for all  $a$  in  $I$ ,  $|\eta(a)| = |a|^r$  with  $r$  real. Applying Lemma 3.11 we see that if  $s + r > \frac{1}{2}$  and  $\varphi_v^0$  is defined

$$\int_{F_v^\times} \left| \varphi_v^0 \left( \begin{pmatrix} a_v & 0 \\ 0 & 1 \end{pmatrix} g_v \right) \right| |a_v|^{s-\frac{1}{2}} d^\times a_v$$

is, for  $g_v$  in  $K_v$ , at most

$$\frac{1}{\left(1 - |\varpi_v|^{s+r-\frac{1}{2}}\right)^2}.$$

Thus if  $\varphi_1$  is of the form (11.1.2) the integral

$$\Psi(g, s, \varphi_1) = \int_I \varphi_1 \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right) |a|^{s-\frac{1}{2}} d^\times a$$

is absolutely convergent and equal to

$$\prod_v \Psi(g_v, s, \varphi_v)$$

for  $\operatorname{Re} s$  sufficiently large. Since  $\Phi(g_v, s, \varphi_v)$  is, by Proposition 3.5, equal to 1 for almost all  $v$  we can set

$$\Phi(g, s, \varphi_1) = \prod_v \Phi(g_v, s, \varphi_v)$$

so that

$$\Psi(g, s, \varphi_1) = L(s, \pi) \Phi(g, s, \varphi_1).$$

We can also introduce

$$\tilde{\Psi}(g, s, \varphi_1) = \int_I \varphi_1 \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right) \eta^{-1}(a) |a|^{s-\frac{1}{2}} d^\times a$$

and show that

$$\tilde{\Psi}(g, s, \varphi_1) = L(s, \tilde{\pi}) \tilde{\Phi}(g, s, \varphi_1)$$

if

$$\tilde{\Phi}(g, s, \varphi_1) = \prod_v \tilde{\Phi}(g_v, s, \varphi_v).$$

**Lemma 11.1.3.** *There is a real number  $s_0$  such that for all  $\varphi_1$  in  $W(\pi, \psi)$  the integrals*

$$\Psi(g, s, \varphi_1) = \int_I \varphi_1 \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right) |a|^{s-\frac{1}{2}} d^\times a$$

$$\tilde{\Psi}(g, s, \varphi_1) = \int_I \varphi_1 \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right) \eta^{-1}(a) |a|^{s-\frac{1}{2}} d^\times a$$

*are absolutely convergent for  $\operatorname{Re} s > s_0$ . The functions  $\Psi(g, s, \varphi_1)$  and  $\tilde{\Psi}(g, s, \varphi_1)$  can both be extended to entire functions of  $s$ . If  $F$  is a number field they are bounded in vertical strips and if  $F$  is a function field they are rational functions of  $q^{-s}$ . Moreover*

$$\tilde{\Psi}(wg, 1-s, \varphi_1) = \Psi(g, s, \varphi_1).$$

We have seen that the first assertion is true for functions of the form (11.1.2). Since they form a basis of  $W(\pi, \psi)$  it is true in general. To show that

$$\varphi(g) = \sum_{\alpha \in F^\times} \varphi_1 \left( \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g \right)$$

we need only show that the series on the right is absolutely convergent. We will do this later on in this paragraph. At the moment we take the equality for granted. Then, for all  $\varphi_1$ ,  $\Psi(g, s, \varphi_1)$  which equals

$$\int_{F^\times \setminus I} \left\{ \sum_{\alpha \in F^\times} \varphi_1 \left( \begin{pmatrix} \alpha a & 0 \\ 0 & 1 \end{pmatrix} g \right) \right\} |a|^{s-\frac{1}{2}} d^\times a$$

is equal to

$$\int_{F^\times \setminus I} \varphi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right) |a|^{s-\frac{1}{2}} d^\times a$$

for  $\operatorname{Re} s$  sufficiently large. Also  $\tilde{\Psi}(g, s, \varphi_1)$  is equal to

$$\int_{F^\times \setminus I} \varphi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right) \eta^{-1}(a) |a|^{s-\frac{1}{2}} d^\times a.$$

We saw in the previous paragraph that, for a given  $g$  and any real number  $M$ ,

$$\left| \varphi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right) \right| = O(|a|^M)$$

as  $|a|$  approaches 0 or  $\infty$ . Thus the two integrals define entire functions of  $s$  which are bounded in vertical strips. If  $F$  is a function field the function

$$a \rightarrow \varphi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right)$$

has compact support on  $F^\times \setminus I$  so that the integral can be expressed as a finite Laurent series in  $q^{-s}$ .

The function  $\tilde{\Psi}(wg, 1-s, \varphi_1)$  is equal to

$$\int \varphi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} wg \right) \eta^{-1}(a) |a|^{\frac{1}{2}-s} d^\times a.$$

Since  $w$  is in  $G_F$  the equality  $\varphi(wg) = \varphi(g)$  holds for all  $g$  in  $G_{\mathbf{A}}$  and this integral is equal to

$$\int \varphi \left( \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} g \right) \eta^{-1}(a) |a|^{\frac{1}{2}-s} d^\times a.$$

Since

$$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

we can change variables in the integral to obtain

$$\int \varphi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right) |a|^{s-\frac{1}{2}} d^\times a$$

which is  $\Psi(g, s, \varphi_1)$ .

If we choose  $\varphi_1$  of the form (11.1.2) we see that  $L(s, \pi)\Phi(g, s, \varphi_1)$  is entire and bounded in vertical strips of finite width. For almost all  $v$  the value of  $\Phi(g_v, s, \varphi_v^0)$  at the identity  $e$  is 1 and for such  $v$  we choose  $\varphi_v = \varphi_v^0$ . At the other places we choose  $\varphi_v$  so that  $\Phi(e, s, \varphi_v)$  is an exponential  $e^{a_v s}$  with real  $a_v$ . Then  $\Phi(e, s, \varphi_1)$  is an exponential. Consequently  $L(s, \pi)$  is also entire and bounded in vertical strips of finite width. If  $F$  is a number field  $\Phi(e, s, \varphi_1)$  will be a power of  $q^{-s}$  so that  $L(s, \pi)$  will be a finite Laurent series in  $q^{-s}$ . Similar considerations apply to  $L(s, \tilde{\pi})$ .

To prove the functional equation we start with the relation

$$L(s, \pi) \prod_v \Phi(e, s, \varphi_v) = L(1-s, \tilde{\pi}) \prod_v \tilde{\Phi}(w, 1-s, \varphi_v).$$

By the local functional equation the right hand side is

$$L(1-s, \tilde{\pi}) \prod_v \{\epsilon(s, \pi_v, \psi_v) \Phi(e, s, \varphi_v)\}.$$

Cancelling the term  $\prod_v \Phi(e, s, \varphi_v)$  we obtain

$$L(s, \pi) = \epsilon(s, \pi) L(1-s, \tilde{\pi}).$$

**Corollary 11.2.** *Suppose  $\pi = \bigotimes_v \pi_v$  is a constituent of  $\mathcal{A}$ . For any quasi-character  $\omega$  of  $F^\times \backslash I$  the products*

$$\prod_v L(s, \omega_v \otimes \pi_v)$$

and

$$\prod_v L(s, \omega_v^{-1} \otimes \tilde{\pi}_v)$$

are absolutely convergent for  $\text{Re } s$  sufficiently large. The functions  $L(s, \omega \otimes \pi)$  and  $L(s, \omega^{-1} \otimes \tilde{\pi})$  they define can be analytically continued to the whole complex plane as meromorphic functions which are bounded at infinity in vertical strips of finite width and have only a finite number of poles. If  $F$  is a function field they are rational functions of  $q^{-s}$ . If  $\pi$  is a constituent of  $\mathcal{A}_0$  they are entire. In all cases they satisfy the functional equation

$$L(s, \omega \otimes \pi) = \epsilon(s, \omega \otimes \pi) L(1-s, \omega^{-1} \otimes \tilde{\pi})$$

if

$$\epsilon(s, \omega \otimes \pi) = \prod_v \epsilon(s, \omega_v \otimes \pi_v, \psi_v).$$

If  $\pi = \bigotimes_v \pi_v$  is a constituent of  $\mathcal{A}$  or  $\mathcal{A}_0$  and  $\omega$  is a quasi-character of  $F^\times \backslash I$  so is  $\omega \otimes \pi$ . Moreover  $\omega \otimes \pi = \bigotimes_v (\omega_v \otimes \pi_v)$ .

The converses to the corollary can take various forms. We consider only the simplest of these. In particular, as far as possible, we restrict ourselves to cusp forms.

**Theorem 11.3.** *Let  $\pi = \bigotimes_v \pi_v$  be a given irreducible representation of  $\mathcal{H}$ . Suppose that the quasi-character  $\eta$  of  $I$  defined by*

$$\pi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \eta(a) I$$

is trivial on  $F^\times$ . Suppose there is a real number  $r$  such that whenever  $\pi_v = \pi(\mu_v, \nu_v)$  the inequalities

$$|\varpi_v|^{-r} \leq |\mu_v(\varpi_v)| \leq |\varpi_v|^r$$

and

$$|\varpi_v|^{-r} \leq |\nu_v(\varpi_v)| \leq |\varpi_v|^r$$

are satisfied. Then for any quasi-character  $\omega$  of  $F^\times \backslash I$  the infinite products

$$L(s, \omega \otimes \pi) = \prod_v L(s, \omega_v \otimes \pi_v)$$



and

$$L(s, \omega^{-1} \otimes \tilde{\pi}) = \prod_v L(s, \omega_v^{-1} \otimes \tilde{\pi}_v)$$

are absolutely convergent for  $\operatorname{Re} s$  large enough. Suppose  $L(s, \omega \otimes \pi)$  and  $L(s, \omega^{-1} \otimes \tilde{\pi})$  are, for all  $\omega$ , entire functions of  $s$  which are bounded in vertical strips and satisfy the functional equation

$$L(s, \omega \otimes \pi) = \epsilon(s, \omega \otimes \pi) L(1-s, \omega^{-1} \otimes \tilde{\pi})$$

If the  $\pi_v$  are all infinite-dimensional  $\pi$  is a constituent of  $\mathcal{A}_0$ .

The absolute convergence of the infinite products is clear. We have to construct a subspace  $U$  of  $\mathcal{A}_0$  which is invariant under  $\mathcal{H}$  and transforms according to the representation  $\pi$ . The space  $W(\pi, \psi)$  transforms according to  $\pi$ . If  $\varphi_1$  belongs to  $W(\pi, \psi)$  set

$$\varphi(g) = \sum_{\alpha \in F^\times} \varphi_1 \left( \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g \right)$$

We shall see later that this series converges absolutely and uniformly on compact subsets of  $G_{\mathbf{A}}$ . Thus  $\varphi$  is a continuous function on  $G_{\mathbf{A}}$ . Since the map  $\varphi_1 \rightarrow \varphi$  commutes with right translations by the elements of  $\mathcal{H}$  we have to show that, for all  $\varphi_1$ ,  $\varphi$  is in  $\mathcal{A}_0$  and that  $\varphi$  is not zero unless  $\varphi_1$  is.

Since  $\psi$  is a character of  $F \backslash \mathbf{A}$

$$\varphi \left( \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} g \right) = \varphi(g)$$

for all  $\xi$  in  $F$ . Thus, for each  $g$ ,

$$\varphi \left( \begin{pmatrix} 1 & x \\ 0 & a \end{pmatrix} g \right)$$

is a function on  $F \backslash \mathbf{A}$ . The constant term of its Fourier expansion is

$$\frac{1}{\text{measure } F \backslash \mathbf{A}} \int_{F \backslash \mathbf{A}} \varphi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx.$$

The integral is equal to

$$\sum_{\alpha} \int_{F \backslash \mathbf{A}} \varphi_1 \left( \begin{pmatrix} 1 & \alpha x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g \right) dx.$$

A typical term of this sum is

$$\varphi_1 \left( \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g \right) \int_{F \backslash \mathbf{A}} \psi(\alpha x) dx = 0.$$

In particular  $\varphi$  is cuspidal. Another simple calculation shows that if  $\beta$  belongs to  $F^\times$

$$\frac{1}{\text{measure } F \backslash \mathbf{A}} \int_{F \backslash \mathbf{A}} \varphi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \psi(-\beta x) dx$$

is equal to

$$\varphi_1 \left( \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} g \right).$$

Thus  $\varphi_1$  is zero if  $\varphi$  is.

By construction

$$\varphi \left( \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g \right) = \varphi(g)$$

if  $\alpha$  is in  $F^\times$ . Moreover, for all  $a$  in  $I$ ,

$$\varphi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g \right) = \eta(a) \varphi(g).$$

If  $a$  is in  $F^\times$  the right side is just  $\varphi(g)$ . Thus  $\varphi$  is invariant under left translations by elements of  $P_F$ , the group of super-triangular matrices in  $G_F$ . Since  $G_F$  is generated by  $P_F$  and  $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  all we need do to show that  $\varphi$  is a function on  $G_F \backslash G_{\mathbf{A}}$  is to show that

$$\varphi(wg) = \varphi(g).$$

By linearity we need only establish this when  $\varphi_1$  has the form (11.1.2). The hypothesis implies as in the direct theorem that the integrals

$$\Psi(g, s, \varphi_1) = \int_I \varphi_1 \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right) |a|^{s-\frac{1}{2}} d^\times a$$

and

$$\tilde{\Psi}(g, s, \varphi_1) = \int_I \varphi_1 \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right) \eta^{-1}(a) |a|^{s-\frac{1}{2}} d^\times a$$

converge absolutely for  $\text{Re } s$  sufficiently large. Moreover

$$\Psi(g, s, \varphi_1) = \prod_v \Psi(g_v, s, \varphi_v) = L(s, \pi) \prod_v \Phi(g_v, s, \varphi_v).$$

Almost all factors in the product on the right are identically 1 so that the product, and therefore  $\Psi(g, s, \varphi_1)$ , is an entire function of  $s$ . In the same way

$$\tilde{\Psi}(g, s, \varphi_1) = L(s, \tilde{\pi}) \prod_v \tilde{\Phi}(g_v, s, \varphi_v)$$

and is entire. Since

$$\tilde{\Phi}(wg_v, 1-s, \varphi_v) = \epsilon(s, \pi_v, \psi_v) \Phi(g_v, s, \varphi_v)$$

the function  $\tilde{\Psi}(wg, 1-s, \varphi_v)$  is equal to

$$L(1-s, \tilde{\pi}) \epsilon(s, \pi) \prod_v \Phi(g_v, s, \varphi_v),$$

which, because of the functional equation assumed for  $L(s, \pi)$ , is equal to  $\Psi(g, s, \varphi_1)$ .

From its integral representation the function  $\Psi(g, s, \varphi_1)$  is bounded in any vertical strip of finite width contained in a certain right half-plane. The equation just established shows that it is also bounded in vertical strips of a left half-plane. To verify that it is

bounded in any vertical strip we just have to check that it grows sufficiently slowly that the Phragmén-Lindelöf principle can be applied.

$$\Psi(g, s, \varphi_1) = L(s, \pi) \prod_v \Phi(g_v, s, \varphi_v).$$

The first term is bounded in any vertical strip by hypothesis. Almost all factors in the infinite product are identically 1. If  $v$  is non-archimedean  $\Phi(g_v, s, \varphi_v)$  is a function of  $|\varpi_v|^s$  and is therefore bounded in any vertical strip. If  $v$  is archimedean

$$\Phi(g_v, s, \varphi_v) = \frac{\Psi(g_v, s, \varphi_v)}{L(s, \pi_v)}$$

We have shown that the numerator is bounded at infinity in vertical strips. The denominator is, apart from an exponential factor, a  $\Gamma$ -function. Stirling's formula shows that it goes to 0 sufficiently slowly at infinity.

If  $\operatorname{Re} s$  is sufficiently large

$$\Psi(g, s, \varphi_1) = \int_{F^\times \setminus I} \sum_{\alpha \in F^\times} \varphi_1 \left( \begin{pmatrix} \alpha a & 0 \\ 0 & 1 \end{pmatrix} g \right) |a|^{s-\frac{1}{2}} d^\times a$$

which is

$$\int_{F^\times \setminus I} \varphi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right) |a|^{s-\frac{1}{2}} d^\times a.$$

This integral converges absolutely when  $\operatorname{Re} s$  is sufficiently large. If  $\operatorname{Re} s$  is large and negative

$$\tilde{\Psi}(wg, 1-s, \varphi_1) = \int_{F^\times \setminus I} \varphi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} wg \right) \eta^{-1}(a) |a|^{\frac{1}{2}-s} d^\times a$$

which equals

$$\int_{F^\times \setminus I} \varphi \left( w \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} g \right) \eta^{-1}(a) |a|^{\frac{1}{2}-s} d^\times a.$$

Using the relation

$$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

and changing variables we see that this integral is equal to

$$\int_{F^\times \setminus I} \varphi \left( w \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right) |a|^{s-\frac{1}{2}} d^\times a.$$

Set

$$f_1(a) = \varphi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right)$$

and

$$f_2(s) = \varphi \left( w \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right).$$

We are trying to show that for any  $g$  the functions  $f_1$  and  $f_2$  are equal. The previous discussion applies to  $\omega \otimes \pi$  as well as to  $\pi$ . If  $\varphi_1$  is in  $W(\pi, \psi)$  the function

$$\varphi'_1(g) = \omega(\det g) \varphi_1(g)$$

is in  $W(\omega \otimes \pi, \psi)$ . When  $\varphi_1$  is replaced by  $\varphi'_1$  the function  $\varphi$  is replaced by

$$\varphi'(g) = \omega(\det g)\varphi(g)$$

and  $f_i$  is replaced by

$$f'_i(a) = \omega(\det g)\omega(a)f_i(a).$$

Thus for any quasi-character  $\omega$  of  $F^\times \backslash I$  the integral

$$\int_{F^\times \backslash I} f_1(a)\omega(a)|a|^{s-\frac{1}{2}} d^\times a$$

is absolutely convergent for  $\operatorname{Re} s$  sufficiently large and the integral

$$\int_{F^\times \backslash I} f_2(a)\omega(a)|a|^{s-\frac{1}{2}} d^\times a$$

is absolutely convergent for  $\operatorname{Re} s$  large and negative. Both integrals represent functions which can be analytically continued to the same entire function. This entire function is bounded in vertical strips of finite width.

The equality of  $f_1$  and  $f_2$  is a result of the following lemma.

**Lemma 11.3.1.** *Let  $f_1$  and  $f_2$  be two continuous functions on  $F^\times \backslash I$ . Assume that there is a constant  $c$  such that for all characters  $\omega$  of  $F^\times \backslash I$  the integral*

$$\int_{F^\times \backslash I} f_1(a)\omega(a)|a|^s d^\times a$$

*is absolutely convergent for  $\operatorname{Re} s > c$  and the integral*

$$\int_{F^\times \backslash I} f_2(a)\omega(a)|a|^s d^\times a$$

*is absolutely convergent for  $\operatorname{Re} s < -c$ . Assume that the functions represented by these integrals can be analytically continued to the same entire function and that this entire function is bounded in vertical strips of finite width. Then  $f_1$  and  $f_2$  are equal.*

Let  $I_0$  be the group of ideles of norm 1. Then  $F^\times \backslash I_0$  is compact. It will be enough to show that for each  $b$  in  $I$  the functions  $f_1(ab)$  and  $f_2(ab)$  on  $F^\times \backslash I_0$  are equal. They are equal if they have the same Fourier expansions. Since any character of  $F^\times \backslash I_0$  can be extended to a character of  $F^\times \backslash I$  we have just to show that for every character  $\omega$  of  $F^\times \backslash I$

$$\widehat{f}_1(\omega, b) = \omega(b) \int_{F^\times \backslash I_0} f_1(ab)\omega(a) d^\times a$$

is equal to

$$\widehat{f}_2(\omega, b) = \omega(b) \int_{F^\times \backslash I_0} f_2(ab)\omega(a) d^\times a.$$

These two functions are functions on  $I_0 \backslash I$  which is isomorphic to  $\mathbf{Z}$  if  $F$  is a number field and to  $\mathbf{R}$  if  $F$  is a function field.

If  $F$  is a function field we have only to verify the following lemma.

**Lemma 11.3.2.** *Suppose  $\{a_1(n) \mid n \in \mathbf{Z}\}$  and  $\{a_2(n) \mid n \in \mathbf{Z}\}$  are two sequences and  $q > 1$  is a real number. Suppose*

$$\sum_n a_1(n)q^{-ns}$$

converges for  $\operatorname{Re} s$  sufficiently large and

$$\sum_n a_2(n) q^{-ns}$$

converges absolutely for  $\operatorname{Re} s$  large and negative. If the functions they represent can be analytically continued to the same entire function of  $s$  the two sequences are equal.

Once stated the lemma is seen to amount to the uniqueness of the Laurent expansion. If  $F$  is a number field the lemma to be proved is a little more complicated.

**Lemma 11.3.3.** *Suppose  $g_1$  and  $g_2$  are two continuous functions on  $\mathbf{R}$ . Suppose there is a constant  $c$  such that*

$$\widehat{g}_1(s) = \int_{\mathbf{R}} g_1(x) e^{sx} dx$$

converges absolutely for  $\operatorname{Re} s > c$  and

$$\widehat{g}_2(s) = \int_{\mathbf{R}} g_2(x) e^{sx} dx$$

converges absolutely for  $\operatorname{Re} s < -c$ . If  $\widehat{g}_1$  and  $\widehat{g}_2$  represent the same entire function and this function is bounded in vertical strips then  $g_1 = g_2$ .

All we need do is show that for every compactly supported infinitely differentiable function  $g$  the functions  $g * g_1$  and  $g * g_2$  are equal. If

$$\widehat{g}(s) = \int_{\mathbf{R}} g(x) e^{sx} dx$$

is the Laplace transform of  $g$  the Laplace transform of  $g * g_i$  is  $\widehat{g}(s)\widehat{g}_i(s)$ . By the inversion formula

$$g * g_i(x) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \widehat{g}(s)\widehat{g}_i(s) e^{-xs} ds$$

where  $b > c$  if  $i = 1$  and  $b < -c$  if  $i = 2$ . The integral converges because  $\widehat{g}$  goes to 0 faster than the inverse of any polynomial in a vertical strip. Cauchy's integral theorem implies that the integral is independent of  $b$ . The lemma follows.

To complete the proof of Theorem 11.3, and Theorem 11.1, we have to show that for any  $\varphi_1$  in  $W(\pi, \psi)$  the series

$$\sum_{\alpha \in F^\times} \varphi_1 \left( \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g \right)$$

is uniformly absolutely convergent for  $g$  in a compact subset of  $G_{\mathbf{A}}$  and that if  $\varphi(g)$  is its sum then, if  $F$  is a number field, for any compact subset  $\Omega$  of  $G_{\mathbf{A}}$  and any  $c > 0$  there are constants  $M_1$  and  $M_2$  such that

$$\left| \varphi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right) \right| \leq M_1 |a|^{M_2}$$

for  $g$  in  $\Omega$  and  $|a| \geq c$ . We prefer to prove these facts in a more general context which will now be described.

For us a divisor is just a formal product of the form

$$D = \prod \mathfrak{p}^{m_{\mathfrak{p}}}.$$

It is taken over all non-archimedean places. The integers  $m_{\mathfrak{p}}$  are non-negative and all but a finite number of them are 0. Let  $S$  be a finite set of non-archimedean places containing all the divisors of  $D$ , that is, all places  $\mathfrak{p}$  for which  $m_{\mathfrak{p}} > 0$ .

If  $a$  belongs to  $I$  we can write  $a$  in a unique manner as a product  $a_S \hat{a}_S$  where the components of  $a_S$  outside  $S$  are 1 and those of  $\hat{a}_S$  inside  $S$  are 1. The idele  $a_S$  belongs to  $I_S = \prod_{v \in S} F_v^\times$ . Let  $I_D^S$  be the set of ideles  $a$  such that, for any  $\mathfrak{p}$  in  $S$ ,  $a_{\mathfrak{p}}$  is a unit which satisfies  $a_{\mathfrak{p}} \equiv 1 \pmod{\mathfrak{p}^{m_{\mathfrak{p}}}}$ . Then  $I = F^\times I_D^S$  and  $F^\times \backslash I$  is isomorphic to  $F^\times \cap I_D^S \backslash I_D^S$ .

If  $\mathfrak{p}$  is in  $S$  let  $K_{\mathfrak{p}}^D$  be the subgroup of all

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in  $K_{\mathfrak{p}}$  for which  $c \equiv 0 \pmod{\mathfrak{p}^{m_{\mathfrak{p}}}}$ . Let  $\hat{K}_{\mathfrak{p}}^D$  be the subgroup of such matrices for which  $a \equiv d \equiv 1 \pmod{\mathfrak{p}^{m_{\mathfrak{p}}}}$ . Set

$$K_S^D = \prod_{\mathfrak{p} \in S} K_{\mathfrak{p}}^D$$

and set

$$\hat{K}_S^D = \prod_{\mathfrak{p} \in S} \hat{K}_{\mathfrak{p}}^D$$

$\hat{K}_S^D$  is a normal subgroup of  $K_S^D$  and the quotient  $K_S^D / \hat{K}_S^D$  is abelian.

Let  $G_D^S$  be the set of all  $g$  in  $G_{\mathbf{A}}$  such that  $g_{\mathfrak{p}}$  is in the group  $K_{\mathfrak{p}}^D$  for all  $\mathfrak{p}$  in  $S$ . Any  $g$  in  $G_{\mathbf{A}}$  may be written as a product  $g_S \hat{g}_S$  where  $g_S$  has component 1 outside of  $S$  and  $\hat{g}_S$  has component 1 inside  $S$ .  $G_S$  is the set of  $g_S$  and  $\hat{G}_S$  is the set of  $\hat{g}_S$ . In particular

$$G_D^S = K_S^D \cdot \hat{G}_S.$$

It is easily seen that

$$G_{\mathbf{A}} = G_F G_D^S.$$

In addition to  $D$  and  $S$  we suppose we are given a non-trivial character  $\psi$  of  $F \backslash \mathbf{A}$ , two characters  $\epsilon$  and  $\hat{\epsilon}$  of  $K_S^D / \hat{K}_S^D$ , two complex valued functions  $\alpha \rightarrow a_{\alpha}$  and  $\alpha \rightarrow \hat{a}_{\alpha}$  on  $F^\times$ , an irreducible representation  $\pi$  of  $\hat{\mathcal{H}}_S = \bigotimes_{v \notin S} \mathcal{H}_v$ , and a quasi-character  $\eta$  of  $F^\times \backslash I$ .

There are a number of conditions to be satisfied. If

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

belongs to  $K_S^D$  then

$$\hat{\epsilon} \left( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right) = \epsilon \left( \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} \right).$$

If  $\alpha$  belongs to  $F^\times$  and  $\beta$  belongs to  $F^\times \cap I_D^S$

$$a_{\alpha\beta} = \epsilon \left( \begin{pmatrix} \beta_S & 0 \\ 0 & 1 \end{pmatrix} \right) a_{\alpha}$$

and

$$\hat{a}_{\alpha\beta} = \hat{\epsilon} \left( \begin{pmatrix} \beta_S & 0 \\ 0 & 1 \end{pmatrix} \right) \hat{a}_{\alpha}.$$

The functions  $\alpha \rightarrow a_\alpha$  and  $\alpha \rightarrow \widehat{a}_\alpha$  are bounded. Moreover  $a_\alpha = \widehat{a}_\alpha = 0$  if for some  $v$  in  $S$  the number  $\alpha$  regarded as an element of  $F_v$  does not lie in the largest ideal on which  $\psi_v$  is trivial. If  $v$  belongs to  $S$  and  $a$  is a unit in  $O_v$

$$\epsilon \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \eta_v(a).$$

Let  $\pi = \bigotimes_{v \notin S} \pi_v$ . Then for  $a$  in  $F_v^\times$

$$\pi_v \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \eta_v(a)I.$$

Because of these two conditions  $\eta$  is determined by  $\pi$  and  $\epsilon$ . There is a real number  $r$  such that if  $\pi_v = \pi(\mu_v, \nu_v)$

$$|\varpi_v|^r \leq |\mu_v(\varpi_v)| \leq |\varpi_v|^{-r}$$

and

$$|\varpi_v|^r \leq |\nu_v(\varpi_v)| \leq |\varpi_v|^{-r}.$$

Finally we suppose that  $\pi_v$  is infinite-dimensional for all  $v$  not in  $S$ .

These conditions are rather complicated. None the less in the next paragraph we shall find ourselves in a situation in which they are satisfied. When  $S$  is empty,  $D = 0$ , and  $a_\alpha = \widehat{a}_\alpha = 1$  for all  $\alpha$  they reduce to those of Theorem 11.3. In particular with the next lemma the proof of that theorem will be complete. We shall use the conditions to construct a space  $U$  of automorphic forms on  $G_{\mathbf{A}}$  such that  $U$  transforms under  $\widehat{\mathcal{H}}_S$  according to  $\pi$  while each  $\varphi$  in  $U$  satisfies

$$\varphi(gh) = \epsilon(h)\varphi(g)$$

for  $h$  in  $K_S^D$ . If  $U$  is such a space then for any  $\varphi$  in  $U$  and any  $a$  in  $I$

$$\varphi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g \right) = \eta(a)\varphi(g).$$

This is clear if  $a$  belongs to  $I_D^S$  and follows in general from the relation  $I = F^\times I_D^S$ .

Recall that  $W(\pi, \psi)$  is the space of functions on  $\widehat{G}_S$  spanned by functions of the form

$$\varphi_1(g) = \prod_{v \notin S} \varphi_v(g_v)$$

where  $\varphi_v$  belongs to  $W(\pi_v, \psi_v)$  for all  $v$  and is equal to  $\varphi_v^0$  for almost all  $v$ .

**Lemma 11.4.** *Suppose  $\varphi_1$  belongs to  $W(\pi, \psi)$ .*

(i) *For any  $g$  in  $G_D^S$  the series*

$$\varphi(g) = \sum_{\alpha \in F^\times} a_\alpha \epsilon(g_S) \varphi_1 \left( \begin{pmatrix} \widehat{\alpha}_S & 0 \\ 0 & 1 \end{pmatrix} \widehat{g}_S \right)$$

*converges absolutely. The convergence is uniform on compact subsets of  $G_D^S$ .*

(ii) *The function  $\varphi$  defined by this series is invariant under left translation by the matrices in  $G_F \cap G_D^S$  of the form*

$$\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}.$$

(iii) Suppose  $F$  is a number field. Let  $\Omega$  be a compact subset of  $G_D^S$ . Then there are positive constants  $M_1$  and  $M_2$  such that

$$|\varphi(g)| \leq M_1 \{|a| + |a|^{-1}\}^{M_2}$$

if

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} h$$

with  $h$  in  $\Omega$ ,  $a$  in  $I_D^S$ , and  $\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$  in  $G_D^S$ .

It is enough to prove these assertions when  $\varphi_1$  has the form

$$\varphi_1(g) = \prod_{v \notin S} \varphi_v(g_v).$$

To establish the first and third assertions we need only consider the series

$$(11.4.1) \quad \sum_{\alpha \in F^\times} \delta(\alpha) \prod_{v \notin S} \left| \varphi_v \left( \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g_v \right) \right|$$

where  $\delta(\alpha) = 0$  if for some  $v$  in  $S$  the number  $\alpha$  regarded as an element of  $F_v$  is not the largest ideal on which  $\psi_v$  is trivial and  $\delta(\alpha) = 1$  otherwise.

We need only consider compact sets  $\Omega$  of the form

$$(11.4.2) \quad \Omega = K_S^D \prod_{v \notin S} \Omega_v$$

where  $\Omega_v$  is a compact subset of  $G_v$  and  $\Omega_v = K_v$  for almost all  $v$ .

**Lemma 11.4.3.** *Suppose  $\Omega$  is of the form (11.4.2). There is a positive number  $\rho$  such that for each non-archimedean place  $v$  which is not in  $S$  there is a constant  $M_v$  such that*

$$\left| \varphi_v \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} h \right) \right| \leq M_v |a|^{-\rho}$$

for  $a$  in  $F_v^\times$  and  $h$  in  $\Omega_v$  and a constant  $c_v$  such that

$$\left| \varphi_v \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} h \right) \right| = 0$$

if  $|a| > c_v$  and  $h$  is in  $\Omega_v$ . Moreover one may take  $M_v = c_v = 1$  for almost all  $v$ .

Since  $\varphi_v$  is invariant under an open subgroup of  $K_v$  for all  $v$  and is invariant under  $K_v$  for almost all  $v$  while  $\Omega_v = K_v$  for almost all  $v$  it is enough to prove the existence of  $M_v$ ,  $c_v$ , and  $\rho$  such that these relations are satisfied when  $h = 1$ . Since the function

$$a \rightarrow \varphi_v \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right)$$

belongs to the space of the Kirillov model the existence of  $c_v$  is clear. The constant  $c_v$  can be taken to be 1 when  $O_v$  is the largest ideal of  $F_v$  on which  $\psi_v$  is trivial and  $\varphi_v = \varphi_v^0$ .



The existence of  $M_v$ , for a given  $v$  and sufficiently large  $\rho$ , is a result of the absolute convergence of the integral defining  $\Psi(e, s, \varphi_v)$ . Thus all we need do is show the existence of a fixed  $\rho$  such that the inequality

$$\left| \varphi_v \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) \right| \leq |a|^{-\rho}$$

is valid for almost all  $v$ . For almost all  $v$  the representation  $\pi_v$  is of the form  $\pi(\mu_v, \nu_v)$  with  $\mu_v$  and  $\nu_v$  unramified,  $O_v$  is the largest ideal of  $F_v$  on which  $\psi_v$  is trivial, and  $\varphi_v = \varphi_v^0$ . Thus, for such  $v$ ,

$$\varphi_v \left( \begin{pmatrix} \epsilon a & 0 \\ 0 & 1 \end{pmatrix} \right) = \varphi_v \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right)$$

if  $\epsilon$  is a unit in  $O_v$  and

$$\sum_n \varphi_v \left( \begin{pmatrix} \varpi_v^n & 0 \\ 0 & 1 \end{pmatrix} \right) |\varpi_v|^{n(s-\frac{1}{2})} = L(s, \pi_v).$$

If  $\rho_v = \mu_v(\varpi_v)$  and  $\sigma_v = \nu_v(\varpi_v)$

$$L(s, \pi_v) = \frac{1}{(1 - \rho_v |\varpi_v|^s)} \frac{1}{(1 - \sigma_v |\varpi_v|^s)}.$$

Since  $|\rho_v| \leq |\varpi_v|^{-r}$  and  $|\sigma_v| \leq |\varpi_v|^{-r}$

$$\left| \varphi_v \left( \begin{pmatrix} \varpi_v^n & 0 \\ 0 & 1 \end{pmatrix} \right) \right| = \left| \frac{\rho_v^{n+1} - \sigma_v^{n+1}}{\rho_v - \sigma_v} \right| \leq (n+1) |\varpi_v|^{-rn}.$$

Since  $|\varpi_v| \leq \frac{1}{2}$  there is a constant  $\epsilon > 0$  such that

$$(n+1) \leq |\varpi_v|^{-\epsilon n}$$

for all  $v$  and all  $n \geq 0$ .

If  $v$  is archimedean the integral representations of the functions in  $W(\pi_v, \psi_v)$  show that there are positive constants  $c_v$ ,  $d_v$ , and  $M_v$  such that

$$\left| \varphi_v \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} h \right) \right| \leq M_v |a|^{-c_v} \exp(-d_v |a|_v^{\epsilon_v})$$

for  $a$  in  $F_v^\times$  and  $h$  in  $\Omega_v$ . The exponent  $\epsilon_v$  is 1 if  $v$  is real and  $\frac{1}{2}$  if  $v$  is complex.

Since we want to prove not only the first assertion but also the third we consider the sum

$$f \left( \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} g \right) = \sum_{\alpha \in F^\times} \delta(\alpha) \prod_{v \notin S} \left| \varphi_v \left( \begin{pmatrix} b_v \alpha & 0 \\ 0 & 1 \end{pmatrix} g_v \right) \right|$$

where  $g$  lies in the set (11.4.2) and  $b$  is an idele such that  $b_v = 1$  for all non-archimedean  $v$ . We also suppose that there is a positive number  $t$  such that  $b_v = t$  for all archimedean  $v$ . If

$\Lambda$  is a set of  $\alpha$  in  $F$  for which  $|\alpha|_v \leq c_v$  for all non-archimedean  $v$  not in  $S$  and  $\delta(\alpha) \neq 0$  then

$$f\left(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

is bounded by

$$\sum_{\substack{\alpha \in \Lambda \\ \alpha \neq 0}} \left\{ \prod_{v \in S_a} M_v |\alpha|_v^{-c_v} \exp(-d_v t |\alpha|_v^{\epsilon_v}) \right\} \left\{ \prod_{v \notin S \cup S_a} M_v |\alpha_v|^{-\rho} \right\}.$$

If  $F$  is a function field  $\Lambda$  is a finite set and there is nothing more to prove. If it is a number field choose for each  $v$  in  $S$  a constant  $c_v$  such that  $\delta(\alpha) = 0$  unless  $|\alpha|_v \leq c_v$ . Since

$$\prod_v |\alpha|_v = 1,$$

$$\prod_{v \notin S \cup S_a} |\alpha|_v^{-\rho} \leq \left\{ \prod_{v \in S} c_v^\rho \right\} \left\{ \prod_{v \in S_a} |\alpha|_v^\rho \right\}.$$

Thus our sum is bounded by a constant times the product of  $\prod_{v \in S_a} t^{-c_v/\epsilon_v}$  and

$$\sum_{\substack{\alpha \in \Lambda \\ \alpha \neq 0}} \prod_{v \in S_a} \left\{ |\alpha|_v^{\rho - c_v} \exp(-d_v t |\alpha|_v^{\epsilon_v}) \right\}.$$

The product  $\prod_{v \in S_a} |\alpha|_v$  is bounded below on  $\Lambda - \{0\}$ . Multiplying each term by the same sufficiently high power of  $\prod_{v \in S_a} |\alpha|_v$  we dominate the series by another series

$$\sum_{\alpha \in \Lambda} \prod_{v \in S_a} \left\{ |\alpha|_v^{\rho_v} \exp(-d_v t |\alpha|_v^{\epsilon_v}) \right\}$$

in which the exponents  $\rho_v$  are non-negative. This in turn is dominated by  $\prod_{v \in S_a} t^{-\rho_v/\epsilon_v}$  times

$$\sum_{\alpha \in \Lambda} \prod_{v \in S_a} \exp\left(\frac{-d_v}{2} t |\alpha|_v^{\epsilon_v}\right).$$

$\Lambda$  may be regarded as a lattice in  $\prod_{v \in S_a} F_v$ . If  $\lambda_1, \dots, \lambda_n$  is a basis of  $\Lambda$  there is a constant  $d$  such that if  $\alpha = \sum a_i \lambda_i$

$$\sum_{v \in S_a} \frac{d_v}{2} |\alpha|_v^{\epsilon_v} \geq d \sum |a_i|.$$

Thus

$$f\left(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

is dominated by some power of  $t$  times a multiple of

$$\left\{ \sum_{a=-\infty}^{\infty} e^{-dt|a|} \right\}^n$$

which is bounded by a multiple of  $(1 + \frac{1}{t})^n$ .

The first assertion is now proved and the third will now follow from the second and the observation that every element of  $I_D^S$  is the product of an element of  $F^\times$ , an idele whose components are 1 at all non-archimedean places and equal to the same positive number at all archimedean places, and an idele which lies in a certain compact set.

Suppose  $\xi$  is in  $F$  and

$$\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}$$

belong to  $G_D^S$ . Then  $\xi$  is integral at each prime of  $S$  and  $\psi_v(\alpha\xi) = 1$  if  $a_\alpha \neq 0$ . If  $g$  belongs to  $G_D^S$  and

$$h = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} g$$

then  $\epsilon(h_S) = \epsilon(g_S)$  and if  $v$  is not in  $S$

$$\varphi_v \left( \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} h_v \right) = \psi_v(\alpha\xi) \varphi_v \left( \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g_v \right).$$

If  $a_\alpha \neq 0$

$$\prod_{v \notin S} \psi_v(\alpha\xi) = \prod_v \psi_v(\alpha\xi) = 1.$$

Consequently

$$\varphi \left( \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} g \right) = \varphi(g).$$

If  $b$  belongs to  $I_D^S$  then

$$\epsilon \left( \begin{pmatrix} b_S & 0 \\ 0 & b_S \end{pmatrix} g_S \right) = \eta(b_S) \epsilon(g_S)$$

and

$$\varphi_1 \left( \begin{pmatrix} \widehat{b}_S \widehat{\alpha}_S & 0 \\ 0 & \widehat{b}_S \end{pmatrix} \widehat{g}_S \right) = \eta(\widehat{b}_S) \varphi_1 \left( \begin{pmatrix} \widehat{\alpha}_S & 0 \\ 0 & 1 \end{pmatrix} \widehat{g}_S \right)$$

so that

$$\varphi \left( \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} g \right) = \eta(b) \varphi(g).$$

In particular if  $\beta$  belongs to  $F^\times \cap I_D^S$

$$\varphi \left( \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix} g \right) = \varphi(g).$$

If  $\beta$  belongs to  $F^\times \cap I_D^S$  and

$$h = \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} g$$

then

$$\epsilon(h_S) = \epsilon \left( \begin{pmatrix} \beta_S & 0 \\ 0 & 1 \end{pmatrix} \right) \epsilon(g_S)$$

and  $\varphi(h)$  is equal to

$$\sum_{\alpha} a_{\alpha} \epsilon \left( \begin{pmatrix} \beta_S & 0 \\ 0 & 1 \end{pmatrix} \right) \epsilon(g_S) \varphi_1 \left( \begin{pmatrix} \hat{\alpha}_S \hat{\beta}_S & 0 \\ 0 & 1 \end{pmatrix} \hat{g}_S \right).$$

Since

$$a_{\alpha\beta} = \epsilon \left( \begin{pmatrix} \beta_S & 0 \\ 0 & 1 \end{pmatrix} \right) a_{\alpha}$$

we can change variables in the summation to see that  $\varphi(h) = \varphi(g)$ .

The lemma is now proved. The function

$$\hat{\varphi}(g) = \sum_{\alpha \in F^{\times}} \hat{a}_{\alpha} \hat{\epsilon}(g_S) \varphi_1 \left( \begin{pmatrix} \hat{\alpha}_S & 0 \\ 0 & 1 \end{pmatrix} \hat{g}_S \right)$$

can be treated in the same fashion.

**Theorem 11.5.** *If  $\omega$  is a quasi-character of  $F^{\times} \backslash I$  such that*

$$\omega_v(a_v) \epsilon \left( \begin{pmatrix} a_v & 0 \\ 0 & 1 \end{pmatrix} \right) = 1$$

*for all units  $a_v$  of  $O_v$  set*

$$\Lambda(s, \omega) = \left\{ \sum_{F^{\times} \cap I_D^S \backslash F^{\times}} a_{\alpha} \omega(\alpha_S) |\alpha_S|^{s-\frac{1}{2}} \right\} \prod_{v \notin S} L(s, \omega_v \otimes \pi_v).$$

*If*

$$\omega_v(a_v) \hat{\epsilon} \left( \begin{pmatrix} a_v & 0 \\ 0 & 1 \end{pmatrix} \right) = 1$$

*for all units  $a_v$  in  $O_v$  set*

$$\hat{\Lambda}(s, \omega) = \left\{ \sum_{F^{\times} \cap I_D^S \backslash F^{\times}} \hat{a}_{\alpha} \omega(\alpha_S) |\alpha_S|^{s-\frac{1}{2}} \right\} \prod_{v \notin S} L(s, \omega_v^{-1} \otimes \tilde{\pi}_v).$$

*Then  $\Lambda(s, \omega)$  and  $\hat{\Lambda}(s, \omega)$  are defined for  $\text{Re } s$  sufficiently large. Suppose that whenever  $\omega$  is such that  $\Lambda(s, \omega)$  or  $\hat{\Lambda}(s, \omega)$  is defined they can be analytically continued to entire functions which are bounded in vertical strips. Assume also that there is an  $A$  in  $F^{\times}$  such that  $|A|_{\mathfrak{p}} = |\varpi_{\mathfrak{p}}|^{m_{\mathfrak{p}}}$  for any  $\mathfrak{p}$  in  $S$  and*

$$\Lambda(s, \omega) = \left\{ \prod_{v \in S} \omega_v(-A) |A|_v^{s-1/2} \right\} \left\{ \prod_{v \notin S} \epsilon(s, \omega_v \otimes \pi_v, \psi_v) \right\} \hat{\Lambda}(1-s, \eta^{-1} \omega^{-1})$$

*whenever  $\Lambda(s, \omega)$  is defined. Then for any  $\varphi_1$  in  $W(\pi, \psi)$  there is an automorphic form  $\varphi$  on  $G_{\mathbf{A}}$  such that*

$$\varphi(g) = \sum a_{\alpha} \epsilon(g_S) \varphi_1 \left( \begin{pmatrix} \hat{\alpha}_S & 0 \\ 0 & 1 \end{pmatrix} \hat{g}_S \right)$$

on  $G_D^S$ .

The infinite products occurring in the definition of  $\Lambda(s, \omega)$  and  $\widehat{\Lambda}(s, \omega)$  certainly converge for  $\operatorname{Re} s$  sufficiently large. To check that the other factors converge one has to check that

$$\sum |\alpha_S|^{s-\frac{1}{2}}$$

converges for  $\operatorname{Re} s$  sufficiently large if the sum is taken over those elements  $\alpha$  of a system of coset representatives of  $F^\times \cap I_D^S \backslash F^\times$  for which  $|\alpha|_v \leq c_v$  for  $v$  in  $S$ . This is easily done.

The idele  $A_S$  has components 1 outside of  $S$  and  $A$  in  $S$ . Since

$$\begin{pmatrix} 0 & 1 \\ A_S & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & A_S^{-1} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & cA_S^{-1} \\ A_S b & a \end{pmatrix}$$

the matrix

$$\begin{pmatrix} 0 & 1 \\ A_S & 0 \end{pmatrix}$$

normalizes  $K_S^D$ . In particular if  $g$  belongs to  $G_D^S$  so does

$$\begin{pmatrix} 0 & 1 \\ A_S & 0 \end{pmatrix} g \begin{pmatrix} 0 & A_S^{-1} \\ 1 & 0 \end{pmatrix}.$$

**Lemma 11.5.1.** *If  $\varphi_1$  is in  $W(\pi, \psi)$  and  $g$  is in  $G_D^S$  then, under the hypotheses of the theorem,*

$$\widehat{\varphi} \left( \begin{pmatrix} 0 & 1 \\ A_S & 0 \end{pmatrix} g \begin{pmatrix} 0 & A_S^{-1} \\ 1 & 0 \end{pmatrix} \right) = \varphi(g).$$

Let  $\varphi'(g)$  be the function on the left. As before all we need do is show that for every character  $\omega$  of  $F^\times \cap I_D^S \backslash I_D^S$  and every  $g$  in  $G_D^S$  the integral

$$(11.5.2) \quad \int_{F^\times \cap I_D^S \backslash I_D^S} \varphi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right) \omega(a) |a|^{s-\frac{1}{2}} d^\times a$$

is absolutely convergent for  $\operatorname{Re} s$  large and positive. The integral

$$(11.5.3) \quad \int_{F^\times \cap I_D^S \backslash I_D^S} \varphi' \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right) \omega(a) |a|^{s-\frac{1}{2}} d^\times a$$

is absolutely convergent for  $\operatorname{Re} s$  large and negative, and they can be analytically continued to the same entire function which is bounded in vertical strips.

If for any  $v$  in  $S$  the character

$$a \rightarrow \omega_v(a) \epsilon \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right)$$

on the group of units of  $O_v$  is not trivial the integrals are 0 when they are convergent. We may thus assume that

$$\omega_v(a) \epsilon \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) = 1$$

for all units in  $O_v$  if  $v$  is in  $S$ .

We discuss the first integral in a formal manner. The manipulations will be justified by the final result. The integrand may be written as a double sum

$$\sum \sum a_{\alpha\gamma} \epsilon \left( \begin{pmatrix} a_S & 0 \\ 0 & 1 \end{pmatrix} g_S \right) \varphi_1 \left( \begin{pmatrix} \widehat{a}_S \widehat{\alpha}_S \widehat{\gamma}_S & 0 \\ 0 & 1 \end{pmatrix} \widehat{g}_S \right) \omega(a) |a|^{s-\frac{1}{2}}.$$

The inner sum is over  $\gamma$  in  $F^\times \cap I_D^S$  and the outer over a set of coset representatives  $\alpha$  of  $F^\times \cap I_D^S \backslash F^\times$ . Since

$$a_{\alpha\gamma} \epsilon \left( \begin{pmatrix} a_S & 0 \\ 0 & 1 \end{pmatrix} \right) = a_\alpha \epsilon \left( \begin{pmatrix} \gamma_S a_S & 0 \\ 0 & 1 \end{pmatrix} \right)$$

and

$$\omega(a) |a|^{s-\frac{1}{2}} = \omega(\gamma a) |\gamma a|^{s-\frac{1}{2}}$$

the integral is equal to  $\epsilon(g_S)$  times the sum over  $\alpha$  of

$$a_\alpha \int_{I_D^S} \varphi_1 \left( \begin{pmatrix} \widehat{\alpha}_S \widehat{a}_S & 0 \\ 0 & 1 \end{pmatrix} \widehat{g}_S \right) \epsilon \left( \begin{pmatrix} a_S & 0 \\ 0 & 1 \end{pmatrix} \right) \omega(a) |a|^{s-\frac{1}{2}} d^\times a.$$

Since  $I_D^S$  is the direct product of

$$\widehat{I}_S = \{ a \in I \mid a_S = 1 \}$$

and a compact group under which the integrand is invariant the previous expression is equal to

$$a_\alpha \int_{\widehat{I}_S} \varphi_1 \left( \begin{pmatrix} \widehat{\alpha}_S a & 0 \\ 0 & 1 \end{pmatrix} \widehat{g}_S \right) \omega(a) |a|^{s-\frac{1}{2}} d^\times a.$$

Changing variables to rid ourselves of the  $\widehat{\alpha}_S$  in the integrand and taking into account the relation

$$1 = \omega(\alpha) |\alpha|^{s-\frac{1}{2}} = \omega(\alpha_S) \omega(\widehat{\alpha}_S) |\alpha_S|^{s-\frac{1}{2}} |\widehat{\alpha}_S|^{s-\frac{1}{2}}$$

we can see that the original integral is equal to

$$\epsilon(g_S) \sum a_\alpha \omega(\alpha_S) |\alpha_S|^{s-\frac{1}{2}} \int_{\widehat{I}_S} \varphi_1 \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \widehat{g}_S \right) \omega(a) |a|^{s-\frac{1}{2}} d^\times a.$$

There is no harm in supposing that  $\varphi_1$  is of the form

$$\varphi_1(\widehat{g}_S) = \prod_{v \notin S} \varphi_v(g_v).$$

We have already seen that, in this case,

$$\int_{\widehat{I}_S} \varphi_1 \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \widehat{g}_S \right) \omega(a) |a|^{s-\frac{1}{2}} d^\times a$$

is convergent for  $\text{Re } s$  large and positive and is equal to

$$\prod_{v \notin S} \int_{F_v^\times} \varphi_v \left( \begin{pmatrix} a_v & 0 \\ 0 & 1 \end{pmatrix} g_v \right) \omega_v(a_v) |a_v|^{s-\frac{1}{2}} d^\times a_v.$$

If  $\varphi'_v$  is the function

$$\varphi'_v(h) = \omega_v(h) \varphi_v(h)$$

in  $W(\omega_v \otimes \pi_v, \psi_v)$  this product is

$$\prod_{v \notin S} \{L(s, \omega_v \otimes \pi_v) \Phi(g_v, s, \varphi'_v) \omega_v^{-1}(\det g_v)\}.$$

Thus the integral (11.5.2) is absolutely convergent for  $\operatorname{Re} s$  large and positive and is equal to

$$\epsilon(g_S) \omega(\det \hat{g}_S) \Lambda(s, \omega) \prod_{v \notin S} \Phi(g_v, s, \varphi'_v).$$

The argument used in the proof of Theorem 11.3 shows that this function is entire.

If

$$h = \begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} 0 & A_S^{-1} \\ 1 & 0 \end{pmatrix}$$

then

$$\hat{\epsilon}(h_S) = \epsilon \left( \begin{pmatrix} a_S & 0 \\ 0 & 1 \end{pmatrix} g_S \right).$$

Thus the integrand in (11.5.3) is equal to

$$\sum \hat{a}_\alpha \epsilon \left( \begin{pmatrix} a_S & 0 \\ 0 & 1 \end{pmatrix} g_S \right) \varphi_1 \left( \begin{pmatrix} \hat{\alpha}_S & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \hat{A}_S & 0 \end{pmatrix} \begin{pmatrix} \hat{a}_S & 0 \\ 0 & 1 \end{pmatrix} \hat{g}_S \right) \omega(a) |a|^{s-\frac{1}{2}}.$$

The sum can again be written as a double sum over  $\gamma$  and  $\alpha$ . Since

$$\hat{a}_{\alpha\gamma} = \epsilon \left( \begin{pmatrix} a_S & 0 \\ 0 & 1 \end{pmatrix} \right) = \hat{a}_\alpha \epsilon \left( \begin{pmatrix} 1 & 0 \\ 0 & \gamma_S \end{pmatrix} \right) \epsilon \left( \begin{pmatrix} a_S & 0 \\ 0 & 1 \end{pmatrix} \right)$$

which equals

$$\hat{a}_\alpha \eta(\gamma_S) \epsilon \left( \begin{pmatrix} \gamma_S^{-1} a_S & 0 \\ 0 & 1 \end{pmatrix} \right)$$

and

$$\varphi_1 \left( \begin{pmatrix} \hat{\alpha}_S \hat{\gamma}_S & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \hat{A}_S & 0 \end{pmatrix} \begin{pmatrix} \hat{a}_S & 0 \\ 0 & 1 \end{pmatrix} \hat{g}_S \right)$$

is equal to

$$\eta(\hat{\gamma}_S) \varphi_1 \left( \begin{pmatrix} \hat{\alpha}_S & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \hat{A}_S & 0 \end{pmatrix} \begin{pmatrix} \hat{\gamma}_S^{-1} \hat{a}_S & 0 \\ 0 & 1 \end{pmatrix} \hat{g}_S \right)$$

we can put the sum over  $F^\times \cap I_D^S$  and the integration over  $F^\times \cap I_D^S \setminus I_D^S$  together to obtain  $\epsilon(g_S)$  times the sum over  $F^\times \cap I_D^S \setminus F^\times$  of

$$\hat{a}_\alpha \int_{I_D^S} \epsilon \left( \begin{pmatrix} a_S & 0 \\ 0 & 1 \end{pmatrix} \right) \varphi_1 \left( \begin{pmatrix} \hat{\alpha}_S & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \hat{A}_S & 0 \end{pmatrix} \begin{pmatrix} \hat{a}_S & 0 \\ 0 & 1 \end{pmatrix} \hat{g}_S \right) \omega(a) |a|^{s-\frac{1}{2}} d^\times a.$$

We write

$$\begin{pmatrix} \hat{\alpha}_S & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \hat{A}_S & 0 \end{pmatrix} \begin{pmatrix} \hat{a}_S & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \hat{\alpha}_S & 0 \\ 0 & \hat{\alpha}_S \end{pmatrix} \begin{pmatrix} -\hat{\alpha}_S^{-1} \hat{A}_S \hat{a}_S & 0 \\ 0 & 1 \end{pmatrix}$$

and then change variables in the integration to obtain the product of  $\omega(-A_S)|A_S|^{s-\frac{1}{2}}$  and

$$\widehat{a}_\alpha \eta^{-1}(\alpha_S) \omega^{-1}(\alpha_S) |\alpha_S|^{\frac{1}{2}-s} \int_{\widehat{I}_S} \varphi_1 \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \widehat{g}_S \right) \omega(a) |a|^{s-\frac{1}{2}} d^\times a.$$

Replacing  $a$  by  $a^{-1}$  and making some simple changes we see that the integral is equal to

$$\int_{\widehat{I}_S} \varphi_1 \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \widehat{g}_S \right) \eta^{-1}(a) \omega^{-1}(a) |a|^{\frac{1}{2}-s} d^\times a$$

which converges for  $\operatorname{Re} s$  large and negative and is equal to

$$\prod_{v \notin S} \{L(1-s, \eta_v^{-1} \omega_v^{-1} \otimes \pi_v) \Phi(wg_v, 1-s, \varphi'_v) \omega_v(\det g_v)\}.$$

Thus the integral 11.5.3 is equal to

$$\epsilon(g_S) \omega(\det \widehat{g}_S) \omega(-A_S) |A_S|^{s-\frac{1}{2}} \widehat{\Lambda}(1-s, \eta^{-1} \omega^{-1}) \prod_{v \notin S} \Phi(wg, 1-s, \varphi'_v)$$

which is entire.

Since

$$\Phi(wg_v, 1-s, \varphi'_v) = \epsilon(s, \omega_v \otimes \pi_v, \psi) \Phi(g_v, s, \varphi'_v)$$

the analytic continuations of (11.5.2) and (11.5.3) are equal. We show as in the proof of Theorem 11.3 that the resultant entire function is bounded in vertical strips of finite width.

There is now a simple lemma to be proved.

**Lemma 11.5.4.** *The group  $G_F \cap G_D^S$  is generated by the matrices in it of the form*

$$\begin{pmatrix} a & \beta \\ 0 & \delta \end{pmatrix}$$

and

$$\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}.$$

This is clear if  $S$  is empty. Suppose that  $S$  is not empty. If

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

belongs to  $G_F \cap G_D^S$  and  $|\alpha|_v = 1$  for all  $v$  in  $S$  then

$$g = \begin{pmatrix} \alpha & 0 \\ \gamma & \delta - \frac{\beta\gamma}{\alpha} \end{pmatrix} \begin{pmatrix} 1 & \frac{\beta}{\gamma} \\ 0 & 1 \end{pmatrix}$$

and both matrices belong to  $G_F \cap G_D^S$ . In general if  $g$  is in  $G_F \cap G_D^S$  then, for each  $v$  in  $S$ ,  $|\alpha|_v \leq 1$ ,  $|\gamma|_v \leq 1$  and either  $|\alpha|_v$  or  $|\gamma|_v$  is 1. Choose  $\xi$  in  $F$  so that, for every  $v$  in  $S$ ,  $|\xi_v| = 1$  if  $|\alpha|_v < 1$  and  $|\xi_v| < 1$  if  $|\alpha|_v = 1$ . Then

$$\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha + \xi\gamma & \beta + \xi\delta \\ \gamma & \delta \end{pmatrix}$$

and  $|\alpha + \xi\gamma|_v = 1$  for all  $v$  in  $S$ . The lemma follows.



We know that if

$$h = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$$

belongs to  $G_F \cap G_D^S$  then  $\varphi(hg) = \varphi(g)$  and  $\widehat{\varphi}(hg) = \widehat{\varphi}(g)$ . Suppose

$$\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$$

is in  $G_F \cap G_D^S$ . Then

$$\varphi\left(\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}g\right) = \widehat{\varphi}\left(\begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}g\begin{pmatrix} 0 & A_S^{-1} \\ 1 & 0 \end{pmatrix}\right).$$

Since the argument on the right can be written

$$\begin{pmatrix} \delta & \gamma A^{-1} \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix}g\begin{pmatrix} 0 & A_S^{-1} \\ 1 & 0 \end{pmatrix}$$

and the first term of this product lies in  $G_F \cap G_D^S$  the right side is equal to

$$\widehat{\varphi}\left(\begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix}g\begin{pmatrix} 0 & A_S^{-1} \\ 1 & 0 \end{pmatrix}\right) = \varphi(g).$$

Thus  $\varphi$  is invariant under  $G_F \cap G_D^S$ . Since  $G_{\mathbf{A}} = G_F G_D^S$  the function  $\varphi$  extends in a unique manner to a function, still denoted  $\varphi$ , on  $G_F \backslash G_{\mathbf{A}}$ . It is clear that  $\varphi$  is  $K$ -finite and continuous and that

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}g\right) = \eta(a)\varphi(g)$$

for all  $a$  in  $I$ . It is not quite so clear that  $\varphi$  is slowly increasing. If  $\Omega$  is a compact subset of  $G_{\mathbf{A}}$  there is a finite set  $\gamma_1, \dots, \gamma_\ell$  in  $G_F$  such that

$$\Omega = \bigcup_{i=1}^{\ell} \Omega \cap \gamma_i^{-1} G_D^S.$$

What we have to show then is that if  $\gamma$  belongs to  $G_F$  and  $c > 0$  is given there are constants  $M_1$  and  $M_2$  such that for all  $g$  in  $\Omega \cap \gamma^{-1} G_D^S$  and all  $a$  in  $I$  for which  $|a| \geq c$

$$\left|\gamma\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}g\right)\right| \leq M_1 |a|^{M_2}.$$

If  $v$  is a place of  $F$ , which is not in  $S$  and is archimedean if  $F$  is a number field, there is a compact set  $C$  in  $I$  such that

$$\{a \in I \mid |a| \geq c\} \subseteq F^\times \{a \in F_v^\times \mid |a| \geq c\} C$$

Thus the inequality has only to be verified for  $a$  in  $F_v^\times$ —of course at the cost of enlarging  $\Omega$ . If

$$\gamma = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$$

then

$$\gamma \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g = \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} \gamma g$$

with  $x = (1 - \alpha)^{\frac{\beta}{\delta}}$  and the conclusion results from Lemma 11.4 and the relation

$$B_{\mathbf{A}} = (B_{\mathbf{A}} \cap G_F)(B_{\mathbf{A}} \cap G_D^S)$$

if

$$B_{\mathbf{A}} = \left\{ \begin{pmatrix} b & y \\ 0 & 1 \end{pmatrix} \in G_{\mathbf{A}} \right\}.$$

Otherwise we write

$$\gamma = \begin{pmatrix} \alpha_1 & \beta_1 \\ 0 & \delta_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \beta_2 \\ 0 & 1 \end{pmatrix}.$$

Then

$$\gamma \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g = \begin{pmatrix} 1 & \frac{\beta_1}{\delta_1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 & 0 \\ 0 & \delta_1 a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{\beta_2}{a} \\ 0 & 1 \end{pmatrix} g.$$

The matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{\beta_2}{a} \\ 0 & 1 \end{pmatrix} g$$

lies in a certain compact set which depends on  $\Omega$ ,  $c$ , and  $\gamma$ . The required inequality again follows from Lemma 11.4.

The space  $U$  of functions  $\varphi$  corresponding to  $\varphi_1$  in  $W(\pi, \psi)$  transforms under  $\widehat{\mathcal{H}}_S$  according to  $\pi$ . Lemma 10.13 implies that every element of  $U$  is an automorphic form. If it is not contained in  $\mathcal{A}_0$ , Lemma 10.12 applied to the functions

$$\varphi_0(g) = \frac{1}{\text{measure } F \backslash \mathbf{A}} \int_{F \backslash \mathbf{A}} \varphi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx$$

with  $\varphi$  in  $U$  shows that there are two quasi-characters  $\mu$  and  $\nu$  on  $F^\times \backslash I$  such that  $\pi_v = \pi(\mu_v, \nu_v)$  for almost all  $v$ .

**Corollary 11.6.** *Suppose there does not exist a pair  $\mu, \nu$  of quasi-characters of  $F^\times \backslash I$  such that  $\pi_v = \pi(\mu_v, \nu_v)$  for almost all  $v$ . Then there is a constituent  $\pi' = \bigotimes \pi'_v$  of  $\mathcal{A}_0$  such that  $\pi_v = \pi'_v$  for all  $v$  not in  $S$ .*

Since  $U$  transforms under  $\widehat{\mathcal{H}}_S$  according to  $\pi$  it is, if  $v$  is not in  $S$ , the direct sum of subspaces transforming under  $\mathcal{H}_v$  according to  $\pi_v$ . By assumption  $U$  is contained in  $\mathcal{A}_0$  and therefore in  $\mathcal{A}_0(\eta)$ . The space  $\mathcal{A}_0(\eta)$  is the direct sum of subspaces invariant and irreducible under  $\mathcal{H}$ . Choose one of these summands  $V$  so that the projection of  $U$  on  $V$  is not 0. If  $\pi' = \bigotimes \pi'_v$  is the representation of  $\mathcal{H}$  on  $V$  it is clear that  $\pi'_v = \pi_v$  if  $v$  is not in  $S$ .

Another way to guarantee that  $U$  lies in the space of cusp forms and therefore that the conclusion of the corollary holds is to assume that for at least one  $v$  not in  $S$  the representation  $\pi_v$  is absolutely cuspidal.

### §12. Some extraordinary representations

In [18] Weil has introduced a generalization of the Artin  $L$ -functions. To define these it is necessary to introduce the Weil groups. These groups are discussed very clearly in the notes of Artin-Tate but we remind the reader of their most important properties. If  $F$  is a local field let  $C_F$  be the multiplicative group of  $F$  and if  $F$  is a global field let  $C_F$  be the idele class group  $F^\times \backslash I$ . If  $K$  is a finite Galois extension of  $F$  the Weil group  $W_{K/F}$  is an extension of  $\mathfrak{G}(K/F)$ , the Galois group of  $K/F$ , by  $C_K$ . Thus there is an exact sequence

$$1 \longrightarrow C_K \longrightarrow W_{K/F} \longrightarrow \mathfrak{G}(K/F) \longrightarrow 1.$$

If  $L/F$  is also Galois and  $L$  contains  $K$  there is a continuous homomorphism  $\tau_{L/F, K/F}$  of  $W_{L/F}$  onto  $W_{K/F}$ . It is determined up to an inner automorphism of  $W_{K/F}$  by an element of  $C_K$ . In particular  $W_{F/F} = C_F$  and the kernel of  $\tau_{K/F, F/F}$  is the commutator subgroup of  $W_{K/F}$ . Also if  $F \subseteq E \subseteq K$  we may regard  $W_{K/F}$  as a subgroup of  $W_{K/E}$ . If  $F$  is global and  $v$  a place of  $F$  we also denote by  $v$  any extension of  $v$  to  $K$ . There is a homomorphism  $\alpha_v$  of  $W_{K_v/F_v}$  into  $W_{K/F}$  which is determined up to an inner automorphism by an element of  $C_K$ .

A representation  $\sigma$  of  $W_{K/F}$  is a continuous homomorphism of  $W_{K/F}$  into the group of invertible linear transformations of a finite-dimensional complex vector space such that  $\sigma(w)$  is diagonalizable for all  $w$  in  $W_{K/F}$ . If  $K$  is contained in  $L$  then  $\sigma \circ \tau_{L/F, K/F}$  is a representation of  $W_{L/F}$  whose equivalence class is determined by that of  $\sigma$ . In particular if  $\omega$  is a generalized character of  $C_F$  then  $\omega \circ \tau_{K/F, F/F}$  is a one-dimensional representation of  $W_{K/F}$  which we also call  $\omega$ . If  $\sigma$  is any other representation  $\omega \otimes \sigma$  has the same dimension as  $\sigma$ . If  $F \subseteq E \subseteq K$  and  $\rho$  is a representation of  $W_{K/E}$  on  $X$  let  $Y$  be the space of functions  $\varphi$  on  $W_{K/F}$  with values in  $X$  which satisfy

$$\varphi(uw) = \rho(u)\varphi(w)$$

for all  $u$  in  $W_{K/E}$ . If  $v \in W_{K/F}$  and  $\varphi \in Y$  let  $\sigma(v)\varphi$  be the function

$$w \rightarrow \varphi(wv)$$

$\sigma(v)\varphi$  also belongs to  $Y$  and  $v \rightarrow \sigma(v)$  is a representation of  $W_{K/F}$ . We write

$$\sigma = \text{Ind}(W_{K/F}, W_{K/E}, \rho).$$

If  $F$  is global and  $\sigma$  is a representation of  $W_{K/F}$  then, for any place  $v$ ,  $\sigma_v = \sigma \circ \alpha_v$  is a representation of  $W_{K_v/F_v}$  whose class is determined by that of  $\sigma$ .

Now we remind ourselves of the definition of the generalized Artin  $L$ -functions. Since we are going to need a substantial amount of detailed information about these functions the best reference is probably [19]. In fact to some extent the purpose of [19] is to provide the background for this chapter and the reader who wants to understand all details will need to be quite familiar with it. If  $F$  is a local field then to every representation  $\sigma$  of  $W_{K/F}$  we can associate a local  $L$ -function  $L(s, \sigma)$ . Moreover if  $\psi_F$  is a non-trivial additive character of  $F$  we can define a local factor  $\epsilon(s, \sigma, \psi_F)$ . The  $L$ -function and the factor  $\epsilon(s, \sigma, \psi_F)$  depend only on the equivalence class of  $\sigma$ .

If  $F$  is a global field we set

$$L(s, \sigma) = \prod_v L(s, \sigma_v)$$

The product converges in a right half-plane and  $L(s, \sigma)$  can be analytically continued to a function meromorphic in the whole complex plane. If  $\psi_F$  is a non-trivial character of  $F \backslash \mathbf{A}$

the functions  $\epsilon(s, \sigma_v, \psi_v)$  are identically 1 for all but a finite number of  $v$ . If

$$\epsilon(s, \sigma) = \prod_v \epsilon(s, \sigma_v, \psi_v)$$

and  $\tilde{\sigma}$  is the representation contragredient to  $\sigma$  the functional equation

$$L(s, \sigma) = \epsilon(s, \sigma) L(1-s, \tilde{\sigma})$$

is satisfied. For all but finitely many places  $v$  the representation  $\sigma_v$  is the direct sum of  $d$ , the dimension of  $\sigma$ , one-dimensional representations. Thus there are generalized characters  $\mu_v^1, \dots, \mu_v^d$  of  $C_{F_v}$  such that  $\sigma_v$  is equivalent to the direct sum of the one-dimensional representations

$$w \mapsto \mu_v^i(\tau_{K_v/F_v, F_v/F_v}(w)).$$

Moreover, for all but finitely many of these  $v$ ,  $\mu_v^1, \dots, \mu_v^d$  are unramified and there is a constant  $r$ , which does not depend on  $v$ , such that

$$|\mu_v^i(\varpi_v)| \leq |\varpi_v|^r \quad 1 \leq i \leq d.$$

If  $F$  is a global or a local field and  $\sigma$  is a representation of  $W_{K/F}$  then  $w \rightarrow \det \sigma(w)$  is a one-dimensional representation and therefore corresponds to a generalized character of  $C_F$ . We denote this character by  $\det \sigma$ .

If  $F$  is a local field,  $\sigma$  is a two-dimensional representation of  $W_{K/F}$ , and  $\psi_F$  is a non-trivial additive character of  $F$  then, as we saw in the first chapter, there is at most one irreducible admissible representation  $\pi$  of  $\mathcal{H}_F$  such that

$$\pi \left( \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \right) = \det \sigma(\alpha) I$$

and, for all generalized characters  $\omega$  of  $C_F$ ,

$$\begin{aligned} L(s, \omega \otimes \pi) &= L(s, \omega \otimes \sigma) \\ L(s, \omega^{-1} \otimes \tilde{\pi}) &= L(s, \omega^{-1} \otimes \sigma) \\ \epsilon(s, \omega \otimes \pi, \psi_F) &= \epsilon(s, \omega \otimes \sigma, \psi_F). \end{aligned}$$

If  $\psi'_F(x) = \psi_F(\beta x)$  then

$$\epsilon(s, \omega \otimes \sigma, \psi'_F) = \det \omega \otimes \sigma(\beta) \epsilon(s, \omega \otimes \sigma, \psi_F)$$

and, since

$$\pi \left( \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \right) = \det \sigma(\alpha) I$$

one also has

$$\epsilon(s, \omega \otimes \pi, \psi'_F) = \det \omega \otimes \sigma(\beta) \epsilon(s, \omega \otimes \pi, \psi_F).$$

Thus  $\pi$ , if it exists at all, is independent of  $\psi_F$ . We write  $\pi = \pi(\sigma)$ .

There are a number of cases in which the existence of  $\pi(\sigma)$  can be verified simply by comparing the definitions of the previous chapter with those of [19]. If  $\mu$  and  $\nu$  are two quasi-characters of  $C_F$  and  $\sigma$  is equivalent to the representation

$$w \rightarrow \begin{pmatrix} \mu(\tau_{K/F, F/F}(w)) & 0 \\ 0 & \nu(\tau_{K/F, F/F}(w)) \end{pmatrix}$$

then  $\pi(\sigma) = \pi(\mu, \nu)$ . If  $K/F$  is a separable quadratic extension,  $\chi$  is a quasi-character of  $C_K = W_{K/K}$ , and

$$\sigma = \text{Ind}(W_{K/F}, W_{K/K}, \chi)$$

then  $\pi(\sigma) = \pi(\chi)$ . Observe that  $\pi(\chi)$  is always infinite-dimensional.

Suppose  $F$  is a global field and  $K$  is a separable quadratic extension of  $F$ . Let  $\chi$  be a quasi-character of  $C_K$  and let

$$\sigma = \text{Ind}(W_{K/F}, W_{K/K}, \chi).$$

If  $v$  does not split in  $K$

$$\sigma_v = \text{Ind}(W_{K_v/F_v}, W_{K_v/K_v}, \chi_v),$$

but if  $v$  splits in  $K$  the representation  $\sigma_v$  is the direct sum of two one-dimensional representations corresponding to quasi-characters  $\mu_v$  and  $\nu_v$  such that  $\mu_v \nu_v^{-1}$  is a character. Thus  $\pi(\sigma_v)$  is defined and infinite-dimensional for all  $v$ .

**Proposition 12.1.** *If there is no quasi-character  $\mu$  of  $C_F$  such that  $\chi(\alpha) = \mu(N_{K/F}\alpha)$  for all  $\alpha$  in  $C_K$  the representation  $\bigotimes_v \pi(\sigma_v)$  is a constituent of  $\mathcal{A}_0$ .*

If  $\omega$  is a generalized character of  $F$  then

$$(\omega \otimes \sigma)_v = \omega_v \otimes \sigma_v.$$

Define a generalized character  $\omega_{K/F}$  of  $C_K$  by

$$\omega_{K/F}(\alpha) = \omega(N_{K/F}(\alpha)).$$

Then

$$\omega \otimes \sigma = \text{Ind}(W_{K/F}, W_{K/K}, \omega_{K/F}\chi)$$

and

$$L(s, \omega \otimes \sigma) = L(s, \omega_{K/F}\chi).$$

The  $L$ -function on the right is the Hecke  $L$ -function associated to the generalized character  $\omega_{K/F}\chi$  of  $C_K$ . It is entire and bounded in vertical strips unless there is a complex number  $r$  such that

$$\omega_{K/F}(\alpha)\chi(\alpha) = |\alpha|^r = |N_{K/F}\alpha|^r.$$

But then

$$\chi(\alpha) = \omega^{-1}(N_{K/F}\alpha)|N_{K/F}\alpha|^r$$

which is contrary to assumption. The function

$$L(s, \omega^{-1} \otimes \tilde{\sigma}) = L(s, \omega_{K/F}^{-1}\chi^{-1})$$

is also entire and bounded in vertical strips. It follows immediately that the collection  $\{\pi(\sigma_v)\}$  satisfies the conditions of Theorem 11.3.

This proposition has a generalization which is one of the principal results of these notes.

**Theorem 12.2.** *Suppose  $F$  is a global field and  $\sigma$  is a two-dimensional representation of  $W_{K/F}$ . Suppose also that for every generalized character  $\omega$  of  $C_F$  both  $L(s, \omega \otimes \sigma)$  and  $L(s, \omega^{-1} \otimes \tilde{\sigma})$  are entire functions which are bounded in vertical strips. Then  $\pi(\sigma_v)$  exists for every place  $v$  and  $\bigotimes_v \pi(\sigma_v)$  is a constituent of  $\mathcal{A}_0$ .*

We observe that the converse to this theorem is an immediate consequence of Theorem 11.1.

We are going to apply Corollary 11.6. There are a large number of conditions which must be verified. We know that  $\pi(\sigma_v)$  is defined for all but a finite number of  $v$ . In particular it is defined for  $v$  archimedean for then  $\sigma_v$  is either induced from a quasi-character of a quadratic extension of  $F_v$  or is the direct sum of two one-dimensional representations. If  $\sigma_v$  is equivalent to the direct sum of two one-dimensional representations corresponding to quasi-characters  $\mu_v$  and  $\nu_v$  then  $\mu_v \nu_v^{-1}$  is a character so that  $\pi(\sigma_v)$  is infinite-dimensional. Let  $S$  be the set of places for which  $\pi(\sigma_v)$  is infinite-dimensional. Let  $S$  be the set of places for which  $\pi(\sigma_v)$  is not defined or, since this is still conceivable, finite-dimensional. We are going to show that  $S$  is empty but at the moment we are at least sure that it is finite. If  $v$  is not in  $S$  set  $\pi_v = \pi(\sigma_v)$ .

If  $v$  is in  $S$  the representation  $\sigma_v$  must be irreducible so that

$$L(s, \omega_v \otimes \sigma_v) = L(s, \omega_v^{-1} \otimes \tilde{\sigma}_v) = 1$$

for every generalized character  $\omega_v$  of  $F_v^\times$ . The Artin conductor  $\mathfrak{p}_v^{m_v}$  of  $\sigma_v$  is defined in the Appendix to [19]. There is a constant  $c_v$ , depending on  $\sigma_v$ , such that if  $\omega_v$  is unramified

$$\epsilon(s, \omega_v \otimes \sigma_v, \psi_v) = c_v \omega_v(\varpi_v)^{m_v+2n} |\varpi_v|^{(m_v+2n_v)(s-\frac{1}{2})}$$

if  $\mathfrak{p}_v^{-n_v}$  is the largest ideal on which  $\psi_v$  is trivial.  $\psi_v$  is the restriction to  $F_v$  of a given non-trivial character of  $F \setminus \mathbf{A}$ .

We take

$$D = \prod_{\mathfrak{p} \in S} \mathfrak{p}^{m_{\mathfrak{p}}}$$

and  $\eta = \det \sigma$ . We define  $\epsilon$  and  $\hat{\epsilon}$  by

$$\epsilon \left( \begin{pmatrix} a_v & 0 \\ 0 & b_v \end{pmatrix} \right) = \det \sigma_v(b_v)$$

and

$$\hat{\epsilon} \left( \begin{pmatrix} a_v & 0 \\ 0 & b_v \end{pmatrix} \right) = \det \sigma_v(a_v)$$

if  $v$  belongs to  $S$  and  $a_v$  and  $b_v$  are units of  $O_v$ . If  $\alpha$  belongs to  $F^\times$  and  $|\alpha|_v = |\varpi_v|^{-n_v}$  for every  $v$  in  $S$  we set  $a_\alpha = 1$  and  $\hat{a}_\alpha = \prod_{v \in S} c_v \det \sigma_v(\alpha)$ ; otherwise we set  $a_\alpha = \hat{a}_\alpha = 0$ .

The function  $\Lambda(s, \omega)$  of Theorem 11.5 is defined only if  $\omega_v$  is unramified at each place of  $S$  and then it equals

$$\left\{ \prod_{v \in S} \omega_v(\varpi_v^{-n_v}) |\varpi_v|^{-n_v(s-\frac{1}{2})} \right\} \left\{ \prod_{v \notin S} L(s, \omega_v \otimes \pi_v) \right\}$$

which is

$$\left\{ \prod_{v \in S} \omega_v(\varpi_v^{-n_v}) |\varpi_v|^{-n_v(s-\frac{1}{2})} \right\} L(s, \omega \otimes \sigma).$$

The function  $\widehat{\Lambda}(s, \omega^{-1}\eta^{-1})$  is also defined if  $\omega_v$  is unramified at each place of  $S$  and is equal to

$$\left\{ \prod_{v \in S} c_v \omega_v(\varpi_v^{n_v}) |\varpi_v|^{-n_v(s-\frac{1}{2})} \right\} L(s, \omega^{-1} \otimes \tilde{\sigma}).$$

Choose  $A$  in  $F^\times$  so that  $|A_v| = |\varpi_v|^{m_v}$  for every  $v$  in  $S$ . Then

$$\prod_{v \in S} \omega_v(-A) |A|_v^{s-\frac{1}{2}} = \prod_{v \in S} \omega_v(\varpi_v)^{m_v} |\varpi_v|^{m_v(s-\frac{1}{2})}.$$

The functional equation asserts that  $L(s, \omega \otimes \sigma)$  is equal to

$$\left\{ \prod_{v \in S} \epsilon(s, \omega_v \otimes \sigma_v, \psi_v) \right\} \left\{ \prod_{v \notin S} \epsilon(s, \omega_v \otimes \sigma_v, \psi_v) \right\} L(1-s, \omega^{-1} \otimes \tilde{\sigma}).$$

The first factor is equal to

$$\left\{ \prod_{v \in S} c_v \omega_v(\varpi_v)^{2n_v} |\varpi_v|^{2n_v(s-\frac{1}{2})} \right\} \left\{ \prod_{v \in S} \omega_v(-A) |A|_v^{s-\frac{1}{2}} \right\}.$$

Therefore  $\Lambda(s, \omega)$  is equal to

$$\left\{ \prod_{v \in S} \omega_v(-A) |A|_v^{s-\frac{1}{2}} \right\} \left\{ \prod_{v \notin S} \epsilon(s, \omega_v \otimes \sigma_v, \psi_v) \right\} \widehat{\Lambda}(1-s, \omega^{-1}\eta^{-1}).$$

The assumptions of Theorem 11.5 are now verified. It remains to verify that of Corollary 11.6. It will be a consequence of the following lemma.

**Lemma 12.3.** *Suppose  $F$  is a global field,  $K$  is a Galois extension of  $F$ , and  $\rho$  and  $\sigma$  are two representations of the Weil group  $W_{K/F}$ . If for all but a finite number of places  $v$  of  $F$  the local representations  $\rho_v$  and  $\sigma_v$  are equivalent then  $\rho$  and  $\sigma$  are equivalent.*

We set

$$L_0(s, \sigma) = \prod_{\mathfrak{p}} L(s, \sigma_{\mathfrak{p}}).$$

The product is taken over all non-archimedean places. We first prove the following lemma.

**Lemma 12.4.** *If  $\sigma$  is unitary the order of the pole of  $L_0(s, \sigma)$  at  $s = 1$  is equal to the multiplicity with which the trivial representation is contained in  $\sigma$ .*

There are fields  $E_1, \dots, E_r$  lying between  $F$  and  $K$ , characters  $\chi_{E_1}, \dots, \chi_{E_r}$ , and integers  $m_1, \dots, m_r$  such that  $\sigma$  is equivalent to

$$\bigoplus_{i=1}^r m_i \text{Ind}(W_{K/F}, W_{K/E_i}, \chi_{E_i})$$

Let  $\delta_i = 1$  if  $\chi_{E_i}$  is trivial and 0 otherwise. Since

$$L_0(s, \sigma) = \prod_{i=1}^r L_0(s, \chi_{E_i})^{m_i}$$

the order of its pole at  $s = 1$  is  $\sum_{i=1}^r m_i \delta_i$ . However

$$\text{Ind}(W_{K/F}, W_{K/E}, \chi_{E_i})$$

contains the trivial representation if and only if  $\chi_{E_i}$  is trivial and then it contains it exactly once. Thus  $\sum_{i=1}^r m_i \delta_i$  is also the number of times the trivial representation occurs in  $\sigma$ .

Observe that if  $T$  is any finite set of non-archimedean primes the order of the pole of

$$\prod_{\mathfrak{p} \notin T} L(s, \sigma_{\mathfrak{p}})$$

at  $s = 1$  is the same as that of  $L_0(s, \sigma)$ .

The first step of the proof of Lemma 12.3 is to reduce it to the case that both  $\rho$  and  $\sigma$  are unitary. Then  $\rho$  and  $\sigma$  certainly have the same degree  $d$ . Let  $\rho$  act on  $X$  and let  $\sigma$  act on  $Y$ . Under the restriction  $\rho$  to  $C_K$  the space  $X$  decomposes into the direct sum of invariant one-dimensional subspaces  $X_1, \dots, X_d$  which transform according to quasi-characters  $\mu^1, \dots, \mu^d$  of  $C_K$ . If  $a$  is a real number let

$$M(a) = \left\{ i \mid \left| \mu^i(\alpha) \right| = |\alpha|^a \text{ for all } \alpha \text{ in } C_K \right\}$$

and let

$$X(a) = \sum_{i \in M(a)} X_i$$

$X(a)$  is invariant under  $W_{K/F}$  and  $X = \bigoplus_a X(a)$ . Let  $\rho(a)$  be the restriction of  $\rho$  to  $X(a)$ . Replacing  $\rho$  by  $\sigma$  and  $X$  by  $Y$  we can define  $\nu^1, \dots, \nu^d$  and  $Y(a)$  in a similar fashion.

We now claim that if  $\rho_v$  is equivalent to  $\sigma_v$  then  $\rho_v(a)$  is equivalent to  $\sigma_v(a)$  for each  $a$ . To see this we need only verify that any linear transformation from  $X$  to  $Y$  which commutes with the action of  $W_{K_v/F_v}$ , or even of  $C_{K_v}$ , takes  $X(a)$  to  $Y(a)$ . Observe that under the restriction of  $\rho_v$  to  $C_{K_v}$  the space  $X_i$  transforms according to the character  $\mu_v^i$  and that  $|\mu_v^i(\alpha)| = |\alpha|^a$  for all  $\alpha$  in  $C_{K_v}$  if and only if  $|\mu^i(\alpha)| = |\alpha|^a$  for all  $\alpha$  in  $C_K$ . Thus  $X(a)$  and  $Y(a)$  can be defined in terms of  $\rho_v$  and  $\sigma_v$  alone. The assertion follows.

Thus we may as well assume that for some real number  $a$

$$\left| \mu^i(\alpha) \right| = \left| \nu^i(\alpha) \right| = |\alpha|^a$$

for all  $i$  and all  $\alpha$  in  $C_K$ . Replacing  $\sigma$  by  $\alpha \rightarrow |\alpha|^{-a} \sigma(\alpha)$  and  $\rho$  by  $\alpha \rightarrow |\alpha|^{-a} \rho(\alpha)$  if necessary we may even assume that  $a = 0$ . Then  $\rho$  and  $\sigma$  will be equivalent to unitary representations and we now suppose them to be unitary.

If  $\tau$  is irreducible and  $\rho \simeq \tau \oplus \rho'$  and  $\sigma \simeq \tau \oplus \sigma'$  then  $\rho'_v$  is equivalent to  $\sigma'_v$  whenever  $\rho_v$  is equivalent to  $\sigma_v$ . Since we can use induction on  $d$  it is enough to show that if  $\tau$  is irreducible and unitary and contained in  $\rho$  then it is contained in  $\sigma$ . Let  $\tilde{\rho}$  and  $\tilde{\sigma}$  be the representations contragredient to  $\rho$  and  $\sigma$ . Certainly  $(\tilde{\rho} \otimes \tau)_v = \tilde{\rho}_v \otimes \tau_v$  is equivalent to  $(\tilde{\sigma} \otimes \tau)_v$  for all but a finite number of  $v$ . Moreover  $\tilde{\rho} \otimes \tau$  contains  $\tilde{\tau} \otimes \tau$  which contains the identity. If  $\tilde{\sigma} \otimes \tau$  contains the identity then, as is well-known and easily verified,  $\sigma$  contains  $\tau$ . On the other hand the orders of the poles of  $L_0(s, \tilde{\rho} \otimes \tau)$  and  $L_0(s, \tilde{\sigma} \otimes \tau)$  at  $s = 1$  are clearly equal so that, by Lemma 12.4,  $\tilde{\sigma} \otimes \tau$  contains the trivial representations if  $\tilde{\rho} \otimes \tau$  does.

We return to the proof of Theorem 12.2. It follows from Lemma 12.3 that if the assumptions of Corollary 11.6 are not satisfied  $\sigma$  is equivalent to the direct sum of two



one-dimensional representations associated to quasi-characters  $\mu$  and  $\nu$  of  $C_F$ . Then

$$L(s, \omega \otimes \sigma) = L(s, \omega\mu)L(s, \omega\nu).$$

The two functions on the right are Hecke  $L$ -functions. The function on the left is entire for every choice of  $\omega$ . Taking  $\omega = \mu^{-1}$  and  $\omega = \nu^{-1}$  we see that  $L(s, \mu^{-1}\nu)$  and  $L(s, \nu^{-1}\mu)$  have a zero at  $s = 1$ . Let  $\mu^{-1}\nu(\alpha) = |\alpha|^r \chi(\alpha)$  where  $\chi$  is a character. Then

$$\begin{aligned} L(s, \mu^{-1}\nu) &= L(s + r, \chi) \\ L(s, \nu^{-1}\mu) &= L(s - r, \chi^{-1}). \end{aligned}$$

Now neither  $L(s, \chi)$  nor  $L(s, \chi^{-1})$  has a zero in the set  $\operatorname{Re} s \geq 1$ . Therefore  $1 + r < 1$  and  $1 - r < 1$ . This is impossible.

We can now apply Corollary 11.6 to assert that there is a constituent  $\pi' = \bigotimes_v \pi'_v$  of  $\mathcal{A}_0$  such that  $\pi'_v = \pi(\sigma_v)$  for  $v$  not in  $S$ . To prove the theorem we need only show that  $\pi'_v = \pi(\sigma_v)$  for  $v$  in  $S$ . Taking the quotient of the two functional equations

$$L(s, \omega \otimes \sigma) = \left\{ \prod_v \epsilon(s, \omega_v \otimes \sigma_v, \psi_v) \right\} L(1 - s, \omega^{-1} \otimes \tilde{\sigma})$$

and

$$L(s, \omega \otimes \pi') = \left\{ \prod_v \epsilon(s, \omega_v \otimes \pi'_v, \psi_v) \right\} L(1 - s, \omega^{-1} \otimes \tilde{\pi}'),$$

we find that

$$\prod_{v \in S} \frac{L(s, \omega_v \otimes \sigma_v)}{L(s, \omega_v \otimes \pi'_v)}$$

is equal to

$$\left\{ \prod_{v \in S} \frac{\epsilon(s, \omega_v \otimes \sigma_v, \psi_v)}{\epsilon(s, \omega_v \otimes \pi'_v, \psi_v)} \right\} \left\{ \prod_{v \in S} \frac{L(1 - s, \omega_v^{-1} \otimes \tilde{\sigma}_v)}{L(1 - s, \omega_v^{-1} \otimes \tilde{\pi}'_v)} \right\}.$$

We need one more lemma. If  $v$  is a non-archimedean place and  $\omega_v$  is a quasi-character of  $F_v^\times$  let  $m(\omega_v)$  be the smallest non-negative integer such that  $\omega_v$  is trivial on the units of  $O_v$  congruent to 1 modulo  $\mathfrak{p}_v^{m(\omega_v)}$ .

**Lemma 12.5.** *Suppose  $S$  is a finite set of non-archimedean places and  $v_0 \in S$ . Suppose that we are given a quasi-character  $\chi_{v_0}$  of  $F_{v_0}^\times$  and for each  $v \neq v_0$  in  $S$  a non-negative integer  $m_v$ . Then there is a quasi-character  $\omega$  of  $C_F$  such that  $\omega_{v_0} = \chi_{v_0}$  and  $m(\omega_v) \geq m_v$  if  $v \neq v_0$  belongs to  $S$ .*

Suppose  $\chi_{v_0}(\alpha) = |\alpha|_{v_0}^a \chi'_{v_0}(\alpha)$  where  $\chi'_{v_0}$  is a character. If  $\omega'$  is a character of  $C_F$  and  $\omega'_{v_0} = \chi'_{v_0}$  while  $m(\omega'_v) \geq m_v$  for  $v \neq v_0$  in  $S$  we may take  $\omega$  to be the generalized character  $\alpha \rightarrow |\alpha|^r \omega'(\alpha)$  of  $C_F$ . In other words we may assume initially that  $\chi_{v_0}$  is a character. Let  $A$  be the group of ideles whose component at places not in  $S$  is 1, whose component of a place  $v \notin v_0$  in  $S$  is congruent to 1 modulo  $\mathfrak{p}_v^{m_v}$ , and whose component at  $v_0$  is arbitrary. Certainly  $F^\times \cap A = \{1\}$ . We claim that  $F^\times A$  is closed in  $I$ . Indeed if  $\alpha \in I$  there is a compact neighbourhood  $X$  of  $\alpha$  on which the norm is bounded above by  $1/\epsilon$  and below by  $\epsilon$  where  $\epsilon$  is a positive constant. If  $\beta \in F^\times$  and  $\gamma \in A$  then  $|\beta\gamma| = |\gamma|$ . Moreover

$$A_\epsilon = \left\{ \gamma \in A \mid \epsilon \leq |\gamma| \leq \frac{1}{\epsilon} \right\}$$

is compact. Since  $F^\times$  is discrete  $F^\times A_\epsilon$  is closed. Since any point has a compact neighbourhood whose intersection with  $F^\times A$  is closed the set  $F^\times A$  is itself closed.

We can certainly find a character of  $A$  which equals  $\chi_{v_0}$  on  $F_{v_0}^\times$  and, for any  $v \neq v_0$  in  $S$ , is non-trivial on the set of units in  $O_v$  congruent to 1 modulo  $\mathfrak{p}_v^{m_v}$ . Extend this character to  $F^\times A$  by setting it equal to 1 on  $F^\times$ . The result can be extended to a character of  $I$  which is necessarily 1 on  $F^\times$ . We take  $\omega$  to be this character.

Let  $\pi'_v \left( \begin{pmatrix} \alpha_v & 0 \\ 0 & \alpha_v \end{pmatrix} \right) = \eta_v(\alpha_v)$ . If  $\eta(\alpha) = \prod_v \eta_v(\alpha_v)$  then  $\eta$  is a quasi-character of  $F^\times \backslash I$ . Since, by construction,  $\eta = \det \sigma$  on  $I_D^s$  the quasi-characters  $\eta$  and  $\det \sigma$  are equal. Therefore  $\eta_v = \det \sigma_v$  for all  $v$ . We know that if  $m(\omega_v)$  is sufficiently large,

$$L(s, \omega_v \otimes \sigma_v) = L(s, \omega_v \otimes \pi'_v) = 1$$

and

$$L(1-s, \omega_v^{-1} \otimes \tilde{\sigma}_v) = L(1-s, \omega_v^{-1} \otimes \tilde{\pi}'_v) = 1.$$

Moreover, by Proposition 3.8

$$\epsilon(s, \omega_v \otimes \pi'_v, \psi_v) = \epsilon(s, \omega_v \eta_v, \psi_v) \epsilon(s, \omega_v, \psi_v).$$

It is shown in the Appendix of [19] that if  $m(\omega_v)$  is sufficiently large

$$\epsilon(s, \omega_v \otimes \sigma_v, \psi_v) = \epsilon(s, \omega_v \det \sigma_v, \psi_v) \epsilon(s, \omega_v, \psi_v).$$

Applying Lemma 12.5 and the equality preceding it we see that if  $v$  is in  $S$  and  $\omega_v$  is any quasi-character of  $F_v^\times$

$$\frac{L(s, \omega_v \otimes \sigma_v)}{L(s, \omega_v \otimes \pi'_v)} = \left\{ \frac{\epsilon(s, \omega_v \otimes \sigma_v, \psi_v)}{\epsilon(s, \omega_v \otimes \pi'_v, \psi_v)} \right\} \left\{ \frac{L(1-s, \omega_v^{-1} \otimes \tilde{\sigma})}{L(1-s, \omega_v^{-1} \otimes \tilde{\pi}'_v)} \right\}.$$

Recalling that

$$L(s, \omega_v \otimes \sigma_v) = L(1-s, \omega_v^{-1} \otimes \tilde{\sigma}_v) = 1$$

for  $v$  in  $S$  we see that

$$(12.5.1) \quad \frac{L(1-s, \omega_v^{-1} \otimes \tilde{\pi}'_v)}{L(s, \omega_v \otimes \pi'_v)} = \frac{\epsilon(s, \omega_v \otimes \sigma_v, \psi_v)}{\epsilon(s, \omega_v \otimes \pi'_v, \psi_v)}.$$

The theorem will follow if we show that

$$L(s, \omega_v \otimes \pi'_v) = L(1-s, \omega_v^{-1} \otimes \pi'_v) = 1$$

for all choices of  $\omega_v$ .

If not, either  $\pi'_v$  is a special representation or there are two quasi-characters  $\mu_v$  and  $\nu_v$  of  $F_v^\times$  such that  $\pi'_v = \pi(\mu_v, \nu_v)$ . According to (12.5.1) the quotient

$$\frac{L(1-s, \omega_v^{-1} \otimes \pi'_v)}{L(s, \omega_v \otimes \pi'_v)}$$

is an entire function of  $s$  for every choice of  $\omega_v$ . If  $\pi'_v = \pi(\mu_v, \nu_v)$  and  $m(\mu_v^{-1} \nu_v)$  is positive

$$\frac{L(1-s, \mu_v \otimes \tilde{\pi}'_v)}{L(s, \mu_v^{-1} \otimes \pi'_v)} = \frac{1 - |\varpi_v|^s}{1 - |\varpi_v|^{1-s}}$$

which has a pole at  $s = 1$ . If  $m(\mu_v^{-1} \nu_v) = 0$

$$\frac{L(1-s, \mu_v \otimes \tilde{\pi}'_v)}{L(s, \mu_v^{-1} \otimes \pi'_v)} = \left\{ \frac{1 - |\varpi_v|^s}{1 - |\varpi_v|^{1-s}} \right\} \left\{ \frac{1 - \mu_v^{-1} \nu_v(\varpi_v) |\varpi_v|^s}{1 - \mu_v \nu_v^{-1}(\varpi_v) |\varpi_v|^{1-s}} \right\}$$

which has a pole at  $s = 1$  unless  $\mu_v \nu_v^{-1}(\varpi_v) = |\varpi_v|$ . But then it has a pole at  $s = 2$ . If  $\pi'_v$  is the special representation associated to the pair of quasi-characters

$$\alpha \rightarrow \mu_v(\alpha)|\alpha|^{1/2} \quad \alpha \rightarrow \mu_v(\alpha)|\alpha|^{-1/2}$$

of  $F_v^\times$  then

$$\frac{L(1-s, \mu_v \otimes \tilde{\pi}'_v)}{L(s, \mu_v^{-1} \otimes \pi'_v)} = \frac{1 - |\varpi_v|^{s+\frac{1}{2}}}{1 - |\varpi_v|^{\frac{1}{2}-s}}$$

which has a pole at  $s = \frac{1}{2}$ .

There is a consequence of the theorem which we want to observe.

**Proposition 12.6.** *Suppose  $E$  is a global field and that for every separable extension  $F$  of  $E$ , every Galois extension  $K$  of  $F$ , and every irreducible two-dimensional representation  $\sigma$  of  $W_{K/F}$  the function  $L(s, \sigma)$  is entire and bounded in vertical strips. Then if  $F_1$  is the completion of  $E$  at some place,  $K_1$  is a Galois extension of  $F_1$ , and  $\sigma_1$  is a two-dimensional representation of  $W_{K_1/F_1}$ , the representation  $\pi(\sigma_1)$  exists.*

We begin with a simple remark. The restriction of  $\sigma_1$  to  $C_{K_1}$  is the direct sum of two one-dimensional representations corresponding to generalized characters  $\chi_1$  and  $\chi_2$  of  $C_{K_1}$ . If  $\tau$  belongs to  $G = \mathfrak{G}(K_1/F_1)$  either  $\chi_1(\tau(\alpha)) = \chi_1(\alpha)$  for all  $\alpha$  in  $C_K$  or  $\chi_1(\tau(\alpha)) = \chi_2(\alpha)$  for all  $\alpha$  in  $C_K$ . If the representation  $\sigma_1$  is irreducible there is at least one  $\tau$  for which  $\chi_1(\tau(\alpha)) = \chi_2(\alpha)$ . If  $\chi_1 \neq \chi_2$ , the fixed field  $L_1$  of

$$H = \left\{ \tau \in G \mid \chi_1(\tau(\alpha)) \equiv \chi_1(\alpha) \right\}$$

is a quadratic extension of  $F$ . The restriction of  $\sigma_1$  to  $W_{K_1/L_1}$  is the direct sum of two one-dimensional representations and therefore is trivial on the commutator subgroup  $W_{K_1/L_1}^c$  which is the kernel of  $\tau_{K_1/F_1, L_1/F_1}$ . With no loss of generality we may suppose that  $K_1$  equals  $L_1$  and is therefore a quadratic extension of  $F_1$ . Then  $\sigma_1$  is equivalent to the representation

$$\text{Ind}(W_{K_1/F}, W_{K_1/K_1}, \chi_1).$$

If  $\sigma_1$  is reducible  $\pi(\sigma_1)$  is defined. The preceding remarks show that it is defined if  $\sigma_1$  is irreducible and  $\sigma_1(\alpha)$  is not a scalar matrix for some  $\alpha$  in  $C_{K_1}$ . The proposition will therefore follow from Theorem 12.2 and the next lemma.

**Lemma 12.7.** *Suppose  $F_1$  is the completion of the field  $E$  at some place,  $K_1$  is a Galois extension of  $F_1$ , and  $\sigma_1$  is an irreducible two-dimensional representation such that  $\sigma_1(\alpha)$  is a scalar matrix for all  $\alpha$  in  $C_{K_1}$ . Then there is a separable extension  $F$  of  $E$ , a Galois extension  $K$  of  $F$ , a place  $v$  of  $K$ , and isomorphism  $\varphi$  of  $K_v$  with  $K_1$  which takes  $F_v$  to  $F_1$ , and an irreducible two-dimensional representation  $\sigma$  of  $W_{K/F}$  such that  $\sigma_v$  is equivalent to  $\sigma_1 \circ \varphi$ .*

Observe that the existence of  $\sigma_1$  forces  $F_1$  to be non-archimedean. We establish a further sequence of lemmas.

**Lemma 12.8.** *Suppose  $V$  is a finite-dimensional real vector space,  $G$  is a finite group of linear transformations of  $V$ , and  $L$  is a lattice in  $V$  invariant under  $G$ . If  $\chi$  is a quasi-character of  $L$  invariant under  $G$  there is a quasi-character  $\chi'$  of  $V$  invariant under  $G$  and a positive integer  $m$  such that the restrictions of  $\chi'$  and  $\chi$  to  $mL$  are equal.*

Let  $\widehat{V}$  be the dual of  $V$  and  $\widehat{V}_{\mathbf{C}}$  its complexification. There is a  $y$  in  $\widehat{V}_{\mathbf{C}}$  such that  $\chi(x) = e^{2\pi i \langle x, y \rangle}$  for all  $x$  in  $L$ . If  $z$  belongs to  $\widehat{V}_{\mathbf{C}}$  the generalized character  $x \rightarrow e^{2\pi i \langle x, z \rangle}$  is trivial on  $L$  if and only if  $z$  belongs to  $\frac{\widehat{L}}{m}$ .  $\widehat{L}$  is the lattice

$$\left\{ v \in \widehat{V} \mid \langle x, v \rangle \in \mathbb{Z} \text{ for all } x \text{ in } L \right\}.$$

Let  $\widehat{G}$  be the group contragredient to  $G$ . We have to establish the existence of an  $m$  and a  $z$  in  $\frac{\widehat{L}}{m}$  such that  $y - z$  is fixed by  $\widehat{G}$ . If  $\sigma$  belongs to  $\widehat{G}$  then  $\sigma y - y = w_{\sigma}$  belongs to  $\widehat{L}$ . Clearly  $\sigma w_{\tau} + w_{\sigma} = w_{\sigma\tau}$ . Set

$$z = \frac{1}{[G : 1]} \sum_{\tau} w_{\tau}.$$

If  $m$  is taken to be  $[G : 1]$  this is the required element.

**Lemma 12.9.** *Suppose  $F$  is a global field,  $K$  is a Galois extension of it, and  $v$  is a place of  $K$ . Suppose also that  $[K_v : F_v] = [K : F]$  and let  $\chi_v$  be a quasi-character of  $C_{K_v}$  invariant under  $G = \mathfrak{G}(K_v/F_v) = \mathfrak{G}(K/F)$ . There is a closed subgroup  $A$  of finite index in  $C_K$  which is invariant under  $G$  and contains  $C_{K_v}$  and a quasi-character  $\chi$  of  $A$  invariant under  $G$  whose restriction to  $C_{K_v}$  is  $\chi_v$ .*

Suppose first that the fields have positive characteristic. We can choose a set of non-negative integers  $n_w$ ,  $w \neq v$ , all but a finite number of which are zero, so that the group

$$B = C_{K_v} \times \prod_{w \neq v} U_{K_w}^{n_w}$$

is invariant under  $G$  and contains no element of  $K^{\times}$  except 1. Here  $U_{K_w}^{n_w}$  is the group of units of  $O_{K_w}$  which are congruent to 1 modulo  $\mathfrak{p}_{K_w}^{n_w}$ . We extend  $\chi_v$  to  $B$  by setting it equal to 1 on

$$\prod_{w \neq v} U_{K_w}^{n_w}$$

and then to  $A = K^{\times} B / K^{\times}$  by setting it equal to 1 on  $K^{\times}$ .

Now let the fields have characteristic 0. Divide places of  $K$  different from  $v$  into two sets,  $S$ , consisting of the archimedean places, and  $T$ , consisting of the non-archimedean ones. Choose a collection of non-negative integers  $n'_w$ ,  $w \in T$ , all but a finite number of which are zero, so that

$$B' = C_{K_v} \times \prod_{w \in S} C_{K_w} \times \prod_{w \in T} U_{K_w}^{n'_w}.$$

is invariant under  $G$  and contains no roots of unity in  $K$  except 1. If  $w$  is archimedean let  $U_{K_w}$  be the elements of norm 1 in  $K_w$  and set

$$B'_1 = \prod_{w \in S} U_{K_w} \times \prod_{w \in T} U_{K_w}^{n'_w}.$$

$B'/B'_1$  is isomorphic to the product of  $C_{K_v}$  and

$$V = \prod_{w \in S} C_{K_w} / U_{K_w}$$

which is a vector group. The projection  $L$  of

$$M = B'_1(B' \cap K^{\times})/B'_1$$

on  $V$  is a lattice in  $V$  and the projection is an isomorphism. Define the quasi-character  $\mu$  of  $L$  so that if  $m$  in  $M$  projects to  $m_1$  in  $C_{K_v}$  and to  $m_2$  in  $V$  then

$$\chi_v(m_1)\mu(m_2) = 1.$$

$\mu$  is invariant under  $G$ . Choose a quasi-character  $\mu'$  of  $V$  and an integer  $n$  so that  $\mu'$  and  $\mu$  are equal on  $nL$ . Let  $\nu'$  be the quasi-character obtained by lifting  $\chi_v \times \mu'$  from  $C_{K_v} \times V$  to  $B'$ . It follows from a theorem of Chevalley ([20, Theorem 1]) that we can choose a collection of non-negative integers  $\{n_w \mid w \in T\}$  all but a finite number of which are zero so that  $n_w \geq n'_w$  for all  $w$  in  $T$ , so that

$$B = C_{K_v} \times \prod_{w \in S} C_{K_w} \times \prod_{w \in T} U_{K_w}^{n_w}$$

is invariant under  $G$ , and so that every element of  $B \cap K^\times$  is an  $n$ th power of some element of  $B' \cap K^\times$ . The restriction  $\nu$  of  $\nu'$  to  $B$  is trivial on  $B \cap K^\times$ . We take  $A = K^\times B / K^\times$  and let  $\chi$  be the quasi-character which is 1 on  $K^\times$  and  $\nu$  on  $B$ .

**Lemma 12.10.** *Suppose  $F_1$  is a completion of the global field  $E$ ,  $K_1$  is a finite Galois extension of  $F_1$  with Galois group  $G_1$ , and  $\chi_{K_1}$  is a quasi-character of  $C_{K_1}$  invariant under  $G_1$ . There is a separable extension  $F$  of  $E$ , a Galois extension  $K$  of  $F$ , a place  $v$  of  $K$  such that  $[K_v : F_v] = [K : F]$ , an isomorphism  $\varphi$  of  $K_v$  with  $K_1$  which takes  $F_v$  to  $F_1$ , and a quasi-character  $\chi$  of  $C_K$  invariant under  $\mathfrak{S}(K/F)$  such that  $\chi_v = \chi_{K_1} \circ \varphi$ .*

We may as well suppose that  $F_1 = E_w$ , where  $w$  is some place of  $E$ . It is known ([8, p. 31]) that there is a polynomial with coefficients in  $E$  such that if  $\theta$  is a root of this polynomial  $E_w(\theta)/E_w$  is isomorphic to  $K_1/F_1$ . Let  $L$  be the splitting field of this polynomial and extend  $w$  to a place of  $L$ . The extended place we also call  $w$ . Replacing  $E$  by the fixed field of the decomposition group of  $w$  if necessary we may suppose that  $F_1 = E_w$ ,  $K_1 = L_w$  and  $[L_w : E_w] = [L : E]$ . Now set  $\chi_w = \chi_{K_1}$  and extend  $\chi_w$  to a quasi-character  $\chi'$  of  $A$  as in the previous lemma.

Let  $K$  be the abelian extension of  $L$  associated to the subgroup  $A$ . Since  $A$  is invariant under  $\mathfrak{S}(L/E)$  the extension  $K/E$  is Galois. Let  $v$  be a place of  $K$  dividing the place  $w$  of  $L$ . Since  $A$  contains  $C_{L_w}$  the fields  $K_v$  and  $L_w$  are equal. Let  $F$  be the fixed field of the image of  $\mathfrak{S}(K_v/E_w)$  in  $\mathfrak{S}(K/E)$ . Let  $v$  also denote the restriction of  $v$  to  $F$ . The fields  $F_v$  and  $E_w$  are the same. The mapping  $N_{K/L} : C_K \rightarrow C_L$  maps  $C_K$  into  $A$ . Let  $\chi = \chi' \circ N_{K/L}$ . Then  $\chi$  is clearly invariant under  $\mathfrak{S}(K/F)$ . Since  $N_{K/L}$  restricted to  $K_v$  is an isomorphism of  $K_v$  with  $L_w$  which takes  $F_v$  onto  $E_w$  the lemma is proved.

To prove Lemma 12.7 we need only show that if  $F$  is a global field,  $K$  is a Galois extension of  $F$ ,  $\chi$  is a quasi-character of  $C_K$  invariant under  $\mathfrak{S}(K/F)$ ,  $v$  is a place of  $K$  such that  $[K : F] = [K_v : F_v]$ , and  $\sigma_1$  is an irreducible two-dimensional representation of  $W_{K_v/F_v}$  such that  $\sigma_v(\alpha) = \chi_v(\alpha)I$  for all  $\alpha$  in  $C_{K_v}$  then there is a two-dimensional representation  $\sigma$  of  $W_{K/F}$  such that  $\sigma_v$  is equivalent to  $\sigma_1$ . The representation  $\sigma$  will be irreducible because  $\sigma_1$  is.

Let  $\sigma_1$  act on  $X$ . Let  $\rho_v$  be the right regular representation of  $W_{K_v/F_v}$  on the space  $V_v$  of functions  $f$  on  $W_{K_v/F_v}$  satisfying

$$f(\alpha w) = \chi_v(\alpha)f(w)$$

for all  $\alpha$  in  $C_{K_v}$  and all  $w$  in  $W_{K_v/F_v}$ . If  $\lambda$  is a non-zero linear functional on  $X$  the map from  $x$  to the function  $\lambda(\sigma_1(w)x)$  is a  $W_{K_v/F_v}$ -invariant isomorphism of  $X$  with a subspace  $Y$  of  $V_v$ .

Let  $V$  be the space of all functions  $f$  on  $W_{K/F}$  satisfying

$$f(\alpha w) = \chi(\alpha)f(w)$$

for all  $\alpha$  in  $C_K$  and all  $w$  in  $W_{K/F}$ . Since  $[K : F] = [K_v : F_v]$  the groups  $\mathfrak{G}(K/F)$  and  $\mathfrak{G}(K_v/F_v)$  are equal. Therefore

$$W_{K/F} = C_K W_{K_v/F_v}.$$

Moreover  $C_{K_v} = C_K \cap W_{K_v/F_v}$ . Thus the restriction of functions in  $V$  to  $W_{K_v/F_v}$  is an isomorphism of  $V$  with  $V_v$ . For simplicity we identify the two spaces. Let  $\rho$  be the right regular representation of  $W_{K/F}$  on  $V$ . If  $\alpha$  belongs to  $C_K$  then

$$f(w\alpha) = \chi(w\alpha w^{-1})f(w) = \chi(\alpha)f(w)$$

because  $\chi$  is  $\mathfrak{G}(K/F)$  invariant. Therefore  $\rho(\alpha) = \chi(\alpha)I$  and a subspace  $V$  is invariant under  $W_{K/F}$  if and only if it is invariant under  $W_{K_v/F_v}$ . If we take for  $\sigma$  the restriction of  $\rho$  to  $Y$  then  $\sigma_v$  will be equivalent to  $\sigma_1$ .

## References for Chapter II

Automorphic forms are discussed in terms of group representations in [3] and [11] as well as:

- [13] Godement, R., *Analyse spectrale des fonctions modulaires*, Seminaire Bourbaki, No. 278.
- [14] Godement, R., *Introduction à la theorie de Langlands*, Seminaire Bourbaki, No. 321.  
As its name implies the Hecke theory is a creation of Hecke.
- [15] Hecke E., *Mathematische Werke*.  
Maass seems to have been the first to consider it outside the classical context.
- [16] Maass, H., *Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen*, Math. Ann. 121 (1944).  
It seems to have been Weil who first used several  $L$ -functions to prove a converse theorem.
- [17] Weil, A., *Über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen*, Math. Ann. 168 (1967).  
His generalizations of the Artin  $L$ -functions are introduced in:
- [18] Weil, A., *Sur la théorie du corps de classes*, Jour. Math. Soc. Japan, vol. 3 (1951).  
For various technical facts used in the twelfth paragraph we refer to:
- [19] Langlands R., *On the functional equation of the Artin  $L$ -functions*, Notes, Yale University (in preparation).  
[Unpublished preprint, 1970]  
We have also had occasion to refer to:
- [20] Chevalley, C., *Deux théorèmes d'arithmétique*, Jour. Math. Soc. Japan, vol. 3 (1951).  
A result more or less the same as Proposition 12.1 is proved in:
- [21] Shalika, J. A. and S. Tanaka, *On an explicit construction of a certain class of automorphic forms*, preprint. [American Journal of Mathematics vol. 91, no. 4, October 1969, pp. 1049–1076.]





## CHAPTER III

### Quaternion Algebras

#### §13. Zeta-functions for $M(2, F)$

In this paragraph  $F$  is again a local field and  $A = M(2, F)$  is the algebra of  $2 \times 2$  matrices with entries from  $F$ . The multiplicative group  $A^\times$  of  $A$  is just  $G_F = \text{GL}(2, F)$ . If  $g$  is in  $G_F$  we set

$$|g|_A = \alpha_A(g) = |\det g|_F^2.$$

Let  $\pi$  be an admissible representation of  $\mathcal{H}_F$  on the space  $V$ . Let the contragredient representation  $\tilde{\pi}$  act on  $\tilde{V}$ . If  $v$  belongs to  $V$  and  $\tilde{v}$  to  $\tilde{V}$  the function

$$\langle \pi(g)v, \tilde{v} \rangle = \langle v, \tilde{\pi}(g^{-1})\tilde{v} \rangle$$

is characterized by the relation

$$\int \langle \pi(gh), v, \tilde{v} \rangle f(h) dh = \langle \pi(g)\pi(f)v, \tilde{v} \rangle$$

for all  $f$  in  $\mathcal{H}_F$ .

If  $\Phi$  belongs to the Schwartz space  $\mathcal{S}(A)$  and  $v$  belongs to  $V$  and  $\tilde{v}$  to  $\tilde{V}$  we set

$$Z(\pi, \Phi, v, \tilde{v}) = \int_{G_F} \Phi(g) \langle \pi(g)v, \tilde{v} \rangle d^\times g$$

and

$$Z(\tilde{\pi}, \Phi, v, \tilde{v}) = \int_{G_F} \Phi(g) \langle v, \tilde{\pi}(g)\tilde{v} \rangle d^\times g$$

The choice of Haar measure is not important provided that it is the same for both integrals.

If  $\omega$  is a quasi-character of  $F^\times$

$$Z(\omega \otimes \pi, \Phi, v, \tilde{v}) = \int_{G_F} \Phi(g) \omega(\det g) \langle \pi(g)v, \tilde{v} \rangle d^\times g$$

The purpose of this paragraph is to prove the following theorem.

**Theorem 13.1.** *Let  $\pi$  be an irreducible admissible representation of  $\mathcal{H}_F$  and  $\tilde{\pi}$  its contragredient. Let  $\pi$  act on  $V$  and  $\tilde{\pi}$  on  $\tilde{V}$ .*

- (i) *For every  $v$  in  $V$ ,  $\tilde{v}$  in  $\tilde{V}$ , and  $\Phi$  in  $\mathcal{S}(A)$  the integrals defining  $Z(\alpha_F^s \otimes \pi, \Phi, v, \tilde{v})$  and  $Z(\alpha_F^s \otimes \tilde{\pi}, \Phi, v, \tilde{v})$  converge absolutely for  $\text{Re } s$  sufficiently large.*
- (ii) *Both functions can be analytically continued to functions which are meromorphic in the whole plane and bounded at infinity in vertical strips of finite width.*
- (iii) *If*

$$Z\left(\alpha_F^{s+\frac{1}{2}} \otimes \pi, \Phi, v, \tilde{v}\right) = L(s, \pi) \Xi(s, \Phi, v, \tilde{v})$$

and

$$Z\left(\alpha_F^{s+\frac{1}{2}} \otimes \tilde{\pi}, \Phi, v, \tilde{v}\right) = L(s, \tilde{\pi}) \tilde{\Xi}(s, \Phi, v, \tilde{v})$$

then  $\Xi(s, \Phi, v, \tilde{v})$  and  $\tilde{\Xi}(s, \Phi, v, \tilde{v})$  are entire.

(iv) There exist  $\phi, v_1, \dots, v_n$  and  $\tilde{v}_1, \dots, \tilde{v}_n$  such that  $\sum_{i=1}^n \Xi(s, \Phi, v_i, \tilde{v}_i)$  is of the form  $ae^{bs}$  with  $a \neq 0$ .

(v) If  $\Phi'$  is the Fourier transform of  $\Phi$  with respect to the character  $\psi_A(x) = \psi_F(\text{tr } x)$  then

$$\tilde{\Xi}(1-s, \Phi', v, \tilde{v}) = \epsilon(s, \pi, \psi_F) \Xi(s, \Phi, v, \tilde{v}).$$

We suppose first that  $F$  is non-archimedean and  $\pi$  is absolutely cuspidal. Then we may take  $\pi$  in the Kirillov form so that  $V$  is just  $\mathcal{S}(F^\times)$ . Since an additive character  $\psi_F = \psi$  is given we will of course want to take the Kirillov model with respect to it. The next lemma is, in the case under consideration, the key to the theorem.

**Lemma 13.1.1.** *If  $\varphi$  belongs to  $\mathcal{S}(F^\times)$ ,  $v$  belongs to  $V$ , and  $\tilde{v}$  belongs to  $\tilde{V}$  set*

$$\Phi(g) = \varphi(\det g) \langle v, \tilde{\pi}(g) \tilde{v} \rangle |\det g|_F^{-1}$$

*if  $g$  belongs to  $G_F$  and set  $\Phi(g) = 0$  if  $g$  in  $A$  is singular. Then  $\Phi$  belongs to  $\mathcal{S}(A)$  and its Fourier transform is given by*

$$\Phi'(g) = \varphi'(\det g) \langle \pi(g)v, \tilde{v} \rangle |\det g|_F^{-1} \eta^{-1}(\det g)$$

*if  $g$  belongs to  $G_F$  and*

$$\Phi'(g) = 0$$

*if  $g$  is singular. Here  $\eta$  is the quasi-character of  $F^\times$  defined by*

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \eta(a)I$$

and

$$\varphi' = \pi\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\varphi.$$

This lemma is more easily appreciated if it is compared with the next one which is simpler but which we do not really need.

**Lemma 13.1.2.** *Let  $\mathcal{S}_0(A)$  be the space of all  $\Phi$  in  $\mathcal{S}(A)$  that vanish on the singular elements and satisfy*

$$\int \Phi\left(g_1 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g_2\right) dx = 0$$

*for  $g_1$  and  $g_2$  in  $G_F$ . If  $\Phi$  is in  $\mathcal{S}_0(A)$  so is its Fourier transform.*

Since  $\mathcal{S}_0(A)$  is stable under left and right translations by the elements of  $G_F$  it is enough to show that

$$\Phi'\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right) = 0$$

for  $a$  in  $F$  and that

$$\int_F \Phi' \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) dx = 0$$

To verify these relations we just calculate the left sides!

$$\Phi' \left( \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \right) = \int_A \Phi(g) \psi_A \left( g \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \right) dg$$

The right side is a positive multiple of

$$\int_{G_F} \Phi(g) \psi_A \left( g \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \right) |\det g|^2 d^\times g$$

which equals

$$\int_{G_F/N_F} \psi_A \left( g \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \right) |\det g|^2 \left\{ \int_F \Phi \left( g \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) dx \right\} d^\times g$$

This is 0 because the inner integral vanishes identically.

$$\int_F \Phi' \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) dx$$

is equal to

$$\int \left\{ \int \Phi \left( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right) \psi_F(\alpha + \delta + \gamma x) d\alpha d\beta d\gamma d\delta \right\} dx$$

which, by the Fourier inversion formula, is equal to

$$\int \Phi \left( \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \right) \psi_F(\alpha + \delta) d\alpha d\delta d\beta$$

which equals

$$\int |\alpha| \psi_F(\alpha + \delta) \left\{ \Phi \left( \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \right) d\beta \right\} d\alpha d\delta$$

and this is 0.

We return to the proof of Lemma 13.1.1 for absolutely cuspidal  $\pi$ . Since  $\langle v, \tilde{\pi}(g)\tilde{v} \rangle$  has compact support on  $G_F$  modulo  $Z_F$  the function  $\Phi(g)$  belongs to  $\mathcal{S}(A)$ . Moreover

$$\int_F \Phi \left( g \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} h \right) dx$$

is equal to

$$\varphi(\det gh) |\det gh|_F^{-1} \int \left\langle \pi(g^{-1})v, \tilde{\pi} \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \tilde{\pi}(h)v \right\rangle dx.$$

Since  $\pi$  is absolutely cuspidal this integral is 0. Thus  $\Phi$  belongs to  $\mathcal{S}_0(A)$  and, in particular,  $\Phi'$  vanishes at the singular elements.

Suppose we can show that for all choice of  $\varphi$ ,  $v$ , and  $\tilde{v}$

$$(13.1.3) \quad \Phi'(e) = \varphi'(1)\langle v, \tilde{v} \rangle.$$

If  $h$  belongs to  $G_F$  set  $\Phi_1(g) = \Phi(h^{-1}g)$ . If  $a = \det h$ ,  $\varphi_1(x) = |a|\varphi(a^{-1}x)$ , and  $v_1 = \pi(h)v$ ,

$$\Phi_1(g) = \varphi_1(\det g)\langle v_1, \tilde{\pi}(g)\tilde{v} \rangle |\det g|_F^{-1}.$$

Then  $\Phi'_1(e)$  is equal to

$$\varphi'_1(e)\langle v_1, v \rangle.$$

On the other hand

$$\varphi'_1 = \pi(w)\varphi = |a|\pi(w)\pi\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi$$

which equals

$$|a|\pi\left(\begin{pmatrix} a^{-1} & 0 \\ 0 & a^{-1} \end{pmatrix}\right)\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\pi(w)\varphi.$$

Thus  $\Phi'(h)$ , which equals  $\Phi'_1(e)|\det h|^{-2}$ , is

$$\varphi'(\det h)\langle \pi(h)v, \tilde{v} \rangle \eta^{-1}(\det h)|\det h|^{-1}.$$

The formula (13.1.3) will be a consequence of the next lemma.

**Lemma 13.1.4.** *Let  $d\epsilon$  be the normalized Haar measure on the group  $U = U_F$ . If  $\nu$  is a character of  $U$  set*

$$\eta(\nu, x) = \int_U \nu(\epsilon)\psi(\epsilon x) d\epsilon$$

*if  $x$  is in  $F$ . Let  $dx$  be the Haar measure on  $F$  which is self-dual with respect to  $\psi$ . Then*

$$\int_F \eta(\nu, x\varpi^n)\psi(ax) dx = 0$$

*unless  $|a| = |\varpi|^n$  but if  $a = \zeta\varpi^n$  with  $\zeta$  in  $U$*

$$\int_F \eta(\nu, x\varpi^n)\psi(ax) dx = \nu(-\zeta)|\varpi|^{-n}c^{-1}$$

*if  $c$  is the measure of  $U$  with respect to  $dx$ .*

The general case results from the case  $n = 0$  by a change of variable; so we suppose  $n = 0$ . In this case the formulae amount to a statement of the Fourier inversion formula for the function which is 0 outside of  $U$  and equal to  $c^{-1}\nu(\epsilon)$  on  $U$ .

Suppose we could show that there is a positive constant  $d$  which does not depend on  $\pi$  such that for all  $\varphi$ ,  $v$ , and  $\tilde{v}$

$$\Phi'(e) = d\varphi'(e)\langle v, \tilde{v} \rangle.$$

Then we would have

$$\Phi'(g) = d\varphi'(\det g)\langle \pi(g)v, \tilde{v} \rangle |\det g|^{-1}\eta^{-1}(\det g).$$

Exchanging  $\pi$  and  $\tilde{\pi}$  and recalling that  $\tilde{\pi} = \eta^{-1} \otimes \pi$  we see that  $\Phi''$ , the Fourier transform of  $\Phi'$ , is given by

$$\Phi''(g) = d^2\varphi''(\det g)\langle v, \tilde{\pi}(g)\tilde{v} \rangle |\det g|_F^{-1}\eta(\det g),$$

where  $\varphi'' = \tilde{\pi}(w)\varphi_1$  if  $\varphi_1(a) = \varphi'(a)\eta^{-1}(a)$ . According to the remarks preceding the statement of Theorem 2.18,  $\varphi''$  is the product of  $\pi(w)\varphi' = \eta(-1)\varphi$  and  $\eta^{-1}(\det g)$ . Thus

$$\Phi''(g) = \eta(-1) d^2 \varphi(\det g) \langle v, \tilde{\pi}(g)\tilde{v} \rangle |\det g|_F^{-1}.$$

Since  $\Phi'' = \Phi(-g) = \eta(-1)\Phi(g)$  the numbers  $d^2$  and  $d$  are both equal to 1. The upshot is that in the proof of the formula (13.1.3) we may ignore all positive constants and in particular do not need to worry about the normalization of Haar measures.

Moreover it is enough to prove the formula for  $\varphi, v, \tilde{v}$  in a basis of the spaces in which they are constrained to lie. Oddly enough the spaces are all the same and equal to  $\mathcal{S}(F^\times)$ . Assume  $\varphi_1 = v$ ,  $\varphi_2 = \tilde{v}$ , and  $\varphi$  are supported respectively by  $\varpi^{n_1}U$ ,  $\varpi^{n_2}U$ , and  $\varpi^n U$  and that, for all  $\epsilon$  in  $U$ ,  $\varphi_1(\varpi^{n_1}\epsilon) = \nu_1^{-1}(\epsilon)$ ,  $\varphi_2(\varpi^{n_2}\epsilon) = \nu_2^{-1}(\epsilon)$  and  $\varphi(\varpi^n \epsilon) = \nu^{-1}(\epsilon)$ . All three of  $\nu, \nu_1$  and  $\nu_2$  are characters of  $U$ .

The formal Mellin transforms of these three functions are  $\hat{\varphi}_1(\mu, t) = \delta(\mu\nu_1^{-1})t^{n_1}$ ,  $\hat{\varphi}_2(\mu, t) = \delta(\mu\nu_2^{-1})t^{n_2}$ , and  $\hat{\varphi}(\mu, t) = \delta(\mu\nu^{-1})t^n$ . Recall that, for example,

$$\hat{\varphi}(\mu, t) = \sum_n t^n \int_U \varphi(\varpi^n \epsilon) \mu(\epsilon) d\epsilon.$$

The scalar product  $\langle \varphi_1, \varphi_2 \rangle$  is equal to

$$\int \varphi_1(a) \varphi_2(-a) d^\times a = \delta(\nu_1 \nu_2) \delta(n_1 - n_2) \nu_2(-1).$$

If  $\eta(\epsilon \varpi^n) = \nu_0(\epsilon) z_0^n$  then

$$\hat{\varphi}'(\mu, t) = C(\mu, t) \hat{\varphi}(\mu^{-1} \nu_0^{-1}, t^{-1} z_0^{-1})$$

which equals

$$\delta(\nu \mu \nu_0) \sum_m C_m(\nu^{-1} \nu_0^{-1}) t^{m-n} z_0^{-n}.$$

Consequently

$$\hat{\varphi}'(1) = C_n(\nu^{-1} \nu_0^{-1}) z_0^{-n}.$$

Thus the formula to be proved reads

$$\Phi'(e) = C_n(\nu^{-1} \nu_0^{-1}) z_0^{-n} \nu_2(-1) \delta(\nu_1 \nu_2) \delta(n_1 - n_2).$$

Almost all  $g$  in  $A$  can be written in the form

$$g = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$$

with  $a$  and  $b$  in  $F^\times$  and  $x$  and  $y$  in  $F$ . The additive Haar measure  $dg$  on  $A$  may be written as

$$dg = |\det g|_F^2 d^\times g = |b^4| d^\times b dx |a| d^\times a dy$$

and for any  $g$  of this form

$$\psi_A(g) = \psi_F(b(x - y))$$

while  $\Phi(g)$  is equal to

$$\eta^{-1}(b) |b^2 a|^{-1} \varphi(b^2 a) \left\langle \pi \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \varphi_1, \tilde{\pi} \left( \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \right) \varphi_2 \right\rangle.$$

Let  $f_1$  and  $f_2$  be the two functions which appear in the scalar product. Their formal Mellin transforms can be calculated by the methods of the second paragraph,

$$\widehat{f}_1(\mu, t) = \nu_0(-1)C(\mu, t)\eta(\mu^{-1}\nu_0^{-1}\nu_1^{-1}, \varpi^{n_1}x)\mu^{-1}\nu_0^{-1}(\zeta)z_0^{-r-n_1}t^{-r-n_2}$$

if  $a = \zeta\varpi^r$  and

$$\widehat{f}_2(\mu, t) = \eta(\mu\nu_2^{-1}, \varpi^{n_2}y)t^{n_2}.$$

The scalar product of  $f_1$  and  $f_2$  is equal to

$$\int f_1(a)f_2(-a) d^\times a$$

which, by the Plancherel theorem for  $F^\times$ , is equal to

$$\sum_{\mu} \mu(-1) \frac{1}{2\pi} \int_0^{2\pi} \widehat{f}_1(\mu, e^{i\theta}) \widehat{f}_2(\mu^{-1}, e^{-i\theta}) d\theta.$$

A typical integral is equal to the product of  $\nu_0(-1)\mu^{-1}\nu_0^{-1}(\zeta)z_0^{-r-n_1}$  and

$$\int_0^{2\pi} C(\mu, e^{i\theta}) e^{-i(r+n_1+n_2)\theta} \eta(\mu^{-1}\nu_0^{-1}\nu_1^{-1}, \varpi^{n_1}x) \eta(\mu^{-1}\nu_2^{-1}, \varpi^{n_2}y) d\theta$$

which equals

$$2\pi C_{r+n_1+n_2}(\mu) \eta(\mu^{-1}\nu_0^{-1}\nu_1^{-1}, \varpi^{n_1}x) \eta(\mu^{-1}\nu_2, \varpi^{n_2}y).$$

Also if  $a = \zeta\varpi^r$

$$\eta^{-1}(b)|b^2a|^{-1}\varphi(b^2a) = \varphi(b^2\varpi^r)\nu^{-1}(\zeta)\eta^{-1}(b)|b^2|^{-1}|a|^{-1}.$$

If we put all this information together we get a rather complicated formula for  $\Phi(g)$  which we have to use to compute  $\Phi'(e)$ . The function  $\Phi'(e)$  is expressed as an integral with respect to  $a$ ,  $b$ ,  $x$ , and  $y$ . We will not try to write down the integrand. The integral with respect to  $a$  is an integration over  $\zeta$  followed by a sum over  $r$ . The integrand is a sum over  $\mu$ . The integration over  $\zeta$  annihilates all but one term, that for which  $\mu\nu\nu_0 = 1$ . We can now attempt to write down the resulting integrand, which has to be integrated over  $b$ ,  $x$ , and  $y$ , and summed over  $r$ . It is the product of

$$\eta^{-1}(b)|b|^2\nu(-1)z_0^{-r-n_1}\varphi(b^2\varpi^r)C_{r+n_1+n_2}(\nu^{-1}\nu_0^{-1})$$

and

$$\eta(\nu\nu_1^{-1}, \varpi^{n_1}x)\eta(\nu\nu_0\nu_2^{-1}, \varpi^{n_2}y)\psi_F(b(x-y)).$$

The second expression can be integrated with respect to  $x$  and  $y$ . Lemma 13.1.4 shows that the result is 0 unless  $|b| = |\varpi|^{n_1} = |\varpi|^{n_2}$ . In particular  $\Phi'(e) = 0$  if  $n_1 \neq n_2$ . If  $n_1 = n_2$  the integration over  $b$  need only be taken over  $\varpi^{n_1}U$ . Then the summation over  $r$  disappears and only the term for which  $r + 2n_1 = n$  remains. Apart from positive constants which depend only on the choices of Haar measure  $\Phi'(e)$  is equal to

$$z_0^{-n}\nu_1(-1)C_n(\nu^{-1}\nu_0^{-1}) \int_U \nu_1^{-1}\nu_2^{-1}(\epsilon) d\epsilon.$$

Since

$$\int_U \nu_1^{-1}\nu_2^{-1}(\epsilon) d\epsilon = \delta(\nu_1\nu_2)$$

the proof of Lemma 13.1.1 is complete.

Since  $L(s, \pi) = L(s, \tilde{\pi}) = 1$  if  $\pi$  is absolutely cuspidal the first three assertions of the theorem are, for such  $\pi$ , consequences of the next lemma.

**Lemma 13.1.5.** *Suppose  $\Phi$  belongs to  $\mathcal{S}(A)$ ,  $v$  belongs to  $V$ , and  $\tilde{v}$  belongs to  $\tilde{V}$ . If  $\pi$  is absolutely cuspidal the integral*

$$\int \Phi(g) \langle \pi(g)v, \tilde{v} \rangle |\det g|^{s+\frac{1}{2}} d^\times g$$

*is absolutely convergent for  $\operatorname{Re} s$  sufficiently large and the functions it defines can be analytically continued to an entire function.*

Suppose the integral is convergent for some  $s$ . If  $\xi$  is an elementary idempotent such that  $\pi(\xi)v = v$  the integral is not changed if  $\Phi$  is replaced by

$$\Phi_1(g) = \int_{\operatorname{GL}(2, O_F)} \Phi(gh^{-1}) \xi(h) dh.$$

Since  $\pi$  is absolutely cuspidal it does not contain the trivial representation of  $\operatorname{GL}(2, O_F)$  and we can choose  $\xi$  to be orthogonal to the constant functions on  $\operatorname{GL}(2, O_F)$ . Then  $\Phi_1(0) = 0$ . Thus, when proving the second assertion of the lemma we can suppose that  $\Phi(0) = 0$ .

The support of  $\langle \pi(g)v, \tilde{v} \rangle$  is contained in a set  $Z_F C$  with  $C$  compact. Moreover there is an open subgroup  $K'$  of  $\operatorname{GL}(2, O_F)$  such that the functions  $\Phi(g)$  and  $\langle \pi(g)v, \tilde{v} \rangle$  are invariant under right translations by the elements of  $K'$ . If

$$C \subseteq \bigcup_{i=1}^p g_i K'$$

the integral is equal to

$$\sum_{i=1}^p \langle \pi(g_i)v, \tilde{v} \rangle |\det g_i|^{s+\frac{1}{2}} \int_{F^\times} \Phi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g_i \right) \eta(a) |a|^{2s+1} d^\times a,$$

if each of the integrals in this sum converges. They are easily seen to converge if  $\operatorname{Re} s$  is sufficiently large and if  $\Phi(0) = 0$  they converge for all  $s$ . The lemma is proved.

Now we verify a special case of the fifth assertion.

**Lemma 13.1.6.** *Suppose  $\varphi$  is in  $\mathcal{S}(F^\times)$  and*

$$\Phi(g) = \varphi(\det g) \langle v, \tilde{\pi}(g)\tilde{v} \rangle |\det g|^{-1}.$$

*Then for all  $u$  in  $V$  and all  $\tilde{u}$  in  $\tilde{V}$*

$$\tilde{\Xi}(1-s, \Phi', u, \tilde{u}) = \epsilon(s, \pi, \psi) \Xi(s, \Phi, u, \tilde{u}).$$

The expression  $\Xi(s, \Phi, u, \tilde{u})$  is the integral over  $G_F$  of

$$|\det g|^{s-\frac{1}{2}} \varphi(\det g) \langle \pi(g)u, \tilde{u} \rangle \langle v, \tilde{\pi}(g)\tilde{v} \rangle.$$

The integral

$$\int_{\operatorname{SL}(2, F)} \langle \pi(gh)u, \tilde{u} \rangle \langle v, \tilde{\pi}(gh)\tilde{v} \rangle dh$$

depends only on  $\det g$ . Set it equal to  $F(\det g)$ . Then  $\Xi(s, \Phi, u, \tilde{u})$  is equal to

$$\int_{F^\times} \varphi(a) F(a) |a|^{s-\frac{1}{2}} d^\times a.$$

By Lemma 13.1.1

$$\Phi'(g) = \varphi'(\det g)|\det g|^{-1}\eta^{-1}(\det g)\langle\pi(g)v, \tilde{v}\rangle$$

so that  $\Xi(s, \Phi', u, \tilde{u})$  is equal to

$$\int_{F^\times} \varphi'(a)\tilde{F}(a)|a|^{s-\frac{1}{2}}\eta^{-1}(a) d^\times a$$

if

$$\tilde{F}(a) \int_{\mathrm{SL}(2, F)} \langle u, \tilde{\pi}(gh)\tilde{u} \rangle \langle \pi(gh)v, \tilde{v} \rangle dh$$

whenever  $a = \det g$ . Since the integrand is not changed when  $g$  is replaced by

$$\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} g$$

we have  $\tilde{F}(b^2a) = \tilde{F}(a)$  and  $\tilde{F}(a) = \tilde{F}(a^{-1})$ . The same relations are valid for  $F$ . Also  $\tilde{F}(a) = F(a^{-1})$  so that  $F = \tilde{F}$ .

We remind ourselves that we are now trying to show that

$$\int_{F^\times} \varphi'(a)\tilde{F}(a)\eta^{-1}(a)|a|^{\frac{1}{2}-s} d^\times a$$

is equal to

$$\epsilon(s, \pi, \psi) \int_{F^\times} \varphi(a)F(a)|a|^{s-\frac{1}{2}} d^\times a.$$

If  $U'$  is an open subgroup of  $U_F$  such that

$$\tilde{\pi}\left(\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}\right)\tilde{u} = \tilde{u}$$

and

$$\pi\left(\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}\right)v = v$$

for  $\epsilon$  in  $U'$  then  $F$  and  $\tilde{F}$  are constant on cosets of  $(F^\times)^2U'$  which is of finite index in  $F^\times$ . Write

$$F(a) = \sum_{i=1}^p c_i \chi_i(a)$$

where  $\chi_i$  are characters of  $F^\times/(F^\times)^2U'$ . We may assume that all  $c_i$  are different from 0. Then

$$F(a^{-1}) = \sum_{i=1}^p c_i \chi_i(a^{-1}).$$

The factor  $\epsilon(s, \pi \otimes \chi_i, \psi)$  was defined so that

$$\int_{F^\times} \varphi'(a)\chi_i^{-1}\eta^{-1}(a)|a|^{\frac{1}{2}-s} d^\times a$$

would be equal to

$$\epsilon(s, \chi_i \otimes \pi, \psi) \int_{F^\times} \varphi(a)\chi_i(a)|a|^{s-\frac{1}{2}} d^\times a.$$



All we need do is show that  $\pi$  and  $\chi_i \otimes \pi$  are equivalent, so that

$$\epsilon(s, \chi_i \otimes \pi, \psi) = \epsilon(s, \pi, \psi).$$

A character  $\chi$  is one of the  $\chi_i$  if and only if  $\chi$  is trivial on  $(F^\times)^2$  and

$$\int_{F^\times / (F^\times)^2} F(a) \chi(a) d^\times a \neq 0.$$

This integral is equal to

$$\int_{G_F / Z_F} \chi(g) \langle \pi(g)u, \tilde{u} \rangle \langle v, \tilde{\pi}(g)\tilde{v} \rangle dg$$

which equals

$$\int_{G_F / Z_F} \langle \chi \otimes \pi(g)u, \tilde{u} \rangle \langle v, \tilde{\pi}(g)\tilde{v} \rangle dg.$$

The integral does not change if  $\pi$  is replaced by  $\omega \otimes \pi$ . Thus the Schur orthogonality relations imply that it is non-zero only if  $\pi$  and  $\chi \otimes \pi$  are equivalent.

If  $\Phi$  belongs to  $\mathcal{S}_0(A)$  the functions  $\Phi(g)|\det g|^{s+\frac{1}{2}}$  belongs to  $\mathcal{H}_F$  and we can form the operator

$$T(s, \Phi) = \int_{G_F} \Phi(g) |\det g|^{s+\frac{1}{2}} \pi(g) d^\times g.$$

If  $\Phi$  has the form of the previous lemma the functional equation may be written as

$$T(1-s, \Phi') = \epsilon(s, \pi, \psi) T(s, \Phi).$$

**Lemma 13.1.7.** *Given a non-zero  $w$  in  $V$ , the set of all  $u$  in  $V$  such that for some  $\Phi$  of the form*

$$\Phi(g) = \varphi(\det g) \langle v, \tilde{\pi}(g)\tilde{v} \rangle |\det g|_F^{-1}$$

*the vector  $T(s, \Phi)w$  is of the form  $e^{bs}u$  is a set that spans  $V$ .*

If the function  $\Phi$  is of this form so is the function  $\Phi'(g) = \Phi(hg)$  and

$$T(s, \Phi')w = |\det h|^{-(s+\frac{1}{2})} \pi(h^{-1})T(s, \Phi)w$$

Since  $\pi$  is irreducible we need only show that there is at least one non-zero vector in the set under consideration. Moreover there is an  $r$  such that  $\alpha_F^r \otimes \pi$  is unitary and we may as well suppose that  $\pi$  itself is unitary. Let  $(u, v)$  be a positive invariant form on  $V$ .

Choose  $v = w$  and  $\tilde{v}$  so that  $\langle u, \tilde{v} \rangle = (u, w)$  for all  $u$ . Let  $\varphi$  be the characteristic function of  $U_F$ . Then

$$\Phi(g) = (w, \pi(g)w)$$

if  $|\det g| = 1$  and is 0 otherwise. If

$$H = \{ g \in G_F \mid |\det g| = 1 \}$$

then

$$T(s, \Phi)w = \int_H (w, \pi(g)w) \pi(g)w d^\times g$$

is independent of  $s$  and is non-zero because

$$(T(s, \Phi)w, w) = \int_H \left| (\pi(g)w, w) \right|^2 d^\times g.$$

The fourth assertion follows immediately and the fifth will now be a consequence of the following lemma.

**Lemma 13.1.8.** *Suppose  $\Phi$  belongs to  $\mathcal{S}(A)$  and  $\Psi$  belongs to  $\mathcal{S}_0(A)$ . There is a vertical strip in which the integrals*

$$\iint \Phi(g)\Psi'(h)\langle\pi(g)v, \tilde{\pi}(h)\tilde{v}\rangle|\det g|^{s+\frac{1}{2}}|\det h|^{\frac{3}{2}-s}d^\times g d^\times h$$

and

$$\iint \Phi'(g)\Psi(h)\langle\pi^{-1}(g)v, \tilde{\pi}(h^{-1})\tilde{v}\rangle|\det g|^{\frac{3}{2}-s}|\det h|^{s+\frac{1}{2}}d^\times g d^\times h$$

exist and are equal.

A little juggling shows that there is no harm in supposing that the quasi-character  $\eta$  defined by

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \eta(a)I$$

is a character. Fix  $v$  and  $\tilde{v}$ . Let  $C$  be a compact subset of  $G_F$  which contains the support of  $\Psi$  and  $\Psi'$ . The set

$$\{\tilde{\pi}(h)\tilde{v} \mid h \in C\}$$

is finite. Thus there is a compact set in  $G_F$  such that for any  $h$  in  $C$  the function

$$g \rightarrow \langle\pi(g)v, \tilde{\pi}(h)\tilde{v}\rangle$$

has its support in  $Z_F C'$ . Moreover these functions are uniformly bounded. The first integral is therefore absolutely convergent for  $\operatorname{Re} s > -\frac{1}{2}$ . The second is convergent for  $\operatorname{Re} s < \frac{3}{2}$ .

If  $-\frac{1}{2} < \operatorname{Re} s < \frac{3}{2}$  the first integral is equal to

$$\int \Psi'(h)|\det h|^{\frac{3}{2}-s}\left\{\int \Phi(g)\langle\pi(g)v, \tilde{\pi}(h)\tilde{v}\rangle|\det g|^{s+\frac{1}{2}}d^\times g\right\}d^\times h.$$

Replacing  $g$  by  $hg$  we obtain

$$\int \Psi'(h)|\det h|^2\left\{\int \Phi(hg)\langle\pi(g)v, \tilde{v}\rangle|\det g|^{s+\frac{1}{2}}d^\times g\right\}d^\times h.$$

If we take the additive Haar measure to be  $dh = |\det h|^2 d^\times h$  this may be written as

$$\int \langle\pi(g)v, \tilde{v}\rangle|\det g|^{s+\frac{1}{2}}\left\{\int \Phi(hg)\Psi'(h)dh\right\}d^\times g.$$

The second integral is

$$\int \Psi(h)|\det h|^{s+\frac{1}{2}}\left\{\int \Phi'(g)\langle\pi^{-1}(g)v, \tilde{\pi}^{-1}(h)\tilde{v}\rangle|\det g|^{\frac{3}{2}-s}d^\times g\right\}d^\times h.$$

After a change of variables this becomes

$$\int \langle\pi^{-1}(g)v, \tilde{v}\rangle|\det g|^{\frac{3}{2}-s}\left\{\int \Phi'(gh)\Psi(h)dh\right\}d^\times g.$$

Replacing  $g$  by  $g^{-1}$  we obtain

$$\int \langle\pi(g)v, \tilde{v}\rangle|\det g|^{s+\frac{1}{2}}\left\{|\det g|^{-2}\int \Phi'(g^{-1}h)\Psi(h)dh\right\}d^\times g.$$

Since

$$\int \Phi(hg)\Psi'(h) dh$$

is equal to

$$|\det g|^{-2} \int \Phi'(g^{-1}h)\Psi(h) dh$$

the lemma follows.

The theorem is now proved when  $\pi$  is absolutely cuspidal. Suppose that it is a constituent of  $\tau = \rho(\mu_1, \mu_2)$ . In this case the field may be archimedean. Although  $\tau$  is not necessarily irreducible it is admissible and its matrix coefficients are defined. The contragredient representation  $\tilde{\tau}$  is  $\rho(\mu_1^{-1}, \mu_2^{-1})$  and the space of  $\tau$  is  $\mathcal{B}(\mu_1, \mu_2)$  while that of  $\tilde{\tau}$  is  $\mathcal{B}(\mu_1^{-1}, \mu_2^{-1})$ . If  $f$  belongs to  $\mathcal{B}(\mu_1, \mu_2)$  and  $\tilde{f}$  belongs to  $\mathcal{B}(\mu_1^{-1}, \mu_2^{-1})$  then

$$\langle \tau(g)f, \tilde{f} \rangle = \int_K f(kg)\tilde{f}(k) dk$$

and

$$\langle f, \tilde{\tau}(g)\tilde{f} \rangle = \int_K f(k)\tilde{f}(kg) dk$$

if  $K$  is the standard maximal compact subgroup of  $G_F$ .

If we set

$$\begin{aligned} L(s, \tau) &= L(s, \mu_1)L(s, \mu_2) \\ L(s, \tilde{\tau}) &= L(s, \mu_1^{-1})L(s, \mu_2^{-1}) \end{aligned}$$

and

$$\epsilon(s, \tau, \psi) = \epsilon(s, \mu_1, \psi)\epsilon(s, \mu_2, \psi)$$

the theorem may be formulated for the representation  $\tau$ . We prove it first for  $\tau$  and then for the irreducible constituents of  $\tau$ .

We use a method of R. Godement. If  $\Phi$  belongs to  $\mathcal{S}(A)$  then for brevity the function  $x \rightarrow \Phi(gxh)$  which also belongs to  $\mathcal{S}(A)$  will be denoted by  $h\Phi g$ . Also let

$$\varphi_\Phi(a_1, a_2) = \int_F \Phi \left( \begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} \right) dx$$

where  $dx$  is the measure which is self dual with respect to  $\psi$ . The function  $\varphi_\Phi$  belongs to  $\mathcal{S}(F^2)$ . The map  $\Phi \rightarrow \varphi_\Phi$  of  $\mathcal{S}(A)$  into  $\mathcal{S}(F^2)$  is certainly continuous.

We are now going to define a kernel  $K_\Phi(h, g, s)$  on  $K \times K$ . We set

$$K_\Phi(e, e, s) = Z(\mu_1\alpha_F^s, \mu_2\alpha_F^s, \varphi_\Phi).$$

Recall that the right-hand side is

$$\iint \varphi_\Phi(a_1, a_2)\mu_1(a_1)|a_1|^s\mu_2(a_2)|a_2|^s d^\times a_1 d^\times a_2.$$

In general

$$K_\Phi(h, g, s) = K_{g\Phi h^{-1}}(e, e, s).$$

We also set

$$\tilde{K}_\Phi(e, e, s) = Z(\mu_1^{-1}\alpha_F^s, \mu_2^{-1}\alpha_F^s, \varphi_\Phi)$$

and

$$\tilde{K}_\Phi(h, g, s) = \tilde{K}_{g\Phi h^{-1}}(e, e, s).$$

The kernels are defined for  $\operatorname{Re} s$  sufficiently large and are continuous in  $h$ ,  $g$ , and  $s$  and, for fixed  $h$  and  $g$ , holomorphic in  $s$ .

We now make some formal computations which will be justified by the result. The expression  $Z\left(\alpha_F^{s+\frac{1}{2}} \otimes \tau, \Phi, f, \tilde{f}\right)$  is equal to

$$\int_{G_F} \Phi(g) \left\{ \int_K f(kg) \tilde{f}(k) dk \right\} |\det g|^{s+\frac{1}{2}} d^\times g$$

which is

$$\int_K \tilde{f}(k) \left\{ \int_{G_F} \Phi(g) f(kg) |\det g|^{s+\frac{1}{2}} d^\times g \right\} dk.$$

Changing variables in the inner integral we obtain

$$\int_K \tilde{f}(k) \left\{ \int_{G_F} \Phi(k^{-1}g) f(g) |\det h|^{s+\frac{1}{2}} d^\times g \right\} dk.$$

Using the Iwasawa decomposition to evaluate the integral over  $G_F$  we see that this is equal to

$$\int_{K \times K} K_\Phi(k_1, k_2, s) f(k_2) \tilde{f}(k_1) dk_1 dk_2.$$

Since we could have put in absolute values and obtained a similar result all the integrals are convergent and equal for  $\operatorname{Re} s$  sufficiently large. A similar computation shows that

$$Z\left(\alpha_F^{s+\frac{1}{2}} \otimes \tilde{\tau}, \Phi, f, \tilde{f}\right)$$

is equal to

$$\int_{K \times K} \tilde{K}_\Phi(k_1, k_2, s) f(k_1) \tilde{f}(k_2) dk_1 dk_2$$

if  $\operatorname{Re} s$  is large enough.

If  $\xi$  is an elementary idempotent such that  $\tau(\xi)f = f$  and  $\tilde{\tau}(\xi)\tilde{f} = \tilde{f}$  then

$$Z\left(\alpha_F^{s+\frac{1}{2}} \otimes \tau, \Phi, f, \tilde{f}\right)$$

is not changed if  $\Phi$  is replaced by

$$\Phi_1(g) = \iint \Phi(k_1 g k_2^{-1}) \xi(k_1) \xi(k_2) dk_1 dk_2.$$

Thus, at least when proving the second and third assertions, we may suppose that  $\Phi$  is  $K$ -finite on both sides and, in fact, transforms according to a fixed finite set of irreducible representations of  $K$ . Then, as  $s$  varies, the functions

$$K_\Phi(k_1, k_2, s)$$

stay in some fixed finite-dimensional space  $U$  of continuous functions on  $K \times K$ . The map

$$F \rightarrow \iint F(k_1, k_2) f(k_2) \tilde{f}(k_1) dk_1 dk_2$$

is a linear form on this space and we can find  $g_1, \dots, g_n$  and  $h_1, \dots, h_n$  in  $K$  such that it can be represented in the form

$$F \rightarrow \sum_{i=1}^n \lambda_i F(g_i, h_i).$$

Thus

$$Z\left(\alpha_F^{s+\frac{1}{2}} \otimes \tau, \Phi, f, \tilde{f}\right) = \sum \lambda_i K_\Phi(g_i, h_i, s).$$

Thus to prove the second and third assertions we need only show that for each  $g$  and  $h$  in  $K$  the function

$$\frac{K_\Phi(g, h, s)}{L(s, \tau)}$$

is entire and  $K_\Phi(g, h, s)$  itself is bounded at infinity in vertical strips. There is certainly no harm in supposing that  $g = h = e$  so that

$$K_\Phi(e, e, s) = Z(\mu_1 \alpha_F^s, \mu_2 \alpha_F^s, \varphi_\Phi)$$

Thus the desired facts are consequences of the results obtained in paragraphs 3, 5, and 6 when proving the local functional equation for constituents of  $\tau$ . Replacing  $\tau$  by its contragredient representation we obtain the same results for  $Z\left(\alpha_F^{s+\frac{1}{2}} \otimes \tilde{\tau}, \Phi, f, \tilde{f}\right)$ .

To prove the functional equation we have to see what happens to the Fourier transform when we pass from the function  $\Phi$  to  $\Phi_1$ . The answer is simple:

$$\Phi'_1(g) = \iint \Phi'(k_1 g k_2^{-1}) \xi(k_1) \xi(k_2) dk_1 dk_2.$$

Thus in proving the functional equation we may suppose that  $\Phi$  is  $K$ -finite on both sides. We may also suppose that if  $F(k_1, k_2)$  is in  $U$  so is  $F'(k_1, k_2) = F(k_2, k_1)$ . Then  $Z\left(\alpha_F^{s+\frac{1}{2}} \otimes \tau, \Phi', f, \tilde{f}\right) = \sum \lambda_i \tilde{K}_{\Phi'}(h_i, g_i, s)$ . To prove the functional equation we have to show that

$$\frac{\tilde{K}_{\Phi'}(h, g, 1-s)}{L(1-s, \tau)} = \epsilon(s, \tau, \psi) \frac{K_\Phi(g, h, s)}{L(s, \tau)}$$

for any  $h$  and  $g$  in  $K$ . Since the Fourier transform of  $g\Phi h^{-1}$  is  $h\Phi'g$  it will be enough to do this for  $h = g = e$ . Then the equality reduces to

$$\frac{Z(\mu_1^{-1} \alpha_F^{1-s}, \mu_2^{-1} \alpha_F^{1-s}, \varphi_{\Phi'})}{L(1-s, \tilde{\tau})} = \epsilon(s, \tau, \psi) \frac{Z(\mu_1 \alpha_F^s, \mu_2 \alpha_F^s, \varphi_\Phi)}{L(s, \tau)}$$

and is a result of the facts proved in the first chapter and the next lemma.

**Lemma 13.2.1.** *The Fourier transform of the function  $\varphi_\Phi$  is the function  $\varphi_{\Phi'}$ .*

The value of  $\Phi'$  at

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is

$$\int \Phi\left(\begin{pmatrix} x & y \\ z & t \end{pmatrix}\right) \psi(\alpha x + \beta z + \gamma y + \delta t) dx dy dz dt$$

if  $dx$ ,  $dy$ ,  $dz$ , and  $dt$  are self-dual with respect to  $\psi$ . Thus  $\varphi_{\Phi'}(\alpha, \delta)$  is equal to

$$\int \left\{ \int \Phi \left( \begin{pmatrix} x & y \\ z & t \end{pmatrix} \right) \psi(\alpha x + \delta t) \psi(\beta z) dx dy dz dt \right\} d\beta$$

Applying the Fourier inversion formula to the pair of variables  $\beta$  and  $z$  we see that this is equal to

$$\int \Phi \left( \begin{pmatrix} x & y \\ 0 & t \end{pmatrix} \right) \psi(\alpha x + \delta t) dx dy dt$$

which is the value of the Fourier transform of  $\varphi_{\Phi}$  at  $(\alpha, \delta)$ .

The theorem, with the exception of the fourth assertion, is now proved for the representation  $\tau$ . We will now deduce it, with the exception of the fourth assertion, for the constituents of  $\tau$ . We will return to the fourth assertion later.

If  $\pi$  is a constituent of  $\tau$  either  $\pi = \pi(\mu_1, \mu_2)$  or  $\pi = \sigma(\mu_1, \mu_2)$ . In the first case there is nothing left to prove. In the second only the third assertion remains in doubt. If  $F$  is the complex field, it is alright because we can always find another pair of quasi-characters  $\mu'_1$  and  $\mu'_2$  such that  $\pi = \pi(\mu'_1, \mu'_2)$ . We ignore this case and suppose that  $F$  is real or non-archimedean.

First take  $F$  to be non-archimedean. We may suppose that  $\mu_1$  and  $\mu_2$  are the form  $\mu_1 = \chi \alpha_F^{1/2}$  and  $\mu_2 = \chi \alpha_F^{-1/2}$ . The one-dimensional representation  $g \rightarrow \chi(\det g)$  is contained in  $\tilde{\tau} = \rho(\mu_1^{-1}, \mu_2^{-1})$  and acts on the function  $g \rightarrow \chi(\det g)$ . The matrix elements for  $\pi$  are the functions

$$g \rightarrow \langle \tau(g)f, \tilde{f} \rangle = \langle \pi(g)f, \tilde{f} \rangle$$

where  $\tilde{f}$  belongs to  $\mathcal{B}(\mu_1^{-1}, \mu_2^{-1})$  and

$$\int_K f(k) \chi(\det k) dk = 0.$$

For such an  $f$  there is an elementary idempotent  $\xi$  such that  $\tau(\xi)f = f$  while

$$\int_K \xi(k) dk = 0$$

The value of  $Z\left(\alpha_F^{s+\frac{1}{2}} \otimes \pi, \Phi, f, \tilde{f}\right)$  is not changed if we replace  $\Phi$  by

$$\Phi_1(g) = \int_K \Phi(gh^{-1}) \xi(h) dh.$$

**Lemma 13.2.2.** *If  $g_1$  and  $g_2$  belong to  $G_F$  then*

$$\iint \Phi_1 \left( g_1 \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} g_2 \right) dx dy = 0.$$

It will be enough to prove this when  $g_1$  is the identity. Let

$$\varphi(x, y) = \Phi_1 \left( \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \right).$$

If  $g_1$  is the identity then, after a change of variables, the integral becomes

$$|\det g_2|^{-1} \iint \varphi(x, y) dx dy$$

so that we can also assume  $g_2$  is the identity. Then the integral equals

$$\int_K \left\{ \iint \Phi \left( \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} k \right) dx dy \right\} \xi(k^{-1}) dk.$$

Changing variables as before we see that the inner integral does not depend on  $K$ . Since

$$\int_K \xi(k^{-1}) dk = 0$$

the lemma follows.

To establish the third assertion for the representation  $\pi$  all we need do is show that for any  $g$  and  $h$  in  $K$  the function

$$\frac{K_\Phi(g, h, s)}{L(s, \pi)}$$

is entire provided

$$\iint \Phi \left( g_1 \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} g_2 \right) dx dy = 0$$

for all  $g_1$  and  $g_2$  in  $G_F$ . As usual we need only consider the case that  $g = h = e$ . Since

$$\int \varphi_\Phi(x, 0) dx = 0$$

and

$$K_\Phi(e, e, s) = Z(\mu_1 \alpha_F^s, \mu_2 \alpha_F^s, \varphi_\Phi)$$

we need only refer to Corollary 3.7.

If  $F$  is the field of real numbers the proof is going to be basically the same but a little more complicated. We may assume that  $\mu_1 \mu_2^{-1}(x) = |x|^{2p+1-m} (\text{sgn } x)^m$ , where  $p$  is a non-negative integer and  $m$  is 0 or 1, and that  $\pi$  acts on  $\mathcal{B}_S(\mu_1, \mu_2)$ . The restriction of  $\pi$  to  $\text{SO}(2, \mathbf{R})$  contains only those representations  $\kappa_n$  for which  $n \equiv 1 - m \pmod{2}$  and  $|n| \geq 2p + 1 - m$ . Let  $\xi_n$  be the elementary idempotent corresponding to the representation  $\kappa_n$ . As before we may suppose that

$$(13.2.3) \quad \int_{\text{SO}(2, \mathbf{R})} \Phi(xk^{-1}) \xi_n(k) dk = 0$$

if  $\kappa_n$  does not occur in the restriction of  $\pi$  to  $\text{SO}(2, \mathbf{R})$ .

**Lemma 13.2.4.** *If  $\Phi$  satisfies (13.2.3), if  $g_1$  and  $g_2$  belong to  $G_F$ , and  $\varphi = \varphi_{g_1 \Phi g_2}$  then*

$$\int_{\mathbf{R}} x^i \frac{\partial^j}{\partial y^j} \varphi(x, 0) dx = 0$$

*if  $i \geq 0$ ,  $j \geq 0$  and  $i + j = 2p - m$ .*

We may assume that  $g_2 = e$ . If  $\varphi = \varphi_\Phi$  let

$$L(\Phi) = \int_{\mathbf{R}} x^i \frac{\partial^j}{\partial y^j} \varphi(x, 0) dx$$

and let

$$F(g) = L(g\Phi).$$

We have to show that, under the hypothesis of the lemma,  $F(g) = 0$  for all  $g$ . However  $F$  is defined for all  $\Phi$  in  $\mathcal{S}(A)$  and if  $\Phi$  is replaced by  $h\Phi$  the function  $F$  is replaced by  $F(gh)$ . Thus to establish the identity

$$F\left(\begin{pmatrix} a_i & z \\ 0 & a_2 \end{pmatrix} g\right) = \eta_1(a_1)\eta_2(a_2)F(g),$$

where  $\eta_1(a_1) = a_1^{-i}|a_1|^{-1}$  and  $\eta_2(a_2) = a_2^j|a_2|^{-1}$ , we need only establish it for  $g = e$ .

Let

$$h = \begin{pmatrix} a_1 & z \\ 0 & a_2 \end{pmatrix}.$$

Then

$$h\Phi\left(\begin{pmatrix} x & u \\ 0 & y \end{pmatrix}\right) = \Phi\left(\begin{pmatrix} a_1y & xz + a_2u \\ 0 & a_2y \end{pmatrix}\right).$$

If  $\varphi = \varphi_\Phi$  and  $\varphi_1 = \varphi_{h\Phi}$  then  $\varphi_1(x, y)$ , which is given by

$$\int \Phi\left(\begin{pmatrix} a_1x & xz + a_2u \\ 0 & a_2y \end{pmatrix}\right) du,$$

is equal to

$$|a_2|^{-1} \int \Phi\left(\begin{pmatrix} a_1x & u \\ 0 & a_2y \end{pmatrix}\right) du = |a_2|^{-1} \varphi(a_1x, a_2y).$$

Moreover  $F(h)$  is equal to

$$\int x^i \frac{\partial^j \varphi_1}{\partial y^j}(x, 0) dx$$

which equals

$$a_1^{-i}|a_1|^{-1}a_2^j|a_2|^{-1} \int x^i \frac{\partial^j \varphi}{\partial y^j}(x, 0) dx$$

as required.

Finally if

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

and  $\varphi = \varphi_{g\Phi}$  then  $F(g)$  is equal to

$$\int x^i \frac{\partial^j \varphi}{\partial y^j}(x, 0) dv$$

and

$$\varphi(x, y) = \int \Phi\left(\begin{pmatrix} \alpha x + \gamma u & \beta x + \delta u \\ y\gamma & y\delta \end{pmatrix}\right) du.$$



Since we can interchange the orders of differentiation and integration,

$$\frac{\partial^j \varphi}{\partial y^j}(x, 0) = \sum_{n=0}^j \lambda_n \gamma^n \delta^{j-n} \int \varphi_n(\alpha x + \gamma u, \beta x + \delta u) du$$

where

$$\varphi_n(x, y) = \frac{\partial^j \Phi}{\partial \gamma^n \partial \delta^{j-n}} \left( \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \right)$$

and the numbers  $\lambda_n$  are constants. Thus  $F(g)$  is a linear combination of the functions

$$\gamma^n \delta^{j-n} \iint x^i \varphi_n(\alpha x + \gamma u, \beta x + \delta u) dx du.$$

If  $\alpha \neq 0$  we may substitute  $x - \frac{\gamma u}{2}$  for  $x$  to obtain

$$\gamma^n \delta^{j-n} \iint \frac{(x - \gamma u)^i}{2} \varphi_n \left( \alpha x, \beta x + \frac{\Delta u}{2} \right) dx du$$

where  $\Delta = \det g$ . Substituting  $u - \frac{\alpha \beta}{\Delta} x$  for  $u$  we obtain

$$\gamma^n \delta^{j-n} \iint \left( x + \frac{\beta \gamma}{\Delta} x - \frac{\gamma u}{2} \right)^i \varphi_n \left( \alpha x, \frac{\Delta u}{2} \right) dx du.$$

After one more change of variables this becomes

$$\Delta^{-i} |\Delta|^{-1} \gamma^n \delta^{j-n} \iint (\delta x - \gamma u)^i \varphi_n(x, u) dx du.$$

In conclusion  $F(g)$  is a function of the form

$$F \left( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right) = \Delta^{-i} |\Delta|^{-1} P(\alpha, \beta, \gamma, \delta)$$

where  $P$  is a polynomial.

Thus the right translates of  $F$  by the elements of  $G_F$  span a finite-dimensional space. In particular it is  $O(2, \mathbf{R})$  finite and if  $\eta_1 = \mu'_1 \alpha_F^{1/2}$  while  $\eta_2 = \mu'_2 \alpha_F^{-1/2}$  it lies in a finite-dimensional invariant subspace of  $\mathcal{B}(\mu'_1, \mu'_2)$ . Thus it lies in  $\mathcal{B}_F(\mu'_1, \mu'_2)$ . Since  $\mu'_1 \mu'_2{}^{-1} = \mu_1^{-1} \mu_2$  no representation of  $SO(2, \mathbf{R})$  occurring in  $\pi(\mu'_1, \mu'_2)$  can occur in  $\pi = \sigma(\mu_1, \mu_2)$ . If  $F$  is not zero then for at least one such representation  $\kappa_n$

$$F_1(g) = \int_{SO(2, \mathbf{R})} f(gk^{-1}) \xi_n(k) dk$$

is not identically 0. But  $F_1$  is the result of replacing  $\Phi$  by

$$\Phi_1(x) = \int_{SO(2, \mathbf{R})} \Phi(xk^{-1}) \xi_n(k) dk$$

in the definition of  $F$ . In particular if  $\Phi$  satisfies the conditions of the lemma both  $\Phi_1$  and  $F_1$  are zero. Therefore  $F$  is also zero and the lemma is proved.

The third assertion can now be verified as in the non-archimedean case by appealing to Lemma 5.17. The fourth has still to be proved in general.

If  $F$  is the real field let  $\mathcal{S}_1(A)$  be the space of functions of the form

$$\Phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \exp(-\pi(a^2 + b^2 + c^2 + d^2))P(a, b, c, d)$$

where  $P$  is a polynomial. If  $F$  is the complex field  $\mathcal{S}_1(A)$  will be the space of functions of the form

$$\Phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \exp(-\pi(a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d}))P(a, \bar{a}, b, \bar{b}, c, \bar{c}, d, \bar{d})$$

where  $P$  is again a polynomial. If  $F$  is non-archimedean  $\mathcal{S}_1(A)$  will just be  $\mathcal{S}(A)$ . The space  $\mathcal{S}_1(F^2)$  is defined in a similar manner.

**Lemma 13.2.5.** *Suppose  $\varphi$  belongs to  $\mathcal{S}_1(F^2)$ . Then there is a  $\Phi$  in  $\mathcal{S}_1(A)$  such that*

$$K_\Phi(e, e, s) = Z(\mu_1\alpha_F^s, \mu_2\alpha_F^s, \varphi)$$

and  $f_1, \dots, f_n$  in  $\mathcal{B}(\mu_1, \mu_2)$  together with  $\tilde{f}_1, \dots, \tilde{f}_n$  in  $\mathcal{B}(\mu_1^{-1}, \mu_2^{-1})$  such that

$$\sum_{i=1}^n \int_{K \times K} K_\Phi(h, g, s) f_i(g) \tilde{f}_i(h) dg dh = K_\Phi(e, e, s).$$

Since there is a  $\varphi$  in  $\mathcal{S}_1(F^2)$  such that

$$Z(\mu_1\alpha_F^s, \mu_2\alpha_F^s, \varphi) = ae^{bs}L(s, \tau)$$

this lemma will imply the fourth assertion for the representation  $\tau$ .

Given  $\varphi$  the existence of  $\Phi$  such that  $\varphi = \varphi_\Phi$  and therefore

$$K_\Phi(e, e, s) = Z(\mu_1\alpha_F^s, \mu_2\alpha_F^s, \varphi_\Phi)$$

is a triviality and we worry only about the existence of  $f_1, \dots, f_n$  and  $\tilde{f}_1, \dots, \tilde{f}_n$ .

It is easily seen that if

$$\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix}$$

and

$$\begin{pmatrix} b_1 & y \\ 0 & b_2 \end{pmatrix}$$

belong to  $K$  then

$$K_\Phi\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix}h, \begin{pmatrix} b_1 & y \\ 0 & b_2 \end{pmatrix}g, s\right)$$

is equal to

$$\mu_1(a_1)\mu_2(a_2)\mu_1^{-1}(b_1)\mu_2^{-1}(b_2)K_\Phi(h, g).$$

Also

$$K_\Phi(hh_1, gg_1, s) = K_{g_1\Phi h_1^{-1}}(h, g, s).$$

Since  $\Phi$  belongs to  $\mathcal{S}_1(A)$  it is  $K$ -finite on the left and right. Thus there is a finite set  $S$  of irreducible representations of  $K$  such that if  $U_1$  is the space of functions  $F$  on  $K$  which satisfy

$$F\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix}h\right) = \mu_1(a_1)\mu_2(a_2)F(h)$$

for all

$$\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix}$$

in  $K$  and can be written as a linear combination of matrix elements of representations in  $S$  and  $U_2$  is the space of functions  $F'$  on  $K$  which satisfy

$$F' \left( \begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} h \right) = \mu_1^{-1}(a_1) \mu_2^{-1}(a_2) F'(h)$$

and can be written as a linear combination of matrix elements of representations in  $S$  then, for every  $s$ , the function

$$(h, g) \rightarrow K_\Phi(h, g, s)$$

belongs to the finite-dimensional space  $U$  spanned by functions of the form  $(h, g) \rightarrow F(h)F'(g)$  with  $F$  in  $U_1$  and  $F'$  in  $U_2$ .

Choose  $F_1, \dots, F_n$  and  $F'_1, \dots, F'_n$  so that for every function  $F$  in  $U$

$$F(e, e) = \sum_{i=1}^n \lambda_i \int_{K \times K} F(h, g) \overline{F}_i(h) \overline{F}'_i(g) dh dg.$$

Since  $\overline{F}_i$  is the restriction to  $K$  of an element of  $\mathcal{B}(\mu_1^{-1}, \mu_2^{-1})$  while  $\overline{F}'_i$  is the restriction to  $K$  of an element of  $\mathcal{B}(\mu_1, \mu_2)$  the lemma follows.

Unfortunately this lemma does not prove the fourth assertion in all cases. Moreover there is a supplementary condition to be verified.

**Lemma 13.2.6.** *Suppose  $F$  is non-archimedean and  $\pi$  is of the form  $\pi = \pi(\mu_1, \mu_2)$  with  $\mu_1$  and  $\mu_2$  unramified. Suppose  $\Phi$  is the characteristic function of  $M(2, O_F)$  in  $M(2, F)$ . If  $v$  and  $\tilde{v}$  are invariant under  $K = \text{GL}(2, O_F)$  and if*

$$\int_K d^\times g = 1$$

then

$$Z \left( \alpha_F^{s+\frac{1}{2}} \otimes \pi, \Phi, v, \tilde{v} \right) = L(s, \pi) \langle v, \tilde{v} \rangle.$$

Suppose  $f$  belongs to  $\mathcal{B}(\mu_1, \mu_2)$  and is identically 1 on  $K$  while  $\tilde{f}$  belongs to  $\mathcal{B}(\mu_1^{-1}, \mu_2^{-1})$  and is identically 1 on  $K$ . Then

$$\langle f, \tilde{f} \rangle = \int_K f(k) \tilde{f}(k) dk = 1$$

and if  $\tau = \rho(\mu_1, \mu_2)$  we are trying to show that

$$Z \left( \alpha_F^{s+\frac{1}{2}} \otimes \tau, \Phi, f, \tilde{f} \right) = L(s, \tau).$$

The left side is equal to

$$\int_{K \times K} K_\Phi(h, g, s) f(h) \tilde{f}(g) dh dg.$$

Since  $\Phi$  is invariant on both sides under  $K$  this is equal to

$$K_\Phi(e, e, s) = Z(\mu_1 \alpha_F^s, \mu_2 \alpha_F^s, \varphi)$$

if

$$\varphi(x, y) = \int \Phi \left( \begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \right) dz.$$

Since we have so normalized the Haar measure on  $G_F$  that

$$\int_{G_F} F(g) dg = \int_K \left\{ \int F \left( \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} k \right) d^\times a_1 d^\times a_2 dx \right\} dk$$

where  $dk$  is the normalized measure on  $K$ ,  $dx$  is the measure on  $F$  which assigns the measure 1 to  $O_F$ , and  $d^\times a$  is the measure on  $F^\times$  which assigns the measure 1 to  $U_F$ , the function  $\varphi$  is the characteristic function of  $O_F \times O_F$  and

$$Z(\mu_1 \alpha_F^s, \mu_2 \alpha_F^s, \varphi) = L(s, \mu_1) L(s, \mu_2)$$

as required.

This lemma incidentally proves the fourth assertion for the one-dimensional representation  $g \rightarrow \chi(\det g)$  if  $\chi$  is unramified. If  $\chi$  is ramified and  $\pi$  corresponds to  $\chi$  then  $\pi = \pi(\mu_1, \mu_2)$  if  $\mu_1(a) = \chi(a)|a|^{1/2}$  and  $\mu_2(a) = \chi(a)|a|^{-1/2}$ . Thus  $L(s, \pi) = 1$ . If  $\Phi$  is the restriction of the function  $\chi^{-1}$  to  $K$  then

$$Z(\pi, \Phi, v, \tilde{v}) = \langle v, \tilde{v} \rangle \int_K d^\times g$$

and the fourth assertion is verified in this case.

Take  $\mu_1$  and  $\mu_2$  of this form with  $\chi$  possibly unramified and suppose that  $\pi = \sigma(\mu_1, \mu_2)$ . Suppose first that  $\chi$  is unramified. Let  $\varphi_0$  be the characteristic function of  $O_F$  in  $F$  and let

$$\varphi_1(x) = \varphi_0(x) - |\varpi^{-1}| \varphi_0(\varpi^{-1}x).$$

It has  $O_F$  for support. Set

$$\Phi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \varphi_1(a) \varphi_0(b) \varphi_0(c) \varphi_0(d).$$

It has  $M(2, O_F)$  for support and depends only on the residues of  $a$ ,  $b$ ,  $c$ , and  $d$  modulo  $\mathfrak{p}_F$ . If

$$K^1 = \{ k \in K \mid k \equiv e \pmod{\mathfrak{p}} \}$$

then  $K_\Phi(h, g, s)$  depends only on the cosets of  $h$  and  $g$  modulo  $K^1$ . Also

$$K_\Phi \left( e, w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, s \right) = 0$$

if  $x$  is in  $O_F$ . To see this we observe first that if

$$\Phi_1(g) = \Phi \left( gw \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right)$$

then  $\varphi_{\Phi_1}(a_1, a_2)$  is equal to

$$\int_F \Phi \left( \begin{pmatrix} -y & a_1 - xy \\ -a_2 & -a_2 x \end{pmatrix} \right) dy$$

which equals

$$\varphi_0(a_2)\varphi_0(a_2x) \int_{O_F} \varphi_1(y)\varphi_0(a_1 - xy) dy.$$

Since  $x$  is in  $O_F$  the function  $\varphi_0(a_1 - xy)$  equals  $\varphi_0(a_1)$  for  $y$  in  $O_F$  and this expression is 0 because

$$\int_{O_F} \varphi_1(y) dy = 0.$$

We choose  $f$  in  $\mathcal{B}_S(\mu_1, \mu_2)$  so that  $f(gk) = f(g)$  if  $k$  belongs to  $K_1$ ,  $f(e) = 1$ , and

$$f(e) + \sum_{x \in O_F/\mathfrak{p}} f\left(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = 0.$$

We choose  $\tilde{f}$  in  $\mathcal{B}(\mu_1, \mu_2)$  so that  $\tilde{f}(gk) = \tilde{f}(g)$  if  $k$  belongs to  $K^1$ ,  $\tilde{f}(e) = 1$ , and

$$\tilde{f}\left(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = 0$$

if  $x$  belongs to  $O_F$ . Then

$$\int_{K \times K} K_\Phi(h, g, s) \tilde{f}(h) f(g) dh dg$$

is equal to

$$\int_K K_\Phi(e, g, s) f(g) dg = K_\Phi(e, e, s)$$

which equals

$$Z(\mu_1 \alpha_F^s, \mu_2 \alpha_F^s, \varphi_\Phi).$$

Moreover

$$\varphi_\Phi(a_1, a_2) = \varphi_1(a_1)\varphi_0(a_2)$$

so that, as we saw when proving Corollary 3.7,  $L(s, \pi)$  is a constant times  $Z(\mu_1 \alpha_F^s, \mu_2 \alpha_F^s, \varphi_\Phi)$ .

If  $\chi$  is ramified  $L(s, \pi) = 1$ . If  $\Phi$  has support in  $K$  then  $Z(\alpha_F^{s+1/2} \otimes \pi, \Phi, v, \tilde{v})$  is equal to

$$\int_K \Phi(k) \langle \pi(k)v, \tilde{v} \rangle dk$$

and we can certainly choose  $v, \tilde{v}$  and  $\Phi$  so that this is not 0.

We are not yet finished. We have yet to take care of the representations not covered by Lemma 13.2.5 when the field is archimedean. If  $F$  is the complex field we have only the finite-dimensional representations to consider. There is a pair of characters  $\mu_1$  and  $\mu_2$  such that  $\pi$  is realized on the subspace  $\mathcal{B}_f(\mu_1, \mu_2)$  of  $\mathcal{B}(\mu_1, \mu_2)$ . There will be positive integers  $p$  and  $q$  such that  $\mu_1 \mu_2^{-1}(z) = z^{-p} \bar{z}^{-q}$ . The representations  $\sigma = \rho_{|q-p|}$  of  $\mathrm{SU}(2, \mathbf{C})$  which is of degree  $|q - p| + 1$  is contained in the restriction of  $\pi$  to  $\mathrm{SU}(2, \mathbf{C})$ . In particular  $\mathcal{B}_f(\mu_1, \mu_2)$  contains all functions  $f$  in  $\mathcal{B}(\mu_1, \mu_2)$  whose restrictions to  $\mathrm{SU}(2, \mathbf{C})$  satisfy

$$f\left(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} k\right) = \mu_1(a_1) \mu_2(a_2) f(k)$$

and transform on the right according to  $\sigma$ .

We are going to use an argument like that used to prove Lemma 13.2.5. Suppose we can find a function  $\Phi$  in  $\mathcal{S}_1(A)$  such that

$$Z(\mu_1 \alpha_F^s, \mu_2 \alpha_F^s, \varphi_\Phi)$$

differs from  $L(s, \pi)$  by an exponential factor and such that  $\Phi$  transforms on the right under  $\mathrm{SU}(2, \mathbf{C})$  according to the representation  $\sigma$ . Then  $K_\Phi(h, g, s)$  will satisfy the same conditions as in Lemma 13.2.5. Moreover the functions  $F'$  in the space we called  $U_2$  can be supposed to transform on the right under  $\mathrm{SU}(2, \mathbf{C})$  according to  $\sigma$ , so that the functions  $\overline{F}'_i$  will correspond to functions  $f_i$  in  $\mathcal{B}_f(\mu_1, \mu_2)$ . Then

$$\int_{K \times K} K_\Phi(h, g, s) \tilde{f}_i(h) f_i(g) dh dg = Z\left(\alpha_F^{s+\frac{1}{2}} \otimes \tau, \Phi, f_i, \tilde{f}_i\right)$$

is equal to

$$Z\left(\alpha_F^{s+\frac{1}{2}} \otimes \pi, \Phi, v_i, \tilde{v}_i\right)$$

if  $v_i = f_i$  and  $\tilde{v}_i$  is the restriction of  $\tilde{f}_i$ , regarded as a linear functional, to  $\mathcal{B}_f(\mu_1, \mu_2)$ .

There are four possible ways of writing  $\mu_1$  and  $\mu_2$ .

- (i)  $\mu_1(z) = z^{m_1}(z\bar{z})^{s_1}$ ,  $\mu_2(z) = z^{m_2}(z\bar{z})^{s_2}$ ,  $m_1 - m_2 = q - p$ .
- (ii)  $\mu_1(z) = z^{m_1}(z\bar{z})^{s_1}$ ,  $\mu_2(z) = \bar{z}^{m_2}(z\bar{z})^{s_2}$ ,  $m_1 + m_2 = q - p$ .
- (iii)  $\mu_1(z) = \bar{z}^{m_1}(z\bar{z})^{s_1}$ ,  $\mu_2(z) = z^{m_2}(z\bar{z})^{s_2}$ ,  $-m_1 - m_2 = q - p$ .
- (iv)  $\mu_1(z) = \bar{z}^{m_1}(z\bar{z})^{s_1}$ ,  $\mu_2(z) = \bar{z}^{m_2}(z\bar{z})^{s_2}$ ,  $m_2 - m_1 = q - p$ .

In all four cases  $m_1$  and  $m_2$  are to be non-negative integers.  $\Phi$  is the product of  $\exp(-\pi(a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d}))$  and a polynomial. We write down the polynomial in all four cases and leave the verifications to the reader.

- (i.a)  $m_1 \geq m_2 : \bar{a}^{m_1-m_2}(\bar{a}\bar{d} - \bar{b}\bar{c})^{m_2}$ .
- (i.b)  $m_1 \leq m_2 : (\bar{a}\bar{d} - \bar{b}\bar{c})^{m_1} \bar{d}^{m_2-m_1}$ .
- (ii)  $\bar{a}^{m_1} d^{m_2}$ .
- (iii)  $a^{m_1} \bar{d}^{m_2}$ .
- (iv.a)  $m_1 \geq m_2 : a^{m_1-m_2}(ad - bc)^{m_2}$ .
- (iv.b)  $m_2 \geq m_1 : (ad - bc)^{m_1} d^{m_2-m_1}$ .

For the real field the situation is similar. Suppose first that  $\pi = \pi(\mu_1, \mu_2)$  is finite-dimensional. If  $\mu_1\mu_2(-1) = 1$  then  $\pi$  contains the trivial representation of  $\mathrm{SO}(2, \mathbf{R})$  and if  $\mu_1\mu_2(-1) = -1$  it contains the representation

$$\kappa_1 : \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \rightarrow e^{i\theta}$$

defined after Lemma 5.5. We list the four possibilities for  $\mu_1$  and  $\mu_2$  and the polynomial  $P$  by which  $\exp(-\pi(a^2 + b^2 + c^2 + d^2))$  is to be multiplied to obtain  $\Phi$ .

- (i)  $\mu_1(-1) = \mu_2(-1) = 1$ :  $P(a, b, c, d) = 1$ .
- (ii)  $\mu_1(-1) = \mu_2(-1) = -1$ :  $P(a, b, c, d) = ad - bc$ .
- (iii)  $\mu_1(-1) = 1, \mu_2(-1) = -1$ :  $P(a, b, c, d) = c - id$ .
- (iv)  $\mu_1(-1) = \mu_2(-1) = -1$ :  $P(a, b, c, d) = a - ib$ .

Only the special representations remain to be considered. We may suppose that  $\pi = \sigma(\mu_1, \mu_2)$  where  $\mu_1$  and  $\mu_2$  are of the form  $\mu_1(x) = |x|^{r+\frac{q}{2}}$  and  $\mu_2(x) = |x|^{r-\frac{q}{2}}(\mathrm{sgn} t)^m$

with  $q = 2p + 1 - m$  and with  $p$  a non-negative integer. Moreover  $m$  is 0 or 1. The function  $L(s, \pi)$  differs from

$$\Gamma\left(\frac{s+r+\frac{q}{2}}{2}\right)\Gamma\left(\frac{s+r+\frac{q}{2}+1}{2}\right)$$

by an exponential as does

$$Z(\mu_1\alpha_F^s, \mu_2\alpha_F^s, \varphi)$$

if

$$\varphi(a_1, a_2) = e^{-\pi(a_1^2+a_2^2)}a_2^{q+1}.$$

Since the representation of  $\kappa_{q+1}$  occurs in the restriction of  $\pi$  to  $\mathrm{SO}(2, \mathbf{R})$  we may take

$$\Phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \exp(-\pi(a^2 + b^2 + c^2 + d^2))(c + id)^{q+1}.$$

### §14. Automorphic forms and quaternion algebras

Let  $F$  be a global field and let  $M'$  be a quaternion algebra over  $F$ . The multiplicative group  $G'$  of  $M'$  may be regarded as an algebraic group over  $F$ . In the ninth paragraph we have introduced the group  $G'_\mathbf{A}$  and the Hecke algebra  $\mathcal{H}'$ . A continuous function  $\varphi$  on  $G'_F \backslash G'_\mathbf{A}$  is said to be an automorphic form if for every elementary idempotent  $\xi$  in  $\mathcal{H}'$  the space

$$\{ \rho(\xi f) \varphi \mid f \in \mathcal{H}' \}$$

is finite-dimensional.

If  $\varphi$  is an automorphic form it is  $Z'_\mathbf{A}$  finite on the left if  $Z'$  is the centre of  $G'$ . Let  $\mathcal{A}'$  be the space of automorphic forms on  $G'_\mathbf{A}$  and if  $\eta$  is a quasi-character of  $F^\times \backslash I$  let  $\mathcal{A}'(\eta)$  be the space of  $\varphi$  in  $\mathcal{A}'$  for which  $\varphi'(ag) = \eta(a)\varphi'(g)$  for all  $a$  in  $Z'_\mathbf{A}$  which, for convenience, we identify with  $I$ . The first assertion of the following lemma is easily proved by the methods of the eighth paragraph. The second is proved by the methods of the tenth. The proof is however a little simpler because  $G'_F Z'_\mathbf{A} \backslash G'_\mathbf{A}$  is compact. Since, at least in the case of number fields, the proof ultimately rests on general facts from the theory of automorphic forms nothing is gained by going into details.

**Lemma 14.1.**

- (i) *If an irreducible admissible representation  $\pi$  of  $\mathcal{H}'$  is a constituent of  $\mathcal{A}'$  then for some  $\eta$  it is a constituent of  $\mathcal{A}'(\eta)$ .*
- (ii) *The space  $\mathcal{A}'(\eta)$  is the direct sum of subspaces irreducible and invariant under  $\mathcal{H}'$ . The representation of  $\mathcal{H}'$  on each of these subspaces is admissible and no representation occurs more than a finite number of times in  $\mathcal{A}'(\eta)$ .*

Now we have to remind ourselves of some facts whose proofs are scattered throughout the previous paragraphs. Suppose  $\pi = \bigotimes_v \pi_v$  is an irreducible admissible representation of  $\mathcal{H}'$ . For each  $v$  the representation  $\pi_v$  of  $\mathcal{H}'_v$  is irreducible and admissible. Suppose  $\psi$  is a non-trivial additive character of  $F \backslash \mathbf{A}$  and  $\psi_v$  is its restriction to  $F_v$ . We have defined  $L(s, \pi_v)$ ,  $L(s, \tilde{\pi}_v)$ , and  $\epsilon(s, \pi_v, \psi_v)$ . If  $u_v$  is in the space of  $\pi_v$  and  $\tilde{u}_v$  in the space of  $\tilde{\pi}_v$  we have set

$$Z\left(\alpha_F^{s+\frac{1}{2}} \otimes \pi_v, \Phi, u_v, \tilde{u}_v\right)$$

equal to

$$\int_{G'_{F_v}} \Phi(g) \langle \pi_v(g) u_v, \tilde{u}_v \rangle |\nu(g)|^{s+\frac{1}{2}} d^\times g.$$

We know that

$$h_v \epsilon(s, \pi_v, \psi_v) \left\{ \frac{Z\left(\alpha_F^{s+\frac{1}{2}} \otimes \pi_v, \Phi, u_v, \tilde{u}_v\right)}{L(s, \pi_v)} \right\}$$

is entire and equals

$$\frac{Z\left(\alpha_F^{\frac{1}{2}-s} \otimes \tilde{\pi}_v, \Phi', u_v, \tilde{u}_v\right)}{L(1-s, \tilde{\pi}_v)}.$$



The factor  $h_v$  is 1 if  $G'_{F_v}$  is isomorphic to  $\mathrm{GL}(2, F_v)$  and is  $-1$  otherwise. The case that  $G'_{F_v}$  is isomorphic to  $\mathrm{GL}(2, F_v)$  was treated in the previous paragraph. The other cases were treated in the fourth and fifth paragraphs.

**Theorem 14.2.** *Suppose  $\pi$  is a constituent of the space of automorphic forms on  $G'_{\mathbf{A}}$ . The infinite products*

$$\prod_v L(s, \pi_v)$$

and

$$\prod_v L(s, \tilde{\pi}_v)$$

are absolutely convergent for  $\mathrm{Re} s$  sufficiently large. The functions  $L(s, \pi)$  and  $L(s, \tilde{\pi})$  defined by them can be analytically continued to the whole complex plane as meromorphic functions. If  $F$  is a number field they will have only a finite number of poles and will be bounded at infinity in vertical strips of finite width. If

$$\epsilon(s, \pi) = \prod_v \epsilon(s, \pi_v, \psi_v)$$

the functional equation

$$L(s, \pi) = \epsilon(s, \pi) L(1 - s, \tilde{\pi})$$

will be satisfied.

We may suppose that  $\pi$  acts on the subspace  $V$  of  $\mathcal{A}'(\eta)$ . Let  $\varphi$  be a non-zero function in  $V$ . For almost all  $v$  the algebra  $M'_v = M' \otimes_F F_v$  is split and  $G'_{F_v} = G'_v$  is isomorphic to  $\mathrm{GL}(2, F_v)$ . Moreover for almost all such  $v$ , say for all  $v$  not in  $S$ ,  $\varphi$  is an eigenfunction of the elements of  $\mathcal{H}'_v = \mathcal{H}'_{F_v}$  which are invariant on both sides under translations by the elements of  $K'_v$ . Thus if  $f$  is such an element and  $\varphi(g) \neq 0$  the corresponding eigenvalue  $\lambda_v(f)$  is

$$\lambda_v(f) = \varphi(g)^{-1} \int_{G'_v} \varphi(gh) f(h) dh.$$

To prove the absolute convergence of the infinite products we have only to refer to Lemma 3.11 as in the proof of Theorem 11.1.

The representation  $\tilde{\pi}$  contragredient to  $\pi$  can be defined. If  $\pi = \bigotimes \pi_v$  acts on  $V = \bigotimes_{u_v^0} V_v$  then  $\tilde{\pi} = \bigotimes \tilde{\pi}_v$  acts on  $\tilde{V} = \bigotimes_{u_v^0} \tilde{V}_v$  where  $\tilde{u}_v^0$  is, for almost all  $v$ , fixed by  $K'_v$  and satisfies  $\langle u_v^0, \tilde{u}_v^0 \rangle = 1$ . The pairing between  $V$  and  $\tilde{V}$  is defined by

$$\left\langle \bigotimes u_v, \bigotimes \tilde{u}_v \right\rangle = \prod_v \langle u_v, \tilde{u}_v \rangle.$$

Almost all terms in the product are equal to 1. If  $u$  is in  $V$  and  $\tilde{u}$  is in  $\tilde{V}$  the matrix element  $\langle \pi(g)u, \tilde{u} \rangle$  can also be introduced. If  $f$  is in  $\mathcal{H}'$

$$\langle \pi(f)u, \tilde{u} \rangle = \int_{G'_{\mathbf{A}}} f(g) \langle \pi(g)u, \tilde{u} \rangle d^\times g.$$

If  $F(g)$  is a linear combination of such matrix elements and  $\Phi$  belongs to the Schwartz space on  $A'_A$  we set<sup>1</sup>

$$Z\left(\alpha_F^{s+\frac{1}{2}}, \Phi, F\right) = \int_{G'_A} \Phi(g) F(g) |\nu(g)|^{s+\frac{1}{2}} d^\times g.$$

The function  $\tilde{F}(g) = F(g^{-1})$  is a linear combination of matrix coefficients for the representation  $\tilde{\pi}$ . We set

$$Z\left(\alpha_F^{s+\frac{1}{2}}, \Phi, \tilde{F}\right) = \int_{G'_A} \Phi(g) \tilde{F}(g) |\nu(g)|^{s+\frac{1}{2}} d^\times g.$$

Before stating the next lemma we observe that if  $\chi$  is a quasi-character of  $F^\times \backslash I$  the one-dimensional representation  $g \rightarrow \chi(\nu(g))$  is certainly a constituent of  $\mathcal{A}'$ .

**Lemma 14.2.1.** *If  $\pi$  is a constituent of  $\mathcal{A}'$ , the integrals defining the functions*

$$Z\left(\alpha_F^{s-\frac{1}{2}}, \Phi, F\right) \quad \text{and} \quad Z\left(\alpha_F^{s-\frac{1}{2}}, \Phi, \tilde{F}\right)$$

*are absolutely convergent for  $\text{Re } s$  large enough. The two functions can be analytically continued to the whole complex plane as meromorphic functions with only a finite number of poles. If  $\pi$  is not of the form  $g \rightarrow \chi(\nu(g))$  they are entire. If  $F$  is a number field they are bounded at infinity in vertical strips of finite width. In all cases they satisfy the functional equation*

$$Z\left(\alpha_F^{s+\frac{1}{2}}, \Phi, F\right) = Z\left(\alpha_F^{\frac{3}{2}-s}, \Phi', \tilde{F}\right)$$

*if  $\Phi'$  is the Fourier transform of  $\Phi$ .*

There is no harm in assuming that  $F$  is of the form

$$F(g) = \prod_v \langle \pi(g_v) u_v, \tilde{u}_v \rangle = \prod_v F_v(g_v)$$

and that  $\Phi$  is of the form

$$\Phi(x) = \prod_v \Phi_v(x_v)$$

where, for almost all  $v$ ,  $\Phi_v$  is the characteristic function of  $M(2, O_v)$ . Recall that for almost all  $v$  we have fixed an isomorphism  $\theta_v$  of  $M'_v$  with  $M(2, F_v)$ .

We know that each of the integrals

$$\int_{G'_v} \Phi_v(g_v) F_v(g_v) |\nu(g_v)|^{s+\frac{1}{2}} d^\times g_v$$

converges absolutely for  $\text{Re } s$  sufficiently large. Let  $S$  be a finite set of primes which contains all archimedean primes such that outside of  $S$  the vector  $u_v$  is  $u_v^0$ , the vector  $\tilde{u}_v$  is  $\tilde{u}_v^0$ ,  $\Phi_v$  is the characteristic function of  $M(2, O_v)$ , and  $\pi_v = \pi_v(\mu_v, \nu_v)$  where  $\mu_v$  and  $\nu_v$  are unramified. Let  $\pi'_v = \pi_v(|\mu_v|, |\nu_v|)$ . If  $v$  is not in  $S$  the integral

$$\int_{K'_v} \Phi(g_v) F_v(g_v) |\nu(g_v)|^{s+\frac{1}{2}} d^\times g_v = 1$$

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<sup>1</sup>Unfortunately the symbol  $F$  plays two quite different roles on this page!

and if  $\sigma = \operatorname{Re} s$

$$\int_{G'_v} |\Phi_v(g_v)| |F_v(g_v)| |\nu(g_v)|^{\sigma+\frac{1}{2}} d^\times g_v$$

is, as we see if we regard  $\pi_v$  as acting on  $\mathcal{B}(\mu_b, \nu_v)$ , at most

$$\int_{G'_v} \Phi_v(g_v) \langle \pi'_v(g_v) f_v, \tilde{f}_v \rangle |\nu(g_v)|^{\sigma+\frac{1}{2}} d^\times g_v$$

if  $f_v$  and  $\tilde{f}_v$  are the unique  $K'_v$ -invariant elements in  $\mathcal{B}(|\mu_v|, |\nu_v|)$  and  $\mathcal{B}(|\mu_v|^{-1}, |\nu_v|^{-1})$  which take the value 1 at the identity. We suppose that the total measure of  $K'_v$  is 1 so that  $\langle f_v, \tilde{f}_v \rangle = 1$ . According to Lemma 13.2.6 the integral is equal to  $L(\sigma, \pi'_v)$ . Since

$$\prod_{v \in S} L(\sigma, \pi'_v)$$

is absolutely convergent for  $\sigma$  sufficiently large the integral defining  $Z\left(\alpha_F^{s+\frac{1}{2}}, \Phi, F\right)$  is also and is equal to

$$\prod_v Z\left(\alpha_{F_v}^{s+\frac{1}{2}} \otimes \pi_v, \Phi_v, u_v, \tilde{u}_v\right)$$

and to

$$L(s, \pi) \prod_v \Xi(s, \Phi_v, u_v, \tilde{u}_v).$$

Notice that  $\Xi(s, \Phi_v, u_v, \tilde{u}_v)$  is identically 1 for almost all  $v$ .  $Z\left(\alpha_F^{s+\frac{1}{2}}, \Phi, \tilde{F}\right)$  may be treated in a similar fashion. If we take  $\pi$  to be the trivial representation we see that

$$\int_{G'_A} \Phi(g) |\nu(g)|^{s+\frac{1}{2}} d^\times g$$

is absolutely convergent for  $\operatorname{Re} s$  sufficiently large.

It will be enough to prove the remaining assertions of the lemma when  $\eta$  is a character. We may also assume that if  $\eta$  is of the form  $\eta(a) = |a|^r$  then  $r = 0$ . We have identified  $V$  with a subspace of  $\mathcal{A}'(\eta)$ . We may take  $\tilde{V}$  to be  $\{\tilde{\varphi} \mid \varphi \in V\}$ . To see this observe that this space is invariant under  $\mathcal{H}'$  and that

$$\langle \varphi_1, \varphi_2 \rangle = \int_{G'_F Z'_A \backslash G'_A} \varphi_1(g) \tilde{\varphi}_2(g) dg$$

is a non-degenerate bilinear form. Here  $\varphi_1$  belongs to  $V$  and  $\tilde{\varphi}_2$  belongs to  $\tilde{V}$ . The remaining assertions need only be verified for functions of the form

$$F(g) = \int_{G'_F Z'_A \backslash G'_A} \varphi(hg) \tilde{\varphi}(h) dh$$

with  $\varphi$  in  $V$  and  $\tilde{\varphi}$  in  $\tilde{V}$ .

For such an  $F$  the function  $Z\left(\alpha_F^{s+\frac{1}{2}}, \Phi, F\right)$  is equal to

$$\int \Phi(g) \left\{ \int \varphi(hg) \tilde{\varphi}(h) dh \right\} |\nu(g)|^{s+\frac{1}{2}} d^\times g.$$

Since  $\varphi$  and  $\tilde{\varphi}$  are bounded this double integral converges absolutely for  $\operatorname{Re} s$  sufficiently large. We first change variables by substituting  $h^{-1}g$  for  $g$ . The integration with respect to  $g$  can then be carried out in three steps. We first sum over  $G'_F$ , then we integrate over  $Z'_F \backslash Z'_A$  which we identify with  $F^\times \backslash I$ , and finally we integrate over  $G'_F Z'_A \backslash G'_A$ . Thus if  $K_\Phi(h_1, h_2, s)$  is

$$|\nu(h_1^{-1})|^{s+\frac{1}{2}} |\nu(h_2)|^{s+\frac{1}{2}} \int_{F^\times \backslash I} \sum_{G'_F} \Phi(h_1^{-1} \xi a h_2) \eta(a) |a|^{2s+1} d^\times a$$

the function  $Z\left(\alpha_F^{s+\frac{1}{2}}, \Phi, F\right)$  is equal to

$$\iint \varphi(h_2) \tilde{\varphi}(h_1) K_\Phi(h_1, h_2, s) dh_1 dh_2.$$

The integrations with respect to  $h_1$  and  $h_2$  are taken over  $G'_F Z'_A \backslash G'_A$ . A similar result is of course valid for  $Z\left(\alpha_F^{s+\frac{1}{2}}, \Phi, \tilde{F}\right)$ . If  $\tilde{K}_\Phi(h_1, h_2, s)$  is

$$|\nu(h_1^{-1})|^{s+\frac{1}{2}} |\nu(h_2)|^{s+\frac{1}{2}} \int_{F^\times \backslash I} \sum_{G'_F} \Phi(h_1^{-1} \xi a h_2) \eta^{-1}(a) |a|^{2s+1} d^\times a$$

then  $Z\left(\alpha_F^{s+\frac{1}{2}}, \Phi, \tilde{F}\right)$  is equal to

$$\iint \varphi(h_2) \tilde{\varphi}(h_2) \tilde{K}_\Phi(h_1, h_2, s) dh_1 dh_2.$$

We first study

$$\theta(s, \Phi) = \int_{F^\times \backslash I} \sum_{\xi \neq 0} \Phi(\xi a) \eta(a) |a|_F^{2s+1} d^\times a$$

and

$$\tilde{\theta}(s, \Phi) = \int_{F^\times \backslash I} \sum_{\xi \neq 0} \Phi(\xi a) \eta^{-1}(a) |a|_F^{2s+1} d^\times a.$$

The sums are taken over  $G'_F$ , the set of non-zero elements of  $M'$ . Choose two non-negative continuous functions  $F_0$  and  $F_1$  on the positive real numbers so that  $F_0(t) + F_1(t) = 1$ ,  $F_1(t) = F_0(t^{-1})$ , and so that  $F_0$  vanishes near zero while  $F_1$  vanishes near infinity. If

$$\theta_i(s, \Phi) = \int_{F^\times \backslash I} \sum_{\xi \neq 0} \Phi(\xi a) \eta(a) |a|^{2s+1} F_i(|a|) d^\times a$$

we have

$$\theta(s, \Phi) = \theta_0(s, \Phi) + \theta_1(s, \Phi).$$

In the same way we may write

$$\tilde{\theta}(s, \Phi) = \tilde{\theta}_0(s, \Phi) + \tilde{\theta}_1(s, \Phi),$$

where  $\theta_0(s, \Phi)$  and  $\tilde{\theta}_0(s, \Phi)$  are entire functions of  $s$  which are bounded in vertical strips.

Applying the Poisson formula we obtain

$$\Phi(0) + \sum_{\xi \neq 0} \Phi(\xi a) = |a|_F^{-4} \left\{ \Phi'(0) + \sum_{\xi \neq 0} \Phi'(\xi a^{-1}) \right\}.$$

Thus, for  $\operatorname{Re} s$  sufficiently large,  $\theta_1(s, \Phi)$  is equal to the sum of

$$\int_{F^\times \setminus I} \sum_{\xi \neq 0} \Phi'(\xi a^{-1}) \eta(a) |a|^{2s-3} F_1(|a|) d^\times a,$$

which, after the substitution of  $a^{-1}$  for  $a$ , is seen to equal  $\tilde{\theta}_0(1-s, \Phi')$ , and

$$\int_{F^\times \setminus I} \{ \Phi'(0) |a|^{-4} - \Phi(0) \} \eta(a) |a|^{2s+1} F_1(|a|) d^\times a.$$

Thus if

$$\lambda(s) = \int_{F^\times \setminus I} |a|^s \eta(a) F_1(|a|) d^\times a$$

the function  $\theta(s, \Phi)$  is equal to

$$\theta_0(s, \Phi) + \tilde{\theta}_0(1-s, \Phi') + \Phi'(0) \lambda(2s-3) - \Phi(0) \lambda(2s+1).$$

A similar result is valid for  $\tilde{\theta}(s, \Phi)$ . The function

$$\theta_0(s, \Phi) + \tilde{\theta}_0(1-s, \Phi')$$

is entire and bounded in vertical strips and does not change when  $s$  and  $\Phi$  are replaced by  $1-s$  and  $\Phi'$ .

If  $\eta$  is not of the form  $\eta(a) = |a|^r$  the function  $\lambda(s)$  vanishes identically. If  $\eta$  is trivial and  $I_0$  is the group of ideles of norm 1

$$\lambda(s) = \int_{F^\times \setminus I} |a|^{2s+1} F_j(|a|) d^\times a.$$

It is shown in [10] that this function is meromorphic in the whole plane and satisfies  $\lambda(s) + \lambda(-s) = 0$ . If  $F$  is a number field, its only pole is at  $s = 0$  and is simple. Moreover it is bounded at infinity in vertical strips of finite width. If  $F$  is a function field its poles are simple and lie at the zeros of  $1 - q^{-s}$ . Here  $q$  is the number of elements in the field of constants.

Thus  $\theta(s, \Phi)$  is meromorphic in the whole plane and is equal to  $\tilde{\theta}(1-s, \Phi')$ . If  $h\Phi g$  is the function  $x \rightarrow \Phi(gxh)$  then

$$K_\Phi(h_1, h_2, s) = |\nu(h_1^{-1})|^{s+\frac{1}{2}} |\nu(h_2)|^{s+\frac{1}{2}} \theta(s, h_2 \Phi h_1^{-1})$$

while

$$\tilde{K}_\Phi(h_1, h_2, s) = |\nu(h_1^{-1})|^{s+\frac{1}{2}} |\nu(h_2)|^{s+\frac{1}{2}} \tilde{\theta}(s, h_2 \Phi h_1^{-1}).$$

Since the Fourier transform of  $h_2 \Phi h_1^{-1}$  is

$$|\nu(h_2)|^{-2} |\nu(h_1)|^2 h_1 \Phi' h_2^{-1}$$

we have

$$K_\Phi(h_1, h_2, s) = \tilde{K}_{\Phi'}(h_2, h_1, s).$$

The functional equation of the lemma follows. So do the other assertions except the fact that the functions  $Z\left(\alpha_F^{s+\frac{1}{2}}, \Phi, F\right)$  and  $Z\left(\alpha_F^{s+\frac{1}{2}}, \Phi, \tilde{F}\right)$  are entire when  $\eta$  is trivial and  $\pi$  is not of the form  $g \rightarrow \chi(\nu(g))$ . In this case the functions  $\varphi$  and  $\tilde{\varphi}$  are orthogonal to the constant functions and the kernels  $K_\Phi(h_1, h_2, s)$  and  $\tilde{K}_{\Phi'}(h_1, h_2, s)$  may be replaced by

$$K'_{\Phi'}(h_1, h_2, s) = \tilde{K}_{\Phi'}(h_1, h_2, s) + \Phi(0)\lambda(2s+1) - \Phi'(0)\lambda(2s-3)$$

and

$$\tilde{K}'_{\Phi'}(h_1, h_2, s) = \tilde{K}_{\Phi'}(h_1, h_2, s) + \Phi'(0)\lambda(2s+1) - \Phi(0)\lambda(2s-3).$$

The functional equation of the kernels is not destroyed but the poles disappear.

The theorem follows easily from the lemma. In fact suppose that the finite set of places  $S$  is so chosen that for  $v$  not in  $S$

$$\Xi(s, \Phi_v^0, u_v^0, \tilde{u}_v^0) = 1$$

if  $\Phi_v^0$  is the characteristic function of  $M(2, O_v)$ . If  $v$  is in  $S$  choose  $\Phi_v^i, u_v^i, \tilde{u}_v^i, 1 \leq i \leq n_v$ , so that

$$\sum_{i=1}^{n_v} \Xi(s, \Phi_v^i, u_v^i, \tilde{u}_v^i) = e^{b_v s}$$

where  $b_v$  is real. If  $\alpha$  is a function from  $S$  to the integers and, for each  $v$  in  $S, 1 \leq \alpha(v) \leq n_v$ , set

$$\Phi_\alpha(g) = \left\{ \prod_{v \in S} \Phi_v^{\alpha(v)}(g_v) \right\} \left\{ \prod_{v \notin S} \Phi_v^0(g_v) \right\}$$

and set

$$F_\alpha(g) = \left\{ \prod_{v \in S} \langle \pi_v(g_v) u_v^{\alpha(v)}, \tilde{u}_v^{\alpha(v)} \rangle \right\} \left\{ \prod_{v \notin S} \langle \pi_v(g_v) u_v^0, \tilde{u}_v^0 \rangle \right\}.$$

Then

$$\sum_{\alpha} Z\left(\alpha_F^{s+\frac{1}{2}}, \Phi_\alpha, F_\alpha\right) = c^{bs} L(s, \pi)$$

where  $b$  is real. The required analytic properties of  $L(s, \pi)$  follow immediately.

To prove the functional equation choose for each  $v$  the function  $\Phi_v$  and the vectors  $u_v$  and  $\tilde{u}_v$  so that

$$\Xi(s, \Phi_v, u_v, \tilde{u}_v)$$

is not identically 0. We may suppose that, for almost all  $v$ ,  $\Phi_v = \Phi_v^0, u_v = u_v^0$ , and  $\tilde{u}_v = \tilde{u}_v^0$ . Let

$$\Phi(g) = \prod_v \Phi_v(g_v)$$

and let

$$F(g) = \prod_v \langle \pi_v(g_v) u_v, \tilde{u}_v \rangle.$$

Then

$$Z\left(\alpha_F^{s+\frac{1}{2}}, \Phi, F\right) = L(s, \pi) \prod_v \Xi(s, \Phi_v, u_v, \tilde{u}_v)$$

and

$$Z\left(\alpha_F^{\frac{3}{2}-s}, \Phi', \tilde{F}\right) = L(1-s, \tilde{\pi}) \prod_v \tilde{\Xi}(1-s, \Phi'_v, u_v, \tilde{u}_v).$$

Since

$$\tilde{\Xi}(1-s, \Phi'_v, u_v, \tilde{u}_v) = h_v \epsilon(s, \pi_v, \psi_v) \Xi(s, \Phi_v, u_v, \tilde{u}_v)$$

the functional equation of the lemma implies that

$$L(s, \pi) = \left\{ \prod_v h_v \right\} \epsilon(s, \pi) L(1-s, \tilde{\pi}).$$

Since, by a well-known theorem, the algebra  $M'$  is split at an even number of places the product  $\prod_v h_v$  equals 1.

**Corollary 14.3.** *If  $\pi$  is a constituent of  $\mathcal{A}'$  which is not of the form  $g \rightarrow \chi(\nu(g))$  then for any quasi-character  $\omega$  of  $F^\times$  the functions  $L(s, \omega \otimes \pi)$  and  $L(s, \omega^{-1} \otimes \tilde{\pi})$  are entire and bounded in vertical strips of finite width. Moreover they satisfy the functional equation*

$$L(s, \omega \otimes \pi) = \epsilon(s, \omega \otimes \pi) L(1-s, \omega^{-1} \otimes \tilde{\pi}).$$

We have only to observe that if  $\pi$  is a constituent of  $\mathcal{A}'$  then  $\omega \otimes \pi$  is also.

Now we change the notation slightly and let  $\pi' = \bigotimes \pi'_v$  be an irreducible admissible representation of  $\mathcal{H}'$ . We want to associate to it a representation  $\pi = \bigotimes \pi_v$  of  $\mathcal{H}$ , the Hecke algebra of  $\mathrm{GL}(2, \mathbf{A})$ . If  $M'_v$  is split then  $\pi_v$  is just the representation corresponding to  $\pi'_v$  by means of the isomorphism  $\theta_v$  of  $G_{F_v}$  and  $G'_{F_v}$ . If  $M'_v$  is not split  $\pi_v$  is the representation  $\pi_v(\pi'_v)$  introduced in the fourth and fifth paragraphs. In both cases  $\pi_v$  is defined unambiguously by the following relations

$$\begin{aligned} L(s, \omega_v \otimes \pi_v) &= L(s, \omega_v \otimes \pi'_v) \\ L(s, \omega_v \otimes \tilde{\pi}_v) &= L(s, \omega_v \otimes \tilde{\pi}'_v) \\ \epsilon(s, \omega_v \otimes \pi_v, \psi_v) &= \epsilon(s, \omega_v \otimes \pi'_v, \psi_v) \end{aligned}$$

which holds for all quasi-characters  $\omega_v$  of  $F_v^\times$ .

Applying the previous corollary and Theorem 11.3 we obtain the following theorem.

**Theorem 14.4.** *If  $\pi'$  is a constituent of  $\mathcal{A}'$  and  $\pi'_v$  is infinite-dimensional at any place where  $M'$  splits then  $\pi$  is a constituent of  $\mathcal{A}_0$ .*

Some comments on the assumptions are necessary. If  $\pi'$  is a constituent of  $\mathcal{A}'$  we can always find a quasi-character  $\omega$  of  $F^\times \setminus I$  such that  $\omega \otimes \pi'$  is unitary. If  $\pi' = \bigotimes \pi'_v$  the same is true of the representations  $\pi'_v$ . In particular if  $M'$  splits at  $v$  the representation  $\pi'_v$  will not be finite-dimensional unless it is one-dimensional. Various density theorems probably prevent this from happening unless  $\pi'$  is of the form  $g \rightarrow \chi(\nu(g))$ . If  $\pi'$  is of this form then all but a finite number of the representations  $\pi_v$  are one-dimensional. But if  $M'$  does not split at  $v$  the representation  $\pi_v$  is infinite-dimensional. Thus  $\pi$  cannot act on a subspace of  $\mathcal{A}$ . However it can still be a constituent of  $\mathcal{A}$ . This is in fact extremely likely. Since the proof we have in mind involves the theory of Eisenstein series we prefer to leave the question unsettled for now.

### §15. Some orthogonality relations

It is of some importance to characterize the range of the map  $\pi' \rightarrow \pi$  from the constituents of  $\mathcal{A}'$  to those of  $\mathcal{A}$  discussed in the last chapter. In this paragraph we take up the corresponding local question. Suppose  $F$  is a local field and  $M'$  is the quaternion algebra over  $F$ . Let  $G'_F$  be the group of invertible elements of  $M'$ . We know how to associate to every irreducible admissible representation  $\pi'$  of  $\mathcal{H}'_F$  an irreducible admissible representation  $\pi = \pi(\pi')$  of  $\mathcal{H}_F$  the Hecke algebra of  $\mathrm{GL}(2, F)$ .

**Theorem 15.1.** *Suppose  $F$  is non-archimedean. Then the map  $\pi' \rightarrow \pi$  is injective and its image is the collection of special representations together with the absolutely cuspidal representations.*

The proof requires some preparation. We need not distinguish between representations of  $G'_F$  and  $\mathcal{H}'_F$  or between representations of  $G_F$  and  $\mathcal{H}_F$ . An irreducible admissible representation  $\pi$  of  $G_F$  is said to be square-integrable if for any two vectors  $u_1$  and  $u_2$  in the space of  $\pi$  and any two vectors  $\tilde{u}_1$  and  $\tilde{u}_2$  in the space of  $\tilde{\pi}$  the integral

$$\int_{Z_F \backslash G_F} \langle \pi(g)u_1, \tilde{u}_1 \rangle \langle u_2, \tilde{\pi}(g)\tilde{u}_2 \rangle dg$$

is absolutely convergent. Since  $\tilde{\pi}$  is equivalent to  $\eta^{-1} \otimes \pi$  if

$$\pi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \eta(a)I$$

this is equivalent to demanding that

$$\int_{Z_F \backslash G_F} \left| \langle \pi(g)u_1, \tilde{u}_1 \rangle \right|^2 |\eta^{-1}(\det g)| dg$$

be finite for every  $u_1$  and  $\tilde{u}_1$ .

If  $\pi$  is square-integrable and  $\omega$  is a quasi-character of  $F^\times$  then  $\omega \otimes \pi$  is square integrable. We can always choose  $\omega$  so that  $\omega^2 \eta$  is a character. If  $\eta$  is a character choose  $u_0$  different from 0 in the space  $V$  of  $\pi$ . Then

$$(u_1, u_2) = \int_{Z_F \backslash G_F} \langle \pi(g)u_1, u_0 \rangle \overline{\langle \pi(g)u_2, u_0 \rangle} dg$$

is a positive-definite form on the space  $V$  of  $\pi$  so that  $\pi$  is unitary and square-integrable in the usual sense.

The Schur orthogonality relations when written in the form

$$\int_{Z_F \backslash G_F} \langle \pi(g)u_1, \tilde{u}_1 \rangle \langle u_2, \tilde{\pi}(g)\tilde{u}_2 \rangle dg = \frac{1}{d(\pi)} \langle u_2, \tilde{u}_1 \rangle \langle u_1, \tilde{u}_2 \rangle$$

are valid not only for representations which are square-integrable in the usual sense but also for representations which are square-integrable in our sense. The formal degree  $d(\pi)$  depends on the choice of Haar measure. Notice that  $d(\omega \otimes \pi) = d(\pi)$ .

The absolutely cuspidal representations are certainly square-integrable because their matrix elements are compactly supported modulo  $Z_F$ .

**Lemma 15.2.** *The special representations are square-integrable.*



Suppose  $\sigma = \sigma(\alpha_F^{1/2}, \alpha_F^{-1/2})$ . Since

$$\chi \otimes \sigma = \sigma(\chi \alpha_F^{1/2}, \chi \alpha_F^{-1/2})$$

it is enough to show that  $\sigma$  is square-integrable. If  $\varphi$  belongs to  $\mathcal{B}_s(\alpha_F^{1/2}, \alpha_F^{-1/2})$  and  $\tilde{\varphi}$  belongs to  $\mathcal{B}(\alpha_F^{-1/2}, \alpha_F^{1/2})$  then

$$f(g) = \langle \varphi, \rho(g^{-1}) \tilde{\varphi} \rangle$$

is the most general matrix coefficient of  $\sigma$ . Here  $\mathcal{B}(\alpha_F^{-1/2}, \alpha_F^{1/2})$  is the space of locally constant functions on  $N_F A_F \backslash G_F$  and  $\mathcal{B}_s(\alpha_F^{1/2}, \alpha_F^{-1/2})$  is the space of locally constant functions  $\varphi$  on  $G_F$  that satisfy

$$\varphi\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right) = \left|\frac{a_1}{a_2}\right| \varphi(g)$$

and

$$\int \varphi\left(w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) dx = 0.$$

Since

$$G_F = \bigcup_{n \geq 0} Z_F K \begin{pmatrix} \varpi^{-n} & 0 \\ 0 & 1 \end{pmatrix} K$$

we can choose the Haar measure on  $Z_F \backslash G_F$  so that

$$\int_{Z_F \backslash G_F} |f(g)|^2 dg$$

is equal to

$$\sum_{n \geq 0} c(n) \int \left| f\left(k_1 \begin{pmatrix} \varpi^{-n} & 0 \\ 0 & 1 \end{pmatrix} k_2\right) \right|^2 dk_1 dk_2$$

where  $c(0) = 1$  and

$$c(n) = q^n \left(1 + \frac{1}{q}\right)$$

if  $n > 0$ . Here  $q = |\varpi|^{-1}$ . Since  $f$  is  $K$ -finite on both sides and its translates are also matrix coefficients we need only show that

$$\sum_{n=0}^{\infty} \left| f\left(\begin{pmatrix} \varpi^{-n} & 0 \\ 0 & 1 \end{pmatrix}\right) \right|^2 q^n$$

is finite. It will be more than enough to show that

$$\Phi(a) = f\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = O(|a|^{-1})$$

as  $a \rightarrow \infty$ .

We recall that

$$\Phi(a) = \int_F \varphi\left(w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \tilde{\varphi}\left(w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) dx.$$

The function

$$\varphi_1(x) = \varphi\left(w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)$$

is integrable and the function

$$\varphi_2(x) = \tilde{\varphi}\left(w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)$$

is bounded and locally constant. Moreover

$$\Phi(a) = \int_F \varphi_1(x) \varphi_2(a^{-1}x) dx.$$

Suppose  $\varphi_2(x) = \varphi_2(0)$  for  $|x| \leq M$ . If  $|a| \geq 1$

$$\Phi(a) = \varphi_2(0) \int_{\{x \mid |x| \leq |a|M\}} \varphi_1(x) dx + \int_{\{x \mid |x| > |a|M\}} \varphi_1(x) \varphi_2(a^{-1}x) dx.$$

Since

$$\int_F \varphi_1(x) dx = 0$$

$\Phi(a)$  is equal to

$$\int_{\{x \mid |x| > |a|M\}} (\varphi_2(a^{-1}x) - \varphi_2(0)) \varphi_1(x) dx.$$

The function  $\varphi_2$  is bounded so we need only check that

$$\int_{\{x \mid |x| > |a|\}} |\varphi_1(x)| dx = O(|a|^{-1})$$

as  $|a| \rightarrow \infty$ . The absolute value of the function  $\varphi$  is certainly bounded by some multiple of the function  $\varphi'$  in  $\mathcal{B}(\alpha_F^{1/2}, \alpha_F^{-1/2})$  defined by

$$\varphi'\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} k\right) = \left|\frac{a_1}{a_2}\right|$$

if  $k$  is in  $\mathrm{GL}(2, O_F)$ . Since

$$w\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -x \end{pmatrix} = \begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} k$$

with  $y$  in  $F$  and  $k$  in  $\mathrm{GL}(2, O_F)$ , if  $|x| > 1$

$$\int_{\{x \mid |x| > |\varpi|^{-n}\}} |\varphi_1(x)| dx = O\left(\sum_{k=n+1}^{\infty} |\varpi|^k\right) = O(|\varpi|^n).$$

Since we need to compare orthogonality relations on the two groups  $G_F = \mathrm{GL}(2, F)$  and  $G'_F$  we have to normalize their Haar measure simultaneously. There are two ways of doing this. We first describe the simplest. Choose a non-trivial additive character  $\psi$  of  $F$ . Then  $\psi_M(x) = \psi(\mathrm{tr} x)$  and  $\psi_{M'}(x) = \psi(\tau(x))$  are non-trivial additive characters of  $M = M(2, F)$  and  $M'$ . Let  $dx$  and  $dx'$  be the Haar measures on  $M$  and  $M'$  self-dual with respect to  $\psi_M$  and  $\psi_{M'}$ . Then

$$d^\times x = |x|_M^{-1} dx = |\det x|_F^{-2} dx$$

and

$$d^\times x' = |x'|_{M'}^{-1} dx' = |\nu(x')|_F^{-2} dx'$$

are Haar measure on  $G_F$  and  $G'_F$ .

The second method takes longer to describe but is more generally applicable and for this reason well worth mentioning. Suppose  $G$  and  $G'$  are two linear groups defined over  $F$  and suppose there is an isomorphism  $\varphi$  of  $G'$  with  $G$  defined over the finite Galois extension  $K$ . Suppose the differential form  $\omega$  on  $G$  is defined over  $F$ . In general the form  $\omega' = \varphi_*\omega$  on  $G'$  is not defined over  $F$ . Suppose however that  $\omega$  is left and right invariant and under an arbitrary isomorphism it is either fixed or changes sign. Suppose moreover that for every  $\sigma$  in  $\mathfrak{G}(K/F)$  the automorphism  $\sigma(\varphi)\varphi^{-1}$  of  $G$  is inner. Then

$$\sigma(\omega') = \sigma(\varphi)_*\sigma\omega = \sigma(\varphi)_*\omega = \varphi_*(\sigma(\varphi)\varphi^{-1})_*\omega = \varphi_*\omega = \omega'$$

and  $\omega'$  is also defined over  $F$ . If  $\xi$  is another such isomorphism of  $G'$  with  $G$  then

$$\xi_*(\omega) = \varphi_*(\xi\varphi^{-1})_*\omega = \pm\varphi_*\omega = \pm\omega'$$

and the measures associated to  $\varphi_*\omega$  and  $\xi_*\omega$  are the same. Thus a Tamagawa measure on  $G_F$  determines one on  $G'_F$ .

We apply this method to the simple case under consideration. If

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a typical element of  $M$  then

$$\mu = da \wedge db \wedge dc \wedge dd$$

is a differential form invariant under translations and the associated measure is self-dual with respect to  $\psi_M$ . If  $\omega = (\det x)^{-2}\mu$  then  $\omega$  is an invariant form on  $G$  and the associated measure is  $d^\times x$ .

If  $K$  is any separable quadratic extension of  $F$  we may imbed  $K$  in both  $M$  and  $M'$ . Let  $\sigma$  be the non-trivial element of  $\mathfrak{G}(K/F)$ . There is a  $u$  in  $M$  and a  $u'$  in  $M'$  such that  $M = K + Ku$  and  $M' = K + Ku'$  while  $uxu^{-1} = x^\sigma$  and  $u'xu'^{-1} = x^\sigma$  for all  $x$  in  $K$ . Moreover  $u^2$  is a square in  $F^\times$  and  $u'^2 = \gamma$  is an element of  $F^\times$  which is not the norm of any element of  $K$ . We may suppose that  $u^2 = 1$ . If we let  $K$  act to the right the algebra  $L = K \otimes_F K$  is an algebra over  $K$ . The automorphism  $\sigma$  acts on  $L$  through its action on the first factor. There is an isomorphism  $L \rightarrow K \oplus K$  which transforms  $\sigma$  into the involution  $(x, y) \rightarrow (y, x)$ . In particular every element of  $K \otimes 1$  is of the form  $\delta\delta^\sigma$  with  $\delta$  in  $L$ . Choose  $\delta$  so that  $\gamma = \delta\delta^\sigma$ . If

$$M'_K = M' \otimes_F K = L \otimes Lu'$$

and

$$M_K = M \otimes_F K = L \otimes Lu,$$

let  $\varphi$  be the linear map from  $M'_K$  to  $M_K$  which sends  $x + yu'$  to  $x + y\delta u$ . The map  $\varphi$  is easily seen to be an isomorphism of  $M'_K$  and  $M_K$  as algebras over  $K$ . Moreover  $\sigma(\varphi)\varphi^{-1}$  takes  $x + yu$  to

$$x + y\delta^\sigma\delta^{-1}u = \delta^{-1}(x + yu)\delta$$

and is therefore inner. Thus  $\varphi$  determines an isomorphism of  $G'$  the multiplicative group of  $M'$  with  $G$  the multiplicative group of  $M$ . The isomorphism  $\varphi$  is defined over  $K$  and  $\sigma(\varphi)\varphi^{-1}$  is inner. Let  $|\omega'|$  be the Haar measure on  $G'_F$  associated to the Haar measure  $|\omega| = d^\times x$  on  $G_F$ . We want to show that  $|\omega'|$  is just  $d^\times x'$ .

Let  $\theta$  be an invariant form on  $K$ . The obvious projections of  $M = K \oplus Ku$  on  $K$  define differential forms  $\theta_1$  and  $\theta_2$  on  $M$ . Let  $\theta_1 \wedge \theta_2 = c\mu$ . In the same way the projections of  $M' = K \oplus Ku'$  on  $K$  define differential forms  $\theta'_1$  and  $\theta'_2$  on  $M'$ . If we extend the scalars from  $F$  to  $K$  we can consider the map  $x \rightarrow x\delta$  of  $L$  into itself. We can also regard  $\theta$  as a form on  $L$  and then its inverse image is  $N(\delta)\theta = \gamma\theta$ . Thus

$$\varphi_*(\theta_1 \wedge \theta_2) = \gamma\theta'_1 \wedge \theta'_2.$$

Thus if  $\mu' = \varphi_*(\mu)$

$$c\mu' = \gamma\theta'_1 \wedge \theta'_2.$$

Suppose  $c_1|\theta|$  is self-dual with respect to the character  $\psi_K(x) = \psi(\tau(x))$  on  $K$ . Then

$$\int \left\{ \int \Phi(a, b) \psi_K(ax + by^\sigma) |\theta(a)| |\theta(b)| \right\} |\theta(x)| |\theta(y)| = c_1^{-4} \Phi(0, 0)$$

and

$$|\gamma|_F^2 \int \left\{ \int \Phi(a, b) \psi_K(ax + by^\sigma \gamma) |\theta(a)| |\theta(b)| \right\} |\theta(x)| |\theta(y)| = c_1^{-4} \Phi(0, 0).$$

If  $x + yu$  belongs to  $M$  with  $x$  and  $y$  in  $K$  then, since  $\tau(u) = 0$ ,

$$\tau(x + yu) = \tau(x) = \text{Tr}_{K/F}(x).$$

In the same way

$$\tau(x + yu') = \text{Tr}_{K/F}(x).$$

Thus

$$\begin{aligned} \psi_M((x + yu)(a + bu)) &= \psi_K(xa + yb^\sigma) \\ \psi_{M'}((x + yu')(a + bu')) &= \psi_K(xa + yb^\sigma \gamma). \end{aligned}$$

Thus  $c_1^2|\theta_1 \wedge \theta_2|$  is self-dual with respect to  $\psi_M$  and  $c_1^2|\gamma|_F|\theta'_1 \wedge \theta'_2|$  is self-dual with respect to  $\psi_{M'}$ . Since  $c_1^2 = |c|_F$  the measure  $|\mu'|$  is self-dual with respect to  $\psi_{M'}$ . Finally  $\omega' = \nu(x')^{-2} dx'$  so that  $|\omega'|$  is just  $d^\times x'$ . Thus the two normalizations lead to the same result.

If  $b$  is in  $M$  or  $M'$  the eigenvalues of  $b$  are the roots  $\alpha_1$  and  $\alpha_2$  of the equation

$$X^2 - \tau(b)X + \nu(b) = 0.$$

If  $b$  is in  $G_F$  or  $G'_F$  it is said to be regular if  $\alpha_1$  and  $\alpha_2$  are distinct; otherwise it is singular. We set

$$\delta(b) = \left| \frac{(\alpha_1 - \alpha_2)^2}{\alpha_1 \alpha_2} \right|_F.$$

The set of singular elements is of measure 0. If  $b$  is regular the subalgebra of  $M$  or  $M'$  generated by  $b$  is a separable quadratic extension  $E$  of  $F$  and the multiplicative group of  $E$  is a Cartan subgroup of  $G_F$  or  $G'_F$ . To obtain a set of representatives for the conjugacy classes of Cartan subgroups of  $G_F$  or  $G'_F$  we choose once and for all a set  $S'$  of representatives for the classes of separable quadratic extensions of  $F$ . We also choose for each  $E$  in  $S'$  an imbedding of  $E$  in  $M$  and in  $M'$ . The multiplicative group of  $E$  may be regarded as a Cartan subgroup  $B_F$  of either  $G_F$  or  $G'_F$ . The symbol  $S'$  will also stand for the collection of Cartan subgroups obtained in this way. It is a complete set of representatives for the conjugacy classes of Cartan subgroups of  $G'_F$ . If  $S$  is the result of adjoining to  $S'$  the group  $A_F$  of diagonal matrices then  $S$  is a complete set of representatives for the conjugacy classes

of Cartan subgroups of  $G_F$ . If  $B_F$  is in  $S'$  we choose the Tamagawa measure  $\mu_B$  on  $B_F$  as in the seventh paragraph. The analogue for  $G'_F$  of the formula (7.2.2) is

$$\int_{Z'_F \backslash G'_F} f(g) \omega'_0(g) = \sum_{S'} \frac{1}{2} \int_{Z_F \backslash B_F} \delta(b) \left\{ \int_{B_F \backslash G'_F} f(g^{-1}bg) \omega'_B(g) \right\} \mu_B^0(b).$$

Let  $\widehat{B}_F$  be the set of regular elements in  $B_F$  and let

$$C = \bigcup_{S'} Z_F \backslash \widehat{B}_F.$$

We may regard  $C$  as the discrete union of the spaces  $Z_F \backslash \widehat{B}_F$ . We introduce on  $C$  the measure  $\mu(c)$  defined by

$$\int_C f(c) \mu(c) = \frac{1}{2} \sum_{S'} \frac{1}{\text{measure}(Z_F \backslash B_F)} \int_{Z_F \backslash \widehat{B}_F} f(b) \delta(b) \mu_B^0(b).$$

**Lemma 15.3.** *Let  $\eta$  be a quasi-character of  $F^\times$  and let  $\Omega'(\eta)$  be the set of equivalence classes of irreducible representations  $\pi$  of  $G'_F$  such that  $\pi(a) = \eta(a)$  for  $a$  in  $Z'_F$ , which we identify with  $F^\times$ . If  $\pi_1$  and  $\pi_2$  belong to  $\Omega'(\eta)$  and*

$$f(g) = \chi_{\pi_1}(g) \chi_{\pi_2}(g)$$

where  $\chi_\pi(g) = \text{Tr } \pi(g)$  then

$$\int_C f(c) \mu(c) = 0$$

if  $\pi_1$  and  $\pi_2$  are not equivalent and

$$\int_C f(c) \mu(c) = 1$$

if they are.

Since  $Z'_F \backslash G'_F$  is compact we may apply the Schur orthogonality relations for characters to see that

$$\frac{1}{\text{measure } Z'_F \backslash G'_F} \int_{Z'_F \backslash G'_F} f(g) \omega'_0(g)$$

is 0 if  $\pi_1$  and  $\pi_2$  are not equivalent and is 1 if they are. According to the integration formula remarked above this expression is equal to

$$\frac{1}{\text{measure } Z'_F \backslash G'_F} \sum_{S'} \frac{1}{2} \int_{Z_F \backslash B_F} f(b) \delta(b) (\text{measure } B_F \backslash G'_F) \mu_b^0(b).$$

Since

$$\text{measure } Z'_F \backslash G'_F = (\text{measure } Z_F \backslash B_F) (\text{measure } B_F \backslash G'_F)$$

the lemma follows. Observe that  $Z_F$  and  $Z'_F$  tend to be confounded.

There is form of this lemma which is valid for  $G_F$ .

**Lemma 15.4.** *Let  $\eta$  be a quasi-character of  $F^\times$ . Let  $\Omega_0(\eta)$  be the set of equivalence classes of irreducible admissible representations  $\pi$  of  $G_F$  which are either special or absolutely cuspidal and satisfy  $\pi(a) = \eta(a)$  for all  $a$  in  $Z_F$ . Suppose  $\pi_1$  and  $\pi_2$  belong to  $\Omega_0(\eta)$ . Let  $f = f_{\pi_1, \pi_2}$  be the function*

$$f(b) = \chi_{\pi_1}(b) \chi_{\pi_2}(b)$$

on  $C$ . Then  $f$  is integrable and

$$\int_C f(c)\mu(c)$$

is 1 if  $\pi_1$  and  $\pi_2$  are equivalent and 0 otherwise.

It is enough to prove the lemma when  $\eta$  is a character. Then  $\chi_{\tilde{\pi}}$  is the complex conjugate of  $\chi_{\pi}$  and  $f_{\pi,\pi}$  is positive. If the functions  $f_{\pi,\pi}$  are integrable then by the Schwarz inequality all the functions  $f_{\pi_1,\pi_2}$  are integrable.

Let  $\Omega(\eta)$  be the set of irreducible admissible representations  $\pi$  of  $G_F$  such that  $\pi(a) = \eta(a)$  for  $a$  in  $Z_F$ . If  $\varphi$  is a locally constant function on  $G_F$  such that

$$\varphi(ag) = \eta^{-1}(a)\varphi(g)$$

for  $a$  in  $Z_F$  and such that the projection of the support of  $\varphi$  on  $Z_F \backslash G_F$  is compact then we define  $\pi(\varphi)$ , if  $\pi$  is in  $\Omega(\eta)$ , by

$$\pi(\varphi) = \int_{Z_F \backslash G_F} \varphi(g)\pi(g)\omega^0(g).$$

It is easily seen that  $\pi(\varphi)$  is an operator of finite rank and that the trace of  $\pi(\varphi)$  is given by the convergent integral

$$\int_{Z_F \backslash G_F} \varphi(g)\chi_{\pi}(g)\omega^0(g).$$

In fact this follows from the observation that there is a  $\varphi_1$  in  $\mathcal{H}_F$  such that

$$\varphi_1(g) = \int_{Z_F} \varphi_1(ag)\eta(a)\mu_Z(a)$$

and the results of the seventh paragraph.

Suppose  $\pi_1$  is absolutely cuspidal and unitary and acts on the space  $V_1$ . Suppose also that  $\pi_1(a) = \eta(a)$  for  $a$  in  $Z_F$ . Choose a unit vector  $u_1$  and  $V_1$  and set

$$\varphi(g) = d(\pi_1)(u_1, \pi_1(g)u_1).$$

Since  $\pi_1$  is integrable it follows from the Schur orthogonality relations that  $\pi_2(\varphi) = 0$  if  $\pi_2$  in  $\Omega(\eta)$  is not equivalent to  $\pi_1$  but that  $\pi_2(\varphi)$  is the orthogonal projection on  $\mathbf{C}u_1$  if  $\pi_2 = \pi_1$ . In the first case  $\text{Tr } \pi_2(\varphi) = 0$  and in the second  $\text{Tr } \pi_2(\varphi) = 1$ .

On the other hand

$$\text{Tr } \pi_2(\varphi) = \int_{Z_F \backslash G_F} \chi_{\pi_2}(g)\varphi(g)\omega^0(g).$$

We apply formula (7.2.2) to the right side to obtain

$$\sum_S \frac{1}{2} \int_{Z_F \backslash B_F} \chi_{\pi_2}(b)\delta(b) \left\{ \int_{B_F \backslash G_F} \varphi(g^{-1}bg)\omega_B(g) \right\} \mu_B^0(g).$$

If  $B_F$  belongs to  $S'$  the inner integral is equal to

$$\frac{1}{\text{measure } Z_F \backslash B_F} d(\pi_1) \int_{Z_F \backslash G_F} (u_1, \pi_1(g^{-1}bg)u_1)\omega_B(g)$$

which by Proposition 7.5 is equal to

$$\frac{1}{\text{measure } Z_F \backslash B_F} \chi_{\tilde{\pi}_1}(b).$$

If  $B_F$  is  $A_F$  the group of diagonal matrices the inner integral is, apart from a constant relating Haar measures, the product of  $d(\pi_1)$  and the integral over  $\mathrm{GL}(2, O_F)$  of

$$\int_F \left( \pi_1 \left( \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} b \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \pi_1(k) u_1, \pi_1(k) u_1 \right) dx.$$

If

$$b = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$$

this is

$$\left| 1 - \frac{\alpha_2}{\alpha_1} \right|^{-1} \int_F \left( \pi_1(b) \pi_1 \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \pi_1(k) u_1, \pi_1(k) u_1 \right) dx$$

which we know is 0. Collecting these facts together we see that  $f = f_{\pi_2, \pi_1}$  is integrable on  $C$  if  $\pi_1$  is absolutely cuspidal and that its integral has the required value.

To complete the proof all we need do is show that if  $\pi = \sigma(\chi_{\alpha_F}^{1/2}, \chi_F^{-1/2})$  is a special representation then  $f = f_{\pi, \pi}$  is integrable on  $C$  and

$$\int_C f(c) \mu(c) = 1.$$

If  $\pi'$  is the one-dimensional representation  $g \rightarrow \chi(\nu(g))$  of  $G'_F$  then  $\pi = \pi(\pi')$ . To prove the existence of  $\chi_\pi$  we had to show in effect that if  $B_F$  was in  $S'$  and  $b$  was in  $\widehat{B}_F$  then

$$\chi_\pi(b) = -\chi_{\pi'}(b).$$

Thus  $f_{\pi, \pi} = f_{\pi', \pi'}$  and the assertion in this case follows from the previous lemma.

The relation just used does not seem to be accidental.

**Proposition 15.5.** *Suppose  $\pi'$  is an irreducible admissible representation of  $G'_F$  and  $\pi = \pi(\pi')$  the corresponding representation of  $G_F$ . If  $B_F$  is in  $S'$  and  $b$  is in  $\widehat{B}_F$*

$$\chi_{\pi'}(b) = -\chi_\pi(b).$$

We may suppose that  $\pi'$  is not one-dimensional and that  $\pi$  is absolutely cuspidal. We may also suppose that they are both unitary. We take  $\pi$  in Kirillov form with respect to some additive character  $\psi$ . If  $\varphi$  is in  $\mathcal{S}(F^\times)$  the function

$$\varphi' = \pi \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \varphi$$

is also.

Since the measures  $\mu$  and  $\mu'$  are self-dual with respect to the characters  $\psi_M$  and  $\psi_{M'}$  Lemma 13.1.1 and Proposition 4.5 show us that for any  $\lambda$  in  $F^\times$

$$(15.5.1) \quad \int_{G_F} \varphi(\det g) (\pi(g^{-1})u, u) |\det g| \psi_M(\lambda g) \omega(g)$$

is equal to

$$\varphi'(\lambda^2) \eta^{-1}(\lambda) |\lambda|^{-2}$$

and that

$$(15.5.2) \quad \int_{G'_F} \varphi(\det g) (\pi'(g^{-1})u', u') |\det g| \psi_{M'}(\lambda g) \omega'(g)$$

is equal to

$$-\varphi'(\lambda^2) \eta^{-1}(\lambda) |\lambda|^{-2}.$$

Here  $u$  is a unit vector in the space of  $\pi$  and  $u'$  a unit vector in the space  $\pi'$ . In any case (15.5.1) is just the negative of (15.5.2).

If we use formula (7.2.1) to express the integral (15.5.1) as a sum over  $S$  we obtain

$$\frac{1}{2} \sum_{S'} \frac{1}{\text{measure } Z_F \backslash B_F} \int_{B_F} \varphi(\det b) |\det b| \frac{\chi_\pi(b^{-1})}{d(\pi)} \delta(b) \psi_M(\lambda b) \mu_B(b).$$

The contribution from  $A_F$  vanishes as in the previous lemma. The other integrals have been simplified by means of Proposition 7.5. There is of course an obvious analogue for the group  $G'_F$  of the formula (7.2.1). If we apply it we see that (15.5.2) is equal to

$$\frac{1}{2} \sum_{S'} \frac{1}{\text{measure } Z_F \backslash B_F} \int_{B_F} \varphi(\det b) |\nu(b)| \frac{\chi_{\pi'}(b^{-1})}{d(\pi')} \delta(b) \psi_{M'}(\lambda b) \mu_B(b)$$

if  $\nu(b)$  is the reduced norm. Of course on  $B_F$  the functions  $\nu(b)$  and  $\det b$  are the same. Choose  $B_F^0$  in  $S'$  and  $b_0$  in  $\widehat{B}_F^0$ . We shall show that

$$\frac{\chi_{\pi'}(b_0^{-1})}{d(\pi')} = \frac{-\chi_\pi(b_0^{-1})}{d(\pi)}.$$

The orthogonality relations of the previous two lemmas will show that  $d(\pi) = d(\pi')$  and we will conclude that

$$\chi_{\pi'}(b_0^{-1}) = -\chi_\pi(b_0^{-1}).$$

The norm and the trace of  $b_0$  are the same whether it is regarded as an element of  $M$  or of  $M'$ . In fact if  $\widehat{B}_F^0$  is the multiplicative group of  $E$  in  $S'$  the norm and the trace are in both cases the norm and the trace of  $b_0$  as an element of  $E$ . Since  $b_0$  and its conjugate in  $E$  are conjugate in  $G_F$  and  $G'_F$  we can choose an open set  $U$  in  $E^\times$  containing both  $b_0$  and its conjugate so that

$$|\nu(b)| \chi_{\pi'}(b^{-1}) \delta(b) = |\nu(b_0)| \chi_{\pi'}(b_0^{-1}) \delta(b_0)$$

if  $b$  is in  $U$ . Lemma 7.4.2 shows that  $\chi_\pi$  is locally constant in  $\widehat{B}_F^0$ . Thus we can also suppose that

$$|\det b| \chi_\pi(b^{-1}) \delta(b) = |\det b_0| \chi_\pi(b_0^{-1}) \delta(b_0)$$

if  $b$  is in  $U$ . Suppose  $\alpha_0$  and  $\beta_0$  are the trace and norm of  $b_0$ . We can choose a positive integer  $m$  so that if  $\alpha - \alpha_0$  and  $\beta - \beta_0$  belong to  $\mathfrak{p}_F^m$  the roots of

$$X^2 - \alpha X + \beta$$

belong to  $E$  and in fact lie in  $U$ .

Let  $\xi(\lambda)$  be the expression (15.5.1) regarded as a function of  $\lambda$ . Keeping in mind the fact that

$$\psi_M(\lambda b) = \psi_{M'}(\lambda b) = \psi(\lambda \text{tr } b),$$



we compute

$$(15.5.3) \quad \frac{1}{\text{measure } \mathfrak{p}_F^{-m-n}} \int_{\mathfrak{p}_F^{-m-n}} \xi(\lambda) \psi(-\lambda \alpha_0) d\lambda$$

where  $\mathfrak{p}_F^{-n}$  is the largest ideal on which  $\psi$  is trivial. Since

$$\frac{1}{\text{measure } \mathfrak{p}_F^{-m-n}} \int_{\mathfrak{p}_F^{-m-n}} \psi(\lambda(\text{tr } b - \alpha_0)) d\lambda$$

is 0, unless  $\text{tr } b - \alpha_0$  belongs to  $\mathfrak{p}_F^m$  when it is 1, the integral (15.5.3) is equal to

$$\frac{1}{2} \sum_{S'} \frac{1}{\text{measure } Z_F \setminus B_F} \int_{V(B_F)} \varphi(\det b) |\det b| \frac{\chi_\pi(b^{-1})}{d(\pi)} \delta(b) \mu_B(b)$$

if

$$V(B_F) = \{ b \in B_F \mid \text{tr } b - \alpha_0 \in \mathfrak{p}_F^m \}.$$

If we take  $\varphi$  to be the characteristic function of

$$\{ \beta \in F \mid \beta - \beta_0 \in \mathfrak{p}_F^m \}$$

the summation disappears and we are left with

$$\frac{1}{2} \cdot \frac{1}{\text{measure } Z_F \setminus B_F} |\det b_0| \frac{\chi_\pi(b_0^{-1})}{d(\pi)} \delta(b_0) \int_{V(B_F^0)} \varphi(\det b) \mu_B(b).$$

If we replace  $\xi(\lambda)$  by the expression (15.5.2) the final result will be

$$\frac{1}{2} \cdot \frac{1}{\text{measure } Z_F \setminus B_F} |\nu(b_0)| \frac{\chi_{\pi'}(b_0^{-1})}{d(\pi)} \delta(b_0) \int_{V(B_F^0)} \varphi(\det b) \mu_B(b).$$

Since these differ only in sign the proposition follows.

We are now in a position to prove Theorem 15.1. The orthogonality relations and the previous lemma show that the map  $\pi' \rightarrow \pi$  is injective because the map takes  $\Omega'(\eta)$  into  $\Omega_0(\eta)$ . It is enough to verify that  $V$  is surjective when  $\eta$  is unitary. Let  $\mathcal{L}^2(\eta)$  be the space of all measurable functions  $f$  on

$$\bigcup_{S'} \widehat{B}_F$$

such that  $f(ab) = \eta(a)f(b)$  if  $a$  is in  $Z_F$  and

$$\int_C |f(c)|^2 \mu(c)$$

is finite. By the Peter-Weyl theorem the set of functions  $\chi_{\pi'}, \pi' \in \Omega'(\eta)$ , form an orthonormal basis of  $\mathcal{L}^2(\eta)$ . The family  $\chi_\pi, \pi \in \Omega_0(\eta)$ , is an orthonormal family in  $\mathcal{L}^2(\eta)$ . By the previous proposition the image of  $\Omega'(\eta)$  in  $\Omega_0(\eta)$  is actually an orthonormal basis and must therefore be the whole family.

We observe that it would be surprising if the relation  $d(\pi) = d(\pi')$  were not also true when  $\pi'$  is one-dimensional. The facts just discussed are also valid when  $F$  is the field of real numbers. They follow immediately from the classification and the remarks at the end of the seventh paragraph.

We conclude this paragraph with some miscellaneous facts which will be used elsewhere. The field  $F$  is again a non-archimedean field. Let  $K = \text{GL}(2, O_F)$  and let  $K_0$  be the set of all matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in  $K$  for which  $c \equiv 0 \pmod{\mathfrak{p}_F}$ . Suppose  $\pi$  is an irreducible admissible representation of  $G_F$  in the space  $V$ . We are interested in the existence of a non-zero vector  $v$  in  $V$  such that

$$\pi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) v = \omega_1(a) \omega_2(d) v$$

for all matrices in  $K_0$  while

$$\pi \left( \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix} \right) v = \omega_0 v$$

$\omega_0$  is a constant and  $\omega_1$  and  $\omega_2$  two characters of  $U_F$ . The coefficient  $\varpi$  is a generator of  $\mathfrak{p}_F$ . Since

$$\begin{pmatrix} 0 & \varpi^{-1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \varpi & 0 \end{pmatrix} = \begin{pmatrix} d & \varpi^{-1}c \\ \varpi b & a \end{pmatrix}$$

such a vector can exist only if  $\omega_1 = \omega_2 = \omega$ .

**Lemma 15.6.** *Suppose  $\omega$  and  $\omega_0$  are given. Let  $\pi$  be  $\rho(\mu_1, \mu_2)$  which may not be irreducible. There is a non-zero vector  $\varphi$  in  $\mathcal{B}(\mu_1, \mu_2)$  satisfying the above conditions if and only if the restrictions of  $\mu_1$  and  $\mu_2$  to  $U_F$ , the group of units of  $O_F$ , are equal to  $\omega$  and*

$$\omega_0^2 = \mu_1(-\varpi) \mu_2(-\varpi)$$

*Moreover  $\varphi$  if it exists is unique apart from a scalar factor.*

It is easily seen that  $K$  is the disjoint union of  $K_0$  and

$$K_0 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} K_0 = K_0 w K_0$$

Let  $\varphi_1$  be the function which is 0 on  $K_0 w K_0$  and on  $K_0$  is given by

$$\varphi_1 \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \omega(ad).$$

Let  $\varphi_2$  be the function which is 0 on  $K_0$  and takes the value  $\omega(a'd'ad)$  at

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If  $\varphi$  in  $\mathcal{B}(\mu_1, \mu_2)$  satisfies

$$\pi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \varphi = \omega(ad) \varphi$$

for all matrices in  $K_0$  then the restrictions of  $\varphi$  to  $K$  must be a linear combination of  $\varphi_1$  and  $\varphi_2$ . This already implies that  $\omega$  is the restriction of  $\mu_1$  and  $\mu_2$  to  $U_F$ . Suppose  $\varphi = a\varphi_1 + b\varphi_2$ . Since

$$\pi \left( \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix} \right) \varphi_1 = |\varpi|^{1/2} \mu_1(\varpi) \mu_2(-1) \varphi_2$$

and

$$\pi \left( \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix} \right) \varphi_2 = |\varpi|^{-1/2} \mu_2(-\varpi) \varphi_1$$

while  $\mu_1(-1) = \mu_2(-1) = \omega(-1)$ , we have

$$\omega_0 b = |\varpi|^{1/2} \mu_1(-\varpi) a$$

and

$$\omega_0 a = |\varpi|^{-1/2} \mu_2(-\varpi) b$$

Apart from scalar factors there is at most one solution of this equation. There is one non-trivial solution if and only if  $\omega_0^2 = \mu_1(-\varpi) \mu_2(-\varpi)$ .

**Lemma 15.7.** *Suppose  $\pi = \sigma(\mu_1, \mu_2)$  is the special representation corresponding to the quasi-characters  $\mu_1 = \chi \alpha_F^{-1/2}$  and  $\mu_2 = \chi \alpha_F^{1/2}$ . There is a non-zero vector  $v$  in the space of  $\pi$  such that*

$$\pi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) v = \omega(ad) v$$

for all matrices in  $K_0$  while

$$\pi \left( \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix} \right) v = \omega_0 v$$

if and only if  $\omega$  is the restriction of  $\chi$  to  $U_F$  and  $\omega_0 = -\chi(-\varpi)$ . If  $v$  exists it is unique apart from a scalar.

We first let  $\pi$  act on  $\mathcal{B}_s(\mu_2, \mu_1)$  a subspace of  $\mathcal{B}(\mu_2, \mu_1)$ . The condition on  $\omega$  follows from the previous lemma which also shows that  $\omega_0$  must be  $\pm\chi(-\varpi)$ . If we take the plus sign we see that  $v$  must correspond to the function whose restriction to  $K$  is constant. Since this function does not lie in  $\mathcal{B}_s(\mu_2, \mu_1)$  only the minus sign is possible. To see the existence we let  $\pi$  act on

$$\mathcal{B}_s(\mu_1, \mu_2) = \mathcal{B}(\mu_1, \mu_2) / \mathcal{B}_f(\mu_1, \mu_2)$$

In  $\mathcal{B}(\mu_1, \mu_2)$  there are two functions satisfying the conditions of the lemma. One with  $\omega_0 = -\chi(-\varpi)$  and one with  $\omega_0 = \chi(-\varpi)$ . One of the two, and we know which, must have a non-zero projection on  $\mathcal{B}_s(\mu_1, \mu_2)$ .

The above lemmas together with the next one sometimes allow us to decide whether or not a given representation is special.

**Lemma 15.8.** *If the absolutely cuspidal representation  $\pi$  acts on  $V$  there is no non-zero vector  $v$  in  $V$  such that*

$$\pi \left( \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix} \right) v = \omega_0 v$$

and

$$\pi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) v = \omega(ad) v$$

for all matrices in  $K_0$ .

We may suppose that  $\pi$  is the Kirillov form with respect to an additive character  $\psi$  such that  $O_F$  is the largest ideal on which  $\psi$  is trivial. Then  $v$  is a function  $\varphi$  in  $\mathcal{S}(F^\times)$ . If  $a$  is in  $U_F$  and  $b$  is in  $F^\times$  we must have  $\varphi(ab) = \omega(a)\varphi(b)$ . Moreover if  $b$  is in  $F^\times$  and  $x$  is in  $O_F$  then  $\varphi(b) = \psi(xb)\varphi(b)$ . Thus  $\varphi(b) = 0$  if  $b$  is not in  $O_F$ . Consequently  $\widehat{\varphi}(\nu, t)$  is 0 if  $\nu \neq \omega^{-1}$  but  $\widehat{\varphi}(\omega^{-1}, t)$  is a polynomial of the form

$$a_m t^m + \cdots + a_n t^n$$

with  $a_m a_n \neq 0$ . If  $\varphi_1(b) = \varphi(-\varpi b)$  then

$$\widehat{\varphi}_1(\omega^{-1}, t) = \frac{\omega(-1)}{t} \widehat{\varphi}(\omega^{-1}, t).$$

Let

$$\pi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \eta(a)I$$

and let  $\nu_0$  be the restriction of  $\eta$  to  $U_F$  while  $z_0 = \eta(\varpi)$ . The character  $\nu_0$  will have to be equal to  $\omega^2$ . The relation

$$\omega_0 \varphi = \pi \left( \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix} \right) \varphi = \pi \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \varphi_1$$

implies that

$$\omega_0 \widehat{\varphi}(\omega^{-1}, t) = C(\omega^{-1}, t) \omega(-1) z_0 t \widehat{\varphi}(\omega^{-1}, z_0^{-1} t^{-1}).$$

By Proposition 2.23,  $C(\omega^{-1}, t)$  is of the form  $ct^{-\ell}$  with  $\ell \geq 2$ . Thus the right side has a pole at 0 not shared by the left. This is a contradiction.

### §16. An application of the Selberg trace formula

In the fourteenth paragraph we saw that if  $\pi' = \bigotimes_v \pi'_v$  is a constituent of  $\mathcal{A}'$  and  $\pi'$  is not of the form  $g \rightarrow \chi(\nu(g))$  where  $\chi$  is a quasi-character of  $F^\times \backslash I$  then  $\pi = \bigotimes_v \pi_v$ , with  $\pi_v = \pi(\pi'_v)$ , is a constituent of  $\mathcal{A}_0$ . Let  $S$  be the set of places at which the quaternion algebra  $M'$  does not split. Given the results of the previous paragraph it is tempting to conjecture that the following theorem is valid.

**Theorem 16.1.** *Suppose  $\pi = \bigotimes \pi_v$  is a constituent of  $\mathcal{A}_0$ . If for every  $v$  in  $S$  the representation  $\pi_v$  is special or absolutely cuspidal then for every  $v$  there is a representation  $\pi'_v$  such that  $\pi_v = \pi(\pi'_v)$  and  $\pi' = \bigotimes \pi'_v$  is a constituent of  $\mathcal{A}'$ .*

The existence of  $\pi'_v$  has been shown. What is not clear is that  $\pi'$  is a constituent of  $\mathcal{A}'$ . It seems to be possible to prove this by means of the Selberg trace formula. Unfortunately a large number of analytical facts need to be verified. We have not yet verified them. However the theorem and its proof seem very beautiful to us; so we decided to include a sketch of the proof with a promise to work out the analytical details and publish them later. We must stress that the sketch is merely a formal argument so that the theorem must remain, for the moment, conjectural.

We first review some general facts about traces and group representations. Suppose  $G$  is a locally compact unimodular group and  $Z$  is a closed subgroup of the centre of  $G$ . Let  $\eta$  be a character of  $Z$ . We introduce the space  $L^1(\eta)$  of all measurable functions  $f$  on  $G$  which satisfy  $f(ag) = \eta^{-1}(a)f(g)$  for all  $a$  in  $Z$  and whose absolute values are integrable on  $Z \backslash G$ . If  $f_1$  and  $f_2$  belong to  $L^1(\eta)$  so does their product  $f_1 * f_2$  which is defined by

$$f_1 * f_2(g) = \int_{Z \backslash G} f_1(gh^{-1})f_2(h) dh$$

If  $f$  belongs to  $L^1(\eta)$  let  $f^*$  be the function  $f^*(g) = \overline{f}(g^{-1})$ . It also belongs to  $L^1(\eta)$ . A subalgebra  $B$  of  $L^1(\eta)$  will be called ample if it is dense and closed under the operation  $f \rightarrow f^*$ .

Let  $\pi$  be a unitary representation of  $G$  on the Hilbert space  $H$  such that  $\pi(a) = \eta(a)I$  for all  $a$  in  $Z$ . We do not suppose that  $\pi$  is irreducible. If  $f$  belongs to  $L^1(\eta)$  we set

$$\pi(f) = \int_{Z \backslash G} f(g)\pi(g) dg$$

If  $\pi(f)$  is compact for all  $f$  in some ample subalgebra  $B$  then  $\pi$  decomposes into the direct sum of irreducible representations no one of which occurs more than a finite number of times.

**Lemma 16.1.1.** *Suppose  $\pi_1$  and  $\pi_2$  are two unitary representations of  $G$  such that  $\pi_1(a) = \eta(a)I$  and  $\pi_2(a) = \eta(a)I$  for all  $a$  in  $Z$ . Suppose there is an ample subalgebra  $B$  of  $L^1(\eta)$  such that  $\pi_1(f)$  and  $\pi_2(f)$  are of Hilbert-Schmidt class for all  $f$  in  $B$ .*

(i) *If for every  $f$  in  $B$*

$$\text{trace } \pi_1(f)\pi_1(f^*) \geq \text{trace } \pi_2(f)\pi_2(f^*)$$

*then  $\pi_2$  is equivalent to a subrepresentation of  $\pi_1$ .*

(ii) *If for every  $f$  in  $B$*

$$\text{trace } \pi_1(f)\pi_1(f^*) = \text{trace } \pi_2(f)\pi_2(f^*)$$

then  $\pi_2$  is equivalent to  $\pi_1$ .

Let  $\pi_1$  act on  $H_1$  and let  $\pi_2$  act on  $H_2$ . A simple application of Zorn's lemma shows that we can choose a pair of closed invariant subspaces  $M_1$  and  $M_2$ , of  $H_1$  and  $H_2$  respectively, such that the restrictions of  $\pi_1$  to  $M_1$  and  $\pi_2$  to  $M_2$  are equivalent and such that the pair  $M_1, M_2$  is maximal with respect to this property. Replacing  $H_1$  and  $H_2$  by the orthogonal complements of  $M_1$  and  $M_2$  we may suppose that  $M_1 = 0$  and that  $M_2 = 0$ . To prove the first assertion of the lemma we have to show that with this assumption  $H_2 = 0$ . If the second condition is satisfied we can reverse the roles of  $\pi_1$  and  $\pi_2$  to see that  $H_1$  is also 0.

Before beginning the proof we make a simple remark. Suppose  $\sigma$  is an irreducible unitary representation of  $G$  on  $L$  and  $\sigma_\alpha, \alpha \in A$ , is an irreducible unitary representation of  $G$  on  $L_\alpha$ . Suppose that  $\sigma(a) = \eta(a)I$  for all  $a$  in  $Z$  and  $\sigma_\alpha(a) = \eta(a)I$  for all  $a$  in  $Z$  and all  $\alpha$  in  $A$ . Suppose that  $\sigma$  is equivalent to none of the  $\sigma_\alpha$  and that a non-zero vector  $x$  in  $L$  and vectors  $x_\alpha$  in  $L_\alpha$  are given. Finally suppose that

$$\sum_{\alpha} \|\sigma_\alpha(f)x_\alpha\|^2$$

is finite for every  $f$  in  $B$ . Then if  $\epsilon$  is any positive number there is an  $f$  in  $B$  such that

$$\sum_{\alpha} \|\sigma_\alpha(f)x_\alpha\|^2 < \epsilon \|\sigma(f)x\|^2.$$

Suppose the contrary and let  $L'$  be the closure in  $\bigoplus_{\alpha} L_{\alpha}$  of

$$\left\{ \bigotimes \sigma_{\alpha}(f)x_{\alpha} \mid f \in B \right\}$$

$L'$  is invariant under  $G$  and the map

$$\bigoplus \sigma_{\alpha}(f)x_{\alpha} \rightarrow \sigma(f)x$$

may be extended to a continuous  $G$ -invariant map  $A'$  of  $L'$  into  $L$ . If  $A'$  were 0 then  $\sigma(f)x = 0$  for all  $f$  in  $B$  which is impossible. Let  $A$  be the linear transformation from  $\bigoplus L_{\alpha}$  to  $L$  which is  $A'$  on  $L'$  and 0 on its orthogonal complement. The transformation  $A$  commutes with  $G$  and is not 0. Let  $A_{\alpha}$  be the restriction of  $A$  to  $L_{\alpha}$ . The transformation  $A_{\alpha}$  is a  $G$ -invariant map of  $L_{\alpha}$  into  $L$  and is therefore 0. Thus  $A$  is 0. This is a contradiction.

Suppose  $H_2$  is not 0. There is an  $h$  in  $B$  such that  $\pi_1(h) = 0$ . If  $f = h * h^*$  then  $\pi_2(f)$  is positive semi-definite and of trace class. It has a positive eigenvalue and with no loss of generality we may suppose that its largest eigenvalue is 1. Let  $\pi_2 = \bigotimes \pi_2^{\beta}$ , where  $\pi_2^{\beta}$  acts on  $H_2^{\beta}$ , be a decomposition of  $\pi_2$  into irreducible representations. There is a  $\beta_0$  and a unit vector  $x$  in  $H_2^{\beta_0}$  such that  $\pi_2(f)x = x$ . Let  $\pi_1 = \bigoplus \pi_1^{\alpha}$ , where  $\pi_1^{\alpha}$  acts on  $H_1^{\alpha}$ , be a decomposition of  $\pi_1$  into irreducible representations. Choose an orthogonal basis  $\{x^{\alpha, \gamma} \mid \gamma \in \Gamma_{\alpha}\}$  of  $H_1^{\alpha}$  consisting of eigenvectors of  $\pi_1(f)$ . Since

$$\text{trace } \pi_1(f) \geq \text{trace } \pi_2(f)$$

the largest eigenvalue of  $\pi_1(f)$  is positive. Let it be  $\lambda$ .

If  $f_1$  belongs to  $B$ ,

$$\sum_{\alpha} \sum_{\gamma} \|\pi_1^{\alpha}(f_1)x^{\alpha, \gamma}\|^2$$

is the Hilbert-Schmidt norm of  $\pi_1(f_1)$  and is therefore finite. By assumption  $\pi_2^{\beta_0}$  is not equivalent to any of the representations  $\pi_1^{\alpha}$  so that we can apply our earlier remark to the

vector  $x$  and the family of representations  $\pi_1^{\alpha,\gamma} = \pi_1^\alpha$  together with the family of vectors  $x^{\alpha,\gamma}$  to infer the existence of an  $f_1$  in  $B$  such that

$$\sum_{\alpha} \sum_{\gamma} \|\pi_1(f_1)x^{\alpha,\gamma}\|^2 < \frac{1}{2\lambda} \|\pi_2(f_1)x\|^2.$$

Then

$$\text{trace } \pi_1(f_1 f) \pi_1^*(f_1 f) = \text{trace } \pi_1^*(f_1 f) \pi_1(f_1 f)$$

is equal to

$$\sum_{\alpha,\gamma} \|\pi_1(f_1) \pi_1(f) x^{\alpha,\gamma}\|^2 \leq \lambda \sum_{\alpha,\gamma} \|\pi_1(f_1) x^{\alpha,\gamma}\|^2.$$

The right side is less than

$$\frac{1}{2} \|\pi_2(f_1)x\|^2 = \frac{1}{2} \|\pi_2(f_1 f)x\|^2$$

which is at most

$$\frac{1}{2} \text{trace } \pi_2(f_1 f) \pi_2^*(f_1 f).$$

This is a contradiction.

The next lemma is a consequence of the results of [35].

**Lemma 16.1.2.** *Suppose  $\eta$  is trivial so that  $L^1(\eta) = L^1(Z \backslash G)$ . Suppose that  $B$  is an ample subalgebra of  $L^1(\eta)$  which is contained in  $L^2(Z \backslash G)$ . If there is a positive constant  $\gamma$  and a unitary representation  $\pi$  of  $Z \backslash G$  such that  $\pi(f)$  is of Hilbert-Schmidt class for all  $f$  in  $B$  and*

$$\text{trace } \pi(f) \pi(f^*) = \gamma \int_{Z \backslash G} |f(g)|^2 dg$$

*then  $Z \backslash G$  is compact.*

In proving the theorem it is better to deal with representations in the adèle groups than to deal with representations of the global Hecke algebras. We have to assume that the reader is sufficiently well acquainted with the theory of group representations to pass back and forth unaided between the two viewpoints.

If  $F$  is a global field,  $\mathbf{A}$  is the adèle ring of  $F$ ,  $G = \text{GL}(2)$ , and  $\eta$  is a character of the idele class group  $F^\times \backslash I$  the space  $A(\eta)$  of all measurable functions  $\varphi$  on  $G_F \backslash G_{\mathbf{A}}$  that satisfy

$$\varphi \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g \right) = \eta(a) \varphi(g)$$

for all  $a$  in  $I$  and whose absolute values are square-integrable on  $G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}}$  is a Hilbert space. If  $\varphi$  belongs to this space

$$\int_{N_F \backslash N_{\mathbf{A}}} \varphi(n g) dn$$

is defined for almost all  $g$ . If it is 0 for almost all  $g$  the function  $\varphi$  is said to be a cusp form. The space  $A_0(\eta)$  of all such cusp forms is closed and invariant under  $G_{\mathbf{A}}$ . It is in fact the closure of  $\mathcal{A}_0(\eta)$ . It decomposes in the same way but now into a direct sum of closed orthogonal subspaces  $V$  on which  $G_{\mathbf{A}}$  acts according to an irreducible representation  $\pi = \bigotimes \pi_v$ . Thus  $V$  is now isomorphic to a tensor product of Hilbert spaces. Of course

the same representations occur now as occurred before. Similar remarks apply to the multiplicative group  $G'$  of a quaternion algebra  $M'$  over  $F$ .

It will be enough to prove the theorem when  $\pi$  is a constituent of some  $\mathcal{A}_0(\eta)$  or  $A_0(\eta)$  and  $\eta$  is a character because we can always take the tensor product of  $\pi$  with a suitable quasi-character. Suppose  $\eta$  is given. Let  $S$  be the set of places at which  $M'$  does not split. Suppose that for each  $v$  in  $S$  we are given an irreducible unitary representation  $\sigma'_v$  of  $G'_{F_v} = G'_v$  such that

$$\sigma'_v(a) = \eta_v(a)I$$

for all  $a$  in  $F_v^*$  which we identify with  $Z'_v = Z'_{F_v}$ . Let  $\sigma_v = \pi(\sigma'_v)$  be the representation of  $G_v$  corresponding to  $\sigma'_v$ . We may take  $\sigma_v$  unitary. Let  $\sigma_v$  act on  $U_v$  and let  $\sigma'_v$  act on  $U'_v$ . Fix a unit vector  $u'_v$  in  $U'_v$  and a unit vector  $u_v$  in  $U_v$  which is  $K_v$ -finite. The vector  $u'_v$  is automatically  $K'_v$ -finite.

Write  $A_0(\eta)$  as the direct sum, in the Hilbert space sense, of mutually orthogonal invariant irreducible subspaces  $V^1, V^2, \dots$ . Let the factorization of the representation  $\pi^i$  on  $V^i$  be  $\bigotimes \pi_v^i$ . Let  $\pi_v^i$  act on  $V_v^i$ . For simplicity of notation we identify  $V^i$  with  $\bigotimes V_v^i$ . We also suppose that if  $v$  is in  $S$  and  $\pi_v^i$  is equivalent to  $\sigma_v$  then  $U_v = V_v^i$  and  $\pi_v^i = \sigma_v$ . Let  $X$  be the set of all  $i$  such that  $\pi_v^i = \sigma_v$  for all  $v$  in  $S$  and if  $i$  belongs to  $X$  let

$$M^i = \left\{ \bigotimes_{v \in S} u_v \right\} \otimes \left\{ \bigotimes_{v \in S} V_v^i \right\}.$$

$M^i$  is invariant and irreducible under the action of

$$\widehat{G}_S = \{ g = (g_v) \mid g_v = 1 \text{ for all } v \text{ in } S \}.$$

Let

$$M = \bigoplus_{i \in X} M^i.$$

$M$  is a Hilbert space and  $\widehat{G}_S$  acts on  $M$ . If at least one of the representations  $\sigma'_v$ ,  $v \in S$ , is not one-dimensional set  $N = M$ . If they are all one-dimensional, let  $N$  be the subspace of  $A_0(\eta)$  spanned, in the Hilbert space sense, by  $M$  and the functions  $g \rightarrow \chi(\det g)$  where  $\chi$  is a character of  $F^\times \backslash I$  such that  $\chi^2 = \eta$  and  $\sigma'_v(g) = \chi_v(\nu(g))$  for all  $g$  in  $G'_v$  if  $v$  is in  $S$ . If  $v$  is non-archimedean this last condition determines  $\chi_v$  uniquely. If  $v$  is real it only determines it on the positive numbers.

Let  $A'(\eta)$  be the space of all measurable functions  $\varphi$  on  $G'_F \backslash G'_\mathbf{A}$  that satisfy  $\varphi(ag) = \eta(a)\varphi(g)$  for all  $a$  in  $I$  and whose absolute values are square integrable on  $G'_F Z'_\mathbf{A} \backslash G'_\mathbf{A}$ . Replacing  $\sigma_v$  by  $\sigma'_v$  and  $u_v$  by  $u'_v$  we define  $N'$  in the same way as we defined  $M$ . If at least one of the representations  $\sigma'_v$ ,  $v \in S$ , is not one-dimensional we set  $M' = N'$ . However if they are all one-dimensional and  $\chi$  is a character of  $F^\times \backslash I$  such that  $\chi^2 = \eta$  and  $\sigma'_v(g) = \chi_v(\nu(g))$  for all  $G$  in  $G'_V$  if  $v$  is in  $S$  then the function  $g \rightarrow \chi(\nu(g))$  belongs to  $N'$ . We let  $M'$  be the orthogonal complement in  $N'$  of the set of such functions. The group  $\widehat{G}'_S$  acts on  $M'$  and  $N'$ . However by means of the local isomorphisms  $\theta_v$  we can define an isomorphism of  $\widehat{G}_S$  and  $\widehat{G}'_S$ . Thus  $\widehat{G}_S$  acts on  $M$  and  $M'$ . To prove the theorem we need only show that the representations on these two spaces are equivalent. To do this we combine Lemma 16.1.1 with the Selberg trace formula.



To apply Lemma 16.1.1 we have to introduce an algebra  $B$ . It will be the linear span of  $B_0$ , the set of functions  $f$  on  $\widehat{G}_S$  of the form

$$f(g) = \prod_{v \notin S} f_v(g_v)$$

where the functions  $f_v$  satisfy the following conditions.

(i) If  $a_v$  belongs to  $F_v^\times$  then

$$f_v(a_v g_v) = \eta_v^{-1}(a_v) f_v(g_v).$$

(ii) The function  $f_v$  is  $K_v$ -finite on both sides and the projection of the support of  $f_v$  on  $Z_v \backslash G_v$  is compact.

(iii) If  $v$  is archimedean,  $f_v$  is infinitely differentiable.

(iv) If  $v$  is non-archimedean,  $f_v$  is locally constant.

(v) For almost all non-archimedean  $v$  the function  $f_v$  is 0 outside of  $Z_v K_v$  but on  $Z_v K_v$  is given by

$$f_v(g) = \omega_v^{-1}(\det g)$$

where  $\omega_v$  is unramified and satisfies  $\omega_v^2 = \eta_v$ .

We introduce  $B'$  in the same way. We may identify  $B$  and  $B'$  and to verify the conditions of the lemma we need only show that if  $f = f_1 * f_2$  with  $f_1$  and  $f_2$  in  $B_0$  then

$$\text{trace } \sigma(f) = \text{trace } \sigma'(f)$$

if  $\sigma$  is the representation on  $M$  and  $\sigma'$  that on  $M'$ . Let  $\tau$  be the representation on  $N$  and  $\tau'$  that on  $N'$ . Since

$$\text{trace } \tau(f) = \text{trace } \sigma(f) + \sum \int_{\widehat{Z}_S \backslash \widehat{G}_S} \chi(g) f(g) dg$$

and

$$\text{trace } \tau'(f) = \text{trace } \sigma'(f) + \sum \int_{\widehat{Z}_S \backslash \widehat{G}_S} \chi(g) f(g) dg$$

we need only show that

$$\text{trace } \tau(f) = \text{trace } \tau'(f).$$

Before beginning the proof we had better describe the relation between the Haar measures on the groups  $Z_{\mathbf{A}} \backslash G_{\mathbf{A}}$  and  $Z'_{\mathbf{A}} \backslash G'_{\mathbf{A}}$ . Choose a non-trivial character  $\psi$  of  $F \backslash \mathbf{A}$ . If  $\omega_0$  is any invariant form of maximal degree on  $Z \backslash G$  defined over  $F$  and therefore over each  $F_v$  we can associate to  $\omega_0$  and  $\psi_v$  a Haar measure  $\omega_0(v)$  on  $Z_v \backslash G_v$ . Then  $\prod_{v \notin S} \omega_0(v)$  determines a Haar measure  $\omega_0$  on  $\widehat{Z}_S \backslash \widehat{G}_S$  and  $\prod_v \omega_0(v)$  determines a Haar measure  $\omega_0$  on  $Z_{\mathbf{A}} \backslash G_{\mathbf{A}}$ . The measure on  $Z_{\mathbf{A}} \backslash G_{\mathbf{A}}$  is independent of  $\psi$  and is called the Tamagawa measure. As in the previous paragraph we can associate to  $\omega_0(v)$  a measure  $\omega'_0(v)$  on  $Z'_v \backslash G'_v$  and therefore to  $\omega_0$  a measure  $\omega'_0$  on  $\widehat{Z}'_S \backslash \widehat{G}'_S$  or  $Z'_{\mathbf{A}} \backslash G'_{\mathbf{A}}$ .

We first take  $f = f_1 * f_2$  in  $B'$  and find a formula for  $\text{trace } \tau'(f)$ . Let  $d(\sigma'_v)$  be the formal degree of  $\sigma'_v$  with respect to the measure  $\omega'_0(v)$  and let  $\xi'_v$  be the function

$$\xi'_v(g) = d(\sigma'_v) \overline{(\sigma'_v(g) u'_v, u'_v)}$$

on  $G'_v$ . Let  $\Phi' = \Phi'_f$  be the function

$$\Phi'(g) = \left\{ \prod_{v \in S} \xi'_v(g_v) \right\} f(\widehat{g}_S)$$

on  $G'_A$ . Here  $\widehat{g}_S$  is the projection of  $g$  on  $\widehat{G}'_S$ . If  $\rho'$  is the representation of  $G'_A$  on  $A'(\eta)$  the restriction of  $\rho'(\Phi')$  to  $N'$  is  $\tau'(f)$  and  $\rho'(\Phi)$  annihilates the orthogonal complement of  $N'$ . Thus

$$\text{trace } \rho'(\Phi') = \text{trace } \tau'(f).$$

If  $\varphi$  is in  $A'(\eta)$  then  $\rho'(\Phi')\varphi(g)$  is equal to

$$\int_{Z'_A \backslash G'_A} \varphi(gh) \Phi'(h) \omega'_0(h) = \int_{Z'_A \backslash G'_A} \varphi(h) \Phi'(g^{-1}h) \omega'_0(h).$$

The integration on the right can be performed by first summing over  $Z'_F \backslash G'_F$  and then integrating over  $Z_A G'_F \backslash G'_A$ . If

$$\Phi'(g, h) = \sum_{Z'_F \backslash G'_F} \Phi'(g^{-1}\gamma h)$$

the result is

$$\int_{Z'_A G'_F \backslash G'_A} \varphi(h) \Phi'(g, h) \omega'_0(h).$$

Thus the trace of  $\rho'(\Phi)$  is equal to

$$\int_{Z'_A G'_F \backslash G'_A} \Phi'(g, g) dg.$$

If we write out the integrand and perform the usual manipulations (cf [29]) we see that this integral is

$$(16.1.3) \quad \sum_{\{\gamma\}} \text{measure}(Z'_A G'_F(\gamma) \backslash G'_A(\gamma)) \int_{G'_A(\gamma) \backslash G'_A} \Phi'(g^{-1}\gamma g).$$

The sum is over a set of representatives of the conjugacy classes in  $G'_F$ . Here  $G'_A(\gamma)$  is the centralizer of  $\gamma$  in  $G'_A$  and  $G'_F(\gamma)$  is its centralizer in  $G'_F$ .

Let  $Q'$  be a set of representatives for the equivalence classes of quadratic extensions  $E$  of  $F$  such that  $E \otimes_F F_v$  is a field for all  $v$  in  $S$ . For each  $E$  in  $Q'$  fix an imbedding of  $E$  in the quaternion algebra  $M'$ . Let  $B_F = B_F(E)$  be the multiplicative group of  $E$ , considered as a subalgebra of  $M'$ , or what is the same the centralizer of  $E$  in  $G'_F$ . Let  $B_A = B_A(E)$  be the centralizer of  $E$  in  $G'_A$ . Let  $Q'_1$  be the separable extensions in  $Q'$  and  $Q'_2$  the inseparable ones if they exist. Then (16.1.3) is the sum of

$$(16.1.4) \quad \text{measure}(Z'_A G'_F \backslash G'_A) \Phi'(e),$$

if  $e$  is the identity,

$$(16.1.5) \quad \frac{1}{2} \sum_{Q'_1} \sum_{\substack{\gamma \in Z'_F \backslash B_F \\ \gamma \notin Z'_F}} \text{measure}(Z'_A B_F \backslash B_A) \int_{B_A \backslash G_A} \Phi'(g^{-1}\gamma g) \omega_B(g)$$

and

$$(16.1.6) \quad \sum_{Q'_2} \sum_{\substack{\gamma \in Z'_F \backslash B_F \\ \gamma \notin Z_F}} \text{measure}(Z'_A B_F \backslash B_A) \int_{B_A \backslash G_A} \Phi'(g^{-1} \gamma g) \omega_B(g).$$

The last sum is deceptive because  $Q'_2$  has at most one element. The measure  $\omega_B$  is the quotient of the measure on  $Z'_A \backslash G'_A$  by that on  $Z'_A \backslash B_A$ . The choice of the measure on  $Z'_A \backslash B_A$  is not too important. We do suppose that it is a product measure.<sup>2</sup>

The expression (16.1.4) is equal to

$$\text{measure}(Z'_A G'_F \backslash G'_A) \left\{ \prod_{v \in S} d(\sigma'_v) \right\} f(e).$$

The integrals of (16.1.5) and (16.1.6) are equal to the product

$$\prod_{v \in S} \frac{\chi_{\sigma'_v}(\gamma^{-1})}{\text{measure}(Z'_v \backslash B_v)}$$

and

$$\int_{\widehat{B}_S \backslash \widehat{G}'_S} f(g^{-1} \gamma g) \omega_B.$$

Now regard  $f = f_1 * f_2$  as an element of  $B$ . We can still introduce for each  $v$  in  $S$  the function

$$\xi_v(g) = d(\sigma_v) \overline{(\sigma_v(g) u_v, u_v)}$$

on  $G_v$ . The factor  $d(\sigma_v)$  is the formal degree of  $\sigma_v$  with respect to the measure  $\omega_0(v)$ . If  $\sigma'_v$  is not one-dimensional  $\xi_v$  is integrable and we can use it to define a function  $\Phi$  to which we can hope to apply the trace formula. When  $\sigma'_v$  is one-dimensional the function  $\xi_v$  is not even integrable so it is of no use to us. However in this case we can find an integrable function  $\zeta_v$  with the following properties:

(i) For all  $a$  in  $F_v$

$$\zeta_v \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g \right) = \eta_v^{-1}(a) \zeta_v(g).$$

(ii) For a suitable choice of  $u_v$  the operator  $\sigma_v(\zeta_v)$  is the orthogonal projection on the space  $\mathbf{C}u_v$ .

(iii) If  $\chi_v$  is a character of  $F_v^\times$  such that  $\chi_v^2 = \eta_v$  then

$$\int_{Z_v \backslash G_v} \chi_v(\det g) \zeta_v(g) \omega_0(v)$$

is  $-1$  if  $\sigma'_v(h) = \chi_v(\nu(h))$  for all  $h$  in  $G'_v$  and is  $0$  otherwise.

(iv) If  $\pi_v$  is a unitary infinite-dimensional irreducible admissible representation of  $G_v$  which is not equivalent to  $\sigma_v$  but satisfies

$$\pi_v \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \eta_v(a) I$$

<sup>2</sup>In (16.1.5) the factor  $\frac{1}{2}$  is not quite correct. If we want to leave it in, both  $\gamma$  and its conjugate must be counted, even if they differ only by an element of  $F$ .

for all  $a$  in  $F_v^\times$  then

$$\text{trace } \pi_v(\zeta_v) = 0.$$

If  $v$  is real we cannot describe  $\zeta_v$  without a great deal more explanation than is desirable at present. However after a few preliminary remarks we will be able to describe it when  $v$  is non-archimedean.

Suppose  $\sigma'_v(g) = \chi_v(\nu(g))$  for  $g$  in  $G'_v$  and  $\pi_v$  is a representation of  $G_v$  such that

$$\pi_v \left( \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right) = \eta_v(a)I$$

for all  $a$  in  $F_v^\times$ . Applying Lemma 3.9 to  $\chi_v^{-1} \otimes \pi_v$  we see that the restriction of  $\pi_v$  to  $K_v$  contains the representation  $k \rightarrow \chi_v(\det k)$  if and only if  $\pi_v = \pi(\mu_v, \nu_v)$ ,  $\mu_v \nu_v = \eta_v$ , and the restrictions of  $\mu_v$  and  $\nu_v$  to  $U_v$ , the group of units of  $F_v$ , are both equal to the restriction of  $\chi_v$ . Let  $\zeta'_v$  be the function on  $G_v$  which is 0 outside of  $Z_v K_v$  but on  $K_v$  is equal to

$$\frac{1}{\text{measure}(Z_v \backslash Z_v K_v)} \chi_v^{-1}(\det g).$$

Let  $H_v$  be the group generated by  $Z_v$ , the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in  $K_v$  for which  $c \equiv 0 \pmod{\mathfrak{p}_v}$ , and

$$\begin{pmatrix} 0 & 1 \\ \varpi_v & 0 \end{pmatrix}.$$

Let  $\omega_v$  be the character  $\omega_v(a) = (-1)^n \chi_v(a)$  if  $|a| = |\varpi_v|^n$ . According to the concluding lemmas of the previous paragraph there is a non-zero vector  $u$  in the space of  $\pi_v$  such that

$$\pi_v(g)u = \omega_v(\det g)u$$

for all  $g$  in  $H_v$  if and only if  $\pi_v$  is equivalent to  $\sigma_v$ ,  $\pi_v = \pi(\mu_v, \nu_v)$  is infinite-dimensional,  $\mu_v \nu_v = \eta_v$ , and the restrictions of  $\mu_v$  and  $\nu_v$  to  $U_v$  are equal to the restriction of  $\chi_v$ , or  $\pi_v$  is the one-dimensional representation

$$g \rightarrow \omega_v(\det g).$$

Let  $\zeta''_v$  be the function which is 0 outside of  $H_v$  and equal to

$$\frac{1}{\text{measure } Z_v \backslash H_v} \omega_v^{-1}(\det g)$$

on  $H_v$ . We may take

$$\zeta_v = \zeta''_v - \zeta'_v.$$

There are some consequences of the four conditions on  $\zeta_v$  which we shall need. If  $\mu_v$  and  $\nu_v$  are two characters of  $F_v^\times$  such that  $\mu_v \nu_v = \eta_v$ , the trace of  $\rho(\zeta_v, \mu_v, \nu_v)$  is a multiple of

$$\int_{Z_v \backslash A_v} \mu_v(\alpha) \nu_v(\beta) \left| \frac{\alpha}{\beta} \right|^{1/2} \left\{ \int_{N_v} \int_{K_v} \zeta_v(k^{-1} a n k) \, dn \, dk \right\} da$$

if

$$a = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Since this is 0 for all possible choice of  $\mu_v$  and  $\nu_v$

$$\int_{N_v} \int_{K_v} \zeta_v(k^{-1}ank) dk dn = 0$$

for all  $a$ . We also observe that if  $\sigma'_v$  is not one-dimensional then

$$\int_{N_v} \int_{K_v} \xi_v(k^{-1}ank) dk dn = 0$$

for all  $a$ .

If  $\pi_v$  is special or absolutely cuspidal trace  $\pi_v(\zeta_v)$  is therefore equal to

$$\frac{1}{2} \sum_{S'} \int_{Z_v \setminus B_v} \left\{ \int_{B_v \setminus G_v} \zeta_v(g^{-1}bg) \omega_b(v) \right\} \chi_{\pi_v}(b) \delta(b) \mu_B^0(b).$$

Since trace  $\pi_v(\zeta_v)$  is 1 if  $\pi_v$  is equivalent to  $\sigma_v$  and 0 otherwise the orthogonality relations imply that

$$\int_{B_v \setminus G_v} \zeta_v(g^{-1}bg) \omega_B(v) = \frac{-1}{\text{measure } Z_v \setminus B_v} \chi_{\sigma_v}(b^{-1})$$

for all regular  $b$  and therefore, by continuity, for all  $b$  whose eigenvalues do not lie in  $F_v$ . It probably also follows from the Plancherel theorem that  $\zeta_v(e) = d(\sigma_v)$ . We do not need this but we shall eventually need to know that  $\zeta_v(e) = d(\sigma'_v)$ . For the moment we content ourselves with observing that if  $\omega_v$  is a character of  $F_v^\times$  and  $\sigma'_v$  is replaced by  $\omega_v \otimes \sigma'_v$  the formal degree does not change and  $\zeta_v$  is replaced by the function  $g \rightarrow \omega_v^{-1}(\det g) \zeta_v(g)$  so that  $\zeta_v(e)$  does not change. Thus the relation  $\zeta_v(e) = d(\sigma'_v)$  need only be proved when  $\sigma'_v$  is trivial.

Let  $S_1$  be the subset of  $v$  in  $S$  for which  $\sigma'_v$  is one-dimensional and let  $S_2$  be the complement of  $S_1$  in  $S$ . Given  $f = f_1 * f_2$  in  $B$  we set

$$\Phi(g) = \left\{ \prod_{v \in S_1} \zeta_v(g_v) \right\} \left\{ \prod_{v \in S_2} \xi_v(g_v) \right\} f(\widehat{g}_S).$$

Let  $\rho_0^+$  be the representation of  $G_{\mathbf{A}}$  on  $A_0^+(\eta)$  the sum, in the Hilbert space sense, of  $A_0(\eta)$  and the functions  $\chi : g \rightarrow \chi(\det g)$  where  $\chi$  is a character of  $F^\times \setminus I$  such that  $\chi^2 = \eta$  and let  $\rho$  be the representation on  $A(\eta)$ . If at least one of the representations  $\rho'_v$  is not one-dimensional  $\rho_0^+(\Phi)$  annihilates the orthogonal complement of  $A_0(\eta)$ . If they are all one-dimensional we apply the third condition on the functions  $\zeta_v$  together with the fact that the number of places in  $S$  is even to see that  $\rho_0^+(\Phi)\chi = 0$  unless  $\sigma'_v(h) = \chi_v(\nu(h))$  for all  $h$  in  $G'_v$  and all  $v$  in  $S$  but that if this is so

$$\rho_0^+(\Phi)\chi = \tau(f)\chi.$$

Recall that  $A_0(\eta)$  is the direct sum of spaces  $V^i$  on which  $G_{\mathbf{A}}$  acts according to representations  $\pi^i = \bigotimes \pi_v^i$ . If at least one of the representations  $\sigma'_v$  is not one-dimensional  $\rho_0^+(\Phi)$  is equal to  $\sigma(f)$  on  $M$  and annihilates the orthogonal complement of  $M$  in  $A_0(\eta)$ . Suppose they are all one-dimensional. If  $i$  belongs to  $X$  the restrictions of  $\rho_0^+(\Phi)$  and  $\sigma(f)$  or  $\tau(f)$  to  $M^i$  are equal and  $\rho_0^+(\Phi)$  annihilates the orthogonal complement of  $M^i$  in  $V^i$ . If  $i$  is not

in  $X$  the trace of the restriction of  $\rho_0^+(\Phi)$  to  $V^i$  is

$$\left\{ \prod_{v \in S} \text{trace } \pi_v^i(\zeta_v) \right\} \left\{ \text{trace } \widehat{\pi}_S^i(f) \right\}$$

if  $\widehat{\pi}_S^i = \bigotimes_{v \notin S} \pi_v^i$ . Since  $\pi_v^i$ ,  $v \in S$ , are all infinite-dimensional and for at least one such  $v$  the representation  $\pi_v^i$  is not equivalent to  $\sigma_v$ ,

$$\prod_{v \in S} \text{trace } \pi_v^i(\zeta_v) = 0.$$

We conclude that

$$\text{trace } \rho_0^+(\Phi) = \text{trace } \tau(f).$$

To show that

$$\text{trace } \tau(f) = \text{trace } \tau'(f)$$

we have to apply the trace formula to find a suitable expression for  $\text{trace } \rho_0^+(\Phi)$ . In order to describe the formula we need to state some results in the theory of Eisenstein series.

Consider the collection of pairs of characters  $\mu, \nu$  of  $F^\times \backslash I$  such that  $\mu\nu = \eta$ . Two such pairs,  $\mu, \nu$  and  $\mu', \nu'$  are said to be equivalent if there is a complex number  $r$  such that  $\mu' = \mu\alpha_F^r$  and  $\nu' = \nu\alpha_F^{-r}$ . If  $a$  belongs to  $I$  then  $\alpha_F^r(a) = |a|^r$ . Let  $P$  be a set of representatives for these equivalence classes.

Suppose  $(\mu, \nu)$  belongs to  $P$ . If  $s$  is a complex number the space  $\mathcal{B}(\mu\alpha_F^{s/2}, \nu\alpha_F^{-s/2})$  of functions on  $N_{\mathbf{A}} \backslash G_{\mathbf{A}}$  is defined as in the tenth paragraph. Since the functions in this space are determined by their restrictions to  $K$  we may think of it as a space of functions on  $K$  in which case it is independent of  $s$ . Thus we have isomorphisms

$$T_s : \mathcal{B}(\mu\alpha_F^{s/2}, \nu\alpha_F^{-s/2}) \rightarrow \mathcal{B}(\mu, \nu).$$

The theory of Eisenstein series provides us with a function  $(\varphi, s) \rightarrow E(\varphi, s)$  from  $\mathcal{B}(\mu, \nu) \times \mathbf{C}$  to  $\mathcal{A}(\eta)$ . Let  $E(g, \varphi, s)$  be the value of  $E(\varphi, s)$  at  $g$ . For a given  $\varphi$  the function  $E(g, \varphi, s)$  is continuous in  $g$  and meromorphic in  $s$ . Moreover there is a discrete set of points in  $\mathbf{C}$  such that outside of this set it is holomorphic in  $s$  for all  $g$  and  $\varphi$ . If  $s$  is not in this set the map  $\varphi \rightarrow E(T_s\varphi, s)$  of  $\mathcal{B}(\mu\alpha_F^{s/2}, \nu\alpha_F^{-s/2})$  into  $\mathcal{A}(\eta)$  commutes with the action of  $\mathcal{H}$ .

If the total measure of  $N_F \backslash N_{\mathbf{A}}$  is taken to be 1 the integral

$$\int_{N_F \backslash N_{\mathbf{A}}} E(ng, T_s\varphi, s) dn$$

is equal to

$$\varphi(g) + (M(s)\varphi)(g),$$

where  $M(s)$  is a linear transformation from  $\mathcal{B}(\mu\alpha_F^{s/2}, \nu\alpha_F^{-s/2})$  to  $\mathcal{B}(\nu\alpha_F^{-s/2}, \mu\alpha_F^{s/2})$  which commutes with the action of  $\mathcal{H}$ . It is meromorphic in the sense that

$$\langle M(s)T_s^{-1}\varphi_1, T_s^{-1}\varphi_2 \rangle$$

is meromorphic if  $\varphi_1$  belongs to  $\mathcal{B}(\mu, \nu)$  and  $\varphi_2$  belongs to  $\mathcal{B}(\nu^{-1}, \mu^{-1})$ . The quotient of  $M(s)$  by

$$\frac{L(1-s, \nu\mu^{-1})}{L(1+s, \mu\nu^{-1})} = \epsilon(1-s, \nu\mu^{-1}) \frac{L(s, \mu\nu^{-1})}{L(1+s, \mu\nu^{-1})}$$

is holomorphic for  $\operatorname{Re} s \geq 0$ . Since the analytic behaviour of  $E(g, \varphi, s)$  is controlled by that of  $M(s)$  it should be possible, as we observed before, to use the Eisenstein series to show that a constituent of  $\mathcal{B}(\mu\alpha_F^{s/2}, \nu\alpha_F^{-s/2})$  is also a constituent of  $\mathcal{A}(\eta)$ .

To indicate the dependence of  $M(s)$  on  $\mu$  and  $\nu$  we write  $M(\mu, \nu, s)$ . Then

$$M(\mu, \nu, s)M(\nu, \mu, -s) = I.$$

If  $s$  is purely imaginary we can introduce the inner product

$$(\varphi_1, \varphi_2) = \int_K \varphi_1(k) \overline{\varphi_2(k)} dk$$

on  $\mathcal{B}(\mu\alpha_F^{s/2}, \nu\alpha_F^{-s/2})$ . Let  $B(\mu\alpha_F^{s/2}, \nu\alpha_F^{-s/2})$  be its completion with respect to this inner product. We may think of  $B(\mu\alpha_F^{s/2}, \nu\alpha_F^{-s/2})$  as a function space on  $G_{\mathbf{A}}$  on which  $G_{\mathbf{A}}$  acts by right translations. The representation of  $G_{\mathbf{A}}$  on  $B(\mu\alpha_F^{s/2}, \nu\alpha_F^{-s/2})$  is unitary. Let  $g$  correspond to the operator  $\rho(g, \mu, \nu, s)$  and if  $f$  is in  $L^1(\eta)$  let

$$\rho(f, \mu, \nu, s) = \int_{Z_{\mathbf{A}} \backslash G_{\mathbf{A}}} f(g) \rho(g, \mu, \nu, s) \omega_0(g)$$

The isomorphism  $T_s$  extends to an isometry, from  $B(\mu\alpha_F^{s/2}, \nu\alpha_F^{-s/2})$  to  $B(\mu, \nu)$  and  $M(\mu, \nu, s)$  extends to an isometry from  $B(\mu\alpha_F^{s/2}, \nu\alpha_F^{-s/2})$  to  $B(\nu\alpha_F^{-s/2}, \mu\alpha_F^{s/2})$ . In particular

$$M^*(\mu, \nu, s) = M(\nu, \mu, -s).$$

Suppose  $(\mu, \nu)$  is in  $P$  and, for some  $r$ ,  $\nu = \mu\alpha_F^r$  and  $\mu = \nu\alpha_F^{-r}$ . Replacing  $\mu$  by  $\mu\alpha_F^{r/2}$  and  $\nu$  by  $\nu\alpha_F^{-r/2}$  if necessary we may suppose that  $\mu = \nu$ . We may also suppose that if  $(\mu, \nu)$  is in  $P$  and is not equivalent to  $(\nu, \mu)$  then  $(\nu, \mu)$  is also in  $P$ . Let  $L$  be the Hilbert space sum

$$\bigoplus_P B(\mu, \nu)$$

and let  $\mathcal{L}$  be the algebraic sum

$$\bigoplus_P \mathcal{B}(\mu, \nu).$$

If we define  $L(s)$  to be

$$\bigoplus_p B(\mu\alpha_F^{s/2}, \nu\alpha_F^{-s/2})$$

and  $\mathcal{L}(s)$  to be

$$\bigoplus_P \mathcal{B}(\mu\alpha_F^{s/2}, \nu\alpha_F^{-s/2})$$

we can again introduce the map

$$T_s : L(s) \rightarrow L.$$

The representation  $g \rightarrow \rho(g, s)$  is the representation

$$g \rightarrow \bigoplus \rho(g, \mu, \nu, s)$$

on  $L(s)$ .  $M(s)$  will be the operator on  $L(s)$  which takes  $\bigoplus \varphi(\mu, \nu)$  to  $\bigoplus \varphi_1(\mu, \nu)$  with

$$\varphi_1(\nu, \mu) = M(\mu, \nu, s) \varphi(\mu, \nu).$$

It is unitary.

If  $F$  has characteristic 0 let  $H$  be the space of all square integrable functions  $\varphi$  from the imaginary axis to  $L$  such that

$$T_{-s}^{-1}\varphi(-s) = M(s)T_s^{-1}\varphi(s)$$

with the norm

$$\frac{c}{\pi} \int_{-i\infty}^{i\infty} \|\varphi(s)\|^2 d|s|,$$

where  $c$  is a positive constant relating various Haar measures. It will be defined more precisely later. If  $F$  is a function field with field of constants  $\mathbf{F}_q$  the functions in  $H$  are to be periodic of period  $\frac{\log q}{2\pi}i$  and the norm is to be

$$\frac{c \log q}{\pi} \int_0^{\frac{2\pi}{\log q}} \|\varphi(s)\|^2 d|s|.$$

On the whole we shall proceed as though  $F$  had characteristic 0 merely remarking from time to time the changes to be made when the characteristic is positive.

If  $\varphi = \bigoplus \varphi(\mu, \nu)$  is in  $\mathcal{L}$  we set

$$E(g, \varphi, s) = \sum E(g, \varphi(\mu, \nu), s).$$

If  $\varphi$  in  $H$  takes values in  $\mathcal{L}$

$$\lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-iT}^{iT} E(g, \varphi(s), s) d|s| = \tilde{\varphi}(g)$$

exists in  $A(\eta)$ . The map  $\varphi \rightarrow \tilde{\varphi}$  extends to an isometry of  $H$  with a subspace  $A_1(\eta)$  of  $A(\eta)$ . If  $g$  is in  $G_{\mathbf{A}}$  and  $\varphi'$  is defined by

$$\varphi'(s) = T_s \rho(g, s) T_s^{-1} \varphi(s)$$

then  $\tilde{\varphi}'$  is  $\rho(g)\tilde{\varphi}$ .

The orthogonal complement of  $A_1(\eta)$  is  $A_0^+(\eta)$ . Thus if  $E$  is the orthogonal projection of  $A(\eta)$  on  $A_1(\eta)$  the trace of  $\rho_0^+(\Phi)$  is the trace of  $\rho(\Phi) - E\rho(\Phi)$  which, according to the Selberg trace formula, is the sum of the following expressions which we first write out and then explain.

(i)

$$\text{measure}(Z_{\mathbf{A}} G_F \backslash G_{\mathbf{A}}) \Phi(e).$$

(ii)

$$\frac{1}{2} \sum_{Q_1} \sum_{\substack{\gamma \in Z_F \backslash B_F \\ \gamma \notin Z_F}} \text{measure}(Z_{\mathbf{A}} B_F \backslash B_{\mathbf{A}}) \int_{B_{\mathbf{A}} \backslash G_{\mathbf{A}}} \Phi(g^{-1} \gamma g) \omega_B(g).$$

(iii)

$$\sum_{Q_2} \sum_{\substack{\gamma \in Z_F \backslash B_F \\ \gamma \notin Z_F}} \text{measure}(Z_{\mathbf{A}} B_F \backslash B_{\mathbf{A}}) \int_{B_{\mathbf{A}} \backslash G_{\mathbf{A}}} \Phi(g^{-1} \gamma g) \omega_B(g).$$

(iv)

$$-c \sum_{\substack{\gamma \in Z_F \backslash A_F \\ \gamma \notin Z_F}} \sum_v \left\{ \prod_{w \neq v} \omega(\gamma, f_w) \right\} \omega_1(\gamma, f_v).$$



(v)

$$c \left[ \lambda_0 \prod_v \theta(0, f_v) + \lambda_{-1} \left\{ \sum_v \theta'(0, f_v) \prod_{w \neq v} \theta(0, f_w) \right\} \right].$$

(vi) If  $F$  is a number field

$$-\frac{1}{4} \text{trace } M(0) \rho(\Phi, 0),$$

but

$$-\frac{\log q}{4} \left\{ \text{trace } M(0) \rho(\Phi, 0) + \text{trace } M\left(\frac{\pi}{\log q}\right) \rho\left(\Phi, \frac{\pi}{\log q}\right) \right\}$$

if  $F$  is a function field.(vii) If  $F$  is a number field

$$\frac{1}{4\pi} \int_{-i\infty}^{i\infty} \text{trace } m^{-1}(s) m'(s) \rho(\Phi, s) d|s|,$$

but

$$\frac{\log q}{4\pi} \int_0^{\frac{2\pi}{\log q}} \text{trace } m^{-1}(s) m'(s) \rho(\Phi, s) d|s|$$

if  $F$  is a function field.(viii) The sum over  $(\mu, \nu)$  and  $v$  of

$$\frac{1}{4\pi} \int_{-i\infty}^{i\infty} \text{tr} \{ R^{-1}(\mu_v, \nu_v, s) R'(\mu_v, \nu_v, s) \rho(f, \mu_v, \nu_v, s) \} \left\{ \prod_{w \neq v} \text{tr} \rho(f_w, \mu_w, \nu_w, s) \right\} d|s|$$

if  $F$  is a number field and of

$$\frac{\log q}{4\pi} \int_0^{\frac{2\pi}{\log q}} \text{tr} \{ R^{-1}(\mu_v, \nu_v, s) R'(\mu_v, \nu_v, s) \rho(f, \mu_v, \nu_v, s) \} \left\{ \prod_{w \neq v} \text{tr} \rho(f_w, \mu_w, \nu_w, s) \right\} d|s|$$

if  $F$  is a function field.The function  $\Phi$  is of the form

$$\Phi(g) = \prod_v f_v(g_v).$$

Let  $Q$  be a set of representatives for the equivalence classes of quadratic extensions of  $F$ . For each  $E$  in  $Q$  fix an imbedding of  $E$  in the matrix algebra  $M = M(2, F)$ . Let  $B_F = B_F(E)$  be the multiplicative group of  $E$ , considered as a subalgebra of  $M$ . It is the centralizer of  $E$  in  $G_F$ . Let  $B_{\mathbf{A}} = B_{\mathbf{A}}(E)$  be the centralizer of  $E$  in  $G_{\mathbf{A}}$ . Let  $Q_1$  be the collection of separable extensions in  $Q$  and  $Q_2$  the collection of inseparable extensions. Let, moreover,  $A_F$  be the group of diagonal matrices in  $G_F$ .

Choose on  $N_{\mathbf{A}}$  that Haar measure which makes the measure of  $N_F \backslash N_{\mathbf{A}}$  equal to 1. Choose on  $K$  the normalized Haar measure. On the compact group  $H$  obtained by taking the quotient of

$$\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in A_{\mathbf{A}} \mid |\alpha| = |\beta| \right\}$$

by  $Z_{\mathbf{A}}A_F$  choose the normalized Haar measure. This group  $H$  is the kernel of the map

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \rightarrow \log \left| \frac{\alpha}{\beta} \right|^{1/2}$$

of  $A_F Z_{\mathbf{A}} \backslash A_{\mathbf{A}}$  onto  $\mathbf{R}$  or  $\log q\mathbf{Z}$ . On  $\mathbf{R}$  one has the standard measure  $dx$  and on  $\log q\mathbf{Z}$  one has the standard measure which assigns the measure 1 to each point. The measures on  $H$  and on  $H \backslash (A_F Z_{\mathbf{A}} \backslash A_{\mathbf{A}})$  together with the measure on  $Z_{\mathbf{A}} \backslash A_F Z_{\mathbf{A}}$  which assigns the measure 1 to each point serve to define a measure  $da$  on  $Z_{\mathbf{A}} \backslash A_{\mathbf{A}}$ . The constant  $c$  is defined by demanding that

$$\int_{Z_{\mathbf{A}} \backslash G_{\mathbf{A}}} f(g) \omega_0(g)$$

be equal to

$$c \int_{Z_{\mathbf{A}} \backslash A_{\mathbf{A}}} \int_{N_{\mathbf{A}}} \int_K f(ank) da dn dk$$

if  $f$  is an integrable function on  $Z_{\mathbf{A}} \backslash G_{\mathbf{A}}$ . We may suppose that the measures on  $Z_{\mathbf{A}} \backslash A_{\mathbf{A}}$ ,  $N_{\mathbf{A}}$ , and  $K$  are given as product measures and in particular that

$$\int_{K_v} dk_v = 1$$

and

$$\int_{N_v} \chi(n_v) dn_v = 1$$

for almost all  $v$  if  $\chi$  is the characteristic function of

$$\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in O_{F_v} \right\}.$$

The factors  $\omega(\gamma, f_v)$  and  $\omega_1(\gamma, f_v)$  appearing in the fourth expression are defined by

$$\omega(\gamma, f_v) = \int_{N_v} \int_{K_v} f_v(k_v^{-1} n_v^{-1} \gamma n_v k_v) dn_v dk_v$$

and

$$\omega_1(\gamma, f_v) = \int_{N_v} \int_{K_v} f_v(k_v^{-1} n_v^{-1} \gamma n_v k_v) \log \lambda(n_v) dn_v dk_v.$$

If

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} n = \begin{pmatrix} \alpha' & 0 \\ 0 & \beta' \end{pmatrix} n' k'$$

then

$$\lambda(n) = \left| \frac{\alpha'}{\beta'} \right|.$$

Set  $\theta(s, f_v)$  equal to

$$\frac{1}{L(1+s, 1_v)} \int_{Z_v \backslash A_v} \int_{K_v} f_v(k_v^{-1} a_v^{-1} n_0 a_v k_v) \left| \frac{\alpha_v}{\beta_v} \right|^{-1-s} da_v dk_v$$

where

$$a_v = \begin{pmatrix} \alpha_v & 0 \\ 0 & \beta_v \end{pmatrix}$$

and

$$n_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We take  $1_v$  to be the trivial character of  $F_v^\times$ . Then  $\theta(s, f_v)$  is analytic at least for  $\operatorname{Re} s > -1$ . Its derivative at 0 is  $\theta'(0, f_v)$ . If

$$L(1+s, 1_F) = \prod_v L(1+s, 1_v)$$

the Laurent expansion of  $L(1+s, 1_F)$  about  $s=0$  is

$$\frac{\lambda_{-1}}{s} + \lambda_0 + \cdots.$$

The operator  $m(s)$  is the operator on  $L(s)$  which for each  $(\mu, \nu)$  multiplies every element of  $B(\mu\alpha_F^{s/2}, \nu\alpha_F^{-s/2})$  by

$$\frac{L(1-s, \nu\mu^{-1})}{L(1+s, \mu\nu^{-1})}.$$

We may represent  $B(\mu\alpha_{F_v}^{s/2}, \nu\alpha_{F_v}^{-s/2})$  as

$$\bigotimes_v B(\mu_v\alpha_{F_v}^{s/2}, \nu_v\alpha_{F_v}^{-s/2})$$

when  $s$  is purely imaginary. If  $\operatorname{Re} s > 0$  let  $R(\mu_v, \nu_v, s)$  be the operator from  $\mathcal{B}(\mu_v\alpha_{F_v}^{s/2}, \nu_v\alpha_{F_v}^{-s/2})$  to  $\mathcal{B}(\nu_v\alpha_{F_v}^{-s/2}, \mu_v\alpha_{F_v}^{s/2})$  defined by setting

$$R(\mu_v, \nu_v, s)\varphi(g)$$

equal to

$$\epsilon(1-s, \mu_v^{-1}\nu_v, \psi_v) \frac{L(1+s, \mu_v\nu_v^{-1})}{L(s, \mu_v\nu_v^{-1})} \int_{N_v} \varphi\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ng\right) dn.$$

These operators can be defined for  $s$  purely imaginary by analytic continuation. They are then scalar multiples of unitary operators and for a given  $\mu, \nu$  are in fact unitary for almost all  $v$ . Thus  $R(\mu_v, \nu_v, s)$  can be defined as an operator  $B(\mu_v\alpha_{F_v}^{s/2}, \nu_v\alpha_{F_v}^{-s/2})$  when  $s$  is purely imaginary and

$$M(s) = \sum_{(\mu, \nu)} \left\{ \bigotimes_v R(\mu_v, \nu_v, s) \right\} \frac{L(1-s, \nu\mu^{-1})}{L(1+s, \mu\nu^{-1})}.$$

Set

$$N(s) = T_S M(s) T_S^{-1}$$

and if  $N'(s)$  is the derivative of  $N(s)$  set

$$M'(s) = T_s^{-1} N'(s) T_s$$

Define  $R'(\mu_v, \nu_v, s)$  in a similar fashion. Then

$$\operatorname{trace} M^{-1}(s) M'(s) \rho(\Phi, s)$$

is the sum of

$$\operatorname{trace} m^{-1}(s) m'(s) \rho(\Phi, s)$$

and

$$\sum_{(\mu, \nu)} \sum_v \left\{ \text{tr } R^{-1}(\mu_v, \nu_v, s) R'(\mu_v, \nu_v, s) \rho(f_v, \mu_v, \nu_v, s) \right\} \left\{ \prod_{w \neq v} \text{tr } \rho(f_w, \mu_w, \nu_w, s) \right\},$$

where  $\rho(f_v, \mu_v, \nu_v, s)$  is the restriction of  $\rho(f_v)$  to  $B(\mu_v \alpha_{F_v}^{s/2}, \nu_v \alpha_{F_v}^{-s/2})$ .

If  $E(\mu, \nu, s)$  is the projection of  $L(s)$  on  $B(\mu \alpha_F^{s/2}, \nu \alpha_F^{-s/2})$  we can write

$$m(s) = \sum a(\mu, \nu, s) E(\mu, \nu, s)$$

where the  $a(\mu, \nu, s)$  are scalars. Thus

$$\text{trace } m^{-1}(s) m'(s) \rho(\Phi, s)$$

is equal to

$$\sum \frac{a'(\mu, \nu, s)}{a(\mu, \nu, s)} \left\{ \prod_v \text{trace } \rho(f_v, \mu_v, \nu_v, s) \right\}.$$

We can also write

$$M(0) = \sum a(\mu, \nu) E(\mu, \nu, 0)$$

so that

$$\text{trace } M(0) \rho(\Phi, 0)$$

is equal to

$$\sum a(\mu, \nu) \left\{ \prod_v \text{trace } \rho(f_v, \mu_v, \nu_v, 0) \right\}.$$

If  $F$  is a function field

$$M\left(\frac{\pi}{\log q}\right) = \sum B(\mu, \nu) E\left(\mu, \nu, \frac{\pi}{\log q}\right).$$

If

$$(16.1.7) \quad \int_{K_v} \int_{N_v} f_v(k^{-1} a n k) dn dk = 0$$

for all  $a$  in  $A_v = A_{F_v}$  then  $\omega(\gamma, f_v) = 0$  for all  $\gamma$ ,  $\theta(0, f_v) = 0$ , and

$$\text{trace } \rho(f_v, \mu_v, \nu_v, s) = 0$$

for all  $\mu_v, \nu_v$ , and  $s$ . In particular if (16.1.7) is satisfied for at least two  $v$  the expressions (iv) to (viii) vanish and the trace formula simplifies considerably.

We now apply this formula to the function

$$\Phi(g) = \left\{ \prod_{v \in S_1} \zeta_v(g_v) \right\} \left\{ \prod_{v \in S_2} \xi_v(g_v) \right\} f(\widehat{g}_S)$$

where  $f = f_1 * f_2$  with  $f_1$  and  $f_2$  in  $B$  is of the form

$$f(\widehat{g}_S) = \prod_{v \notin S} f_v(g_v).$$

Since  $S$  has at least two elements and the functions  $\zeta_v$  and  $\xi_v$  satisfy (16.1.7), only the expressions (i) to (iii) do not vanish identically. The expression (i) is now equal to

$$\left\{ \prod_{v \in S_1} \zeta_v(e) \right\} \left\{ \prod_{v \in S_2} d(\sigma_v) \right\} f(e)$$

We recall that  $d(\sigma_v) = d(\sigma'_v)$  if  $v$  is in  $S_2$ .

We may suppose that  $Q_2$  is equal to  $Q'_2$  and that  $Q'_1$  is a subset of  $Q_1$ . If  $E$  is in  $Q_1$  or  $Q_2$  and  $\gamma$  is in  $B_F = B_F(E)$  but not in  $Z_F$

$$\int_{B_{\mathbf{A}} \backslash G_{\mathbf{A}}} \Phi(g^{-1}\gamma g) \omega_B(g)$$

is equal to the product of

$$\left\{ \prod_{v \in S_1} \int_{B_v \backslash G_v} \zeta_v(g_v^{-1}\gamma g_v) \omega_B(v) \right\} \left\{ \prod_{v \in S_2} \int_{B_v \backslash G_v} \xi_v(g_v^{-1}\gamma g_v) \omega_B(v) \right\}$$

and

$$\int_{\widehat{B}_S \backslash \widehat{G}_S} f(g^{-1}\gamma g) \omega_B.$$

If  $v$  is in  $S$  and  $E \otimes_F F_v$  is not a field so that  $B_v$  is conjugate to  $A_v$ , the corresponding factor in the first of these two expressions vanishes. Thus the sum in (ii) need only be taken over  $Q'_1$ . If  $E$  is in  $Q'_1$  or  $Q_2$  the first of these two expressions is equal to

$$\prod_{v \in S} \frac{\chi_{\sigma_v}(\gamma^{-1})}{\text{measure } Z_v \backslash B_v}.$$

Thus, in the special case under consideration, (ii) is equal to (16.1.5) and (iii) is equal to (16.1.6) so that

$$\text{trace } \tau(f) - \left\{ \prod_{v \in S_1} \zeta_v(e) \right\} \left\{ \prod_{v \in S_2} d(\sigma_v) \right\} \text{measure}(Z_{\mathbf{A}} G_F \backslash G_{\mathbf{A}}) f(e)$$

is equal to

$$\text{trace } \tau'(f) - \left\{ \prod_{v \in S} d(\sigma'_v) \right\} \text{measure}(Z'_{\mathbf{A}} G'_F \backslash G'_{\mathbf{A}}) f(e).$$

We may take  $\eta$  to be trivial and apply Lemmas 16.1.1 and 16.1.2 to see that, in this case,

$$\text{trace } \tau(f) = \text{trace } \tau'(f)$$

and

$$\left\{ \prod_{v \in S} d(\sigma'_v) \right\} \text{measure}(Z'_{\mathbf{A}} G'_F \backslash G'_{\mathbf{A}})$$

is equal to

$$\left\{ \prod_{v \in S_1} \zeta_v(e) \right\} \left\{ \prod_{v \in S_2} d(\sigma_v) \right\} \text{measure}(Z_{\mathbf{A}} G_F \backslash G_{\mathbf{A}}).$$

Still taking  $\eta$  trivial we choose the  $\sigma'_v$  so that none of them are one-dimensional and conclude that

$$(16.1.8) \quad \text{measure}(Z'_\mathbf{A} G'_F \backslash G'_\mathbf{A}) = \text{measure}(Z_\mathbf{A} G_F \backslash G_\mathbf{A}).$$

Then we take exactly one of them to be one-dimensional and conclude that  $\zeta_v(e) = d(\sigma'_v)$ . Thus  $\zeta_v(e) = d(\sigma'_v)$  and

$$\text{trace } \tau(f) = \text{trace } \tau'(f)$$

in general.

The relation (16.1.8) is well-known. One can hope however that the proof of it just given can eventually be used to show that the Tamagawa numbers of two groups which differ only by an inner twisting are the same or at least differ only by an explicitly given factor. Since the method of [33] can probably be used to evaluate the Tamagawa numbers of quasi-split groups the problem of evaluating the Tamagawa numbers of reductive groups would then be solved. However a great deal of work on the representation theory of groups over local fields remains to be done before this suggestion can be carried out.

To complete our formal argument we need to sketch a proof of the trace formula itself. One must use a bootstrap method. The first step, which is all we shall discuss, is to prove it for some simple class of functions  $\Phi$ . We take  $\Phi$  of the form  $\Phi = f' * f''$  with

$$f'(g) = \prod_v f'_v(g_v)$$

and

$$f''(g) = \prod_v f''_v(g_v)$$

where  $f'_v$  and  $f''_v$  satisfy the five conditions on page 267. The function  $f_v$  is  $f'_v * f''_v$ .

Suppose  $\varphi$  is a  $K$ -finite compactly supported function in  $A(\eta)$ . For each purely imaginary  $s$  define  $\tilde{\varphi}(s)$  in  $\mathcal{L}$  by demanding that

$$\frac{1}{2c} \int_{G_F Z_\mathbf{A} \backslash G_\mathbf{A}} \varphi(g) \overline{E}(g, \varphi', s) \omega_0(g) = (\tilde{\varphi}(s), \varphi')$$

be valid for all  $\varphi'$  in  $\mathcal{L}$ . The map  $\varphi \rightarrow \tilde{\varphi}(s)$  extends to a continuous map of  $A(\eta)$  onto  $H$ ,  $\tilde{\varphi}(s)$  being the function in  $H$  corresponding to  $E\varphi$  in  $A_1(\eta)$ .

For each  $(\mu, \nu)$  in  $P$  choose an orthonormal basis  $\{\varphi_i(\mu, \nu)\}$  of  $\mathcal{B}(\mu, \nu)$ . We may suppose that any elementary idempotent in  $\mathcal{H}$  annihilates all but finitely many elements of this basis. If

$$\tilde{\varphi}(s) = \sum_{(\mu, \nu)} \sum_i a_i(\mu, \nu, s) \varphi_i(\mu, \nu)$$

then

$$a_i(\mu, \nu, s) = \frac{1}{2c} \int_{G_F Z_\mathbf{A} \backslash G_\mathbf{A}} \varphi(g) \overline{E}(g, \varphi_i(\mu, \nu), s) \omega_0(g).$$

Let

$$\rho(\Phi, s) T_S^{-1} \varphi_i(\mu, \nu) = \sum_j \rho_{ji}(\Phi, \mu, \nu, s) T_S^{-1} \varphi_j(\mu, \nu).$$

For all but finitely many  $\mu, \nu, i$  and  $j$  the functions  $\rho_{ji}(\Phi, \mu, \nu, s)$  vanish identically.  $E\rho(\Phi)\varphi$  is equal to

$$\lim_{T \rightarrow \infty} \sum_{\mu, \nu} \sum_{i, j} \frac{1}{4\pi c} \int_{-iT}^{iT} \rho_{ij}(\Phi, \mu, \nu, s) a_j(\mu, \nu, s) E(g, \varphi_i(\mu, \nu), s) d|s|.$$

A typical one of these integrals is equal to the integral over  $G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}}$  of the product of  $\varphi(g)$  and

$$\int_{-iT}^{iT} \rho_{ij}(\Phi, \mu, \nu, s) E(g, \varphi_i(\mu, \nu), s) \overline{E}(h, \varphi_j(\mu, \nu), s) d|s|.$$

Thus the kernel of  $E\rho(\Phi)$  is the sum over  $(\mu, \nu)$  and  $i, j$  of

$$\frac{1}{4\pi c} \int_{-i\infty}^{i\infty} \rho_{ij}(\Phi, \mu, \nu, s) E(g, \varphi_i(\mu, \nu), s) \overline{E}(h, \varphi_j(\mu, \nu), s) d|s|.$$

The kernel of  $\rho(\Phi)$  is

$$\Phi(g, h) = \sum_{Z_F \backslash G_F} \Phi(g^{-1} \gamma h).$$

To compute the trace of  $\rho(\Phi) - E\rho(\Phi)$  we integrate the difference of these two kernels over the diagonal.

The function  $\Phi(g, g)$  may be written as the sum of

$$(16.2.1) \quad \sum_{\delta \in P_F \backslash G_F} \sum_{\substack{\gamma \in N_F \\ \gamma \neq e}} \Phi(g^{-1} \delta^{-1} \gamma \delta g),$$

where  $P_F$  is the group of super-triangular matrices in  $G_F$ ,

$$(16.2.2) \quad \frac{1}{2} \sum_{\substack{\gamma \in Z_F \backslash A_F \\ \gamma \notin Z_F}} \sum_{\delta \in A_F \backslash G_F} \Phi(g^{-1} \delta^{-1} \gamma \delta g),$$

where  $A_F$  is the group of diagonal matrices in  $G_F$ ,

$$(16.2.3) \quad \frac{1}{2} \sum_{Q_1} \sum_{\substack{\gamma \in Z_F \backslash B_F \\ \gamma \notin Z_F}} \sum_{\delta \in B_F \backslash G_F} \Phi(g^{-1} \delta^{-1} \gamma \delta g)$$

and

$$(16.2.4) \quad \sum_{Q_2} \sum_{\substack{\gamma \in Z_F \backslash B_F \\ \gamma \notin Z_F}} \sum_{\delta \in B_F \backslash G_F} \Phi(g^{-1} \delta^{-1} \gamma \delta g)$$

together with

$$(16.2.5) \quad \Phi(e).$$

The constant  $\Phi(e)$  can be integrated over  $G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}}$  immediately to give the first term of the trace formula. The standard manipulations convert (16.2.3) and (16.2.4) into the second and third terms of the trace formula.

The expressions (16.2.1) and (16.2.2) have to be treated in a more subtle fashion. We can choose a constant  $e_1 > 0$  so that if

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} k,$$

with  $x$  in  $\mathbf{A}$ ,  $\alpha$  and  $\beta$  in  $I$  such that  $\left|\frac{\alpha}{\beta}\right| \geq c_1$ , and  $k$  in  $K$ , and if

$$\gamma g = \begin{pmatrix} 1 & x' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha' & 0 \\ 0 & \beta' \end{pmatrix} k',$$

with  $\gamma$  in  $G_F$ ,  $x'$  in  $\mathbf{A}$ ,  $\alpha'$  and  $\beta'$  in  $I$  such that  $\left|\frac{\alpha'}{\beta'}\right| \geq c_1$ , and  $k'$  in  $K$ , then  $\gamma$  belongs to  $P_F$ . Let  $\chi$  be the characteristic function of

$$\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} k \mid \left|\frac{\alpha}{\beta}\right| \geq c_1 \right\}.$$

The expression (16.2.2) is the sum of

$$\frac{1}{2} \sum_{\delta \in P_F \backslash G_F} \sum_{\substack{\gamma \in Z_F \backslash P_F \\ \gamma \notin Z_F N_F}} \Phi(g^{-1} \delta^{-1} \gamma \delta g) (\chi(\delta g) + \chi(\epsilon(\gamma) \delta g))$$

and

$$\frac{1}{2} \sum_{\delta \in P_F \backslash G_F} \sum_{\substack{\gamma \in Z_F \backslash P_F \\ \gamma \notin Z_F N_F}} \Phi(g^{-1} \delta^{-1} \gamma \delta g) (1 - \chi(\delta g) - \chi(\epsilon(\gamma) \delta g)).$$

Here  $\epsilon(\gamma)$  is any element of  $G_F$  not in  $P_F$  such that

$$\epsilon(\gamma) \gamma \epsilon^{-1}(\gamma) \in P_F.$$

There is always at least one such  $\epsilon(\gamma)$ . The integral of the second sum over  $G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}}$  converges. It is equal to

$$\frac{1}{2} \int_{Z_{\mathbf{A}} P_F \backslash G_{\mathbf{A}}} \sum_{\substack{\gamma \in Z_F \backslash P_F \\ \gamma \notin Z_F N_F}} \Phi(g^{-1} \gamma g) (1 - \chi(g) - \chi(\epsilon(\gamma) g)) \omega_0(g).$$

Every  $\gamma$  occurring in the sum can be written as  $\delta^{-1} \gamma_0 \delta$  with  $\gamma_0$  in  $A_F$  and  $\delta$  in  $P_F$ . Then

$$(\delta^{-1} \epsilon(\gamma_0) \delta) (\delta^{-1} \gamma_0 \delta) (\delta^{-1} \epsilon(\gamma_0) \delta)^{-1} = \delta^{-1} (\epsilon(\gamma_0) \gamma_0 \epsilon^{-1}(\gamma_0)) \delta,$$

so that we can take  $\epsilon(\gamma) = \delta^{-1} \epsilon(\gamma_0) \delta$ . We take

$$\epsilon(\gamma_0) = w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since  $\chi(\delta g) = \chi(g)$  and

$$\chi(\delta^{-1} w \delta g) = \chi(w \delta g)$$

the integrand is

$$\sum_{\substack{\gamma \in Z_F \backslash A_F \\ \gamma \notin Z_F}} \sum_{\delta \in A_F \backslash P_F} \Phi(g^{-1} \delta^{-1} \gamma \delta g) (1 - \chi(\delta g) - \chi(w \delta g)).$$

The integral itself is equal to

$$\frac{1}{2} \sum_{\substack{\gamma \in Z_F \backslash A_F \\ \gamma \notin Z_F}} \int_{Z_{\mathbf{A}} A_F \backslash G_{\mathbf{A}}} \Phi(g^{-1} \gamma g) (1 - \chi(g) - \chi(w g)) \omega_0(g).$$

All but a finite number of the integrals in this sum are 0.



It is convenient to write each of them in another form. If

$$g = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} nk$$

then  $\chi(g)$  is 1 if  $\left|\frac{\alpha}{\beta}\right| \geq c_1$  and is 0 if  $\left|\frac{\alpha}{\beta}\right| < c_1$ . If

$$wn = \begin{pmatrix} \alpha' & 0 \\ 0 & \beta' \end{pmatrix} n'k'$$

and  $\lambda(n)$  is  $\left|\frac{\alpha'}{\beta'}\right|$  then  $\chi(wg)$  is 1 if  $\left|\frac{\alpha}{\beta}\right| \leq \frac{\lambda(n)}{c_1}$  and is 0 if  $\left|\frac{\alpha}{\beta}\right| > \frac{\lambda(n)}{c_1}$ . It is easily seen that  $\lambda(n) \leq 1$ . Thus if  $c_1 > 1$ , as we may suppose, one of  $\chi(g)$  and  $\chi(wg)$  is always 0. The integral

$$\int_{Z_{\mathbf{A}} A_F \backslash G_{\mathbf{A}}} \Phi(g^{-1} \gamma g) (1 - \chi(g) - \chi(wg)) \omega_0(g)$$

is equal to

$$c \int_{N_{\mathbf{A}}} \int_K \Phi(k^{-1} n^{-1} \gamma n k) (2 \log c_1 - \log \lambda(n)) \, dn \, dk$$

which we write as the sum of

$$(16.2.6) \quad 2c \log c_1 \int_{N_{\mathbf{A}}} \int_K \Phi(k^{-1} n^{-1} \gamma n k) \, dn \, dk$$

and

$$- \sum_v c \int_{N_{\mathbf{A}}} \int_K \Phi(k^{-1} n^{-1} \gamma n k) \log \lambda(n_v) \, dn \, dk.$$

If we express each of the integrals in the second expression as a product of local integrals we obtain the fourth term of the trace formula. All but a finite number of the integrals are 0 so that the sum is really finite. We will return to (16.2.6) later. If  $F$  is a function field over  $\mathbf{F}_q$  it is best to take  $c_1$  to be a power of  $q^n$  of  $q$ . Then  $2 \log c_1$  is replaced by  $2n - 1$ .

The expression (16.2.1) is the sum of

$$\sum_{\delta \in P_F \backslash G_F} \sum_{\substack{\gamma \in N_F \\ \gamma \neq e}} \Phi(g^{-1} \delta^{-1} \gamma \delta g) \chi(\delta g)$$

and

$$\sum_{\delta \in P_F \backslash G_F} \sum_{\substack{\gamma \in N_F \\ \gamma \neq e}} \Phi(g^{-1} \delta^{-1} \gamma \delta g) (1 - \chi(\delta g)).$$

The integral of the second expression over  $G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}}$  converges. It is equal to

$$\int_{P_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}}} \sum_{\substack{\gamma \in N_F \\ \gamma \neq e}} \Phi(g^{-1} \gamma g) (1 - \chi(g)) \omega_0(g).$$

If

$$n_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

the integrand is equal to

$$\sum_{N_F Z_F \backslash P_F} \Phi(g^{-1} \delta^{-1} n_0 \delta g) (1 - \chi(\delta g)),$$

so that the integral itself is equal to

$$\int_{N_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}}} \Phi(g^{-1} n_0 g) (1 - \chi(g)) \omega_0(g)$$

which is

$$c \int_{Z_{\mathbf{A}} \backslash A_{\mathbf{A}}} \int_K \Phi(k^{-1} a^{-1} n_0 a k) (1 - \chi(a)) \left| \frac{\alpha}{\beta} \right|^{-1} da dk$$

if

$$a = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

The integrand vanishes outside of a compact set. Thus the integral is the limit as  $s$  approaches 0 from above of

$$c \int_{Z_{\mathbf{A}} \backslash A_{\mathbf{A}}} \int_K \Phi(k^{-1} a^{-1} n_0 a k) (1 - \chi(a)) \left| \frac{\alpha}{\beta} \right|^{-1-s} da dk,$$

which is the difference of

$$c \int_{Z_{\mathbf{A}} \backslash A_{\mathbf{A}}} \int_K \Phi(k^{-1} a^{-1} n_0 a k) \left| \frac{\alpha}{\beta} \right|^{-1-s} da dk$$

and

$$c \int_{Z_{\mathbf{A}} \backslash A_{\mathbf{A}}} \int_K \Phi(k^{-1} a^{-1} n_0 a k) \left| \frac{\alpha}{\beta} \right|^{-1-s} \chi(a) da dk.$$

The first of these two expressions is equal to

$$c \left\{ \prod_v \int_{Z_v \backslash A_v} \int_{K_v} f_v(k_v^{-1} a_v^{-1} n_0 a_v k_v) \left| \frac{\alpha_v}{\beta_v} \right|^{-1-s} da_v dk_v \right\}$$

which is

$$(16.2.7) \quad cL(1+s, 1_F) \left\{ \prod_v \theta(s, f_v) \right\}.$$

Observe that if  $v$  is non-archimedean and  $f_v$  is 0 outside of  $Z_v K_v$  and is 1 on the elements of  $Z_v K_v$  of determinant 1 then

$$\int_{Z_v \backslash A_v} \int_{K_v} f_v(k_v^{-1} a_v^{-1} n_0 a_v k_v) \left| \frac{\alpha_v}{\beta_v} \right|^{-1-s} da_v dk_v$$

is the product of the measure of

$$\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in Z_v \backslash A_v \mid |\alpha| = |\beta| \right\}$$

and

$$\sum_{n=0}^{\infty} |\varpi_v^n|^{1+s} = L(1+s, 1_v),$$

so that

$$\prod_v \theta(s, f_v) = \theta(s, \Phi)$$

is analytic for  $\operatorname{Re} s > -1$  and its derivative at 0 is

$$\sum_v \theta'(s, f_v) \left\{ \prod_{w \neq v} \theta(s, f_w) \right\}.$$

The function (16.2.7) has a simple pole at  $s = 0$ . The constant term in its Laurent expansion is

$$c \left[ \lambda_0 \prod_v \theta(0, f_v) + \lambda_{-1} \left\{ \sum_v \theta'(0, f_v) \prod_{w \neq v} \theta(0, f_w) \right\} \right],$$

which is the fifth term of the trace formula.

The expression

$$c \int_{Z_{\mathbf{A}} \backslash A_{\mathbf{A}}} \int_K \Phi(k^{-1} a^{-1} n_0 a k) \left| \frac{\alpha}{\beta} \right|^{-1-s} \chi(a) da dk$$

is equal to

$$c \int_{Z_{\mathbf{A}} A_F \backslash A_{\mathbf{A}}} \int_K \sum_{\substack{\gamma \in N_F \\ \gamma \neq e}} \Phi(k^{-1} a^{-1} \gamma a k) \left| \frac{\alpha}{\beta} \right|^{-1-s} \chi(a) da dk.$$

Choose a non-trivial character  $\psi$  of  $F \backslash \mathbf{A}$  and let

$$\Psi(y, g) = \int_{\mathbf{A}} \Phi \left( g^{-1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \psi(xy) dx.$$

Then

$$\Psi(y, ag) = \left| \frac{\beta}{\alpha} \right|^{-1} \Psi \left( \frac{\alpha}{\beta} y, g \right).$$

Moreover by the Poisson summation formula

$$\sum_{\substack{\gamma \in N_F \\ \gamma \neq e}} \Phi(k^{-1} a^{-1} \gamma a k)$$

is equal to

$$\sum_{y \neq 0} \left| \frac{\beta}{\alpha} \right|^{-1} \Psi \left( \frac{\alpha}{\beta} y, k \right) + \left| \frac{\beta}{\alpha} \right|^{-1} \Psi(0, k) - \Phi(e).$$

The integral

$$c \int_{Z_{\mathbf{A}} A_F \backslash A_{\mathbf{A}}} \int_K \left| \frac{\alpha}{\beta} \right|^{-s} \chi(a) \left\{ \sum_{y \neq 0} \Psi \left( \frac{\alpha}{\beta} y, k \right) \right\} da dk$$

is a holomorphic function of  $s$  and its value at  $s = 0$  approaches 0 as  $c_1$  approaches  $\infty$ . Since we shall eventually let  $c_1$  approach  $\infty$  it contributes nothing to the trace formula. If  $F$  is a number field

$$c \int_{Z_{\mathbf{A}} A_F \backslash A_{\mathbf{A}}} \int_K \Phi(e) \left| \frac{\alpha}{\beta} \right|^{-1-s} \chi(a) da dk$$

is a multiple of

$$\frac{1}{1+s} \cdot \frac{1}{c_1^{1+s}}$$

which is defined at  $s = 0$ . Its value there approaches 0 as  $c_1$  approaches  $\infty$ . Finally

$$c \int_{Z_{\mathbf{A}} A_F \backslash A_{\mathbf{A}}} \int_K \Psi(0, k) \left| \frac{\alpha}{\beta} \right|^{-s} \chi(a) da dk$$

is equal to

$$\frac{c}{s c_1^s} \int_K \Psi(0, k) dk.$$

The pole of this function at  $s = 0$  must cancel that of (16.2.7). Consequently

$$\int_K \Psi(0, k) dk = \lambda_{-1} \theta(0, \Phi).$$

The constant term in its Laurent expansion about 0 is

$$-c \log c_1 \int_K \Psi(0, k) dk.$$

Not this expression but its negative

$$(16.2.8) \quad c \log c_1 \int_K \Psi(0, k) dk$$

enters into the integral of the kernel of  $\rho(\Phi) - E\rho(\Phi)$  over the diagonal. If  $F$  is a function field  $\frac{c}{s c_1^s}$  is to be replaced by

$$\frac{c q^{-ns}}{1 - q^{-s}}$$

and  $\log c_1$  by  $n - \frac{1}{2}$ .

The Poisson summation formula can be used to simplify the remaining part of (16.2.1). We recall that it is

$$\sum_{\delta \in P_F \backslash G_F} \sum_{\substack{\gamma \in N_F \\ \gamma \neq e}} \Phi(g^{-1} \delta^{-1} \gamma \delta g) \chi(\delta g).$$

We subtract from this

$$\sum_{\delta \in P_F \backslash G_F} \Psi(0, \delta g) \chi(\delta g)$$

to obtain the difference between

$$\sum_{\delta \in P_F \backslash G_F} \sum_{y \neq 0} \Psi(y, \delta g) \chi(\delta g)$$

and

$$\sum_{\delta \in P_F \backslash G_F} \Phi(e) \chi(\delta g).$$

The integrals of both these functions over  $Z_{\mathbf{A}} G_F \backslash G_{\mathbf{A}}$  converge and approach 0 as  $c_1$  approaches  $\infty$ . They may be ignored.

The remaining part of (16.2.2) is the sum of

$$\frac{1}{2} \sum_{\delta \in P_F \backslash G_F} \sum_{\substack{\gamma \in Z_F \backslash P_F \\ \gamma \notin Z_F N_F}} \Phi(g^{-1} \delta^{-1} \gamma \delta g) \chi(\delta g)$$

and

$$\frac{1}{2} \sum_{\delta \in P_F \backslash G_F} \sum_{\substack{\gamma \in Z_F \backslash P_F \\ \gamma \notin Z_F N_F}} \Phi(g^{-1} \delta^{-1} \gamma \delta g) \chi(\epsilon(\gamma) \delta g).$$

These two sums may be written as

$$\frac{1}{2} \sum_{\substack{\gamma \in Z_F \backslash A_F \\ \gamma \notin Z_F}} \sum_{A_F \backslash G_F} \Phi(g^{-1} \delta^{-1} \gamma \delta g) \chi(\delta g)$$

and

$$\frac{1}{2} \sum_{\substack{\gamma \in Z_F \backslash A_F \\ \gamma \notin Z_F}} \sum_{A_F \backslash G_F} \Phi(g^{-1} \delta^{-1} \gamma \delta g) \chi(w \delta g).$$

Replacing  $\delta$  by  $w^{-1} \delta$  in the second sum we see that the two expressions are equal. Their sum is equal to twice the first which we write as

$$\sum_{\substack{\gamma_1 \in Z_F \backslash A_F \\ \gamma_1 \notin Z_F}} \sum_{\delta \in P_F \backslash G_F} \sum_{\gamma_2 \in N_F} \Phi(g^{-1} \delta^{-1} \gamma_1 \gamma_2 \delta g) \chi(\delta g).$$

For a given  $\Phi$  all but finitely many of the sums

$$(16.2.9) \quad \sum_{\delta \in P_F \backslash G_F} \sum_{\gamma_2 \in N_F} \Phi(g^{-1} \delta^{-1} \gamma_1 \gamma_2 \delta g) \chi(\delta g)$$

are zero. Set

$$\Psi(y, \gamma_1, g) = \int_{\mathbf{A}} \Phi \left( g^{-1} \gamma_1 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \psi(xy) dx.$$

The expression (16.2.9) is the sum of

$$\sum_{\delta \in P_F \backslash G_F} \sum_{y \neq 0} \Psi(y, \gamma_1, \delta g) \chi(\delta g)$$

and

$$\sum_{\delta \in P_F \backslash G_F} \Psi(0, \gamma_1, \delta g) \chi(\delta g).$$

The first of these two expressions is integrable on  $G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}}$  and its integral approaches 0 as  $c_1$  approaches  $\infty$ .

Since  $\Psi(0, g) = \Psi(0, e, g)$  we have expressed  $\Phi(g, g)$  as the sum of

$$(16.2.10) \quad \sum_{\delta \in P_F \backslash G_F} \sum_{\gamma \in Z_F \backslash A_F} \Psi(0, \gamma, \delta g) \chi(\delta g)$$

and a function which can be integrated over  $G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}}$  to give the first five terms of the trace formula, the sum of (16.2.8) and one-half of the sum over  $\gamma$  in  $Z_F \backslash A_F$  but not in  $Z_F$  of (16.2.6) which is

$$(16.2.11) \quad c \log c_1 \sum_{\gamma \in Z_F \backslash A_F} \int_{N_{\mathbf{A}}} \int_K \Phi(k^{-1} \gamma n k) dn dk,$$

and an expression which goes to 0 as  $c_1$  approaches  $\infty$ .

Now we discuss the kernel of  $E\rho(\Phi)$  in the same way. Set  $H(g; \mu, \nu, i, j, s)$  equal to

$$\rho_{ij}(\Phi, \mu, \nu, s) E(g, \varphi_i(\mu, \nu), s) \bar{E}(g, \varphi_j(\mu, \nu), s).$$

On the diagonal the kernel of  $E\rho(\Phi)$  is equal to

$$\sum_{\mu, \nu} \sum_{i, j} \frac{1}{4\pi c} \int_{-i\infty}^{i\infty} H(g; \mu, \nu, i, j, s) d|s|$$

if  $F$  is a number field and to

$$\sum_{\mu, \nu} \sum_{i, j} \frac{\log q}{4\pi c} \int_0^{\frac{2\pi}{\log q}} H(g; \mu, \nu, i, j, s) d|s|$$

if  $F$  is a function field. We set  $E_1(g, \varphi, s)$  equal to

$$\sum_{P_F \backslash G_F} \{T_s^{-1} \varphi(\delta g) + M(s) T_s^{-1} \varphi(\delta g)\} \chi(\delta g)$$

and let

$$E_2(g, \varphi, s) = E(g, \varphi, s) - E_1(g, \varphi, s).$$

If, for  $m = 1, 2, n = 1, 2$ ,  $H_{mn}(g; \mu, \nu, i, j, s)$  is

$$\rho_{ij}(\Phi, \mu, \nu, s) E_m(g, \varphi_i(\mu, \nu), s) \bar{E}_n(g, \varphi_j(\mu, \nu), s)$$

and  $\Phi_{mn}(g)$  is, at least when  $F$  is a number field,

$$\sum_{\mu, \nu} \sum_{i, j} \frac{1}{4\pi c} \int_{-i\infty}^{i\infty} H_{m,n}(g; \mu, \nu, i, j, s) d|s|,$$

the kernel of  $E\rho(\Phi)$  is

$$\sum_{m=1}^n \sum_{n=1}^2 \Phi_{mn}(g)$$

on the diagonal.

If  $m$  or  $n$  is 2

$$\int_{G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}}} \Phi_{mn}(g) \omega_0(g)$$

is equal to

$$(16.2.12) \quad \frac{1}{4\pi c} \int_{-i\infty}^{i\infty} \sum_{\mu, \nu} \sum_{i, j} \left\{ \int_{G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}}} H_{m,n}(g, \mu, \nu, i, j, s) \omega_0(g) \right\} d|s|.$$

Take first  $m = n = 2$ . If  $F$  is a number field a formula for the inner product

$$\int_{G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}}} E_2(g, \varphi_1, s) \bar{E}_2(g, \varphi_2, s) \omega_0(g)$$

can be inferred from the formulae of [26] and [27]. The result is the sum of

$$c \lim_{t \searrow 0} \frac{1}{2t} \left\{ c_1^{2t}(\varphi_1, \varphi_2) - c_1^{-2t}(N(t+s)\varphi_1, N(t+s)\varphi_2) \right\},$$

where  $N(t+s) = T_{t+s}M(t+s)T_{t+s}^{-1}$ , and

$$c \lim_{t \searrow 0} \frac{1}{2s} \left\{ c_1^{2s}(\varphi_1, N(t+s)\varphi_2) - c_1^{-2s}(N(t+s)\varphi_1, \varphi_2) \right\}.$$

The second expression is equal to

$$\frac{c}{2s} \left\{ c_1^{2s}(\varphi_1, N(s)\varphi_2) - c_1^{-2s}(N(s)\varphi_1, \varphi_2) \right\}.$$

The first is the sum of

$$2c \log c_1(\varphi_1, \varphi_2)$$

and

$$-\frac{c}{2} \left\{ (N^{-1}(s)N'(s)\varphi_1, \varphi_2) + (\varphi_1, N^{-1}(s)N'(s)\varphi_2) \right\}.$$

If  $F$  is a function field over  $\mathbf{F}_q$  and  $c_1 = q^n$  the inner product is the sum of

$$c \log q \left\{ \frac{1 - q^s + q^{-s}}{1 - q^{-2s}} (\varphi_1, N(s)\varphi_2) q^{2(n-1)s} + \frac{1 - q^{-s} + q^s}{1 - q^{2s}} (N(s)\varphi_1, \varphi_2) q^{-2(n-1)s} \right\}$$

and

$$(2n-1)c(\varphi_1, \varphi_2)$$

and

$$-\frac{c}{2} \left\{ (N^{-1}(s)N'(s)\varphi_1, \varphi_2) + (\varphi_1, N^{-1}(s)N'(s)\varphi_2) \right\}.$$

Certainly

$$\sum_{\mu, \nu} \sum_{i, j} \rho_{ij}(\Phi, \mu, \nu, s) (\varphi_i(\mu, \nu), \varphi_j(\mu, \nu)) = \text{trace } \rho(\Phi, s)$$

which equals

$$\sum_{\mu, \nu} c \int_{N_{\mathbf{A}}} \int_{Z_{\mathbf{A}} \backslash A_{\mathbf{A}}} \int_K \Phi(k^{-1}ank) \mu(\alpha) \nu(\beta) \left| \frac{\alpha}{\beta} \right|^{\frac{s+1}{2}} dn da dk$$

or

$$\sum_{\mu, \nu} c \int_{N_{\mathbf{A}}} \int_{Z_{\mathbf{A}} A_F \backslash A_{\mathbf{A}}} \int_K \sum_{\gamma \in Z_F \backslash A_F} \Phi(k^{-1}a\gamma nk) \mu(\alpha) \nu(\beta) \left| \frac{\alpha}{\beta} \right|^{\frac{s+1}{2}} dn da dk.$$

Thus if  $H$  is the set of all

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

in  $Z_{\mathbf{A}} A_F \backslash A_{\mathbf{A}}$  for which  $|\alpha| = |\beta|$

$$(16.2.13) \quad \frac{1}{4\pi c} \int_{-i\infty}^{i\infty} \text{trace } \rho(\Phi, s) d|s|$$

is equal to

$$\frac{1}{2} \sum_{\mu, \nu} \int_H \int_{N_{\mathbf{A}}} \int_K \sum_{\gamma} \Phi(k^{-1}a\gamma nk) \mu(\alpha) \nu(\beta) dn dk da$$

which is

$$\frac{1}{2} \int_{N_A} \int_K \Phi(k^{-1} \gamma n k) dn dk.$$

When multiplied by  $2c \log c_1$  the effect of this is to cancel the term (16.2.11). If  $F$  is a function field (16.2.13) is said to be replaced by

$$\frac{\log q}{4\pi c} \int_0^{\frac{2\pi}{\log q}} \text{trace } \rho(\Phi, s) d|s|$$

but the conclusion is the same.

The expression

$$\sum_{\mu, \nu} \sum_{i, j} \rho_{ij}(\Phi, \mu, \nu, s) (\varphi_i(\mu, \nu), N(s) \varphi_j(\mu, \nu))$$

is equal to

$$\text{trace } M^{-1}(s) \rho(\Phi, s)$$

when  $s$  is purely imaginary and

$$\sum_{\mu, \nu} \sum_{i, j} \rho_{ij}(\Phi, \mu, \nu, s) (N(s) \varphi_i(\mu, \nu), \varphi_j(\mu, \nu))$$

is equal to

$$\text{trace } M(s) \rho(\Phi, s).$$

Since  $M(0) = M^{-1}(0)$

$$\lim_{c_1 \rightarrow \infty} \frac{1}{8\pi} \int_{-i\infty}^{i\infty} \frac{1}{s} \{ c_1^{2s} \text{trace } M^{-1}(s) \rho(\Phi, s) - c_1^{-2s} \text{trace } M(s) \rho(\Phi, s) \} d|s|$$

is equal to

$$\frac{1}{4} \text{trace } M(0) \rho(\Phi, 0).$$

When multiplied by  $-1$  this is the sixth term of the trace formula. For a function field it is to be replaced by

$$\frac{\log q}{4} \left\{ \text{trace } M(0) \rho(\Phi, 0) + \text{trace } M\left(\frac{\pi}{\log q}\right) \rho\left(\Phi, \frac{\pi}{\log q}\right) \right\}.$$

When  $s$  is purely imaginary

$$(N^{-1}(s) N'(s) \varphi_1, \varphi_2) = (\varphi_1, N^{-1}(s) N'(s) \varphi_2).$$

Moreover

$$\sum_{\mu, \nu} \sum_{i, j} \rho_{ij}(\Phi, \mu, \nu, s) (N^{-1}(s) N'(s) \varphi_i(\mu, \nu), \varphi_j(\mu, \nu))$$

is equal to

$$\text{trace } M^{-1}(s) M'(s) \rho(\Phi, s).$$

Thus

$$\frac{1}{4\pi} \int_{-i\infty}^{i\infty} \text{trace } M^{-1}(s) M'(s) \rho(\Phi, s) d|s|$$

is to be added to the trace formula. It gives the seventh and eighth terms.



Next we consider (16.2.12) when  $m = 2$  and  $n = 1$ . If  $\varphi'_2 = T_s^{-1}\varphi_2$  and  $\varphi''_2 = M(s)T_s^{-1}\varphi_2$  the integral

$$(16.2.14) \quad \int_{G_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}}} E_2(g, \varphi_1, s) \overline{E}_1(g, \varphi_2, s) \omega_0(g)$$

is the sum of

$$\int_{P_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}}} E_2(g, \varphi_1, s) \overline{\varphi}'_2(g) \chi(g) \omega_0(g)$$

and

$$\int_{P_F Z_{\mathbf{A}} \backslash G_{\mathbf{A}}} E_2(g, \varphi_1, s) \overline{\varphi}''_2(g) \chi(g) \omega_0(g).$$

Since  $\varphi'_2$ ,  $\varphi''_2$  and  $\chi$  are all functions on  $Z_{\mathbf{A}} N_{\mathbf{A}} P_F \backslash G_{\mathbf{A}}$  while, as is known,

$$\chi(g) \int_{N_{\mathbf{A}}} E_2(ng, \varphi_1, s) dn = 0$$

when  $c_1$  is sufficiently large, the integral (16.2.14) is 0. Thus (16.2.12) is 0 when  $m = 2$  and  $n = 1$  and also when  $m = 1$  and  $n = 2$ .

Set

$$F(g, \varphi, s) = T_s^{-1}\varphi(g) + M(s)T_s^{-1}\varphi(g)$$

and set  $H_0(g, \mu, \nu, i, j, s)$  equal to

$$\rho_{ij}(\Phi, \mu, \nu, s) F(g, \varphi_i(\mu, \nu), s) \overline{F}(g, \varphi_j(\mu, \nu), s) \chi(g).$$

If  $c_1$  is so large that  $\chi(\delta_1 g) \chi(\delta_2 g) = 0$  when  $\delta_1$  and  $\delta_2$  do not belong to the same coset of  $P_F$  the function  $\Phi_{1,1}(g)$  is equal to

$$\sum_{\mu, \nu} \sum_{i, j} \sum_{P_F \backslash G_F} \frac{1}{4\pi c} \int_{-i\infty}^{i\infty} H_0(\delta g, \mu, \nu, i, j, s) d|s|.$$

If  $\varphi'_i(g, \mu, \nu)$  is the value of  $T_s^{-1}\varphi_i(\mu, \nu)$  at  $g$  then

$$\sum_{i, j} \rho_{ij}(\Phi, \mu, \nu, s) \varphi'_i(h, \mu, \nu) \overline{\varphi}'_j(g, \mu, \nu)$$

is the kernel of  $\rho(\Phi, \mu, \nu, s)$  which is

$$c \int_{N_{\mathbf{A}}} \int_{Z_{\mathbf{A}} \backslash A_{\mathbf{A}}} \Phi(g^{-1} a n h) \left| \frac{\alpha}{\beta} \right|^{\frac{s+1}{2}} \mu(\alpha) \nu(\beta) dn da.$$

If we set  $h = g$ , divide by  $4\pi c$ , integrate from  $-i\infty$  to  $i\infty$ , and then sum over  $\mu$  and  $\nu$  we obtain

$$\frac{1}{2} \sum_{\gamma \in Z_F \backslash A_F} \Psi(0, \gamma, g).$$

If  $\varphi''_i(g, \mu, \nu)$  is the value of  $M(s)T_s^{-1}\varphi_i(\mu, \nu)$  at  $g$

$$\sum_{i, j} \rho_{ij}(\Phi, \mu, \nu, s) \varphi''_i(h, \mu, \nu) \overline{\varphi}''_j(g, \mu, \nu)$$

is the kernel of

$$M(\mu, \nu, s) \rho(\Phi, \mu, \nu, s) M(\nu, \mu, -s) = \rho(\Phi, \nu, \mu, -s).$$

Thus  $\Phi_{1,1}(g)$  is the sum of

$$(16.2.15) \quad \sum_{\delta \in P_F \backslash G_F} \sum_{\gamma \in Z_F \backslash A_F} \Psi(0, \gamma, \delta g) \chi(\delta g)$$

and

$$\sum_{\mu, \nu} \sum_{i, j} \sum_{P_F \backslash G_F} \frac{\chi(\delta g)}{4\pi c} \int_{-i\infty}^{i\infty} \{H_1(\delta g, \mu, \nu, i, j, s) + H_2(\delta g, \mu, \nu, i, j, s)\} d|s|$$

where  $H_1(g, \mu, \nu, i, j, s)$  is

$$\rho_{ij}(\Phi, \mu, \nu, s) \varphi'_i(g, \mu, \nu) \overline{\varphi''_j}(g, \mu, \nu)$$

and  $H_2(g, \mu, \nu, i, j, s)$  is

$$\rho_{ij}(\Phi, \mu, \nu, s) \varphi''_i(g, \mu, \nu) \overline{\varphi'_j}(g, \mu, \nu).$$

The expression (16.2.15) cancels (16.2.10). If  $g = nak$  with

$$a = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},$$

$H_1(g, \mu, \nu, i, j, s)$  is equal to

$$\rho_{ij}(\Phi, \mu, \nu, s) \mu \left( \frac{\alpha}{\beta} \right) \nu \left( \frac{\beta}{\alpha} \right) \left| \frac{\alpha}{\beta} \right|^{s+1} \varphi'_i(k) \overline{\varphi''_j}(k).$$

The functions  $\rho_{ij}(\Phi, \mu, \nu, s)$  are infinitely differentiable on the imaginary axis. Thus

$$\frac{1}{4\pi c} \int_{-i\infty}^{i\infty} H_1(g, \mu, \nu, i, j, s) d|s|$$

is  $O\left(\left|\frac{\alpha}{\beta}\right|^M\right)$  as  $\left|\frac{\alpha}{\beta}\right| \rightarrow \infty$  for any real  $M$ . Thus if this expression is multiplied by  $\chi(g)$  and averaged over  $P_F \backslash G_F$  the result is integrable on  $Z_{\mathbf{A}} G_F \backslash G_{\mathbf{A}}$  and its integral approaches 0 as  $c_1$  approaches  $\infty$ . Thus it contributes nothing to the trace. Nor do the analogous integrals for  $H_2(g, \mu, \nu, i, j, s)$ .

## References for Chapter III

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The theorem of §16 can still be stated and proved if  $M$  is replaced by a quaternion algebra which splits everywhere that  $M'$  does. The proof is in fact rather easier. However these apparently more general theorems are immediate consequences of the proof of the original theorem. Theorems very similar to that of §16 and its extensions have been proved by Shimizu. Our methods differ little from his.
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