SPECTRA OF METRIC GRAPHS AND SUMMATION FORMULAE

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JOINT WORK WITH P. KURASOV.

X A COMPACT RIEMANNIAN MANIFOLD

Δ THE LAPLACIAN ON FUNCTIONS ON X.

SPECTRUM: $\Delta\phi + k^2\phi = 0$

$$\mathrm{Spec}(X) = \{k\} \subset \mathbb{R} \quad \text{DISCRETE}.$$

• CHAZARAIN; DUISTERMAAT/GUILLEMIN DETERMINE THE SINGULAR SUPPORT OF THE TEMPERED DISTRIBUTION

$$\widehat{\mu_X}(t) = \text{TRACE}\left(2\cos(\sqrt{\Delta}\,t)\right); \; \mu_X := \sum_{k \in \mathrm{spec}(X)} \delta_k .$$

IN TERMS OF THE FIXED SETS OF THE GEODESIC FLOW ON $T_1^*(X)$ AT TIME t.

• IF X HAS A BOUNDARY OR IS SINGULAR OR IN THE CASE OF INFINITE VOLUME (WITH POLES REPLACING EIGENVALUES) THE ANALYSIS OF THE PROPOGATION OF SINGULARITIES IS MUCH MORE SUBTLE. IT WAS CARRIED OUT BY GUILLEMIN/MELROSE AND MELROSE...

"MELROSE TRACE FORMULA"

"MELROSE POISSON SUM FORMULA".

EXAMPLE $X = S^1 = \mathbb{R}/\mathbb{Z}$ WITH ARC LENGTH

$\mathrm{spec}(X) = \mathbb{Z}$; $\phi_m(x) = e^{2\pi i m x}$.

SUMMATION FORMULA IS THE CLASSICAL POISSON SUM

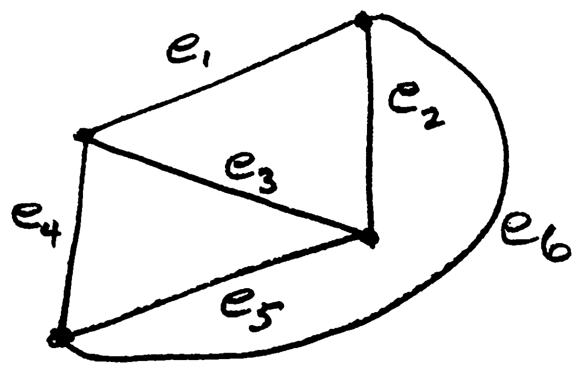
$$\widehat{\sum_{k\in\mathbb{Z}} \delta_k} = \sum_{m\in\mathbb{Z}} \delta_m \text{ ;}$$ ARITHMETIC PROGRESSIONS.

IN GENERAL IT IS RARE THAT $\widehat{\mu_X}$ IS A SUM OF A DISCRETE SET OF POINT MASSES ; WHAT IS CALLED A "CRYTALLINE MEASURE" (MEYER).

• SELBERG'S TRACE FORMULA FOR LOCALLY SYMMETRIC X's GIVES THE FULL DISTRIBUTION $\widehat{\mu_X}$ EXPLICITLY ; THE RIEMANN - GUINAND EXPLICIT FORMULA IN THE THEORY OF ZETA FUNCTIONS GIVES SUCH A CRYTALLINE LIKE STRUCTURE IF "RH" HOLDS.

• WE STICK TO X ONE DIMENSIONAL AND ALLOW IT TO HAVE A FINITE NUMBER OF POINT SINGULARITIES.

METRIC OR QUANTUM GRAPHS:



G COMBINATORIAL CONNECTED GRAPH
N EDGES e_j
M VERTICES v_k

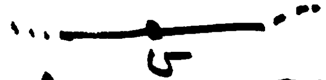
EQUIP THE EDGES WITH LENGTHS l_j, $j = 1, 2, \ldots N$ TO GET A METRIC GRAPH X WHICH IS SMOOTH ON THE EDGES (INTERIOR) SINGULAR AT THE VERTICES.

$$\Delta = \frac{d^2}{dx_j^2}$$ ON FUNCTIONS ϕ ON THE EDGES W.R.T x_j

FOR THE BOUNDARY CONDITIONS AT THE VERTICES WE CHOOSE THE NEUMANN OR KIRCHOFF CONDITION:

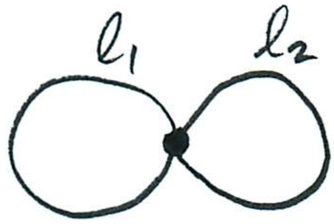
- ϕ IS CONTINUOUS AT THE v's
- $\sum_e \partial_e \phi(v) = 0$ FOR EACH VERTEX v AND e IS INWARD EDGE TO v.

WITH THIS A DEGREE ONE VERTEX  CORRESPONDS TO THE USUAL NEUMANN CONDITION.

A DEGREE TWO VERTEX  HAS A REMOVABLE SINGULARITY; SO ASSUME THERE ARE NO DEGREE TWO VERTICES.

Δ IS SELF ADJOINT AND HAS DISCRETE k SPECTRUM IN $\mathbb{R}$.

- IT IS CONVENIENT TO DEFINE SPEC(X) TO TO BE THE NON-ZERO 'k-SPECTRUM OF Δ' AND TO INCLUDE 0 WITH MULTIPLICITY $2+N-M$.

EXAMPLE: $X =$ 

; FIGURE EIGHT, $N=2, M=1$

$$\mathrm{Spec}(X) = \left\{ \frac{2\pi k_1}{\ell_1}, \frac{2\pi k_2}{\ell_2}, \frac{2\pi k_3}{\ell_1+\ell_2} : k_1, k_2, k_3 \in \mathbb{Z} \right\}$$

WEYL LAW: FOR ANY X AS ABOVE

$$\#\left\{ \mathrm{Spec}(X) \cap [-T,T] \right\} = \frac{2(\ell_1+\ell_2+\dots+\ell_N)}{\pi} T + O(1)$$

AS $T \to \infty$.

SO SPEC(X) HAS A DENSITY IN $\mathbb{R}$ WHICH IS THAT OF AN ARITHMETIC PROGRESSION AND μ_X IS LOCALLY UNIFORMLY BOUNDED — THE NUMBER OF ATOMS IN AN INTERVAL OF FIXED LENGTH IS BOUNDED FROM ABOVE.

COMPUTING SPEC(X):

ON THE EDGES AN EIGENCTION TAKES THE FORM $\phi(x_j) = a e^{k x_j} + b e^{-k x_j}$; OUR BOUNDARY CONDITIONS LEAD TO THE SECULAR DETERMINANT (KOTTOS/SMILANSKY)

. GIVEN THE UNDERLYING GRAPH G DEFINE THE 2N BY 2N MATRICES INDEXED BY THE ORIENTED EDGES $e_1, \hat{e}_1, e_2, \hat{e}_2, \ldots, e_N, \hat{e}_N$

$$U(z_1, \ldots, z_N) = (u_{fg}) \quad ; \quad u_{fg} = z_f \delta_{fg}$$

AND THE SCATTERING MATRIX

$$S = (s_{f,g}) \quad ; \quad s_{fg} = \begin{cases} -\delta_{f\hat{g}} + \dfrac{2}{\deg(v)} & \text{if } g \text{ follows } f \text{ through } v \\ 0 & \text{otherwise} \end{cases}$$

HERE deg(v) is its degree.

S IS UNITARY.

SPECTRAL OR SECULAR POLYNOMIAL OF G:

$$P_G(z_1, z_2, \ldots, z_N) := \det\left(I - U(z_1, .., z_n) S\right).$$

WHICH WE CONSIDER AS A LAURENT POLYNOMIAL IN $z_1, \ldots, z_N$.

IMMEDIATE PROPERTIES OF P_G:

(i) $P_G(z)$ IS DEGREE $2N$ AND IS OF DEGREE TWO IN EACH z_j.

(ii) LET $P^{\iota}(z_1, \dots, z_N) = P(1/z_1, 1/z_2, \dots, 1/z_N)$. THEN BOTH P_G AND P_G^{ι} ARE "$D = \{z : |z| < 1\}$ STABLE" THAT IS THEY DON'T VANISH FOR z WITH $z_j \in D$, FOR ALL j (FOLLOWS FROM THE UNITARITY OF S).

- THE CONNECTION TO COMPUTING SPEC(X) IS: (BARRA/GASPARD)

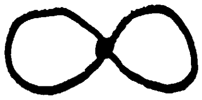
$$\operatorname{spec}(X) = \{\text{ZEROS WITH MULTIPLICITY OF } k \longrightarrow P_G(e^{ikl_1}, e^{ikl_2}, \dots, e^{ikl_N})\}.$$

CLEARLY THE ALGEBRAIC VARIETY

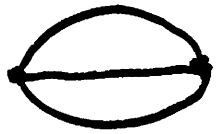
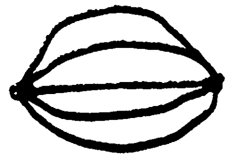
$$Z_G = \{z : P_G(z) = 0\} \subset (\mathbb{C}^*)^N$$

PLAYS A CENTRAL ROLE AND IN PARTICULAR THE QUESTION OF ~~ITS~~ THE FACTORIZATION OF P_G (OVER $\mathbb{C}$).

SPECIAL EXAMPLES:

$G =$  FIGURE EIGHT; $P_G(z_1, z_2) = (z_1 - 1)(z_2 - 1)(z_1 z_2 - 1)$

Z_G IS A UNION OF THREE SUBTORI.

$G =$  W_3 ; OR MORE GENERALLY W_N : 

$$P_G(z_1, z_2, z_3) = \left(z_1 z_2 z_3 + \tfrac{1}{3}(z_1 z_2 + z_1 z_3 + z_2 z_3) - \tfrac{1}{3}(z_1 + z_2 + z_3) - 1\right)\left(z_1 z_2 z_3 - \tfrac{1}{3}(z_1 z_2 + z_1 z_3 + z_2 z_3) - \tfrac{1}{3}(z_1 + z_2 + z_3) + 1\right)$$

FACTORIZATION CORRESPONDS TO THE SYMMETRY: REFLECTION THRU THE MIDPOINT OF EACH EDGE.

THEOREM 1 (KURASOV-S):

ASSUME THAT G IS NOT W_N THEN

(i) $$P_G(z) = Q_G(z) . \prod_{e \text{ A LOOP}} (z_e - 1)$$

WHERE THE PRODUCT IS OVER ALL LOOP EDGES IN G, AND $Q_G(z)$ IS ABSOLUTELY IRREDUCIBLE.

(ii) Z_{Q_G} DOES NOT CONTAIN AN $N-1$ DIMENSIONAL SUBTORUS OR TRANSLATE THEREOF UNLESS G IS THE FIGURE EIGHT.

REMARK: PART (i) WAS CONJECTURED BY COLIN DE VERDIERE.

THEOREM 2 (K–S) ADDITIVE STRUCTURE OF SPEC(X)

X IS A METRIC GRAPH ON G

(i) $\mathrm{Spec}(X) = L_1(X) \sqcup L_2(X) \cdots L_\nu(X) \sqcup N(X)$ (WITH MULT)

WHERE $L_j(X)$ IS A FULL INFINITE ARITHMETIC PROGRESSION AND THE NON-STRUCTURED PART, IF NON-EMPTY SATISFIES

- $\#(N(X) \cap [-T,T]) = \alpha T + O(1)$ AS $T \to \infty$

  WITH $\alpha = \frac{2}{\pi}(\ell_1 + \cdots + \ell_N) - \left(\frac{1}{d_1} + \cdots \frac{1}{d_\nu}\right)$; d_j's THE COMMON DIFF.

  AND $\alpha > 0$.

- THERE IS $c = c(G) < \infty$ SUCH THAT FOR ANY ARITHMETIC PROGRESSION $P \subset \mathbb{R}$

  $$\#(N(X) \cap P) \leq c(G)$$

- IN PARTICULAR $N(X)$ CONTAINS NO ARITHMETIC PROGRESSION LONGER THAN $c(G)$.

- $$\dim_{\mathbb{Q}} \mathrm{span}(N(X)) = \infty .$$

(ii) IF $\ell_1, \ell_2, \ldots \ell_N \in \mathbb{Q}$ (PROJECTIVELY) THEN $N(X) = \phi$.

IF $\ell_1, \ell_2, \ldots \ell_N$ ARE LINEARLY INDEPENDENT OVER $\mathbb{Q}$, THEN EXCEPT FOR THE FIGURE EIGHT ν IS EQUAL TO THE NUMBER OF LOOPS IN G, $\dim_{\mathbb{Q}}(\mathrm{spec}\, X) = \infty$, AND IF G HAS NO LOOPS, $\mathrm{spec}(X) = N(X)$.

<u>REMARK:</u> THE STRUCTURED PART $L_j(X)$ $j = 1, 2, \ldots, \nu$ AND $c(G)$ CAN BE DETERMINED EFFECTIVELY.

SUMMATION FORMULA FOR X

FOR METRIC GRAPHS THE SUMMATION FORMULA TAKES AN EXACT FORM (ROTH, KOTTOS/SMILANSKY, KURASOV)

$$\widehat{\sum_{k \in \mathrm{spec}(X)} \delta_k} = \frac{2(\ell_1 + \cdots \ell_n)}{\pi}\delta_0 + \frac{1}{\pi}\sum_{p \in P} \ell(\mathrm{prim}\, p)\left[S_v(p)\,\delta_{\ell(p)} + \overline{S_v(p)}\,\delta_{-\ell(p)}\right]$$

WHERE:

- P IS THE SET OF ORIENTED PERIODIC PATHS IN G UP TO CYCLIC EQUIVALENCE (BACKTRACKING ALLOWED)
- $\ell(p)$ IS THE LENGTH OF THE PATH
- $\mathrm{prim}(p)$ IS THE PRIMITIVE PART OF p (GOING AROUND ONCE)
- $S_v(p)$ IS THE PRODUCT OF THE SCATTERING COEFF ~~AT~~ THE VERTICES ENCOUNTERED ON TRAVERSING p.

$\widehat{\mu_X}$ IS SUPPORTED IN

$$\Lambda = \{ m_1\ell_1 + m_2\ell_2 + \cdots + m_N\ell_N : m_j \geq 0 \text{ IN } \mathbb{Z} \}$$

WHICH IS A DISCRETE SET, BUT NOT LOCALLY UNIFORMLY BOUNDED.

$\Rightarrow \mu_X$ IS A CRYSTALLINE MEASURE SATISFYING EXOTIC PROPERTIES

(a) $\dim_{\mathbb{Q}}(\mathrm{supp}\,\mu_X) = \infty$, $\dim_{\mathbb{Q}}(\mathrm{supp}\,\hat{\mu}_X) < \infty$

(b) μ_X IS LOCALLY UNIFORMLY BOUNDED (AND POSITIVE)

NOTE THAT $\hat{\mu}_X$ CANNOT BE LOCALLY BOUNDED

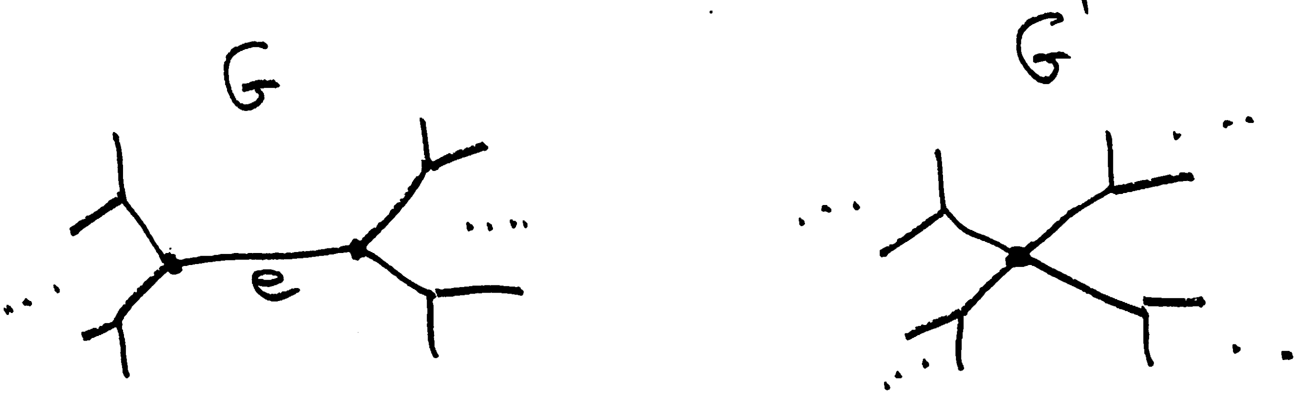
<u>THEOREM</u> (LEV / OLEVSKII): IF μ IS A CRYSTALLINE MEASURE WITH BOTH μ AND $\hat{\mu}$ LOCALLY UNIFORMLY BOUNDED THEN μ CORRESPONDS TO A FINITE UNION OF ARITHMETIC PROGRESSIONS.

• ONE CAN PRODUCE SIMILAR SUCH EXOTIC CRYSTALLINE MEASURES USING ANY $P(z_1, z_2, \ldots, z_N)$ FOR WHICH $\underline{P}$ AND $\underline{P}'$ ARE D-STABLE. FOR EXAMPLE FROM THOSE ARISING IN THE LEE-YANG THEOREM AND THE THEORY OF HYPERBOLIC POLYNOMIALS WHERE THE PROOF OF STABILITY IS NOT A CONSEQUENCE OF A DETERMINANTAL FORMULA AND UNITARITY.

• IN THIS VEIN THE CRYSTALLINE MEASURES ARISING FROM THE EXPLICIT FORMULA IN THE THEORY OF PRIME NUMBERS LIES DEEPER IN A WAY THAT NEEDS EXAMINATION; AS IT IS EQUIVALENT TO "RH".

OUTLINE OF PROOFS:

THE PROOF OF THEOREM 1 IS BASED ON EDGE CONTRACTION



G IS CONTRACTED TO G' BY REMOVING e AND IDENTIFYING THE END POINTS. WE ALLOW THE INTRODUCTION DEGREE TWO VERTICES, LOOPS...

- THE KEY LEMMA ASSERTS THAT IN SUCH A CONTRACTION P_G AND $P_{G'}$ ARE RELATED BY SPECIALIZING THE VARIABLE z_e TO 1.

IN THIS WAY ONE CAN FOLLOW THE FACTORIZATION PROPERTIES OF P_G UNDER REPEATED CONTRACTION. THE "WATER MELLON" GRAPHS WN APPEAR AS END POINTS THAT NEED SPECIAL ATTENTION, AND OTHERWISE ONE NAVIGATES, THE CONTRACTIONS TO A FINITE (SMALL) NUMBER OF CONFIGURATIONS THAT ARE EXAMINED DIRECTLY.

THEOREM 2 IS BASED ON SOME ADVANCED RESULTS IN DIOPHANTINE ANALYSIS ON TORI.

LANG'S $\mathbb{G}_m$ CONJECTURES:

THERE ARE TWO FLAVORS; VERTICAL AND HORIZONTAL, WE NEED BOTH.

$\mathbb{G}_m$ = MULTIPLICATIVE GROUP $\mathbb{C}^*$

$T = (\mathbb{C}^*)^N$ IS AN N-TORUS, IT IS AN ALGEBRAIC GROUP UNDER COORDINATE PRODUCT.

$V \subset (\mathbb{C}^*)^N$ AN ALGEBRAIC SUBVARIETY

GIVEN BY THE ZERO SET OF LAURENT POLYNOMIALS.

$$\mathrm{tor}(T) = \{ (z_1, \ldots, z_N) : z_j \text{ IS A ROOT OF UNITY FOR ALL } j = 1, \ldots, N \}$$

$\mathrm{tor}(T)$ CONSISTS OF ALL POINTS IN T OF FINITE ORDER.

VERTICAL LANG CONJ:

GIVEN $V \subset T$ AS ABOVE, THERE ARE FINITELY MANY SUBTORI OR TRANSLATES THEREOF BY TORSION POINTS, $T_1, T_2, \ldots, T_\nu$ CONTAINED IN V SUCH THAT

$$\mathrm{tor}(T) \cap V = \mathrm{tor}(T) \cap (T_1 \cup T_2 \cdots \cup T_\nu).$$

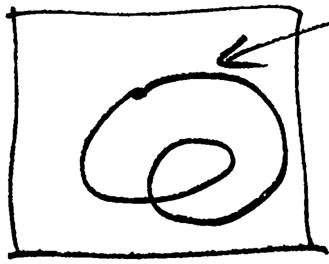
SO WHAT APPEARS TO BE A NON-LINEAR COMPLICATED PROBLEM IS IN FACT VERY STRUCTURED IN THAT TORSION POINTS CAN ONLY LIE ON A FINITE NUMBER OF COSETS OF SUBGROUPS. NOTE THE T_j's MAY BE ZERO DIMENSIONAL IN WHICH CASE THEY ARE TORSION POINTS.

THERE ARE A NUMBER OF PROOFS OF THIS VERTICAL CASE AND THE PROOF CAN BE MADE EFFECTIVE IN THAT THE T_j's ARE DETERMINED.

ONE PROOF PROCEEDS AS FOLLOWS:

N=2:

$V \cap (S^1 \times S^1) \subset V \cap T$



V

IF $\rho = (\rho_1, \rho_2) \in \mathrm{tor}(T) \cap V$, $\rho_1^{m_1} = 1$, $\rho_2^{m_2} = 1$

AND $\sigma \in \mathrm{GAL}(K(\rho_1, \rho_2)/K)$ WHERE K IS THE FIELD OF DEFINITION OF V ; THEN

$$\sigma((\rho_1, \rho_2)) \in \mathrm{tor}(T) \cap V.$$

NOW THESE GALOIS ORBITS GROW FAST AS THE ORDER OF ρ INCREASES

$$\deg[\mathbb{Q}(\rho_m) : \mathbb{Q}] = \phi(m) \gg m^{1-\varepsilon}.$$

HENCE IF ONE CAN ESTABLISH A SUITABLE NON TRIVIAL UPPER BOUND FOR THE NUMBER OF TORSION POINTS OF SUCH ORDER ON V (ASSUMING V DOES NOT CONTAIN SUBTORI) THEN ONE IS LED TO THERE BEING NO SUCH POINTS OF LARGE ORDER.

SUCH UPPER BOUNDS CAN BE GIVEN IN THIS TORUS CASE BY ELEMENTARY METHODS.

THIS UPPER BOUND VS GALOIS ORBIT METHOD HAS PROVEN TO BE ROBUST FOR OTHER VERTICAL PROBLEMS:

- BOMBIERI-PILA; GIVE UPPER BOUNDS SHARP UP TO EXPONENT FOR TRANSCENDETAL CURVES IN THE PLANE; FOR RATIONAL POINTS

- PILA-WILKIE GIVE SHARP UPPER BOUNDS FOR RATIONAL POINTS ~~POINTS~~ ON THE TRANSCENTAL PARTS OF DEFINABLE SETS IN O-MINIMAL STRUCTURES IN $\mathbb{R}^n$.

- PILA-ZANNIER PROVE THE VERTICAL ABELIAN VARIETY VERSION OF LANG'S CONJ, ALSO KNOWN AS THE MANIN-MUMFORD CONJ.

- THE VERTICAL ANALOGUE IN SHIMURA VARIETIES OF TORSION POINTS ARE "CM-POINTS" AND THESE LIE ON FINITELY MANY SHIMURA SUBVARIETIES "ANDRE-OORT" CONJ.
- PROVED FOR PRODUCTS OF MODULAR CURVES BY PILA
- PROVED FOR $\mathcal{A}_g$ BY PILA AND TSIMERMAN.

HORIZONTAL LANG G_m CONJ FOR $T=(\mathbb{C}^*)^N$

IF $V \subset T$ IS AS ABOVE AND Γ IS A FINITELY GENERATED SUBGROUP OF T, THERE FINITELY MANY TRANSLATES OF SUBTORI $T_1, T_2, \ldots, T_\nu$ IN V, SUCH THAT

$$\Gamma \cap V = \Gamma \cap (T_1 \cup T_2 \cdots \cup T_\nu).$$

THIS LIES DEEPER AND IT WAS PROVEN BY M. LAURENT. THE KEY INPUT IS THE SCHMIDT SUBSPACE THEOREM WHICH IS A STRIKING HIGHER DIMENSIONAL VERSION OF THE THUE-SIEGEL-ROTH THEOREM.

SIMPLEST VERSION (SCHMIDT)

LET $L_1(x), L_2(x), \ldots, L_n(x)$ BE n linearly INDEPENDENT LINEAR FORMS IN $(x_1, \ldots x_n) = x$ WITH REAL ALGEBRAIC COEFFICIENTS;

THEN FOR $\varepsilon > 0$ THE SET OF SOLUTIONS WITH $x \in \mathbb{Z}^n$ OF

$$|L_1(x) L_2(x) \cdots L_n(x)| < \|x\|^{-\varepsilon}$$

LIE IN FINITELY MANY PROPER $\mathbb{Q}$-LINEAR SUBSPACES OF $\mathbb{Q}^n$.

NOTE: THE PROOF YIELDS AN EFFECTIVE BOUND FOR THE NUMBER OF SUBSPACES BUT NOT FOR THEIR DETERMINATION

VERTICAL AND HORIZONTAL:

TO COMBINE THE TWO LET $\overline{\Gamma}$ BE THE DIVISION GROUP OF Γ

$$\overline{\Gamma} = \{ z \in T : z^{\ell} \in \Gamma \text{ FOR SOME } \ell \geq 1 \}$$

(SO $\overline{1} = \mathrm{tor}(T)$).

THE ULTIMATE VERSION WHICH IS ALSO UNIFORM OVER THE DEFINING FIELDS AND QUANTITATIVE IN THE ~~rank~~ r OF Γ IS DUE TO EVERSTE/SCHLICKEWEI/SCHMIDT:

THEOREM: $V \subset (\mathbb{C}^*)^N$, Γ A FINITELY GENERATED SUBGROUP OF RANK r; THERE ARE $T_1, T_2, \dots T_\nu$ TRANSLATES OF SUBTORI CONTAINED IN V SUCH THAT

$$\overline{\Gamma} \cap V = \overline{\Gamma} \cap (T_1 \cup T_2 \dots \cup T_\nu)$$

AND

$$\nu \leq (C(V))^{r}.$$

REMARK: THE CONSTANT $C(V)$ CAN BE GIVEN EXPLICITLY, HOWEVER THE ACTUAL SAY ZERO DIMENSIONAL T_j'S CANNOT IN GENERAL BE DETERMINED BY THIS PROOF.

THE PROOF INVOLVES SPECIALIZATION ARGUMENTS REDUCING TO $\Gamma \subset T(\bar{\mathbb{Q}})$ AND ABSOLUTE VERSIONS OF THE SCHMIDT SUBSPACE THEOREM, AS WELL AS A STUDY OF POINTS OF SMALL HEIGHT AND LARGE HEIGHT . . .

AFTER ANALYZING OUR SUBVARIETIES Z_G AND APPLYING THE DIOPHANTINE ANALYSIS WE ARRIVE AT:

- GIVEN G THERE IS $\varepsilon(G) > 0$ SUCH THAT FOR ANY t DISTINCT POINTS IN $N(X)$, $x_1, x_2, \dots, x_t$

$$\dim_{\mathbb{Q}} \operatorname{span}(x_1, \dots, x_t) \geq \varepsilon(G) \log t .$$

WHICH LEADS TO THEOREM 2.