On the Bezout problem for entire analytic sets

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1. Introduction

It is well-known that the fields of algebraic geometry and complex analysis frequently have a parallel development and also frequently share the same underlying general principles. For example, the *local theory* of algebraic or of analytic varieties is roughly the same. The global theory of *affine varieties* or of *Stein manifolds* again follows the same general patterns.

A somewhat less obvious parallel arises in the study of the *zeroes of an entire holomorphic function* $f(z)$ ($z \in C$). Here the growth of the *maximum modulus function*

$$M(f, r) = \max_{|z| \leq r} \log |f(z)|$$

plays the role of the *degree* of a polynomial. Assuming that $f(0) = 1$, a fundamental result is the bound

$$n(f, r) \leq CM(f, 2r) \tag{1}$$

on the number $n(f, r)$ of zeroes of f in $|z| \leq r$, generalizing the obvious bound on the number of zeroes of a polynomial. Suitably interpreted, the estimate (1) carries over to bound the size of the analytic hypersurface $f(z) = 0$ where $f \in \mathcal{O}(C^n)$ is an entire holomorphic function of n variables

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(cf. § 3 in [6]).

The purpose of this paper is to discuss one instance where the analogy between analytic and algebraic geometry seemingly breaks down. This is the *transcendental Bezout problem* of estimating the number $n(f_1, f_2; r)$ of common zeroes in $|z_1|, |z_2| \leq r$ of two entire functions $f_1(z_1, z_2)$ and $f_2(z_1, z_2)$ in terms say of $M(f_1, r)$ and $M(f_2, r)$. Cornalba and Shiffman [5] have given examples to show that there is generally no estimate of the form (1), even though such an estimate does hold "on the average" (Stoll [15]). The purpose of this paper is to attempt to clarify the problem in the belief that the analogy between algebraic and analytic geometry might, if properly understood, continue to hold in a suitable form.

More specifically, we consider the Bezout problem in the formulation of trying to estimate the "size" of the intersection

$$V \cap \mathbf{C}^k$$

of analytic subvariety $V \subset \mathbf{C}^n$ in terms of the "size" of V. In this form, the Bezout problem is already solved in case $\operatorname{codim}(V) = 1$ (cf. § 4 below), and so the first interesting case is that of an *analytic curve* $V \subset \mathbf{C}^3$. Our main theorem (Theorem 1 in § 3) is an estimate on the size of the intersection

$$V \cap \mathbf{C}^2$$

which is *independent* of the particular $\mathbf{C}^2$. The estimate is in terms of the growth of the areas of V and of the *dual variety* V^*, and in terms of the number and position of the *inflection points*[1] on V. An examination of the Cornalba-Shiffman example shows that consideration of the inflection points is essential.*

2. Formulation of the problem

a) Let $f_1(z), \cdots, f_k(z) \in \mathcal{O}(\mathbf{C}^n)$ be entire holomorphic functions. For a point $a = (a_1, \cdots, a_k) \in \mathbf{C}^k$ we let $V_a \subset \mathbf{C}^n$ be the analytic set defined by

$$\begin{cases} f_1(z) = a_1 , \\ \quad\vdots \\ f_k(z) = a_k . \end{cases}$$

We assume throughout that $\operatorname{codim}_z(V_a) = k$ at every point $z \in V_a$. Letting $V_a[r] = \{z \in V_a : |z| \leq r\}$, the *Bezout problem* is to estimate the "size" of $V_a[r]$

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[1] If V is locally given by $\zeta \to z(\zeta)$ ($\zeta \in \mathbf{C}$, $z(\zeta) \in \mathbf{C}^3$), then the inflection points are defined by $z(\zeta) \wedge z'(\zeta) \wedge z''(\zeta) = 0$ in case $z(\zeta) \neq 0$, $z(\zeta) \wedge z'(\zeta) \neq 0$, and by $3\,\mathrm{ord}(z(\zeta) \wedge z'(\zeta)) < 3\,\mathrm{ord}(z(\zeta)) + \mathrm{ord}(z(\zeta) \wedge z'(\zeta) \wedge z''(\zeta))$ in the general case (the factors of 3 appear because of homogeneity restrictions).

in terms of the f_j's *independently of the point* $a \in \mathbf{C}^k$. By the "size" of $V_a[r]$, we mean as a beginning the Euclidean area

$$v(V_a, r) = \int_{V_a[r]} \omega^{n-k} \tag{2}$$

where

$$\omega = dd^c |z|^2 = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\left(\sum_{i=1}^n |z_i|^2\right)$$

is the standard Kähler metric on $\mathbf{C}^n$. In the same way that the growth of

$$M(f, r) = \max_{|z|=r} \log |f(z)|$$

generalizes the degree of a polynomial, the growth of

$$\mu(V_a, r) = \frac{v(V_a, r)}{r^{2n-2k}} \tag{3}$$

generalizes *the degree of an algebraic set* in $\mathbf{C}^n$. In fact, it is a fundamental theorem of Stoll ([12] and [13]) that for any codimension k analytic set $V \subset \mathbf{C}^n$, the function

$$\mu(V, r) = \frac{1}{r^{2n-2k}} \int_{V[r]} \omega^{2n-2k}$$

is increasing with r and

$$\mu(V, r) \leqq d \qquad (d \in \mathbf{Z}^+)$$

if, and only if, V is an algebraic set of degree $\leqq d$ in $\mathbf{C}^n$.[3] This result will be discussed below, and deeper insight into it should follow from an understanding of the Bezout problem.

b) A more general form of the Bezout problem is the following: Let $V, W \subset \mathbf{C}^n$ be analytic sets of pure codimensions k, l respectively. Assuming that the intersection $Z = V \cap W$ has pure codimension $k + l$, we want to estimate the size of Z in terms of V and W. Indeed, this version of the problem is closer to the usual algebro-geometric statement.

Two comments are relevant. The first is that the size of $V \times W \subset \mathbf{C}^n \times \mathbf{C}^n$ may be estimated by that of V, W, and thus writing

$$Z = V \cap W \cong (V \times W) \cap \mathbf{C}^n_\Delta$$

where $\mathbf{C}^n_\Delta \subset \mathbf{C}^n \times \mathbf{C}^n$ is $\mathbf{C}^n$ embedded as the *diagonal* in $\mathbf{C}^n \times \mathbf{C}^n$, we see that it will suffice to take W to be a linear subspace of $\mathbf{C}^n$. Secondly, refering to

[3] Kneser [7] first proved the monotonicity of $\mu(V, r)$ using a form of Stokes' theorem not yet available. Lelong [8] gave another complete proof of monotonicity. In case $0 \neq V$, Stokes' theorem gives $\mu(V, r) = \int_{V[r]} (dd^c \log |z|^2)^{n-k}$ which makes the monotonicity evident, since $dd^c \log |z|^2 \geqq 0$.

the first form of the Bezout problem discussed above, if we let

$$W = \{(z; f_1(z), \cdots, f_k(z)) : z \in \mathbf{C}^n\} \subset \mathbf{C}^n \times \mathbf{C}^k$$

be the graph of $(f_1, \cdots, f_k)$, then on the one hand

$$V_0 \cong W \cap (\mathbf{C}^n \times \{0\}),$$

while on the other hand the growth of W and that of the f_j's is roughly the same (cf. the Ahlfors-Shimizu form of the Nevanlinna characteristic function given in [9, pp. 171–177]). Thus the more general form of the Bezout problem does indeed contain the first question as a special case.

On the basis of this discussion, then, we shall concentrate on the Bezout problem in the following form: Let $G(l, n)$ be the *Grassman manifold* of $n - l$ planes through the origin in $\mathbf{C}^n$. Letting $V \subset \mathbf{C}^n$ be an analytic set of codimension k, and denoting points of $G(l, n)$ by $A, B, \cdots$, we want an estimate on the size of the intersection

$$A \cap V$$

which is independent of A, always assuming that $\operatorname{codim}_z(A \cap V) = k + l$ at all points $z \in A \cap V$.

A recent theorem of Shiffman gives us an *average Bezout Theorem* analogous to the classical *Crofton's Formula* in integral geometry. Letting dA be the measure on $G(l, n)$ invariant under the unitary group and suitably normalized, Shiffman's result is the formula

$$\mu(V, r) = \int_{A \in G(l,n)} \mu(V \cap A, r) dA\,, \tag{4}$$

where $k + l \leqq n$ and μ is given by (3), and where we assume for simplicity that V does not pass through the origin.

In particular taking $l = n - k$, (4) becomes

$$\mu(V, r) = \int_{A \in G(n-k,n)} n(A \cap V[r]) dA \tag{5}$$

where $n(A \cap V[r])$ is the *number of points of intersection* (counted with multiplicities) of A and V in $|z| \leqq r$. If V is algebraic of degree d, then $\mu(V, r) \uparrow d$ as $r \to \infty$, and Stoll's Theorem is the converse to this statement.

3. The first main theorem

Let $V \subset \mathbf{C}^{n+1}$ be an analytic set of pure dimension m (note the $n + 1$ on $\mathbf{C}^{n+1}$). To estimate the size of $V \cap A$ where A is a linear space, it is sufficient to treat the case when A is a hyperplane and then proceed inductively downward (this is also necessary when using Crofton's Formula (4)). Henceforth, we will thus assume that $A, B, \cdots$, are *hyperplanes in* $\mathbf{C}^{n+1}$, and we view

$A, B, \cdots$, as points in the projective space $(\mathbf{P}^n)^*$ dual to the projective space $\mathbf{P}^n$ of lines through the origin in $\mathbf{C}^{n+1}$.

A technique for approaching this problem is to apply *Nevanlinna Theory* to the *residual mapping* (assuming that $o \notin V$)

$$\begin{cases} f: V \longrightarrow \mathbf{P}^n , \\ f(z) = \text{line } \vec{oz} \quad \text{for} \quad z \in V . \end{cases}$$

This is the method used by Stoll [13] in the original proof of his theorem mentioned above; and even though other proofs of this result have since been given ([2], [11], and [16]), Stoll's argument still has global geometric appeal and also offers one possible means of understanding the general Bezout problem. To state the first main theorem (abbreviated F.M.T.) of Nevanlinna Theory in the present context, we use the following notations:

$$(6)\quad \begin{cases} \tau = \log|z|: V \longrightarrow \mathbf{R} & (\textit{exhaustion function} \text{ for } V), \\ \Omega = dd^c\tau & (\textit{Levi form} \text{ of } \tau), \\ V[r] = \{z \in V: \tau(z) \leqq \log r\}, & \\ T(r) = \int_0^r \mu(V, t)\dfrac{dt}{t} & (\textit{order function} \text{ for } V), \\ N(A, r) = \int_0^r \mu(A \cap V, t)\dfrac{dt}{t} & (\textit{counting function} \text{ for } A \cap V), \\ |z, A| = |\langle z, A\rangle| & (z \in \mathbf{C}^{n+1},\ A \in (\mathbf{C}^{n+1})^*), \\ \rho_A(z) = \dfrac{|z|\,|A|}{|z, A|} & (1 \leqq \rho_A \leqq +\infty), \\ m(A, r) = \int_{\partial V[r]} \log \rho_A d^c\tau \wedge \Omega^{m-1} \geqq 0 & (\textit{proximity form}), \\ S(A, r) = \int_{V[r]} \log \rho_A \Omega^m & (\textit{remainder term}) . \end{cases}$$

The F.M.T. now reads (cf. [6, Proposition 5.14] and [14])

$$(7)\qquad N(A, r) + m(A, r) = T(r) + S(A, r) .$$

Remarks. (1) First I want to apologize for the flood of notations and terminology, which anyone familiar with value distribution theory will recognize as standard. In one form or another, the F.M.T. is the technique for proving all known versions of Bezout-type theorems. With reference to (6), the proximity form $m(A, r) \geqq 0$ and so (7) gives an inequality

$$(8)\qquad N(A, r) \leqq T(r) + S(A, r) .$$

If we interpret "size" of an analytic variety Z as meaning growth of $\mu(Z, r)$, then the Bezout problem amounts to estimating $N(A, r)$ in terms of $T(r)$

and other quantities independent of A. Inequality (8) suggests that we attempt to estimate $S(A, r)$. This will be done in two cases.

(2) In the counting function $N(A, r)$, *multiplicities* must be taken into account. In case V is smooth, $A \cap V$ is the zero set of a holomorphic function and then it is clear how to define multiplicities. Our Main Theorem deals with the case when V is an analytic curve, and thus has a canonical desingularization (=normalization) $\tilde{V}$; in this case, multiplicities are defined by pulling back the holomorphic function A to $\tilde{V}$. The general case may be treated using recent results of Tung [17], where the F.M.T. for singular varieties is treated.

4. The hypersurface case

With the notation of Section 3, the one case in which the Bezout problem always has an affirmative answer is when $V \subset \mathbf{C}^{n+1}$ is a hypersurface. We will briefly discuss three proofs of this, all of which already exist in the literature.

First proof. We begin by remarking that

$$\int_{V[r]} \Omega^n = \mu(V, r) . \tag{9}$$

This follows from Crofton's Formula (4) applied to the projective space $\mathbf{P}^n$ of lines through the origin in $\mathbf{C}^{n+1}$ because

(i) $\Omega = dd^c \log |z|$ is the pull-back under f of the Kähler metric on $\mathbf{P}^n$, and

(ii) We evidently have the equality

$$\int_{V[r]} \Omega^n = \int_{A \in \mathbf{P}^{n*}} n(A \cap V[r]) dA .$$

Next, since the residual mapping $f: V \to \mathbf{P}^n$ is *equidimensional* and the form

$$\rho_A \Omega^n$$

is integrable on $\mathbf{P}^n$, we have for the remainder

$$S(A, r) = \mu(V, r)\left\{\frac{1}{\mu(V, r)}\int_{V[r]} \log \rho_A \Omega^n\right\}$$
$$\leq \mu(V, r) \log \left\{\frac{1}{\mu(V, r)}\int_{V[r]} \rho_A \Omega^n\right\}$$

by the *concavity of the logarithm.* Now the integral

$$\int_{V[r]} \rho_A \Omega^n$$

may be directly estimated by standard methods in Nevanlinna Theory (cf.

the second proof of Proposition 4.1 in [6]). This leads to a bound on $S(A, r)$ which is independent of A, and thence to an estimate

$$N(A, r) \leqq CT(r)^{1+\epsilon} + C', \qquad (\epsilon > 0). \tag{10}$$

(Actually, (10) is an easily stated but crude form of the estimate in [6].)

Second proof. For each point $z \in \mathbf{C}^{n+1} - \{0\}$, we let $L_z = \vec{oz}$ be the line connecting z to the origin, and we consider the counting function

$$N(z, r) = N(L_z \cap V, r|z|) = \int_0^{r|z|} n(L_z \cap V, t)\frac{dt}{t}$$

which measures the number of points on $L_z \cap V$ in the disc of radius $r|z|$. By Crofton's Formula (4) we can estimate the average

$$\int_{|z|\leq 1} N(z, r)d\mu(z)$$

($d\mu(z) =$ Euclidean measure) of $N(z, r)$ over $|z| \leqq 1$ in terms of

$$\int_0^r \mu(V, t)\frac{dt}{t}.$$

On the other hand, $N(z, r)$ is a *pluri-subharmonic* function of z a fact which may most easily be seen as follows: By the second problem of Cousin for $\mathbf{C}^{n+1}$, we may write $V = \{z: f(z) = 0\}$ for some entire function $f \in \mathcal{O}(\mathbf{C}^{n+1})$ with $f(0) = 1$. Then by *Jensen's theorem* ([6] and [15])

$$N(z, r) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta}z)|d\theta,$$

which is obviously pluri-subharmonic in z. Now pluri-subharmonic functions satisfy the *sub-mean-value property*, and thus the value of $N(z, r)$ may be estimated by its average over a ball around z. This coupled with the estimate on $\int_{|z|\leq 1} N(z, r)d\mu(z)$ leads to an estimate on $N(z, r)$ in terms of $T(2r)$ for all z with $|z| = 1$. The inequality (10) follows by averaging over z with $z \in A$, $|z| = 1$.

Third proof. Given V, we may solve the second problem of Cousin with growth conditions (cf. Skoda [11]) to write $V = \{z: f(z) = 0\}$ where $f \in \mathcal{O}(\mathbf{C}^{n+1})$ satisfies an estimate

$$M(f, r) \leqq CT(2r)(\log r)^2 + C'.$$

Given a line $A \in \mathbf{P}^n$, we then apply Jensen's theorem to $f|A$ to obtain the variant of (10):

$$N(A, r) \leqq M(f, r) \leqq CT(2r)(\log r)^2 + C'.$$

With further work, the referee remarks that the estimate

$$N(A, r) \leq CT(2r) + C'$$

can also be proved.

Remark. The Bezout estimate (10) is easily stated but somewhat crude. A more careful argument gives the inequality (cf. § 4 in [6]).

$$N(A, r) \leq T(r) + o(T(r)) \quad // \tag{10'}$$

where the meaning of "//" is explained preceding the statement of Theorem I in Section 5 below. Using (10′) we may prove Stoll's theorem for hypersurfaces as follows: Assuming that $\mu(V, r) \leq d$, we have $T(r) \leq d \log r + c$, and (10′) then gives

$$\mu(A \cap V) \leq d$$

for *all* hyperplanes $A \in (\mathbf{P}^n)^*$. Iterating this we find that V meets every line through the origin in $\leq d$ points, from which it easily follows that V is algebraic of degree $\leq d$ (cf. the proof of Chow's theorem given in §2 of [6]).

5. Curves in $\mathbf{C}^3$; statement of the main theorem

Cornalba and Shiffman [5] have given an example of an analytic curve $V \subset \mathbf{C}^3$ such that $\mu(V, r) = O(r^\varepsilon)$ for every $\varepsilon > 0$, but for which there is a 2-plane $A \in (\mathbf{P}^3)^*$ with

$$n(A \cap V[r]) \neq O(r^N)$$

for any N. Thus, in the terminology of the theory of functions of finite order, V has *order zero* but $A \cap V$ has *infinite order*. Observe that V is necessarily *non-degenerate* in the sense that it is not contained in a $\mathbf{C}^2$, since otherwise the Bezout Theorem for hypersurfaces could be used. This suggests that we seek a Bezout estimate which involves not only the growth of the area of V, but also the size of the *osculating variety* V^* associated to V. More precisely, we consider the residual mapping

$$f: V \longrightarrow \mathbf{P}^2 .$$

Associated to this *holomorphic curve* is the *dual curve*

$$f^*: V \longrightarrow (\mathbf{P}^2)^*$$

which associates to each point $z \in V$ the 2-plane $f^*(z)$ spanned by z and the tangent line to V at z. If φ is the standard Kähler metric on a projective space, the (1, 1)-forms

$$\begin{cases} \Omega_0 = f^*\varphi \, , \\ \Omega_1 = (f^*)^*\varphi \end{cases}$$

are the fundamental local invariants for the metric geometry of $f(V)$. The order functions

$$\text{(11)}\qquad \begin{cases} T_0(r) = \int_0^r \left\{ \int_{V[t]} \Omega_0 \right\} \frac{dt}{t}\,, \\ T_1(r) = \int_0^r \left\{ \int_{V[t]} \Omega_1 \right\} \frac{dt}{t} \end{cases}$$

measure the growth of V and of the osculating variety V^* respectively. If

$$\text{(12)}\qquad \Omega_0 = h\Omega_1\,,$$

the non-negative function h becomes infinite at the *inflection points* of V (cf. (26) in §7 below). The quantity

$$\text{(13)}\qquad S(r) = \int_{V[r]} \log^+ h \cdot \Omega_0 \geqq 0$$

is intrinsically associated to V and would appear to become large if V has many inflection points (cf. the example in §7 below).

To state our main result, we suppose given $\varepsilon > 0$, and use the notation

$$\theta(r) \leqq \psi(r) \qquad //$$

to mean that the stated inequality holds outside an open set $E \subset \mathbf{R}^+$ such that $\int_E dt/t < \infty$.

THEOREM I. *There exist constants C, C', C'' such that for any $A \in (\mathbf{P}^2)^*$, the counting function $N(A, r)$ satisfies an estimate*

$$\text{(14)}\qquad N(A, r) \leqq C\{T_0(r)^{1+\varepsilon} + T_1(r)^\varepsilon\} + C'S(r) + C'' \qquad //\,.$$

An easy corollary results by assuming that the holomorphic curve V is *bounded away from having inflection points* in the sense that $h \leqq c < \infty$ in (13). Then $S(r) = O(T_0(r))$ and so we obtain the

(15) COROLLARY. *If $V \subset \mathbf{C}^3$ is an analytic curve which is bounded away from having inflection points, then*

$$N(A, r) \leqq C\{T_0(r)^{1+\varepsilon} + T_1(r)^\varepsilon\} + C'$$

for all 2-planes $A \in (\mathbf{P}^2)^*$.

Remark. In the Cornalba-Shiffman example, both $T_0(r)$ and $T_1(r)$ are $O(r^{2+\delta})$ for any $\delta > 0$ (cf. §7 below), but V has many inflection points.

There are two criticisms of the above theorem. The first is that the result fails to yield the Stoll theorem, which we recall is the statement that V is an algebraic curve, if, and only if, $T_0(r) \leqq d \log r + c$ (cf. [12]). The second is that the quantity $S(r)$ given by (13) does not have a direct geometric interpretation, although it does appear somewhat naturally in the second main theorem for $V \subset \mathbf{C}^3$ (not discussed here). Consequently, Theorem I should be thought of as indicating that some understanding of the Bezout problem is perhaps possible rather than as a definitive result.

Finally, it is possible to prove a result similar to Theorem I for an analytic curve $V \subset \mathbf{C}^n$ for any n. The estimate will bound $N(A, r)$ in terms of

(a) $$T_0(r)^{1+\epsilon},\ T_1(r)^{\epsilon},\ \cdots,\ T_{n-2}(r)^{\epsilon}$$

where $T_{k-1}(r)$ is the order function for the k^{th} *associated curve* [4], and

(b) A quantity $S(r)$ which measures the number of *stationary points* of order k for $1 \leq k \leq n$.

I do not know even a conjectural statement in case $\dim V > 1$, $\operatorname{codim} V > 1$, and this is another indication that the correct result is yet to be found.*

6. Proof of the main theorem

(a) We will apply the theory of holomorphic curves, especially the so-called *Ahlfors' inequalities* [1], to the residual mapping $f: V \to \mathbf{P}^2$. In doing this we will follow the terminology of the paper by Chern [4], and will also use his notation with the following two exceptions:

(i) We use z instead of Chern's Z to denote a point in $\mathbf{C}^3$;

(ii) Hyperplanes in $\mathbf{P}^2$ will be denoted by A instead of α.

Given a local holomorphic coordinate ζ on V, the corresponding point in $\mathbf{C}^3$ will be denoted by $z(\zeta)$. Following Chern, we then use *Frenet frames* z_1, z_2, z_3 where

$$\begin{cases} z_1 = \dfrac{z}{|z|}\,, & \\ z_0 \wedge z_1 = \dfrac{z \wedge z'}{|z \wedge z'|} & \left(z' = \dfrac{dz}{d\zeta}\right), \\ z_0 \wedge z_1 \wedge z_2 = \dfrac{z \wedge z' \wedge z''}{|z \wedge z' \wedge z''|} & ((z'' = (z')')\,. \end{cases}$$

For a hyperplane $A \in (\mathbf{P}^2)^*$, we set

$$\begin{cases} \varphi_0 = |z_0, A|^2 = |\langle z_0, A\rangle|^2 & (\text{assuming } |A| = 1)\,, \\ \varphi_1 = |z_0, A|^2 + |z_1, A|^2\,. & \end{cases}$$

Then $\varphi_0 = 0$ at the points of intersection $A \cap V$ and $\varphi_1 = 0$ at the points of $A \cap V$ where V is tangent to A.

Referring to the F.M.T. (7), we have an inequality (for $0 < \lambda < 1$)

* (Footnote added in proof) See W. Stoll, Deficit and Bezout estimates Tulane Conf. in *Value Distribution Theory*, I, Marcel Dekker, 1973.