ON CARTAN'S METHOD OF LIE GROUPS AND MOVING FRAMES AS APPLIED TO UNIQUENESS AND EXISTENCE QUESTIONS IN DIFFERENTIAL GEOMETRY

P. GRIFFITHS

TABLE OF CONTENTS

0. INTRODUCTION.

These notes are an exposition of the philosophy due to Elie Cartan that, via the use of moving frames, the theory of Lie groups constitutes a powerful and elegant method for studying uniqueness and existence questions for submanifolds of a homogeneous space. This philosophy, as expounded in his beautiful book "Groupes finis et continus et la géométrie différentielle", Gauthier–Villars (Paris), is perhaps not as widely appreciated as it should be, especially as regards the higher order invariants of a submanifold. A possible reason for this is that, even though the basic Lie group statements underlying the theory are of a rather general nature, their application to geometry seems at present more adapted to special cases depending on subtle conditions of non-degeneracy, rather than constituting a vast general theory. It is the intricacy and beauty of these special cases which in the end justifies the general approach. Our purpose here is to present a somewhat updated and hopefully clear exposition of a portion of the Cartan philosopy, together with a few traditional and some new applications to geometry. In particular, we emphasize the case of holo-

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morphic curves in locally homogeneous complex manifolds. It is here that the non-degeneracy assumptions become most easily dealt with analytically. Moreover, as will be further discussed below, we have in mind eventual applications of these methods to variations of Hodge structure, and also perhaps to value distribution theory.

We now give a brief outline of this paper. In Section 1, we state and prove the two propositions about uniqueness and existence of mappings of a manifold into a Lie group G which underlie the theory. Both results are phrased in terms of the left invariant Maurer–Cartan forms on G, and they essentially boil down to the standard existence and uniqueness for ordinary differential equations as given by the Frobenius theorem.

In Section 2, we discuss a few examples of how a Lie group G may be frequently interpreted as the set of "frames" on a homogeneous space G/H. When this is done, the Maurer–Cartan forms appear in the structure equations of a "moving frame", and the Maurer–Cartan equations give a complete set of relations for the structure equations of a moving frame. The question of describing the position of a submanifold M of G/H may then be thought of as attaching to M a "natural frame", or, equivalently, a cross-section of the fibration $G \to G/H$ over M. The Maurer–Cartan forms for G, when restricted to this natural frame, become a complete set of invariants for M in G/H.

Before proceeding to some remarks on the general theory, we thought it worthwhile to mention some classical examples. Section 3 is therefore devoted to Euclidean differential geometry, in particular, to a proof of the standard uniqueness and existence theorems for curves and hypersurfaces in $\mathbf{R}^n$, both presented in the general philosophy of this paper. Of course, there are many more rigidity theorems than those discussed here, but generally speaking they seem to involve either global considerations or thorny algebraic problems.

In Section 4 we take up holomorphic curves in complex projective space. Via the use of Frenet frames, the Cartan structure equations for the unitary group immediately yield the (unintegrated) Second Main Theorem of H. and J. Weyl. Next, it is observed that, by use of the general lemmas on Lie groups, the Second Main Theorem easily implies Calabi's striking result (Ann. of Math., vol. 58 (1953), pp. 1–23) that a non-degenerate holomorphic curve in $\mathbf{P}^n$ is uniquely determined, up to a rigid motion, by its first fundamental form alone, this being in strong contrast to the real case. Following this, we give a new and easily stated existence theorem for when a metric Riemann surface can be isometrically mapped into $\mathbf{P}^n$. Section 4 concludes with a brief discussion concerning the local character of the classical Plücker relations. By Calabi's result, these Plücker relations exist locally, at least in principle, and they can be made explicit by the Second Main Theorem.

Section 5 is perhaps the most important one. In it we discuss the related questions of *rigidity* and *contact* of submanifolds of a homogeneous space. These constitute the central theme of Cartan's book cited above. The idea is that, by going to a sufficiently high order jet or contact element of a submanifold

M of a homogeneous space, there will naturally appear a good frame over M in a similar manner to the appearance of the Frenet frames of a curve in Euclidean space. Restricting the Maurer–Cartan forms to this "natural frame" then gives a complete set of invariants of M. Moreover, these invariants may be arbitrarily prescribed when $\dim M = 1$, and may be prescribed subject only to the integrability conditions arising from the Maurer–Cartan equations when $\dim M > 1$. The effective use of frames in specific cases involves subtle questions of higher order geometry, and goes far beyond the somewhat common notion that "frames are essentially the same as studying connections in the principal bundle of the tangent bundle." It is the analysis of higher order contact that necessitates non-degeneracy assumptions on the submanifold, and it is perhaps for this reason that general results in higher order geometry seem to me less interesting than special cases.

In the second part of Section 5, we illustrate the general philosophy by proving a rigidity theorem for non-degenerate curves in the Grassmannian $G(n, 2n)$ of oriented n-planes in $\mathbf{R}^{2n}$. We find that a non-degenerate curve Λ is uniquely determined, up to rigid motion, by the *second* order information along Λ, and moreover the n first order and $n(n - 1)$ second order invariants may be arbitrarily prescribed.

Having in mind the rigidity of holomorphic curves in $\mathbf{P}^n$ (§4) and second order behavior of real curves in the Grassmannian $G(n, 2n)$ (§5), we turn in Section 6 to the rigidity question for non-degenerate holomorphic curves in the complex Grassmannian $\mathbf{G}(n, 2n)$. Such a non-degenerate curve $\Lambda(\zeta)$ turns out to be uniquely given by its second order behavior. However, the holomorphic nature of $\Lambda(\zeta)$ implies that the number of independent second order invariants is *at most* $n(n - 1)/2$, which is one-half the number in the real case. Although we think it is likely, we are unable to definitely prove that these invariants are of second and not first order. In §6(b) we give a geometric interpretation of a regular point on a non-degenerate holomorphic curve $\Lambda(\zeta)$ in $\mathbf{G}(2, 4)$. Thinking of $\Lambda(\zeta)$ as a family of lines in $\mathbf{P}^3$, the condition that Λ be degenerate is that the ruled surface S_Λ traced out by the family of lines $\Lambda(\zeta)$ be developable. A point ζ_0 on a non-degenerate curve Λ is regular in case the lines $\Lambda(\zeta_0 + \epsilon)$ do not meet the line $\Lambda(\zeta_0)$ for small ϵ. We then characterize regular points according to the Schubert hyperplanes in $\mathbf{G}(2, 4)$ which meet $\Lambda(\zeta)$ to second order at $\zeta = \zeta_0$.

To conclude this introduction, we should like to make a few comments of a general nature leading to a brief discussion of the problem which provided the underlying motivation for this study.

The first remark is that the discussion of higher order invariants seems to be considerably simpler for submanifolds of a homogeneous space G/H where H is compact. In this connection, the reader is invited to compare the Grassmannian example given in §5 with the case of curves in the affine or projective plane discussed in the book by Cartan, which require jets of orders 3 or 5 respectively to frame the situation.

Now in the case of a holomorphic curve in $\mathbf{P}^n$ or $\mathbf{G}(n, 2n)$, we may use either the complex linear group, which is more natural for the study of analytic or geometric invariants, or the unitary group which relates to metric invariants. So far as *first* order behavior is concerned, both approaches lead essentially to the same results, essentially because of the *infinitesimal Wirtinger principle* expressing the analytic invariants in metric terms. However, when one passes to higher order contact, the relationship between analytic and metric invariants seems less intimate. For holomorphic curves in $\mathbf{P}^n$, this doesn't matter because of Calabi's rigidity theorem mentioned above. On the other hand, for holomorphic curves in $\mathbf{G}(n, 2n)$ the lack of a good analytic interpretation of the 2nd order invariants may cause difficulty in trying to study the deeper analytic properties, such as the value distribution theory, of these curves.

Now one case in which metrics are *intrinsic* to the analytic situation is that encountered in variation of Hodge structure.[1] Here, one may think of being given an analytic family $\{V_\zeta\}_{\zeta \in M}$ of smooth, projective algebraic varieties V_ζ, and then the periods of the integrals on V_ζ, or equivalently the Hodge decomposition of the cohomology $H^*(V_\zeta, \mathbf{C})$, generates a holomorphic *period mapping*

$$\Lambda : M \to \Gamma \backslash G/H \tag{0.1}$$

where G/H is a *period matrix domain* or, equivalently, *classifying space for Hodge structures*, and Γ is the *monodromy group*. The Riemann–Hodge bilinear relations induce intrinsic metrics in the situation (0.1), and it seems of interest to find a complete set of local invariants for the period mapping Λ. As a first guess, we would suggest that, as a consequence of the infinitesimal period relation (cf., Griffiths–Schmid, loc. cit.), the period mapping is determined up to rigid motion by its first order information.

Looking beyond the local situation, among the deepest and most interesting results in the study of Hodge structures are those concerning the global monodromy group Γ in case M is an algebraic variety. The basic outstanding problem is whether or not Γ is of finite index in its arithmetic closure $\Gamma_{\mathbf{Z}}$. One possible approach to this question is the following: Let $\tilde{M}$ be the universal covering of M and

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{\Lambda}} & G/H \\ \downarrow & & \downarrow \\ M & \xrightarrow{\Lambda} & \Gamma\backslash G/H \end{array}$$

the lifted period mapping. By definition, the image $\tilde{\Lambda}(\tilde{M})$ is Γ-invariant, and, as a consequence of the finiteness of the volume of $\Lambda(M)$, *if* in addition

[1] An expository account of this subject is given in *Variation of Hodge Structure: A Discussion of Recent Results and Methods of Proof* by P. Griffiths and W. Schmid, to appear in Proc. Tata Institute Conference on Discrete Groups and Moduli, Bombay (1973).

the image is Γ_z-invariant, then Γ is of finite index in Γ_z. However, in general $\tilde{\Lambda}(\tilde{M})$ will not be Γ_z-invariant, but the "span" in G/H of $\tilde{\Lambda}(\tilde{M})$ should be. Now the "span" of a subvariety of a homogeneous space has not yet been defined precisely, but intuitively it should be the homogeneous subspace G'/H' ($G' \subset G$, $H' = H \cap G'$) generated by the osculating frames of sufficiently high order of the subspace. In particular, the "span" should be all of G/H in case Λ is "non-degenerate", and by Deligne's semi-simplicity theorem for the global monodromy group Γ, the span of $\tilde{\Lambda}(\tilde{M})$ should be all of G/H in case Γ is irreducible. Restricting to this situation, one might hope that a suitable modification of the finite volume argument would yield the general result.

This is, of course, extremely speculative, but it seemed worthwhile to mention a possible global implication of the study of the local higher order behavior of a subvariety of a homogeneous space. In any event, as regards the period mapping (0.1), it seems to us that either (i) there are no higher order invariants, or (ii) if there turn out to be higher order invariants, then these should be of algebro-geometric interest.

1. TWO LEMMAS ON MAURER–CARTAN FORMS.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. The left-invariant *Maurer–Cartan forms* on G may be considered collectively as a $\mathfrak{g}$-valued 1-form on G which satisfies the *Maurer–Cartan equation*

$$\boxed{d\omega = \tfrac{1}{2}[\omega, \omega]} \tag{1.1}$$

More precisely, if we let $X_1, \cdots, X_n$ be a basis for the Lie algebra $\mathfrak{g}$ of left invariant vector fields on G and $\omega_1, \cdots, \omega_n$ the dual basis for $\mathfrak{g}^*$, then $\omega = \sum_{i=1}^n X_i \otimes \omega_i$. Setting $d(X_i \otimes \omega_i) = X_i \otimes d\omega_i$ and $[X_i \otimes \omega_i, X_j \otimes \omega_j] = [X_i, X_j] \otimes \omega_i \wedge \omega_j$, equation (1.1) is satisfied.

For example, in the general linear group $GL(n, \mathbf{R})$ we let $g = (g_{ij})$ be a variable non-singular matrix, and then

$$\omega = g^{-1}\, dg \tag{1.2}$$

is the Maurer–Cartan form, which satisfies the equation (1.1) for $GL(n, \mathbf{R})$

$$d\omega = \omega \wedge \omega.$$

In general, if $G \subset GL(n, \mathbf{R})$ is a closed linear group, then the Maurer–Cartan forms for G are spanned by the restrictions to G of the matrix entries ω_{ij} in (1.2). For instance, if $G = O(n)$ is the orthogonal group, then the Maurer–Cartan form ω for G will be thought of as being given by (1.2), subject to the relation

$$\omega + {}^t\omega = 0.$$

We shall give two simple and essentially local results concerning smooth maps of a manifold M into a Lie group G. The first of these is

(1.3) *Let f, $\tilde{f}M \to G$ be two smooth maps of a connected manifold M into G. Then*

$$f(x) = g \cdot \tilde{f}(x)$$

for fixed $g \in G$ if, and only if,

$$f^*\omega = \tilde{f}^*\omega$$

where ω is the Maurer–Cartan form on G.

Proof. In the linear case we write

$$f(x) = g(x)\tilde{f}(x),$$

and differentiate to obtain

$$f^{-1}\,df = f^{-1}\,dg\tilde{f} + \tilde{f}^{-1}\,d\tilde{f},$$

from which it is obvious using (1.2) that

$$dg = 0 \Leftrightarrow f^*\omega = \tilde{f}^*\omega,$$

thus proving our result in this case.

In the general case, we first remark that a smooth mapping $h \cdot M \to G$ is constant if, and only if, $h^*\omega \equiv 0$ since ω gives a basis for the cotangent space at all points of G. We shall apply this to

$$h(x) = f(x)\tilde{f}(x)^{-1}$$

to show that, if $f^*\omega = \tilde{f}^*\omega$, then $h^*\omega = 0$ at an arbitrary point $x_0 \in M$. Changing $f(x)$ and $\tilde{f}(x)$ into $f(x_0)^{-1}f(x)$ and $\tilde{f}(x_0)^{-1}\tilde{f}(x)$, respectively, does not change the assumption $f^*\omega = \tilde{f}^*\omega$ nor the condition that $h^*\omega = 0$ at x_0, and thus we may assume that $f(x_0) = \tilde{f}(x_0) = e$. If M is an open interval on the real line, and if ξ, $\eta \in T_e(G)$ are the respective tangent vectors to the curves $f(x)$, $\tilde{f}(x)$ at $x = x_0$, then

$$\xi - \eta \in T_e(G)$$

is the tangent to the product curve $f(x) \cdot \tilde{f}(x)^{-1}$ at $x = x_0$. From this it follows that, at x_0,

$$h^*\omega = f^*\omega - \tilde{f}^*\omega,$$

which then proves (1.3). Q.E.D.

Our second lemma is a well-known existence theorem:

(1.4) *Suppose that φ is a $\mathfrak{g}$-valued 1-form on a connected and simply connected manifold M. Then there exists a C^∞ map $f : M \to G$ with $f^*\omega = \varphi$ if, and only if,*

$$d\varphi = \tfrac{1}{2}[\varphi, \varphi].$$

Moreover, the resulting map is unique up to left translation.

Proof. We first prove (1.4) locally around a point $x_0 \in V$ by using the Frobenius theorem to construct the graph of f. To insure that the submanifold of $M \times G$ to be constructed is indeed a graph, some preliminary considerations are necessary. Denoting small neighborhoods of x_0 by U, it is easy, using the exponential map for G, to find $g : U \to G$ such that $g(x_0) = e$ and $g^*\omega = \varphi$ *at* x_0. Now write the desired mapping f in the form

$$f(x) = h(x) \cdot g(x),$$

and we shall seek to find h with

$$h^*\omega = \psi = Ad\, g(\varphi - g^*\omega), \tag{1.5}$$

since (1.5) implies that $f^*\omega = \varphi$ as required. Note that, by construction,

$$\psi(x_0) = 0, \tag{1.6}$$

and the integrability condition

$$d\psi = \tfrac{1}{2}[\psi, \psi]$$

follows from that for φ and (1.5).

On $U \times G$ we consider the differential system given by the $\mathfrak{g}$-valued 1-form

$$\theta = \psi - \omega.$$

This system is completely integrable, since

$$\begin{aligned} d\theta &= \tfrac{1}{2}[\varphi, \varphi] - \tfrac{1}{2}[\omega, \omega] \\ &= \tfrac{1}{2}[\theta, \varphi] + \tfrac{1}{2}[\omega, \theta] \\ &\equiv 0 \text{ (modulo } \theta). \end{aligned}$$

Moreover, in the tangent space to $U \times G$ at (x_0, e) the equation

$$\theta(x_0) = 0$$

defines the tangent space to $U \times \{e\}$ by (1.6). Consequently, both the complete integrability and rank conditions required by the *Frobenius theorem* are satisfied, and so we may find a maximal integral manifold V of the differential system $\theta = 0$ passing through (x_0, e). Since

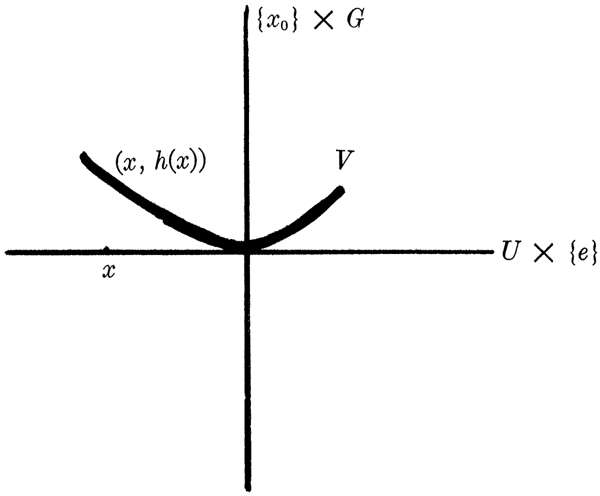
$$T_{(x_0, e)}(V) = T_{(x_0, e)}(U \times \{e\}),$$

V is the graph of a map $h : U \to G$.

By construction, $h^*\omega = \varphi$ and so (1.4) has been proved locally.

Now we cover M by open sets U_α in which there are maps $f_\alpha : U_\alpha \to G$ with $f_\alpha^*\omega = \varphi$. In $U_\alpha \cap U_\beta$,

$$g_{\alpha\beta} = f_\alpha \cdot f_\beta^{-1}$$



give *constant* maps into G, and thus define a flat principal bundle with fibre G. Since M is simply-connected, this bundle has a global flat cross-section $\{g_\alpha\}$, and replacing f_α by $g_\alpha^{-1} f_\alpha$ gives a global map $f : M \to G$ satisfying $f^*\omega = \varphi$.
Q.E.D.

2. LIE GROUPS AND MOVING FRAMES.

In geometry one frequently encounters the *homogeneous manifold* G/H of cosets of a closed subgroup H of Lie group G, and one is interested in the geometric properties of submanifolds M of G/H which are invariant under G. The philosophy of Elie Cartan is that, in many cases, G may be identified with a set of "frames" on G/H, and then associated to a submanifold M of G/H will be a natural set of frames or, if one likes, cross-sections of $G \to G/H$ over M. In this situation the Maurer–Cartan forms on G when restricted to this natural set of frames over M yield a complete set of invariants to which the two Lie group lemmas of §1 may be applied. In this section we will give a few examples of Lie groups interpreted as frames on a homogeneous space, and in the following paragraphs these considerations will be applied to geometry.

Example 1. On the Euclidean space $\mathbf{R}^n$ we define a *frame* F to be a set of vectors

$$F = (x; e_1, \cdots, e_n)$$

where $x \in \mathbf{R}^n$ is a position vector and $e_1, \cdots, e_n$ is an orthonormal basis for $\mathbf{R}^n$. In an obvious way the set of all such frames F may be identified with the Lie group $E(n)$ of Euclidean motions, x being the translation component and $e_1, \cdots, e_n$ the rotation part of a general Euclidean motion.

In this language, obtaining the left-invariant Maurer–Cartan forms on $E(n)$

is equivalent to the Cartan method of "moving frames". More precisely, given a smooth mapping

$$\psi \cdot E(n) \to \mathbf{R}^n$$

which is *equivariant* in the sense that

$$\psi(F \cdot F') = F \cdot \psi(F')$$

for frames $F, F' \in E(n)$, the differential $d\psi$ is an $\mathbf{R}^n$-valued 1-form on $E(n)$ which may be written as

$$d\psi(F) = \sum_{i=1}^{n} \psi_i(F) e_i$$

at a frame $F = (x; e_1, \cdots, e_n)$. It follows immediately that, *writing the differential $d\psi$ at the frame F in terms of the basis for $\mathbf{R}^n$ determined by F leads to left invariant 1-forms ψ_i on $E(n)$.* This is the Cartan method of moving frames, and when ψ is taken to be x or one of the e_i's, we are led to a basis for the Maurer–Cartan forms on $E(n)$. Explicitly, we write

$$\boxed{\begin{cases} dx = \sum_i \omega_i e_i \\ de_i = \sum_j \omega_{ij} e_j, \qquad \omega_{ij} + \omega_{ji} = 0 \end{cases}} \tag{2.1}$$

and then the ω_i and ω_{ij} $(i < j)$ give a basis for the Maurer–Cartan forms on the Euclidean group. Taking the exterior derivatives of the equations (2.1) yields

$$\boxed{\begin{cases} d\omega_i = \sum_j \omega_{ij} \wedge \omega_j \\ d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} \end{cases}} \tag{2.2}$$

which are the Maurer–Cartan equations (1.1) for the group $E(n)$.

From the fibre bundle point of view, the position vector map

$$x : E(n) \to \mathbf{R}^n \tag{2.3}$$

gives the principal fibration $E(n) \to E(n)/O(n)$ with group $O(n)$. The forms ω_i are horizontal in the fibration, and the quadratic differential form

$$I = \sum_i (\omega_i)^2$$

is the pull-back under x of the usual metric on $\mathbf{R}^n$. The fibration (2.3) may be identified with the principal bundle of orthonormal tangent frames for this metric. When this is done, the first equation in (2.2) says that the ω_{ij} form the connection matrix for the Riemannian connection on $\mathbf{R}^n$, and the second equation states that this connection has no curvature.

Example 2. We shall represent points of complex projective n-space $\mathbf{P}^n$ by non-zero homogeneous coordinate vectors $Z = [z_0, \cdots, z_n]$ in $\mathbf{C}^{n+1}$. A frame F for $\mathbf{P}^n$ is a unitary basis

$$F = \{Z_0, Z_1, \cdots, Z_n\}$$

for $\mathbf{C}^{n+1}$; thus the set of all such frames is the unitary group $U(n+1)$. Writing as before

$$\boxed{dZ_i = \sum_j \theta_{ij} Z_j, \qquad \theta_{ij} + \bar{\theta}_{ji} = 0} \tag{2.4}$$

gives the Maurer–Cartan forms θ_{ij} on $U(n+1)$, and taking the exterior derivative of (2.4) gives the Maurer–Cartan equation

$$\boxed{d\theta_{ij} = \sum_k \theta_{ik} \wedge \theta_{kj}} \tag{2.5}$$

for the unitary group.

The mapping

$$Z_0 : U(n+1) \to \mathbf{P}^n \tag{2.6}$$

gives a principal fibration with fibre $U(1) \times U(n)$, the $U(1)$ factor corresponding to all rotations

$$Z_0 \to e^{\sqrt{-1}\psi_0} Z_0$$

in the line $\overrightarrow{OZ_0}$ in $\mathbf{C}^{n+1}$, and the $U(n)$ factor being the rotations in the orthogonal plane $Z_0^{\perp}$ to Z_0. The corresponding vector bundles are the universal line bundle

$$L \to \mathbf{P}^n$$

with fibre $L_{Z_0} = \overrightarrow{OZ_0}$, and universal quotient bundle

$$Q \to \mathbf{P}^n$$

with fibre $Q_{Z_0} = \mathbf{C}^{n+1}/L_{Z_0}$.

Using the index range $0 \leq i, j, k \leq n$ and $1 \leq \alpha, \beta, \gamma \leq n$, the 1-forms

$$\theta_{0\alpha}$$

are horizontal for the fibration (2.6), and we claim that they are of type (1, 0) for the usual complex manifold structure on $\mathbf{P}^n$. To verify this, let $\zeta \to Z(\zeta)$ be a holomorphic mapping of the ζ-disc into $\mathbf{P}^n$ and

$$Z_0(\zeta), Z_1(\zeta), \cdots, Z_n(\zeta)$$

a C^∞ lifting of this map into $U(n+1)$. Then

$$Z_0(\zeta) = e^{\sqrt{-1}\psi_0} \frac{Z(\zeta)}{||Z(\zeta)||},$$

and consequently $\bar{\partial} Z_0$ is a multiple of Z_0 since $\bar{\partial} Z(\zeta) = 0$. It follows that

$$dZ_0(\zeta) = \theta_{00} Z_0 + \sum_\alpha \theta_{0\alpha} Z_\alpha$$

where all $\theta_{0\alpha}$ are of type (1, 0).

The (1, 1) form

$$\Omega_0 = \frac{\sqrt{-1}}{2} \{ \sum_\alpha \theta_{0\alpha} \wedge \bar{\theta}_{0\alpha} \}$$

is the pull-back to $U(n+1)$ in (2.6) of the standard Kähler form on $\mathbf{P}^n$ associated to the Fubini-Study metric. The matrices of 1-forms

$$\{\theta_{00}\}, \quad \{\theta_{\alpha\beta}\}$$

are the connection matrices for the metric connections in the universal bundles L and Q respectively. Setting

$$\varphi_{\alpha\beta} = \delta_{\alpha\beta}\theta_{00} - \theta_{\beta\alpha},$$

it follows from (2.5) that

$$\begin{cases} d\theta_{0\alpha} = \sum_\beta \varphi_{\alpha\beta} \wedge \theta_{0\beta} \\ \varphi_{\alpha\beta} + \bar{\varphi}_{\beta\alpha} = 0, \end{cases}$$

and thus $\{\varphi_{\alpha\beta}\}$ is the connection matrix for the Kähler metric

$$I = \sum_\alpha \theta_{0\alpha} \bar{\theta}_{0\alpha}$$

on $\mathbf{P}^n$.

Example 3. We denote by $G(k, n)$ the Grassmann manifold of oriented k-planes in $\mathbf{R}^n$.[1] Points of $G(k, n)$ will be denoted by Λ, and a frame F lying over Λ is a set

$$F = (e_1, \cdots, e_k; e_{k+1}, \cdots, e_n)$$

of vectors such that $e_1, \cdots, e_k$ forms an oriented basis for Λ and $e_1, \cdots, e_n$ forms an oriented basis for $\mathbf{R}^n$. The set of all such frames is clearly the proper orthogonal group $SO(n)$, and the fibration

$$SO(n) \to SO(n)/SO(k) \times SO(n-k) \cong G(k, n) \tag{2.7}$$

sends F to $\Lambda = e_1 \wedge \cdots \wedge e_k$.

The structure equations for this frame manifold are

[1] Since the considerations in this paper are primarily local, questions or orientation will not concern us.

$$\boxed{\begin{aligned} de_i &= \sum_j \omega_{ij} e_j \\ d\omega_{ij} &= \sum_k \omega_{ik} \wedge \omega_{kj} \end{aligned}} \tag{2.8}$$

If we use the index range $1 \leq \alpha, \beta \leq k$, $k + 1 \leq \mu, \nu \leq n$, then the 1-forms

$$\omega_{\alpha\mu}$$

are horizontal for the fibration (2.7). The quadratic differential form

$$I = \sum_{\alpha,\mu} (\omega_{\alpha\mu})^2$$

is the standard invariant metric on $G(k, n)$, and the matrices

$$\{\omega_{\alpha\beta}\}, \quad \{\omega_{\mu\nu}\}$$

of 1-forms are the connection matrices for the obvious universal vector bundles over the Grassmannian.

In Section 5 we shall use the *Plücker embedding*

$$p : G(k, n) \to \bigwedge^k \mathbf{R}^n \cong \mathbf{R}^{\binom{n}{k}}$$

obtained by sending a k-plane Λ into

$$e_1 \wedge \cdots \wedge e_k \in \bigwedge^k \mathbf{R}^n$$

where $e_1, \cdots, e_k$ forms an oriented basis for Λ. From (2.8) we find the relation

$$d(e_1 \wedge \cdots \wedge e_k) = \sum_{\alpha,\mu} \pm (e_1 \wedge \cdots \wedge \hat{e}_\alpha \wedge \cdots \wedge e_k \wedge e_\mu)\omega_{\alpha\mu}$$

which implies that: *p is an equivariant isometric embedding.*

Similar considerations apply also to the Grassmannian $PG(k, n) \cong G(k + 1, n + 1)$ of oriented k-planes in real projective space, and to the complex Grassmannians $\mathbf{G}(k, n)$ of complex k-planes in $\mathbf{C}^n$ and $\mathbf{P}G(k, n) \cong \mathbf{G}(k + 1, n + 1)$ of k-planes in $\mathbf{P}^n$. This latter example will be extensively discussed in §6 below.

3. EUCLIDEAN GEOMETRY.

(a) *Curves in $\mathbf{R}^n$.* We shall see that, through the use of Frenet frames, (1.3) and (1.4) may be interpreted as the classical statements that the curvature, torsion, etc. of a curve in Euclidean space uniquely determine the curve up to rigid motion, and that these quantities may be arbitrarily prescribed. Let $x(s)$ be a curve in $\mathbf{R}^n$, which for convenience we assume parametrized by arc length so that $\|x'(s)\| = 1$. The curve is *non-degenerate* in case it does not lie in a linear subspace. This is expressed analytically by saying that the *Wronskian*

$$W(s) = x(s) \wedge x'(s) \wedge \cdots \wedge x^{(n-1)}(s)$$

is not identically zero. The point s_0 is *regular* in case $W(s_0) \not\equiv 0$, and in a neighborhood of a regular point we may define a *Frenet frame*

$$(x(s), e_1(s), \cdots, e_n(s))$$

inductively by the condition

$$e_1(s) \wedge \cdots \wedge e_k(s) = \pm \frac{x'(s) \wedge \cdots \wedge x^{(k)}(s)}{\|x'(s) \wedge \cdots \wedge x^{(k)}(s)\|}.$$

Geometrically, the vectors $e_1(s), \cdots, e_k(s)$ span the *kth osculating space* to the curve, and they are uniquely determined up to sign. The Frenet frame gives a distinguished lifting of the curve to the frame manifold, and a complete set of invariants is obtained by restricting the Maurer–Cartan forms to this lifting.

More precisely, since $x'(s) = e_1(s)$ we observe that, on the Frenet frame,

$$\omega_1 = ds \quad \text{and} \quad \omega_2 = \cdots = \omega_n = 0.$$

Next, since $e_k(s)$ is a linear combination of $x'(s), \cdots, x^{(k)}(s)$, de_k is a linear combination of $x'(s), \cdots, x^{(k+1)}(s)$ and consequently

$$\omega_{kl} = 0 \qquad \text{for} \quad l > k + 1$$

Using the symmetry $\omega_{kl} + \omega_{lk} = 0$ and writing

$$\omega_{k,k+1} = \kappa_k(s)\, ds,$$

the structure equations (2.1) restricted to the Frenet frame become the *Frenet equations*

$$\boxed{\begin{aligned} \frac{dx}{ds} &= e_1 \\ \frac{de_k}{ds} &= -\kappa_{k-1}(s)e_{k-1} + \kappa_{k+1}(s)e_{k+1} \end{aligned}} \tag{3.1}$$

For $n = 3$, $\kappa_1(s)$ is the *curvature* and $\kappa_2(s)$ is the *torsion.* In general, (1.3) and (1.4) imply the classical statements: *The "curvatures" $\kappa_k(s)$ ($k = 1, \cdots, n - 1$) uniquely determine the curve $x(s)$ up to rigid motion,* and by setting $\omega_1 = ds$, $\omega_2 = \cdots = \omega_n = 0$, $\omega_{k,k+1} = \kappa_k(s)\, ds$, and $\omega_{kl} = 0$ for $l > k + 1$, *a curve $x(s)$ exists with preassigned curvature functions $\kappa_k(s)$.*

(b) *Hypersurfaces in Euclidean space.* Let $M \subset \mathbf{R}^{n+1}$ be a connected hypersurface. Associated to M are the *Darboux frames*

$$(x; e_1, \cdots, e_n, e_{n+1})$$

where $x \in M$, the vectors $e_1, \cdots, e_n$ constitute an orthonormal tangent frame to M at x, and e_{n+1} is a unit normal. Using the index range

$$\begin{cases} 1 \leq i, j \leq n + 1 \\ 1 \leq \alpha, \beta \leq n, \end{cases}$$

the e_α are determined up to orthogonal transformation and e_{n+1} is determined up to sign. By restricting the Maurer–Cartan forms on $E(n + 1)$ to the manifold of Darboux frames, we obtain a complete set of invariants of the hypersurface M. We shall interpret these as the traditional first and second fundamental forms of M, and upon doing this the Maurer–Cartan equations (2.2) will become the Gauss and Codazzi equations.

More precisely, since dx is tangent to M and is thus a linear combination of the e_α ,

$$\omega_{n+1} = 0 \tag{3.2}$$

on the manifold of Darboux frames.[1] The quadratic differential form

$$I = \sum_\alpha (\omega_\alpha)^2$$

is the *first fundamental form* of M. To obtain the second fundamental form, we use (3.2) and (2.2) to have

$$0 = d\omega_{n+1} = \sum_\alpha \omega_{n+1,\alpha} \wedge \omega_\alpha \tag{3.3}$$

Applying the *Cartan lemma*[2] to (3.3) gives

$$\omega_{\alpha,n+1} = \sum_\beta b_{\alpha\beta}\omega_\beta , \qquad b_{\alpha\beta} = b_{\beta\alpha} ,$$

and the *second fundamental form* is defined by

$$II = \sum_{\alpha,\beta} b_{\alpha\beta}\omega_\alpha\omega_\beta \tag{3.4}$$

The Maurer–Cartan equations (2.2) are

$$\boxed{\begin{aligned} d\omega_\alpha &= \sum_\beta \omega_{\alpha\beta} \wedge \omega_\beta \\ d\omega_{\alpha\beta} - \sum \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} &= \omega_{\alpha,n+1} \wedge \omega_{n+1,\beta} \quad \text{(Gauss)} \\ d\omega_{\alpha,n+1} &= \sum_\beta \omega_{\alpha\beta} \wedge \omega_{\beta,n+1} \quad \text{(Codazzi)} \end{aligned}} \tag{3.5}$$

We now use (1.3) to prove that: *M is determined up to rigid motion by its first and second fundamental forms.* To do this it is convenient to think of M as an abstract manifold together with a map

[1] Moreover, it is clear that (3.2) is the only additional relation beyond the Maurer–Cartan equations.

[2] The Cartan lemma states that if $\omega_1, \cdots, \omega_n$ are linearly independent 1-forms on a manifold N, and if $\varphi_1, \cdots, \varphi_n$ are 1-forms on N which satisfy

$$\sum_\alpha \varphi_\alpha \wedge \omega_\alpha = 0,$$

then $\varphi_\alpha = \sum a_{\alpha\beta}\omega_\beta$ where $a_{\alpha\beta} = a_{\beta\alpha}$.

$$M \xrightarrow{x} \mathbf{R}^{n+1}.$$

Given another map

$$M \xrightarrow{\tilde{x}} \mathbf{R}^{n+1}$$

such that $I = \tilde{I}$, the manifold of Darboux frames is the same for both embeddings. Thus

$$\begin{cases} \omega_\alpha = \tilde{\omega}_\alpha & (\alpha = 1, \cdots, n) \\ \omega_{n+1} = \tilde{\omega}_{n+1} = 0 \end{cases} . \tag{3.6}$$

We next claim that, as a consequence of (3.6),

$$\omega_{\alpha\beta} = \tilde{\omega}_{\alpha\beta} \qquad \text{(Gauss theorem)} . \tag{3.7}$$

To prove this, use the Gauss equation in (3.5) to obtain

$$\sum_\beta (\omega_{\alpha\beta} - \tilde{\omega}_{\alpha\beta}) \wedge \omega_\beta = 0 .$$

By the Cartan lemma,

$$\omega_{\alpha\beta} - \tilde{\omega}_{\alpha\beta} = \sum a_{\alpha\beta\gamma}\omega_\gamma , \qquad a_{\alpha\beta\gamma} = a_{\alpha\gamma\beta} .$$

But $a_{\alpha\beta\gamma} = -a_{\beta\alpha\gamma}$ since $\omega_{\alpha\beta} + \omega_{\beta\alpha} = 0$, and this easily implies that all $a_{\alpha\beta\gamma} = 0$. If now $II = \tilde{II}$, then

$$\omega_{\alpha,n+1} = \tilde{\omega}_{\alpha,n+1} \tag{3.8}$$

by (3.4). Combining (3.6)–(3.8) we see that all Maurer–Cartan forms agree on the frame bundle of M, and rigidity follows from (1.3).

Suppose now that M is a simply-connected n-manifold on which we have a Riemannian metric I and quadratic differential form II. On the orthonormal frame bundle for M, global 1-forms ω_α and $\omega_{\alpha\beta}$ are uniquely defined by

$$\begin{cases} I = \sum (\omega_\alpha)^2 \\ d\omega_\alpha = \sum_\beta \omega_{\alpha\beta} \wedge \omega_\beta , \qquad \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0 . \end{cases}$$

If we define $\omega_{\alpha,n+1} = -\omega_{n+1,\alpha}$ by

$$\omega_{\alpha,n+1} = \sum_\beta b_{\alpha\beta}\omega_\beta$$

where

$$II = \sum_{\alpha,\beta} b_{\alpha\beta}\omega_\alpha\omega_\beta ,$$

and if the Gauss and Codazzi equations (3.7) are subsequently satisfied, then from (1.4) we obtain: *Given a simply-connected n-manifold M with Riemannian metric I and quadratic form II such that the equations of Gauss and Codazzi are* *satisfied, there exists an immersion $M \xrightarrow{x} \mathbf{R}^{n+1}$ realizing I and II as first and second fundamental forms.*

4. HOLOMORPHIC CURVES.

(a) *Frenet frames and the Second Main Theorem.* A *holomorphic curve* is a holomorphic map $Z : M \to \mathbf{P}^n$ from a Riemann surface M into complex projective space. The holomorphic curve is *non-degenerate* in case the image does not lie in a proper linear subspace of $\mathbf{P}^n$. In terms of a local holomorphic coordinate ζ on M, the holomorphic curve may be given by a holomorphic homogeneous coordinate vector

$$Z(\zeta) = [z_0(\zeta), \cdots, z_n(\zeta)].$$

Analytically, non-degeneracy is expressed by saying that the Wronskian

$$W(\zeta) = Z(\zeta) \wedge \cdots \wedge Z^{(n)}(\zeta)$$

is not identically zero. As in the real case, ζ_0 is a *regular point* in case $W(\zeta_0) \neq 0$. In the neighborhood of a regular point we may define *Frenet frames* $Z_0(\zeta), \cdots, Z_n(\zeta)$ by the conditions

$$\frac{Z(\zeta) \wedge \cdots \wedge Z^{(k)}(\zeta)}{||Z(\zeta) \wedge \cdots \wedge Z^{(k)}(\zeta)||} = e^{i\varphi} Z_0(\zeta) \wedge \cdots \wedge Z_k(\zeta). \tag{4.1}$$

Each $Z_\alpha(\zeta)$ is then determined up to a rotation

$$Z_k(\zeta) \to e^{i\varphi_k} Z_k(\zeta). \tag{4.2}$$

Geometrically, $W_k(\zeta) = Z(\zeta) \wedge \cdots \wedge Z^{(k)}(\zeta)$ in the point in the Grassmannian $\mathbf{P}G(k, n)$ of k-planes in $\mathbf{P}^n$ determined by the kth osculating space to the holomorphic curve. Since $Z_k(\zeta)$ is a linear combination of $Z(\zeta), \cdots, Z^{(k)}(\zeta)$ and $\bar\partial Z^{(k)}(\zeta) = 0$, it follows as in the real case that

$$\begin{cases} \theta_{kl} = 0 & \text{if} \quad l > k+1 \\ \theta_{k,k+1} & \text{is of type} \quad (1, 0) \end{cases} \tag{4.3}$$

Thus the *Frenet equations* for a holomorphic curve are

$$\boxed{dZ_k = \theta_{k,k-1} Z_{k-1} + \theta_{k,k} Z_k + \theta_{k,k+1} Z_{k+1}} \tag{4.4}$$

At this juncture it is convenient to recall the structure equations of local Hermitian geometry. Let $ds^2 = h^2\, d\zeta\, d\bar\zeta$ be a conformal metric on M with associated (1, 1) form

$$\Omega = \frac{\sqrt{-1}}{2} h^2\, d\zeta \wedge d\bar\zeta.$$

The *Ricci form* is defined by

$$\operatorname{Ric} \Omega = \sqrt{-1}\, \partial\bar\partial \log h; \tag{4.5}$$

equivalently,

$$\operatorname{Ric}\Omega = -2K\Omega$$

where K is the *Gaussian curvature* of the metric. To give an alternate means for calculating Ric Ω, we write

$$\Omega = \frac{\sqrt{-1}}{2}\,\theta \wedge \bar{\theta}$$

where θ is a (1, 0) form determined up to rotation $\theta \rightarrow e^{\sqrt{-1}\psi}\theta$. Then there exists a unique 1-form φ (the connection form) which satisfies

$$\text{(4.6)} \qquad \begin{cases} d\theta = \varphi \wedge \theta \\ \varphi + \bar{\varphi} = 0, \end{cases}$$

and one verifies that

$$\text{(4.7)} \qquad \operatorname{Ric}\Omega = \frac{\sqrt{-1}}{2}\,d\varphi.$$

Explicitly, if we choose $\theta = h\,d\zeta$ then $\varphi = -\partial \log h + \bar{\partial} \log h$. Rotating θ into $e^{\sqrt{-1}\psi}\theta$ changes the connection form φ into $\varphi + d\psi$, and this obviously does not change Ric Ω.

Returning to our holomorphic curve, we set

$$\text{(4.8)} \qquad \Omega_k = \frac{\sqrt{-1}}{2}\,\theta_{k,k+1} \wedge \bar{\theta}_{k,k+1}\,.$$

Clearly, Ω_k is the pull-back under W_k of the standard Kähler form on $\mathbf{P}G(k, n)$. To calculate the Ricci form of Ω_k, we shall use (4.6) to verify that the connection form for the metric determined by Ω_k is

$$\text{(4.9)} \qquad \varphi_k = \theta_{k,k} - \theta_{k+1,k+1}\,.$$

Indeed, by (2.5) and (4.3)

$$\begin{aligned} d\theta_{k,k+1} &= \theta_{k,k} \wedge \theta_{k,k+1} - \theta_{k,k+1} \wedge \theta_{k+1,k} \\ &= \varphi_k \wedge \theta_{k,k+1}\,, \end{aligned}$$

while obviously $\varphi_k + \bar{\varphi}_k = 0$. Using (4.6) it follows that the connection form for Ω_k is given by (4.9). By (4.7) and (2.5)

$$\begin{aligned} \operatorname{Ric}\Omega_k &= \frac{\sqrt{-1}}{2}\,d\varphi_k \\ &= \frac{\sqrt{-1}}{2}\,(\theta_{k,k-1} \wedge \theta_{k-1,k} + \theta_{k,k+1} \wedge \theta_{k+1,k} \\ &\quad - \theta_{k+1,k} \wedge \theta_{k,k+1} - \theta_{k+1,k+2} \wedge \theta_{k+2,k+1}) \\ &= \Omega_{k-1} + \Omega_{k+1} - 2\Omega_k \end{aligned}$$

The basic formula

$$\boxed{\operatorname{Ric}\Omega_k = \Omega_{k-1} + \Omega_{k+1} - 2\Omega_k} \tag{4.10}$$

is called the *Second Main Theorem* in the theory of holomorphic curves. The first few equations arising from (4.10) are

$$\begin{cases} \operatorname{Ric}\Omega_0 = -2\Omega_0 + \Omega_1 \\ \operatorname{Ric}\Omega_1 = \Omega_0 - 2\Omega_1 + \Omega_2 \\ \operatorname{Ric}\Omega_2 = \Omega_1 - 2\Omega_2 + \Omega_3 , \end{cases}$$

etc., and from this we see that

(4.11) *Given a non-degenerate holomorphic curve $Z(\zeta)$, the osculating metrics Ω_k $(k = 1, \cdots, k-1)$ are uniquely determined by Ω_0 using the Second Main Theorem* (4.10).

As an application of (4.10), we shall derive the theorem of Blaschke that: *The Poincaré metric $(d\zeta\, d\bar{\zeta})/(1 - |\zeta|^2)^2$ on the unit disc cannot be obtained by an isometric embedding into $\mathbf{P}^n$.*

Proof. We shall give the argument when $n = 2$, the general case being the same. Setting

$$\Omega = \frac{\sqrt{-1}}{2} \frac{d\zeta \wedge d\bar{\zeta}}{(1 - |\zeta|^2)^2} ,$$

we have

$$\operatorname{Ric}\Omega = 2\Omega.$$

If $\Omega = \Omega_0$ for some embedding in $\mathbf{P}^2$, then

$$2\Omega = \operatorname{Ric}\Omega = \operatorname{Ric}\Omega_0 = -2\Omega_0 + \Omega_1$$

or

$$\Omega_1 = 4\Omega_0 .$$

Then by (4.10)

$$\begin{cases} \operatorname{Ric}\Omega_1 = -2\Omega_1 + \Omega_0 = -3\Omega_0 , \quad \text{and} \\ \operatorname{Ric}\Omega_1 = \operatorname{Ric}\Omega_0 = 2\Omega_0 , \end{cases}$$

which is a contradiction.

(b) *Uniqueness and existence of holomorphic curves.* Using (1.3) and (4.1) we shall prove the following result of Calabi (*Isometric Imbedding of Complex Manifolds*, Ann. of Math., vol. 58 (1953), pp. 1–23):

(4.12) *A non-degenerate holomorphic curve is uniquely determined up to rigid motion by its first fundamental form Ω_0.*

Proof. Suppose that we are given two holomorphic curves

$$Z, \tilde{Z} : M \to \mathbf{P}^n$$

such that, with the obvious notation,

$$\Omega_0 = \tilde{\Omega}_0 .$$

Then, by (4.11) we obtain

$$\Omega_k = \tilde{\Omega}_k \qquad (k = 0, \cdots, n-1). \tag{4.13}$$

Assuming now (4.13), we wish to determine Frenet frames Z_k, $\tilde{Z}_k$ for Z, $\tilde{Z}$ respectively such that the Maurer–Cartan forms satisfy

$$\theta_{k,l} = \tilde{\theta}_{k,l} \qquad (k, l = 0, \cdots n). \tag{4.14}$$

The implication "(4.13) $\Rightarrow$ (4.14)" is the complex analogue of the real theorem that the arc-length, curvature, torsion, etc. determine a curve in $\mathbf{R}^n$ (§3(a)).

To begin with, rotating a given Frenet frame by

$$Z_k \to e^{\sqrt{-1}\varphi_k} Z_k$$

induces the changes

$$\begin{cases} \theta_{k,k+1} \to e^{\sqrt{-1}\varphi_k}\theta_{k,k+1} \\ \theta_{k,k} \to \sqrt{-1}\, d\varphi_k + \theta_{k,k} \end{cases} \tag{4.15}$$

on the Maurer–Cartan forms. By (4.13) and the Cartan structure equations,

$$\begin{cases} \dfrac{\sqrt{-1}}{2}\, \theta_{k,k+1} \wedge \bar{\theta}_{k,k+1} = \Omega_k = \tilde{\Omega}_k = \dfrac{\sqrt{-1}}{2}\, \tilde{\theta}_{k,k+1} \wedge \bar{\tilde{\theta}}_{k,k+1} \\ d(\theta_{k,k} - \theta_{k+1,k+1}) = \theta_{k,k+1} \wedge (\theta_{k,k} - \theta_{k+1,k+1}) \\ d\theta_{k,k} = \theta_{k,k-1} \wedge \theta_{k-1,k} + \theta_{k,k+1} \wedge \theta_{k+1,k} = d\tilde{\theta}_{k,k} \end{cases} \tag{4.16}$$

Using the last equation in (4.16) for $k = n$ gives $d(\theta_{n,n} - \tilde{\theta}_{n,n}) = 0$, and so locally on M

$$\theta_{n,n} - \tilde{\theta}_{n,n} = \sqrt{-1}\, d\varphi_n .$$

For a real function φ_n. Rotating Z_n through angle $-\varphi_n$ gives, by (4.15), $\theta_{n,n} = \tilde{\theta}_{n,n}$. The first relation in (4.16) yields

$$\theta_{k,k+1} = e^{\sqrt{-1}\varphi_k}\tilde{\theta}_{k,k+1} \qquad (k = 0, \cdots, n-1),$$

and thus rotating $Z_0, \cdots, Z_{n-1}$ through angles $\varphi_0, \cdots, \varphi_{n-1}$ and using (4.15) again gives the relations

$$\begin{cases} \theta_{k,k+1} = \tilde{\theta}_{k,k+1} \qquad (k = 0, \cdots, n-1) \\ \theta_{n,n} = \tilde{\theta}_{n,n} \end{cases} . \tag{4.17}$$

Finally, the second equation in (4.16), when combined with the first relation in (4.17), implies

$$\theta_{k,k} - \theta_{k+1,k+1} = \tilde{\theta}_{k,k} - \tilde{\theta}_{k+1,k+1} \qquad (k = 0, \cdots, n-1)$$

which when taken with the other equation in (4.17) implies (4.14). Q.E.D.

We shall now use (1.4) and the Second Main Theorem 4.10 to obtain an existence theorem for non-degenerate holomorphic curves with preassigned metric. Given a positive (1, 1) form Ω_0 on a Riemann surface M, we define a (1, 1) form Ω_1 by

$$\mathrm{Ric}\,\Omega_0 + 2\Omega_0 = \Omega_1 .$$

In case Ω_1 is positive, which is equivalent to the Gaussian curvature of Ω_0 being <1, we may define a (1, 1) form Ω_2 by

$$\mathrm{Ric}\,\Omega_1 + 2\Omega_1 - \Omega_0 = \Omega_2 .$$

If $\Omega_2 > 0$, we may then continue and define Ω_3 by (4.10) and so forth.

Definition. We say that Ω_0 *satisfies the Second Main Theorem in dimension* n in case positive (1, 1) forms $\Omega_0, \cdots, \Omega_{n-1}$ can be inductively defined by (4.10), and when this is done

$$\mathrm{Ric}\,\Omega_{n-1} = -2\Omega_{n-1} + \Omega_{n-2} .$$

(4.18) *A metric Ω_0 on a simply connected Riemann surface M is induced by a holomorphic mapping $Z : M \to \mathbf{P}^n$ if, and only if, Ω_0 satisfies the Second Main Theorem in dimension n.*

Proof. Given $\Omega_0, \cdots, \Omega_{n-1}$ satisfying (4.10), we write

$$\Omega_k = \frac{\sqrt{-1}}{2}\,\theta_{k,k+1} \wedge \bar{\theta}_{k,k+1}$$

for (1, 0) forms $\theta_{k,k+1}$. Then 1-forms φ_k satisfying

$$\text{(4.19)} \qquad \begin{cases} d\theta_{k,k+1} = \varphi_k \wedge \varphi_{k,k+1} \\ \varphi_k + \bar{\varphi}_k = 0 \end{cases}$$

are uniquely defined, and by (4.10)

$$\text{(4.20)} \qquad d\varphi_k = 2\theta_{k,k+1} \wedge \theta_{k+1,k} - \theta_{k-1,k} \wedge \theta_{k,k-1} - \theta_{k+1,k+2} \wedge \theta_{k+1,k+2} .$$

Using the Poincaré lemma, define θ_{00} by

$$\text{(4.21)} \qquad d\theta_{00} = \theta_{01} \wedge \theta_{10}$$

and set

$$\theta_{11} = \theta_{00} - \varphi_0$$
$$\theta_{22} = \theta_{11} - \varphi_1$$
$$\cdot$$

If we define $\theta_{ij} = -\bar{\theta}_{ji}$ and $\theta_{ij} = 0$ for $j > i + 1$, then (4.19)–(4.21) imply the Maurer–Cartan equations

$$d\theta_{ij} = \sum_k \theta_{ik} \wedge \theta_{kj} ,$$

and we may apply (1.4) to conclude (4.18). Q.E.D.

(c) *Discussion of the Plücker relations.* The Second Main Theorem (4.10) is one of the basic facts instrumental in the extension of the value distribution theory of an entire meromorphic function to a non-degenerate entire holomorphic curve

$$Z : \mathbf{C} \to \mathbf{P}^n$$

(c.f., "Holomorphic curves and metrics of negative curvature" by M. Cowen and the present author to appear in Jour. d'Analyse). The relation (4.10) was originally due to H. and J. Weyl, who remarked that the Second Main Theorem essentially constitutes an unintegrated Plücker formula.[1] Here we wish to point out that *the Plücker formulae are of a local nature, and are essentially reflections of the Cartan equations for the unitary group when applied to a Frenet frame.*

More precisely, we let $\mathbf{P}G(k, n)$ be the Grassmannian of k-planes in $\mathbf{P}^n$. For a fixed $(n - k - 1)$-plane Λ, the set of all k-planes W which meet Λ constitutes a divisor D_Λ on $\mathbf{P}G(k, n)$ (these are the *Schubert hyperplane sections* of the Grassmannian). The unitary group $U(n + 1)$ acts transitively on the space $\mathbf{P}G(n - k - 1, n)$ of such divisors D_Λ , and we denote by $d\Lambda$ the unique invariant measure of total volume one.

Given a (possibly non-compact) Riemann surface $\tilde{M}$ and non-degenerate holomorphic mapping $Z : \tilde{M} \to \mathbf{P}^n$, we consider the associated curves $W_k : \tilde{M} \to \mathbf{P}G(k, n)$, and denote by Ω_k the pull-back under W_k of the standard Kähler metric on $\mathbf{P}G(k, n)$. Given a relatively compact open set $M \subset \tilde{M}$, we then define the *mean degree*

$$\delta_k(M) = \int_{\Lambda \in \mathbf{P}G(n-k-1,n)} n(M, \Lambda)\, d\Lambda$$

to be the average of the number $n(M, \Lambda)$ of points of intersection of $W_k(M)$ with the Schubert hyperplanes D_Λ . In case $\tilde{M}$ is compact, we may take $M = \tilde{M}$, and then $n(M, \Lambda)$ is the same for all Λ, and $\delta_k(M)$ is *degree* of $W_k(M)$ in the usual sense of algebraic geometry. If we let

$$v_k(M) = \frac{1}{\pi} \int_M \Omega_k$$

[1] The classical Plücker formulae for an algebraic curve C in $\mathbf{P}^n$ give linear relationships between the *degrees*, as defined in algebraic geometry, of the various osculating curves $C^{(k)}$ $(k = 0, \cdots, n - 1, C^{(0)} = C)$, a formula in which the genus of C and the singularities of the various $C^{(k)}$ appear also linearly. We will discuss this formula in some detail for plane curves with ordinary singularities.

be the *area* of $W_k(M)$, then *Crofton's formula from integral geometry* gives the relation

$$\boxed{(4.23) \qquad v_k(M) = \delta_k(M),}$$

expressing the area of $W_k(M)$ as the average number of points of intersection of this curve with Schubert hyperplanes. If $M = \tilde{M}$ is compact, (4.23) is the classical *Wirtinger theorem,* equating the degree (a projective invariant) with the area (a metric invariant).

The Plücker formulae are linear relations among the mean degrees $\delta_k(M)$. They are obtained by applying the Gauss–Bonnet theorem (for singular metrics) to the Second Main Theorem (4.10). In the non-compact case, an integral over the boundary of M in $\tilde{M}$ appears, but in some cases, such as $\tilde{M} = \mathbf{C}$ and $M =$ disc of radius r, this may be estimated (cf., the paper by Cowen–Griffiths cited above). In the compact case $M = \tilde{M}$, there is no boundary term and one proceeds as follows: A *pseudo-metric* on M is a C^∞ non-negative (1, 1) form such that, given a point $x \in M$ and local holomorphic coordinate ζ centered at x,

$$\Omega = \frac{\sqrt{-1}}{2} |\zeta|^{2\mu(x)} h_0(\zeta)\, d\zeta \wedge d\bar{\zeta}$$

where h_0 is a positive C^∞ function. The singularity index $\mu(\Omega)$ and Ricci form Ric Ω are defined by

$$(4.24) \qquad \begin{cases} \mu(\Omega) = \sum_{x \in M} \mu(x) \\ \operatorname{Ric} \Omega = \dfrac{\sqrt{-1}}{2} \partial\bar{\partial} \log h_0 \, . \end{cases}$$

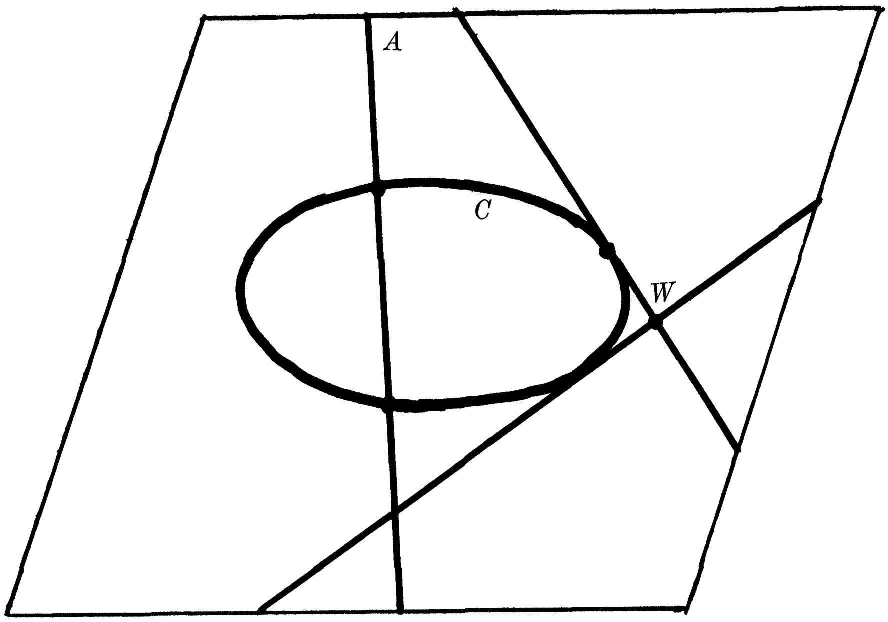
The Gauss–Bonnet theorem for singular matrics is the formula

$$(4.25) \qquad \int_M \operatorname{Ric} \Omega + \chi(M) + \mu(\Omega) = 0,$$

where $\chi(M) = 2 - 2g$ (g = genus of M) is the Euler–Poincarè characteristic. Setting $\mu_k = \mu(\Omega_k)$ and applying (4.25) to (4.10) gives the general *Plücker formulae*

$$\boxed{(4.26) \qquad \mu_k + \delta_{k-1}(M) + \delta_{k+1}(M) = 2\delta_k(M) + 2g - 2.}$$

It is perhaps worthwhile to conclude by specializing (4.26) to plane curves with ordinary singularities, which will now be explained. Given a compact Rieman surface M and non-degenerate holomorphic mapping $Z : M \to \mathbf{P}^2$, the image $Z(M)$ will be an algebraic curve C of degree d in $\mathbf{P}^2$, where d is the number of points of intersection of C with a general line A. The first associated curve will be denoted by $Z^* : M \to \mathbf{P}^{2^*}$, where $\mathbf{P}^{2^*}$ is the dual projective space of lines in $\mathbf{P}^2$. The image curve $C^* = Z^*(M)$ is the *dual curve* to C, whose degree d^* is the number of tangent lines to C passing through a general point $W \in \mathbf{P}^2$



The degree d^* of C^* is traditionally called the *class* of C. The relation

$$(Z^*)^* = Z$$

is easy to verify.

Given a point $x \in M$ and local holomorphic coordinate ζ centered at x, we may make a linear change of coordinates in $\mathbf{P}^2$ such that Z has the form

$$Z(\zeta) = [1, \zeta^{\alpha+1} + \cdots, \zeta^{\alpha+\beta+2} + \cdots] \tag{4.27}$$

for non-negative integers α, β. Using that $Z^*(\zeta) = Z(\zeta) \wedge Z'(\zeta)$, we find that

$$\begin{aligned} Z^*(\zeta) &= [(\alpha+1)\zeta^{\alpha} + \cdots, (\alpha+\beta+2)\zeta^{\alpha+\beta+1} + \cdots, -(\beta+1)\zeta^{2\alpha+\beta+2}] \\ &= \left[1, \left(\frac{\alpha+\beta+2}{\alpha+1}\right)\zeta^{\beta+1} + \cdots, \frac{-(\beta+1)}{\alpha+1}\zeta^{\alpha+\beta+2} + \cdots\right]. \end{aligned} \tag{4.27*}$$

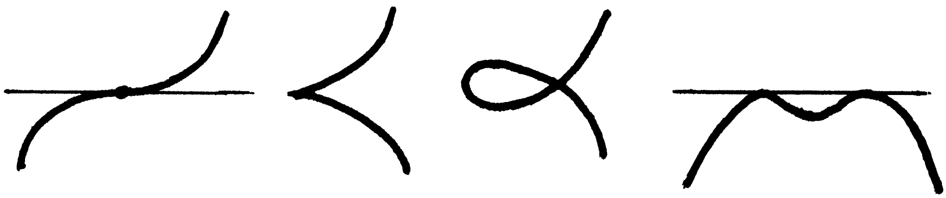
This together with (4.27) imply that:

$$\begin{cases} \alpha = \mu_0(x) \text{ is the ramification index of } Z \text{ at } x; \\ \beta = \mu_1(x) \text{ is the ramification index of } Z^* \text{ at } x. \end{cases} \tag{4.28}$$

The curve $Z : M \to \mathbf{P}^2$ is said to have ordinary singularities in every point $x \in M$ is either:

(i)	a *regular point*	$(\alpha = \beta = 0)$;
(ii)	a *flex*	$(\alpha = 0, \beta = 1)$;
(iii)	a *cusp*	$(\alpha = 1, \beta = 0)$;
(iv)	a *double point*	($Z(x) = Z(x')$ for some $x' \neq x$, and where the two branches of C passing through $Z(x)$ have distinct tangents); or
(v)	a *bitangent*	(the tangent line to C at $Z(x)$ is also tangent at some other point $Z(x')$).

The pictures are:



flex cusp double point bitangent

Using (4.27) and (4.27)*, it follows that:

$$(4.29)\qquad \begin{cases} Z(x) \text{ is a flex on } C \Leftrightarrow Z^*(x) \text{ is a cusp on } C^*, \text{ and} \\ Z(x) \text{ is a bitangent on } C \Leftrightarrow Z^*(x) \text{ is a double point on } C^*. \end{cases}$$

In particular, Z has ordinary singularities $\Leftrightarrow$ Z^* has ordinary singularities. Using the notations:

$$\begin{cases} \delta = \# \text{ double points on } C \\ k = \# \text{ cusps on } C \\ f = \# \text{ flexes on } C \\ b = \# \text{ bitangents on } C, \end{cases}$$

and similarly δ^*, k^*, f^*, b^* for C^*, then by (4.29)

$$(4.30)\qquad \begin{cases} f = k^*, \quad f^* = k \\ b = \delta^*, \quad b^* = \delta. \end{cases}$$

The Plücker formulae (4.26) for $k = 0, 1$ are

$$(4.31)\qquad \begin{cases} k + d^* = 2d + 2g - 2 \\ f + d = 2d^* + 2g - 2, \end{cases}$$

where (4.28) has been used to calculate μ_0 and μ_1. If we use the *genus formula*

$$g = \frac{(d-1)(d-2)}{2} - \delta - k = \frac{(d^*-1)(d^*-2)}{2} - \delta^* - k^*$$

in (4.31), we obtain the classical Plücker formulae

$$(4.32) \qquad \boxed{\begin{cases} d^* = d(d-1) - 2\delta - 3k \\ d = d^*(d^*-1) - 2b - 3f. \end{cases}}$$

It is, I think, quite remarkable that these beautiful relations are simply reflections of the Cartan structure equations for the unitary group $U(3)$.

5. RIGIDITY AND CONTACT.

(a) *The general problem.* Let G be a Lie group, $H \subset G$ a closed subgroup, and G/H the resulting homogeneous space. We consider smooth mappings

$$f : M \to G/H$$

of manifold M into G/H.

Definition. Two such mappings

$$f, \tilde{f} : M \to G/H$$

have *contact of order* μ if, for each $x \in M$, there exists a $g_x \in G$ *depending on* x such that

$$f \quad \text{and} \quad g_x \circ \tilde{f}$$

agree up to order μ at x.[1]

Thus any two mappings have contact of order zero since G acts transitively on G/H. If $G = E(n)$ is the Euclidean group and $H = O(n)$ so that $G/H = \mathbf{R}^n$, then f and $\tilde{f}$ have *contact of order one* if, and only if, the first fundamental forms induced by f and $\tilde{f}$ are the same. They have *contact of order two* exactly when the first and second fundamental forms are the same, etc. Although I have never seen a formal proof, it is presumably the case that

(5.1) *Given $f : M \to G/H$ which satisfies a suitable non-degeneracy condition, there exists an integer $\mu = \mu(f, G, H)$ such that f and $\tilde{f}$ are congruent by a fixed $g \in G$ if, and only if they have contact of order μ.*[2]

In any event, the special cases of (5.1) are in some ways more interesting than the general principle. Thus, the uniqueness results of Secs. 4, 5 may be rephrased as:

(5.2) *Two non-degenerate curves $x(s)$, $\tilde{x}(s)$ in $\mathbf{R}^n$ are congruent by a rigid motion if, and only if, they have contact of order n;*

Two hypersurfaces $x, \tilde{x} : M \to \mathbf{R}^{n+1}$ are congruent if, and only if, they have contact of order two;

[1] In general, for our purposes, it will be convenient to assume that the Jacobian of f has maximal rank.

[2] There is a general discussion of sorts in Chapter VI of the book by Cartan. In the review of this book by Hermann Weyl (Bull. A.M.S., vol. 44(1938), pps. 598–607), he attempts to clarify the general problem of framing a submanifold of a homogeneous space and subsequent rigidity question, but seems to feel that the argument, as it stands, is incomplete.

(5.3) *Two non-degenerate holomorphic curves $Z, \tilde{Z} : M \rightarrow \mathbf{P}^n$ (M being a Riemann surface) are congruent if, and only if, they have contact of order one.*

The way one generally proves results such as (5.2)–(5.3) is, by using contact elements of sufficiently high order, to associate to the mapping

$$f : M \rightarrow G/H$$

a "natural" lifting or set of liftings

$$\begin{tikzcd} & G \arrow[d] \\ M \arrow[ur, dashed, "F"] \arrow[r, "f"'] & G/H \end{tikzcd} \tag{5.4}$$

and then (1.3) may be applied to F. *Moreover, the generalized Frenet or Darboux frame F should be adapted to the geometry of the original mapping f, so as to make it apparent that if f and $\tilde{f}$ have contact of order $\mu = \mu(f, G, H)$, then*

$$F^*\omega = \tilde{F}^*\omega$$

where ω is the Maurer–Cartan form on G.

To explain this a little further, let us assume that $\dim M = 1$ and discuss first how *not* to frame M. In this case, an Ad H-invariant splitting

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$$

induced a G-invariant connection in the principle bundle

$$G \rightarrow G/H,$$

and a lifting F in (5.4) may always be found by simply using this connection; i.e., by requiring that $F_*(\zeta) \in \mathfrak{p}$ for all tangent vectors ζ to M. This *naive lifting* seems to be almost never the correct one to use! For example, in the case of a curve

$$x(s) = (x_1(s), \cdots, x_n(s))$$

in $\mathbf{R}^n$, the naive lifting requires that the frame

$$(x(s); e_1(s), \cdots, e_n(s))$$

satisfy $de_i \equiv 0$. Then the Maurer–Cartan forms pulled back under F are just

$$dx_1, \cdots, dx_n,$$

and we obtain an uninteresting rigidity statement which tells us nothing about contact.

As a rule of thumb, a "good lifting" F in (5.4) seems to have the following properties, at least when $\dim M = 1$:

(i) *Being able to uniquely choose F depends on the mapping $f : M \rightarrow G/H$ being "non-degenerate" in a suitable sense,*

(ii) *The integer* μ *such that contact of order* μ *implies rigidity as in* (5.1) *should be easily discernible from* F;

(iii) *The number of independent* 1-*forms appearing in* $F^*(\omega)$ *should be equal to* $d = \dim (G/H)$, *since a general (real) curve in* G/H *has* d *degrees of freedom.*

As an indication of the subtlety of finding F in special cases, we remark that framing a curve in the real projective plane requires the consideration of contact of order *six*! In the second half of this section we shall give a simpler but non-classical example of a good lifting F.

(b) *Curves in Grassmannians.* We consider the Grassmannian $G(n, 2n)$ of oriented n-planes in $\mathbf{R}^{2n}$. This is the homogeneous manifold $SO(2n)/SO(n) \times SO(n)$ of dimension n^2, and the interpretation of $SO(2n)$ as the set of frames for $G(n, 2n)$ was discussed in Section 2. Our rigidity statement is

(5.5) *Two non-degenerate curves*[3] Λ, $\tilde{\Lambda}$ *in* $G(n, 2n)$ *are congruent if, and only if, they have contact of order two.*

For notational simplicity, we shall give the proof when $n = 2$; the general argument is essentially the same. An additional advantage for doing this is that, for $n = 2$, there are only a small number (in fact one) of exceptional cases, and we are able to completely discuss these.

We begin by choosing an arbitrary frame $e_1^*(s)$, $e_2^*(s)$, $e_3^*(s)$, $e_4^*(s)$ such that

$$\begin{cases} e_1^*(s) \wedge e_2^*(s) = \Lambda(s) \\ e_3^*(s) \wedge e_4^*(s) = \Lambda(s)^{\perp} \end{cases}$$

The vectors $e_1^*(s)$, $e_2^*(s)$ and $e_3^*(s)$, $e_4^*(s)$ are then determined up to general rotations in $SO(2)$. Writing

$$\frac{de_\alpha^*}{ds} \equiv \sum_\beta A_{\alpha\beta} e_{2+\beta}^* \quad (\text{modulo } e_1^*, e_2^*)$$

where $1 \leq \alpha, \beta \leq 2$, the condition that

$$A = (A_{\alpha\beta}) \in \operatorname{Hom}(\Lambda, \Lambda^{\perp})$$

be an isomorphism is independent of our choice of frames.

Definition. We define $\Lambda(s)$ to be *non-degenerate* in case the transformation A is an isomorphism.[4]

As mentioned, we shall try to justify this definition by examining the degenerate case below. Assuming that Λ is non-degenerate, we shall put A in canonical form.

In general, given two Euclidean vector spaces V, W and a linear isomorphism

$$T : V \rightarrow W,$$

[3] Geometrically, $\Lambda(s)$ is non-degenerate in case the n-planes $\Lambda(s)$ and $\Lambda(s + \Delta s)$ do not intersect outside the origin. The precise definition of non-degeneracy will be given below—of course, a "general" curve is non-degenerate.

[4] An interpretation of non-degeneracy vis á vis the Schubert hyper-planes in $G(2, 4)$ will be discussed in Sec. 6(b).

the image of the unit sphere in V is an ellipse in W, and we may use the axis vectors for this ellipse to put T in standard form. Algebraically, using the identification of W with its dual, one considers the positive definite symmetric isomorphism

$$S = {}^tTT : V \to V.$$

Choose an orthonormal basis $v_1, \cdots, v_n$ for V such that

$$Sv_\alpha = \lambda_\alpha^2 v_\alpha, \qquad \lambda_\alpha > 0,$$

and set

$$\lambda_\alpha w_\alpha = Tv_\alpha .$$

Then from

$$\lambda_\alpha \lambda_\beta (w_\alpha, w_\beta) = (Tv_\alpha, Tv_\beta) = (Sv_\alpha, v_\beta) = \delta_{\alpha\beta}\lambda_\alpha\lambda_\beta$$

it follows that $w_1, \cdots, w_n$ is an orthonormal basis for W.

Applying these considerations to Λ, $\Lambda^\perp$ and A allows us to select a distinguished frame $e_1(s)$, $e_2(s)$, $e_3(s)$, $e_4(s)$ for $\Lambda(s)$ where

$$\boxed{\frac{de_\alpha}{ds} = \sum_\beta \frac{\omega_{\alpha\beta}}{ds} e_\beta + \lambda_\alpha e_{2+\alpha} .} \tag{5.6}$$

At points where $\lambda_1 \neq \lambda_2$ this frame is unique up to signs and is smooth; otherwise, it is only continuous. It will turn out to be the good lifting F discussed in Sec. 5(a) above.

We are now ready to prove (5.5). If two non-degenerate curves $\Lambda(s)$ and $\tilde{\Lambda}(s)$ osculate to $2nd$ order at $s = s_0$, then, using the *Plücker embedding* (Sec. 2(c)), we have

$$e_1(s) \wedge e_2(s) \equiv \tilde{e}_1(s) \wedge \tilde{e}_2(s) \qquad (\text{modulo } (s - s_0)^2) \tag{5.7}$$

where the $e_i(s)$ and $\tilde{e}_i(s)$ are the natural frames constructed above. Letting "$'$" denote the derivative with respect to s, from

$$\begin{cases} e_1 \wedge (e_1 \wedge e_2)' = -\lambda_1 e_1 \wedge e_2 \wedge e_3 \\ e_2 \wedge (e_1 \wedge e_2)' = \lambda_2 e_1 \wedge e_2 \wedge e_4 \end{cases}$$

and the similar equations for $\tilde{e}_i$, it follows that

$$\begin{cases} e_i(s_0) = \tilde{e}_i(s_0) & (i = 1, \cdots, 4) \\ \lambda_\alpha(s_0) = \tilde{\lambda}_\alpha(s_0) & (\alpha = 1, 2) . \end{cases} \tag{5.8}$$

As a consequence of (5.6) and (5.8), we see that if Λ and $\tilde{\Lambda}$ osculate to first order, then

$$\omega_{\alpha,2+\beta} = \tilde{\omega}_{\alpha,2+\beta} \qquad (\alpha, \beta = 1, 2). \tag{5.9}$$

It remains to show that $2nd$ order contact leads to

$$\begin{cases}\omega_{12} = \tilde{\omega}_{12} \\ \omega_{34} = \tilde{\omega}_{34} .\end{cases} \tag{5.10}$$

For this we calculate:

$$(e_1 \wedge e_2)' = -\lambda_1 e_2 \wedge e_3 + \lambda_2 e_1 \wedge e_4$$

$$\begin{aligned}(e_1 \wedge e_2)'' &= \left(\lambda_1 \frac{\omega_{12}}{ds} - \lambda_2 \frac{\omega_{34}}{ds}\right) e_1 \wedge e_3 \\ &+ \left(\lambda_2 \frac{\omega_{12}}{ds} - \lambda_1 \frac{\omega_{34}}{ds}\right) e_2 \wedge e_4 \\ &+ (\cdots)\end{aligned}$$

where $(\cdots)$ are terms involving the other $e_i \wedge e_j$. Comparing this with (5.7), (5.8) gives at $s = s_0$

$$\begin{cases}\lambda_1\omega_{12} - \lambda_2\omega_{34} = \lambda_1\tilde{\omega}_{12} - \lambda_2\tilde{\omega}_{34} \\ \lambda_2\omega_{12} - \lambda_1\omega_{34} = \lambda_2\tilde{\omega}_{12} - \lambda_1\tilde{\omega}_{34} .\end{cases} \tag{5.11}$$

In the general case when $\lambda_1 \neq \lambda_2$, (5.11) immediately yields (5.10).

In the exceptional case where $\lambda_1 = \lambda_2 = \lambda$ on an open subset of M, we first make $\lambda \equiv 1$ by a change of variables $s = s(t)$. Then the structure equations become

$$\begin{cases}\dfrac{de_1}{ds} = \dfrac{\omega_{12}}{ds} e_2 + e_3 \\[2ex] \dfrac{de_2}{ds} = \dfrac{-\omega_{12}}{ds} e_1 + e_4 .\end{cases} \tag{5.12}$$

Applying a rotation through angle φ to both frames e_1, e_2 and e_3, e_4 preserves the equations (5.12), and solving

$$-\omega_{12} = d\varphi$$

allows us to make $\omega_{12} = 0$ for the rotated frame. From (5.11) it now follows that

$$\begin{cases}\omega_{12} = 0 = \tilde{\omega}_{12} \\ \omega_{34} = \tilde{\omega}_{34} ,\end{cases}$$

thereby proving (5.10). Q.E.D.

We now examine the degenerate case. For this we first consider a curve $e(s)$ on the 3-sphere in $\mathbf{R}^4$. Setting

$$\begin{cases}e_1 = e \\ e_2 = e' \\ e_1 \wedge e_2 \wedge e_3 = \dfrac{e \wedge e' \wedge e''}{|e \wedge e' \wedge e''|}\end{cases}$$

defines a Frenet frame (e_1, e_2, e_3, e_4) associated to the curve e, and taking

$$\Lambda(s) = e_1 \wedge e_2 \tag{5.13}$$

gives a curve in $G(2, 4)$. This curve is degenerate since

$$e_1' \wedge e_2' = 0 \quad \text{in} \quad \Lambda^{\perp},$$

and we shall show that any degenerate curve is of this form.

Geometrically, any curve Λ in $G(2, 4)$ gives a *ruled surface* S_Λ in the real projective space $\mathbf{R}P^3$, since $G(2, 4)$ is just the set of oriented lines in $\mathbf{R}P^3$. Surfaces of the form (5.13) correspond to the locus of tangent lines to a curve in $\mathbf{R}P^3$, and are said to be *developable.* So what we are about to prove is that any degenerate ruled surface is developable.

Given a degenerate curve

$$\Lambda(s) = e_1^*(s) \wedge e_2^*(s),$$

we have by assumption that

$$\frac{de_1^*}{ds} \equiv \lambda e_3^* \quad (\text{modulo } \Lambda)$$

$$\frac{de_2^*}{ds} \equiv \mu e_3^* \quad (\text{modulo } \Lambda) \ .$$

Changing variables by $s = s(t)$ allows us to assume that $\lambda^2 + \mu^2 = 1$, and then we write

$$\begin{cases} \lambda = -\sin \varphi \\ \mu = \cos \varphi \ . \end{cases}$$

Rotating the frame e_1^*, e_2^* through angle φ then gives

$$\frac{de_1}{ds} \equiv 0 \quad (\text{modulo } \Lambda)$$

$$\frac{de_2}{ds} \equiv e_3 \quad (\text{modulo } \Lambda).$$

This implies the relations

$$\begin{cases} de_1 = \omega_{12} e_2 \\ de_2 = -\omega_{12} e_1 + \omega_{23} e_3 \\ de_3 = -\omega_{23} e_2 + \omega_{34} e_4 \\ de_4 = -\omega_{34} e_4 \ , \end{cases}$$

from which it is clear that the curve Λ in $G(2, 4)$ is part of the Frenet frame of a curve in the 3-sphere. Q.E.D.

Remark. Three observations concerning this example may be of interest.

The first is that, whereas the classical examples (§§3, 4) of Euclidean and non-Euclidean geometry are symmetric spaces of rank one, the Grassmannian $G(n, 2n)$ is a symmetric space of rank n. Thus, the use of frames in studying contact may be thought of as having wider applicability than just to the classical Euclidean and non-Euclidean geometries.

The second is that, even though $G(n, 2n)$ has dimension n^2, contact of order *two* is sufficient to insure rigidity. Geometrically, this is clear from the proof, and presumably one may also give a purely group theoretic explanation.

A final remark is that the $2[n(n-1)/2] + n = n^2$ differential forms

$$\{\omega_{\alpha\beta}, \omega_{n+\alpha,n+\beta}, \omega_{\alpha,n+\alpha}\}$$

arising from the natural frame of non-degenerate curve Λ in $G(n, 2n)$ are generically independent. Conversely, they may be prescribed arbitrarily, thus constructing a curve in $G(n, 2n)$. This is because of (1.4) and since $\dim G(n, 2n) = n^2$.

6. HOLOMORPHIC CURVES IN A GRASSMANNIAN.

(a) *Frames and rigidity for ruled surfaces.* Let $\Lambda : M \to \mathbf{G}(n, 2n)$ be a holomorphic mapping of a Riemann surface M into the Grassmannian of n-planes in $\mathbf{C}^{2n}$. We denote by $\Lambda(\zeta)$ the image of a local coordinate $\zeta \in M$, and shall think of $\Lambda(\zeta)$ as a holomorphic curve in $\mathbf{G}(n, 2n)$. Locally $\Lambda(\zeta)$ is spanned by holomorphic vectors $Z_1(\zeta), \cdots, Z_n(\zeta)$, and we say that the curve is *non-degenerate* in case

$$Z_1 \wedge \cdots \wedge Z_n \wedge \frac{dZ_1}{d\zeta} \wedge \cdots \wedge \frac{dZ_n}{d\zeta} \not\equiv 0. \tag{6.1}$$

Comparing (4.12) and (5.5), it seems quite likely that such a non-degenerate curve should be determined up to rigid motion by either its first or second order behavior, and it is of interest to determine which possibility actually occurs. Although unable to settle this question definitively, we are able to show that (i) second order contact-implies rigidity, and (ii) the Cauchy–Riemann equations imply a large number of relations among the second order invariants, so that in any case the number of independent ones is much less than in the real case.

Before stating precisely what we are able to find out, we shall give some preliminary definitions. As in the real case, we shall restrict ourselves to the case $n = 2$ although the main result (6.3) is valid in general. Then $\Lambda(\zeta)$ is a holomorphic curve in $\mathbf{G}(2, 4) \cong \mathbf{P}G(1, 3)$, and may thus be interpreted as a complex analytic *ruled surface* S_Λ traced out by the family of lines $\Lambda(\zeta)$ in $\mathbf{P}^3$. The ruled surface is *developable* in case it is the set of tangent lines to a holomorphic curve $Z(\zeta)$ in $\mathbf{P}^3$, and as in the real case one may easily prove

(6.2) Λ *is degenerate* $\Leftrightarrow$ S_Λ *is developable.*

In addition to developable surfaces, another special class of ruled surfaces arises as follows: Let L_1, L_2 be two orthogonal lines in $\mathbf{P}^3$, and

$$Z_i(\zeta) \in L_i \qquad (i = 1, 2)$$

holomorphic mappings to these lines. Then the family of lines in $\mathbf{P}^3$ joining $Z_1(\zeta)$ to $Z_2(\zeta)$ traces out what we shall call a *special ruled surface*. Our result is:

(6.3) *Two non-degenerate holomorphic curves Λ, $\tilde{\Lambda}$ which have second order contact are congruent. Moreover, associated to Λ is a second order invariant θ, which is a 1-form on Λ with the property that $\theta = 0 \Leftrightarrow S_\Lambda$ is a special ruled surface.*

We are *not* able to determine whether θ is in fact a first order invariant, although calculations by John Adams indicate that at least part of θ is first order. Regarding the rigidity question for holomorphic curves in general algebraic varieties, Mark Green has pointed out the following immediate consequence of (4.12):

(6.4) *Let $V \subset \mathbf{P}^N$ be an irreducible projective algebraic variety and $\Lambda : M \to V$ a holomorphic curve which does not lie in a proper algebraic subvariety of V. Then Λ is determined up to rigid motion of V by its first fundamental form.*

In case V is $\mathbf{G}(2, 4)$ embedded in $\mathbf{P}^5$ by Plücker, (6.4) says that an arbitrary holomorphic curve in $\mathbf{G}(2, 4)$ is either determined up to rigid motion of the Grassmannian by its first order behavior, or else Λ lies on the intersection of two non-singular quadrics in $\mathbf{P}^5$. It is not clear just how this algebraic degeneracy is related to the analytic non-degeneracy assumption (6.1).

To prove (6.3), we begin as usual by considering unitary frame fields $Z_1(\zeta)$, $Z_2(\zeta)$, $Z_3(\zeta)$, $Z_4(\zeta)$ such that

$$\Lambda(\zeta) = Z_1(\zeta) \wedge Z_2(\zeta),$$

and we try to determine a natural such framing. Following the procedure in the real case, we define a linear map

$$A_\zeta : \Lambda(\zeta) \to \mathbf{C}^4/\Lambda(\zeta)$$

as follows: Given $Z \in \Lambda(\zeta_0)$, choose a holomorphically varying $Z(\zeta) \in \Lambda(\zeta)$ for $|\zeta - \zeta_0| < \epsilon$ and with $Z(\zeta_0) = Z$, and set

$$A_\zeta(Z) = \left.\frac{dZ(\zeta)}{d\zeta}\right|_{\zeta=\zeta_0} .$$

It is clear that A_ζ varies holomorphically with ζ, and moreover

$$\text{rank } A_\zeta \leq 1 \quad \text{for all} \quad \zeta \Leftrightarrow \Lambda \quad \text{is degenerate.}$$

Assuming non-degeneracy, A_ζ is an isomorphism outside a discrete set. Restricting our attention to such *regular points* where rank $A_\zeta = 2$, we may select *natural frames* $Z_1, \cdots, Z_4$ with the property that

$$\begin{cases} A_\zeta Z_1 = \lambda Z_3 \\ A_\zeta Z_2 = \mu Z_4 . \end{cases} \tag{6.5}$$

By rotating Z_1 and Z_2, we may further assume that λ, μ are real and positive. In case $\lambda \neq \mu$, such frames are then determined up to rotations

$$\begin{cases} Z_1 \to e^{i\varphi} Z_1 , & Z_3 \to e^{i\varphi} Z_3 \\ Z_2 \to e^{i\psi} Z_2 , & Z_4 \to e^{i\psi} Z_4 . \end{cases} \tag{6.6}$$

At *exceptional points* where $\lambda = \mu$, Z_1, Z_2 and Z_3, Z_4 are only determined up to the same unitary transformation in two variables.

Using (6.5), the structure equations of a natural frame field are

$$\boxed{\begin{cases} dZ_1 = \theta_{11} Z_1 + \theta_{12} Z_2 + \theta_{13} Z_3 \\ dZ_2 = \theta_{21} Z_1 + \theta_{22} Z_2 + \theta_{24} Z_4 , \quad \text{where} \\ \theta_{13} = \lambda\, d\zeta \quad \text{and} \quad \theta_{24} = \mu\, d\zeta \end{cases}} \tag{6.7}$$

are of type (1, 0) since $\Lambda(\zeta)$ is holomorphic. From (6.7) we obtain

$$d(Z_1 \wedge Z_2) = (\theta_{11} + \theta_{22}) Z_1 \wedge Z_2 - \theta_{13} Z_2 \wedge Z_3 + \theta_{14} Z_2 \wedge Z_4 , \tag{6.8}$$

from which it follows that

$$\lambda^2 + \mu^2, \lambda\mu$$

are first order invariants of Λ. In particular,

$$\omega = (\lambda + \mu)\, d\zeta = \theta_{13} + \theta_{24}$$

is well-defined, and

$$\omega \wedge \bar{\omega} = (\lambda + \mu)^2\, d\zeta \wedge d\bar{\zeta}$$

is a positive (1, 1) form on the curve. Setting

$$\varphi = (\theta_{11} + \theta_{22}) - (\theta_{33} + \theta_{44}), \tag{6.9}$$

it follows from the Cartan structure equations that

$$\begin{cases} d\omega = \varphi \wedge \omega \\ \varphi + \bar{\varphi} = 0. \end{cases}$$

Thus φ is uniquely determined by ω, and is a first order invariant.

To obtain further information, we go to second order. For this we rewrite (6.8) in the form

$$(Z_1 \wedge Z_2)_\zeta = \left(\frac{\theta'_{11}}{d\zeta} + \frac{\theta'_{22}}{d\zeta}\right) Z_1 \wedge Z_2 - \lambda Z_2 \wedge Z_3 + \mu Z_1 \wedge Z_4 \tag{6.10}$$

where $(-)_\zeta$ is the ζ-derivative and

$$\theta = \theta' + \theta''$$

is the type decomposition of a 1-form into (1, 0) and (0, 1) components. Since the curve $\Lambda(\zeta)$ is holomorphic,

(6.11) $\quad (Z_1 \wedge Z_2)_{\bar{\zeta}\zeta}$ is a linear combination of $Z_1 \wedge Z_2$, $(Z_1 \wedge Z_2)_\zeta$.

Computing modulo $Z_1 \wedge Z_2$, we obtain from (6.10)

$$(6.12) \quad \begin{aligned} (Z_1 \wedge Z_2)_{\bar{\zeta}\zeta} \equiv {} & \left(-\lambda_{\bar{\zeta}} - \lambda \frac{\theta''_{22}}{d\bar{\zeta}} - \lambda \frac{\theta''_{33}}{d\bar{\zeta}}\right) Z_2 \wedge Z_3 \\ & + \left(\mu_{\bar{\zeta}} + \mu \frac{\theta''_{11}}{d\bar{\zeta}} + \mu \frac{\theta''_{44}}{d\bar{\zeta}}\right) Z_1 \wedge Z_4 \\ & + \left(-\lambda \frac{\theta''_{21}}{d\bar{\zeta}} + \mu \frac{\theta''_{43}}{d\bar{\zeta}}\right) Z_1 \wedge Z_3 \\ & + \left(-\lambda \frac{\theta''_{34}}{d\bar{\zeta}} + \mu \frac{\theta''_{12}}{d\bar{\zeta}}\right) Z_2 \wedge Z_4 . \end{aligned}$$

Combining (6.10)–(6.12) yields the two relations

$$(6.13) \quad \boxed{(\partial - \bar{\partial}) \log \frac{\mu}{\lambda} = (\theta_{22} + \theta_{33}) - (\theta_{11} + \theta_{44})}$$

$$(6.14) \quad \boxed{\theta_{34} = \frac{\lambda}{\mu} \theta'_{12} + \frac{\mu}{\lambda} \theta''_{12}}$$

Adding (6.9) and (6.13) gives that:

(6.15) $\quad \theta_{11} - \theta_{33}$, $\theta_{22} - \theta_{44}$ are first order invariants.

Here is another proof of (6.14). Writing

$$\theta_{12} = \alpha\, d\zeta + \beta\, d\bar{\zeta}, \quad \theta_{34} = \gamma\, d\zeta + \delta\, d\bar{\zeta},$$

we have from the Cartan structure equation

$$\begin{aligned} 0 = d\theta_{14} &= \theta_{12} \wedge \theta_{24} + \theta_{13} \wedge \theta_{34} \\ &= (-\beta\mu + \delta\lambda)\, d\zeta \wedge d\bar{\zeta}; \\ 0 = d\theta_{23} &= \theta_{21} \wedge \theta_{13} + \theta_{24} \wedge \theta_{43} \\ &= (\lambda\bar{\alpha} - \mu\bar{\gamma})\, d\zeta \wedge d\bar{\zeta}, \end{aligned}$$

which gives

$$\delta = (\mu/\lambda)\beta, \quad \gamma = (\lambda/\mu)\alpha$$

John Adams has pointed out that, if we take the exterior derivative of $\theta_{11} - \theta_{33}$ in (6.15) and let "$\equiv$" denote "congruence modulo first order terms",

$$d(\theta_{11} - \theta_{33}) = \theta_{12} \wedge \theta_{21} + \theta_{13} \wedge \theta_{31} - \theta_{31} \wedge \theta_{13} - \theta_{34} \wedge \theta_{43} ,$$

or

$$d(\theta_{11} - \theta_{33}) \equiv \left(\frac{\lambda^2 - \mu^2}{\mu^2}\right)(|\alpha|^2 + |\beta|^2)\, d\zeta \wedge d\bar{\zeta}. \tag{6.17}$$

Since θ_{12} is only determined up to multiplication by

$$e^{\sqrt{-1}(\varphi - \psi)}$$

via a rotation (6.6), this means that θ_{12} has at most two real degrees of freedom modulo first order invariants.

To prove that θ_{12} is indeed a second order invariant, we compute $(Z_1 \wedge Z_2)_{\zeta\zeta}$ modulo $Z_1 \wedge Z_2$ and $(Z_1 \wedge Z_2)_\zeta$ to obtain

$$(Z_1 \wedge Z_2)_{\zeta\zeta} \equiv \left(-\lambda \frac{\theta'_{21}}{d\zeta} + \mu \frac{\theta'_{43}}{d\zeta}\right)Z_1 \wedge Z_3 + \left(-\lambda \frac{\theta'_{34}}{d\zeta} + \mu \frac{\theta'_{12}}{d\zeta}\right)Z_2 \wedge Z_4 . \tag{6.17}$$

Comparing (6.14) and (6.17) gives:

(6.18) *If $\lambda \neq \mu$, then θ_{12} and θ_{34} are determined up to multiplication by $e^{i(\varphi-\psi)}$ by the second order behavior of Λ.*

We are now ready to prove (6.3). The function $(\lambda - \mu)^2$ is real-analytic, and thus is either identically zero or vanishes on a lower dimensional set. We first consider the case $\lambda \not\equiv \mu$, and may restrict our attention to *nonexceptional points* where $\lambda \neq \mu$. If $\Lambda(\zeta)$ and $\tilde{\Lambda}(\zeta)$ have first order contact, then by the above discussion

$$\begin{cases} \theta_{13} = \tilde{\theta}_{13} , \quad \theta_{24} = \tilde{\theta}_{24} , \quad \theta_{14} = \theta_{23} = \tilde{\theta}_{14} = \tilde{\theta}_{23} = 0 \\ \theta_{11} - \theta_{33} = \tilde{\theta}_{11} - \tilde{\theta}_{33} , \quad \theta_{22} - \theta_{44} = \tilde{\theta}_{22} - \tilde{\theta}_{44} . \end{cases} \tag{6.19}$$

Using the Cartan structure equations,

$$\begin{aligned} d(\theta_{11} - \tilde{\theta}_{11}) &= \theta_{12} \wedge \theta_{21} + \theta_{13} \wedge \theta_{31} - \tilde{\theta}_{12} \wedge \tilde{\theta}_{21} - \tilde{\theta}_{13} \wedge \tilde{\theta}_{31} \\ &= 0 \end{aligned}$$

since $\theta_{12} \wedge \theta_{21} = (-\,|\alpha|^2 + |\beta|^2)\, d\zeta \wedge d\bar{\zeta}$ is a first order invariant. Similarly

$$d(\theta_{44} - \tilde{\theta}_{44}) = 0.$$

Solving the equations

$$d(\theta_{11} - \tilde{\theta}_{11}) = i\, d\varphi$$

$$d(\theta_{44} - \tilde{\theta}_{44}) = i\, d\psi$$

and rotating Z_1, Z_2 through angles φ, ψ gives

$$\theta_{11} = \tilde{\theta}_{11} , \; \theta_{44} = \tilde{\theta}_{44} .$$

Comparing with (6.19) we obtain:

(6.20) *If Λ, $\tilde{\Lambda}$ have first order contact, then*

$$\theta_{ij} = \tilde{\theta}_{ij} \qquad (i \leq j)$$

for all $(i, j) \neq (1, 2)$ or $(3, 4)$.

We now prove that $\theta_{12} = \tilde{\theta}_{12}$ in case Λ, $\tilde{\Lambda}$ have second order contact. In any case, by (6.18),

$$\theta_{12} = e^{i\rho}\tilde{\theta}_{12} .$$

Now, on the one hand

$$d\theta_{12} = (\theta_{11} - \theta_{22}) \wedge \theta_{12} = (\tilde{\theta}_{11} - \tilde{\theta}_{22}) \wedge e^{i\rho}\tilde{\theta}_{12} \tag{6.21}$$

while on the other hand

$$d(e^{i\rho}\tilde{\theta}_{12}) = i\, d\rho \wedge \tilde{\theta}_{12} + (\tilde{\theta}_{11} - \tilde{\theta}_{22}) \wedge e^{i\rho}\tilde{\theta}_{12} . \tag{6.22}$$

Combining (6.21) and (6.22) together with the same relations for θ_{34}, and using (6.14) gives

$$\begin{cases} \partial\rho \wedge \theta_{12}'' + \bar{\partial}\rho \wedge \theta_{12}' = 0 \\ \lambda^2\, \partial\rho \wedge \theta_{12}'' + \mu^2\, \bar{\partial}\rho \wedge \theta_{12}' = 0. \end{cases}$$

If $\theta_{12} \neq 0$, then since $\lambda \neq \mu$ we obtain

$$d\rho = 0.$$

Rotating Z_1 and Z_3 through the *constant* angle ρ does not change (6.20), and gives $\theta_{12} = \tilde{\theta}_{12}$, and then $\theta_{34} = \tilde{\theta}_{34}$ by (6.14). Thus natural frames for Λ and $\tilde{\Lambda}$ have been chosen such that all Maurer–Cartan forms agree, and rigidity is proven.

In the exceptional case where $\lambda \equiv \mu$, we will show that $Z_1, \cdots, Z_4$ may be chosen so that

$$\theta_{12} \equiv 0, \tag{6.23}$$

and this clearly implies rigidity. For this it is useful to use the language of vector bundles. The frame Z_1, Z_2 is a unitary frame for the universal vector bundle S over Λ, and $\{\theta_{\alpha\beta}\}$ $(1 \leq \alpha, \beta \leq 2)$ is the connection matrix. By the Cartan structure equation, the curvature is

$$\Theta = \begin{pmatrix} \theta_{13} \wedge \theta_{31} & 0 \\ 0 & \theta_{24} \wedge \theta_{42} \end{pmatrix} = -\lambda^2 \begin{pmatrix} d\zeta \wedge d\bar{\zeta} & 0 \\ 0 & d\zeta \wedge d\bar{\zeta} \end{pmatrix}.$$

On the other hand, $\theta_{11} + \theta_{22}$ is the connection form for the line bundle $L = \det S$ relative to the frame $Z_1 \wedge Z_2$, and the associated curvature is $d(\theta_{11} + \theta_{22}) = -2\lambda^2\, d\zeta \wedge d\bar{\zeta}$. Thus

$$S \otimes L^{-1/2}$$

is a *flat* vector bundle, and applying a suitable unitary charge to the frame $Z_1 \otimes (Z_1 \wedge Z_2)^{-1/2}$, $Z_2 \otimes (Z_1 \wedge Z_2)^{-1/2}$ gives a new frame W_1, W_2 with zero connection matrix. Then $Z_1^* = W_1 \otimes (Z_1 \wedge Z_2)^{1/2}$, $Z_2^* = W_2 \otimes (Z_1 \wedge Z_2)^{1/2}$ is a unitary frame for S with connection matrix

$$\begin{bmatrix} \frac{1}{2}(\theta_{11} + \theta_{22}) & 0 \\ 0 & \frac{1}{2}(\theta_{11} + \theta_{22}) \end{bmatrix}.$$

For this frame $\theta_{12}^* \equiv 0$, and (6.23) is satisfied.

Finally, suppose we can choose a natural frame for Λ so that the second order invariant

$$\theta = \theta_{12} \equiv 0. \tag{6.24}$$

By (6.14), $\theta_{34} \equiv 0$ and thus

$$\begin{cases} Z_1 \wedge (Z_1)_\zeta \wedge (Z_1)_{\zeta\zeta} \equiv 0 \\ Z_2 \wedge (Z_2)_\zeta \wedge (Z_2)_{\zeta\zeta} \equiv 0. \end{cases}$$

Consequently, both holomorphic curves $Z_1(\zeta)$, $Z_2(\zeta)$ lie in a line, and it follows that S_Λ is a special ruled surface. This completes the proof of (6.3).

(b) *Non-degeneracy and Schubert hyperplanes.* Let $\Lambda(\zeta)$ be a holomorphic curve in $\mathbf{G}(2, 4)$ given locally by two holomorphic vectors $Z_1(\zeta)$, $Z_2(\zeta)$ with $\Lambda(\zeta) = Z_1(\zeta) \wedge Z_2(\zeta)$. Setting

$$\Delta(\zeta) = Z_1(\zeta) \wedge Z_2(\zeta) \wedge Z_1'(\zeta) \wedge Z_2'(\zeta),$$

the holomorphic curve in non-degenerate in case $\Delta(\zeta) \not\equiv 0$, and ζ_0 is a regular point in case $\Delta(\zeta_0) \neq 0$. In Sec. 6(a) we remarked that $\Lambda(\zeta)$ is degenerate $\Leftrightarrow$ the ruled surface S_Λ is developable, and we now wish to give a geometric interpretation of the regular points on Λ.

Before doing this it is convenient to recall the analogous statements for a holomorphic curve $Z(\zeta) \in \mathbf{P}^n$. Setting

$$W(\zeta) = Z(\zeta) \wedge Z'(\zeta) \wedge \cdots \wedge Z^{(n)}(\zeta),$$

the curve is non-degenerate in case $W(\zeta) \not\equiv 0$, and ζ_0 is a regular point in case $W(\zeta_0) \neq 0$. Now then, $Z(\zeta)$ is non-degenerate $\Leftrightarrow$ the curve does not lie in a $\mathbf{P}^{n-1}$, and ζ_0 is a regular point $\Leftrightarrow$ there is a unique hyperplane H_{ζ_0} having contact of order exactly $n - 1$ with $Z(\zeta)$ at ζ_0 ; indeed

$$H_{\zeta_0} = Z(\zeta_0) \wedge Z'(\zeta_0) \wedge \cdots \wedge Z^{(n-1)}(\zeta_0).$$

For $\mathbf{G}(2, 4)$, we consider the Plücker embedding

$$\mathbf{G}(2, 4) \hookrightarrow \mathbf{P}^5.$$

Among the hyperplane sections of $\mathbf{G}(2, 4)$ are special ones, called *Schubert hyperplanes*, which are defined as follows:

Thinking of $\mathbf{G}(2, 4) = \mathbf{P}G(1, 3)$ as the lines in $\mathbf{P}^3$, for a fixed line L we set

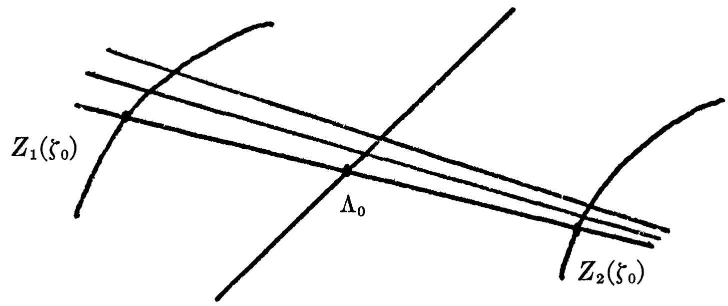
$$H_L = \{L' \in \mathbf{P}G(1, 3) : L \cdot L' \neq \phi\}. \tag{6.25}$$

The incidence relation (6.25) then defines the Schubert hyperplane H_L in $\mathbf{G}(2, 4)$. Given such a line L, we choose orthogonal unit vectors W_1, W_2 which span L, and, upon setting $\Theta = W_1 \wedge W_2$, the Schubert hyperplane section of our holomorphic curve is defined by

(6.26) $$\Lambda(\zeta) \wedge \Theta = 0.$$

Our characterization of regular points is:

(6.27) *The point ζ_0 is regular $\Leftrightarrow$ for each point $\Lambda_0 = \rho Z_1(\zeta_0) + \sigma Z_2(\zeta_0)$ on the line $\Lambda(\zeta_0)$, there exists a unique line $L = L(\Lambda_0)$ passing through Λ_0 and such that $\Lambda(\zeta)$ has second order contact with H_L at ζ_0.*



Proof. Following the notations of §6(a) we consider natural frames Z_1, Z_2, Z_3, Z_4 for $\Lambda(\zeta)$, but where λ, μ in (6.5) are allowed to be complex numbers. Then arbitrary rotations

$$Z_\alpha \to e^{i\varphi_\alpha} Z_\alpha$$

are permissible, and so we may choose φ_α so that all

$$\theta_{\alpha\alpha}(\zeta_0) = 0.$$

Setting

(6.28) $$W_1 = \rho Z_1(\zeta_0) + \sigma Z_2(\zeta_0) = \Lambda_0,$$

we seek to uniquely determine

(6.29) $$W_2 = a Z_1(\zeta_0) + b Z_2(\zeta_0) + c Z_3(\zeta_0) + d Z_4(\zeta_0)$$

such that

$$\Lambda(\zeta) \wedge \Theta$$

vanishes to exactly *2nd* order at $\zeta = \zeta_0$. The orthogonality conditions on W_1, W_2 are

(6.30) $$\begin{cases} |\rho|^2 + |\sigma|^2 = 1, & |a|^2 + |b|^2 + |c|^2 + |d|^2 = 1 \\ a\bar{\rho} + b\bar{\sigma} = 0 . & \end{cases}$$

Writing $\theta_{12} = \alpha\, d\zeta + \beta\, d\bar{\zeta}$ and computing modulo $Z_1 \wedge Z_2$, we have from (6.17) that, at $\zeta = \zeta_0$,

$$(Z_1 \wedge Z_2)_\zeta \equiv -\lambda Z_2 \wedge Z_3 + \mu Z_1 \wedge Z_4 \tag{6.31}$$

$$\begin{aligned}(Z_1 \wedge Z_2)_{\zeta\zeta} \equiv{}& -\lambda_\zeta Z_2 \wedge Z_3 + \mu_\zeta Z_1 \wedge Z_4 \\ &+ \frac{\bar{\beta}}{\lambda}(\lambda^2 - \mu^2)Z_1 \wedge Z_3 + \frac{\alpha}{\mu}(\mu^2 - \lambda^2)Z_2 \wedge Z_4 \\ &+ 2\lambda\mu Z_3 \wedge Z_4 .\end{aligned}\tag{6.32}$$

The last term in (6.32) is non-zero $\Leftrightarrow$ ζ_0 is a regular point, and this observation is the basis of (6.27).

By (6.28) and (6.31), the condition

$$(Z_1 \wedge Z_2)_\zeta \wedge \Theta = 0$$

at $\zeta = \zeta_0$ is

$$(-\lambda\rho Z_1 \wedge Z_2 \wedge Z_3 - \mu\sigma Z_1 \wedge Z_2 \wedge Z_4) \wedge W_2 = 0,$$

which using (6.29) is

$$-\lambda\rho\, d + \mu\sigma c = 0. \tag{6.33}$$

Similarly, letting $\chi_1, \chi_2, \cdots$ denote coefficients whose explicit form is irrelevant, we have from (6.32) that

$$\begin{aligned}(Z_1 \wedge Z_2)_{\zeta\zeta} \wedge W_1 ={}& \chi_1 Z_1 \wedge Z_2 \wedge Z_3 + \chi_2 Z_1 \wedge Z_2 \wedge Z_4 \\ &+ 2\lambda\mu\rho Z_1 \wedge Z_3 \wedge Z_4 + 2\lambda\mu\sigma Z_2 \wedge Z_3 \wedge Z_4 .\end{aligned}$$

Using (6.29), the condition

$$(Z_1 \wedge Z_2)_{\zeta\zeta} \wedge \Theta = 0$$

is

$$\chi_1 d - \chi_2 c + 2\lambda\mu\rho b - 2\lambda\mu\sigma a = 0. \tag{6.34}$$

If $\sigma = 0$, then $a = d = 0$ and

$$b = \frac{\chi_2}{2\lambda\mu\rho}\, c$$

so that L is uniquely determined. If $\sigma \neq 0$, then $a \neq 0$ and by (6.30) and (6.33)

$$\begin{cases} b = -\left(\dfrac{\bar{\rho}}{\bar{\sigma}}\right)a \\[2ex] c = \left(\dfrac{\lambda\rho}{\mu\sigma}\right) d, \end{cases}$$

and plugging this into (6.34) and using $|\rho|^2 + |\sigma|^2 = 1$ gives

$$\chi_3\, d = \frac{2\lambda\mu}{\bar{\sigma}}\, a$$

Thus the ratios

$$\left\{\frac{b}{a} = -\frac{\bar{\rho}}{\bar{\sigma}}, \qquad \frac{d}{a} = \frac{2\lambda\mu}{\bar{\sigma}\chi_3}, \qquad \frac{c}{a} = \frac{2\lambda^2\rho}{|\sigma|^2\,\chi_3}\right\}$$

are uniquely defined, and so L is uniquely determined by the condition of second order contact. Q.E.D.

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MASSACHUSETTS 02138