topology. In the case of the Martin exit boundary of the set $\Omega = \{1, 2, 3, \ldots\}$, the usual topology and uniform structure are discrete, i.e., points form open sets and $I$ is a uniformity.

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# ON CERTAIN HOMOGENEOUS COMPLEX MANIFOLDS

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The purpose of this note is to discuss some results on homogeneous vector bundles over homogeneous complex manifolds; complete proofs together with some more results and applications are to appear later. We refer to the papers of Wang[6] and Bott[3] for the terminology and basic results of the theory.

Let, $G,U$ be complex Lie groups such that $X = G/U$ is a $C$-space and let $\rho: U \to GL(E^\rho)$ be a holomorphic representation of $U$; then we may form the *homogeneous vector bundle* $E^\rho \to \mathbf{E}^\rho = G \times_U E^\rho \to X$. We recall that any $C$-space $X$ fibers over a rational $C$-space $X^*$ with a complex $a$-torus as fiber: $T^{2a} \to X \to X^*$. The maximal compact subgroup $M$ of $G$ acts transitively on $X$ and transitively on the fibers of $\mathbf{E}^\rho$; letting $\hat{\mathbf{E}}^\rho$ be the sheaf associated to $\mathbf{E}^\rho$, $H^*(X, \hat{\mathbf{E}}^\rho)$ is an $M$-module. We denote this induced representation by $\rho^*$,

$$\rho^*: M \to GL(H^*(X, \hat{\mathbf{E}}^\rho));$$

it is our problem to study the transformation $\rho \to \rho^*$.

Let $\mathbf{u}$ = complex Lie algebra of $U$ and $\mathbf{h}_U$ = maximal abelian subalgebra of $\mathbf{u}$; then $\mathbf{h}_U \subset \mathbf{h}$, where $\mathbf{h}$ is a Cartan subalgebra of $\mathbf{g}$ = complex Lie algebra of $G$. If the weights of $\rho$ on $\mathbf{h}_U$ are the restrictions of weights on $\mathbf{h}$ to $\mathbf{h}_U$, we call $\rho$ *rational;* otherwise $\rho$ is *irrational.* If $\rho$ is rational, we may describe an element $J\rho$ giving the highest weight of an irreducible representation of $M$ as follows: we assume that $\rho$ is irreducible so that it is determined by its highest weight, again denoted by $\rho$. Then $\rho$ is a weight on $\mathbf{h}$ which is dominant for $\mathbf{u}$ but may well not be dominant for

**g**; we let $g$ = one half the sum of the positive roots of $(\mathbf{g}, \mathbf{h})$ and look at $\rho + g$. If $\rho + g$ is singular, we set $J\rho = \mathbf{0}$, the zero representation of $M$. If $\rho + g$ is regular, there exists a unique $\tau$ in the Weyl group of $M$ such that $\tau(\rho + g) - g$ is dominant for **g** and thus determines an irreducible representation $J\rho$ of $M$; this describes the mapping $\rho \rightarrow J\rho$; and we write $J\rho$: $M \rightarrow GL(V^{J\rho})$. Also associated to $\rho$, we may define an integer $\langle\rho\rangle$, where $\langle\rho\rangle$ = index of the element $\tau$ defined above.

THEOREM 1. *If $\rho$ is irreducible and irrational, $H^*(X, \tilde{\mathbf{E}}^\rho) = 0$. If $\rho$ is rational and irreducible, then $H^q(X, \tilde{\mathbf{E}}^\rho) = 0$ for $q < \langle\rho\rangle$, $H^{q+\langle\rho\rangle}(X, \tilde{\mathbf{E}}^\rho) \cong V^{J\rho} \otimes \mathbf{C}^{\binom{a}{q}}$, and $\rho^* = (J\rho) \otimes 1$.* (Here we set $\binom{a}{q} = 0$ for $q > a$.)

*Application 1:* Taking $a = 0$ (i.e., $X = X^*$), we recover the main theorem of Bott (Theorem IV′).

*Application 2:* If $a = 0$, then $\chi(X^*, \tilde{\mathbf{E}}^\rho) = (-1)^{\langle\rho\rangle} \dim V^{J\rho}$ (Borel-Hirzebruch), and if $a > 0$, $\chi(X, \tilde{\mathbf{E}}^\rho) = 0$ for any representation $\rho$ (Bott). Here $\chi$ is the sheaf Euler characteristic.

We state the next theorem for the case when $U$ is solvable; the result holds in general but requires a more complicated statement. Let $\rho_1, \ldots, \rho_n$ be the weights of a rational representation $\rho$. We say that $\rho$ satisfies *condition S* if either of the following hold:

$$\text{(i)} \quad \langle\rho_i\rangle = \langle\rho_j\rangle \quad \text{for all } i, j,$$

$$\text{(ii)} \quad \langle\rho_i\rangle > \langle\rho_j\rangle \implies \langle\rho_i\rangle - \langle\rho_j\rangle > 1.$$

Since $U$ is solvable, each $\rho_j$ will give a one-dimensional representation $\rho_j: U \rightarrow GL(E^\rho{}_j)$, and we have

THEOREM 2. *If $\rho$ satisfies condition S, then as an M-module,*

$$H^*(X, \tilde{\mathbf{E}}^\rho) = \oplus_{j=1}^{n} H^*(X, \tilde{\mathbf{E}}^{\rho_j}).$$

Let $\Omega^q$ = sheaf of germs of holomorphic $q$-forms on $X$. Then using some properties of the roots together with Theorem 2, one has

*Application 3:* $H^*(X, \Omega^q)$ is a trivial $M$-module, $H^p(X, \Omega^q) = 0$ for $p < q$, and $H^{p+q}(X, \Omega^q) \cong \mathbf{C}^{\langle q\rangle} \otimes \mathbf{C}^{\binom{a}{p}}$, where $\langle q\rangle$ = number of elements in the Weyl group of $M$ of index $q$. Taking $a = 0$, we get Theorem 22.8 in Borel-Hirzebruch.[2]

In order to derive further applications, we consider the Atiyah sequence (ref. 1, equation 8.1) $0 \rightarrow \mathbf{L} \rightarrow \mathbf{Q} \rightarrow \mathbf{\Theta} \rightarrow 0$. Using Theorem 2, application 3, and a few exact sequences, we have for $X^*$

*Application 4:* $H^q(X^*, \tilde{\mathbf{L}}) = 0$ for all $q$. From this and the Atiyah sequence, it follows that $H^q(X^*, \tilde{\mathbf{\Theta}}) = 0$ for $q > 0$ and $H^0(X^*, \tilde{\mathbf{\Theta}}) \cong \mathbf{g}$. Thus, the analytic structure of $X^*$ is infinitesimally rigid (Bott) and the connected automorphism group $A^0(X^*)$ of $X^*$ is $G$ (Matsushima).

We now let $X$ be a non-Kähler $C$-space with fibering $T^{2a} \rightarrow X \rightarrow X^*$; also we set $X^a = X^* \times T^{2a}$.

THEOREM 3. *If $Y = X$ or $X^a$, then $H^q(Y, \tilde{\mathbf{\Theta}}) \cong A^q \oplus B^q$, where $A^q = \mathbf{g} \otimes \mathbf{C}^{\binom{a}{q}}$ and $B^q = \mathbf{C}^a \otimes \mathbf{C}^{\binom{a}{q}}$. The representation of M on $A^q$ is Ad ⊗ 1 and on $B^q$ is 1 ⊗ 1 (1 is the trivial representation).*

*Application 5:* $H^0(Y, \tilde{\mathbf{\Theta}}) \cong' \mathbf{g} \otimes \mathbf{C}^a$ and $A^0(Y) \cong G \times T^{2a}$. If $Y = X^a$, $A^0(Y)$ acts in the obvious way; if $Y = X$, then the fibering $T^{2a} \rightarrow X \rightarrow X^*$ is a homogeneous fibering over $X^*$, $T^{2a}$ acts as structure group in this fibering, and $G$ acts by lifting the action of $G$ on $X^*$. (This generalizes results in refs. 4 and 6.)

THEOREM 4. *If $a = 1$, $H^1(Y, \bar{\Theta})$ parametrizes an infinitesimal deformation space of $Y$ (cf. ref.5). If $a > 1$, then there are primary obstructions to deformation but there are no secondary obstructions and an infinitesimal deformation space is given by* $(\mathbf{h} \otimes \mathbf{C}^a) \oplus (\mathbf{C}^a \otimes \mathbf{C}^a)$.

*Application 6:* The homogeneous variations of the structure of $Y$ lie in $B^1$ (using the above notation) and are given by varying the toral structure.

*Application 7:* We construct explicitly the nonhomogeneous variations of structure of $X^* \times T^{2a}$ where $a > 1$ and $X^* = G/U^*$. These deformations are global and are parametrized by $\mathbf{h} \otimes \mathbf{C}^a \cong \mathbf{h} \otimes H^{0,1}(T^{2a}, \mathbf{C})$. Let $\omega \in H^{0,1}(T^{2a}, \mathbf{C})$, $h \in \mathbf{h}$; define a representation $\Phi(h, \omega)$: $H^1(T^{2a}, \mathbf{Z}) \rightarrow G$ by $\Phi(h, \omega)(z) = \exp\left(\left(\int_z \omega\right)h\right)$ for $z \in H_1(T^{2a}, \mathbf{Z})$. If $\mathbf{C}^a$ is the universal covering of $T^{2a}$, we have the fibering $H_1(T^{2a}, \mathbf{Z}) \rightarrow \mathbf{C}^a \rightarrow T^{2a}$, and we may form the associated bundle $G \rightarrow P(h, \omega) \rightarrow T^{2a}$, where $P(h, \omega) = \mathbf{C}^a \times_{H_1(T^{2a}, \mathbf{Z})} G$ and $H_1(T^{2a}, \mathbf{Z})$ acts on $G$ by $\Phi(h, \omega)$. The nonhomogeneous deformations of $G/U^* \times T^{2a}$ are given by the manifolds $P(h, \omega)/U^*$.

One may also construct a local family of complex structures in the non-Kähler case; this local family corresponds to the infinitesimal family given in Theorem 4. We shall not enter into the details here.

Let $\mathbf{E}_1$, $\mathbf{E}_2$ be analytic vector bundles over a complex manifold $Y$; we define Ext $(\mathbf{E}_2, \mathbf{E}_1)$ to be the set of analytic vector bundles $\mathbf{E}$ such that we have $0 \rightarrow \mathbf{E}_1 \rightarrow \mathbf{E} \rightarrow \mathbf{E}_2 \rightarrow 0$. There is an isomorphism $t$: Ext $(\mathbf{E}_2, \mathbf{E}_1) \rightarrow H^1(Y, \text{Hom}\,(\bar{\mathbf{E}}_2, \bar{\mathbf{E}}_1))$ (see ref. 1).

THEOREM 5. *Let $X$ be a C-space and $\mathbf{E}^\psi$, $\mathbf{E}^\xi$ be homogeneous vector bundles over $X$. Then $\mathbf{E} \in \text{Ext}(\mathbf{E}^\xi, \mathbf{E}^\psi)$ is a homogeneous vector bundle $\iff t(\mathbf{E}) \in H^1(X, \text{Hom}(\bar{\mathbf{E}}^\xi, \bar{\mathbf{E}}^\psi))$ is acted upon trivially by $M$.*

Using Theorems 5 and 7 and the fact that all line bundles over $X$ are homogeneous (this follows easily from Theorem 1), we have

*Application 8:* All line bundles over X with nilpotent structure group are homogeneous.

*Application 9:* Let $X^* = M/V^*$ be algebraic and let $(\tau_1, \ldots, \tau_r)$ be a set of simple roots of $M$ such that $(\tau_1, \ldots, \tau_s)$ are simple roots of $V^*$. Then there exists an indecomposable vector bundle $E \rightarrow \mathbf{E} \rightarrow X^*$ with solvable structure group and dim $E > 1 \iff$ there exists a $j$ with $s < j \leq r$ such that $(\tau_j, \tau_i) = 0$ for $1 \leq i \leq s$, where $(\ ,\ )$ = Killing form.

COROLLARY (Ise). *If the second Betti number $b_2(X^*) = 1$, then every bundle with solvable structure group is a sum of line bundles.*

Finally, we give the following result which was conjectured in reference 4.

THEOREM 6. *If $\psi$ is irreducible as a representation, then $\mathbf{E}^\psi$ is indecomposable.*

It is known (ref. 3, Theorem 1) that for any $\psi$, $H^*(X, \bar{\mathbf{E}}^\psi)$ may be written in terms of Lie algebra cohomology. Using this fact coupled with Theorem 1, Theorems 2–6 and the applications are proven using a variety of standard techniques in Lie algebra cohomology, representation theory, complex manifold theory, etc. The proof of Theorem 1 may be done in a couple of ways; we outline one proof which is done in two steps. First, we obtain a generalization to the non-Kähler case of the Kodaira criterion for the vanishing of sheaf cohomology groups. For $C$-spaces, this goes as follows:

THEOREM O. *Let* $E^{\psi} \rightarrow \mathbf{E}^{\psi} \rightarrow X$ *be a homogeneous line bundle over a C-space of complex dimension n. Suppose that the first Chern class* $c_1(\mathbf{E}^{\psi})$ *is given by a negative semi-definite quadratic form of index* $k \leq n$. *Then* $H^q(X, \check{\mathbf{E}}^{\psi}) = 0$ *for* $q < k$. (If $k = n$, we have again Kodaira's theorem.)

To apply Theorem O to $C$-spaces, a fairly extensive study of the differential geometry of homogeneous vector bundles is useful; these results may be of independent interest. The reason is that the Atiyah construction of the Chern classes in terms of forms does not work in the non-Kähler case and so one must use a curvature tensor in order to construct the forms.

Using Theorem O and the fact that $H^*(X, \check{\mathbf{E}}^{\psi})$ may be written in terms of Lie algebra cohomology, Theorem 1 is completed using several spectral sequences in Lie algebra cohomology. The Leray spectral sequence used by Bott does not seem to give the complete information here. As mentioned above, the details of the proofs together with other results and applications will appear later.

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# *DUALITY, HAAR PROGRAMS, AND FINITE SEQUENCE SPACES*

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Conjectured by von Neumann and proved by Gale, Kuhn, and Tucker,[1] the dual theorem of linear programming has been unique among dual extremal (or variational) principles (see, for example, K. Friedrichs[2] for classical mathematical physics principles and J. B. Dennis[3] and W. S. Dorn[4] for more recent use of Legendre transformations to establish dual "quadratic" programming principles) applying to general systems of constraints involving a finite number of variables in that neither principle contains the variables associated with the other. The theorem has also been shown to be as fundamental for the theory of linear inequalities (see particularly Charnes and Cooper[5] for this approach) as the classic Farkas-Minkowski lemma.

Generalizations to linear mappings between linear topological spaces were forthcoming from S. Karlin and H. F. Bohnenblust[6] (also L. Hurwicz[7]) for the Farkas-Minkowski lemma and from D. Bratton (also recently K. Kretschmer[8]) for the dual theorem in a brilliant unpublished but *well-known* paper.[9] As expected, these