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CHARACTERISTIC COHOMOLOGY OF DIFFERENTIAL SYSTEMS II: CONSERVATION LAWS FOR A CLASS OF PARABOLIC EQUATIONS

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CONTENTS

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Introduction. In Part I of this series of papers, we developed several aspects of the general theory of the characteristic cohomology of an exterior differential system (EDS). In the local involutive case, we proved that this characteristic cohomology $\bar{H}^q$ vanishes in the range $0 < q < n - \ell$, where ℓ is a geometric invariant of the system (in the "unmixed" case, ℓ is the codimension of the complex characteristic variety). We then defined the first potentially nonvanishing group $\bar{H}^{n-\ell}$ to be the space $\mathscr{C}$ of conservation laws associated to the EDS. It was shown that the space of conservation laws has a "structure", in that $\mathscr{C}$ is naturally identifiable with the kernel of a canonically defined linear differential operator. Moreover, the highest order part of any $\Phi \in \mathscr{C}$ has a canonical expression derived from the symbol of the EDS.

In this second part, we shall refine this analysis of conservation laws for a class of exterior differential systems that we call *parabolic systems*. As we shall explain, by *parabolic system* we mean an exterior differential system on a 7-dimensional manifold that is locally (but generally not globally) equivalent to the exterior differential system arising from a second-order parabolic equation[1] for one unknown function of two independent variables. Examples of such systems arising in geometry include gradient flows associated to locally defined, first-order functionals given on curves on a surface or surfaces immersed in a Riemannian 3-manifold with one of the principal curvatures being constant.

According to the general theory developed in Part I, the local conservation laws of such a system are isomorphic to a certain space $\mathscr{C}$ of closed 2-forms in the infinitely prolonged differential ideal, modulo nothing. Moreover, the general form that any such closed 2-form must take is known from the symbol and sub-principal symbol of the EDS.

A particularly nice feature of the parabolic case is that the representing 2-forms in $\mathscr{C}$ turn out to be well defined on the original 7-manifold, i.e., it turns out not to be necessary to pass to the infinite prolongation after all. This should, perhaps,

[1] By *parabolic equation* we simply mean one with multiple characteristics in the classical sense. This includes, as a special case, the class of so-called "evolutionary" equations (such as the heat equation) which are the most commonly considered examples of parabolic equations in the PDE literature.

not be surprising, since for the (simpler) case of parabolic evolution equations this is well known and easy to prove by the methods used by Vinogradov, Olver, Tsujishita, and Mikhailov et al. This is, of course, not true for hyperbolic or elliptic equations, for which the classification of the conservation laws is considerably more subtle.

The questions that we shall address include the following.

What is the "geometry" of parabolic systems whose space of conservation laws have a given structure?

Determine the conditions on a parabolic EDS in order that it admit a given number of independent conservation laws.

Give a "dimension count" and normal form for all parabolic systems which have at least k independent conservation laws.

The word "determine" should be understood to mean "give an algorithmic method that is applicable in practice to examples", such as those mentioned above. What we are really after is, using the general theory as a guide, to begin to address the question of how one may *effectively* determine whether a given EDS has conservation laws, what those laws look like, and so forth. As will be seen below, we are able to carry this out completely for parabolic systems. Indeed, for these systems the problem turns out to have an underlying geometry that is surprisingly rich.

The "dimension count" that emerges from our study is the following: First, the set Σ of all parabolic systems is locally parametrized by one arbitrary function of six variables (loosely speaking, we may say that Σ has "transcendental dimension six"). If we then let Σ_k denote the classes of parabolic system for which $\dim \mathscr{C} \geqslant k$, then we shall see that

Σ_1 depends on one arbitrary function of five variables

Σ_2 depends on one arbitrary function of four variables

Σ_3 depends on one arbitrary function of three variables

$\Sigma_4 = \Sigma_5 = \cdots =$ the classes of linear equations.

The last statement means that: *A parabolic system has at least four independent conservation laws if and only if it is locally equivalent to a linear PDE system.* Thus, a posteriori we see that the imposition of each additional conservation law reduces the "transcendental dimension" by one at each step until we reach Σ_4, which has transcendental dimension two, that being the same as the transcendental dimension of the classes of linear equations.

Moreover, we shall be able to give a local normal form for all systems with $\dim \mathscr{C} = 1, 2, 3, \geqslant 4$ which exhibits and makes precise the meaning of the phrase "depends on one arbitrary function of k variables". We shall also give an effective algorithm—to be stated in a moment—which gives a way of "de-prolonging" parabolic systems in each of the sets Σ_k and of determining their conservation laws. Finally, underlying this algorithm is a beautiful and unexpected geometry.

In order to describe this algorithm, we recall the following terminology from the theory of exterior differential systems (see [BCG[3]]). A *Pfaffian* differential ideal $\mathscr{I}$ is one which is generated as a differential ideal by 1-forms. Setting

$$\Theta = \mathscr{I} \cap \Omega^1(M)$$

means that $\mathscr{I}$ is generated algebraically by the forms θ, $d\theta$ where $\theta \in \Theta$.

There are two natural constructions associated to Pfaffian differential ideals. The first is the *derived flag*, defined as the sequence of Pfaffian differential ideals

$$\mathscr{I} = \mathscr{I}_1 \supset \mathscr{I}_2 \supset \mathscr{I}_3 \supset \cdots,$$

where $\mathscr{I}_k$ has generating 1-forms $\Theta_k = \mathscr{I}_k \cap \Omega^1(M)$ defined inductively for $k \geqslant 1$ by

$$\Theta_k = \{\theta \in \Theta_{k-1} | d\theta \equiv 0 \bmod \Theta_{k-1}\}.$$

The second construction is the *Cartan system* of $\mathscr{I}$, which may be thought of as the smallest submodule of $\Omega^1(M)$ needed to express all the forms θ and $d\theta$ for $\theta \in \Theta$. The Cartan system of $\mathscr{I}$ is completely integrable and therefore defines a local foliation $M \to N$. This foliation has the property that there exists a differential ideal $\bar{\mathscr{I}}$ on N so that $\mathscr{I}$ is generated on M by the pullbacks of generators of $\bar{\mathscr{I}}$ on N. In practice, both the derived flag and Cartan system of a given Pfaffian differential ideal are readily computable.

With these preliminaries out of the way, here is how our algorithm for determining the conservation laws may be described: We first observe that a parabolic system on a 7-manifold M is a Pfaffian differential ideal $\mathscr{I}$ of rank 3, and the assumption of parabolicity allows us to canonically define a certain Pfaffian system $\mathscr{M}_1$ of rank 4 which contains $\mathscr{I}$ and restricts to each integral surface of $\mathscr{I}$ to define the characteristic foliation. We denote the derived flag of $\mathscr{M}_1$ by

$$\mathscr{M}_1 \supset \mathscr{M}_2 \supset \mathscr{M}_3 \supset \mathscr{M}_4.$$

Each of the successive $\mathscr{M}_{k+1}$ has codimension at least one in $\mathscr{M}_k$ with equality for $k = 1$. A necessary condition that $\dim \mathscr{C} \geqslant 1$ is that $\mathscr{M}_3$ have codimension one in $\mathscr{M}_2$, and a necessary condition that $\dim \mathscr{C} \geqslant 2$ is that $\mathscr{M}_4$ have codimension one in $\mathscr{M}_3$. Assuming this to be the case, the Cartan system $\mathscr{N}$ of $\mathscr{M}_3$ is a completely integrable system of rank 4 and defines a local foliation

$$M^7 \to N^4.$$

The space of conservation laws of the original system is then isomorphic to the space of closed 2-forms on N whose pullbacks to M are congruent to zero modulo $\mathscr{I}$. The foliation $M \to N$ has the geometric meaning of expressing the given exterior differential system $\mathscr{I}$ as a prolongation of an exterior differential system of a special Monge-Ampere type.

The proof that this algorithm does indeed yield the conservation laws for parabolic systems is based on the equivalence method of É. Cartan. This method is a technique for determining the invariants of an EDS by successive coframe adaptations and use of the so-called "structure equations". Quite often, the equivalence method very quickly leads to massive computations.[2] However, in the situation at hand, guided by the geometric problem of determining the conservation laws, we shall find that each step has geometric meaning and that only three successive frame adaptations are necessary.

For example, the first-frame adaptation comes by restricting to the class of *dispersive* parabolic systems, defined to be those whose *Goursat invariant* (defined in §0) is not zero. Next, we find that, in order for there to be nonzero conservation laws, the *Monge-Ampere invariant* Ψ must vanish, which implies that the system is locally of Monge-Ampere type. This leads to the second-frame adaptation, and, subsequently, to two further relative invariants T and U with the properties that:

$$T \neq 0 \Rightarrow \dim \mathscr{C} \leqq 1$$

$$T = 0, U \neq 0 \Rightarrow \mathscr{C} = 0.$$

These invariants have geometric meaning, and we will take some pains to explain this.

The question naturally arises of how effectively this geometrically formulated algorithm may be applied to an explicit equation

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0.$$

The first, crude, answer is that the invariants associated to this equation are given by (generally complicated) differential polynomials in F and its derivatives, in much the same way that the Riemann curvature tensor is given by formulas in terms of the components of the metric tensor in a local coordinate system. In principal, these polynomials could be written out explicitly, but in practice (just as in the case of Riemannian geometry), one rarely has to resort to the explicit formulas. A more refined answer is that, in many "natural" classes of equations, it

[2] The experience of using the equivalence method may usefully be compared to the famous description of obstruction theory in algebraic topology: The method is like a flashlight for the miner in an underground mine which keeps branching. Using the method allows the miner only to see a few feet ahead and therefore one does not know where a branch may eventually lead (if anywhere). In our class of equations, the problem of determining the conservation laws serves to put a sign at each branch, telling us which way to proceed.

seems to be the case that special features of the situation enable one to quite effectively apply the geometric algorithm given above. This is quite analogous to the way that the curvature of a metric is usually determined by a combination of geometric considerations and computation (although one may compute it by brute force if necessary).

In order to illustrate how the general theory applies to an important special class of equations, in §5 we discuss the conservation laws for parabolic evolution equations, i.e., parabolic equations of the form

$$u_t = f(x, u, u_x, u_{xx}).$$

A first result is that if this equation possesses a conservation law, then $f(x, u, p, r)$ must be linear fractional in r. In this case, it turns out that there is associated a very lovely geometry on a suitable 3-manifold which leads to a normal form for evolution equations having either one or two independent conservation laws. We finally show that such an equation has three independent conservation laws if and only if it is linearizable.

Although we frequently work locally in order to make the appropriate calculations, our conclusions have global significance. This is quite unlike the case, for example, of a classical mechanical system in Hamiltonian form, since by Darboux's theorem such a system is always locally completely integrable. Indeed, in our situation, except in the case of linearizable equations, there are enough local invariants to insure that any local conservation law will automatically have global significance.

More precisely, let $(M, \mathscr{I})$ be a global parabolic system and assume that the rank of suitable linear bundle mappings over M are locally constant. Our discussion below gives the structure of the space of conservation laws in a neighborhood of any point $x \in M$. In precise (but perhaps unappealing) language, we determine the stalk $\underline{\mathscr{C}}_x$ of the sheaf $\underline{\mathscr{C}}$ of conservation laws. It turns out that in the nonlinear case $\underline{\mathscr{C}}$ is what is called a *local system*. As a consequence, if M is connected and simply-connected, then each local conservation law is the restriction of a unique global conservation law. In the general case, the action of the fundamental group must be taken into account. It is in this sense that our local computations have global significance.

A very interesting question arises concerning the "field of definition" of the conservation laws. To illustrate this, suppose that our parabolic system is given in the category of real algebraic varieties. Then, aside from the linear case, we will see that the conservation laws will be given by solving total differential equations of the form

$$(1) \qquad d\begin{pmatrix} f_1 \\ \vdots \\ f_k \end{pmatrix} = \begin{pmatrix} \omega_{11} & \cdots & \omega_{1k} \\ \vdots & & \\ \omega_{k1} & \cdots & \omega_{kk} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_k \end{pmatrix}$$

where each ω_{ij} is a rational 1-form and the matrix $\Omega = \|\omega_{ij}\|$ satisfies the integrability condition $d\Omega = \Omega \wedge \Omega$. In the language of differential algebra, the conservation laws live in differential field extensions defined by algebraic differential equations (1) of the field of rational functions. We will see that nonalgebraic field extensions are indeed necessary (this happens for heat-shrinking plane curves on surfaces of constant nonzero Gauss curvature). Thus conservation laws are expressed in terms of transcendental functions of a familiar and well-known kind, but they need not be differential polynomials, as is the case for many of the classical integrable equations.

This discussion leads naturally into the concept introduced in §6 of an *integrable extension* of a parabolic system. The conservation laws we discuss in §0–§5 are what are known classically as *local conservation laws*. Intuitively, a local conservation law is some universal expression in terms of the unknown functions and a finite number of their derivatives which induce closed differential forms whenever a solution to the PDE is substituted into the expression. Examples show that an equation with a finite number of local conservation laws may nevertheless have an infinite number of conservation laws expressible in a suitable differential extension of the ring of coordinate functions on the original manifold. Such differential extensions are essentially obtained by adjoining the "functions" $f_1, \dots, f_k$ in a system of the type (1) above but where the integrability conditions are of the form $d\Omega \equiv \Omega \wedge \Omega \bmod \mathscr{I}$. The "functions" f_i may not exist in the usual sense, and the appropriate concept is that of an integrable extension of an exterior differential system. In §6 this concept is introduced, and a first structure theorem for integrable extensions of parabolic systems is proved and illustrated.

To conclude this introduction, we would like to offer a contextual observation and, in particular, to say what we are not doing. These two papers are intended to help further the general objective of developing a geometric understanding of partial differential equations.[3] Developing such an understanding means in part to study the geometry[4] associated to a particular PDE or to a class of equations, and this is the main thrust of the present work. It is also a means to shed new light on their solutions, and this is only indirectly done here. Our opinion is that developing geometric understanding of partial differential equations will necessitate integrating these two aspects, the intrinsic geometry of the PDE and solving the PDE. Although appealing, this principle has yet to be firmly established.

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[3] This is related to, but not the same as, using partial differential equations to study problems in geometry.

[4] By "geometry" we mean the structures (including conservation laws, a connection if it exists, etc.) intrinsically associated to the equation.

0. Basics

Second-order contact geometry and structure equations for PDE. In classical notation, a second-order partial differential equation for a single unknown function $u(x, y)$ is an equation of the form

$$F(x, y, u, p, q, r, s, t) = 0 \tag{0}$$

where, as usual, p, q, r, s, and t denote u_x, u_y, u_{xx}, u_{xy}, and u_{yy}, respectively. We shall assume that F is a smooth function of its eight arguments, at least near the locus $F = 0$. The assumption of *nondegeneracy* (which we make henceforth) is that the triple of functions (F_r, F_s, F_t) does not have a common zero on the locus $F = 0$.

We want to rewrite this equation as an exterior differential system. Before doing so, we first review the structure equations of the contact system in the space of variables (x, y, u, p, q, r, s, t). On this space, the *second-order contact system* is generated by the three 1-forms

$$\begin{aligned} \underline{\theta}_0 &= du - p\,dx - q\,dy \\ \underline{\theta}_1 &= dp - r\,dx - s\,dy \\ \underline{\theta}_2 &= dq - s\,dx - t\,dy, \end{aligned} \tag{1}$$

which satisfy the structure equations

$$\begin{aligned} d\underline{\theta}_0 &\equiv -\underline{\theta}_1 \wedge dx - \underline{\theta}_2 \wedge dy \quad \operatorname{mod} \underline{\theta}_0 \\ \left.\begin{aligned} d\underline{\theta}_1 &\equiv -dr \wedge dx - ds \wedge dy \\ d\underline{\theta}_2 &\equiv -ds \wedge dx - dt \wedge dy \end{aligned}\right\} &\operatorname{mod} \underline{\theta}_0, \underline{\theta}_1, \underline{\theta}_2 . \end{aligned}$$

Writing $\underline{\omega}^1 = dx$, $\underline{\omega}^2 = dy$, $\underline{\pi}_{11} = dr$, $\underline{\pi}_{12} = \underline{\pi}_{21} = ds$, and $\underline{\pi}_{22} = dt$, these structure equations take the form

$$\begin{aligned} d\underline{\theta}_0 &\equiv -\underline{\theta}_i \wedge \underline{\omega}^i \operatorname{mod} \underline{\theta}_0 \\ d\underline{\theta}_i &\equiv -\underline{\pi}_{ij} \wedge \underline{\omega}^j \operatorname{mod} \underline{\theta}_0, \underline{\theta}_1, \underline{\theta}_2, \end{aligned} \tag{2}$$

where $1 \leqslant i, j \leqslant 2$, and we have employed the summation convention. An *admissible coframing* is a local coframing $(\theta_0, \theta_1, \theta_2, \omega^1, \omega^2, \pi_{11}, \pi_{12}, \pi_{22})$ on this space which satisfies $\operatorname{span}\{\theta_0, \theta_1, \theta_2\} = \operatorname{span}\{\underline{\theta}_0, \underline{\theta}_1, \underline{\theta}_2\}$ and for which the analogue of the structure equations (2) holds, i.e.,

$$\begin{aligned} d\theta_0 &\equiv -\theta_i \wedge \omega^i \operatorname{mod} \theta_0 \\ d\theta_i &\equiv -\pi_{ij} \wedge \omega^j \operatorname{mod} \theta_0, \theta_1, \theta_2 . \end{aligned}$$

It is not hard to see that for any admissible coframing there are functions A_0, B^i_j and A^i_j so that

$$\theta_0 = A_0\underline{\theta}_0$$

$$\theta_i = B^j_i\underline{\theta}_j \bmod \underline{\theta}_0$$

$$\omega^i \equiv A^i_j\underline{\omega}^j \bmod \underline{\theta}_0, \underline{\theta}_1, \underline{\theta}_2 .$$

In fact, it is not hard to show that $B = A_0 A^{-1}$ and

$$\pi_{ij} \equiv A_0^{-1} B^k_i B^\ell_j \underline{\pi}_{k\ell} \bmod \underline{\theta}_0, \underline{\theta}_1, \underline{\theta}_2, \underline{\omega}^1, \underline{\omega}^2 .$$

Returning to the partial differential equation (0), since the eight coordinate functions are subject to the relation $F = 0$, there is a single relation among their differentials of the form

(3) $$F_r\, dr + F_s\, ds + F_t\, t \equiv 0 \bmod dx, dy, \theta_0, \theta_1, \theta_2 .$$

According to the classical theory of characteristics for second-order equations, the characteristic covectors on any solution of the given equation are given by the factors of the expression

(4) $$Q = F_t\, dx^2 - F_s\, dx\, dy + F_r\, dy^2 .$$

The equation $F = 0$ is said to be *parabolic* if the quadratic form Q is of rank 1 (instead of 2), along the entire locus $F = 0$. We warn the reader that this is a slightly more general notion of parabolicity than is frequently encountered in the literature. For example, by this definition, the equation $u_{xx} = 0$ is parabolic even though it is too degenerate (in a sense to be made precise below) to be regarded as a parabolic equation in the PDE literature. Moreover, most of the literature on parabolic equations concentrates on the case of parabolic *evolution* equations; in our case this would be an equation of the form $u_t = F(x, u, u_x, u_{xx})$, where the partial of F with respect to its last variable (i.e., u_{xx}) is positive. Our parabolic equations will not generally be equivalent to equations of this special kind, even locally up to contact transformations. For example, the equation

$$u_{xx} - 2uu_{xy} + u^2 u_{yy} = u_y$$

is parabolic in the above sense (and is even nondegenerate in the sense we will describe below), but it cannot be put in evolutionary form, even locally.

Henceforth, we shall assume that $F = 0$ is a parabolic equation. We denote the locus $F = 0$ (which is a smooth hypersurface in $\mathbb{R}^8$) by M^7, and we let $\mathcal{I}$ denote the rank-3 Pfaffian system on M generated by the restrictions of the three 1-forms $\{\underline{\theta}_0, \underline{\theta}_1, \underline{\theta}_2\}$.

In a general admissible coframing, the linear relation (3) is expressible in the form

$$Q^{ij}\pi_{ij} \equiv 0 \bmod \theta_0, \theta_1, \theta_2, \omega^1, \omega^2,$$

where $Q^{ij} = Q^{ji}$. Taking into account the possibilities for an admissible coframing, our assumption that Q has rank 1 means that we may find such an admissible coframing so that

$$(Q^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and the relation (3) becomes simply

$$\pi_{11} \equiv 0 \bmod \theta_0, \theta_1, \theta_2, \omega^1, \omega^2.$$

With this in mind, we can simplify the notation somewhat since we are restricting to a 7-manifold. A coframing $\Phi = (\theta_0, \theta_1, \theta_2, \omega^1, \omega^2, \pi_3, \pi_4)$ on an open subset U in M is said to be *0-adapted* to $\mathscr{I}$ if the 1-forms θ_0, θ_1, and θ_2 span the restriction of $\mathscr{I}$ to U and the following structure equations hold:

(5)
$$\begin{aligned} d\theta_0 &\equiv -\theta_1 \wedge \omega^1 - \theta_2 \wedge \omega^2 \quad \bmod \theta_0 \\ d\theta_1 &\equiv -\pi_3 \wedge \omega^2 \\ d\theta_2 &\equiv -\pi_3 \wedge \omega^1 - \pi_4 \wedge \omega^2 \end{aligned} \left.\vphantom{\begin{aligned} a \\ b \end{aligned}}\right\} \bmod \theta_0, \theta_1, \theta_2.$$

In the language of differential systems, a coframing is 0-adapted to $\mathscr{I}$ if, in the structure equation, the *principal symbol* of the exterior differential system is normalized and the *torsion* has been absorbed. The assumption that $F = 0$ is a parabolic equation implies that M can be covered by open sets U on which such 0-adapted coframings exist (in fact, the existence of such local coframings is *equivalent* to parabolicity).

We observe also that a coframing is 0-adapted if and only if the symbol of its structure equations is normalized to be that of an equation

$$u_{xx} = f(x, t, u, u_t, u_x),$$

and the torsion has been absorbed, which is expressed in our notation by the condition

$$\pi_{11} \equiv 0 \bmod \theta_0, \theta_1, \theta_2.$$

(Note, however, that we are not asserting that every parabolic equation is actually contact equivalent to an equation of the above form. This is false.)

Conversely, the structure equations (5) characterize second-order parabolic PDE in the following sense: Suppose that M^7 is a 7-manifold on which there exists a coframing Φ satisfying the structure equations (5). The Cartan system of the form θ_0 is clearly generated by $\{\theta_0, \theta_1, \theta_2, \omega^1, \omega^2\}$. It follows from the Pfaff normal form theorem that every point of M has a neighborhood U on which there exists a submersion

$$f = (x, y, u, p, q)\colon U \to \mathbb{R}^5$$

with the property that the form $f^*(du - p\,dx - q\,dy)$ is a multiple of θ_0. For any point $m \in U$, let $P_m \subset T_m U$ be the 4-plane annihilated by the forms $\{\theta_0, \theta_1, \theta_2\}$. Clearly the image $F(m) = f_*(P_m) \subset T_{f(m)}\mathbb{R}^5$ is a 2-plane which is a contact element for the contact form $du - p\,dx - q\,dy$. Let $\mathscr{V}(\mathbb{R}^5)$ denote the (8-dimensional) space of such contact elements.

The structure equations (5) imply that the map $F\colon U \to \mathscr{V}(\mathbb{R}^5)$ is an immersion of U into $\mathscr{V}(\mathbb{R}^5)$. By unwinding definitions, it can be verified that F pulls back the canonical system on $\mathscr{V}(\mathbb{R}^5)$ to be the system spanned by $\{\theta_0, \theta_1, \theta_2\}$, and, moreover, the independence condition is described by the forms $\{\omega^1, \omega^2\}$. Locally, the image of F can be described by a second-order partial differential equation which the structure equations (5) imply to be parabolic. Thus, the coframing Φ defines a second-order PDE uniquely up to contact equivalence.

For this reason we give the following.

Definition. A *parabolic system* is given by a pair $(M, \mathscr{I})$ where M is a 7-manifold and $\mathscr{I}$ is a rank-3 Pfaffian system, and every point of M has a neighborhood U in which there is a coframing $\theta_0, \theta_1, \theta_2, \omega^1, \omega^2, \pi_3, \pi_4$ such that $\mathscr{I}|_U$ is generated by $\theta_0, \theta_1, \theta_2$ and the structure equations (5) are satisfied.

For each parabolic system $(M, \mathscr{I})$ there is, lying over M, a principal bundle with structure group a certain $G_0 \subset \mathrm{GL}(7, \mathbb{R})$

$$\mathscr{F} \to M$$

whose local cross-sections over any open set $U \subset M$ consist of all coframings with domain U which are 0-adapted to $\mathscr{I}$. The automorphisms of the exterior differential system $(M, \mathscr{I})$ can be identified with the bundle automorphisms of this fibration which preserve the so-called structure equations (see below). Moreover, as we shall see, the geometry of the conservation laws of the underlying partial differential equation can be studied in terms of the "intrinsic" (i.e., contact-invariant) geometry of $\mathscr{I}$.

Our tool for getting at the intrinsic geometry of $\mathscr{I}$ will be the equivalence method of É. Cartan, which will determine the geometric invariants of the $\mathscr{I}$ by introducing intrinsic "partial connections" on the fibration $\mathscr{F}$ whose "curvatures" and "torsions" may be interpreted as relative invariants and tensorial quantities. Normalizing these will then lead to a reduction of the structure group of $\mathscr{F} \to M$, and further invariants.

From the point of view of exterior differential systems and partial differential equations, the objects of interest for a given parabolic system $\mathcal{I}$ are the surfaces $S \subset M$ on which the forms in $\mathcal{I}$ vanish, the so-called *integral manifolds* of $\mathcal{I}$. We are also usually interested in imposing the natural *independence condition*; namely, we are interested in the integral manifolds S of $\mathcal{I}$ on which the 2-form $\omega^1 \wedge \omega^2$ is nonzero. (Note that this condition does not depend on the choice of 0-adapted coframing Φ in which it is expressed.) Since the structure equations imply that for any 0-adapted coframing, the 2-form $\pi_3 \wedge \omega^2$ must vanish on each integral manifold S of $(\mathcal{I}, \omega^1 \wedge \omega^2)$ and since ω^2 does not vanish on S, it follows that there must be a function λ on S so that $\pi_3 = \lambda\omega^2$. In particular, S is foliated by integral curves of the so-called *characteristic system* $\mathcal{M}$, which is defined (in the domain of any local 0-adapted coframing) to be the span of the 1-forms θ_0, θ_1, θ_2, π_3, and ω^2. The integral curves of the system $(\mathcal{M}, \omega^1)$ are called the *characteristic curves* of the system $\mathcal{I}$. Thus, every integral surface of $(\mathcal{I}, \omega^1 \wedge \omega^2)$ is foliated by characteristic curves of $\mathcal{I}$.

The initial value problem which is most commonly of interest for parabolic equations is the characteristic initial value problem: Given a characteristic curve of $\mathcal{I}$, when does it lie in or form one boundary of an integral surface of $(\mathcal{I}, \omega^1 \wedge \omega^2)$? How many such surfaces are there? (Note that this should be contrasted with the case of the hyperbolic theory, where one is interested in posing *noncharacteristic* initial value problems or the case of the elliptic theory, where one is generally not interested in initial value problems at all.) However, in actual geometric problems, one does not usually consider arbitrary characteristic curves but, in addition, one imposes some sort of compactness or completeness assumption on the characteristic initial curves γ that one considers. We will not generally worry much about this point since our study will mainly be local. Nevertheless, it is worth remarking that there are interesting problems in this regard. For example, one could consider the problem of defining a reasonable notion of completeness for γ (and $\mathcal{I}$) which would suffice to guarantee existence and/or uniqueness of the surface S. Also interesting are cases where there are closed characteristic curves.

As we have seen, any parabolic system can be locally realized as a second-order parabolic equation in the plane. However, this realization may not be possible globally.

Example 1: The heat equation for curves on Riemannian surfaces. Let $(S, d\sigma^2)$ be an oriented surface with a smooth Riemannian metric $d\sigma^2$ specified. We want to discuss the partial differential equation for smooth immersed curves $\gamma\colon N^1 \to S$ known as the "heat equation shrinking curves on S". This is the equation for a 1-parameter family $\Gamma\colon N^1 \times [0, T] \to S$ of immersed curves which satisfy the condition

$$\frac{\partial \Gamma}{\partial t}(u, t) = \kappa(u, t) N(u, t),$$

where, at each $(u, t) \in N^1 \times [0, T]$, $\kappa(u, t)$ and $N(u, t)$ represent the geodesic curvature and oriented unit normal at $u \in N^1$ to the immersed curve $\gamma_t\colon N \to S$ given by $\gamma_t(u) = \Gamma(u, t)$.

We will now show how this partial differential equation can be expressed as a parabolic system on a 7-manifold. Let $F \to S$ be the oriented orthonormal frame bundle of S with respect to $d\sigma^2$. Thus, an element $f \in F$ is of the form $f = (s; e_1, e_2)$ where $s \in S$ and (e_1, e_2) are an oriented orthonormal basis of T_sS. Let η_1 and η_2 denote the dual 1-forms on F and let η_{21} denote the connection form. One has the structure equations

$$\begin{aligned} d\eta_1 &= \eta_{21} \wedge \eta_2 \\ d\eta_2 &= -\eta_{21} \wedge \eta_1 \end{aligned} \qquad d\eta_{21} = -K\eta_1 \wedge \eta_2,$$

where K is the Gaussian curvature of S regarded as a function on F.

If Γ is a solution of the heat equation for curves in S, then Γ has a natural lift $\tilde{\Gamma}\colon N \times [0, T] \to F$ given by

$$\tilde{\Gamma}(u, t) = (\Gamma(u, t); T(u, t), N(u, t))$$

where $T(u, t)$ and $N(u, t)$ are the unit tangent and normal vectors to the curve γ_t at $u \in N$. By the very definition of the tautological forms on F, it follows that there are formulas

$$\tilde{\Gamma}^*(\eta_{21}) = \kappa\tilde{\Gamma}^*(\eta_1) - \lambda\, dt \qquad \tilde{\Gamma}^*(\eta_2) = \kappa\, dt,$$

where κ is as defined above and λ is some function on $N \times [0, T]$. Conversely any map $\tilde{\Gamma}\colon N \times [0, T] \to F$ is the canonical lift of a solution Γ of the heat equation shrinking curves on S provided that it both satisfies these identities for some functions κ and λ and satisfies the open condition that $\tilde{\Gamma}^*(\eta_1) \wedge dt \neq 0$.

With this in mind, set $M = F \times \mathbb{R}^4$ with coordinates t, u_2, u_3, and u_4 on $\mathbb{R}^4$ and define

$$\begin{aligned} \theta_0 &= \eta_2 - u_2\, dt \\ \theta_1 &= \eta_{21} - u_2\eta_1 - u_3\, dt \\ \theta_2 &= du_2 - u_3\eta_1 - u_4\, dt \end{aligned} \qquad \begin{aligned} \omega^2 &= dt \\ \omega^1 &= \eta_1 \end{aligned} \qquad \begin{aligned} \pi_3 &= du_3 - (u_4 - u_2(u_2^2 + K))\eta_1 \\ \pi_4 &= du_4 + u_2^2 u_3\eta_1 . \end{aligned}$$

One readily sees that these forms satisfy the structure equations (5). Thus, the system $\mathscr{I}$ generated by θ_0, θ_1, and θ_2 is a parabolic system. Moreover, the integral manifolds of $(\mathscr{I}, \omega^1 \wedge \omega^2)$ are clearly the $\tilde{\Gamma}$ which are the canonical lifts of solutions of the heat equation shrinking curves on S.

Although the heat equation shrinking curves on S is a parabolic system, it is not difficult to show that it cannot be globally realized as a parabolic second-order partial differential equation. We will return to this example in §4, where we will show that unless the Gauss curvature K of the metric $d\sigma^2$ is constant, there are no nontrivial conservation laws for this equation, while for the case where K is constant, there is exactly one nontrivial conservation law. (As a consequence, we will show that there is no local integral formula for the "vanishing point" to which an embedded closed curve in the plane shrinks under this flow.)

From now on, we will treat the slightly more general case of a parabolic system $\mathscr{I}$ on a γ-manifold M.

The Goursat invariant. Conside two 0-adapted local coframings with the same domain, say, $\Phi = (\theta_0, \theta_1, \theta_2, \omega^1, \omega^2, \pi_3, \pi_4)$, and $\tilde{\Phi} = (\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_2, \tilde{\omega}^1, \tilde{\omega}^2, \tilde{\pi}_3, \tilde{\pi}_4)$. These two coframings are easily seen to be related by a "transition matrix" of the form

$$\begin{bmatrix} \tilde{\theta}_0 \\ \tilde{\theta}_1 \\ \tilde{\theta}_2 \\ \tilde{\omega}^2 \\ \tilde{\omega}^1 \\ \tilde{\pi}_3 \\ \tilde{\pi}_4 \end{bmatrix} = g \cdot \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \omega^2 \\ \omega^1 \\ \pi_3 \\ \pi_4 \end{bmatrix}, \tag{6}$$

where g is a matrix of the form

$$\begin{bmatrix} a_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & a_1 & 0 & 0 & 0 & 0 & 0 \\ * & a_1 b_1 & a_2 & 0 & 0 & 0 & 0 \\ * & a_1(c_1 + b_2 b_1) & a_2 b_2 & a_0/a_2 & 0 & 0 & 0 \\ * & * & a_2 c_1 & -(a_0/a_2) b_1 & a_0/a_1 & 0 & 0 \\ * & * & * & (a_0/a_2) c_2 & 0 & a_1 a_2/a_0 & 0 \\ * & * & * & * & (a_0/a_1) c_2 & 2(a_1 a_2/a_0) b_1 & (a_2)^2/a_0 \end{bmatrix},$$

where the functions a_0, a_1, and a_2 are nonzero, the functions b_1, b_2, c_1, and c_2 are arbitrary, and the entries marked by a $*$ are also arbitrary. Conversely, if Φ is a 0-adapted coframing and $\tilde{\Phi}$ is related to Φ by such a matrix, then $\tilde{\Phi}$ is also 0-adapted. It follows that the 0-adapted coframings associated to a parabolic system are the local sections of the principal G_0-bundle $\mathscr{F} \to P$ where $G_0 \subset \mathrm{GL}(7, \mathbb{R})$ is the lower triangular group of dimension 19 described in Equation (6). We note the zero in the (5, 6) position, expressing the fact that we cannot add to π_3 a multiple of ω^1 while preserving the structure equations (5); this in turn ultimately reflects the geometric fact that 0-adapted frames already contain the information of the characteristic directions on integral surfaces of the system.

Let us consider a 0-adapted coframing in local coordinates. Starting with the original parabolic equation $F = 0$, one can always perform a contact transformation in a neighborhood of any point of M so as to get a new equation which can be locally solved for r. Thus, we can write the equation in the form

$$F(x, y, u, p, q, r, s, t) = r - E(x, y, u, p, q, s, t) = 0.$$

The parabolicity condition then takes the form $E_t = -(1/4)E_s^2$. In this case, it can be verified that the following is a 0-adapted coframing:

$$\begin{aligned}
\theta_0 &= du - p\,dx - q\,dy \\
\theta_1 &= dp - E\,dx - s\,dy - \frac{1}{2}E_s(dq - s\,dx - t\,dy) \\
\theta_2 &= dq - s\,dx - t\,dy \\
\omega^1 &= dx \\
\omega^2 &= dy + \frac{1}{2}E_s\,dx \\
\pi_3 &= ds - \frac{1}{2}E_s\,dt - (E_y + qE_u + sE_p + tE_q)\,dx \\
\pi_4 &= dt.
\end{aligned} \tag{7}$$

It is easy to compute that, for the coordinate 0-adapted coframing (7), the following structure equation holds:

$$d\theta_1 \equiv -\pi_3 \wedge \omega^2 - \theta_2 \wedge (A\pi_3 + B\omega^2 + C\omega^1) \bmod \theta_0, \theta_1, \tag{8}$$

where

$$A = \frac{1}{2}E_{ss}$$

$$B = -\frac{1}{2}(E_{sy} + qE_{su} + sE_{sp} + tE_{sq})$$

$$C = E_q + \frac{1}{2}E_pE_s - \frac{1}{2}E_{sx} + \frac{1}{4}E_s(E_{sy} + qE_{su} + sE_{sp} + tE_{sq})$$

$$- \frac{1}{2}(pE_{su} + EE_{sp} + sE_{sq} + (E_y + qE_u + sE_p + tE_q)E_{ss}).$$

Indeed, for any 0-adapted coframing, the structure equations (5) imply, after some calculation, that there is an equation of the form (8) for some functions A, B, and C; the explicit coordinate coframing (7) serves to identify these coefficients explicitly in local coordinates.

Now, adding to ω^2 the term $A\theta_2$ eliminates the term $-\theta_2 \wedge A\pi_3$ in $d\theta_1$, and then adding to π_3 the term $-B\theta_2$ eliminates the term $-\theta_2 \wedge B\omega^2$, resulting in the equation

(9) $$d\theta_1 \equiv -\pi_3 \wedge \omega^2 - C\theta_2 \wedge \omega^1 \bmod \theta_0, \theta_1.$$

We now find that we cannot eliminate the $\theta_2 \wedge \omega^1$ term from this equation by further admissible changes of coframings that preserve the condition $A = B = 0$. This suggests that C may be a tensorial quantity. Indeed, for any coframing $\tilde{\Phi}$ related to Φ as in Equation (6), we find that

$$\tilde{C} = (a_1^2/(a_0a_2))C.$$

Thus, the quantity C is an example of what is known classically as a *relative invariant* of the G-structure; it was first identified explicitly by E. Goursat [Go]. We shall henceforth refer to C as the *Goursat invariant* of the system $\mathcal{I}$. The complicated formula for the Goursat invariant of an explicitly known equation illustrates one of the difficulties of working with geometric quantities in coordinates. Because of Goursat's work on parabolic equations for which C vanishes identically, Cartan [Ca1] calls such equations by the name *equations of Goursat type.*

Example 1 (continued). For the parabolic system which describes the heat equation shrinking curves on a Riemannian surface, it is easy to compute that, in the coframing described above, we have $d\theta_1 \equiv -\theta_2 \wedge \omega^1 - \pi_3 \wedge \omega^2 \bmod \theta_0, \theta_1$. Thus, the Goursat invariant for this system is nonzero.

Example 2. As might be expected, the Goursat invariant for the equation $u_{xx} = 0$ vanishes while the Goursat invariant for the classical heat equation $u_{xx} - u_t = 0$ is nonzero.

Example 3: Weingarten surfaces. Let (N^3, ds^2) be an oriented Riemannian 3-manifold. On any oriented surface $S \subset N$, one can define the principal curvatures κ_1 and κ_2. The surface S is said to be a *Weingarten surface* if there is a nontrivial functional relation of the form $F(\kappa_1, \kappa_2) = 0$.

For any specified $F(\kappa_1, \kappa_2)$, the Weingarten condition defines a second-order differential equation for surfaces in N, which we shall now describe. For simplicity, we shall assume that the surfaces S that we consider are free of umbilics, i.e., that $\kappa_1 \neq \kappa_2$ at any point of S. Let $F \to N$ be the oriented orthonormal frame bundle of N, and let η_i and $\eta_{ij} = -\eta_{ji}$ for $1 \leqslant i, j \leqslant 3$ be the tautological and connection forms on F. They satisfy the structure equations (summation convention assumed)

$$d\eta_i = -\eta_{ij} \wedge \eta_j$$

$$d\eta_{ij} = -\eta_{ik} \wedge \eta_{kj} + \frac{1}{2} R_{ijkl}\eta_k \wedge \eta_l,$$

where the functions R_{ijkl} are the components of the Riemann curvature tensor.

Let $f_1(\kappa)$ and $f_2(\kappa)$ be two functions of κ which satisfy $F(f_1(\kappa), f_2(\kappa)) = 0$. We shall assume that $f_1(\kappa) \neq f_2(\kappa)$ for any κ and that f_1' and f_2' do not simultaneously vanish. Let $M = F \times \mathbb{R}$, with coordinate κ on the second factor. Define the 1-forms

$$\begin{aligned} &\theta_0 = \eta_3 && \omega^1 = \eta_1 && \pi_2 = f_1'(\kappa)\,d\kappa \\ &\theta_1 = \eta_{31} - f_1(\kappa)\eta_1 && \omega^2 = \eta_2 && \pi_3 = (f_1(\kappa) - f_2(\kappa))\eta_{21} \\ &\theta_2 = \eta_{32} - f_2(\kappa)\eta_2 && && \pi_4 = f_2'(\kappa)\,d\kappa. \end{aligned}$$

It is easy to see that they satisfy the equations

$$d\theta_0 \equiv -\theta_1 \wedge \omega^1 - \theta_2 \wedge \omega^2 \qquad \text{mod } \theta_0$$

$$\left.\begin{aligned} d\theta_1 &\equiv -\pi_2 \wedge \omega^1 - \pi_3 \wedge \omega^2 + R_{3112}\omega^1 \wedge \omega^2 \\ d\theta_2 &\equiv -\pi_3 \wedge \omega^1 - \pi_4 \wedge \omega^2 + R_{3212}\omega^1 \wedge \omega^2 \end{aligned}\right\} \text{mod } \theta_0, \theta_1, \theta_2$$

and the one relation $f_2'(\kappa)\pi_2 - f_1'(\kappa)\pi_4 = 0$. It follows that the differential system $\mathcal{I}$ generated by θ_0, θ_1, and θ_2 with independence condition $\omega^1 \wedge \omega^2 \neq 0$ (whose integrals are the nonumbilic surfaces which satisfy the Weingarten relation) is of parabolic type if and only if one of f_1' or f_2' is identically zero. Thus, the parabolic Weingarten relations are those which restrict one of the principal curvatures to be constant.

By symmetry, we may assume that $f_1(\kappa) \equiv c$ for some constant c and that $f_2(\kappa) = \kappa + c$. Then the nondegeneracy condition is simply $\kappa \neq 0$. It then follows that the coframing $\Phi = (\theta_0, \theta_1, \theta_2, \omega^2, \omega^1, \pi_3, \pi_4)$ is a 0-adapted coframing for this parabolic system on M. Redefining ω^2 and π_3 to be $\eta_2 + \kappa^{-1}\theta_2$ and $\kappa\eta_{12} -$

$R_{3112}\eta_1$, respectively, the refined structure equation for $d\theta_1$ becomes

$$d\theta_1 \equiv -\pi_3 \wedge \omega^2 - \kappa^{-1}R_{1312}\theta_2 \wedge \omega^1 \bmod \theta_0, \theta_1 .$$

Thus, the Goursat invariant in this coframing is $C = \kappa^{-1}R_{1312}$. Notice that this vanishes identically on M if and only if R_{1312} vanishes identically on F. In turn, it is known that this happens if and only if the metric ds^2 has constant sectional curvature.

Equations of Goursat type. It is possible to interpret the Goursat invariant as a way of measuring the "dispersive" nature of the characteristic initial value problem for a parabolic equation. Since this is the essential characteristic of parabolic equations in the usual PDE studies, the equations for which the Goursat invariant vanishes, i.e., the equations of Goursat type, are of less interest. Indeed, the bulk of this paper will consider only the non-Goursat case.

In fact, it turns out that parabolic equations of Goursat type belong more to the study of ordinary differential equations. In [Ca1], Cartan shows how equations of Goursat type can be integrated using only techniques from ordinary differential equations. In the interest of completeness, and because it is interesting, we will now indicate how this is done.

Assume that $C \equiv 0$. Then the structure equation for θ_1 becomes

$$d\theta_1 \equiv -\pi_3 \wedge \omega^2 - \theta_2 \wedge (A\pi_3 + B\omega^2) \equiv -(\pi_3 + B\theta_2) \wedge (\omega^2 - A\theta_2) \bmod \theta_0, \theta_1 .$$

It follows that, replacing ω^2 by $\omega^2 - A\theta_2$ and π_3 by $\pi_3 + B\theta_2$, one may arrange that the structure equations take the form

$$\left.\begin{aligned} d\theta_0 &\equiv -\theta_2 \wedge \omega^2 \\ d\theta_1 &\equiv -\pi_3 \wedge \omega^2 \end{aligned}\right\} \bmod \theta_0, \theta_1 .$$

It follows that the Cartan system of the rank-2 Pfaffian system $\mathscr{K} = \{\theta_0, \theta_1\}$ is the rank-5 Pfaffian system $\mathscr{M} = \{\theta_0, \theta_1, \theta_2, \omega^2, \pi_3\}$. In particular, the characteristic system $\mathscr{M}$ is completely integrable.

It follows that every point of M lies in an open set U on which there exists a submersion $f: U \to \mathbb{R}^5$ whose fibers are the leaves in U of the system $\mathscr{M}$. Moreover, there exists a rank-2 Pfaffian system $\overline{\mathscr{K}}$ on $f(U) \subset \mathbb{R}^5$ which pulls back under f to be the system $\mathscr{K}$ restricted to U. From the structure equations, it follows that there is a well-defined rank-3 Pfaffian system $\overline{\mathscr{K}}^+$ which contains $\overline{\mathscr{K}}$ and pulls back via f to be spanned by the forms $\{\theta_0, \theta_1, \omega^2\}$.

Let $N \subset U$ be any integral manifold of $(\mathscr{I}, \omega^1 \wedge \omega^2)$. Since θ_0, θ_1, θ_2, and $\omega^2 \wedge \pi_3$ all vanish on N, it follows that the image $C = f(N)$ has rank 1 and is hence a curve in $\mathbb{R}^5$. By the admissibility condition, it follows that this curve C is an integral curve of $\overline{\mathscr{K}}$ but is not an integral curve of $\overline{\mathscr{K}}^+$. (Note also that the intersection of N with any fiber of f is a characteristic curve.)

Suppose now that we are given a *noncharacteristic* integral curve $P \subset U$ of the system $\{\theta_0, \theta_1, \theta_2\}$. Since P is noncharacteristic, ω^2 does not vanish on P. It follows that the image $C = f(P)$ is an integral curve of $\overline{\mathscr{K}}$ but not an integral curve of $\overline{\mathscr{K}}^+$. Let $P^+ = f^{-1}(C)$. Then P^+ is a 3-dimensional submanifold of U. (Clearly, P^+ is simply the union of the fibers of f which intersect P.) By construction, θ_0 and θ_1 vanish on P^+ while ω^2 does not. On the other hand $\theta_2 \wedge \omega^2$ clearly does vanish on P^+ since $d\theta_0 \equiv -\theta_2 \wedge \omega^2 \bmod \theta_0, \theta_1$. Hence there exists a function g on P^+ so that $\theta_2 = g\omega^2$ on P^+. The function g vanishes along $P \subset P^+$ since P is an integral manifold of θ_0, θ_1, and θ_2. Finally, the structure equation $d\theta_2 \equiv -\pi_3 \wedge \omega^1 - \pi_4 \wedge \omega^2 \bmod \theta_0, \theta_1, \theta_2$ implies that dg does not vanish on P^+. In particular, the locus $g = 0$ is a smooth surface in P^+ which contains P. By its very construction, it is the *unique* integral manifold of $\{\theta_0, \theta_1, \theta_2\}$ which contains P. Thus, the noncharacteristic initial value problem is solved.

Example 2 (continued). Consider the parabolic equation $u_{xx} = 0$. Of course, its general solution is of the form $u(x, y) = f(y) + xg(y)$, where f and g are arbitrary functions of y. It is easy to compute that the characteristic curves of $\mathscr{I}$ are of the form

$$(x, y, u, p, q, r, s, t) = (x, y_0, u_0 + p_0 x, p_0, q_0 + s_0 x, 0, s_0, f(x)),$$

where x_0, y_0, u_0, p_0, q_0, and s_0 are constants while f is an arbitrary function of x. Clearly such a curve does not lie in an integral surface of $\mathscr{I}$ unless f is a linear function of x. Thus, the characteristic initial value problem for this (very trivial) equation is not well posed; one has neither existence nor uniqueness.

This ill-posedness of the characteristic initial value problem is a general feature of parabolic equations of Goursat type. On the other hand, for the system associated to the (non-Goursat) classical heat equation $u_{xx} = u_y$, the characteristic curves of $\mathscr{I}$ are of the form

$$(x, y, u, p, q, r, s, t) = (x, y_0, f(x), f'(x), f''(x), f''(x), f'''(x), f^{(iv)}(x)),$$

where y_0 is any constant and f is any function of x. Of course, in this case, provided either that f is periodic or decays sufficiently rapidly at infinity, one has existence and uniqueness of solutions of the characteristic initial value problem.

Example 3 (continued). In a space N of constant sectional curvature, the above method leads to the result that the surfaces $S \subset N$ which have one principal curvature equal to a constant c are the normal tubes (of constant radius) of curves in N. The characteristic curves are the "normal circles". Note that in this case, the Frenet frame of any curve of constant geodesic curvature c yields an integral curve of the system $\mathscr{I}$, but that only the ones with vanishing torsion lie in integral surfaces.

1. Conservation laws for equations not of Goursat type: First steps

Preliminary structure equations on the infinite prolongation. In Part I we introduced the infinite prolongation of an involutive exterior differential system. Denoting by Ω^* the forms on the infinite prolongation and by $\mathscr{I} \subset \Omega^*$ the infinitely prolonged differential ideal, we saw that Ω^* is filtered by subalgebras Ω^*_k such that each $\mathscr{I}_k = \Omega^*_k \cap \mathscr{I}$ is a Pfaffian differential ideal with the following property: Setting $\Theta_k = \mathscr{I}_k \cap \Omega^1$, then

$$d\Theta_k \equiv 0 \bmod \{\Theta_{k+1}\}. \tag{0}$$

That is, the extensions of differential ideals $\mathscr{I}_k \subset \mathscr{I}_{k+1}$ may be thought of as adjoining new 1-forms to "close up" $\mathscr{I}_k$ relative to the Frobenius integrability condition. This construction then ultimately led to the weight filtrations and generalized Spencer cohomology, which gave us a first approximation to the characteristic cohomology and, in particular, to the conservation laws of the system.

In this section, we will first derive the structure equations for an infinitely prolonged parabolic system that is not of Goursat type, i.e, a system whose Goursat invariant is nonvanishing. This will be done after choosing coframings for which Goursat's relative invariant satisfies $C = 1$. Beyond normalizing the principal symbol, this frame reduction has one implication on the structure equations that we should like to mention. Namely, the condition $C \neq 0$ is a nondegeneracy condition on the *subprincipal symbol* and restricting to adapted frames where $C = 1$ amounts to normalizing the subprincipal as well as the principal symbol. When this is done we have, in a manner of speaking, looked at the geometry one level below the highest order terms, and this is manifest in that the Frobenius extension condition (0) is replaced by the 2-step Frobenius extension condition expressed by Equation (2) below.

From now on, we assume that the relative invariant C never vanishes on M. We may then define a more restricted class of local coframings than the 0-adapted ones by refining the structure equation for $d\theta_1$.

From the calculations in the previous section, we know that for any 0-adapted coframing Φ, there exist functions A, B, and C so that

$$d\theta_1 \equiv -\pi_3 \wedge \omega^2 - \theta_2 \wedge (A\pi_3 + B\omega^2 + C\omega^1) \bmod \theta_0, \theta_1.$$

Replacing ω^2 and π_3 in Φ by the forms $\omega^2 + A\theta_2$ and $\pi_3 - B\theta_2$, respectively, we get a new coframing $\tilde{\Phi}$ which is easily seen to be 0-adapted and in which A and B are zero. Thus, let us assume that

$$d\theta_1 \equiv -C\theta_2 \wedge \omega^1 - \pi_3 \wedge \omega^2 \bmod \theta_0, \theta_1.$$

Since the function C is nonzero, the following is also a 0-adapted coframing:

$$\tilde{\Phi} = (C\theta_0, \theta_1, C\theta_2, \omega^2, C\omega^1, \pi_3, C\pi_4).$$

Moreover, this 0-adapted coframing has $\tilde{C} = 1$.

This motivates the following definition: A coframing $\Phi = (\theta_0, \theta_1, \theta_2, \omega^1, \omega^2, \pi_3, \pi_4)$ is said to be *1-adapted* if it is 0-adapted and also satisfies the condition

$$d\theta_1 \equiv -\theta_2 \wedge \omega^1 - \pi_3 \wedge \omega^2 \bmod \theta_0, \theta_1 .$$

As noted above, 1-adapted coframings are those for which both the principal symbol and subprincipal symbol have been normalized. It is not difficult to show that any two 1-adapted local coframings Φ and $\tilde{\Phi}$ on the same domain are related by a transition matrix of the form

$$\begin{bmatrix} \tilde{\theta}_0 \\ \tilde{\theta}_1 \\ \tilde{\theta}_2 \\ \tilde{\omega}^2 \\ \tilde{\omega}^1 \\ \tilde{\pi}_3 \\ \tilde{\pi}_4 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 \\ ab_1 & a/r & 0 & 0 & 0 & 0 & 0 \\ * & ab_2/r & a/r^2 & 0 & 0 & 0 & 0 \\ * & ab_3/r & 0 & r^2 & 0 & 0 & 0 \\ * & * & ab_3/r^2 & -b_2 r^2 & r & 0 & 0 \\ * & * & a(b_1+b_2)/r^2 & b_4 r^2 & 0 & a/r^3 & 0 \\ * & * & * & * & b_4 r & 2ab_2/r^3 & a/r^4 \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \omega^2 \\ \omega^1 \\ \pi_3 \\ \pi_4 \end{bmatrix}$$

where the functions a and r are nonzero, the functions b_1, b_2, b_3, and b_4 are arbitrary, and the entries marked by an $*$ are also arbitrary.

Conversely, if Φ is a 1-adapted coframing and $\tilde{\Phi}$ is related to Φ by such a matrix, then $\tilde{\Phi}$ is also 1-adapted. It follows that the 1-adapted coframings are the local sections of a G_1-structure on the underlying 7-manifold M, where $G_1 \subset \mathrm{GL}(7, \mathbb{R})$ is a lower triangular subgroup of dimension 16.

Note, in particular, the "transition" relation $\tilde{\omega}^2 \equiv r^2\omega^2 \bmod \theta_0, \theta_1$. This implies that the foliation of any integral manifold N of $(\mathcal{I}, \omega^1 \wedge \omega^2)$ by characteristic curves ($\omega^2 = 0$) is transversely oriented. Thus, there is a "positive" sense to motion transverse to the characteristic curves. This is characteristic of non-Goursat parabolics: there is a well-defined sense of "increasing time".

We now want to derive a crude version of the structure equations for the successive prolongations of a non-Goursat parabolic system. We begin with the structure equations of a (local) 1-adapted coframing:

$$\begin{aligned} & d\theta_0 \equiv -\theta_1 \wedge \omega^1 - \theta_2 \wedge \omega^2 \bmod \theta_0 \\ (1) \qquad & d\theta_1 \equiv -\theta_2 \wedge \omega^1 - \pi_3 \wedge \omega^2 \bmod \theta_0, \theta_1 \\ & d\theta_2 \equiv -\pi_3 \wedge \omega^1 - \pi_4 \wedge \omega^2 \bmod \theta_0, \theta_1, \theta_2 . \end{aligned}$$

All of the constructions to be carried out in this section can be "globalized" straightforwardly, so for simplicity we will assume that the domain of the coframing is M itself. The reader can make the necessary changes to describe the global structure if this is desired.

First, we describe the space of integral elements of $(\mathscr{I}, \omega^1 \wedge \omega^2)$ at a general point of M. For any two numbers p_5 and p_6, the 2-plane $E(p, p_5, p_6) \subset T_pM$, defined as the set of $v \in T_pM$ which satisfy the relations

$$\theta_0(v) = \theta_1(v) = \theta_2(v) = (\pi_3 - p_5\omega^2)(v) = (\pi_4 - p_5\omega^1 - p_6\omega^2)(v) = 0$$

is an integral element of $(\mathscr{I}, \omega^1 \wedge \omega^2)$. Conversely every integral element of $(\mathscr{I}, \omega^1 \wedge \omega^2)$ is of this form.

Thus, the first prolongation space of $(\mathscr{I}, \omega^1 \wedge \omega^2)$ can be described as follows. The underlying manifold is $M^{\langle 1 \rangle} = M \times \mathbb{R}^2$ (with coordinates p_5 and p_6 on the $\mathbb{R}^2$ factor). The system $\mathscr{I}^{\langle 1 \rangle}$ on $M^{\langle 1 \rangle}$ is generated by θ_0, θ_1, θ_2, and the two 1-forms

$$\theta_3 = \pi_3 - p_5\omega^2$$

$$\theta_4 = \pi_4 - p_5\omega^1 - p_6\omega^2.$$

This yields the equations

$$d\theta_0 \equiv -\theta_1 \wedge \omega^1 - \theta_2 \wedge \omega^2 \bmod \theta_0,$$

$$d\theta_1 \equiv -\theta_2 \wedge \omega^1 - \theta_3 \wedge \omega^2 \bmod \theta_0, \theta_1,$$

$$d\theta_2 \equiv -\theta_3 \wedge \omega^1 - \theta_4 \wedge \omega^2 \bmod \theta_0, \theta_1, \theta_2.$$

Taking the exterior derivative of the second of these equations and then reducing modulo $\Theta_3 = \{\theta_0, \theta_1, \theta_2, \theta_3\}$ yields

$$0 \equiv -d\theta_2 \wedge \omega^1 - d\theta_3 \wedge \omega^2 \equiv (\theta_4 \wedge \omega^2) \wedge \omega^1 - d\theta_3 \wedge \omega^2$$

$$\equiv -(d\theta_3 + \theta_4 \wedge \omega^1) \wedge \omega^2 \bmod \Theta_3.$$

It follows that there exists a 1-form π_5 on $M^{\langle 1 \rangle}$ so that

$$d\theta_3 \equiv -\theta_4 \wedge \omega^1 - \pi_5 \wedge \omega^2 \bmod \Theta_3.$$

Computing $0 = d(d\theta_2)$, reducing modulo $\Theta_4 = \{\theta_0, \theta_1, \theta_2, \theta_3, \theta_4\}$, and using the relation just obtained yields

$$0 \equiv -(d\theta_4 + \pi_5 \wedge \omega^1) \wedge \omega^2 \bmod \Theta_4.$$

Of course, this implies that there must exist a 1-form π_6 on $M^{\langle 1 \rangle}$ so that

$$d\theta_4 \equiv -\pi_5 \wedge \omega^1 - \pi_6 \wedge \omega^2 \bmod \Theta_4 .$$

Considering how θ_3 and θ_4 were defined, it easily follows that

$$\left.\begin{aligned} \pi_5 &\equiv dp_5 \\ \pi_6 &\equiv dp_6 \end{aligned}\right\} \bmod \Theta_4 \cup \{\omega^1, \omega^2\}.$$

In particular, π_5 and π_6 are independent on the fibers of the projection $M^{\langle 1 \rangle} \to M$.

It is clear that the process just described can be continued at each prolongation. Thus, on the kth prolongation space $M^{\langle k \rangle} = M^{\langle k-1 \rangle} \times \mathbb{R}^2$, the system $\mathscr{I}^{\langle k \rangle}$ is spanned by 1-forms $\theta_0, \theta_1, \dots, \theta_{2k+2}$ which generate the kth prolonged system and satisfy

$$d\theta_j \equiv -\theta_{j+1} \wedge \omega^1 - \theta_{j+2} \wedge \omega^2 \bmod \Theta_j = \{\theta_0, \theta_1, \dots, \theta_j\},$$

for $j \leqslant 2k$ while

$$d\theta_{2k+1} \equiv -\theta_{2k+2} \wedge \omega^1 - \pi_{2j+3} \wedge \omega^2 \bmod \Theta_{2k+1}$$

$$d\theta_{2k+2} \equiv -\pi_{2k+3} \wedge \omega^1 - \pi_{2j+4} \wedge \omega^2 \bmod \Theta_{2k+2},$$

where π_{2j+3} and π_{2j+4} are independent when restricted to the fibers of the projection $M^{\langle k \rangle} \to M^{\langle k-1 \rangle}$.

Passing to the infinite prolongation $M^{\langle \infty \rangle}$, one has the "infinitely prolonged" system $\mathscr{I}^{\langle \infty \rangle} = \Theta_\infty = \{\theta_0, \theta_1, \dots\}$ which satisfies the "crude structure equations":

$$(2) \qquad d\theta_j \equiv -\theta_{j+1} \wedge \omega^1 - \theta_{j+2} \wedge \omega^2 \bmod \Theta_j = \{\theta_0, \theta_1, \dots, \theta_j\}.$$

The forms in $\mathscr{I}^{\langle \infty \rangle}$, together with the forms ω^1 and ω^2 suffice to generate the full exterior ideal of forms on $M^{\langle \infty \rangle}$. The adjective "crude" refers to the fact that these equations reflect normalizing the principal and subprincipal symbols; they do not reflect the lower-order invariants of the system.

Conservation laws: General form. We set $\Omega^* = \Omega^*(M^{(\infty)})$ and let $\mathscr{I}$ denote the differential ideal generated by Θ_∞. By definition the space $\mathscr{C}$ of conservation laws is given by

$$\mathscr{C} = \frac{\{\varphi \in \Omega^1 | d\varphi \equiv 0 \bmod \mathscr{I}\}}{\{df | f \in \Omega^0\} \cup \Theta_\infty}.$$

Using the general formalism from Part I, we will establish two results (Proposition 1 and 2) which give the general form of any conservation law for a parabolic system. Following the proofs of these two results, we will give an alternate proof of the second proposition, one which does not explicitly use the language of Spencer cohomology.

PROPOSITION 1. *Under the assumption that $H^q(M) = 0$ for $q > 0$,*

$$\mathscr{C} \cong \{\overline{\Phi} = A(\omega^1 \wedge \theta_0 + \omega^2 \wedge \theta_1) + B\omega^2 \wedge \theta_0 | d\overline{\Phi} \equiv 0 \bmod \mathscr{I} \wedge \mathscr{I}\}.$$

Moreover, the function B is a linear combination of A and its first derivatives, and therefore any such $\overline{\Phi}$ is uniquely determined by A. The condition $d\overline{\Phi} \equiv 0 \bmod \mathscr{I} \wedge \mathscr{I}$ is a linear PDE for A whose "highest order part looks like the backwards heat equation."

We will explain what this latter statement means during the course of the proof; cf. Equation (7) below.

Proof. We first recall what the general constructions from Part I give in this case. With the notations

$$F^p\Omega^* = \text{image of } \{\underbrace{\mathscr{I} \wedge \cdots \wedge \mathscr{I}}_{p} \wedge \Omega^* \to \Omega^*\}$$

$$\overline{\Omega}^{p,*} = F^p\Omega^*/F^{p+1}\Omega^* \text{ with induced differential } \bar{d}$$

$$\overline{H}^{p,*} = \text{cohomology of } \{\overline{\Omega}^{p,*}, \bar{d}\},$$

we have by definition

$$\mathscr{C} = \overline{H}^{0,1}.$$

On the other hand, the spectral sequence $E_r^{p,q}$ of the filtered complex $\{\Omega^*, F^p, d\}$ has

$$E_1^{p,*} = \overline{H}^{p,*}$$

$$E_\infty^{p,q} = 0 \text{ for } p + q > 0$$

since we have assumed that M, and therefore also $M^{(\infty)}$, has no topology. Moreover, since the exterior differential system $\mathscr{I}$ has characteristic number $\ell = 1$ (see §4.2 in Part I), it follows from general considerations (loc. cit.) that

$$E_1^{p,0} = 0 \text{ for } p > 0.$$

Combining this with the above, we obtain

$$\mathscr{C} \cong \ker d_1 \colon E_1^{1,1} \to E_1^{2,1}. \tag{3}$$

We will see that (3) translates into the assertion of the proposition. Indeed, in the case at hand we will show that

$$E_1^{2,1} = 0. \tag{4}$$

Since it is a general fact that

$$E_1^{1,1} = \{\ker \nabla \colon E \to F\},$$

where E, F are canonically defined vector bundles (of rank 1 in our particular case), and ∇ is a canonical linear first-order differential operator, the proposition will result from (4) and the identification of E, F, and ∇ using a modified Spencer-type of cohomology.

We now turn to specific calculations based on the crude structure equations

$$\begin{cases} d\theta_j \equiv -\theta_{j+1} \wedge \omega^1 - \theta_{j+2} \wedge \omega^2 \bmod \Theta_j \\ d\omega^i \equiv \alpha^i \omega^1 \wedge \omega^2 \bmod \Theta_\infty . \end{cases} \tag{5}$$

We define weights by setting

$$\begin{cases} w(\theta_j) = j \\ w(\omega^1) = -1 \\ w(\omega^2) = -2 \end{cases}$$

and denote by $\bar{F}_k$ the induced weight filtration on $\bar{\Omega}^{p,*}$ with $\bar{\Omega}_k^{p,*} = \bar{F}_k \bar{\Omega}^{p,*} / \bar{F}_{k-1} \Omega^{p,*}$ (see §2.4 in Part I for further details). It follows from (5) that $\bar{d}$ preserves $\bar{F}_k$ and that the induced differential

$$\bar{\bar{d}} \colon \bar{\Omega}_k^{p,q} \to \bar{\Omega}_k^{p,q+1}$$

is *linear over the functions* and satisfies

$$\begin{cases} \bar{\bar{d}}\theta_j = -\theta_{j+1} \wedge \omega^1 - \theta_{j+2} \wedge \omega^2 \\ \bar{\bar{d}}\omega^i = 0 . \end{cases}$$

Thus the $\bar{\bar{d}}$ cohomology is purely algebraic and is therefore given by sections of a vector bundle whose fibers "look like" the fibers in the "flat" case $u_t - u_{xx} = 0$. This will compute the E_1-term of the weight spectral sequence $\bar{E}_r^{k,*}$, and, since for $p > 0$ that spectral sequence abuts to $E_1^{p,*}$, we will obtain a description of this latter group. Writing this out explicitly will give the proposition. Here are the details.

Denote by $\{\Lambda = \bigoplus_{p,q} \Lambda^{p,q}, \delta\}$ the bigraded complex in the flat linear constant coefficient case. Thus

$$\Lambda = \Lambda\{\theta_0, \theta_1, \theta_2, \ldots; \omega^1, \omega^2\}$$

$$\delta\theta_i = -\theta_{i+1} \wedge \omega^1 - \theta_{i+2} \wedge \omega^2$$

$$\delta\omega^i = 0.$$

We let $\mathscr{H}^{p,q}$ denote the cohomology in bidegree (p, q) of $\{\Lambda, \delta\}$ and shall prove that

(i) $\mathscr{H}^{1,1}$ has a basis $\theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2$;
(ii) $\mathscr{H}^{1,2}$ has a basis $\theta_0 \wedge \omega^1 \wedge \omega^2$;
(iii) $\mathscr{H}^{2,1} = 0$.

To do this, we set $\Lambda' = \Lambda \wedge \omega^2$ and $\Lambda'' = \Lambda/\Lambda'$ and observe that each of Λ' and Λ'' is isomorphic to the constant coefficient unconstrained case in one independent and one dependent variable. Thus we have the exact cohomology sequence

$$0 \to H^1(\Lambda) \to H^1(\Lambda'') \xrightarrow{\Delta} H^2(\Lambda') \to H^2(\Lambda) \to 0. \tag{6}$$

Taking $p = 1$ we have

$$H^1(\Lambda'') \cong \mathbb{F}\theta_0 \wedge \omega^1$$

$$H^2(\Lambda') \cong \mathbb{F}\theta_0 \wedge \omega^1 \wedge \omega^2$$

$$\begin{aligned}\Delta(\theta_0 \wedge \omega^1) &= \text{class of } \delta(\theta_0 \wedge \omega^1)\\ &= \text{class of } \theta_2 \wedge \omega^1 \wedge \omega^2\\ &= \text{class of } \delta(-\theta_1 \wedge \omega^2)\\ &= \text{class of } (-\delta\theta_1) \wedge \omega^2\\ &= 0.\end{aligned}$$

This tells us not only that $\mathscr{H}^{1,1}$ and $\mathscr{H}^{1,2}$ have rank 1, but it also tells us how to lift $\theta_0 \wedge \omega^1$ to the δ-closed form $\theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2$. In fact, it tells us more. In the general nonflat case, denote by $\underline{H}^1(\Lambda)$ and $\underline{H}^2(\Lambda)$ the vector bundles with fibers $H^1(\Lambda)$ and $H^2(\Lambda)$, respectively, and define a linear differential operator $\underline{\nabla}$: $\underline{H}^1(\Lambda) \to \underline{H}^2(\Lambda)$ by the formula

$$\underline{\nabla}(A(\theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2)) = \text{class of } \bar{d}(A(\theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2))$$

$$=: (\nabla A)\theta_0 \wedge \omega^1 \wedge \omega^2.$$

Writing

$$\bar{d}A =: D_1 A\omega^1 + D_2 A\omega^2$$

we have

$$\bar{d}(A(\theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2)) = (D_2 A)\theta_0 \wedge \omega^1 \wedge \omega^2 - (D_1 A)\theta_1 \wedge \omega^1 \wedge \omega^1$$

$$+ A\bar{d}(\theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2).$$

Also, for some functions α and β, we have

$$\bar{d}(\theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2) = \alpha\theta_0 \wedge \omega^1 \wedge \omega^2 - \beta\theta_1 \wedge \omega^1 \wedge \omega^2.$$

Thus

$$\bar{d}(A(\theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2)) = (D_2 A + \alpha A)\theta_0 \wedge \omega^1 \wedge \omega^2 - (D_1 A + \beta A)\theta_1 \wedge \omega^1 \wedge \omega^2.$$

By the general theory we may reduce the second term to a multiple of $\theta_0 \wedge \omega^1 \wedge \omega^2$. Explicitly, using the fact that $\bar{d}(\theta_0 \wedge \omega^2) = -\theta_1 \wedge \omega^1 \wedge \omega^2 + \varepsilon\theta_0 \wedge \omega^1 \wedge \omega^2$ for some ε, we have

$$\bar{d}((D_1 A + \beta A)(\theta_0 \wedge \omega^2)) = (-D_{11}A - \beta D_1 A - \beta_1 A)\theta_0 \wedge \omega^1 \wedge \omega^2$$

$$- (D_1 A + \beta A)\theta_1 \wedge \omega^1 \wedge \omega^2$$

$$+ \varepsilon(D_1 A + \beta A)\theta_0 \wedge \omega^1 \wedge \omega^2,$$

which implies that there exist functions λ and μ so that

$$\bar{d}(A(\theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2)) \equiv (D_2 A + D_{11}A + \lambda D_1 A + \mu A)\theta_0 \wedge \omega^1 \wedge \omega^2$$

$$\times \bmod \mathscr{I} \wedge \mathscr{I} + \operatorname{image}(\bar{d}).$$

This gives

$$\nabla A = D_2 A + D_{11} A + \lambda D_1 A + \mu A, \tag{7}$$

where λ, μ depend on lower-order invariants of the parabolic system. Finally, we have

$$E_1^{1,1} \cong \{\overline{\Phi} = A(\omega^1 \wedge \theta_0 + \omega^2 \wedge \theta_1) - (D_1 A + \beta A)\omega^2 \wedge \theta_0 | \nabla A = 0\}. \tag{8}$$

The proposition will thus follow from (iii).

For this we use (6) in the case $p = 2$ together with our knowledge of $\Lambda^2_\Delta \mathbb{F}[x]$. Thus, referring to Proposition 2 in §3.2 of Part I, we have that

$$H^1(\Lambda'') \cong \{\theta_i \wedge \theta_0 \wedge \omega^1 | i \text{ odd}\}$$

$$H^2(\Lambda') \cong \{\theta_i \wedge \theta_0 \wedge \omega^1 \wedge \omega^2 | i \text{ odd}\}.$$

Now comes the interesting point where signs and weights are important:

$$\begin{aligned}\Delta(\theta_i \wedge \theta_0 \wedge \omega^1) &= \text{class of } \delta(\theta_i \wedge \theta_0 \wedge \omega^1)\\ &= \text{class of } (-\theta_{i+2} \wedge \theta_0 - \theta_i \wedge \theta_2) \wedge \omega^1 \wedge \omega^2\\ &= \text{class of } (-\theta_{i+2} \wedge \theta_0 + \theta_{i+1} \wedge \theta_1) \wedge \omega^1 \wedge \omega^2\end{aligned}$$

(since $\delta(\theta_i \wedge \theta_1 \wedge \omega^2) = (\theta_{i+1} \wedge \theta_1 + \theta_i \wedge \theta_2) \wedge \omega^1 \wedge \omega^2$)

$$= \text{class of } (-\theta_{i+2} \wedge \theta_0 - \theta_{i+2} \wedge \theta_0) \wedge \omega^1 \wedge \omega^2$$

(similar calculation). Thus, it follows that

$$\Delta(\theta_i \wedge \theta_0 \wedge \omega^1) = -2(\theta_{i+2} \wedge \theta_0 \wedge \omega^1 \wedge \omega^2).$$

Thus Δ is injective and this implies (iii). □

At this stage we know that

$$\mathscr{C} \cong \{\text{functions } A \text{ satisfying } \nabla A = 0 \text{ in (7)}\}.$$

That is, any conservation law is uniquely represented by a form

$$\Phi = A(\omega^1 \wedge \theta_0 + \omega^2 \wedge \theta_1) - (D_1 A + \beta A)\omega^2 \wedge \theta_0 + \sum_{i<j} B^{ij}\theta_i \wedge \theta_j \tag{9}$$

satisfying

$$d\Phi = 0. \tag{10}$$

The condition that Φ be closed implies that $\nabla A = 0$; conversely, if $\nabla A = 0$ then by the general theory we may determine the B^{ij} in (9) such that (10) is satisfied. In fact, for $\overline{\Phi} \in E_1^{1,1}$ given by (8) we have $d_1 \overline{\Phi} = 0$ since $E_1^{2,1} = 0$. But, by definition, $d_1 \overline{\Phi} =$ class of $d\overline{\Phi}$ and, since $\bar{d}\overline{\Phi} = 0$, it follows that $d\overline{\Phi} \in F^2\Omega^2$ is quadratic in the θ_i. Since the cohomology class of $d\overline{\Phi}$ is zero we must have

$$d\overline{\Phi} = -\bar{d}(\Sigma B^{ij}\theta_i \wedge \theta_j)$$

for some functions B^{ij}. Then

$$d(\overline{\Phi} + \Sigma B^{ij}\theta_i \wedge \theta_j) \in F^3\Omega^*$$

is cubic in the θ_i and is closed. Since $\mathscr{H}^{3,0} = E^{3,0} = 0$, it follows that $d(\overline{\Phi} + \Sigma B^{ij}\theta_i \wedge \theta_j) = 0$.

In fact, more is true. Recall that a function on the infinite prolongation $M^{(\infty)}$ is, by definition, a function on some finite prolongation $M^{(k)}$. The vanishing of $E^{2,1}$ and $E^{3,0}$ can be exploited to prove that, for A satisfying (7) and well defined on $M^{(k)}$, the B^{ij} are given by some universal linear differential operators

$$B^{ij} = L_k^{ij}(A, \partial A, \ldots, \partial^{\ell} A).$$

One question naturally arises: *For a function A satisfying (7), what order of jet does A depend on?* That is, what is the smallest k such that A is defined on $M^{(k)}$? Similarly, *is there a fixed k_0 such that any function A satisfying (7) is already defined on $M^{(k_0)}$? Or do we keep adding new conservation laws as k increases?* The latter is, of course, the case for famous completely integrable equations like KDV. Thus, it is reasonable to ask if this phenomenon can occur for our parabolic equations.

To discuss this question, we set

$$dA = \sum_{i=1,2} A_i \omega^i + \sum_{j \geqq 0} A^j \theta_j$$

and define

$$\mathscr{C}_k = \{A \mid 0 = \nabla A = A^{k+1} = A^{k+2} = \cdots\}.$$

Then $\mathscr{C}_k \subset \mathscr{C}_{k+1}$ and the least integer k such that $A \in \mathscr{C}_k$ reflects the order of jet that A depends on (roughly speaking, that order is $[k/2] + 1$). If

$$\Phi = A(\omega^1 \wedge \theta_0 + \omega^2 \wedge \theta_1) - (D_1 A + \beta A)\omega^2 \wedge \theta_0 + \sum_{\substack{i+j \leqq s \\ i<j}} B^{ij}\theta_i \wedge \theta_j$$

satisfies $d\Phi = 0$, then it is clear that A lies in $\mathscr{C}_k$ for some $k \leqq s - 1$. Thus, the following result gives a fixed bound on k for all nontrivial conservation laws.

PROPOSITION 2. *Any Φ of the form (9) satisfying (10) is of the form*

$$(11)\quad \Phi = A(\omega^1 \wedge \theta_0 + \omega^2 \wedge \theta_1) - (D_1 A + \beta A)\omega^2 \wedge \theta_0 + C^1\theta_0 \wedge \theta_1 + C^2\theta_0 \wedge \theta_2 + C^3(\theta_0 \wedge \theta_3 + \theta_1 \wedge \theta_2).$$

In particular, $\mathscr{C}_2 = \mathscr{C}$.

Therefore, local conservation laws for our second-order parabolic systems can depend at most on u, ∂u, $\partial^2 u$, where u is the unknown function. In particular, *the "KDV phenomenon" cannot occur for such systems.*

Proof. For Φ of the form (9) we let s be the largest integer so that there exists a nonzero B^{ij} with $i + j = s$. Thus s is the highest weight appearing in Φ. For each $k \geqq 0$ let Γ_k denote the *algebraic* ideal generated by θ_0, θ_1 and all of the quadratic terms $\theta_i \wedge \theta_j$ where $i + j \leqq k$. Expanding $d\Phi = 0$ and reducing modulo Γ_s yields the relations

$$\sum_{\substack{i+j=s-1\\ i<j}} B^{ij}[(-\theta_{i+2} \wedge \omega^2) \wedge \theta_j - \theta_i \wedge (-\theta_{j+2} \wedge \omega^2)] \equiv 0$$

$$\sum_{\substack{i+j=s\\ i<j}} B^{ij}[(-\theta_{i+1} \wedge \omega^1 - \theta_{i+2} \wedge \omega^2) \wedge \theta_j - \theta_i \wedge (-\theta_{j+1} \wedge \omega^1 - \theta_{j+2} \wedge \omega^2)] \equiv 0,$$

where the congruences are modulo θ_0 and θ_1. The second of these uncouples into the relations

$$\sum_{\substack{i+j=s\\ i<j}} B^{ij}(\theta_{i+1} \wedge \theta_j + \theta_i \wedge \theta_{j+1}) \equiv \sum_{\substack{i+j=s\\ i<j}} B^{ij}(\theta_{i+2} \wedge \theta_j + \theta_i \wedge \theta_{j+2}) \equiv 0,$$

where, again, the congruences are taken θ_0 and θ_1. If $s > 3$, it is easy to see that these relations imply that $B^{ij} = 0$ for all $i + j = s$. Moreover, if $s = 3$ then these relations imply that $B^{12} = B^{03}$. This implies our result. □

We shall now give an alternative, more direct argument for the local normal form of conservation laws for parabolic systems. It is based on the identification

$$\mathscr{C} \cong \text{closed forms in } E_1^{1,1} = \frac{\{\Phi \in F^1\Omega^2 | d\Phi = 0\}}{d(F^1\Omega^1)}.$$

The result is the following.

PROPOSITION 2′. *Any closed 2-form Φ in $F^1\Omega^2$ can be written uniquely in the form*

$$A(\omega^1 \wedge \theta_0 + \omega^2 \wedge \theta_1) + B\omega^2 \wedge \theta_0 + C^1\theta_0 \wedge \theta_1 + C^2\theta_0 \wedge \theta_2 + C^3(\theta_0 \wedge \theta_3 + \theta_1 \wedge \theta_2) + d(\lambda^0\theta_0 + \cdots + \lambda^p\theta_p).$$

In particular, the space of conservation laws for $\mathcal{I}$ is isomorphic to the vector space consisting of the closed 2-forms Φ of the form

$$A(\omega^1 \wedge \theta_0 + \omega^2 \wedge \theta_1) + B\omega^2 \wedge \theta_0 + C^1\theta_0 \wedge \theta_1 + C^2\theta_0 \wedge \theta_2 + C^3(\theta_0 \wedge \theta_3 + \theta_1 \wedge \theta_2).$$

Proof. Suppose that Φ is a closed 2-form in $F^1\Omega^2$.

First, we will prove the uniqueness of the claimed representation for Φ. Suppose that Φ can be represented as the proposition claims, and that we also have a representation of the form

$$\Phi = \tilde{A}(\omega^1 \wedge \theta_0 + \omega^2 \wedge \theta_1) + \tilde{B}\omega^2 \wedge \theta_0 + \sum_{0 \leqslant i < j \leqslant \tilde{q}} \tilde{c}^{ij}\theta_i \wedge \theta_j + d(\tilde{\lambda}^0\theta_0 + \cdots + \tilde{\lambda}^{\tilde{p}}\theta_{\tilde{p}}).$$

Clearly, we may suppose that $p = \tilde{p}$ and $q = \tilde{q}$. (Simply replace p and $\tilde{p}$ by their maximum, etc., and set the new coefficients in the appropriate representation equal to 0.) Taking the difference of the two representations and setting $A - \tilde{A} = \hat{A}$, etc., yields

$$0 = \hat{A}(\omega^1 \wedge \theta_0 + \omega^2 \wedge \theta_1) + \hat{B}\omega^2 \wedge \theta_0 + \sum_{0 \leqslant i < j \leqslant q} \hat{c}^{ij}\theta_i \wedge \theta_j + d(\hat{\lambda}^0\theta_0 + \cdots + \hat{\lambda}^{p}\theta_{p}).$$

Expanding out the exterior derivative of the right-hand side of this relation and then reducing modulo Θ_{p+1} and terms quadratic in the θ_i, we see that $\hat{\lambda}^p = 0$. Since $\hat{\lambda}^p$ could have been supposed to be the last nonzero $\hat{\lambda}$, it follows that all of the $\hat{\lambda}^i$ must vanish. It now follows that $\hat{A} = \hat{B} = \hat{c}^{ij} = 0$ as well. Thus, the representation is unique, as claimed.

Second, we prove existence of such a representation for Φ. Now, since Φ is in the ideal generated by Θ_∞, there exist integers p and q sufficiently large together with functions a^i, b^i, and c^{ij}, so that

$$\Phi = \sum_{j=0}^{p-1} a^j\omega^1 \wedge \theta_j + \sum_{j=0}^{p} b^j\omega^2 \wedge \theta_j + \sum_{0 \leqslant i < j \leqslant q} c^{ij}\theta_i \wedge \theta_j.$$

Taking the exterior derivative of both sides and reducing modulo Θ_p and the "quadratic ideal" $(\Theta_\infty)^2$ yields the congruence

$$0 \equiv (a^{p-1}\omega^1 + b^{p-1}\omega^2) \wedge (\theta_{p+1} \wedge \omega^2) + (b^p\omega^2) \wedge (\theta_{p+1} \wedge \omega^1 + \theta_{p+2} \wedge \omega^2).$$

It follows that $a^{p-1} = b^p$.

If $p < 2$, a slight relabeling yields

$$\Phi = A(\omega^1 \wedge \theta_0 + \omega^2 \wedge \theta_1) + B\omega^2 \wedge \theta_0 + \sum_{0 \leqslant i < j \leqslant q} c^{ij}\theta_i \wedge \theta_j.$$

On the other hand, if $p \geqslant 2$, then the relation $a^{p-1} = b^p$ implies that Φ can be rewritten in the form

$$\Phi = \sum_{j=0}^{p-2} a^j\omega^1 \wedge \theta_j + \sum_{j=0}^{p-1} b^j\omega^2 \wedge \theta_j + \sum_{0 \leqslant i < j \leqslant q} c^{ij}\theta_i \wedge \theta_j$$
$$+ a^{p-1}(\omega^1 \wedge \theta_{p-1} + \omega^2 \wedge \theta_p).$$

Since $d\theta_{p-2} \equiv \omega^1 \wedge \theta_{p-1} + \omega^2 \wedge \theta_p \bmod \Theta_{p-2}$, it follows that, by modifying a^i and b^i for $0 \leqslant i \leqslant p-2$ and the appropriate c^{ij} as well as possibly raising q, we may express Φ in the form

$$\Phi = \sum_{j=0}^{p-2} a^j\omega^1 \wedge \theta_j + \sum_{j=0}^{p-1} b^j\omega^2 \wedge \theta_j + \sum_{0 \leqslant i < j \leqslant q} c^{ij}\theta_i \wedge \theta_j + d(a^{p-1}\theta_{p-2}).$$

Now, if $p > 2$, this argument can be repeated on $\Phi - d(a^{p-1}\theta_{p-2})$. In fact, it is now clear that, by repeating this construction at most $p-2$ times, we can write Φ in the form

$$\Phi = a^0\omega^1 \wedge \theta_0 + b^0\omega^2 \wedge \theta_0 + b^1\omega^2 \wedge \theta_1 + \sum_{0 \leqslant i < j \leqslant q} c^{ij}\theta_i \wedge \theta_j$$
$$+ d(a^1\theta_0 + \cdots + a^{p-1}\theta_{p-2}).$$

Differentiating once more and reducing modulo θ_0, θ_1, and the quadratic ideal as before yields that $a^0 = b^1$. Thus, with a slight relabeling, we have

$$\Phi = A(\omega^1 \wedge \theta_0 + \omega^2 \wedge \theta_1) + B\omega^2 \wedge \theta_0 + \sum_{0 \leqslant i < j \leqslant q} c^{ij}\theta_i \wedge \theta_j$$
$$+ d(a^1\theta_0 + \cdots + a^{p-1}\theta_{p-2}).$$

At this point the proof proceeds as before. We will derive further limitations on the c^{ij}, and for this purpose, we may drop the exact differential term and suppose that our closed 2-form Φ is of the form

$$\Phi = A(\omega^1 \wedge \theta_0 + \omega^2 \wedge \theta_1) + B\omega^2 \wedge \theta_0 + \sum_{0 \leqslant i < j \leqslant q} c^{ij}\theta_i \wedge \theta_j.$$

Let $s \geqslant 0$ be the largest integer so that there exists a nonzero c^{ij} so that $i + j = s$. For each $k \geqslant 0$ let Υ_k denote the *algebraic* ideal generated by θ_0, θ_1, and all of the quadratic terms $\theta_i \wedge \theta_j$ where $i + j \leqslant k$. Expanding $d\Phi = 0$ and reducing modulo Υ_s yields the relations

$$\sum_{\substack{i+j=s-1\\ i<j}} c^{ij}[(-\theta_{i+2} \wedge \omega^2) \wedge \theta_j - \theta_i \wedge (-\theta_{j+2} \wedge \omega^2)] \equiv 0$$

$$\sum_{\substack{i+j=s\\ i<j}} c^{ij}[(-\theta_{i+1} \wedge \omega^1 - \theta_{i+2} \wedge \omega^2) \wedge \theta_j - \theta_i \wedge (-\theta_{j+1} \wedge \omega^1 - \theta_{j+2} \wedge \omega^2)] \equiv 0,$$

where the congruences are modulo θ_0 and θ_1. The second of these two relations uncouples into the relations

$$\sum_{\substack{i+j=s\\ i<j}} c^{ij}(\theta_{i+1} \wedge \theta_j + \theta_i \wedge \theta_{j+1}) \equiv \sum_{\substack{i+j=s\\ i<j}} c^{ij}(\theta_{i+2} \wedge \theta_j + \theta_i \wedge \theta_{j+2}) \equiv 0$$

where, again, the congruences are taken modulo θ_0 and θ_1. If $s > 3$, it is easy to see that these relations imply that $c^{ij} = 0$ for all $i + j = s$. Moreover, if $s = 3$, then these relations imply that $c^{12} = c^{03}$. Writing C^i in place of c^{0i}, we now have Φ in the desired form. □

2. The non-Goursat equivalence problem. In order to make further progress in understanding the space of conservation laws for a given parabolic system, we will need to develop an understanding of the invariants of parabolic systems in general. For this purpose, we will use É. Cartan's method to study the equivalence problem for parabolic systems. (For the convenience of the reader, a summary of the "recipe" for applying this method is given in Appendix 1 to this section.)

Implemented blindly, the equivalence method frequently leads to unmanageable calculations or unintelligible results. However, we will see that, when motivated by the geometric problem of understanding conservation laws, we are led to study special cases and make normalizations in such a way that the equivalence method works very nicely. Before beginning the detailed calculations, we will now explain in outline how this will go.

Consider a pair of 1-adapted coframings

$$\Phi = (\theta_0, \theta_1, \theta_2, \omega^1, \omega^2, \pi_3, \pi_4)$$

$$\tilde{\Phi} = (\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\theta}_2, \tilde{\omega}^1, \tilde{\omega}^2, \tilde{\pi}_3, \tilde{\pi}_4)$$

on some domain U in M which satisfy the transition relations given at the beginning of §1. Extend each as described in §1 to 1-adapted coframings on the infinite prolongation $U^{(\infty)} \subset M^{(\infty)}$ so that they satisfy the crude structure equations (2) of §1. As we saw in that section, any conservation law has a unique representing 2-form Υ on $U^{(\infty)}$ of the form (9). Thus, there are coefficient functions A, $\tilde{A}$, B, $\tilde{B}$, etc., so that

$$\Upsilon = A(\theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2) + B\theta_0 \wedge \omega^2 + C^1\theta_0 \wedge \theta_1 + C^2\theta_0 \wedge \theta_2 + C^3(\theta_0 \wedge \theta_3 + \theta_1 \wedge \theta_2)$$
$$= \tilde{A}(\tilde{\theta}_0 \wedge \tilde{\omega}^1 + \tilde{\theta}_1 \wedge \tilde{\omega}^2) + \tilde{B}\tilde{\theta}_0 \wedge \tilde{\omega}^2 + \tilde{C}^1\tilde{\theta}_0 \wedge \tilde{\theta}_1 + \tilde{C}^2\tilde{\theta}_0 \wedge \tilde{\theta}_2 + \tilde{C}^3(\tilde{\theta}_0 \wedge \tilde{\theta}_3 + \tilde{\theta}_1 \wedge \tilde{\theta}_2). \tag{1}$$

By the transition formulas given in §1, we know that $\tilde{\theta}_0 = a\theta_0$ and $\tilde{\theta}_3 \equiv (a/r^3)\theta_3 \bmod \Theta_2$. It follows easily that

$$C^3 = (a^2/r^3)\tilde{C}^3. \tag{2}$$

In particular, whether or not $C^3 = 0$ for a given conservation law Υ is independent of the choice of 1-adapted coframing in which Υ is expanded.

One of the first results we will get from the calculations below is that $C^3 = 0$ for all conservation laws of a parabolic system.

Second, the equivalence method leads us to introduce an invariant of a non-Goursat parabolic system (called the *Monge-Ampere invariant*) whose vanishing is necessary in order that the system have nontrivial conservation laws. (See Appendix 2 of this section for a review of the notion of a Monge-Ampere system.) Thus, we will pursue the equivalence problem calculations only in the case of Monge-Ampere systems.

Third, we show that a Monge-Ampere admits a more restricted class of coframings, which we call *2-adapted*, characterized by the vanishing of certain expressions computable for any 1-adapted coframing. These 2-adapted coframings are the sections of a certain principal G_2-subbundle of the bundle of 1-adapted coframes. We then show that, among these 2-adapted coframings there is a certain subclass, the *3-adapted* coframings, characterized by the vanishing of certain expressions computable for any 2-adapted coframing. These 3-adapted coframings are the sections of a certain principal G_3-subbundle of the bundle of 2-adapted coframes.

Finally, we show that, in any 3-adapted coframing, the identities

$$C^1 = C^2 = 0 \tag{3}$$

hold for any closed 2-form Υ of the form (1).

In conclusion, the assumption that $\mathscr{C} \neq \varnothing$ will imply that a set of invariants of the system must vanish and, for such parabolic systems, we may reduce the structure group to a set of 3-adapted coframes relative to which any conservation law is of the form

$$(4)\qquad \begin{cases} \Phi = A(\omega^1 \wedge \theta_0 + \omega^2 \wedge \theta_1) + B\omega^2 \wedge \theta_0 \\ B = -(D_1 + \beta)A. \end{cases}$$

Under changes of coframing we will have

$$\begin{cases} A = ar\tilde{A} \\ B = ar^2\tilde{B} - \dfrac{ab}{r}A \end{cases}$$

where a, b are parameters in the group G_3 given below. Using (4), in the next section we will be able to determine a normal form for all parabolic systems for which $\mathscr{C} \neq \varnothing$.

The Monge-Ampere invariant. We will now carry out this process explicitly. Let $\mathscr{F} \to M$ denote the bundle of 1-adapted coframes over M. As explained before, $\mathscr{F}$ is a principal right G_1 bundle over M. The structure equations of $\mathscr{F}$ can be written in the form

$$(5)\qquad \begin{bmatrix} d\theta_0 \\ d\theta_1 \\ d\theta_2 \\ d\omega^2 \\ d\omega^1 \\ d\pi_3 \\ d\pi_4 \end{bmatrix} = - \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_1 & \alpha - \rho & 0 & 0 & 0 & 0 & 0 \\ \nu_1 & \beta_2 & \alpha - 2\rho & 0 & 0 & 0 & 0 \\ \nu_2 & \beta_3 & 0 & 2\rho & 0 & 0 & 0 \\ \nu_3 & \nu_4 & \beta_3 & -\beta_2 & \rho & 0 & 0 \\ \nu_5 & \nu_6 & \beta_1 + \beta_2 & \theta_5 & 0 & \alpha - 3\rho & 0 \\ \nu_7 & \nu_8 & \nu_9 & \theta_6 & \theta_5 & 2\beta_2 & \alpha - 4\rho \end{bmatrix}$$

$$\wedge \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \omega^2 \\ \omega^1 \\ \pi_3 \\ \pi_4 \end{bmatrix} + \begin{bmatrix} \Theta_0 \\ \Theta_1 \\ \Theta_2 \\ \Omega^2 \\ \Omega^1 \\ \Pi_3 \\ \Pi_4 \end{bmatrix}.$$

The square matrix of 1-forms in (5) (whose components are henceforth to be referred to as the *pseudoconnection* forms) assumes values in the Lie algebra of G_1. However, this matrix is not uniquely defined by this condition.

Since

$$
\begin{aligned}
& d\theta_0 \equiv -\theta_1 \wedge \omega^1 - \theta_2 \wedge \omega^2 \bmod \theta_0 \\
(6) \qquad & d\theta_1 \equiv -\theta_2 \wedge \omega^1 - \pi_3 \wedge \omega^2 \bmod \theta_0, \theta_1 \\
& d\theta_2 \equiv -\pi_3 \wedge \omega^1 - \pi_4 \wedge \omega^2 \bmod \theta_0, \theta_1, \theta_2
\end{aligned}
$$

it follows that we may choose the forms α, ρ, and β_1 so that

$$
\begin{aligned}
\Theta_0 &= -\theta_1 \wedge \omega^1 - \theta_2 \wedge \omega^2 \\
\Theta_1 &= -\theta_2 \wedge \omega^1 - \pi_3 \wedge \omega^2 .
\end{aligned}
$$

This uniquely determines α modulo θ_0 and ρ and β_1 modulo $\{\theta_0, \theta_1\}$. It then follows that we may choose ν_1 and β_2 so that

$$\Theta_2 = -\pi_3 \wedge \omega^1 - \pi_4 \wedge \omega^2 - \varepsilon \wedge \theta_2$$

for some 1-form ε which is a linear combination of the basic forms. In fact, by modifying ν_1 and β_2 appropriately, we may assume that ε is a linear combination of ω^1, ω^2, π_3, and π_4, so we do. Now, taking the exterior derivative of $d\theta_0$ and reducing modulo θ_0 gives the relation

$$\theta_1 \wedge \Omega^1 + \theta_2 \wedge (\Omega^2 - \varepsilon \wedge \omega^2) \equiv 0 \bmod \theta_0 .$$

In particular, it follows that $\Omega^2 \equiv \varepsilon \wedge \omega^2 \bmod \theta_0, \theta_1, \theta_2$, so by modifying ν_2 and β_3, it follows that we may arrange that

$$\Omega^2 = \varepsilon \wedge \omega^2 + \varphi \wedge \theta_2$$

for some 1-form φ. By suitably modifying ν_2 and β_3, we may assume that φ is a linear combination of ω^1, ω^2, π_3, and π_4. This then implies that $\theta_1 \wedge \Omega^1 \equiv 0 \bmod \theta_0$, so Ω^1 is a linear combination of θ_0 and θ_1. Now, by modifying ν_3 and ν_4 suitably, we may arrange that $\Omega^1 = 0$, which we do.

Differentiating $d\theta_1$ and reducing modulo θ_0 and θ_1 yields the identity

$$(\varphi \wedge \pi_3 + \varepsilon \wedge \omega^1) \wedge \theta_2 + (\Pi_3 + \pi_4 \wedge \omega^1 + \varepsilon \wedge \pi_3) \wedge \omega^2 \equiv 0 \bmod \theta_0, \theta_1 .$$

It follows that $\Pi_3 \equiv -\pi_4 \wedge \omega^1 - \varepsilon \wedge \pi_3 \bmod \theta_0, \theta_1, \theta_2, \omega^2$; so modifying ν_5, ν_6, and θ_5 appropriately, we may assume that

$$\Pi_3 = -\pi_4 \wedge \omega^1 - \varepsilon \wedge \pi_3 - \gamma \wedge \theta_2$$

for some 1-form γ which is a linear combination of ω^1, π_3, and π_4. Substituting this back into the above identity, we get

$$(\gamma \wedge \omega^2 + \varepsilon \wedge \omega^1 + \varphi \wedge \pi_3) \wedge \theta_2 \equiv 0 \bmod \theta_0, \theta_1 .$$

Since none of γ, ε, or φ contain any terms involving θ_0, θ_1, or θ_2, it follows that we actually have the relation

$$\gamma \wedge \omega^2 + \varepsilon \wedge \omega^1 + \varphi \wedge \pi_3 = 0.$$

Finally, computing the exterior derivative of $d\theta_2$ and reducing modulo θ_0, θ_1, and θ_2 yields the identity

$$(\Pi_4 + 2\varepsilon \wedge \pi_4) \wedge \omega^2 \equiv 0 \bmod \theta_0, \theta_1, \theta_2 .$$

Hence, by suitably modifying the 1-forms ν_7, ν_8, ν_9, and θ_6, we may assume, as we shall henceforth, that

$$\Pi_4 = -2\varepsilon \wedge \pi_4 .$$

Our work so far has resulted in structure equations of the form

$$\begin{bmatrix} d\theta_0 \\ d\theta_1 \\ d\theta_2 \\ d\omega^2 \\ d\omega^1 \\ d\pi_3 \\ d\pi_4 \end{bmatrix} = - \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_1 & \alpha - \rho & 0 & 0 & 0 & 0 & 0 \\ \nu_1 & \beta_2 & \alpha - 2\rho & 0 & 0 & 0 & 0 \\ \nu_2 & \beta_3 & 0 & 2\rho & 0 & 0 & 0 \\ \nu_3 & \nu_4 & \beta_3 & -\beta_2 & \rho & 0 & 0 \\ \nu_5 & \nu_6 & \beta_1 + \beta_2 & \theta_5 & 0 & \alpha - 3\rho & 0 \\ \nu_7 & \nu_8 & \nu_9 & \theta_6 & \theta_5 & 2\beta_2 & \alpha - 4\rho \end{bmatrix} \wedge \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \omega^2 \\ \omega^1 \\ \pi_3 \\ \pi_4 \end{bmatrix}$$
$$+ \begin{bmatrix} -\theta_1 \wedge \omega^1 - \theta_2 \wedge \omega^2 \\ -\theta_2 \wedge \omega^1 - \pi_3 \wedge \omega^2 \\ -\pi_3 \wedge \omega^1 - \pi_4 \wedge \omega^2 - \varepsilon \wedge \theta_2 \\ \varepsilon \wedge \omega^2 + \varphi \wedge \theta_2 \\ 0 \\ -\pi_4 \wedge \omega^1 - \varepsilon \wedge \pi_3 - \gamma \wedge \theta_2 \\ -2\varepsilon \wedge \pi_4 \end{bmatrix},$$

where

$$\gamma \wedge \omega^2 + \varepsilon \wedge \omega^1 + \varphi \wedge \pi_3 = 0$$

and γ has no ω^2 component. It follows from Cartan's lemma that there are functions $S_0, \ldots, S_4$ on $\mathcal{F}$ so that

$$\begin{bmatrix} \varphi \\ \varepsilon \\ \gamma \end{bmatrix} = \begin{bmatrix} S_0 & S_1 & S_3 \\ S_1 & S_2 & S_4 \\ S_3 & S_4 & 0 \end{bmatrix} \begin{bmatrix} \pi_3 \\ \omega^1 \\ \omega^2 \end{bmatrix} .$$

In order to perform further structure reductions, it is important to understand how the functions S_i vary on the fibers of $\mathcal{F}$ over M. To determine this, we differentiate the structure equations.

First, expanding the identity $d(d\theta_0) = 0$ yields

$$(\nu_3 \wedge \theta_1 + \nu_2 \wedge \theta_2 - \nu_1 \wedge \omega^2 - \beta_1 \wedge \omega^1 - d\alpha) \wedge \theta_0 = 0.$$

Thus, there exists a 1-form π_0 so that

$$d\alpha = -\pi_0 \wedge \theta_0 + \nu_3 \wedge \theta_1 + \nu_2 \wedge \theta_2 - \nu_1 \wedge \omega^2 - \beta_1 \wedge \omega^1 .$$

Next, expanding the identity $d(d\theta_1) \equiv 0 \bmod \theta_0$ yields

$$(d\rho + (\nu_4 - \nu_2) \wedge \theta_2 + (\nu_1 - \nu_6) \wedge \omega^2 + (2\beta_1 - \beta_2) \wedge \omega^1 + \beta_3 \wedge \pi_3) \wedge \theta_1$$
$$\equiv 0 \bmod \theta_0 .$$

Thus, there exist 1-forms π_1 and χ_0 so that

$$d\rho = -\chi_0 \wedge \theta_0 - \pi_1 \wedge \theta_1 - (\nu_4 - \nu_2) \wedge \theta_2 - (\nu_1 - \nu_6) \wedge \omega^2 - (2\beta_1 - \beta_2) \wedge \omega^1$$
$$- \beta_3 \wedge \pi_3 .$$

Now, the identity $d(d\theta_2) \equiv 0 \bmod \theta_0, \theta_1$ takes the form $-E \wedge \theta_2 \equiv 0 \bmod \theta_0, \theta_1$ where

$$E = (S_3 + S_2 S_1)\pi_3 \wedge \omega^1 + 2S_4 S_1 \pi_3 \wedge \omega^2 + S_2 S_4 \omega^1 \wedge \omega^2$$
$$+ (dS_4 - 2S_4\rho - S_3\pi_4 + S_2\beta_2 - S_1\theta_5 - 2\nu_6 + \nu_9) \wedge \omega^2$$
$$+ (dS_2 - S_2\rho - 2S_1\pi_4 + 4\beta_1 - 2\beta_2) \wedge \omega^1 + (dS_1 + S_1(3\rho - \alpha) - S_0\pi_4 + \beta_3)$$
$$\wedge \pi_3 .$$

Finally, the identity $d(d\omega^2) \equiv 0 \bmod \theta_0, \theta_1$ takes the form $E \wedge \omega^2 + \Upsilon \wedge \theta_2 \equiv$

0 mod θ_0, θ_1, where

$$\Upsilon = -2S_0S_4\omega^2 \wedge \pi_3 + (3S_2S_0 - 2S_1^2)\pi_3 \wedge \omega^1 + (S_3S_2 - 2S_4S_1)\omega^2 \wedge \omega^1$$
$$+ (dS_3 + S_3(2\rho - \alpha) + S_2\beta_3 + S_1(\beta_1 + 2\beta_2) - S_0\theta_5 - 2\nu_4 + 3\nu_2) \wedge \omega^2$$
$$+ (dS_1 + S_1(3\rho - \alpha) - S_0\pi_4 + \beta_3) \wedge \omega^1 + (dS_0 - S_0(2\alpha - 7\rho)) \wedge \pi_3 .$$

These relations imply the formulas

$$\begin{aligned} dS_0 &\equiv S_0(2\alpha - 7\rho) \\ dS_1 &\equiv S_1(\alpha - 3\rho) - \beta_3 \\ dS_2 &\equiv S_2(\rho) - 4\beta_1 + 2\beta_2 \\ dS_3 &\equiv S_3(\alpha - 2\rho) - S_2\beta_3 - S_1(\beta_1 + 2\beta_2) + S_0\theta_5 + 2\nu_4 - 3\nu_2 \\ dS_4 &\equiv S_4(2\rho) - S_2\beta_2 + S_1\theta_5 - \nu_9 + 2\nu_6 , \end{aligned} \tag{7}$$

where the congruences are taken modulo the span of the semibasic 1-forms θ_0, θ_1, θ_2, π_3, π_4, ω^1, and ω^2. It follows that the 2-form $\Psi = S_0\pi_3 \wedge \pi_4$ modulo the forms $\{\theta_0, \theta_1, \theta_2, \omega^1, \omega^2\}$ is well defined on M. Note that Ψ therefore restricts to each (2-dimensional) leaf of the system $\{\theta_0, \theta_1, \theta_2, \omega^1, \omega^2\}$ to be a well-defined area form. We call this form the *Monge-Ampere invariant* because of the following result.

PROPOSITION 1. *The invariant Ψ vanishes identically if and only if the parabolic system is locally equivalent to an equation of Monge-Ampere type.*

Proof. First, suppose that Ψ vanishes identically, i.e., that $S_0 \equiv 0$. Then the structure equations imply

$$\left.\begin{aligned} d\theta_0 &\equiv 0 \\ d\theta_1 &\equiv \omega^1 \wedge \theta_2 \\ d\omega^2 &\equiv S_1\omega^1 \wedge \theta_2 \end{aligned}\right\} \bmod \theta_0, \theta_1, \omega^2 .$$

Thus, the Cartan system $\mathscr{L}$ of the system $\mathscr{K} = \{\theta_0, \theta_1, \omega^2\}$ is the same as the Cartan system of the single 1-form θ_0, namely $\mathscr{L} = \{\theta_0, \theta_1, \theta_2, \omega^2, \omega^1\}$. It follows that M can be covered by open sets U on which there can be defined a submersion $f\colon U \to \mathbb{R}^5$ whose fibers are the leaves of the system $\mathscr{L}$ restricted to U. Moreover, there is a differential ideal $\mathscr{I}$ defined on $\mathbb{R}^5$ whose pull-back under f is generated by the forms $\{\theta_0, d\theta_0, \theta_1 \wedge \omega^2\}$. From the discussion in Appendix 2 to this section, we see that this ideal is a parabolic system of Monge-Ampere type. Let $\mathscr{V}_2(\mathscr{I})$ denote the space of 2-dimensional integral elements of $\mathscr{I}$. There is a natural map $F: U \to \mathscr{V}_2(\mathscr{I})$ defined by letting $F(p)$ denote the image 2-plane $f'(E_p) \subset T_{f(p)}\mathbb{R}^5$ where $E_p \subset T_pU$ is the 4-plane on which θ_0, θ_1, and θ_2 vanish. Using the structure equations, it is easy to see that F is a local diffeomorphism and that F pulls the canonical Pfaffian system on $\mathscr{V}_2(\mathscr{I})$ back to be the system on U generated by θ_0, θ_1, and θ_2.

The converse, that any parabolic Monge-Ampere system has its invariant Ψ vanish identically, can be left to the reader. □

Note. If one is interested in isolating the Monge-Ampere invariant without going through the equivalence method, this may be done as follows. Taking the exterior derivative of the middle equation in (6) gives $0 \equiv \pi_3 \wedge d\omega^2 \bmod \theta_0, \theta_1, \theta_2, \omega^1, \omega^2$, which implies that $d\omega^2 \equiv S_0\theta_2 \wedge \pi_3 \bmod \theta_0, \theta_1, \omega^1, \omega^2$. Under a change of 1-adapted coframing of the above form, we have

$$d\tilde{\omega}^2 \equiv r^2\, d\omega^2 \bmod \theta_0, \theta_1, \omega^1, \omega^2,$$

the point here being that, when we change coframings, no multiple of ω^1 appears in $\tilde{\omega}^2$. It follows again from the equations of coframe rotation in G_1 that

$$S_0 = \frac{a^2}{r^7}\tilde{S}_0,$$

so that

$$\Psi \equiv S_0 \pi_3 \wedge \pi_4 \bmod\{\theta_0, \theta_1, \theta_2, \omega^1, \omega^2\}$$

is well defined. (The reason one might have expected invariants to turn up in $d\omega^2$ is that, on integral manifolds of $\mathscr{I}$, the equation $\omega^2 = 0$ defines the characteristic foliation, which has an invariant meaning.)

It is easy to show that, for a parabolic equation of the form $r = E(x, y, u, p, q, s, t)$ (in the classical notation), the quantity S_0 is a nonzero multiple of E_{sss}, so that this equation is Monge-Ampere if and only if it is at most quadratic in s.

With the finer structure equations we now have at our disposal, we are now ready for the first refinement of Proposition 1. The following calculation takes place up on the infinite prolongation, which we recall means that we are up on the kth prolongation $M^{(k)}$ for some k. The 1-forms θ_0, θ_1, θ_2, θ_3, θ_4, ω^1, ω^2 are defined on $M^{(1)}$ and are semibasic relative to the projection $M^{(k)} \to M^{(1)}$; the coefficients A, B, C^1, C^2, C^3 are functions on $M^{(k)}$.

PROPOSITION 2. *Relative to the prolongation of any 1-adapted local coframing, any closed 2-form Φ of the form*

$$\Phi = A(\omega^1 \wedge \theta_0 + \omega^2 \wedge \theta_1) + B\omega^2 \wedge \theta_0 + C^1\theta_0 \wedge \theta_1 + C^2\theta_0 \wedge \theta_2 + C^3(\theta_0 \wedge \theta_3 + \theta_1 \wedge \theta_2)$$

must satisfy $C^3 = 0$. *Moreover, on the open set in* M *where the Monge-Ampere invariant is nonzero, the only such closed 2-form is* $\Phi = 0$. *In particular, if a parabolic equation is "everywhere non-Monge-Ampere", then its space* $\mathscr{C}$ *of conservation laws is trivial.*

Proof. Suppose, as usual, that $(\theta_0, \theta_1, \theta_2, \omega^1, \omega^2, \pi_3, \pi_4)$ is a 1-adapted coframing on an open set U in M. Then by our equivalence problem calculations, there are pseudoconnection forms on U so that the structure equations hold.

We then define θ_3, θ_4 on $M^{(1)}$ as in Section 1; explicitly, $\theta_3 = \pi_3 - p_5\omega^2$ and $\theta_4 = \pi_4 - p_5\omega^1 - p_6\omega^2$. Assume that Φ is closed and of the form given in the proposition. Then a short calculation shows that $d\Phi \equiv \Psi \wedge \theta_1 \bmod \theta_0$ where

$$\begin{aligned}\Psi = (dA - A(\alpha + \rho) - C^1\theta_2 + A\omega^1 + C^3\theta_4 + B\varepsilon) \wedge \omega^2 \\ - (dC^3 - C^3(2\alpha - 3\rho) + C^2\omega^1 - A\phi - C^3\varepsilon) \wedge \theta_2 + 2C^3\theta_3 \wedge \omega^1.\end{aligned}$$

Since Φ is closed, we have $\Psi \equiv 0 \bmod \theta_0, \theta_1$. Since $\Psi \equiv 2C^3\theta_3 \wedge \omega^1 \bmod \theta_0, \theta_1, \theta_2, \omega^2$, it follows that $C^3 \equiv 0$. This proves the first part of the proposition.

In order to prove the second part, note that, with $C^3 = 0$, the formula for Ψ simplifies to

$$\Psi = (dA - A(\alpha + \rho) - C^1\theta_2 + B\omega^1 + A\varepsilon) \wedge \omega^2 - (C^2\omega^1 - A\phi) \wedge \theta_2.$$

Of course, this implies that $A\phi - C^2\omega^1 \equiv 0 \bmod \theta_0, \theta_1, \theta_2, \omega^2$. However, using the fact that $\phi = S_0\pi_3 + S_1\omega^1 + S_3\omega^2$, this clearly implies that $AS_0 = 0$ and $C^2 = AS_1$.

Now if $\Phi \neq 0$ then $A \neq 0$ and hence $S_0 = 0$ as required. □

Second and third reductions. In light of Proposition 2, there is no point in pursuing the study of the non-Monge-Ampere case if one is interested in finding equations with nontrivial conservation laws. Thus, let us assume that $S_0 \equiv 0$. Examining the relations (7) with this new assumption in mind, we see that there exists a principal subbundle $\mathscr{F}'$ of $\mathscr{F}$ which is defined by the equations $S_1 = S_2 = S_3 = S_4 = 0$. The structure group of $\mathscr{F}'$ is of codimension 4 in the structure group of $\mathscr{F}$. We shall denote this 12-dimensional lower triangular subgroup of GL(7, $\mathbb{R}$) by G_2. (This group bears no relation to the famous simple group of the same notation.) The group G_2 consists of the matrices of the form

$$\begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 \\ ab & a/r & 0 & 0 & 0 & 0 & 0 \\ * & 2ab/r & a/r^2 & 0 & 0 & 0 & 0 \\ 2ac & 0 & 0 & r^2 & 0 & 0 & 0 \\ * & 3ac/r & 0 & -2br^2 & r & 0 & 0 \\ * & ad/r & 3ab/r^2 & er^2 & 0 & a/r^3 & 0 \\ * & * & 2ad/r^2 & * & er & 4ab/r^3 & a/r^4 \end{pmatrix},$$

where a and r are nonzero; b, c, d, and e are arbitrary; and the entries marked by an asterisk are also arbitrary.

Restricting the structure equations found so far from $\mathscr{F}$ to $\mathscr{F}'$, we have $0 = d(d\theta_2) \equiv -E \wedge \theta_2 \bmod \theta_0, \theta_1$, where E simplifies to

$$E = (\nu_9 - 2\nu_6) \wedge \omega^2 + (4\beta_1 - 2\beta_2) \wedge \omega^1 + (\beta_3) \wedge \pi_3 .$$

Thus, we must have $E \equiv 0 \bmod \theta_0, \theta_1, \theta_2$. Also, $0 = d(d\omega^2) \equiv E \wedge \omega^2 + \Upsilon \wedge \theta_2 \bmod \theta_0, \theta_1$ where now Υ simplifies to

$$\Upsilon = (3\nu_2 - 2\nu_4) \wedge \omega^2 + (\beta_3) \wedge \omega^1 .$$

Using these relations and taking advantage of the remaining ambiguity of the pseudoconnection forms in the structure equations, a little work shows that it is possible to modify these forms in such a way as to have

$$\begin{aligned} \beta_2 &= 2\beta_1 + R_0\theta_2 + R_1\omega^1 + R_2\omega^2 + R_3\pi_3 \\ \nu_9 &= 2\nu_6 - 2R_2\omega^1 \\ \beta_3 &= 2T\theta_2 - 2R_3\omega^1 \\ \nu_4 &= \frac{3}{2}\nu_2 + U\theta_2 + T\pi_3 - R_0\omega^1 \end{aligned}$$

for some (unique) functions R_0, R_1, R_2, R_3, T, and U on $\mathscr{F}'$. Setting $\beta_1 = \beta$, $\nu_2 = 2\gamma$, and $\nu_6 = \delta$, the structure equations take the form

$$\begin{bmatrix} d\theta_0 \\ d\theta_1 \\ d\theta_2 \\ d\omega^2 \\ d\omega^1 \\ d\pi_3 \\ d\pi_4 \end{bmatrix} = - \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta & \alpha - \rho & 0 & 0 & 0 & 0 & 0 \\ \nu_1 & 2\beta & \alpha - 2\rho & 0 & 0 & 0 & 0 \\ 2\gamma & 0 & 0 & 2\rho & 0 & 0 & 0 \\ \nu_3 & 3\gamma & 0 & -2\beta & \rho & 0 & 0 \\ \nu_5 & \delta & 3\beta & \theta_5 & 0 & \alpha - 3\rho & 0 \\ \nu_7 & \nu_8 & 2\delta & \theta_6 & \theta_5 & 4\beta & \alpha - 4\rho \end{bmatrix} \wedge \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \omega^2 \\ \omega^1 \\ \pi_3 \\ \pi_4 \end{bmatrix}$$

$$+ \begin{bmatrix} -\theta_1 \wedge \omega^1 - \theta_2 \wedge \omega^2 \\ -\theta_2 \wedge \omega^1 - \pi_3 \wedge \omega^2 \\ -\pi_3 \wedge \omega^1 - \pi_4 \wedge \omega^2 - R \wedge \theta_1 \\ 2(R_3\omega^1 - T\theta_2) \wedge \theta_1 \\ \Omega^1 \\ -\pi_4 \wedge \omega^1 - R \wedge \theta_2 \\ 2R_2\omega^1 \wedge \theta_2 - 2R \wedge \pi_3 \end{bmatrix},$$

where $\Omega^1 = R \wedge \omega^2 + 2R_3\omega^1 \wedge \theta_2 + (R_0\omega^1 - U\theta_2 - T\pi_3) \wedge \theta_1$ and $R = R_0\theta_2 + R_1\omega^1 + R_2\omega^2 + R_3\pi_3$.

We shall say that a coframing which is a section of $\mathscr{F}'$ is *2-adapted*. By computations similar to those done to determine how the functions S_i varied on $\mathscr{F}$, it is not difficult to show that

$$(8)\quad \left.\begin{aligned} dT &\equiv T(2\alpha - 5\rho) \\ dU &\equiv U(2\alpha - 4\rho) + 4T\beta + 6R_3\gamma \\ dR_0 &\equiv R_0(\alpha - \rho) + 7R_3\beta - \nu_3 \\ dR_1 &\equiv R_1(2\rho) + 3\nu_1 - \delta \\ dR_2 &\equiv R_2(3\rho) - 2R_1\beta + R_3\theta_5 + 2\nu_5 - \nu_8 \\ dR_3 &\equiv R_3(\alpha - 2\rho) - \gamma \end{aligned}\right\} \bmod \theta_0, \theta_1, \theta_2, \pi_3, \pi_4, \omega^1, \omega^2.$$

It follows that T is a relative invariant. It is the fundamental invariant of Monge-Ampere systems.

It also follows that a principal subbundle $\mathscr{F}''$ of codimension 4 in $\mathscr{F}'$ can be defined by the equations $R_0 = R_1 = R_2 = R_3 = 0$ (i.e., as the locus where these functions vanish). Although this is somewhat bad form (since we have not normalized T and U as well), we will refer to $\mathscr{F}''$ as the third-order structure bundle of the parabolic system. A coframing on M which is a section of $\mathscr{F}''$ will be said to be 3-adapted.

The structure group of the bundle $\mathscr{F}''$ will be denoted G_3. By the relations (8), we know the Lie algebra of the group G_3, and, from this, it is not difficult to show that the group G_3 consists of the matrices of the form

$$\begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 \\ ab & a/r & 0 & 0 & 0 & 0 & 0 \\ ac & 2ab/r & a/r^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2br^2 & r & 0 & 0 \\ ae & 3ac/r & 3ab/r^2 & dr^2 & 0 & a/r^3 & 0 \\ gf & 2ae'/r & 6ac/r^2 & fr^2 & dr & 4ab/r^3 & a/r^4 \end{bmatrix},$$

where a and r are nonzero, b, c, d, e, f, and g are arbitrary, and $e' = e + 3bc - 2b^3$. (The method of getting these formulas is to use the above relations to compute the Lie algebra of G_3 and then exponentiate.) In particular, G_3 has dimension 8, as expected.

Now, setting $\delta - 3v_1 = \kappa$, $v_3 = \phi$, $v_8 - 2v_5 = \mu$, and $v_1 = v$, the structure equations become

$$\text{(9)} \quad \begin{bmatrix} d\theta_0 \\ d\theta_1 \\ d\theta_2 \\ d\omega^2 \\ d\omega^1 \\ d\pi_3 \\ d\pi_4 \end{bmatrix} = - \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta & \alpha - \rho & 0 & 0 & 0 & 0 & 0 \\ v & 2\beta & \alpha - 2\rho & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\rho & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\beta & \rho & 0 & 0 \\ v_5 & 3v & 3\beta & \theta_5 & 0 & \alpha - 3\rho & 0 \\ v_7 & 2v_5 & 6v & \theta_6 & \theta_5 & 4\beta & \alpha - 4\rho \end{bmatrix}$$

$$\wedge \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \omega^2 \\ \omega^1 \\ \pi_3 \\ \pi_4 \end{bmatrix} + \begin{bmatrix} -\theta_1 \wedge \omega^1 - \theta_2 \wedge \omega^2 \\ -\theta_2 \wedge \omega^1 - \pi_3 \wedge \omega^2 \\ -\pi_3 \wedge \omega^1 - \pi_4 \wedge \omega^2 \\ -2T\theta_2 \wedge \theta_1 - 2\gamma \wedge \theta_0 \\ -\phi \wedge \theta_0 - (T\pi_3 + U\theta_2 + 3\gamma) \wedge \theta_1 \\ -\pi_4 \wedge \omega^1 - \kappa \wedge \theta_1 \\ -\mu \wedge \theta_1 - 2\kappa \wedge \theta_2 \end{bmatrix}.$$

Now expanding out the identity $d(d\theta_0) = 0$, we see that there must exist a 1-form π_0 so that

$$d\alpha = -\pi_0 \wedge \theta_0 - v \wedge \omega^2 - \beta \wedge \omega^1 + \phi \wedge \theta_1 + 2\gamma \wedge \theta_2.$$

Similarly, expanding out the identity $d(d\theta_1) = 0$ yields that there must exist 1-forms π_1, χ, and ξ, so that

$$d\rho = -\chi \wedge \theta_0 - \pi_1 \wedge \theta_1 + 2v \wedge \omega^2 + \kappa \wedge \omega^2 - \gamma \wedge \theta_2 - T\theta_2 \wedge \pi_3$$

$$d\beta = -\xi \wedge \theta_0 - (\pi_0 - \chi) \wedge \theta_1 + \rho \wedge \beta - v_5 \wedge \omega^2 - v \wedge \omega^1 + \phi \wedge \theta_2 + 2\gamma \wedge \pi_3.$$

Next, expanding out the identity $d(d\theta_2) = 0$ and reducing modulo θ_0 yields the formula

$$[\omega^2 \wedge \mu + \omega^1 \wedge \kappa + \theta_2 \wedge (\phi - 2\pi_1 + 2T\pi_4 + U\pi_3) + \pi_3 \wedge \gamma] \wedge \theta_1 \equiv 0 \bmod \theta_0.$$

Thus, we must have

$$\omega^2 \wedge \mu + \omega^1 \wedge \kappa + \theta_2 \wedge (\phi - 2\pi_1 + 2T\pi_4 + U\pi_3) + \pi_3 \wedge \gamma \equiv 0 \bmod \theta_0, \theta_1.$$

Expanding out the identity $d(d\omega^2) = 0$ and reducing modulo θ_0 yields the formula

$$[(T\pi_4 - \pi_1) \wedge \omega^2 + (\gamma + T\pi_3) \wedge \omega^1 - (dT - T(2\alpha - 5\rho)) \wedge \theta_2] \wedge \theta_1 \equiv 0 \bmod \theta_0,$$

or, equivalently,

$$(T\pi_4 - \pi_1) \wedge \omega^2 + (\gamma + T\pi_3) \wedge \omega^1 - (dT - T(2\alpha - 5\rho)) \wedge \theta_2 \equiv 0 \bmod \theta_0, \theta_1 .$$

In particular, it follows from this that there must exist functions T_0, T_1, T_2, P_2, and P_1 so that

$$dT = T(2\alpha - 5\rho) + T_0\theta_0 + T_1\theta_1 + T_2\theta_2 + P_1\omega^1 + P_2\omega^2 .$$

Substituting this back into the relations we already have yields the following relations among the "torsion forms" γ, ϕ, κ, μ, and π_1:

$$\left.\begin{aligned} \omega^2 \wedge \mu + \omega^1 \wedge \kappa + \theta_2 \wedge (\phi - 2\pi_1 + 2T\pi_4 + U\pi_3) + \pi_3 \wedge \gamma &\equiv 0 \\ (T\pi_4 + P_2\theta_2 - \pi_1) \wedge \omega^2 + (\gamma + T\pi_3 + P_1\theta_2) \wedge \omega^1 &\equiv 0 \end{aligned}\right\} \bmod \theta_0, \theta_1 .$$

Keeping in mind these relations and again taking advantage of the ambiguity in the remaining pseudoconnection forms, it is not hard to show that we can modify the pseudoconnection forms so as to have the following formulas:

$$(10) \qquad \begin{cases} \gamma = G\theta_1 + H\omega^1 - P_1\theta_2 - T\pi_3 \\ \pi_1 = P\omega^2 + P_2\theta_2 + T\pi_4 \\ \phi = F_1\theta_1 + F_2\theta_2 + D\omega^1 - (P_1 + U)\pi_3 \\ \kappa = D\theta_2 + K\omega^1 + H\pi_3 \\ \mu = -2P\theta_2 \end{cases}$$

for some unique functions G, H, P, F_1, F_2, D, and K on $\mathscr{F}''$. These normalizations make α, ρ, and β unique and make ν unique modulo θ_0.

Reduction past the third order becomes complicated unless one makes fairly stringent assumptions about these 14 functions. Fortunately, we will not need to carry the reduction any further in this generality.

Here is a sample of the sort of information we can get from the equivalence problem calculations. Again we are back up on the prolongation $M^{(k)}$ for some k.

PROPOSITION 3. *Relative to any 3-adapted local coframing, any closed 2-form Φ of the form*

$$\Phi = A(\omega^1 \wedge \theta_0 + \omega^2 \wedge \theta_1) + B\omega^2 \wedge \theta_0 + C^1\theta_0 \wedge \theta_1 + C^2\theta_0 \wedge \theta_2$$

must satisfy $C^1 = C^2 = 0$. Moreover, on the open set in M where the relative invariant T is nonzero, the space $\mathscr{C}$ of conservation laws is at most of dimension 1. If $T = 0$ but U is nonzero, the space $\mathscr{C}$ of conservation laws is trivial.

Proof. Let Φ have the stated form. Expanding out the relation $d\Phi = 0$ and reducing modulo θ_0 gives

$$0 \equiv d\Phi \equiv (dA - A(\alpha + \rho) + B\omega^1 - C^1\theta_2) \wedge \omega^2 \wedge \theta_1 - C^2\theta_1 \wedge \omega^1 \wedge \theta_2 \bmod \theta_0.$$

Of course, now reducing modulo ω^2 shows that we must have $C^2 = 0$. Substituting this back into the above equation implies that there must exist functions B_0, B_1, and B_2 so that

$$dA = A(\alpha + \rho) + B_0\theta_0 + B_1\theta_1 + B_2\omega^2 - B\omega^1 + C^1\theta_2.$$

Substituting this back into the relation $d\Phi = 0$ and then reducing modulo ω^2 and θ_1 yields the relation

$$0 \equiv d\Phi \equiv 2C^1\theta_0 \wedge \theta_2 \wedge \omega^1 \bmod \omega^2, \theta_1.$$

Of course, this implies that $C^1 = 0$. Thus, the first part of the proposition is demonstrated.

Next, let us substitute the relation $C^1 = 0$ into the formula for $d\Phi$. This yields, after some simplification,

$$0 \equiv d\Phi \equiv (A(P_1 - U) - 2BT)\theta_2 \wedge \theta_1 \wedge \theta_0 - (B_1 + AH)\omega^1 \wedge \theta_1 \wedge \theta_0 \bmod \omega^2.$$

In particular, it follows that we must have $A(P_1 - U) - 2BT = 0$.

Let us now restrict attention to the open set where T is nonzero. Then our argument so far shows that, in order to be closed, Φ must be a multiple of the 2-form

$$\Phi_0 = 2T(\omega^1 \wedge \theta_0 + \omega^2 \wedge \theta_1) + (P_1 - U)\omega^2 \wedge \theta_0.$$

In other words, $\Phi = L\Phi_0$ for some function L, which, in order to avoid triviality, we may assume is nonzero. The condition $d\Phi = 0$ is then equivalent to $d\Phi_0 = -(dL/L) \wedge \Phi_0$, which is a differential equation for L. Now, since Φ_0 is a 2-form of rank 2, there can be at most one 1-form λ so that $d\Phi_0 = -\lambda \wedge \Phi_0$. If no such λ exists, then clearly there is no nonzero function L which satisfies $d\Phi_0 = -(dL/L)$

$\wedge\, \Phi_0$. If such a λ does exist, then it must be closed in order for the equation $dL/L = \lambda$ to have any solutions. In any case, it is clear that there is at most a 1-dimensional space of solutions L of this equation.

Finally, suppose that $T = 0$. Then by the definition of P_1 as the coefficient of ω^1 in dT we must have $P_1 = 0$. Then

$$AU = 0,$$

and if U is nonzero then $A = \Phi = 0$. □

At this stage, motivated by the problem of calculating conservation laws, we have introduced a number of invariants of non-Goursat parabolic systems. The first invariant Ψ must vanish if there are to be any nontrivial conservation laws at all. In the Monge-Ampere case, the invariants T, G, H, etc., all have geometric or physical meaning (we have commented on T).

We will close this section with an illustration of the interpretation of these invariants. Suppose that we define a parabolic system to be *quasi-evolutionary* in case it is locally equivalent to the exterior differential system arising from a PDE of the particular form

$$u_t = F(x, t, u, u_x, u_{xx}), \tag{11}$$

where $F_{u_{xx}} \neq 0$. (To be simply *evolutionary* means that F does not depend on t.) Note that (11) is clearly dispersive and of Monge-Ampere type.

PROPOSITION 4. *The necessary and sufficient conditions that a non-Goursat parabolic system be quasi-evolutionary are that, first, the system must be locally Monge-Ampere (so that $S_0 \equiv 0$) and, second, that $T = H = 0$.*

Here, in outline is how this proposition may be proved: For the EDS arising from (11) we may take $\omega^2 = dt$ and then, after some computation, we see that in the structure equations for a 3-adapted coframing, we have

$$T = \gamma = 0, \tag{12}$$

which by (10) gives $G = H = 0$. (Note that the formula we derived for dT coupled with $T = 0$ implies that $P_1 = 0$.)

Conversely, if our system is dispersive and of Monge-Ampere type, then the structure equations show that ω^2 will be integrable (i.e., $\omega^2 \wedge d\omega^2 = 0$) in case (12) holds. As we already noted, $T = 0$ implies that $P_1 = 0$ and it can be shown that, under these conditions, $H = 0$ implies $G = 0$. Thus, under the conditions of the proposition, ω^2 is integrable and writing $\omega^2 = e^f\, dt$ singles out a "time" coordinate. This is the main step in the proof of the proposition.

Further interpretations of T and H will be given in Theorems 1 and 2 in §3, and in fact Proposition 4 follows from the discussion given there.

Appendix 1. The equivalence method. In the preceding two sections we have made extensive use of É. Cartan's equivalence method to determine the invariants of a parabolic exterior differential system. Aside from Cartan's own exposition [Ca2], there are several other sources for this material, notably Chern [Ch] and Gardner [Ga]. For the convenience of the reader, however, we will now summarize (without proofs) the "recipe" for the equivalence method.

Let V be an n-dimensional vector space over the reals and let $G \subseteq GL(n, \mathbb{R})$ be a Lie subgroup with Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V) \cong V \otimes V^*$. The *first prolongation* $\mathfrak{g}^{(1)}$ and the *Spencer cohomology group* $H^{0,1}(\mathfrak{g})$ of the subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(V)$ are defined by the exact sequence

$$0 \to \mathfrak{g}^{(1)} \to \mathfrak{g} \otimes V^* \xrightarrow{\delta} V \otimes \Lambda^2 V^* \to H^{0,1}(\mathfrak{g}) \to 0,$$

where the mapping δ is the composition of the inclusion $\mathfrak{g} \otimes V^* \hookrightarrow V \otimes V^* \otimes V^*$ with the natural skew-symmetrization mapping $V \otimes V^* \otimes V^* \to V \otimes \Lambda^2 V^*$. (The two spaces $\mathfrak{g}^{(1)}$ and $H^{0,1}(\mathfrak{g})$ depend on the way that $\mathfrak{g}$ is realized as a subalgebra of $\mathfrak{gl}(V)$, not just on the abstract algebra $\mathfrak{g}$.)

Throughout this appendix, we will, in fact, fix an identification of V with $\mathbb{R}^n$, thought of as column vectors of height n. However, for certain purposes in this discussion, it is important to distinguish between V and its dual vectors space V^*. Since it is more convenient to write V^* than $(\mathbb{R}^n)^*$, we maintain the abstract notation.

A local V-coframing on an n-manifold M is a V-valued 1-form η defined on an open set $U \subset M$ with the property that $\eta_x: T_xM \to V$ is an isomorphism for all $x \in U$. Recalling our identification of V with $\mathbb{R}^n$, we may write η in the form

$$\eta = \begin{pmatrix} \eta^1 \\ \vdots \\ \eta^n \end{pmatrix},$$

where the η^i are ordinary 1-forms on U which are linearly independent at every point of U. If $\tilde{\eta}$ is another local V-coframing with domain $\tilde{U} \subset M$, then the *transition matrix* from η to $\tilde{\eta}$ is the function $g: U \cap \tilde{U} \to GL(V)$ which satisfies $\tilde{\eta} = g\eta$.

A *G-structure* on a manifold M of dimension n can be defined as a collection of local V-coframings, the union of whose domains cover M, and whose transition matrices have values in G. In most applications of the method of equivalence, a G-structure arises as a collection of local coframings $\eta_\alpha: TU_\alpha \to V$ where the $\{U_\alpha\}$ form an open cover of M and the transition matrices $g_{\alpha\beta}$ defined by

$$\eta_\beta = g_{\alpha\beta}^{-1} \eta_\alpha$$

have values in G. Often the (local) coframings arise as the coframes which satisfy some geometric properties associated to another geometric object.

For example, in §0, we associated to each parabolic system on a 7-manifold M, the family of 0-adapted (local) coframes, and showed that they had the property that the transition matrix between any two of them took values in a certain 19-dimensional subgroup $G_0 \subset \mathrm{GL}(7, \mathbb{R})$. (In fact, G_0 was a subgroup of the group of lower triangular matrices.) Thus, these coframes define a G-structure on M^7 which is associated to and, in fact, defines the underlying parabolic structure. In §1, we defined, for parabolic structures for which the Goursat invariant was nonzero, the 1-adapted local coframes and noted that their transition functions lay in a certain 16-dimensional subgroup $G_1 \subset G$. Thus, the 1-adapted coframes constituted a G_1-structure which was a "reduction" of the original G-structure.

A G-structure $\{\eta_\alpha \colon TU_\alpha \to V | \alpha \in A\}$ gives rise in a natural way to the principal right G-bundle $\pi\colon P \to M$ of all G-coframes of the G-structure. This is the bundle whose local sections with domain U are simply the local coframings $\eta\colon TU \to V$ whose transitions to the coframings η_α have values in G. For each $g \in G$, the right action $R_g\colon P \to P$ is defined by the rule $R_g^*(\eta) = g^{-1}\eta$ for any local section η.

There is a canonical V-valued 1-form ω defined on P. It is characterized by the property that, in the local trivialization $\tau\colon \pi^{-1}(U) \to U \times G$ associated to any section η of P, we have $\omega = \tau^*(g^{-1}\eta)$. Note that ω is π-semibasic, i.e., $\omega(v) = 0$ for all vectors $v \in TP$ which are tangent to the fibers of π. It also manifestly satisfies the G-equivariance property $R_g^*(\omega) = g^{-1}\omega$.

Two G-structures with associated principal bundles $P \to M$ and $\bar{P} \to \bar{M}$ are *equivalent* if there is a diffeomorphism $f\colon M \to \bar{M}$ inducing a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{F} & \bar{P} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\bar{\pi}} \\ M & \xrightarrow{f} & \bar{M}, \end{array}$$

where $F = (f^*)^{-1}$. It can be shown that this is equivalent to the existence of a diffeomorphism

$$F\colon P \to \bar{P}$$

which satisfies

$$F^*(\bar{\omega}) = \omega.$$

The graph of a local equivalence between the G-structures P and $\bar{P}$ is therefore an integral manifold of the exterior differential system on $P \times \bar{P}$ defined by the relations

$$\omega - \bar{\omega} = 0.$$

Of course, this problem may be treated by the general methods of the theory of exterior differential systems. According to these methods, one determines the torsion of the exterior differential system and sets this equal to zero to define the sublocus of $P \times \bar{P}$ where there may be integral elements of the system. Then, one repeats this process on this sublocus (assumed to be a submanifold), and so forth. Having eliminated the torsion of the system at the first level, one then applies Cartan's test to check if the system is now in involution; if not, then one prolongs and starts the process over.

Cartan developed a method of taking into account the special features of this problem which are due to the built-in G-equivariance. This method, the *equivalence method*, distinguishes G-structures (and computes their automorphism groups) by finding the so-called "(differential) invariants" of a G-structure. (We will be more precise about what these "invariants" are below.)

For example, when $G = O(n)$, a G-structure amounts to a Riemannian metric and the components of the Riemann curvature tensor, viewed as functions on P, can be combined in various ways (e.g., the scalar curvature) to give the well-known invariants of a metric. (These are second-order invariants—in the case of a Riemannian metric, there are no first-order invariants.)

The equivalence method will find analogous quantities to attach to any G-structure, providing certain nondegeneracy conditions are met. Moreover, it tries to find these invariants by studying the possible connections (in a suitably general sense) on a G-structure. Again, this will generalize the use of the Levi-Civita connection in Riemannian geometry.

A *pseudoconnection* for a G-structure $P \to M$ is given by a $\mathfrak{g}$-valued 1-form α whose restriction to each fiber is the Mauer-Cartan form—thus $\alpha|_{\pi^{-1}(x)} = g^{-1}\,dg$ for $x \in M$.[1] By exterior differentiation of the equation

$$\omega|_{\pi^{-1}(U)} = \tau^*(g^{-1}\eta),$$

we infer that

$$d\omega = -\alpha \wedge \omega + \Omega,$$

where Ω is a semibasic V-valued 2-form that we may therefore write as

$$\Omega = \frac{1}{2}T(\omega \wedge \omega)$$

where T is a $V \otimes \Lambda^2 V^*$-valued function on P.

[1] Note that we do *not* require the usual equivariance condition $R_g^*(\alpha) = Ad_{g^{-1}}(\alpha)$. Unless G is reductive, the equivalence method will *not* generally work if one imposes the equivariance condition. For this reason, É. Cartan used a broader notion of connection than that in current use.

If $\bar{\alpha}$ is another pseudoconnection on P, then $\alpha - \bar{\alpha}$ is semibasic and is therefore of the form

$$\alpha - \bar{\alpha} = S(\omega)$$

where S is a $\mathfrak{g} \otimes V^*$-valued function on P. It follows that

$$T = \bar{T} + \delta S,$$

so that the reduced mapping

$$[T]: P \to H^{0,1}(\mathfrak{g})$$

is well defined, independent of the choice of pseudoconnection α. It is easy to see that even though the pseudoconnection α may not be $\mathrm{Ad}(G)$-equivariant as is the case for true connections, nevertheless we will have

$$R_g^*[T] = g^{-1} \cdot [T],$$

where the action on the right-hand side is the natural action of G on $H^{0,1}(\mathfrak{g})$. Thus, canonically associated to the G-structure is a function

$$\tau: M \to H^{0,1}(\mathfrak{g})/G$$

called the *torsion* of the G-structure.[2] It represents the basic first-order invariant of the G-structure.

We now want to describe the procedure of *reduction* which is the first of two processes central to the method of equivalence. A submanifold $W \subset H^{0,1}(\mathfrak{g})$ will be said to be a G-cross-section if, for all $w \in W$, we have $W \cap (G \cdot w) = w$ and $T_w W \cap T_w(G \cdot w) = 0$ for all $w \in W$. We say that a cross-section W is of constant type G_1 if the G-stabilizer of w is $G_1 \subset G$ for all $w \in W$. Most of the G-cross-sections encountered in practice are "natural" linear or affine subspaces of $H^{0,1}(\mathfrak{g})$ and have constant type.

In "favorable" cases for application of the method of equivalence, the image $[T](P)$ will lie in a set of the form $G \cdot W$ where W is a G-cross-section of some constant type G_1. In this case, one can canonically define a (smooth) G_1-substructure on M by letting $P_1 = [T]^{-1}(W)$. This step of passing to a canonical substructure is known as *reduction.*

For example, when one considers the G-structure associated to the 0-adapted coframes as defined in §0, the G-orbits of the points $[T(u)]$ were of dimension 2 or

[2] This torsion is only indirectly related to the torsion of the EDS $\omega - \bar{\omega} = 0$ on $P \times \bar{P}$ mentioned above since, in the present situaton, we are only dealing with a single G-structure.

3 and were "coordinatized" by the functions A, B, and C. The stabilizer types of these orbits depended only on whether C was zero or not. For G-structures with $C \neq 0$ (i.e., the structures associated to non-Goursat systems), the affine subspace $W \subset H^{0,1}(\mathfrak{g})$ defined by the equations $A = 0$, $B = 0$, and $C = 1$ gave a G-cross-section of type G_1 (as defined in §1). This gave us the bundle of 1-adapted coframes which we associated to any non-Goursat parabolic system. On the other hand, for equations of Goursat type, we would have had $C \equiv 0$ and the appropriate G-cross-section would have been the linear subspace W' defined by the equations $A = B = C = 0$.

To continue with the general case, if G_1 is a proper subgroup of G, then one may begin the process again, consider the intrinsic torsion $[T_1]: P_1 \to H^{0,1}(\mathfrak{g}_1)$ and look for an appropriate G_1-cross-section.

Again, for example, when one considers the G_1-structure associated to the 1-adapted coframes of a non-Goursat system as defined in §1, the G_1-orbits of the points $[T_1(u)]$ were of dimension 4 or 5 and were "parameterized" by the coordinates S_0 through S_4. Again, the stabilizer types of these orbits depended only on whether the coordinate S_0 was zero or not. For G_1-structures with $S_0 \neq 0$ (i.e., the non-Monge-Ampere equations), the appropriate cross-section would have been the subspace defined by $S_0 = 1$ and $S_1 = S_2 = S_3 = S_4 = 0$. However, it turned out that we were only interested in Monge-Ampere systems (because of Proposition 2), so we used instead the subspace W_1 defined by $S_0 = S_1 = S_2 = S_3 = S_4 = 0$, whose points were stabilized by the subgroup G_2 as defined in §2.

Clearly, this process can be repeated as long as we are in the "favorable" case of being able to find a suitable cross-section and as long as the new stabilizer G_k is a proper subgroup of the group G_{k-1}. However, unless one chooses one's problem carefully, it often does not take long to either run out of favorable cases or (more rarely) to reduce to the case where $G_k = G_{k+1} = \cdots$.

Suppose that at, say, the kth stage, this process stabilizes. Then one invokes the other main idea in the method of equivalence, that of *prolongation*. Although we did not need to get into this in this paper, we shall say a few words about how this process goes. For convenience of notation, we relabel and set $G_k = G$ and $P_k = P$.

Having normalized the torsion, we may seek to normalize the pseudoconnection α. For this we must choose a splitting $j: H^{0,1}(\mathfrak{g}) \to V \otimes \Lambda^2 V^*$ of the surjection

$$V \otimes \Lambda^2 V^* \to H^{0,1}(\mathfrak{g}),$$

which, as previously noted, it may not be possible to do in a G-equivariant manner. In any case, we may then choose a pseudoconnection α so that $T = j([T])$. We then have the equation

$$d\omega = -\alpha \wedge \omega + \frac{1}{2} T(\omega \wedge \omega),$$

where T now takes values in $j(H^{0,1}(\mathfrak{g})) \subset V \otimes \Lambda^2 V^*$. This condition determines α up to a $\mathfrak{g}^{(1)}$-valued function, and this is all the normalization that is possible at this stage.

To see why we must use pseudoconnections, suppose, for example, that G acts trivially on $H^{0,1}(\mathfrak{g})$. Unless G is reductive, it can very well happen that there is no G-invariant complement to $\delta(\mathfrak{g} \otimes V^*)$ on which G acts trivially. In such cases, there simply will not be any G-equivariant connection with the torsion normalized as above.

Now, even having normalized the torsion, we will not have a unique pseudoconnection α unless the first prolongation $\mathfrak{g}^{(1)}$ of $\mathfrak{g}$ is zero (as happens, for example, in the Riemannian case). On P we therefore consider the set of all $(V \oplus \mathfrak{g})$-valued coframings of the form

$$\omega^{(1)} = \begin{pmatrix} \omega \\ \alpha + S(\omega) \end{pmatrix}, \qquad S \in \mathfrak{g}^{(1)} \subset V \otimes V^* \otimes V^*,$$

that is, where the first n components of $\omega^{(1)}$ are just the components of ω, and the remaining dim $\mathfrak{g}$ components are the components of the pseudoconnection α well defined up to the addition of a term of the form $S(\omega)$ where S takes values in $\mathfrak{g}^{(1)}$. Since $S(\omega) \wedge \omega = 0$, this modification does not affect the normalized torsion T; moreover, it is the most general such modification. The set of such coframings therefore defines a $\mathfrak{g}^{(1)}$-structure on P, with an associated coframe bundle $P^{(1)} \to P$. Since $\mathfrak{g}^{(1)}$ is an abelian group (written additively), its Lie algebra is simply $\mathfrak{g}^{(1)}$, embedded into $\mathfrak{gl}(V \oplus \mathfrak{g})$ as the "matrices" of the form

$$\left(\begin{array}{c|c} 0 & 0 \\ \hline \mathfrak{g}^{(1)} & 0 \end{array}\right).$$

We may now repeat the process that we went through for our original G-structure on M, arriving at the differential invariants on the second-order frame bundle $P^{(1)}$. Note that $P^{(1)}$ has the structure of a principal bundle over M with structure group $G^{(1)} = \mathfrak{g}^{(1)} \times_\rho G$, where ρ is the natural representation of G on $\mathfrak{g}^{(1)}$.

For the Riemannian case $\mathfrak{g}^{(1)} = (0)$ and the torsion may be normalized to zero, giving an intrinsic connection and resulting I-structure on P (I is the group with only the unit matrix). The torsion of this I-structure then contains the components of the Riemannian curvature tensor as second-order differential invariants.

In general, matters are not so simple (or perhaps, in the general case, they are sometimes more interesting?). The main "result" of the equivalence method, which seems to have not been completely formulated and proved except in special cases, is that the above is a finite process: after some finite sequence of applications of the reduction and prolongation procedures, we will have, in some sense, a "complete, generating" set of differential invariants. (In suitably nonsingular cases, one can prove a form of this finiteness theorem by an application of the Cartan-Kuranishi prolongation theorem.)

In the present paper we only found it necessary to carry out three structure group reductions. The first came by setting $C = 1$ and $A = B = 0$ for dispersive parabolic systems. The second came for Monge-Ampere systems (those with $S_0 = 0$) by setting the components S_0, S_1, S_2, S_3, S_4 of the torsion equal to zero (which, once $S_0 = 0$, is possible by the transformation rules given by equation (7) in §2). The third reduction came by setting $R_0 = R_1 = R_2 = R_3 = 0$ (cf. equation (8) in §2). Fortunately, no further torsion normalization nor consideration of higher-order frame bundles will prove necessary in this paper.

In concluding this discussion of a recipe for the equivalence method, we want to explain what is behind the equations (7) and (8) just referred to.

First a general remark. On P we may take the exterior derivative of the equation

$$d\omega = -\alpha \wedge \omega + \frac{1}{2} T(\omega \wedge \omega)$$

to have, after simplification,

$$-(d\alpha + \alpha \wedge \alpha) \wedge \omega + DT(\omega \wedge \omega) = 0,$$

where DT is defined by

$$\frac{1}{2}\alpha \wedge T(\omega \wedge \omega) + d\left(\frac{1}{2} T(\omega \wedge \omega)\right) = (DT)(\omega \wedge \omega).$$

Were α a connection in the usual sense, then $d\alpha + \alpha \wedge \alpha$ would be its curvature, which is entirely semibasic, and DT would be the covariant derivative of the torsion tensor T. In general, by the Maurer-Cartan equation, $d\alpha + \alpha \wedge \alpha$ restricts to zero on the fibers of $P \to M$ and thus is in the ideal $\mathscr{I}$ generated by the semibasic forms. It is clear that $d\alpha + \alpha \wedge \alpha$ takes its values in $\mathfrak{g} \otimes V^* \otimes$ (1-forms mod semibasic 1-forms). If we denote by Δ the operation "d mod semibasic 1-forms", then the above equation allows us to solve for ΔT and obtain an equation of the form

$$\Delta T = \beta(T) + \gamma,$$

where β is a 1-form, defined mod semibasic forms, that is linear in T, and γ is a 1-form defined mod semibasic forms. In fact, γ comes from the $(d\alpha + \alpha \wedge \alpha) \wedge \omega$ term. Now ΔT represents the infinitesimal variation of T along a fiber and the γ part may easily be seen to be in the image of the mapping $\mathfrak{g} \otimes V^* \to V \otimes \Lambda^2 V^*$. Thus the variation of the Spencer cohomology class is given by

$$\Delta[T] = [\beta(T)],$$

and this tells us in practice what the representation of G on $H^{0,1}(\mathfrak{g})$ is (of course, we *could*, in principle, find this directly but, in practice, this "infinitesimal method" is often quicker).

More importantly, in practice we will have normalized certain of the components of $[T]$. For example, for parabolic systems, part of $[T]$ will be normalized simply by the original structure equations (§0), another part by assuming the system is dispersive and then setting $C = 1$, leading to the crude structure equations (§1), and so forth. This means that the remaining components of $[T]$ will transform *affine* linearly (as is apparent in equations (7) and (8) in §2). This then allows us to further restrict the structure group by requiring that these components vanish. It is this process that is occurring in the second and third reductions above.

Appendix 2. Monge-Ampere systems. In this appendix, we want to recall the definition and some of the special properties of Monge-Ampere equations. Particularly important for us will be their characterization in terms of special properties of the exterior differential systems which are used to model them. This way of looking at Monge-Ampere equations is by no means new, having been developed extensively, beginning in the 1920's, by Goursat, Cartan, Lepage, and de Donder, among others. All of what we outline below is to be found in the works of these authors. The interested reader may consult the references for some leads into this literature.

A partial differential equation

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$

is rewritten as an exterior differential system $\mathscr{I}$ on a 7-manifold M in the usual way by introducing coordinates (x, y, u, p, q, r, s, t) in the jet manifold $J^2(\mathbb{R}^2, \mathbb{R})$, taking M to be the hypersurface

$$F(x, y, u, p, q, r, s, t) = 0$$

(assumed to be a manifold in the open set under question), and taking $\mathscr{I}$ to be the Pfaffian system generated by the contact forms

$$\theta_0 = du - p\,dx - q\,dy$$

$$\theta_1 = dp - r\,dx - s\,dy$$

$$\theta_2 = dq - s\,dx - t\,dy.$$

One then proceeds to study $(M, \mathscr{I})$ by the methods of the theory of exterior differential systems. Thus, the first derived system of $\mathscr{I}$ is generated by θ_0; the Cartan system of θ_0, θ_1, θ_2 consists of all of $\Omega^1(M)$, one may introduce the invariants of $\mathcal{I}$ by the equivalence method, and so forth. However, if the equation is known to have a certain form, one can often simplify this process considerably. Because of its importance in this paper, we want to explain how this goes for a particular special case.

Classically, a *Monge-Ampere equation* is defined to be one of the form

$$E(u_{xx}u_{yy} - u_{xy}^2) + Au_{xx} + 2Bu_{xy} + Cu_{yy} + D = 0,$$

where A, B, C, D, E are given functions of x, y, u, u_x, u_y. Although it is not obvious, the class of Monge-Ampere equations is invariant under contact transformation, and hence forms a geometrically natural class of second-order partial differential equations.

Moreover, in contrast to the general case, Monge-Ampere equations can be modeled by an exterior differential system on a 5-manifold, and we want to briefly explain how this goes. On the space $J = J^1(\mathbb{R}^2, \mathbb{R})$ with coordinates (x, y, u, p, q), we introduce the 1-form

$$\theta = du - p\,dx - q\,dy$$

and 2-form

$$\Omega = E\,dp \wedge dq + A\,dp \wedge dy + B(dq \wedge dy + dx \wedge dp) + C\,dx \wedge dq + D\,dx \wedge dy,$$

and we denote by $\mathcal{J}$ the exterior differential system generated by θ and Ω. It is clear that the integral surfaces of $\mathcal{J}$ on which $dx \wedge dy \neq 0$ are locally in one-to-one correspondence with the solutions to the Monge-Ampere equations given above.

The relationship between the first, more general EDS construction and this one is that *$(M, \mathcal{I})$ is the first prolongation of $(J, \mathcal{J})$.*

We now want to slightly widen the notion of a Monge-Ampere *equation* to that of a Monge-Ampere *system.*

Definition. A *Monge-Ampere system* is an exterior differential system $\mathcal{J}$ given on a 5-manifold J, where $\mathcal{J}$ is locally generated by a 1-form θ and 2-form Ω satisfying the conditions

(i) $\theta \wedge (d\theta)^2 \neq 0$;

(ii) $d\theta$ and Ω are linearly independent mod θ.

While it is not difficult to give examples of Monge-Ampere systems which are not globally equivalent to the ones which arise from Monge-Ampere equations, at the local level there is no difference, as the following proposition shows.

PROPOSITION. *Any Monge-Ampere system is locally equivalent to one induced by a Monge-Ampere equation.*

Proof. By the Pfaff-Darboux theorem, we may locally find coordinates (x, y, u, p, q) on J such that

$$\theta = du - p\,dx - q\,dy.$$

Since $\{dx, dy, \theta, dp, dq\}$ forms a coframing, we may write

$$\Omega = E\,dp \wedge dq + A\,dp \wedge dy + B(dq \wedge dy + dx \wedge dp) + C\,dx \wedge dq + D\,dx \wedge dy$$
$$+ G(dx \wedge dp + dy \wedge dq) + \alpha \wedge \theta,$$

where A, B, C, D, E, G are functions and α is a 1-form. Setting

$$\Omega' = \Omega - G\,d\theta - \alpha \wedge \theta$$

we see that θ and Ω' locally generate the exterior differential system $\mathcal{J}$, while clearly

$$\theta = \Omega' = 0, \qquad dx \wedge dy \neq 0$$

defines a partial differential equation of Monge-Ampere type. □

We want to explain one aspect of the basic geometry of Monge-Ampere systems, a classification into types which corresponds to the classification of second-order partial differential equations into hyperbolic, elliptic, and parabolic types.

First, relative to any choice of generators (θ, Ω) of $\mathcal{J}$, we define a quadratic form in two variables $Q(\xi, \eta)$ by

$$Q(\xi, \eta)\theta \wedge (d\theta)^2 = (\xi\,d\theta + \eta\Omega)^2 \wedge \theta.$$

We may write

$$Q(\xi, \eta) = \xi^2 + 2f\xi\eta + g\eta^2,$$

where f and g are functions on the domain of the local generators (θ, Ω).

Now, if $(\tilde{\theta}, \tilde{\Omega})$ is another set of local generators of $\mathcal{J}$ as above, then we have

$$\tilde{\theta} = a\theta$$
$$\tilde{\Omega} = b\Omega + c\,d\theta + \alpha \wedge \theta,$$

where $a \neq 0$, $b \neq 0$, and c are functions and α is a 1-form. The corresponding quadratic form is given by

$$\tilde{Q}(\xi, \eta) = Q(\xi + (c/a)\eta, (b/a)\eta).$$

Thus, the sign of the discriminant

$$\Delta = g - f^2$$

is independent of the choice of local generators, and we say that the system is
(i) hyperbolic if $\Delta < 0$;
(ii) elliptic if $\Delta > 0$;
(iii) parabolic if $\Delta = 0$.
Since

$$\Delta = (AC - DE - B^2)$$

for the EDS arising from a Monge-Ampere PDE, our terminology agrees with the classical one.

Finally, in this paper we are concerned with parabolic Monge-Ampere systems, and we show how to choose generators of such a system in a form that makes contact with Proposition 1 in §2 above. To do this, we choose generators (θ_0, Ω) so that $Q(\xi, \eta) = \xi^2$. (The above transformation rule for the quadratic form Q clearly implies that this can be done.) This translates into the equations

$$d\theta_0 \wedge \Omega \wedge \theta_0 = \Omega^2 \wedge \theta_0 = 0.$$

From the second equation, we see that $\Omega \bmod \theta_0$ is decomposable; consequently there exist 1-forms θ_1, ω^2 such that

$$\Omega \equiv \theta_1 \wedge \omega^2 \bmod \theta_0 .$$

From the first equation, we infer that there exist 1-forms ω^1, θ_2 so that

$$d\theta_0 \equiv -\theta_1 \wedge \omega^1 - \theta_2 \wedge \omega^2 \bmod \theta_0 .$$

From $\theta_0 \wedge (d\theta_0)^2 \neq 0$ we know that θ_0, θ_1, θ_2, ω^1, ω^2 is a local coframing. Then $\mathcal{I}$ is generated algebraically by

$$\theta_0, d\theta_0, \Omega = \theta_1 \wedge \omega^2,$$

and this choice of notation aligns with that in the proof of Proposition 1.

3. Normal forms for parabolic Monge-Ampere systems admitting a conservation law

Parabolic Monge-Ampere systems. Since the local structure of parabolic Monge-Ampere systems is not entirely clear, we will give a discussion of their representation in local coordinates. For this purpose, we will make the following general definition. (The reader should compare the discussion in the proof of Proposition 1 in §2 above.)

Definition. *A parabolic Monge-Ampere system* on a 5-manifold M^5 is a differential system $\mathcal{I}$ on M with the property that M can be covered by open sets U on which there exist coframings (called *0-adapted* coframings) $\Sigma = (\theta_0, \theta_1, \theta_2, \omega^1, \omega^2)$ which satisfy the conditions that $d\theta_0 \equiv -\theta_1 \wedge \omega^1 - \theta_2 \wedge \omega^2 \bmod \theta_0$, and that $\mathcal{I}$ restricted to U is generated by θ_0, $d\theta_0$, and $\Omega = \theta_1 \wedge \omega^2$.

The following result gives a local description of all of the parabolic Monge-Ampere systems in the real-analytic case. Presumably, this result is also true without the hypothesis of real-analyticity, but our proof uses the Cartan-Kähler theorem in an essential way.[1]

THEOREM 1. *Let $\mathcal{I}$ be a real-analytic, parabolic Monge-Ampere system on a manifold M^5. Then $\mathcal{I}$ is locally equivalent to the Monge-Ampere system generated by a (parabolic) quasi-linear equation of the form*

$$u_{xx} + 2B(x, y, u, u_x, u_y)u_{xy} + (B(x, y, u, u_x, u_y))^2 u_{yy} + D(x, y, u, u_x, u_y) = 0.$$

Conversely, for any functions B and D of x, y, u, u_x, and u_y, the above equation describes a parabolic Monge-Ampere differential system.

Proof. Suppose that $\mathcal{I}$ is a parabolic Monge-Ampere system on M^5. Let $\Sigma = (\theta_0, \theta_1, \theta_2, \omega^1, \omega^2)$ be a 0-adapted coframing for $\mathcal{I}$ on an open set U in M.

The first step in the proof is to construct a Frobenius system $\mathcal{K}$ of rank 3 on U which has the property that $\mathcal{I}$ is contained in the ideal generated by $\mathcal{K}$. (This step requires the use of the Cartan-Kähler theorem, hence the assumption of real-analyticity.) Let b_1 and b_2 be coordinates on $\mathbb{R}^2$ and let $X = U \times \mathbb{R}^2$. Define the following 1-forms on X:

$$\eta_0 = \theta_0$$

$$\eta_1 = \omega^1 + b_1\theta_1 + b_2\theta_2$$

$$\eta_2 = \omega^2 + b_2\theta_1 .$$

Note that, modulo the η_i, all of the forms on U may be written as linear combinations of θ_1 and θ_2. In particular, there are formulas of the form

$$\left.\begin{aligned} d\omega^1 &\equiv T^1\theta_1 \wedge \theta_2 \\ d\omega^2 &\equiv T^2\theta_1 \wedge \theta_2 \\ d\theta_1 &\equiv T^3\theta_1 \wedge \theta_2 \\ d\theta_2 &\equiv T^4\theta_1 \wedge \theta_2 \end{aligned}\right\} \bmod \eta_0, \eta_1, \eta_2,$$

[1] While the analogous result for hyperbolic Monge-Ampere systems was apparently known to Lie, we have not been able to find this statement for the parabolic case in the literature.

where, of course, the functions T^i depend on the variables b_1 and b_2 as well. Using these formulas, it is easy to see that the following structure equations hold:

$$\left.\begin{aligned} d\eta_0 &\equiv 0 \\ d\eta_1 &\equiv \beta_1 \wedge \theta_1 + \beta_2 \wedge \theta_2 \\ d\eta_2 &\equiv \beta_2 \wedge \theta_1 \end{aligned}\right\} \mod \eta_0, \eta_1, \eta_2,$$

where

$$\beta_1 \equiv db_1 - (T^1 + b_1 T^3 + b_2 T^4)\theta_2$$

$$\beta_2 \equiv db_2 - (T^2 + b_2 T^3)\theta_2.$$

Clearly, $\Sigma_+ = (\theta_0, \theta_1, \theta_2, \omega^1, \omega^2, \beta_1, \beta_2)$ is a coframing of the 7-manifold X.

Let $\mathscr{I}$ be the differential system generated by the two 5-forms

$$\Upsilon_1 = d\eta_1 \wedge \eta_0 \wedge \eta_1 \wedge \eta_2 = (\beta_1 \wedge \theta_1 + \beta_2 \wedge \theta_2) \wedge \eta_0 \wedge \eta_1 \wedge \eta_2$$

$$\Upsilon_2 = d\eta_2 \wedge \eta_0 \wedge \eta_1 \wedge \eta_2 = (\beta_2 \wedge \theta_1) \wedge \eta_0 \wedge \eta_1 \wedge \eta_2.$$

Take the independence condition to be the 5-form $\Omega = \theta_0 \wedge \theta_1 \wedge \theta_2 \wedge \omega^1 \wedge \omega^2$. Then any integral of $(\mathscr{I}, \Omega)$ is described locally as the graph of a mapping (b_1, b_2): $U \to \mathbb{R}^2$ where the functions b_1 and b_2 satisfy the condition that the rank-3 system $\mathscr{K}$ generated by the 1-forms $\{\theta_0, \omega^1 + b_1\theta_1 + b_2\theta_2, \omega^2 + b_2\theta_1\}$ should be a Frobenius system.

Now, examining the formulas for the generators Υ_i in terms of the coframing Σ_+, it is immediate that the reduced Cartan characters are $(s_0', s_1', s_2', s_3', s_4', s_5') = (0, 0, 0, 0, 2, 0)$. Moreover, it is also easy to see that the space of integral elements of $(\mathscr{I}, \Omega)$ at each point is of dimension $8 = 4s_4'$. Hence, Cartan's Test is satisfied, and the system $(\mathscr{I}, \Omega)$ is involutive. Since we have assumed the original system $\mathscr{I}$ to be real-analytic, it follows that the desired integral manifolds exist and (locally) depend on two functions of four variables. This completes the first step. (See Chapter III in [BCG³] for a discussion of the Cartan characters and Cartan's test.)

Fix a point $m \in U$. Applying the construction from the first step, choose a rank-3 Frobenius system $\mathscr{K}$ in a neighborhood of m with generators of the form $\eta_0 = \theta_0$, $\eta_1 = \omega^1 + b_1\theta_1 + b_2\theta_2$, and $\eta_2 = \omega^2 + b_2\theta_1$. Let u, x, and y denote three independent first integrals of $\mathscr{K}$ on a neighborhood of m. We assume (as we clearly may) that $\theta_0 \wedge dx \wedge dy \neq 0$.

Since θ_0 lies in $\mathscr{K}$ and hence is a linear combination of du, dx, and dy, we may divide the θ_i by an appropriate nonzero function so as to arrange that

$$\theta_0 = du - p\,dx - q\,dy$$

for some functions p and q on a neighborhood of m. Now, θ_0 is a contact form on U, so it follows that the functions x, y, u, p, and q are independent on a neighborhood of m. Restricting this neighborhood if necessary, we may assume that these five functions actually form a local coordinate system.

Now, clearly there are functions A, B, C, D, E, and F on a neighborhood of m so that

$$\theta_1 \wedge \omega^2 \equiv E\,dp \wedge dq + A\,dp \wedge dy + B\,dq \wedge dy + C\,dx \wedge dq + D\,dx \wedge dy + F\,dx \wedge dp \mod \theta_0 .$$

However, by construction, we have both $(\theta_1 \wedge \omega^2) \wedge d\theta_0 \wedge \theta_0 = 0$ and $(\theta_1 \wedge \omega^2) \wedge dx \wedge dy \wedge \theta_0 = 0$. The reader may easily calculate that these equations force $E = 0$ and $F = B$. Thus, we may assume that

$$\theta_1 \wedge \omega^2 = A\,dp \wedge dy + B(dq \wedge dy + dx \wedge dp) + C\,dx \wedge dq + D\,dx \wedge dy.$$

Since the form on the left-hand side of this equation is decomposable, it follows that $AC - B^2 = 0$. Now, it is not difficult to show that, by slightly rearranging the variables if necessary, we may assume that $A \neq 0$. Replacing ω^2 and θ_2 by $A\omega^2$ and $A^{-1}\theta_2$, respectively, we may clearly arrange that $A = 1$ and hence that $C = B^2$.

It is now clear that the integral manifolds of $\mathcal{I} = (\theta_0, d\theta_0, \theta_1 \wedge \omega^2)$ may be described locally near m as the "graphs" of the form $u = f(x, y)$, $p = f_x(x, y)$, and $q = f_y(x, y)$, where f is a function of x and y which satisfies the quasi-linear parabolic differential equation

$$f_{xx} + 2B(x, y, f, f_x, f_y)f_{xy} + (B(x, y, f, f_x, f_y))^2 f_{yy} + D(x, y, f, f_x, f_y) = 0.$$

The converse is easy and is left to the reader. □

Note that a corollary of Theorem 1 is that any real-analytic parabolic Monge-Ampere system is locally contact equivalent to a quasi-linear parabolic equation.

A "count of functions" shows that the contact equivalence classes of parabolic Monge-Ampere systems depend on two functions of 5 variables. Thus, the "normal form" described in Theorem 1 is likely to be optimal. Certainly, there will not be changes of variables which allow one to normalize the functions B and D much further.

However, in the case that we know more about the invariants of the parabolic Monge-Ampere system, we can considerably tighten the normal form of Theorem 1. As a sample of the sort of result we have in mind, we present the following theorem.

THEOREM 2. *Let $\mathcal{I}$ be a parabolic Monge-Ampere system on a manifold M^5. Suppose that the relative invariant T vanishes identically. Then, the function H*

defined by equation (10) in §2 is a relative invariant. On the open set where H is nonzero, $\mathcal{I}$ is locally equivalent to the Monge-Ampere system generated by a (parabolic) quasi-linear equation of the form

$$u_{xx} + 2uu_{xy} + u^2 u_{yy} + D(x, y, u, u_x, u_y) = 0.$$

On any open set where the relative invariant H vanishes identically, $\mathcal{I}$ is locally equivalent to the Monge-Ampere system generated by a (parabolic) quasi-linear equation of the form

$$u_{xx} + D(x, y, u, u_x, u_y) = 0.$$

Proof. Let Φ be a 3-adapted local coframe. Since T vanishes identically, we know that the function P_1 must also vanish identically. The structure equations (9) and (10) in the preceding section show that we have congruences of the form

$$d\omega^2 \equiv 2\theta_0 \wedge (G\theta_1 + H\omega^1) \bmod \omega^2$$

$$d\theta_0 \equiv -\theta_1 \wedge \omega^1 \bmod \omega^2, \theta_0$$

$$d\theta_1 \equiv -\theta_2 \wedge \omega^1 \bmod \omega^2, \theta_0, \theta_1 .$$

Under a frame rotation in the group G_3 we may verify that $H = ar\tilde{H}$, and thus H is a relative invariant. Suppose that $H \neq 0$. Upon dividing ω^2 by $-2H$ and replacing ω^1 by $\omega^1 - (G/H)\theta_1$, the above equations simplify to

$$d\omega^2 \equiv -\theta_0 \wedge \omega^1 \bmod \omega^2$$

$$d\theta_0 \equiv -\theta_1 \wedge \omega^1 \bmod \omega^2, \theta_0$$

$$d\theta_1 \equiv -\theta_2 \wedge \omega^1 \bmod \omega^2, \theta_0, \theta_1 .$$

These congruences imply that the system generated by $\{\omega^2, \theta_0, \theta_1\}$ is a Pfaffian system of the type described by Goursat's Normal Form Theorem. (cf. page 54 in [BCG^3]). According to this result, there exist local coordinates (x, y, v_0, v_1, v_2) in a neighborhood of any point so that

$$\omega^2 = \lambda(dy - v_0\, dx)$$

$$\theta_0 = \lambda(dv_0 - v_1\, dx) - \mu(dy - v_0\, dx)$$

$$\theta_1 = \lambda(dv_1 - v_2\, dx) - \kappa(dv_0 - v_1\, dx) - \nu(dy - v_0\, dx)$$

for some functions $\lambda \neq 0$, μ, κ, and ν. Clearly, by scaling in the coframe, we may assume that $\lambda = 1$. Set $u = v_0$, $p = v_1 - \mu v_0$, and $q = \mu$. Then $\theta_0 = du - p\, dx -$

$q\,dy$. Since θ_0 is a contact form, it follows that the functions (x, y, u, p, q) are independent and hence form a local coordinate system on a neighborhood of any point in their domain.

Now, the system $\mathscr{I}$ is generated by θ_0, $d\theta_0$ and $\theta_1 \wedge \omega^2$. It follows by a short calculation that, setting $D = q(p + uq) - v_2$, the system $\mathscr{I}$ is generated by the 1-form θ_0 and the two 2-forms $d\theta_0$ and

$$\Upsilon = (dp + u\,dq + D\,dx) \wedge (dy - u\,dx).$$

The normal form in the first part of the theorem now follows immediately.

Now let us assume, instead, that H vanishes identically. If G were nonzero on an open set, then the Cartan system of the 1-form ω^2 would clearly be $\{\omega^2, \theta_0, \theta_1\}$, and hence this latter system would be completely integrable. However, the congruence $d\theta_1 \equiv -\theta_2 \wedge \omega^1 \bmod \theta_0, \theta_1, \omega^2$ shows that this is not the case. Thus, G vanishes identically. It then follows by Goursat Normal Form that there must exist local coordinates (x, y, v_0, v_1, v_2) so that

$$\omega^2 = \xi\,dy$$

$$\theta_0 = \lambda(dv_0 - v_1\,dx) - \mu\,dy$$

$$\theta_1 = \lambda(dv_1 - v_2\,dx) - \kappa(dv_0 - v_1\,dx) - \nu\,dy,$$

where $\xi \neq 0$, $\lambda \neq 0$, μ, κ, and ν are functions on the domain of the local coordinates. Again, we may assume that $\xi = \lambda = 1$. Setting $v_0 = u$, $v_1 = p$, $\mu = q$, and $D = -v_2$, we see that $\mathscr{I}$ is locally generated by the 1-form $\theta_0 = du - p\,dx - q\,dy$ and the two 2-forms $d\theta_0$ and

$$\Upsilon = (dp + D\,dx) \wedge dy.$$

The second half of the theorem is now obvious. □

It may be worth remarking that, for an equation of the first type, the Goursat relative invariant is proportional to $C = D_q - uD_p - (p + uq)$ while, for an equation of the second type, it is proportional to $C = D_q$. Note also that it is precisely the non-Goursat equations of the second type which can be locally placed in the "quasi-evolutionary" form

$$u_t = F(x, t, u, u_x, u_{xx}).$$

That is, *for dispersive parabolic systems we have introduced a sequence of relative invariants*

$$S_0, T, H$$

where T becomes a relative invariant only when $S_0 = 0$ and H becomes a relative invariant only when $S_0 = T = 0$. As stated in Proposition 4 in §2, those systems which can locally be put in the above evolutionary form are precisely those for which $S_0 = T = H = 0$. In §5 below we shall completely analyze conservation laws for classical evolution equations.

Equations admitting a conservation law. We now want to describe a method of "constructing" all of the non-Goursat parabolic equations which admit at least one conservation law. The result of this discussion is Theorem 3 which, roughly speaking, says that the set of local contact equivalence classes of such equations "depends" on one function of five variables. Since contact equivalence classes of parabolic equations locally depend on one function of six variables,[2] and since the condition to be non-Goursat is open, we may say that *the dispersive parabolic systems admitting a conservation law are "transcendental codimension 1" among all such systems.*

Suppose that $\mathscr{I}$ is a non-Goursat Monge-Ampere system on M^7 which admits a nontrivial conservation law. Let Φ be a nonzero closed 2-form representing this conservation law. Then on the third-order frame bundle $\mathscr{F}''$, the 2-form Φ can be expanded in the form

$$\Phi = A(\omega^1 \wedge \theta_0 + \omega^2 \wedge \theta_1) + B\omega^2 \wedge \theta_0,$$

where A and B are functions on $\mathscr{F}''$ with $B = -(D_1 + \beta)A$. We have already seen that there cannot be any open set where A vanishes but Φ does not. Thus, we shall restrict attention to the open set where A is nonzero and, accordingly, assume henceforth that $A \neq 0$.

Referring to the transformation laws given at the beginning of §2, under a frame rotation in the group G_3 we have

$$A = ar\tilde{A}$$

$$B = ar^2\tilde{B} - 2abr^2\tilde{A}.$$

It follows that we may make a frame adaptation, *depending on the particular conservation law* Φ, in order to have

$$A = 1, B = 0.$$

It follows that, on any open set where Φ is nonzero, there exists a 3-adapted coframe in which Ψ has the expression

$$\Phi = \omega^1 \wedge \theta_0 + \omega^2 \wedge \theta_1.$$

[2] That is, second-order equations are given by hypersurfaces in open sets $U \subset J^2(\mathbb{R}^2, \mathbb{R}) \cong \mathbb{R}^8$ and therefore are locally parametrized by one arbitrary function of seven variables. However, parabolic equations are hypersurfaces in $U \cap V$, where V is the hypersurface in $J^2(\mathbb{R}^2, \mathbb{R})$ given by $rt - s^2 = 0$; as such, parabolics depend on one arbitrary function of six variables.

Now, the Cartan system of the 1-form θ_0 is the rank-5 system $\mathscr{A} = \{\theta_0, \theta_1, \theta_2, \omega^1, \omega^2\}$. Since the system $\mathscr{A}$ is Frobenius, the 7-manifold M can be covered by open sets U on which there exist submersions $\sigma: U \to \sigma(U) \subset \mathbb{R}^5$ whose (connected) fibers are the leaves of $\mathscr{A}$ in U. The 1-form θ_0 is well defined up to a multiple on $\sigma(U)$. Moreover, because Φ is a closed 2-form and is expressed in terms of the 1-forms in $\mathscr{A}$, it follows that there is a well-defined 2-form Υ on $\sigma(U)$ which satisfies $\sigma^*(\Upsilon) = \Phi$.

The differential system $\mathscr{I}$ on $\sigma(U)$ generated by $\{\theta_0, d\theta_0, \Upsilon\}$ is clearly a parabolic Monge-Ampere system, and the submersion $\sigma: U \to \sigma(U)$ is easily seen to be locally isomorphic to the first prolongation of $\mathscr{I}$.

Thus, in order to classify the non-Goursat parabolic systems which admit a conservation law, it suffices to consider parabolic Monge-Ampere systems $\mathscr{I}$ on 5-manifolds which can be generated by a contact form θ_0 and a *closed* nondegenerate 2-form Υ.

THEOREM 3. *Let $\mathscr{I}$ be a parabolic Monge-Ampere system on a 5-manifold M which is non-Goursat and which admits at least one nontrivial conservation law. Then M can be covered by open sets U on which there exists a local coordinate system (x, t, u_0, u_1, u_2) and a function f so that $\mathscr{I} = \{\theta, d\theta, du_0 \wedge dx + du_1 \wedge dt\}$ where*

$$\theta = du_0 - 2f\,dx - u_2\,dt - \frac{\partial f}{\partial u_2}(du_1 - u_2\,dx).$$

Conversely, if f is any function on an open set $U \subset \mathbb{R}^5$ endowed with coordinates (x, t, u_0, u_1, u_2) so that θ defined by the above formula satisfies $\theta \wedge (d\theta)^2 \neq 0$, then the system $\mathscr{I} = \{\theta, d\theta, du_0 \wedge dx + du_1 \wedge dt\}$ is a parabolic, non-Goursat, Monge-Ampere system on U which admits a conservation law. Moreover, the Monge-Ampere invariant T vanishes if and only if f is at most quadratic in u_2.

Proof. As we have already seen, a parabolic Monge-Ampere system $\mathscr{I}$ on M^5 admits a conservation law if and only if it can be generated algebraically by a set $\{\theta, d\theta, \Upsilon\}$ where θ is a contact form and Υ is a closed, nondecomposable 2-form which satisfies the conditions:

$$\theta \wedge (\Upsilon)^2 = \theta \wedge d\theta \wedge \Upsilon = 0.$$

By Darboux's theorem, it is always possible to choose local independent functions x, t, u_0, and u_1 so that

$$\Upsilon = du_0 \wedge dx + du_1 \wedge dt.$$

(The ambiguity in the choice of such functions is parameterized by the pseudo group of local symplectic transformations.) Since $\theta \wedge (\Upsilon)^2 = 0$, it follows that θ is in the linear span of the 1-forms $\{dx, dt, du_0, du_1\}$. By making the choice of symplectic coordinates x, t, u_0, and u_1 sufficiently generic, we may assume that $\theta \wedge dx \wedge dt \wedge du_1 \neq 0$, so by rescaling θ we may write θ in the form

$$\theta = du_0 - p_1\,dx - p_2\,dt - p_3\,du_1$$

for some local functions p_1, p_2, and p_3. Since θ is a contact form, at least one of the functions p_i has its differential independent from $\{dx, dt, du_0, du_1\}$. Again, by making a suitably generic choice of local symplectic coordinates, we may assume that the 1-forms $\{dx, dt, du_0, du_1, dp_2\}$ are linearly independent. Let us rename p_2 as u_2. For reasons which will become clear in a moment, we will also write $p_3 = g$ and $p_1 + u_2 p_3 = 2f$. Then θ takes the form

$$\theta = du_0 - 2f\,dx - u_2\,dt - g(du_1 - u_2\,dx),$$

where f and g are (at the moment) arbitrary functions of the five coordinate functions (x, t, u_0, u_1, u_2).

Expanding the condition $\theta \wedge d\theta \wedge \Upsilon = 0$ in coordinates yields

$$0 = 2(df - g\,du_2) \wedge du_0 \wedge du_1 \wedge dt \wedge dx.$$

It follows that g must be the partial derivative of f with respect to u_2. This establishes the first part of the theorem.

To establish the converse, suppose that f is a function on $U \subset \mathbb{R}^5$ endowed with the coordinates (x, t, u_0, u_1, u_2) and that f satisfies the open condition that, with θ defined as above, one has $\theta \wedge (d\theta)^2 \neq 0$. Then the system $\mathscr{I} = \{\theta, d\theta, \Upsilon\}$, where $\Upsilon = du_0 \wedge dx + du_1 \wedge dt$, is easily shown to satisfy the necessary conditions to be a parabolic Monge-Ampere system on U. Namely, θ is a contact form, $d\theta$ and Υ are linearly independent modulo θ, and the following two identities hold:

$$\theta \wedge (\Upsilon)^2 = \theta \wedge d\theta \wedge \Upsilon = 0.$$

It remains to show that this system is not of Goursat type. To see this, first note that

$$\Upsilon \equiv (du_1 - u_2\,dx) \wedge (dt + f'\,dx) \bmod \theta,$$

where we have written f' to denote the partial derivative of f with respect to u_2. It can be shown by straightforward calculation that the condition that $\mathscr{I}$ have nonvanishing Goursat invariant is simply that the rank-3 Pfaffian system $\mathscr{M} = \{\theta, du_1 - u_2\,dx, dt + f'\,dx\}$ not be completely integrable. However, since

$$d(du_1 - u_2\,dx) = -du_2 \wedge dx \not\equiv 0 \bmod \mathscr{M},$$

it follows that $\mathscr{M}$ is not integrable.

Finally, an easy computation shows that the first derived system $\mathcal{M}'$ of $\mathcal{M}$ is spanned by θ and

$$\omega = dt + f'\,dx + f''(du_1 - u_2\,dx),$$

where f'' denotes the second derivative of f with respect to u_2. One then computes that

$$\left.\begin{aligned} d\theta &\equiv A\,du_1 \wedge dx \\ d\omega &\equiv B\,du_1 \wedge dx + f'''\,du_2 \wedge du_1 - u_2 f'''\,du_2 \wedge dx \end{aligned}\right\} \bmod \theta, \omega,$$

where $A \neq 0$ and f''' is the third derivative of f with respect to u_2. It follows that the Cartan system of $\mathcal{M}'$ is of rank 4 if and only if $f''' \equiv 0$. However, by the structure equations, the vanishing of T is exactly the condition that the Cartan system of $\mathcal{M}'$ have rank 4. □

For the sake of explicitness, we note that the condition that $\theta \wedge (d\theta)^2 \neq 0$ (i.e., that θ be a contact form on U) is equivalent to the (open) condition on f that $Df \neq 0$ where

$$Df = (u_2 f' - 2f)\frac{\partial f'}{\partial u_0} - u_2\frac{\partial f'}{\partial u_1} - 2\left(\frac{\partial f}{\partial t} + u_2\frac{\partial f}{\partial u_0}\right)\frac{\partial f'}{\partial u_2} - \frac{\partial f'}{\partial x} + f'\frac{\partial f'}{\partial t} + 2f'\frac{\partial f}{\partial u_0} + 2\frac{\partial f}{\partial u_1}.$$

It is worth remarking that, in arriving at the normal form of Theorem 3, we essentially made one "generic" choice of symplectic coordinates with respect to the symplectic form Υ. Thus, once a conservation law has been chosen for a given parabolic Monge-Ampere system, the ambiguity in the normal form is that of a choice of symplectic coordinates. Now, it is well known that the local symplectomorphisms in four dimensions depend on one function of four variables, the so-called "generating function" of the canonical transformation. Thus, the normal form is determined up to a choice of one function of four variables. Since the set of allowable functions f in the normal form depends on one function of five variables, it follows that, even up to equivalence, the set of pairs $(\mathcal{I}, \Upsilon)$ depends essentially on one function of five variables.

4. Multiple conservation laws. We now want to study the parabolic equations whose space of conservation laws $\mathcal{C}$ is of dimension greater than 1. By Proposition 3 in §2, such equations would have to satisfy $T = 0$ and $U = 0$. Before beginning a further analysis of these systems, we examine the geometric meaning of the conditions $T = U = 0$.

PROPOSITION 1. *Let $\mathscr{I}$ be a parabolic Monge-Ampere system on a 7-manifold M. Then the system $\mathscr{N} = \{\theta_0, \theta_1, \omega^1, \omega^2\}$ defined relative to any 3-adapted coframing is a Frobenius system. Let $V \subset M$ be any open set for which there exists a submersion σ: $V \to N^4$ whose fibers are the leaves of $\mathscr{N}$ restricted to V. Then the vanishing of T and U on V is the necessary and sufficient condition that there exist a rank-2 subbundle $S \subset \Lambda^2(N)$ whose sections pull back via σ to be linear combinations of the 2-forms*

$$\Upsilon_0 = \theta_0 \wedge \omega^2$$

$$\Upsilon_1 = \theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2.$$

Proof. That the system $\mathscr{N}$ is well defined on M and completely integrable follows immediately from the structure equations on the third-order frame bundle $\mathscr{F}''$. Moreover, these structure equations also imply that, for Υ_0 and Υ_1 as defined in the proposition,

$$\begin{pmatrix} d\Upsilon_0 \\ d\Upsilon_1 \end{pmatrix} = -\begin{pmatrix} \alpha + 2\rho & -\omega^1 \\ -\beta & \alpha + \rho - H\theta_1 \end{pmatrix} \wedge \begin{pmatrix} \Upsilon_0 \\ \Upsilon_1 \end{pmatrix} + \begin{pmatrix} -2T\theta_0 \wedge \theta_1 \wedge \theta_2 \\ (P_1 - U)\theta_0 \wedge \theta_1 \wedge \theta_2 \end{pmatrix}.$$

It follows that there is a well-defined "push-down" of the span of the forms Υ_0 and Υ_1 onto the leaf space of $\mathscr{N}$ if and only if $T = 0$ and $U = P_1$. However, by definition, P_1 is the coefficient of ω^2 in dT; thus $T \equiv 0$ implies $P_1 \equiv 0$. It follows that the "push-down" exists if and only if T and U vanish identically on V. □

Henceforth in this section, we will assume that T and U vanish identically.

Let us say that an open set $V \subset M$ is *admissible* if the leaf space of $\mathscr{N}$ restricted to V is Hausdorff. Clearly, M can be covered by admissible open sets. Since our arguments are local, we may as well assume that M itself is admissible and that there exists a smooth submersion σ: $M \to N^4$ whose fibers are the leaves of $\mathscr{N}$.

PROPOSITION 2. *Suppose that $\mathscr{I}$ is a (non-Goursat) parabolic Monge-Ampere system on M^7 and that $\mathscr{I}$ satisfies $T = U = 0$. Let σ: $M \to N^4$ be the submersion of M onto a 4-manifold N whose fibers are the leaves of $\mathscr{N}$, and let $S \subset \Lambda^2(N)$ denote the "push-down" of the space spanned by the forms Υ_0 and Υ_1. Then the space $\mathscr{C}$ of conservation laws for $\mathscr{I}$ on M is isomorphic to the space of those sections of S over N which, when regarded as 2-forms, are closed.*

Proof. It follows from Proposition 3 in §2 and Proposition 1 above that the space of conservation laws is the space of linear combinations $A\Upsilon_1 + B\Upsilon_0$ which are closed forms. □

With Proposition 2 in mind, we make the following definition.

Definition. A *parabolic structure* on a 4-manifold N is a (smooth) rank-2 subbundle $S \subset \Lambda^2(N)$ with the property that it can be generated locally by a pair of nonzero 2-forms Υ_0 and Υ_1 which satisfy the conditions $(\Upsilon_0)^2 = \Upsilon_0 \wedge \Upsilon_1 = 0$ while $(\Upsilon_1)^2$ is nonzero. We say that S is *non-Goursat* if the rank-2 Pfaffian system associated to the decomposable 2-form Υ_0 is nonintegrable.

Since N is a 4-manifold, there is a natural conformal quadratic form Q on $\Lambda^2(N)$. A parabolic structure is given by a rank-2 subbundle $S \subset \Lambda^2(N)$ such that $Q_{|s}$ has rank equal to 1. Denoting by $L \subset S$ the null line subbundle for Q, the dispersive or non-Goursat condition is that the 2-plane field determined by L should be nonintegrable.

Proposition 2 implies that a non-Goursat parabolic Monge-Ampere system $\mathscr{I}$ on M^7 which satisfies $T = U = 0$ induces a parabolic structure S on the leaf space N of the Frobenius system $\mathscr{N}$. It is easy to see from the structure equations (see below) that S is non-Goursat. Conversely, we claim that a non-Goursat parabolic structure S on a 4-manifold N determines a non-Goursat parabolic Monge-Ampere system $\mathscr{I}$ on an appropriate 7-manifold M.

To see this, suppose that a parabolic non-Goursat structure $S \subset \Lambda^2(N)$ has been specified and let Υ_0 and Υ_1 be local generators for S which satisfy the algebraic conditions as above. Since $\Upsilon_0^2 = 0$ while $\Upsilon_0 \neq 0$, it follows that, locally, there exist 1-forms η_1 and η_2 so that $\Upsilon_0 = \eta_1 \wedge \eta_2$. Since $\Upsilon_1 \wedge \Upsilon_0 = 0$, it follows that there exist 1-forms η_3 and η_4 so that $\Upsilon_1 = \eta_3 \wedge \eta_1 + \eta_4 \wedge \eta_2$. Since $(\Upsilon_1)^2 \neq 0$, it follows that $(\eta_1, \eta_2, \eta_3, \eta_4)$ is a local coframing of N.

The hypothesis that S be non-Goursat is that the system $\{\eta_1, \eta_2\}$ be nonintegrable. In particular, we may assume, by a change of basis, that $d\eta_1 \wedge \eta_1 \wedge \eta_2 = 0$ while $d\eta_2 \wedge \eta_1 \wedge \eta_2$ is nonvanishing. By scaling the generators appropriately, we may even assume that $d\eta_2 \equiv \eta_3 \wedge \eta_4 \mod \eta_1, \eta_2$.

Now introduce a new variable q and let $\theta = \eta_2 - q\eta_1$. Then on the 5-manifold $N \times \mathbb{R}$, it is easy to see that θ is a contact form and that the differential system $\mathscr{I}$ generated by $\{\theta, d\theta, \Upsilon_1\}$ is a (non-Goursat) parabolic Monge-Ampere system. (Basically, it is a partial prolongation of the differential system $\mathscr{S}$ generated on N by the sections of S.)

For all practical purposes, the system $\mathscr{S}$ is the "complete deprolongation" of the original parabolic system $\mathscr{I}$. Calculating with $\mathscr{S}$ is generally simpler than calculating with the original system because the calculations only involve the four "essential" variables.

A handy algorithm. Since we are going to present several examples below, we want to streamline the process of computing the conservation laws. As it stands, computing $\mathscr{S}$ for a given non-Goursat parabolic system $\mathscr{I}$ requires that we set up the coframe bundle $\mathscr{F}$ over a 7-manifold M, compute the structure reduction to 3-adapted coframes and then apply the above discussion. However, it is not really necessary to go through such a roundabout process. We are now going to explain a simple way of computing the "deprolongation" of a parabolic system without setting up the equivalence problem.

First, note that the system $\mathscr{M}_1 = \{\theta_0, \theta_1, \theta_2, \omega^2\}$ is well defined relative to any

0-adapted coframing. This system is not completely integrable however, since we have $d\theta_2 \equiv -\pi_3 \wedge \omega^1 \bmod \mathcal{M}_1$.

The derived system of $\mathcal{M}_1$, denoted $\mathcal{M}_2$, is therefore well defined. The structure equations of a 1-adapted coframe show that for any such coframing, $\mathcal{M}_2 = \{\theta_0, \theta_1, \omega^2\}$. Moreover, in any 1-adapted coframing,

$$\left.\begin{aligned} d\theta_0 &\equiv 0 \\ d\theta_1 &\equiv -\theta_2 \wedge \omega^1 \\ d\omega^2 &\equiv (S_0\theta_3 + S_1\omega^1) \wedge \theta_2 \end{aligned}\right\} \bmod \theta_0, \theta_1, \omega^2.$$

It follows that the Monge-Ampere invariant S_0 vanishes if and only if the derived system of $\mathcal{M}_2$ has rank 2.

Accordingly, let us assume that $\mathcal{I}$ is Monge-Ampere. Then the derived system of $\mathcal{M}_2$, denoted $\mathcal{M}_3$, is of rank 2. Moreover, relative to any 2-adapted coframing, $\mathcal{M}_3 = \{\theta_0, \omega^2\}$. Indeed, in any 2-adapted coframing,

$$\left.\begin{aligned} d\theta_0 &\equiv -\theta_1 \wedge \omega^1 \\ d\omega^2 &\equiv 2(R_3\omega^1 - T\theta_2) \wedge \theta_1 \end{aligned}\right\} \bmod \theta_0, \omega^2.$$

It follows that the derived system of $\mathcal{M}_3$ has rank 1 if and only if T vanishes.

Accordingly, let us assume that T does vanish. Then, in any 3-adapted coframing, ω^2 spans the derived system of $\mathcal{M}_3$. Moreover in any 3-adapted coframing, the Cartan system of $\mathcal{M}_3$ is $\mathcal{N} = \{\omega^2, \theta_0, \omega^1, \theta_1\}$.

Finally, consider the space $\mathcal{Z}$ of 2-forms which are quadratic in $\mathcal{N}$ and which are congruent to zero modulo the systems $\mathcal{M}_3$ and $\mathcal{I}' = \{\theta_0, \theta_1\}$. Relative to a 3-adapted coframing, such a 2-form is of the form

$$\Phi = A\omega^2 \wedge \theta_0 + B_1\omega^1 \wedge \theta_0 + B_2\omega^2 \wedge \theta_1 + C\theta_0 \wedge \theta_1 .$$

The condition that $d\Phi \equiv 0 \bmod \theta_0, \theta_1$ implies that $B_1 = B_2$. Set $B = B_1 = B_2$. It is easy to see now that we have $d\Phi \equiv 0 \bmod \omega_2, \theta_0$. Moreover, the condition that $d\Phi \equiv 0 \bmod \omega^2, \theta_1$ forces $C = 0$. Thus, these two conditions reduce us to a subspace $\mathcal{S} \subset \mathcal{Z}$ which is the set of linear combinations of two 2-forms. By our earlier calculations, this space $\mathcal{S}$ is well defined in the leaf space of $\mathcal{N}$ if and only if the invariant U vanishes.

Thus, to recapitulate, the following algorithm will compute the space $\mathcal{S}$ without (direct) recourse to the equivalence method.

First, write down the rank-4 system $\mathcal{M}_1$ *relative to any 0-adapted coframing. Next, compute the first, second, and third derived systems of* $\mathcal{M}_1$*, labeling them* $\mathcal{M}_2$*,* $\mathcal{M}_3$*, and* $\mathcal{M}_4$*, respectively. Each of these derived systems must have rank one* *less than the preceding one or else $\mathscr{I}$ is not a Monge-Ampere system with $T = 0$. Let $\mathscr{N}$ be the Cartan system of $\mathscr{M}_3$. Let $\mathscr{Z}$ denote the space of 2-forms which are quadratic in $\mathscr{N}$ and are congruent to zero modulo $\mathscr{M}_3$ and $\mathscr{I}' = \{\theta_0, \theta_1\}$. Finally, let $\mathscr{S} \subset \mathscr{Z}$ denote the subspace consisting of those forms $\Phi \in \mathscr{Z}$ which satisfy $d\Phi \equiv 0 \bmod \theta_0, \theta_1$ and $d\Phi \equiv 0 \bmod \omega^2, \theta_1$. This $\mathscr{S}$ is the desired space, and "pushes down" to the leaf space of $\mathscr{N}$ if and only if U also vanishes.*

The upshot of all this is that the handy algorithm gives us a simple method of computing a 2-dimensional span of 2-forms in which all of the conservation laws lie. (In fact, they are precisely the closed 2-forms in the span of these two.) We will now apply this several times in the following examples.

Example 1 (continued). We return to the case of the heat equation shrinking curves on Riemannian surfaces begun in §0. Keeping the notations of that example, we know that any conservation law is of the form

$$\begin{aligned}\Psi &= A[\eta_1 \wedge (\eta_2 - u_2\,dt) + dt \wedge (\eta_{21} - u_2\eta_1 - u_3\,dt)] + B[dt \wedge (\eta_2 - u_2\,dt)] \\ &\quad + (\text{quadratic terms in } \mathscr{I}) \\ &= A(\eta_1 \wedge \eta_2 + dt \wedge \eta_{21}) + B(dt \wedge \eta_2) + (\text{quadratics}).\end{aligned}$$

Rather than go through the equivalence method to eliminate the "quadratics", we shall apply the above algorithm.

To begin with, it is easy to see that $\mathscr{M}_3 = \{\eta_2, dt\}$ with Cartan system $\mathscr{C}(\mathscr{M}_3) = \{\eta_2, dt, \eta_1, \eta_{21}\}$, and that

$$\mathscr{I}' = \{\eta_2 - u_2\,dt, \eta_{21} - u_2\eta_1 - u_3\,dt\}.$$

It follows that the space $\mathscr{Z}$ consists of the set of 2-forms Φ of the form

$$\begin{aligned}\Phi &= B\eta_2 \wedge dt + A_1(\eta_2 - u_2\,dt) \wedge \eta_1 + A_2(\eta_{21} - u_2\eta_1) \wedge dt + C(\eta_2 - u_2\,dt) \\ &\quad \wedge (\eta_{21} - u_2\eta_1 - u_3\,dt).\end{aligned}$$

The condition $d\Phi \equiv 0 \bmod \theta_0, \theta_1$ yields $A_1 = A_2$. Let us denote this common function by A. Then the formula for Φ simplifies to

$$\Phi = B\eta_2 \wedge dt + A(\eta_2 \wedge \eta_1 + \eta_{21} \wedge dt) + C(\eta_2 - u_2\,dt) \wedge (\eta_{21} - u_2\eta_1 - u_3\,dt).$$

The condition that $d\Phi \equiv 0 \bmod \omega^2, \theta_1$ then implies that $C = 0$. Hence, the space $\mathscr{S}$ consists of the 2-forms of the form

$$\Phi = A(\eta_2 \wedge \eta_1 + \eta_{21} \wedge dt) + B\eta_2 \wedge dt.$$

That is, $\mathscr{S}$ is spanned by the 2-forms $\Upsilon_0 = \eta_2 \wedge dt$ and $\Upsilon_1 = \eta_2 \wedge \eta_1 + \eta_{21} \wedge dt$. Note that these are well defined on the 4-manifold $N = F \times \mathbb{R}$.

(In hindsight, it is clear that we could have begun with these forms, since these are the 2-forms which vanish on the "graph" in $F \times \mathbb{R}$ of any solution of the heat equation shrinking curves on the Riemannian surface.)

Now let us compute the conservation laws. The formula for $d\Phi$ becomes

$$d\Phi = dA \wedge (\eta_2 \wedge \eta_1 + \eta_{21} \wedge dt) - AK\eta_1 \wedge \eta_2 \wedge dt + dB \wedge \eta_2 \wedge dt$$
$$- B\eta_{21} \wedge \eta_1 \wedge dt.$$

Reducing first modulo η_2 and then by dt shows that there must exist functions A_0 and A_2 so that

$$dA = A_0\eta_2 + A_2\, dt - B\eta_1 .$$

Substituting this back into the formula for $d\Phi$ shows that there must exist functions B_0 and B_2 so that

$$dB = B_0\eta_2 + B_2\, dt + A_0\eta_{21} + (A_2 + AK)\eta_1 .$$

Now, computing $d(dA) \equiv 0 \bmod \eta_2,\ dt$ yields that $A_0 = 0$. Thus, $dA = A_2\, dt - B\eta_1$. Using this simplification, computing $d(dA) \equiv 0 \bmod dt$ yields $B = B_0 = 0$. Substituting these relations into the above formula for $d(B)$ yields $A_2 = -AK$. Consequently, $dA = -AK\, dt$, and hence $d(dA) = -A\, dK \wedge dt$. Of course, this latter equation implies either that $A = 0$ (in which case, there are no conservation laws) or else that $dK = 0$, i.e., that the metric on the surface has constant Gauss curvature.

Thus, the final result of our calculations is that the parabolic system which represents the heat equation shrinking curves on a Riemannian surface either has no conservation laws (if the surface does not have constant Gauss curvature) or else has a 1-dimensional space of conservation laws represented by

$$\Phi = e^{-Kt}(\eta_1 \wedge \eta_2 + dt \wedge \eta_{21})$$

if the Gauss curvature K is constant. (Note that this matches the above formula for Φ with $A = e^{-Kt}$ and $B = 0$. For comparison with the formulas in §1, note that, according to (9) in §1, one has, in general, that $B = -(D_1 A + \beta A)$. Since, in this example, $D_1 A = 0$ and $\beta = 0$, this is in accord with the general theory.)

When $K \neq 0$ is constant, we see that $\Phi = (-1/K)d(e^{-Kt}\eta_{21})$, and this has the interpretation that, for any solution Γ_t of the heat equation shrinking a curve on such a surface, the integral

$$e^{-Kt}\int_{\Gamma_t} \kappa\, ds$$

remains constant.

When $K = 0$ and S is the flat xy-plane, the conservation law has the following meaning: We use coordinates (x, y, θ) in F so that

$$\begin{aligned} \eta_1 \wedge \eta_2 &= dx \wedge dy \\ \eta_{21} &= +d\theta \\ \Phi &= dx \wedge dy + dt \wedge d\theta \\ &= d\left[\frac{1}{2}(x\,dy - y\,dx) + t\,d\theta\right]. \end{aligned}$$

If $\Gamma_t \subset \mathbb{E}^2$ is in a family of curves evolving according to the equation

$$\frac{\partial \Gamma_t}{\partial t} = \kappa N \tag{1}$$

and with fixed endpoints, then the actual conservation law is given by

$$\frac{d}{dt}\left[\frac{1}{2}\int_{\Gamma_t}(x\,dy - y\,dx) + t\int_{\Gamma_t} d\theta\right] = 0.$$

In particular, if Γ_t is an embedded closed curve with enclosed area A_t we obtain

$$\frac{dA_t}{dt} = -2\pi,$$

so

$$A_t = A_0 - 2\pi t.$$

This "conservation law" was discussed by Gauge and Hamilton [GH] in connection with the problem of studying the PDE (1). In [Gr], it is proved that such a Γ_t shrinks to fixed point $P_0 = (x_0, y_0)$ in finite time $t_0 = A_0/2\pi$.

An open question has been whether x_0 and y_0 may be expressed as an integral around Γ_t of some expression in the components of the position vector and their derivatives. In other words, does there exist a formula

$$x_0 = \int_{\Gamma_t} F(x, y, x', y', \ldots, x^{(k)}, y^{(k)}, t)\,ds \tag{2}$$

and similarly for y_0? We are thus asking whether P_0 is some sort of generalized "center of mass" knowable from local information along Γ_t? The answer is *no*, since if this were the case then (2) would give a second independent conservation law for (1), and we have proved that there is exactly one such law up to constant multiples.

Example 4: Linear parabolics. Consider a linear parabolic equation of the form

$$u_t = A(x, t)u_{xx} + B(x, t)u_x + C(x, t)u,$$

where we assume that A is positive everywhere. It is not difficult to see that by changing the independent variable x and appropriately rescaling u, we can locally arrange that the equation simplifies to $u_t = u_{xx} + Cu$.

We want to compute the space of conservation laws for this equation. Let p stand for u_x; then the solutions of this equation are the integral manifolds of the following pair of 2-forms on $\mathbb{R}^4$:

$$\Upsilon_0 = (du - p\,dx) \wedge dt$$

$$\Upsilon_1 = (du - Cu\,dt) \wedge dx + dp \wedge dt$$

(this result could also have been obtained by computing the deprolongation of the "natural" system on $\mathbb{R}^7$ associated to this equation). A conservation law will then be represented by a closed 2-form of the form

$$\Phi = A((du - Cu\,dt) \wedge dx + dp \wedge dt) + B(du - p\,dx) \wedge dt.$$

We have

$$d\Phi = dB \wedge (du - p\,dx) \wedge dt - B\,dp \wedge dx \wedge dt + dA$$
$$\wedge ((du - Cu\,dt) \wedge dx + dp \wedge dt) - AC\,du \wedge dt \wedge dx.$$

Reducing modulo dt, we see that $dA \equiv 0 \bmod du, dx, dt$. Reducing modulo $du - p\,dx - Cu\,dt$, we see that $B = -(A_x + pA_u)$. Substituting this back into the formula for Φ, we see that the coefficient of the $dp \wedge du \wedge dt$ term is $-2A_u$. Thus, we must have $A_u = 0$. Substituting this back into the formula, we finally get that

$$0 = d\Phi = (A_t + A_{xx} + CA)\,du \wedge dx \wedge dt.$$

Thus, the space $\mathscr{C}$ of conservation laws for this equation is isomorphic to the space of 2-forms of the form

$$\Phi = -A_x(du - p\,dx) \wedge dt + A((du - Cu\,dt) \wedge dx + dp \wedge dt),$$

where A is a function of x and t which satisfies the "backwards" equation $A_t + A_{xx} + CA = 0$. In particular, note that $\mathscr{C}$ is infinite-dimensional.

Example 5: Two conservation laws. In this example, we exhibit a parabolic structure on $\mathbb{R}^4$ which admits a 2-dimensional space of conservation laws and no more.

Let N be the simply connected Lie group of dimension 4 which possesses a basis $(\omega^2, \theta_0, \omega^1, \theta_1)$ of left-invariant 1-forms which satisfy the structure equations

$$d\omega^2 = -\theta^1 \wedge \theta_0 \qquad d\omega^1 = 0$$

$$d\theta_0 = -\theta_1 \wedge \omega^1 \qquad d\theta_1 = 0.$$

Clearly, N is diffeomorphic to $\mathbb{R}^4$. Let $S \subset \Lambda^2(N)$ be the rank-2 subbundle for which the forms $\Upsilon_0 = \theta_0 \wedge \omega^2$ and $\Upsilon_1 = \theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2$ give a basis for the sections. Then S clearly satisfies our hypotheses to be a non-Goursat parabolic structure on N. We are going to show that S admits a 2-dimensional space of conservation laws. Set

$$\Phi = A(\theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2) + B\theta_0 \wedge \omega^2,$$

and assume that $d\Phi = 0$. It is easy to compute that

$$d\Phi = dA \wedge (\theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2) + dB \wedge \theta_0 \wedge \omega^2 - B\theta_1 \wedge \omega^1 \wedge \omega^2.$$

In particular, if $d\Phi = 0$, it follows that $d\Phi \wedge \omega^2 = dA \wedge \theta_0 \wedge \omega^1 \wedge \omega^2 = 0$, so dA must be a linear combination of θ_0, ω^1, and ω^2. Moreover, when we substitute such a combination for dA into the equation $d\Phi = 0$, we find that the coefficient of ω^1 must be $-B$. Thus, there exist functions A_0 and A_2 so that

$$dA = A_0\theta_0 + A_2\omega^2 - B\omega^1.$$

Substituting this back into $d\Phi = 0$ yields that there must exist functions B_0 and B_2 so that

$$dB = B_0\theta_0 + B_2\omega^2 + A_0\theta_1 + A_2\omega^1.$$

Now, we compute that

$$\begin{aligned} 0 &= d(dA) \\ &= dA_0 \wedge \theta_0 + dA_2 \wedge \omega^2 - A_0\theta_1 \wedge \omega^1 - A_2\theta_1 \wedge \theta_0 \\ &\quad - (B_0\theta_0 + B_2\omega^2 + A_0\theta_1) \wedge \omega^1. \end{aligned}$$

Of course, reducing modulo θ_0 and ω^2 implies that $A_0 = 0$. The formula now simplifies to

$$0 = dA_2 \wedge \omega^2 - A_2\theta_1 \wedge \theta_0 - (B_0\theta_0 + B_2\omega^2) \wedge \omega^1.$$

Reducing this modulo ω^2 shows that $B_0 = A_2 = 0$. Finally, substituting this back into the above equation shows that $B_2 = 0$ as well. The equations now reduce to $dA = -B\omega^1$ and $dB = 0$. Of course, these equations are compatible, and they possess a 2-dimensional space of solutions (A, B). Thus, the space of conservation laws for the parabolic system $\mathscr{S} = \{\Upsilon_0, \Upsilon_1\}$ is of dimension 2.

Example 6: Three conservation laws. In this example, we exhibit a parabolic structure on $\mathbb{R}^4$ which admits a 3-dimensional space of conservation laws and no more. (By Theorem 1 below, no equation can admit more than three conservation laws unless it is linear.)

Let N be the simply connected Lie group of dimension 4 which possesses a basis $(\omega^2, \theta_0, \omega^1, \theta_1)$ of left-invariant 1-forms which satisfy the structure equations

$$d\omega^2 = -2\omega^1 \wedge \theta_0 \qquad d\omega^1 = -\theta_1 \wedge \theta_0$$

$$d\theta_0 = -\theta_1 \wedge \omega^1 \qquad d\theta_1 = 0.$$

(Since N is solvable and simply connected, it is diffeomorphic to $\mathbb{R}^4$.) Let $S \subset \Lambda^2(N)$ be the rank-2 subbundle for which the forms $\Upsilon_0 = \theta_0 \wedge \omega^2$ and $\Upsilon_1 = \theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2$ give a basis for the sections. The structure equations on N imply that S is a non-Goursat parabolic structure on N.

We are going to show that S admits a 3-dimensional space of conservation laws. Set

$$\Phi = A(\theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2) + B\theta_0 \wedge \omega^2,$$

and assume that $d\Phi = 0$. It is easy to compute that

$$\begin{aligned} 0 &= d\Phi \\ &= dA \wedge (\theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2) + 2A\theta_1 \wedge \omega^1 \wedge \theta_0 \\ &\quad + dB \wedge \theta_0 \wedge \omega^2 - B\theta_1 \wedge \omega^1 \wedge \omega^2. \end{aligned}$$

Reducing first modulo θ_0 and then modulo ω^2 shows that there must exist functions A_0 and A_2 so that

$$dA = A_0\theta_0 + A_2\omega^2 - B\omega^1 + 2A\theta_1.$$

Substituting this back into the formula for $d\Phi$ yields that there must exist functions B_0 and B_2 so that

$$dB = B_0\theta_0 + B_2\omega^2 + A_0\theta_1 + A_2\omega^1.$$

Now, we compute that

$$\begin{aligned} 0 &= d(dA) \\ &= dA_0 \wedge \theta_0 + dA_2 \wedge \omega^2 - A_0\theta_1 \wedge \omega^1 - 2A_2\omega^1 \wedge \theta_0 \\ &\quad - (B_0\theta_0 + B_2\omega^2 + A_0\theta_1) \wedge \omega^1 + B\theta_1 \wedge \theta_0 + 2(A_0\theta_0 + A_2\omega^2 - B\omega^1) \wedge \theta_1 . \end{aligned}$$

Reducing this modulo θ_0 and ω^2 yields $A_0 = B$. The formula now simplifies to

$$\begin{aligned} 0 = (B_2\omega^2 + A_2\omega^1) \wedge \theta_0 + dA_2 \wedge \omega^2 - 2A_2\omega^1 \wedge \theta_0 - (B_0\theta_0 + B_2\omega^2) \wedge \omega^1 \\ + 2A_2\omega^2 \wedge \theta_1 . \end{aligned}$$

Reducing this modulo ω^2 yields $B_0 = A_2$. Substituting this back into the above equation yields

$$0 = (dA_2 - 2A_2\theta_1 + B_2(\omega^1 - \theta_0)) \wedge \omega^2.$$

Thus, there must exist a function A_3 so that

$$dA_2 = 2A_2\theta_1 - B_2(\omega^1 - \theta_0) + A_3\omega^2.$$

Meanwhile, we have the formula $dB = B_2\omega^2 + B\theta_1 + A_2(\omega^1 + \theta_0)$. Differentiating this relation and reducing modulo ω^2 yields the relation

$$0 = d(dB) \equiv -4B_2\omega^1 \wedge \theta_0 .$$

Thus, $B_2 = 0$. Substituting this back into $d(dB)$ 0 yields $A_3 = 0$. Setting $A_2 = C$, we get the final formulas

$$\begin{aligned} dA &= 2A\theta_1 + B(\theta_0 - \omega^1) + C\omega^2 \\ dB &= B\theta_1 + C(\theta_0 + \omega^1) \\ dC &= 2C\theta_1 . \end{aligned}$$

This is a Frobenius system. Thus, the space $\mathscr{C}$ of conservation laws of $\mathscr{S}$ has dimension equal to 3, as promised.

Four conservation laws imply linearizability. At this juncture, we have found examples of parabolic systems with $\dim \mathscr{C}$ equal to 0, 1, 2, 3, or ∞. We will now prove the following theorem.

THEOREM 1. *Let $\mathscr{I}$ be a parabolic system on a 7-manifold M. Suppose that $\dim \mathscr{C} \geqslant 4$. Then $\mathscr{I}$ is contact-equivalent to the exterior differential system arising from a linear parabolic partial differential equation.*

By Proposition 2 and the ensuing discussion, we are reduced to considering a non-Goursat, Monge-Ampere parabolic structure on a 4-manifold N. We shall investigate the implications on the structure equations of a system for which $\dim \mathscr{C} \geqslant 2$, 3, or 4. The last case will clearly put the most conditions on the system, and these conditions will suffice to give the proof of the linearization result.

The calculations done in the course of the proof will then be utilized to derive a normal form for those parabolic systems for which $\dim \mathscr{C} = 3$. Although we are able to say "how many" systems there are with $\dim \mathscr{C} = 2$, we do not yet have a complete normal for such systems. However, see below for a "rough" normal form.

Proof. We will use coframings $\{\omega^2, \theta_0, \omega^1, \theta_1\}$ on N, where the conditions that define these coframings will be given momentarily. The assumptions $\dim \mathscr{C} \geqslant 2, 3, 4$ will lead to successive adaptations of these coframings.

The notation is chosen for the following reason: The original parabolic system $\mathscr{I}$ was given on a 7-manifold M on which we have defined the class of 3-adapted coframings. Under the projection $\pi\colon M \to N$ and after our frame adaptations on N, the frames $\{\omega^2, \theta_0, \omega^1, \theta_1\}$ will pull back to that part of a 3-adapted coframing $\{\theta_0, \theta_1, \theta_2, \theta_3, \omega^1, \omega^2\}$ on M that is indicated by the notation (thus $\pi^*(\omega^2) = \omega^2$, etc.).

On N, we choose two linearly independent closed 2-forms Υ_1, Υ_2 in the parabolic structure and a function L such that

$$\Omega = \Upsilon_1 - L\Upsilon_2$$

is decomposable. Since the parabolic system on N is assumed to be non-Goursat, it follows that $d\Omega$ is not a multiple of the 2-form Ω.

(*Proof.* Writing $\Omega = \eta_1 \wedge \eta_2$, if $d\Omega = \alpha \wedge \Omega$, then $d\eta_1 \wedge \eta_2 - \eta_1 \wedge d\eta_2 = \alpha \wedge \eta_1 \wedge \eta_2$, which gives $d\eta_1 \wedge \eta_1 \wedge \eta_2 = 0$ or $d\eta_1 \equiv 0 \bmod \eta_1, \eta_2$, and similarly for η_2. This contradicts the nonintegrability of the Pfaffian system $\{\eta_1, \eta_2\}$.)

We now set $\Upsilon = \Upsilon_2$ and choose our coframing on N so that

$$\Omega = \theta_0 \wedge \omega^2$$

$$\Upsilon = \theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2.$$

Using $\Omega \wedge \Upsilon = 0$ and $\Upsilon \wedge \Upsilon \neq 0$, it is easy to see that this may be done. The Pfaffian system $\{\theta_0, \omega^2\}$ is nonintegrable, and therefore its first derived system has rank 1 (the only possibilities are rank 1 and rank 2). Choosing ω^2 as a generator for this system, we have

$$\left.\begin{aligned} d\omega^2 &\equiv 0 \\ d\theta_0 &\equiv M\theta_1 \wedge \omega^1 \end{aligned}\right\} \bmod \theta_0, \omega^2 \tag{3}$$

for some nonzero function M. Since

$$\Upsilon \equiv \theta_0 \wedge \omega^1 \bmod \omega^2,$$

we infer that the flag

$$\{\omega^2\} \subset \{\omega^2, \theta_0\} \subset \{\omega^2, \theta_0, \omega^1\} \subset \{\omega^2, \theta_0, \omega^1, \theta_1\}$$

is well defined. Allowable frame changes preserving these coframing adaptations (especially the expressions for Ω and Υ) are of the form

$$\begin{bmatrix} \hat{\omega}^2 \\ \hat{\theta}_0 \\ \hat{\omega}^1 \\ \hat{\theta}_1 \end{bmatrix} = \begin{bmatrix} r & 0 & 0 & 0 \\ b & r^{-1} & 0 & 0 \\ c & * & r & 0 \\ * & c' & b & r^{-1} \end{bmatrix} \begin{bmatrix} \omega^2 \\ \theta_0 \\ \omega^1 \\ \theta_1 \end{bmatrix}. \tag{4}$$

It follows that

$$\hat{M} = r^{-1}M,$$

and since $M \neq 0$, we may further adapt the coframings so as to have $M = -1$, which we do. It follows that the remaining allowable coframe changes in (4) satisfy $r = 1$. Now, from equations (3), it follows that there exists a 1-form ψ so that

$$d(\Omega) = d(\theta_0 \wedge \omega^2) = -\theta_1 \wedge \omega^1 \wedge \omega^2 + \psi \wedge \theta_0 \wedge \omega^2.$$

Using this and the relations

$$d\Upsilon_1 = d(\Omega + L\Upsilon) = 0$$

$$d\Upsilon_2 = d\Upsilon = 0,$$

we infer that

$$0 = d\Omega + dL \wedge \Upsilon = \psi \wedge \theta_0 \wedge \omega^2 + (dL + \omega^1) \wedge (\theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2).$$

Wedging this latter equation first with θ_0 and then with ω^2 shows that there must be a congruence of the form

$$dL + \omega^1 \equiv 0 \bmod \theta_0, \omega^2.$$

It follows that we may now further restrict our coframings by requiring that

$$\omega^1 = -dL. \tag{5}$$

The remaining admissible changes of coframing are then

$$\begin{bmatrix} \hat{\omega}^2 \\ \hat{\theta}_0 \\ \hat{\omega}^1 \\ \hat{\theta}_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ s & 0 & b & 1 \end{bmatrix} \begin{bmatrix} \omega^2 \\ \theta_0 \\ \omega^1 \\ \theta_1 \end{bmatrix}. \tag{6}$$

(The "0" in the bottom row results from $\hat{\Upsilon} = \Upsilon$.)

We will now deduce restrictions on the structure equations of any such coframing. From (3) and (5), we have

$$\begin{aligned} d\omega^2 &= \rho \wedge \omega^2 + (G\theta_1 + H\omega^1) \wedge \theta_0 \\ d\theta_0 &= \beta \wedge \omega^2 + \alpha \wedge \theta_0 - \theta_1 \wedge \omega^1 \\ d\omega^1 &= 0 \\ d\theta_1 &= \sigma \wedge \omega^2 + \gamma \wedge \theta_0 + \varepsilon \wedge \omega^1 + \phi \wedge \theta_1 \end{aligned}$$

where α, β, γ, ε, ϕ, ρ, and σ are (so far) nonuniquely defined 1-forms and G and H are functions. From $d\Upsilon_1 = d\Upsilon = 0$ we deduce the relations

$$(\alpha + \rho) \wedge \theta_0 \wedge \omega^2 = 0$$

$$\gamma \wedge \theta_0 \wedge \omega^2 + (\varepsilon - \beta) \wedge \omega^1 \wedge \omega^2 + (\alpha + H\theta_1) \wedge \theta_0 \wedge \omega^1 + (\phi + \rho) \wedge \theta_1 \wedge \omega^2 = 0.$$

Wedging the second equation with ω^2, we see that

$$\alpha + H\theta_1 \equiv 0 \bmod \theta_0, \omega^1, \omega^2.$$

Now, in $d\theta_0$, we may absorb all of the ω^2-terms into $\beta \wedge \omega^2$, and thus, we may assume that

$$\alpha = -F\omega^1 - H\theta_1$$

$$\rho = R\theta_0 + F\omega^1 + H\theta_1$$

for some functions F and R. This makes β well defined modulo ω^2, and the second closure condition simplifies to

$$(\gamma - R\theta_1) \wedge \theta_0 + (\varepsilon - \beta) \wedge \omega^1 + (\phi + F\omega^1) \wedge \theta_1 \equiv 0 \bmod \omega^2.$$

By absorbing all of the ω^2-terms in $d\theta_1$ into $\sigma \wedge \omega^2$, we may assume that this equation holds identically, not just modulo ω^2. This then gives

$$\gamma \wedge \theta_0 + \varepsilon \wedge \omega^1 + \phi \wedge \theta_1 = R\theta_1 \wedge \theta_0 + \beta \wedge \omega^1 - F\omega^1 \wedge \theta_1 .$$

The above structure equations now become

$$\begin{aligned}
d\omega^2 &= (R\theta_0 + F\omega^1 + H\theta_1) \wedge \omega^2 + (G\theta_1 + H\omega^1) \wedge \theta_0 \\
d\theta_0 &= \beta \wedge \omega^2 - (F\omega^1 + H\theta_1) \wedge \theta_0 - \theta_1 \wedge \omega^1 \\
d\omega^1 &= 0 \\
d\theta_1 &= \sigma \wedge \omega^2 + \beta \wedge \omega^1 + R\theta_1 \wedge \theta_0 - F\omega^1 \wedge \theta_1 .
\end{aligned}$$

Under a frame change of the form (6), we have

$$\begin{aligned}
G &= \hat{G} \\
H &= \hat{H} + b\hat{G} \\
F &= \hat{F} + 2b\hat{H} + b^2\hat{G} \\
R &= \hat{R} - s\hat{G}.
\end{aligned}$$

It follows that G is an invariant. We shall see below that if $\dim \mathscr{C} \geqslant 3$, then necessarily $G = 0$ and then each of H and R becomes an invariant.

We will now proceed to the relevant calculations. Suppose that

$$\Phi = A\Upsilon + B\Omega$$

is a closed 2-form. This is an overdetermined system of four linear equations for the two unknown functions A and B. By assumption/construction, we know that

$(A, B) = (1, 0)$ and $(A, B) = (L, 1)$ give two linearly independent solutions. In general, we have, for the exterior derivative of Φ,

$$
\begin{aligned}
d\Phi &= dB \wedge \theta_0 \wedge \omega^2 + Bd(\theta_0 \wedge \omega^2) + dA \wedge (\theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2) \\
&= dB \wedge \theta_0 \wedge \omega^2 + B((-(F\omega^1 + H\theta_1) \wedge \theta_0 - \theta_1 \wedge \omega^1) \wedge \omega^2 \\
&\quad - \theta_0 \wedge (F\omega^1 + H\theta_1) \wedge \omega^2) + dA \wedge (\theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2) \\
&= dB \wedge \theta_0 \wedge \omega^2 + (B\omega^1 + dA) \wedge (\theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2).
\end{aligned}
$$

The vanishing of $d\Phi$ is therefore equivalent to the existence of functions A_0, A_2, B_0, and B_2 so that

$$
\begin{aligned}
dA &= A_0\theta_0 + A_2\omega^2 - B\omega^1 \\
dB &= B_0\theta_0 + B_2\omega^2 + A_2\omega^1 + A_0\theta_1 .
\end{aligned}
$$

This is an overdetermined system of linear differential equations for A and B. It is the integrability implications of assuming that this system has at least four linearly independent solutions that we shall be investigating.

Taking the exterior derivative of both sides of the equations $dA = A_0\theta_0 + A_2\omega^2 - B\omega^1$ and reducing modulo θ_0 and ω^2 gives

$$0 \equiv -2A_0\theta_1 \wedge \omega^1 \bmod \theta_0, \omega^2,$$

which implies

$$A_0 = 0.$$

Thus

$$
\begin{aligned}
dA &= A_2\omega^2 - B\omega^1 \\
dB &= B_0\theta_0 + B_2\omega^2 + A_2\omega^1 .
\end{aligned}
$$

Repeating this calculation, but now only reducing modulo ω^2 gives

$$(A_2H + B_0)\omega^1 \wedge \theta_0 + A_2G\theta_1 \wedge \theta_0 \equiv 0 \bmod \omega^2,$$

which implies

$$A_2G = A_2H + B_0 = 0.$$

The case $G \neq 0$. This is the general case; as Example 5 above shows, it may occur for a parabolic system with exactly two conservation laws. Since $G \neq 0$, the last two equations above imply that

$$A_2 = B_0 = 0.$$

From $d(dA) = 0$ we then get $B_2 = 0$, which implies that B must be constant. It now follows easily that Φ is a linear combination (with constant coefficients) of Υ_1 and Υ. In particular, $\dim \mathscr{C} = 2$.

The case $G = 0$. Then we still have

$$B_0 = -A_2 H. \tag{7}$$

From $d(dA) = 0$, we now have

$$(dA_2 + A_2 R\theta_0 + A_2 H\theta_1 + (A_2 F + B_2)\omega^1) \wedge \omega^2 = 0,$$

and thus there is a function C so that

$$dA_2 = C\omega^2 - A_2(R\theta_0 + F\omega^1 + H\theta_1) - B_2\omega^1. \tag{8}$$

Since $G = 0$, it follows that H is an invariant (as is R), and we shall investigate dH. From $d(d\omega^2) \equiv 0 \bmod \omega^2$, we have

$$(dH - 2H^2\theta_1) \wedge \omega^1 \wedge \theta_0 \equiv 0 \bmod \omega^2,$$

which implies that

$$dH = H_2\omega^2 + H_0\theta_0 + H_1\omega^1 + 2H^2\theta_1$$

for some functions H_0, H_1, and H_2. Next, the identity $d(dB) \equiv 0 \bmod \omega^2$ expands to

$$(2B_2 H + A_2(R + 2HF - H_1))\omega^1 \wedge \theta_0 \equiv 0 \bmod \omega^2,$$

which implies

$$2B_2 H + A_2(R + 2HF - H_1) = 0. \tag{9}$$

Now, under a change of frame of the form (6), we have

$$H_1 = \hat{H}_1 + 2\hat{H}^2 b$$

$$F = \hat{F} + 2\hat{H} b.$$

The subcase $H \neq 0$. Under the assumption that $H \neq 0$, we may adapt our coframings so that

$$H_1 = 2HF + R.$$

Then (9) simplifies to $2B_2H = 0$, so $B_2 = 0$. Our equations now simplify to

$$dA = A_2\omega^2 - B\omega^1$$

$$dB = A_2(\omega^1 - H\theta_0).$$

Taking the exterior derivative of these equations, we infer that there is an equation of the form

$$dA_2 = A_2\psi,$$

where ψ is a 1-form computed in terms of the coframing only (i.e., in this coframe, it is the same for all solutions of the conservation law equations). Combining everything gives

$$d\begin{bmatrix} A \\ B \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 & -\omega^1 & \omega^2 \\ 0 & 0 & \omega^1 - H\theta_0 \\ 0 & 0 & \psi \end{bmatrix} \begin{bmatrix} A \\ B \\ A_2 \end{bmatrix}.$$

This system of total differential equations can have at most a 3-dimensional space of solutions, and hence we see that, in the case where $G = 0$ and $H \neq 0$, we must have $\dim \mathscr{C} \leqslant 3$. We will return to this case following completion of the analysis when $\dim \mathscr{C} \geqslant 4$.

The subcase $H = 0$. We now suppose that $H = 0$. Suppose first that R were nonzero. Then (9) would imply that $A_2 = 0$, and then from (7) and (8) we would be able to conclude that $B_0 = B_2 = 0$, which would in turn imply that B is a constant and that $dA = -B\omega^1$. It would, of course, then follow that $\dim \mathscr{C} = 2$.

Thus, if we are to have $\dim \mathscr{C} \geqslant 3$ in this subcase, we must assume that $R = 0$, which we do from now on. With $G = H = R = 0$, the structure equations take the form

$$d\omega^2 = F\omega^1 \wedge \omega^2$$

$$d\theta_0 = \beta \wedge \omega^2 - F\omega^1 \wedge \theta_0 - \theta_1 \wedge \omega^1$$

$$d\omega^1 = 0$$

$$d\theta_1 = \sigma \wedge \omega^2 + \beta \wedge \omega^1 - F\omega^1 \wedge \theta_1.$$

We will show that these equations may be "integrated" leading to a linear parabolic partial differential equation.

We begin by setting $\omega^1 = dx$. From $d\omega^2 \wedge \omega^2 = 0$, we infer that there exist functions f and t such that $\omega^2 = e^f\, dt$. Then

$$d\omega^2 = df \wedge \omega^2 = F\omega^1 \wedge \omega^2 = Fe^f\, dx \wedge dt,$$

which gives

$$(df - F\,dx) \wedge dt = 0,$$

implying

$$df \equiv F\,dx \bmod dt.$$

It follows that $f = f(x, t)$ is a function of x and t and that $F = f_x$.

Next, we have

$$\begin{aligned} d(e^f\theta_0) &= e^f(\beta \wedge \omega^2 - F\omega^1 \wedge \theta_0 - \theta_1 \wedge \omega^1) + (F\,dx + f_t\,dt) \wedge e^f\theta_0 \\ &= (e^f\beta - f_t\theta_0) \wedge \omega^2 - (e^f\theta_1) \wedge \omega^1 \\ &= \tilde{\beta} \wedge dt - \tilde{\theta}_1 \wedge dx, \end{aligned}$$

where $\tilde{\beta} = e^{2f}\beta - e^f f_t\theta_0$ and $\tilde{\theta}_1 = e^f\theta_1$. Since

$$\tilde{\beta} \wedge dt - \tilde{\theta}_1 \wedge dx = d(e^f\theta_0)$$

is closed and of maximal rank, it follows from Darboux's Theorem that there exist functions p and q such that

$$\tilde{\theta}_1 \wedge dx - \tilde{\beta} \wedge dt = dp \wedge dx + dq \wedge dt.$$

By elementary linear algebra (in the form of Cartan's Lemma), there are functions s_1, s_2, and s_3, so that

$$\begin{pmatrix} \tilde{\theta}_1 \\ -\tilde{\beta} \end{pmatrix} = \begin{pmatrix} dp \\ dq \end{pmatrix} + \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix} \begin{pmatrix} dx \\ dt \end{pmatrix}.$$

Moreover, from $d(e^f\theta_0 + p\,dx + q\,dt) = 0$, it follows that

$$e^f\theta_0 = du - p\,dx - q\,dt$$

for some function u. This gives

$$\theta_1 = e^{-f}(dp + s_1\,dx + s_2\,dt)$$

$$\theta_0 \wedge \omega^2 = (du - p\,dx) \wedge dt$$

$$\begin{aligned}\theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2 &= e^{-f}(du - q\,dt) \wedge dx + (dp + s_1\,dx) \wedge dt\\ &= e^{-f}\,du \wedge dx + dp \wedge dt + (s_1 + e^{-f}q)\,dx \wedge dt.\end{aligned}$$

From the structure equations,

$$\begin{aligned}d\theta_1 &\equiv (F\theta_1 + \beta) \wedge dx \bmod dt\\ &\equiv (Fe^{-f}\,dp + \beta) \wedge dx \bmod dt.\end{aligned}$$

But also, by differentiating the above expression for θ_1,

$$\begin{aligned}d\theta_1 &\equiv -e^{-f}(F\,dx) \wedge (dp + s_1\,dx) + e^{-f}\,ds_1 \wedge dx \bmod dt\\ &\equiv (Fe^{-f}\,dp + e^{-f}\,ds_1) \wedge dx \bmod dt.\end{aligned}$$

Comparing these formulas for $d\theta_1$ mod dt, we see that

$$\begin{aligned}ds_1 &\equiv e^{f}\beta \bmod dx, dt\\ &\equiv e^{-f}(e^{2f}\beta) \bmod dx, dt\\ &\equiv e^{-f}(\tilde{\beta} + e^{f}f_t\theta_0) \bmod dx, dt\\ &\equiv e^{-f}(-dq + f_t\,du) \bmod dx, dt\\ &\equiv d(f_te^{-f}u - e^{-f}q) \bmod dx, dt,\end{aligned}$$

and then

$$s_1 = f_te^{-f}u - e^{-f}q + h,$$

where $h = h(x, t)$. Combining everything gives

$$\theta_0 \wedge \omega^2 = (du - p\,dx) \wedge dt$$

$$\theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2 = e^{-f}\,du \wedge dx + dp \wedge dt + (e^{-f}f_tu + h)\,dx \wedge dt.$$

Now the first of these two 2-forms vanishes on a surface of the form $(x, t, u(x, t), p(x, t))$ with $dx \wedge dt \neq 0$ if and only if $p(x, t) = u_x(x, t)$, and the vanishing of the second 2-form as well is equivalent to the condition that $u(x, t)$ satisfy the partial differential equation

$$e^{-f}u_t = u_{xx} + (f_t e^{-f})u + h.$$

Replacing h by $e^{-f}h$, this equation can be rewritten in the form

$$u_t = e^f u_{xx} + f_t u + h. \tag{10}$$

We have thus shown that a parabolic exterior differential system on a 7-manifold M for which $\dim \mathscr{C} \geqslant 4$ is contact-equivalent to the prolongation of a Monge-Ampere exterior differential system on the 4-manifold N associated to the partial differential equation (10). In any case, we have completed the proof of Theorem 1. □

By adding to u a solution of the inhomogeneous equation (10), we may reduce to

$$u_t = e^{f(x,t)}u_{xx} + f_t u,$$

which, by setting $u = e^f v$, gives

$$v_t = (e^f v)_{xx}. \tag{11}$$

We note that (11) is determined by one arbitrary function of two variables, which confirms our "dimension count" for parabolic systems with at least four independent conservation laws.

Parabolic systems with two conservation laws. In this section, we will derive a normal form for parabolic systems which admit two conservation laws. For technical reasons which will become apparent during this discussion, it is helpful to restrict our attention to the real-analytic case.

As we have already seen, any parabolic system on a 7-manifold which admits more than one conservation law can be canonically "deprolonged" (at least locally) to a parabolic structure on a 4-manifold N. Therefore, it suffices to consider the case of parabolic structures $S \subset \Lambda^2(N)$. We have the following theorem.

THEOREM. *Let N be a 4-manifold and suppose that there are two closed, real-analytic 2-forms Υ_1 and Υ_2 defined on N which are everywhere linearly independent and which generate a parabolic structure S on N. Without loss of generality, we may assume that $\Upsilon_1 \wedge \Upsilon_1 \neq 0$. Then every point of N has a neighborhood U on which there exist coordinates (x, y, z, w) and functions Z and W satisfying the partial differential equation*

$$(Z_z + W_w)^2 = 4(Z_z W_w - Z_w W_z)$$

so that

$$\Upsilon_1 = dz \wedge dx + dw \wedge dy$$

$$\Upsilon_2 = dZ \wedge dx + dW \wedge dy.$$

Conversely, given any two functions Z and W on an open set U in (x, y, z, w)-space which satisfy the above differential equation as well as an open condition on their first derivatives which ensures that Υ_1 *and* Υ_2 *as defined above are linearly independent, the system* $S \subset \Lambda^2(U)$ *which they generate is a parabolic structure on U which admits two conservation laws. Moreover, if Z and W satisfy a further open condition on their second derivatives, the system S will be non-Goursat.*

Proof. Let $S \subset \Lambda^2(N)$ be a parabolic structure on the 4-manifold N. The crux of our proof (and the one place where we will need the assumption of real-analyticity) is in showing that every point of N lies in an open set on which there exists a foliation $\mathscr{F}$ which is simultaneously Lagrangian for each of the 2-forms Υ_i.

Let us assume for the moment that such a foliation $\mathscr{F}$ exists on a neighborhood of some point $p \in N$. Then there are independent functions x and y on a neighborhood U of p such that the leaves of $\mathscr{F}$ restricted to U are given by equations of the form $(x, y) = (x_0, y_0)$. It follows that

$$\Upsilon_1 \equiv \Upsilon_0 \equiv 0 \bmod dx, dy.$$

Let $\mathscr{I} \subset \Omega^*(U)$ denote the ideal generated by dx and dy. It is easy to see that the complex $(\mathscr{I}, d)$ is locally exact in degrees above 1. Since each Υ_i is a closed 2-form in $\mathscr{I}$ (after restriction to U), it follows that there exist functions z, w, Z, and W so that

$$\Upsilon_1 = d(z\,dx + w\,dy) = dz \wedge dx + dw \wedge dy$$

$$\Upsilon_2 = d(Z\,dx + W\,dy) = dZ \wedge dx + dW \wedge dy.$$

Since $\Upsilon_1 \wedge \Upsilon_1 \neq 0$, it follows that (x, y, z, w) forms a coordinate system on some neighborhood of p.

Now, the condition that $\{\Upsilon_1, \Upsilon_2\}$ generate a parabolic system on N is equivalent to there being a function L so that

$$\Upsilon_2 \wedge \Upsilon_1 = L\Upsilon_2 \wedge \Upsilon_2 \quad \text{and} \quad \Upsilon_1 \wedge \Upsilon_1 = L^2\Upsilon_2 \wedge \Upsilon_2.$$

Since $\Upsilon_1 \wedge \Upsilon_2 = (Z_z + W_w)/2\Upsilon_1 \wedge \Upsilon_1$ and $\Upsilon_2 \wedge \Upsilon_2 = (Z_z W_w - Z_w W_z)\Upsilon_1 \wedge \Upsilon_1$, it follows that the necessary and sufficient condition that these two closed 2-forms generate a parabolic system is that, first, the partial differential equation given in the statement of the theorem be satisfied and, second, that Z and W satisfy the open condition which ensures that Υ_1 and Υ_2 be linearly independent.

It remains to prove the existence of the desired local Lagrangian foliations. We will show that, in fact, given *any* real-analytic parabolic structure $S \subset \Lambda^2(N)$ on a 4-manifold N, there always exists a covering of N by open sets U on which there exist S-Lagrangian foliations.

To see this, first note that the hypothesis that S define a real-analytic parabolic structure on N implies that N can be covered by open sets U on which there exist real-analytic coframings $(\eta^1, \eta^2, \eta^3, \eta^4)$ so that S restricted to U is generated by the pair of 2-forms $\{\eta^1 \wedge \eta^3 + \eta^2 \wedge \eta^4, \eta^1 \wedge \eta^2\}$.

For any functions a and b on U, the forms $\eta^1 + a\eta^2$ and $\eta^4 - a\eta^3 + b\eta^2$ clearly generate a rank-2 Pfaffian system with the property that

$$\eta^1 \wedge \eta^3 + \eta^2 \wedge \eta^4 \equiv \eta^1 \wedge \eta^2 \equiv 0 \bmod \eta^1 + a\eta^2, \eta^4 - a\eta^3 + b\eta^2.$$

Thus, it suffices to show that a and b can be chosen so that the Pfaffian system $\{\eta^1 + a\eta^2, \eta^4 - a\eta^3 + b\eta^2\}$ is completely integrable. This is where the Cartan-Kähler theorem is needed.

Let $X = U \times \mathbb{R}^2$ and let a and b be regarded as coordinates on the $\mathbb{R}^2$-factor. Consider the exterior differential system generated on X by the two 4-forms

$$\Theta_1 = d(\eta^1 + a\eta^2) \wedge (\eta^1 + a\eta^2) \wedge (\eta^4 - a\eta^3 + b\eta^2)$$

$$\Theta_2 = d(\eta^4 - a\eta^3 + b\eta^2) \wedge (\eta^1 + a\eta^2) \wedge (\eta^4 - a\eta^3 + b\eta^2).$$

It is easy to see that there are 1-forms α and β on X which satisfy $\alpha \equiv da \bmod \eta^i$ and $\beta \equiv db \bmod \eta^i$ so that

$$\Theta_1 = \alpha \wedge \eta^2 \wedge (\eta^1 + a\eta^2) \wedge (\eta^4 - a\eta^3 + b\eta^2)$$

$$\Theta_2 = (-\alpha \wedge \eta^3 + \beta \wedge \eta^2) \wedge (\eta^1 + a\eta^2) \wedge (\eta^4 - a\eta^3 + b\eta^2).$$

Of course, it immediately follows that the system generated by Θ_1 and Θ_2 with independence condition $\Omega = \eta^1 \wedge \eta^2 \wedge \eta^3 \wedge \eta^4$ is in involution with Cartan characters given by $(s_1, s_2, s_3, s_4) = (0, 0, 2, 0)$.

By the Cartan-Kähler theorem there are integral manifolds of this system with independence condition passing through every point of X. Of course, any such integral manifold may be regarded locally as the graph in X of two functions a and b on an open set in U which have the property that

$$0 = d(\eta^1 + a\eta^2) \wedge (\eta^1 + a\eta^2) \wedge (\eta^4 - a\eta^3 + b\eta^2)$$

$$0 = d(\eta^4 - a\eta^3 + b\eta^2) \wedge (\eta^1 + a\eta^2) \wedge (\eta^4 - a\eta^3 + b\eta^2).$$

In other words, the Pfaffian system $\{\eta^1 + a\eta^2, \eta^4 - a\eta^3 + b\eta^2\}$ is completely integrable, as desired. □

Note that the choice of a local Lagrangian foliation for a parabolic system S depends on the choice of two functions of 3 variables. It is easy to see that, once the Lagrangian foliation is chosen, the choice of the functions x, y, z, w, Z, and W as described in the proof of the theorem depend only on choices of functions of 2 variables. Thus, the ambiguity in the choice of the normal form has the generality of functions of 3 variables. However, going the other way, once the coordinates have been established, the choice of the functions Z and W is subject to a single first-order partial differential equation plus some open conditions. It follows that the set of such choices is locally parametrized by a choice of one function of 4 variables. (For example, one may choose Z arbitrarily and then solve the resulting first-order partial differential equation for W by the method of characteristics.) Thus, it is reasonable to say that the equivalence classes of local parabolic systems with two conservation laws "depend" on one function of 4 variables.

Discussion. In partial differential equations, the concept of *a PDE system defined by conservation laws* has great importance [La]. For PDE's with independent variables x, y and dependent variables u, v a system defined by conservation laws is of the form

$$F_x - G_y = 0, \tag{$*$}$$

where F, G are $\mathbb{R}^2$-valued functions of (x, y, u, v). Here, of course,

$$F(x, y, u, v)_x = F_x + F_u u_x + F_v v_x$$

and similarly for G, and we assume that the vectors

$$F_u, F_v, G_u, G_v$$

everywhere span $\mathbb{R}^2$. Clearly $(*)$ is equivalent to the condition that the $\mathbb{R}^2$-valued 2-form

$$\Upsilon = dF \wedge dy + dG \wedge dx$$

vanish on graphs $(x, y) \to (x, y, u(x, y), v(x, y))$. By a change of independent and dependent variables we may assume that

$$F = \begin{pmatrix} u \\ U \end{pmatrix}, \qquad G = \begin{pmatrix} v \\ V \end{pmatrix}$$

and then

$$\Upsilon = \begin{pmatrix} \Upsilon_1 \\ \Upsilon_2 \end{pmatrix},$$

where Υ_1 and Υ_2 are given in the above theorem with z, w replacing u, v. In order that the exterior differential system generated by (*), which is the same as

$$\Upsilon = 0, \qquad dx \wedge dy \neq 0,$$

define a parabolic system, the condition is

$$(U_u + V_u)^2 = 4(U_u V_v - U_v V_u).$$

The theorem thus states that any parabolic EDS having two independent conservation laws is locally equivalent to the exterior differential system arising from a parabolic PDE system defined by conservation laws.

A normal form for parabolic systems with three independent conservation laws. We now want to further analyze the case where $\dim \mathscr{C} = 3$. The conclusion is given at the end of this subsection. Examining the proof of Theorem 1, we see that this case can only arise when, in our previous notation, we have $G = 0$ and $H \neq 0$, so we make these assumptions. Since $G = 0$, we know that H and R are now invariants. The structure equations are

$$\begin{aligned}
d\omega^2 &= (R\theta_0 + F\omega^1 + H\theta_1) \wedge \omega^2 + H\omega^1 \wedge \theta_0 \\
d\theta_0 &= \beta \wedge \omega^2 - (F\omega^1 + H\theta_1) \wedge \theta_0 - \theta_1 \wedge \omega^1 \\
d\omega^1 &= 0 \text{ (in fact, } \omega^1 = -dL) \\
d\theta_1 &= \sigma \wedge \omega^2 + \beta \wedge \omega^1 + R\theta_1 \wedge \theta_0 - F\omega^1 \wedge \theta_1 .
\end{aligned}$$

Admissible changes of coframing are given by (6) above and under such a change, we have

$$\begin{aligned}
H &= \hat{H} \\
F &= \hat{F} + 2b\hat{H} \\
R &= \hat{R}.
\end{aligned}$$

The "pseudoconnection" forms β and σ appearing in the structure equations are not unique, but rather are determined up to substitutions

$$\begin{aligned}
&\beta \to \beta + p\omega^2 \\
&\sigma \to \sigma + p\omega^1 + q\omega^2 .
\end{aligned} \tag{12}$$

As before, we will consider the differential dH. Since $H \neq 0$, it is natural to modify our notation so that we consider $d(\log H)$ instead of dH. Now, we have already seen that $dH \equiv 2H^2\theta_1 \mod \omega^2, \theta_0, \omega^1$. It follows that we may write

$$dH = H(H_2\omega^2 + H_0\theta_0 + H_1\omega^1 + 2H\theta_1)$$

for some functions H_0, H_1, and H_2. It is easy to show that, under a frame change of the form (6), we have

$$\begin{aligned} H_0 &= \hat{H}_0 \\ H_1 &= \hat{H}_1 + 2b\hat{H} \\ H_2 &= \hat{H}_2 + 2s\hat{H} + b\hat{H}_0 . \end{aligned} \tag{13}$$

With these formulas in hand, we are now ready to return to the calculation of the conservation laws. As before, conservation laws are given by 2-forms

$$\Phi = A\Upsilon + B\Omega$$

which satisfy

$$d\Phi = 0.$$

By construction, $(A, B) = (1, 0)$ and $(A, B) = (L, 1)$ are solutions, and, by assumption, there will be a third solution linearly independent from these two. (Since $H \neq 0$ we have already seen that there can be at most one such "extra" conservation law.)

In addition to dH, we will need to consider dA and dB. From the preceding considerations, we know that

$$\begin{aligned} dA &= A_0\theta_0 + A_2\omega^2 - B\omega^1 \\ dB &= B_0\theta_0 + B_2\omega^2 + A_2\omega^1 + A_0\theta_1 \\ dA_2 &= C\omega^2 - A_2(R\theta_0 + F\omega^1 + H\theta_1) - B_2\omega^1, \end{aligned}$$

where

$$2B_2H + A_2(R + 2HF - HH_1) = 0.$$

Now, we have

$$0 = d(dA) \equiv -2A_0\theta_1 \wedge \omega^1 \mod \theta_0, \omega^2,$$

which, of course, implies that $A_0 = 0$. Substituting this back into the equation for dA, we have

$$0 = d(dA) \equiv (A_2 H + B_0)\omega^1 \wedge \theta_0 \bmod \omega^2,$$

which implies that $B_0 = -A_2 H$. Thus, the first two of the above equations simplify to

$$\begin{aligned} dA &= A_2\omega^2 - B\omega^1 \\ dB &= -A_2 H\theta_0 + B_2\omega^2 + A_2\omega^1. \end{aligned}$$

Next, turning to the relation $2B_2 H + A_2(R + 2HF - HH_1) = 0$, since $H \neq 0$, we may determine b in (13) so that

$$R = HH_1 - 2HF,$$

and this implies that we must have $B_2 = 0$.

If, for the moment, we write, for any function f,

$$df = f_1\omega^1 + f_2\omega^2 + f^0\theta_0 + f^1\theta_1,$$

then the partial differential equation system for A includes the equations

$$A_{11} + A_2 = A^0 = A^1 = (A_1)^0 - A_2 H = (A_1)^1 = 0.$$

Of course, this is highly overdetermined.

At this stage, we have further reduced the structure group so that the admissible changes of coframing are given by

$$(14)\qquad \begin{pmatrix} \hat\omega^2 \\ \hat\theta_0 \\ \hat\omega^1 \\ \hat\theta_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ s & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega^2 \\ \theta_0 \\ \omega^1 \\ \theta_1 \end{pmatrix}.$$

The "pseudoconnection" form β is now reduced to a linear combination of ω^2, θ_0, ω^1, and θ_1. By (12), we may eliminate its ω^2 component, so that $\beta \equiv 0 \bmod \theta_0, \omega^1, \theta_1$.

Now we go back and use this information in $d(dB) = 0$ to infer that

$$(C(\omega^1 - H\theta_0) + A_2 H(\beta - H_2\theta_0)) \wedge \omega^2 = 0,$$

which now implies

$$(15)\qquad C(\omega^1 - H\theta_0) + A_2 H(\beta - H_2\theta_0) = 0$$

by our choice of β. Since $H \neq 0$, this gives

$$A_2(\beta - H_2\theta_0) \wedge (\omega^1 - H\theta_0) = 0.$$

Now, if $(\beta - H_2\theta_0) \wedge (\omega - H\theta_0) \neq 0$, then we would have $A_2 = 0$. From the formula for dA_2, this forces $C = B_2 = 0$ and, since $B_0 = -A_2 H$, we would also have $B_0 = 0$. But then, the equations

$$dA = -B\omega^1$$
$$dB = 0$$

imply that the space of conservation laws has dimension 2. Thus, when $\dim \mathscr{C} = 3$ we must have

$$(\beta - H_2\theta_0) \wedge (\omega^1 - H\theta_0) = 0,$$

which implies that there is some function Z so that

$$\beta = H_2\theta_0 - Z(\omega^1 - H\theta_0).$$

Going back to (15), we obtain $C = A_2 HZ$, which implies

$$dA_2 = -A_2(-ZH\omega^2 + F\omega^1 + H(H_1 - 2F)\theta_0 + H\theta_1).$$

In particular, since A_2 is not identically zero, we see that the 1-form

$$\psi = -ZH\omega^2 + F\omega^1 - H(H_1 - 2F)\theta_0 + H\theta_1$$

must be closed. Referring to (14), and the definition of Z, we see that under a change of coframing,

$$Z = \hat{Z} - 2s.$$

Clearly, we may choose s so as to make $Z = 0$, thereby reducing the structure group to the identity. It follows that

$$\psi = F\omega^1 + H(H_1 - 2F)\theta_0 + H\theta_1.$$

Setting $E = A_2$, we have the system of total differential equations

$$\begin{aligned} dA &= E\omega^2 - B\omega^1 \\ dB &= E(\omega^1 - H\theta_0) \\ dE &= -E(F\omega^1 + H(H_1 - 2F)\theta_0 + H\theta_1), \end{aligned} \tag{16}$$

and the structure equations for the coframing become

$$\begin{aligned}
d\omega^2 &= (H(H_1 - 2F)\theta_0 + F\omega^1 + H\theta_1) \wedge \omega^2 + H\omega^1 \wedge \theta_0 \\
d\theta_0 &= H_2\theta_0 \wedge \omega^2 - (F\omega^1 + H\theta_1) \wedge \theta_0 - \theta_1 \wedge \omega^1 \\
d\omega^1 &= 0 \\
d\theta_1 &= \sigma \wedge \omega^2 + H_2\theta_0 \wedge \omega^1 + H(H_1 - 2F)\theta_1 \wedge \theta_0 - F\omega^1 \wedge \theta_1 .
\end{aligned}$$

The 1-form σ is a linear combination of $\{\omega^2, \theta_0, \omega^1, \theta_1\}$. We write (16) as

$$d\begin{bmatrix} A \\ B \\ E \end{bmatrix} = \begin{bmatrix} 0 & -\omega^1 & \omega^2 \\ 0 & 0 & \omega^1 + \nu_0 \\ 0 & 0 & \nu_1 \end{bmatrix} \begin{bmatrix} A \\ B \\ E \end{bmatrix}, \tag{17}$$

where

$$\begin{aligned}
\nu_0 &= -H\theta_0 \\
\nu_1 &= -F\omega^1 - H(H_1 - 2F)\theta_0 - H\theta_1 .
\end{aligned}$$

Denote by κ the 3-by-3 matrix of 1-forms in (17). The condition that our system have three independent conservation laws is equivalent to the complete integrability of the system (17), and this is expressed by the equation

$$d\kappa - \kappa \wedge \kappa = 0.$$

Explicitly, this is

$$\begin{aligned}
d\omega^1 &= 0 \\
d\omega^2 &= -\nu_1 \wedge \omega^2 + \nu_0 \wedge \omega^1 \\
d\nu_0 &= -\nu_1 \wedge (\nu_0 + \omega^1) \\
d\nu_1 &= 0.
\end{aligned}$$

Observing that

$$\omega^1 \wedge \omega^2 \wedge \nu_0 \wedge \nu_1 = \omega^1 \wedge \omega^2 \wedge (-H\theta_0) \wedge (-H\theta_1) \neq 0,$$

we see that $(\omega^1, \omega^2, \nu_0, \nu_1)$ is a coframing of a 4-dimensional Lie group. Elementary considerations show that on any simply connected region, we may choose local coordinates (x, y, u, p) with $p \neq 0$ so that

$$\omega^1 = -dx$$

$$\omega^2 = -p^{-1}(dy - u\,dx)$$

$$\nu_0 = -p^{-1}(du - p\,dx)$$

$$\nu_1 = p^{-1}\,dp.$$

In terms of these coordinates, we may integrate the system (17) to get the 3-parameter family of explicit solutions

$$A = c_0 + c_1 x + c_2 y$$

$$B = c_1 + c_2 u.$$

Now we can solve for θ_0 and θ_1 in terms of the Lie coframing as follows:

$$\theta_0 = (-1/H)\nu_0$$

$$\theta_1 = (-1/H)\nu_1 - (1/H)(H_1 - 2F)\nu_0 - (F/H)\omega^1.$$

Let us write $-1/H = Ip^2$ for some unknown function I. This leads us to write

$$\Omega = \theta_0 \wedge \omega^2 = Ip^2(-1/p)(du - p\,dx) \wedge (-1/p)(dy - u\,dx)$$

$$= I(du - p\,dx) \wedge (dy - u\,dx)$$

and

$$\Upsilon = \theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2$$

$$= Ip(du - p\,dx) \wedge dx - (I\,dp - J(du - p\,dx) - K\,dx) \wedge (dy - u\,dx)$$

$$= Ip\,du \wedge dx - (I\,dp - J(du - p\,dx) - K\,dx) \wedge (dy - u\,dx),$$

where J and K are the appropriate combinations of p, H, F, and H_1.

Now the conditions on I, J, and K are that the three 2-forms

$$\Phi_0 = \Upsilon, \quad \Phi_1 = \Omega + x\Upsilon, \quad \text{and} \quad \Phi_2 = u\Omega + y\Upsilon$$

should all be closed. Straightforward expansion of the equations $d\Phi_0 = d\Phi_1 = d\Phi_2 = 0$ then yield the following results:

1. I is a function of $\{x, y, u\}$ alone;
2. $J = -(I_x + uI_y + pI_u)$;
3. $K = -M - 2p(I_x + uI_y) - p^2 I_u$, where M is a function of $\{x, y, u\}$ alone;
4. $I_{xx} + 2uI_{xy} + u^2 I_{yy} = M_u$.

Conversely, if these four conditions are satisfied, then $d\Phi_0 = d\Phi_1 = d\Phi_2 = 0$, so the parabolic system generated by $\{\Omega, \Upsilon\}$ has a 3-dimensional space of conservation laws.

We can even explicitly solve the condition (4) by writing

$$I = F_{uuu}$$

$$M = F_{uuxx} - 2F_{uxy} + 2F_{yy} + 2u(F_{uuxy} - F_{uyy}) + u^2 F_{uuyy},$$

where F is an arbitrary function of x, y and u with $F_{uuu} \neq 0$. Thus, these systems depend on the choice of one arbitrary function of three variables, thus confirming the "count" for such systems described in the introduction.

Discussion. In the preceding section we have locally identified parabolic systems having two independent conservation laws with systems arising from parabolic PDE's *defined* by a system of conservation laws. Among these is the remarkable class of (nonlinear) parabolic systems admitting exactly three independent conservation laws. These are defined in (x, y, u, p) space by the EDS

$$\Theta_1 = (du - p\,dx) \wedge (dy - u\,dx) = 0$$

$$\Theta_2 = p\,du \wedge dx - (dp - L\,dx) \wedge (dy - u\,dx) = 0,$$

where $L = L(x, y, u, p)$ is a suitable function. We first note that for any L the system $\Theta_1 = \Theta_2 = 0$ is parabolic, since

$$\Theta_1 \wedge \Theta_1 = 0, \quad \Theta_1 \wedge \Theta_2 = 0, \quad \Theta_2 \wedge \Theta_2 \neq 0.$$

The condition that this system admit three conservation laws is an overdetermined system of partial differential equations on L. In the course of the above discussion, we have integrated these equations, expressing L in terms of an arbitrary function $F(x, y, u)$. In fact, in the above notation, noting that

$$\Omega = I\Theta_1, \qquad \Upsilon \equiv I\Theta_2 \bmod \Theta_1,$$

we have $L = K/I$. In particular, noting the above formulas for K and I, we see that $L_{ppp} = 0$, so that L has an expression of the form

$$L = A_0(x, y, u) + A_1(x, y, u)p + A_2(x, y, u)p^2.$$

The remaining equations on L allow us, via (1–4) above, to express A_0, A_1, A_2 rationally in terms of an arbitrary function $F(x, y, u)$ and its derivatives.

Since $H = -1/Ip^2 \neq 0$, no such parabolic system can be equivalent to a quasi-evolutionary PDE, which is perhaps one reason that the expression for L is somewhat complicated.

Taking $F = u^3$ gives an explicit example of a parabolic PDE admitting exactly three independent conservation laws:

$$u_{xx} + 2uu_{xy} + u^2u_{yy} + 2u_y(u_x + uu_y) = 0.$$

This equation has the 4-parameter group of symmetrices

$$\begin{cases} x \to ax + b, & a \neq 0 \\ y \to cy + d, & c \neq 0 \\ u \to (c/a)u. \end{cases}$$

5. Conservation laws for parabolic evolution equations. If one comes to the study of conservation laws from the perspective of partial differential equations given in traditional form, then in most direct terms the issue is this: Given an explicit PDE

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0, \tag{1}$$

how can one in practice determine the conservation laws? The long answer, given in the previous sections, is to

(i) write (1) as an exterior differential system,
(ii) calculate the invariants $C, S_0, T, U, \ldots$ of the EDS by the procedure given in §2–§3,
(iii) in terms of these invariants, one may then determine how many independent conservation laws there are.[1]

A shorter answer is to apply the "handy algorithm", which not only gives the dimension of the space of conservation laws, but also gives their expression. The question still remains, however, of more directly giving formulas for the conservation laws in terms of F, at least for interesting special classes of equations.

In this section we shall show how to do this for equations

$$u_t = F(x, u, u_x, u_{xx}) \tag{2}$$

of classical evolutionary type with time-translation symmetry. The reader might compare our treatment with that of Mikhailov et al [MSS], where a result (their Lemma 2.1) similar to Proposition 1 below is obtained.

[1] The invariants are given by complicated algebraic expressions in F and its derivatives (e.g., see the formula given in §0 for the Goursat invariant C), in much the same way and for a similar reason that the formula for the Gaussian curvature of a metric $ds^2 = a(x, y)\,dx^2 + 2b(x, y)\,dx\,dy + c(x, y)\,dy^2$ is given by a complicated formula in the derivatives of its coefficients.

We shall treat first a special case before dealing with the general equation (2).

Example 7: (Burger-type equations). Burger's equation

$$u_t = u_{xx} + 2uu_x = (u_x + u^2)_x \tag{3}$$

has been studied by a number of authors (see Vinogradov [Vi]). It is known (loc. cit.) that the only local conservation laws are constant multiples of the obvious one

$$\varphi = u\,dx + (u_x + u^2)\,dt.$$

We shall easily derive this result by applying the handy algorithm. For the first part of the calculation, it will be clearer to consider the slightly more general case of an evolution equation in divergence form

$$u_t = (f(u, u_x))_x. \tag{4}$$

In this calculation, we shall assume that $f_p(u, p) \neq 0$ so as to avoid the trivial case of a first-order equation. In the space with coordinates (x, t, u, p) we set

$$\theta_0 = du - p\,dx$$

and consider the pair of 2-forms

$$\Upsilon = du \wedge dx + df \wedge dt = \theta_0 \wedge dx + df \wedge dt$$

$$\Omega = \theta_0 \wedge dt.$$

Integral surfaces of the exterior differential system $\Omega = \Upsilon = 0$ on which $dx \wedge dt \neq 0$ are locally in one-to-one correspondence with solutions to (4).

According to the handy algorithm, conservation laws are given by linear combinations $\Phi = A\Upsilon + B\Omega$ which satisfy

$$d\Phi = 0. \tag{5}$$

We note that $d\Upsilon = 0$ and that the values $(A, B) \equiv (1, 0)$ correspond to the obvious conservation law

$$\varphi_0 = u\,dx + f(u, p)\,dt.$$

As noted above, $d\Upsilon = 0$, and we shall first show that

$$d\Omega = \frac{dx}{f_p} \wedge (\Upsilon - f_u\Omega).$$

Proof

$$d\Omega = -dp \wedge dx \wedge dt = dx \wedge dp \wedge dt$$

$$df \wedge dt = f_p\, dp \wedge dt + f_u\, du \wedge dt$$

$$= f_p\, dp \wedge dt + f_u(\theta_0 + p\, dx) \wedge dt$$

$$\Rightarrow dp \wedge dt \equiv \frac{1}{f_p} df \wedge dt - \frac{f_u}{f_p}\theta_0 \wedge dt \bmod dx$$

$$\equiv \frac{1}{f_p}\Upsilon - \frac{f_u}{f_p}\Omega \bmod dx,$$

from which the result follows.

The equation (5) now gives

$$\left(dA + \frac{B\, dx}{f_p}\right) \wedge \Upsilon + \left(dB - \frac{f_u}{f_p} B\, dx\right) \wedge \Omega = 0. \tag{6}$$

We shall successively reduce (6) modulo dt, dx, θ_0, and dp to reach our conclusion. For any function C we write

$$dC = C_0\theta_0 + C_1\, dx + C_2\, dt + C_3\, dp.$$

Reducing (6) modulo dt gives

$$dA \wedge \theta_0 \wedge dx \equiv 0 \bmod dt$$

$$\Rightarrow A_3 = 0.$$

Next, reducing (6) modulo dx and using $A_3 = 0$ gives

$$-f_p A_0\, dp \wedge \theta_0 \wedge dt + B_3\, dp \wedge \theta_0 \wedge dt \equiv 0 \bmod dx$$

$$\Rightarrow B_3 = f_p A_0.$$

Similarly, we find

$$A_1 f_p\, dx \wedge dp \wedge dt + B\, dx \wedge dp \wedge dt \equiv 0 \bmod \theta_0$$

$$\Rightarrow A_1 = -\frac{B}{f_p}. \tag{7}$$

At this stage we have

$$dA = A_0\theta_0 - \frac{B\,dx}{f_p} + A_2\,dt.$$

Then taking exterior derivatives and reducing modulo θ_0 and dt gives

$$\left(2A_0 + \frac{f_{pp}}{f_p^2}B\right) dp \wedge dx \equiv 0 \bmod \theta_0, dt$$

$$\Rightarrow A_0 = -\frac{1}{2}\frac{f_{pp}}{f_p^2}B.$$

At this juncture, the analysis breaks into the two cases $f_{pp} = 0$, $f_{pp} \neq 0$. We shall restrict to the former case since it includes Burger's equation, and the latter case will be imbedded in our analysis of the general evolution equation (2). With this assumption we obtain $A_0 = 0$ and

$$dA = -\frac{B}{f_p}\,dx + A_2\,dt$$

$$\Rightarrow A = A(x, t) \text{ is a function of } x, t \text{ alone}.$$

Reducing (6) modulo dp, we have

$$\left(dA + \frac{B\,dx}{f_p}\right) \wedge (\theta_0 \wedge dx + f_u\,du \wedge dt) + \left(dB - \frac{f_u}{f_p}B\,dx\right) \wedge \theta_0 \wedge dt \equiv 0 \bmod dp$$

$$\Rightarrow \left(A_1\,dx + A_2\,dt - \frac{B\,dx}{f_p}\right) \wedge (\theta_0 \wedge dx + f_u(\theta_0 + p\,dx) \wedge dt) + A_1 f_u\,dx \wedge \theta_0 \wedge dt$$

$$\equiv 0 \bmod dp$$

$$\Rightarrow (A_2 - f_u A_1 - B_1)\,dx \wedge \theta_0 \wedge dt \equiv 0 \bmod dp$$

$$\Rightarrow B_1 = A_2 + \frac{f_u}{f_p}B$$

using (7).

For Burger's equation,

$$f(u, p) = p + u^2,$$

and we have

$$A_3 = A_0 = 0$$

$$A_1 = -B$$

$$B_1 = A_2 - 2uA_1 .$$

But $A_1 = \partial A/\partial x$ and $A_2 = \partial A/\partial t$ are functions of x, t alone, as then are B and $B_1 = \partial B/\partial x$. From the third equation we have $A_1 = 0$, which by the second equation gives $B = 0$. Then $A_2 = 0$ and A must be a constant. □

Remark. More generally, if

$$f(u, p) = p + g(u),$$

then the above analysis and conclusion (A must be constant and $B = 0$) applies unless $g'(u) = C$ is constant. In this case the original equation is linear and the conservation laws are given by solutions to the "backwards" linear equation

$$A_{xx} + A_t = CA_x .$$

The general evolution equation. We now turn to analysis of the general evolutionary equation (2). Our first result is the following.

PROPOSITION 1. *If the equation*

$$u_t = F(x, u, u_x, u_{xx}), \qquad F_{u_{xx}} \neq 0,$$

has a nontrivial conservation law, then F is linear fractional in the variable u_{xx},[2] *i.e.,*

[2] In the appendix to this section we shall show that the equation

(i) $$u_t = \frac{a(x, u, u_x) + b(x, u, u_x)u_{xx}}{c(x, u, u_x) + e(x, u, u_x)u_{xx}}$$

is locally contact equivalent to an equation in simpler form:

(ii) $$v_t = F(y, v, v_y)(v_{yy} + A(y, v, v_y)), \qquad F \neq 0 .$$

The linearization of (i) in its highest order terms *cannot* be accomplished by a classical gauge transformation

$$\begin{cases} v = v(x, u) \\ y = y(x). \end{cases}$$

In other words, there are fewer equivalence classes of equations admitting a conservation law than it might appear from a classical perspective.

F is of the form

$$F(x, u, u_x, u_{xx}) = \frac{a(x, u, u_x) + b(x, u, u_x)u_{xx}}{c(x, u, u_x) + e(x, u, u_x)u_{xx}}. \tag{8}$$

We shall give the proof below. Assuming this result, we restrict to the open set where $ae - bc \neq 0$. By reversing t if necessary, we may assume that $ae - bc < 0$, which is the condition that the initial value problem be well posed for increasing time. Next, we may clearly scale a, b, c, e simultaneously so as to have

$$ae - bc = -1,$$

which we shall suppose from now on.

Initially, the problem has been posed in a 5-dimensional space with variables (t, x, u, p, r) (recall the classical convention that $p = u_x$ and $r = u_{xx}$). Under the assumption that F has the form (8) we may reduce to a 4-dimensional space, as follows: In the space with coordinates (t, x, u, p), we consider the pair of 2-forms

$$\begin{aligned} \Upsilon &= (du - p\,dx) \wedge (c\,dx + e\,dp) + (a\,dx + b\,dp) \wedge dt \\ \Omega &= (du - p\,dx) \wedge dt, \end{aligned} \tag{9}$$

where a, b, c, e are the given functions of (x, u, p). The integral surfaces of the exterior differential system

$$\Omega = \Upsilon = 0, \tag{10}$$

on which $dx \wedge dt \neq 0$, are locally in one-to-one correspondence with the solutions to the PDE of Proposition 1, with F of the form (8). This EDS has a symmetry vector field given by $T = \partial/\partial t$, and we shall show that the reduced EDS (which "lies" on a 3-dimensional space) has a canonical G-structure where

$$G = \left\{ \begin{bmatrix} \ell & 0 & 0 \\ m & \pm 1 & 0 \\ n & 0 & \pm\ell \end{bmatrix} \middle| \ell \neq 0 \right\}.$$

The invariants or "curvatures" of this G-structure will be labelled

$$W, G, Y, E, R_0.$$

These invariants can be given explicitly as algebraic expressions in the functions a, b, c, e and their derivatives. We will then show that:

(i) If W is nonzero then there can be at most one conservation law, and there is an explicit formula for it if it exists. Indeed, if there is a conservation law, then the PDE turns out to be locally contact equivalent to an equation

$$u_t = (h(x, u, u_x))_x - Cu \tag{11}$$

where $h(x, u, p)$ is an arbitrary function and C is a constant. The conservation law is

$$\varphi = e^{Ct}(u\, dx + h(x, u, u_x)\, dt).$$

(ii) If W is zero and G is nonzero, then there cannot be any conservation law.

(iii) If W and G both vanish and Z is nonzero, then there can be at most two conservation laws, and the form of these can be determined. In fact, the general form of such an equation is

$$u_t = u^2(u_{xx} + g_0(x, u) + 2g_1(x, u)u_x + g_2(x, u)u_x^2),$$

where g_0, g_1, g_2 are all expressed in terms of one arbitrary function $m(x, u)$ of two variables, one arbitrary function $f(x)$ of one variable, and a number of constants C_0. In fact

$$g_2(x, u) = m_u(x, u)$$

$$g_1(x, u) = m_x(x, u) + \frac{f(x)}{u^2}.$$

The expression for $g_0(x, u)$ is more complicated and will be given later.

Thus, evolution equations with one conservation law depend on one arbitrary function of three variables, those with two conservation laws depend on one arbitrary function of two variables (plus some "lower-dimensional" stuff), and we shall see below that those having three conservation laws are linearizable and depend on one arbitrary function of one variable.

(iv) If W, G and Z all vanish but E does not, there can be at most one conservation law; the condition that this happen and the form of the conservation law can be determined.

(v) If W, G, Z and E all vanish but R_0 does not, then there is no conservation law.

(vi) Finally, if W, G, Z, E and R_0 all vanish, then the equation is linearizable and the space of conservation laws is infinite dimensional.

This analysis has the following corollary.

COROLLARY. *If an evolution equation (2) has three independent conservation laws, then it is linearizable.*

This corollary also follows from the normal form given in §4 of parabolic systems having three independent conservation laws; such equations have a nonzero relative invariant (called G in §4) which clearly vanishes for equations of the form $u_t = F(x, t, u, u_x, u_{xx})$.

Before turning to the proofs, we remark that the interesting equation

$$u_t = u^2(u_{xx} + u),$$

which arises independently from applied mathematics and from geometry, will be seen to have exactly two independent conservation laws.

Proof of Proposition 1. In the space with variables (t, x, u, p, r, s) we introduce the 1-forms

$$\theta_0 = du - p\,dx - F(x, u, p, r)\,dt$$

$$\theta_1 = dp - r\,dx - s\,dt$$

$$\omega^2 = dt$$

$$\omega^1 = dx.$$

We claim that any conservation law is given by a closed 2-form of the form

$$\Phi = A_1\omega^1 \wedge \theta_0 + A_2\omega^2 \wedge \theta_1 + B\omega^2 \wedge \theta_0 + C\theta_0 \wedge \theta_1.$$

Proof. From the general theory we know that any conservation law is represented by a closed 2-form Φ that satisfies the following.

(i) Φ is in the algebraic ideal generated by $\theta_0, \theta_1, \theta_2, \ldots$.

(ii) For a 3-adapted coframing, Φ is in the algebraic ideal generated by θ_0 and ω^2.

(iii) Φ is quadratic in the differentials dx, dt, du, dp, dq (where $dq = dF$ for evolution equations)—i.e., Φ is semibasic relative to the mapping $M \to J^1(\mathbb{R}^2, \mathbb{R})$ induced by the inclusion $M \subset J^2(\mathbb{R}^2, \mathbb{R})$ followed by the projection $J^2(\mathbb{R}^2, \mathbb{R}) \to J^1(\mathbb{R}^2, \mathbb{R})$.

From (iii) we infer that Φ does not involve $\theta_2, \theta_3, \ldots$, and is therefore of the above form except that possibly there may be an additional term $D\omega^1 \wedge \theta_1$, since it is not obvious that the 1-forms θ_0, θ_1, ω^2, ω^1 given above are part of a 3-adapted coframing. But in the successive coframe adaptations given in §1 and §2, the θ_0 remains unchanged up to multiples, and the ω^2 is arranged so as to first have the characteristics defined intrinsically on integral manifolds by $\omega^2 = 0$ (this eliminated being able to add an ω^1 term to ω^2) and then to have $d\omega^2$ as simple as possible. Since $\omega^2 = dt$ for our present coframings, both these conditions are satisfied and so condition (ii) is a posteriori satisfied.

The argument is now similar to—but somewhat more complicated than—that given in the above analysis of Burger-type equations. We compute $d\Phi$ and reduce modulo θ_0 and θ_1 to have

$$0 = d\Phi \equiv (A_1 F_r - A_2)\, dr \wedge dt \wedge dx \mod \theta_0, \theta_1,$$

which implies that we must have $A_2 = F_r A_1$ for any conservation law. We now set $A_1 = A$ so that

$$\Phi = A(\omega^1 \wedge \theta_0 + F_r \omega^2 \wedge \theta_1) + B\omega^2 \wedge \theta_0 + C\theta_0 \wedge \theta_1 .$$

Next we compute $d\Phi$ and reduce modulo θ_1, ω^2, yielding

$$0 = d\Phi \equiv (dA + C\, dr) \wedge dx \wedge du \mod \theta_1, \omega^2 .$$

This implies that A is a function only of the variables (x, t, u, p, r) and that $C = -A_r$ (and of course C is a function of (x, t, u, p, r) also).

Having determined that A does not depend on higher-order jets, we may set

$$dA = A_0\, du + A_1\, dp + A_2\, dt + A_3\, dx - C\, dr$$

and reduce $d\Phi$ modulo $\omega^2 = dt$ to obtain

$$0 = d\Phi \equiv (dC + A_1\, dx) \wedge (du - p\, dx) \wedge (dp - r\, dx) \mod dt,$$

which clearly implies that C is a function only of the variables (x, t, u, p). In particular, from $C = -A_r$ we see that $A_{rr} = 0$, i.e., A must be linear in r. Next we reduce $d\Phi$ modulo dp and dx to get

$$0 = d\Phi \equiv (dB + C\, ds + s\, dC) \wedge dt \wedge du \mod dp, dx$$

$$\Rightarrow B + Cs = \text{function of } x, t, u, p,$$

and we may set

$$dB = B_0\, du + B_1\, dp + B_2\, dt + B_3\, dx - C\, ds - s\, dC.$$

Finally, working modulo cubic terms in dx, dt, du, dp, we find that the B_i's drop out and

$$0 = d\Phi \equiv (AF_{rr} - 2CF_r)\, dr \wedge dt \wedge (dp - r\, dx) \mod \Lambda^3\{dx, dt, du, dp\},$$

which implies that $AF_{rr} = 2CF_r$. When coupled with $C = -A_r$ and the fact that A

is linear in r, this identity yields the relation

$$(A^2F_r)_r = 0.$$

This easily implies that F is linear fractional in r with coefficients being functions of (x, t, u, p). Elementary arguments then show that since F does not depend on t, we may assume that the coefficients in the linear fractional expression also have no t-dependence. □

We now turn to analysis of the evolution equation

$$u_t = \frac{a(x, u, u_x) + b(x, u, u_x)u_{xx}}{c(x, u, u_x) + e(x, u, u_x)u_{xx}}$$

where $ae - bc = -1$. To set this up, we work in the space N with variables (t, x, u, p) and let $\mathcal{I}$ be the differential ideal generated by the pair of 2-forms Υ and Ω given by (9) above. We set $T = \partial/\partial t$ and consider the coframing

$$\begin{aligned} \bar\varpi^2 &= dt \\ \bar\theta_0 &= du - p\,dx \\ \bar\varpi^1 &= c(x, u, p)\,dx + e(x, u, p)\,dp \\ \bar\theta_1 &= a(x, u, p)\,dx + b(x, u, p)\,dp. \end{aligned}$$

This coframing satisfies the following geometric properties:
(i) $\mathcal{I}$ is generated by $\bar\theta_0 \wedge \bar\varpi^2$ and $\bar\theta_0 \wedge \bar\varpi^1 + \bar\theta_1 \wedge \bar\varpi^2$;
(ii) $\bar\varpi^2 = 0$ defines the characteristic foliation on integral surfaces of $\mathcal{I}$ on which $\bar\varpi^1 \wedge \bar\varpi^2 \neq 0$, and moreover $\bar\varpi^2(T) = 1$;
(iii) $\bar\theta_0(T) = \bar\theta_1(T) = \bar\varpi^1(T) = 0$;
(iv) $d\bar\theta_0 \equiv -\bar\theta_1 \wedge \bar\varpi^1 \mod \bar\theta_0$.
Over N we consider the principal G-bundle of all coframings whose smooth sections satisfy (i)–(iv). Since any two coframings defined over the same open set are easily seen to be related by

$$\begin{bmatrix} \omega^2 \\ \theta_0 \\ \omega^1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \ell & 0 & 0 \\ 0 & m & \pm 1 & 0 \\ 0 & n & 0 & \pm\ell \end{bmatrix} \begin{bmatrix} \bar\varpi^2 \\ \bar\theta_0 \\ \bar\varpi^1 \\ \bar\theta_1 \end{bmatrix}, \tag{12}$$

we see that G is the group of matrices of the form

$$\begin{bmatrix} \ell & 0 & 0 \\ m & \pm 1 & 0 \\ n & 0 & \pm \ell \end{bmatrix}, \qquad \ell \neq 0,$$

where the $\pm$ sign is the same on 1 and ℓ. We shall carry out the equivalence method for this G-structure (cf. Appendix 1 in §2).

Now it is easy to see that up on the principal G-bundle space we have structure equations of the form

$$\begin{bmatrix} d\theta_0 \\ d\omega^1 \\ d\theta_1 \end{bmatrix} = - \begin{bmatrix} \alpha & 0 & 0 \\ \beta & 0 & 0 \\ \gamma & 0 & \alpha \end{bmatrix} \wedge \begin{bmatrix} \theta_0 \\ \omega^1 \\ \theta_1 \end{bmatrix} - \begin{bmatrix} \theta_1 \wedge \omega^1 \\ P\theta_1 \wedge \omega^1 \\ Q\theta_1 \wedge \omega^1 \end{bmatrix},$$

where P and Q are functions on the coframe bundle and α, β, γ are pseudoconnection forms (not uniquely defined). The point here is that any $\theta_0 \wedge \omega^1$ or $\theta_0 \wedge \theta_1$ terms in $d\omega^1$ and $d\theta_1$ may be absorbed using the pseudoconnection forms. To find out how P and Q vary on the fibers, we compute

$$0 = d^2\theta_0 = -d\alpha \wedge \theta_0 - \gamma \wedge \omega^1 \wedge \theta_0 - \theta_1 \wedge \beta \wedge \theta_0$$

$$\Rightarrow d\alpha \equiv -\gamma \wedge \omega^1 + \beta \wedge \theta_1 \bmod \theta_0 .$$

Similarly,

$$0 = d^2\omega^1 \equiv -(dP - P\alpha + \beta) \wedge \theta_1 \wedge \omega^1 \bmod \theta_0$$

$$0 = d^2\theta_1 \equiv -(dQ + 2\gamma) \wedge \theta_1 \wedge \omega^1 \bmod \theta_0 ,$$

from which we conclude that

$$\left.\begin{aligned} dP &\equiv P\alpha - \beta \\ dQ &\equiv -2\gamma \end{aligned}\right\} \bmod \theta_0, \omega^1, \theta_1 .$$

It follows that there is a principal subbundle on which $P = Q = 0$, and whose group consists of the matrices

$$\begin{bmatrix} \ell & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm \ell \end{bmatrix}.$$

On this reduced frame bundle there is now a unique choice of α such that the structure equations take the form

$$\begin{bmatrix} d\theta_0 \\ d\omega^1 \\ d\theta_1 \end{bmatrix} = -\begin{bmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \end{bmatrix} \wedge \begin{bmatrix} \theta_0 \\ \omega^1 \\ \theta_1 \end{bmatrix} - \begin{bmatrix} \theta_1 \wedge \omega^1 \\ (Z\omega^1 + W\theta_1) \wedge \theta_0 \\ R\omega^1 \wedge \theta_0 \end{bmatrix}$$

for some functions Z, W, and R on the reduced coframe bundle. The point here is that we have used the previous ambiguity in α up to multiples of θ_0 to eliminate any $\theta_0 \wedge \theta_1$ terms in $d\theta_1$. Since α is uniquely defined by geometric properties, it follows that it is a connection—not just a pseudoconnection—whose curvature is given by

$$d\alpha = E\theta_0 \wedge \omega^1 + G\theta_0 \wedge \theta_1 + Z\omega^1 \wedge \theta_1$$

for some other functions E and G.

Proof. From the structure equations above, $0 = d^2\theta_0 \equiv -d\alpha \wedge \theta_0 + \theta_1 \wedge d\omega^1 \bmod \alpha$ enables us to solve for the coefficient of $\omega^1 \wedge \theta_1$ in $d\alpha$.

Further differentiation of the structure equations yields equations of the form

$$\begin{bmatrix} dR \\ dZ \\ dW \end{bmatrix} = \begin{bmatrix} 0 \\ Z\alpha \\ 2W\alpha \end{bmatrix} + \begin{bmatrix} R_0 & R_1 & E \\ Z_0 & Z_1 & Y \\ W_0 & Y & W_1 \end{bmatrix} \begin{bmatrix} \theta_0 \\ \omega^1 \\ \theta_1 \end{bmatrix}.$$

It follows that R and Z^2/W are absolute invariants. Moreover, if Z is nonzero we may reduce coframes completely by requiring that $Z = 1$; if W is nonzero we may reduce by requiring that $W = \pm 1$. However, we will not proceed this way but rather will go directly to our search for conservation laws.

We are looking for closed 2-forms

$$\Phi = A\Upsilon + B\Omega.$$

Calculation of $d\Upsilon$ and $d\Omega$ (cf. the above analysis of Burger-type equations) gives that

$$d\Phi = (dA - A\alpha + B\omega^1 + AR\omega^2) \wedge \Upsilon + (dB - B\alpha) \wedge \Omega.$$

Imposing the condition $d\Phi = 0$ gives that there are expressions

$$dA = A\alpha + (A_2 - AR)\omega^2 + A_0\theta_0 - B\omega^1$$

$$dB = B\alpha + B_2\omega^2 + B_0\theta_0 + A_2\omega^1 + A_0\theta_1.$$

Next we have

$$0 = d^2A \equiv (AZ + 2A_0)\omega^1 \wedge \theta_1 \bmod \theta_0, \omega^2, \alpha$$

$$\Rightarrow A_0 = -\frac{1}{2}AZ.$$

Using this we then calculate

$$\begin{aligned} 0 &= d^2A \\ &= \left(dA_2 - A_2\alpha - \left(AR_0 - \frac{1}{2}ZA_2\right)\theta_0 + (RB + B_2 - AR_1)\omega^1 - AE\theta_1\right) \wedge \omega^2 \\ &\quad + \left(-\frac{1}{2}AZ_1 - AE + \frac{3}{2}BZ + B_0\right)\omega^1 \wedge \theta_0 \\ &\quad + \left(-\frac{1}{2}AY - AG + BW\right)\theta_1 \wedge \theta_0, \end{aligned}$$

which implies the equations

$$\text{(13)} \qquad \begin{cases} B_0 = \left(E + \frac{1}{2}Z_1\right)A - \frac{3}{2}BZ \\ WB = \left(G + \frac{1}{2}Y\right)A. \end{cases}$$

The equation $d^2A = 0$ above also gives

$$dA_2 \equiv A_2\alpha + \left(AR_0 - \frac{1}{2}ZA_2\right)\theta_0 - (RB + B_2 - AR_1)\omega^1 + AE\theta_1 \bmod \omega^2,$$

which implies

$$dA_2 = A_2\alpha + C\omega^2 + \left(AR_0 - \frac{1}{2}ZA_2\right)\theta_0 - (RB + B_2 - AR_1)\omega^1 + AE\theta_1$$

for some function C. We now proceed to analyze cases, following the numbering given at the beginning of this section.

Case (i). $W \neq 0$. From (13) any conservation law must be of the form

$$\Phi = M[W(\theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2) + (G + Y/2)\theta_0 \wedge \omega^2].$$

Denote by Φ_0 the 2-form in brackets. Since $W \neq 0$ we have that $\Phi_0 \wedge \Phi_0 \neq 0$, and consequently

$$d\Phi_0 = \mu \wedge \Phi_0$$

for a unique 1-form (wedging with a nondegenerate 2-form on a 4-manifold gives an isomorphism from the space of 1-forms to the space of 3-forms). Clearly, μ is an expression in the invariants of the EDS and their covariant derivatives. The condition that there be a conservation law is

$$d\mu = 0. \tag{14}$$

In fact, writing $M = e^{-g}$ for some unknown function g, the condition $d\Phi = 0$ gives $dg = \mu$. Thus, the space of conservation laws is of at most dimension 1 and is equal to 1 exactly when the equation (14) (which is an expression in the invariants of the EDS) is satisfied.

We will now derive the normal form (11). For this we see that taking $W = 1$ and replacing θ_1 by $\theta_1 + (G + Y/2)\theta_0$ gives a canonical section of the original coframe bundle such that

$$\Phi = M(\theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2).$$

The action by the time-translation symmetry induces a representation on the space of conservation laws, so that

$$\exp^*_{tT}(\Phi) = e^{Ct}\Phi$$

for some constant C. Taking the t-dependence out of M, we have a new coframing so that

$$\Phi = e^{Ct}(\bar{\theta}_0 \wedge \bar{\omega}^1 + \bar{\theta}_1 \wedge \bar{\omega}^2), \qquad \bar{\omega}^2 = \omega^2 = dt,$$

where $\bar{\theta}_0, \bar{\omega}^1, \bar{\theta}_1$ are defined on (x, u, p) space. Since Φ is closed, we obtain

$$d\bar{\theta}_1 = -C\bar{\theta}_0 \wedge \bar{\omega}^1$$

$$d(\bar{\theta}_0 \wedge \bar{\omega}^1) = 0.$$

It follows that there are (possibly new) coordinates x, u, p and a nonzero function $f(x, u, p)$ such that

$$\bar{\theta}_0 \wedge \bar{\omega}^1 = du \wedge dx \Rightarrow \begin{cases} \bar{\theta}_0 = f(x, u, p)(du - p\,dx) \\ \bar{\omega}^1 = (f(x, u, p))^{-1}\,dx \end{cases}$$

(recall that $d\bar{\theta}_0 = -\bar{\theta}_1 \wedge \bar{\omega}^1$, so that the Pfaff-Darboux theorem applies). Since $d\bar{\theta}_1 = -C\,du \wedge dx$, we see that there is a function $h(x, u, p)$ so that $\bar{\theta}_1 = dh - Cu\,dx$. Thus the system is generated by the pair of 2-forms $(du - p\,dx) \wedge dt$ and $du \wedge dx + (dh - Cu\,dx) \wedge dt$, and hence represents the equation (11).

By calculation, one finds that $h_p = f^2$ and that the invariant W vanishes if and only if $f\,f_{pp} - 2f_p^2 = 0$, which is the condition that f be linear fractional in p. In summary, *the equation,*

$$u_t = (h(x, u, u_x))_x - Cu$$

where h is not linear fractional in u_x, describes all of the equations in Case (i) for which there is a conservation law.

Case (ii). $W = 0$ *and* $G \neq 0$. Since $W = 0$ implies $Y = 0$, we see from (13) that in this case there are no conservation laws.

Case (iii). $W = G = 0$, $Z \neq 0$. This case is the most difficult, and the analysis will be broken into several steps.

Step 1. We shall show that *an equation with the invariant* $W = 0$ *and* $Z \neq 0$ *is contact equivalent to one of the form*

$$u_t = u^2(u_{xx} + g(x, u, u_x)). \tag{15}$$

Proof. We can uniquely adapt coframes so that $Z = -1$, which then gives from the structure equation that

$$d\omega^1 = \omega^1 \wedge \theta_0.$$

It follows that there exist functions u and x with $u \neq 0$ such that

$$\omega^1 = u^{-1}\,dx,$$

where x is unique up to a change of variables of the form $X = X(x)$, and the corresponding U is given by $U = u/X'(x)$ so that $dX/U = dx/u$. Next, we infer from $d(u^{-1}\,dx) = u^{-1}\,dx \wedge u^{-1}\,du = \omega^1 \wedge \theta_0$ that

$$\theta_0 = u^{-1}(du - p\,dx)$$

for some function p. Finally, from

$$d\theta_0 = -\theta_1 \wedge \omega^1 \bmod \theta_0$$

we have that t, x, u, p form a local coordinate system and that

$$\theta_1 = dp + g(x, u, p)\,dx + h(x, u, p)\theta_0.$$

Now

$$\Omega = \theta_0 \wedge dt = u^{-1}(du - p\,dx) \wedge dt$$

$$\Upsilon = \theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2$$

$$\equiv u^{-2}\,du \wedge dx + (dp + g(x, u, p)\,dx) \wedge dt \bmod \Omega$$

so that $\Omega = \Upsilon = 0$, $dx \wedge dt \neq 0$, defines the PDE

$$u^{-2}u_t = u_{xx} + g(x, u, u_x).$$

Step 2. Next we shall show that *if the invariant G vanishes, then g is of the form*

$$g(x, u, p) = g_0(x, u) + 2g_1(x, p)p + g_2(x, u)p^2, \tag{16}$$

i.e., g is at most quadratic in p.

Proof. We have

$$\theta_0 = u^{-1}(du - p\,dx)$$

$$\omega^1 = u^{-1}\,dx$$

$$\theta_1 = dp + g(x, u, p)\,dx + uh(x, u, p)\theta_0$$

(this is a different h than in the previous step). Then

$$d\theta_0 = -\alpha \wedge \theta_0 - \theta_1 \wedge \omega^1 \Rightarrow \alpha = u^{-1}\,du + h(x, u, p)\,dx + k(x, u, p)\theta_0$$

for some function $k(x, u, p)$. Next

$$d\theta_1 \equiv -\alpha \wedge \theta_1 \bmod \theta_0 \Rightarrow h = \frac{1}{2}(g_p - p/u).$$

Then the structure equation $d\theta_1 = -\alpha \wedge \theta_1 + R\theta_0 \wedge \omega^1$ for some R implies that

$$k = \frac{1}{2}(ug_{pp} - 3).$$

Finally, recalling that G is the coefficient of $\theta_0 \wedge \theta_1$ in the curvature $d\alpha$, the vanishing of G is equivalent to

$$\begin{aligned} 0 = d\alpha \wedge \omega^1 &= d(k\theta_0) \wedge \omega^1 \\ &= dk \wedge \frac{du}{u} \wedge \frac{dx}{u} \\ &= \frac{1}{2u}(g_{ppp}\, dp \wedge du \wedge dx). \end{aligned}$$

Thus, $g_{ppp} \neq 0$ and the general form of an evolution equation with $W = G = 0$, $Z \neq 0$ is

$$u_t = u^2(u_{xx} + g_0(x, u) + 2g_1(x, u)u_x + g_2(x, u)u_x^2). \tag{17}$$

It remains to derive the conditions on the three functions g_i so that there do, in fact, exist conservation laws.

Before continuing this analysis, we want to pause and analyze a particular interesting equation of the above type.

Example 8. Curvature heat flow. In Example 1 in §0, we introduced the heat flow equation for a curve Γ on a Riemannian surface. In case Γ is a closed convex plane curve, we may use the method of support functions to give Γ by giving the curvature as a function of the angle that the tangent line makes with, say, the x-axis. Calling the curvature u and the angle x, the resulting PDE for the curvature heat flow evolution of the curve is

$$u_t = u^2(u_{xx} + u). \tag{18}$$

We shall show that *this equation has exactly two independent conservation laws.*

Proof. Following our usual notation, we have

$$\begin{aligned} \Omega &= \theta_0 \wedge dt \qquad (\theta_0 = du - p\, dx) \\ \Upsilon &= \theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2 \\ &= du \wedge dx + u^2(dp + u\, dx) \wedge dt. \end{aligned}$$

The EDS

$$\Omega = \Upsilon = 0, \qquad dx \wedge dt \neq 0$$

clearly defines the equation (18). We look for closed 2-forms

$$\Phi = A\Upsilon + B\Omega.$$

This gives

$$0 = d\Phi \equiv dA \wedge du \wedge dx \bmod dt$$

and straightforward calculation gives that

$$0 = d\Phi \equiv (u^2\, dA + (B + 2upA)\, dx) \wedge (dp + u\, dx) \wedge dt \bmod \theta_0$$

$$\Rightarrow dA = A_2\, dt + A_0\theta_0 - u^{-2}(B + 2upA)\, dx$$

for functions A_0 and A_2. Then we obtain

$$0 = d\Phi = [dB - (u^2A_0 + 2uA)\, dp - (u^3A_0 + 3u^2A + A_2)\, dx] \wedge \theta_0 \wedge dt$$

$$\Rightarrow dB = B_2\, dt + B_0\theta_0 + (u^3A_0 + 3u^2A + A_2)\, dx + (u^2A_0 + 2uA)\, dp.$$

With these expressions for dA and dB, the equation $d\Phi = 0$ is an identity. We must now impose the integrability conditions $d^2A = d^2B = 0$.

First we have

$$0 = d^2A \equiv -2u^{-1}(uA_0 + 2A)\, dp \wedge dx \bmod \theta_0, dt,$$

which implies that $A_0 = -(2/u)A$. Then the formula for d^2A simplifies to

$$0 = d^2A \equiv 2u^{-2}B_0\, du \wedge dx \bmod dt,$$

which implies that $B_0 = 0$. Summarizing,

$$d(u^2A) = u^2A_2\, dt - B\, dx$$

$$dB = B_2\, dt + (u^2A + A_2)\, dx.$$

Thus u^2A is a function of x and t alone, as are u^2A_2, B, B_2, and $u^2A + A_2$. This easily implies that $A_2 = 0$, and hence that u^2A and B are functions of x alone. Hence $B_2 = 0$ and the above equations reduce to

$$d(u^2A) = -B\, dx$$

$$dB = u^2A\, dx.$$

In turn this implies that we must have $A = u^{-2}f(x)$ and $B = -f'(x)$ for some $f(x)$ satisfying

$$f''(x) + f(x) = 0.$$

Thus the space of conservation laws has dimension 2 with basis

$$\Phi_0 = u^{-2} \cos x \, \Upsilon + \sin x \, \Omega$$

$$\Phi_1 = u^{-2} \sin x \, \Upsilon - \cos x \, \Omega.$$

The undifferentiated conservation laws are

$$\varphi_0 = \frac{\cos x}{u} \, dx - (\sin x \, u + \cos x \, u_x) \, dt$$

$$\varphi_1 = \frac{\sin x}{u} \, dx + (\cos x \, u - \sin x \, u_x) \, dt.$$

These correspond to the differentials of the coordinate functions, which must have $\int \varphi_0 = \int \varphi_1 = 0$ for fixed t since the curve remains closed under time evolution.

Our result implies that there are no other local conservation laws, beyond these two obvious ones. As will be seen below, if we pass to the integrable extension obtained by adjoining the "primitives" of φ_0 and φ_1, then we obtain a new conservation law, which is essentially the one we have already found for equation (18).

Step 3. We return to our analysis of equation (17). Setting

$$\Upsilon = u^{-2} \, du \wedge dx + [dp + (g_0 + 2g_1 p + g_2 p^2) \, dx] \wedge dt,$$

where $g_i = g_i(x, u)$,

$$\Omega = (du - p \, dx) \wedge dt,$$

and

$$\Phi = A\Upsilon + B\Omega,$$

we are studying the conditions on the coefficients g_0, g_1, g_2 that the equation

$$d\Phi = 0$$

have two independent solutions in the space of functions A, B. This is a set of four linear equations in two unknowns, hence overdetermined, and we may expect that the assumption of a 2-dimensional solution space will impose very stringent conditions on the g_i. *We will first determine the condition that there exist 1-forms*

v_{ij} *such that*

$$dA = v_{11}A + v_{12}B \tag{19}$$
$$dB = v_{21}A + v_{22}B.$$

Then the condition that we have two independent conservation laws is that the matrix $v = \|v_{ij}\|$ be integrable, i.e., satisfies $dv = v \wedge v$.

The method of calculation is the one we have used repeatedly in computation of examples and analysis of previous cases: We introduce new functions for the derivatives of A, B, and the g_i and determine the conditions on these new functions that there exist solutions to $d\Phi = 0$. With these conditions inserted, we then repeatedly impose the integrability equations of equality of mixed partials. Our calculations were actually carried out by Maple 4.2, and we shall only list the steps, not writing out in full gory detail some of the lengthy intermediate expressions that arise.

First, the equation $d\Phi = 0$ will be satisfied exactly when there are functions A_0, A_2, B_0, and B_2 so that we have the formulas

$$dA = u^2 A_2\, dt + A_0(du - p\, dx) + ((pg_2 + g_1)A - B)\, dx$$

$$\begin{aligned} dB = B_2\, dt + B_0(du - p\, dx) &+ [(g_0 + 2g_1 p + g_2 p^2)A_0 + A_2 + (pg_2 + g_1)B \\ &+ (g_{0,u} - g_{1,x} - g_1^2 + p(g_{1,u} - g_{2,x} - 2g_1 g_2) - p^2 g_2^2)A]\, dx \\ &+ (A_0 - g_2 A)\, dp. \end{aligned}$$

This says in particular that $A = A(t, x, u)$ does not depend on p. The condition $d^2A = 0$ now gives successively

$$A_0 = g_2 A$$

$$B_0 = (g_{1,u} - g_{2,x})A + g_2 B$$

and

$$dA_2 = C\, dt + A_2(g_2 - 2/u)\, du + (g_1 A_2 - B_2/u^2)\, dx$$

for some function C. One observes that equations of the form (19) are beginning to emerge.

We next turn to the condition $d^2B = 0$. This implies first that

$$A_2 = \left(\frac{1}{2}(g_0 g_{2,u} + g_2 g_{0,u} + g_{0,uu} + g_{2,xx}) - g_{1,xu} - g_1 g_{1,u}\right)uA + (g_{1,u} - g_{2,x})uB.$$

At this stage we have $dA = v_{11}A + v_{12}B$. Moreover, we may go back to $d^2A = 0$ using the above expressions for A_0 and A_2. The result is

$$0 = d^2A = E_1\, dx \wedge dt + E_2\, du \wedge dt,$$

where E_1 is an expression of the form $E_1 = B_2 - Q_1A - Q_2B$ where Q_1 and Q_2 are certain (rather ungainly) polynomials in u and the functions g_i and their first and second derivatives, and

$$E_2 = (u^2F_2 + F_1)A + F_1B,$$

where

$$F_1 = 3u^2g_{1,u} + u^3g_{1,uu} - 3u^2g_{2,x} - u^3g_{2,xu},$$

and F_2 is a similar (but larger) expression. Now E_1 and E_2 must both vanish. The vanishing of E_1 gives us an equation of the form $B_2 = Q_1A + Q_2B$. Thus, we finally have (19) where the v_{ij} are expressed in terms of the g_i and their first and second derivatives. From the above formula for E_2, we see that we must have

$$F_1 = F_2 = 0$$

or else the ratio $[A:B]$ will be determined, and there will be at most one conservation law.

Step 4. It remains to impose the integrability conditions and then integrate the resulting equations (19). This will be done by first going back to complete the analysis of the condition $d^2B = 0$. Before doing this we note that $F_1 = 0$ is simply

$$[u^3(g_{1,u} - g_{2,x})]_u = 0.$$

This equation may be solved by introducing arbitrary functions $m(x, u)$ and $f(x)$ and setting

$$\text{(20)} \qquad \begin{cases} g_2(x, u) = m_u(x, u) \\ g_1(x, u) = m_x(x, u) + u^{-2}f(x). \end{cases}$$

Future expressions will be in terms of $g_0(x, u)$, $m(x, u)$ and $f(x)$.

Maple now gives that

$$0 = d^2B = G_1\, du \wedge dt + G_2\, du \wedge dt + G_3\, dx \wedge dt,$$

where

$$G_1 = F_2 A$$

$$G_2 = F_3 A + u^2 F_2 B$$

$$G_3 = F_5 A + u^2 F_4 B,$$

and where F_2, F_3, F_4, F_5 are complicated differential polynomials in g_0, m, and f (this is the same F_2 as above—by construction we have $F_1 = 0$).

Now, we must have $G_1 = G_2 = G_3 = 0$. If we did not have $F_2 = F_3 = F_4 = F_5 = 0$, then the vanishing of the G_i will impose at least one linear relation between A and B. Since we are assuming that there are two linearly independent conservation laws, it follows that we must have $F_2 = F_3 = F_4 = F_5 = 0$. This is a system of partial differential equations for the functions $g_0(x, u)$, $m(x, u)$ and $f(x)$. We will not give the details of the analysis of this system, just the conclusion. It turns out (after much Maple calculation) that these equations are satisfied if and only if there exists a constant C_0 and function $n(x)$ such that the equations

$$2n_x f + 4nf_x - f_{xxx} = 0$$

$$n - u^{-2}C_0 - 3u^{-2}f_x - m_{xx} - 2u^{-2}fm_x - m_x^2 + m_u g_0 + g_{0,u} = 0$$

are satisfied. At this stage, once f is chosen, we may solve the first equation for n by simple integration. In fact, it has a first integral

$$-f_x^2 + 2ff_{xx} - 4f^2 n = C_1 .$$

Once n is known, the second equation (which is first-order and linear in g_0) can be solved once m is chosen by using the method of characteristics to obtain an expression for g_0 in terms of m_1, m_u and functions of x above. In fact, the reader will note that this equation has an integrating factor as well. However, this is not a satisfactory final form, and we will not insist on it. It is enough to know that the equations have solutions depending on one arbitrary function of two variables ($m(x, u)$) and a certain number of arbitrary functions of one variable plus constants.

Going back to the expressions for dA and dB and using the above equations, we have

$$dA = [(2f^2/u^2 - f_x - C_0)\,dt + (m_x + f/u^2)\,dx + m_u\,du]A + [-dx - 2f\,dt]B$$

$$dB = [(f_{xx} - 2fn - 2ff_x/u^2 + 2f^3/u^4)\,dt + (f^2/u^4 - n + f_x/u^2)\,dx - 2f/u^3\,du]A$$
$$+ [-(2f^2/u^2 - f_x + C_0)\,dt + (m_x - f/u^2)\,dx + m_u\,du]B.$$

Inspection of the form of these relations suggests that we define new functions

a, b by the rules

$$a = Ae^{-(m-C_0 t)}$$

$$b = (B - f/u^2 A)e^{-(m-C_0 t)}.$$

Indeed, with the help of Maple we find that the above integrable system for A and B is equivalent to the integrable system

$$\begin{pmatrix} da \\ db \end{pmatrix} = -\begin{pmatrix} f_x\, dt & 2f\, dt + dx \\ (2fn - f_{xx})\, dt + n\, dx & -f_x\, dt \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix},$$

where f and n are related by the above ODE. (Note, in particular, that this system does not involve u at all.)

To proceed further we observe that we have some freedom of choice in our x-coordinate. Indeed, as noted in Step 1, we may change x by any function $X(x)$ with $X' \neq 0$, and then the corresponding U is determined by requiring that $u^{-1}\, dx = U^{-1}\, dX$. From (17) we may calculate how g_1 and g_2 transform, and then from (20), we deduce that the quantity

$$Q = f(x)\, dx^2$$

is invariant under admissible coordinate change. If $Q \equiv 0$ then $f = 0$. The differential equation relating f and n then degenerates to an identity and we have

(21) $$d\begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} 0 & dx \\ n\, dx & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}.$$

Of course, in this case, this means that a and b are functions of x alone, that $b = -a'$, and that a is a solution of the ODE $a'' - na = 0$.

If, on the other hand, $Q \neq 0$, then we introduce a new coordinate X such that $Q = \pm(dX)^2$. Relabeling, we may thus assume that f is constant. Of course, by the ODE relating f and n, this implies that n is also constant. We then have a matrix equation of the form

(21′) $$d\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}.$$

The matrix $\eta = \|\eta_{ij}\|$ will be of the form

$$\eta = \eta_0(dx + 2f\, dt),$$

where η_0 is a constant matrix. In this case, (21′) may be integrated by simple exponentiation.

We see no particular advantage to writing this process out explicitly, but rather note in conclusion that the result will be *to express A and B in terms of the arbitrary function m(x, u) and a number of constants.*

In this way we have given a prescription for all evolutionary equations with two conservation laws together with a prescription for those conservation laws, all of this in terms of the arbitrary function $m(x, u)$ together with a small amount of additional "lower-dimensional" data.

In closing the discussion of this case we note that the time-translation 1-parameter group acts by a 2-dimensional representation on the space of conservation laws with

$$\begin{aligned} \text{Trace} &= -2C_0 \\ \text{Determinant} &= C_0^2 - f_x^2 - 2ff_{xx} + 4f^2 n \\ &= C_0^2 - C_1 . \end{aligned}$$

In particular, the eigenvalues may be purely imaginary, so that even after a contact transformation the equation cannot be put in divergence form

$$u_t = (h(x, u, u_x))_x - Cu .$$

A specific example of this phenomenon is

$$u_t = u^2(u_{xx} + u) + 2u_x .$$

Thus, *the general evolutionary equation with two conservation laws is not obtained by imposing one further conservation law on the general equation with one conservation law.*

Case (iv). $W = G = Z = 0$, $E \neq 0$. Just prior to the analysis of Case (i) we have found the consequences of $d^2A = 0$. Assuming only that $W = G = 0$, we shall find the consequences of $d^2\alpha = d^2Z = d^2B = 0$. First, we have

$$\begin{aligned} 0 = d^2\alpha &= -(dE - E\alpha + (Z^2 + Z_0)\theta_1) \wedge \omega^1 \wedge \theta_0 \\ \Rightarrow dE &= E\alpha + E_0\theta_0 + E_1\omega^1 - (Z^2 + Z_0)\theta_1 . \end{aligned}$$

Similarly,

$$\begin{aligned} 0 &= d^2Z \\ &= (dZ_0 - 2Z_0\alpha - Z(E + Z_1)\omega^1) \wedge \theta_0 + (dZ_1 - Z_1\alpha - (Z^2 + Z_0)\theta_1) \wedge \omega^1 \end{aligned}$$

$$\Rightarrow \begin{cases} dZ_0 = 2Z_0\alpha + Z_2\theta_0 + (Z_3 + Z(E + Z_1))\omega^1 \\ dZ_1 = Z_1\alpha + Z_3\theta_0 + Z_4\omega^1 + (Z^2 + Z_0)\theta_1 \end{cases}$$

for suitable functions Z_2, Z_3 and Z_4. Next, assuming these equations, we have

$$0 = d^2B$$

$$\equiv \left[\left(E_1 + \frac{1}{2}Z_4 - R_0 + \frac{1}{2}ZR\right)A - 2(E + Z_1)B - 2ZA_2\right]\omega^1 \wedge \theta_0 \bmod dt$$

$$\Rightarrow \left(E_1 + \frac{1}{2}Z^4 - R_0 + \frac{1}{2}ZR\right)A - 2(E + Z_1)B - 2ZA_2 = 0.$$

If now $Z = 0$ then this last equation simplifies to

$$(E_1 - R_0)A = 2EB.$$

Thus, if $E \neq 0$ the ratio $A : B$ is determined, and there can be at most one conservation law. Although we shall not do so here, in this case, as in Case (i), the form of the equation can be determined.

Case (v). $W = G = Z = E = 0$, $R_0 \neq 0$. In this case the last equation above gives $R_0 A = 0$, and hence there are no conservation laws.

Case (vi). $W = G = Z = E = R_0 = 0$. In this case $d\alpha = 0$, and writing $\alpha = dk$ we may replace θ_0 and θ_1 by $e^{-k}\theta_0$ and $e^{-k}\theta_1$ to have $\alpha = 0$ (by our hypotheses, the curvature of $d\alpha$ vanishes, so the connection α is flat, and we can change to a frame in which $\alpha = 0$). The structure equations then simplify to

$$\begin{cases} d\theta_0 = -\theta_1 \wedge \omega^1 \\ d\omega^1 = 0 \\ d\theta_1 = R\omega^1 \wedge \theta_0 . \end{cases}$$

We may then find a function x such that $\omega^1 = dx$, and then $dR = R_1\omega^1$ implies that $R = R(x)$. Next, the structure equations imply that $d(\omega^1 \wedge \theta_0) = 0$. It follows that

$$\theta_0 \wedge \omega^1 = du \wedge dx$$

for some function u, and hence $\theta_0 = du - p\,dx$ for some function p. The first structure equation gives that $\theta_1 = dp - H\,dx$ for some function H, and then the third structure equation gives

$$-dH \wedge dx = R(x)\,dx \wedge du$$

$$\Rightarrow d(H - R(x)u) \wedge dx = 0$$

$$\Rightarrow H = R(x)u + T(x)$$

for some function $T(x)$. This finally gives

$$\Omega = \theta_0 \wedge \omega^2 = (du - p\,dx) \wedge dt$$

$$\begin{aligned}\Upsilon &= \theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2 \\ &= du \wedge dx + (dp - (R(x)u + T(x))\,dx) \wedge dt\end{aligned}$$

so that the original partial differential equation is contact equivalent to the linear equation

$$u_t = u_{xx} - R(x)u - T(x).$$

Of course, by replacing u by $u + S(x)$ where $S'' - RS = T$, we get an equivalent equation with $T = 0$. Thus, the equation in this case actually depends on only one function of one variable.

Appendix: An example of contact equivalence. As mentioned on a number of occasions, in this paper we are considering exterior differential systems up to contact equivalence—i.e., diffeomorphism of the underlying manifold preserving the differential ideal. Thus the two PDE's (i) and (ii) given in footnote (2) above are contact equivalent.

From the point of view of traditional PDE, one might say that this fact is interesting but not useful in that the solution to (i) may develop a singularity in finite time whereas (ii) has long time solutions. Of course, this is true, but from a geometric point of view, one should distinguish between singularities which arise as intrinsic properties of the contact-equivalence class of an equation and those which are due to singularities of the contact transformation used to put a given equation in a particularly nice form. The constant coefficient case considered below gives a concrete illustration of this phenomena.

We consider the evolution equation

(i) $$u_t = \frac{a(x, u, u_x) + b(x, u, u_x)u_{xx}}{c(x, u, u_x) + e(x, u, u_x)u_{xx}}.$$

PROPOSITION. *This equation is locally contact equivalent to an equation*

(ii) $$v_t = F(y, v, v_y)(v_{yy} + A(y, v, v_y)), \qquad F > 0.$$

Proof. We retain the notations $\varpi^2 = dt$, $\bar\theta_0 = du - p\,dx$, $\varpi^1 = c(x, u, p)\,dx + e(x, u, p)\,dp$ and $\bar\theta_1 = a(x, u, p)\,dx + b(x, u, p)\,dp$ with

$$d\bar\theta_0 \equiv -\bar\theta_1 \wedge \varpi^1 \bmod \bar\theta_0$$

used above. We shall locally determine a change of coframing

$$\begin{bmatrix} \theta_0 \\ \omega^1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ b & 1 & 0 \\ c & 0 & a \end{bmatrix} \begin{bmatrix} \bar{\theta}_0 \\ \bar{\omega}^1 \\ \bar{\theta}_1 \end{bmatrix}$$

such that

(iii) $$d\omega^1 \wedge \omega^1 = 0 \qquad \text{(i.e., } \omega^1 \text{ is integrable)}$$

(iv) $$d(\theta_0 \wedge \omega^1) = 0.$$

The first equation is

$$d(\bar{\omega}^1 + b\bar{\theta}_0) \wedge (\bar{\omega}^1 + b\bar{\theta}_0) = 0,$$

which (since we are in 3-space) is a single first-order PDE for the unknown function b. It is well known (via the method of characteristics) that such an equation has local solutions. Similarly, the second equation is

$$d(a\bar{\theta}_0 \wedge \bar{\omega}^1) = 0,$$

which again is a single first-order PDE for the function a, and hence will have local solutions.

From (iii) there are functions f and y with $f \neq 0$ such that

$$\omega^1 = f\, dy.$$

From (iv) there is another function v such that $\theta_0 \wedge \omega^1 = dv \wedge dy$, and hence there is yet a third function q so that $\theta_0 = f^{-1}(dv - q\, dy)$. Since $\theta_0 \wedge d\theta_0 \neq 0$, we see that (y, v, q) forms a local coodinate system, and the transformation

$$(x, u, p) \to (y, v, q)$$

is a contact transformation since θ_0 pulls back into a multiple of $\bar{\theta}_0$. Now we have

$$d\theta_0 \equiv -f^{-1}\, dq \wedge dy \bmod \theta_0$$

$$\equiv -\theta_1 \wedge f\, dy \bmod \theta_0,$$

where the second congruence is by the structure equation $d\theta_0 \equiv -\theta_1 \wedge \omega^1 \bmod \theta_0$

which is preserved by our change of coframing. It follows that

$$\theta_1 = f^{-2}(dq + A\,dy + B\theta_0)$$

$$\Omega = \theta_0 \wedge dt = f^{-1}(dv - q\,dy) \wedge dt$$

$$\Upsilon = \theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2 = dv \wedge dy + f^{-2}(dq + A\,dy) \wedge dt$$

and $\Omega = \Upsilon = 0$, $dy \wedge dt \neq 0$ is just the quasi-linear PDE (ii) with $F = f^{-2} > 0$. □

Suppose now that a, b, c, e are constants. We shall show that there is a (global) contact transformation $(x, u, p) \to (X, U, P)$ which converts

(i)′ $$u_t = \frac{a + bu_{xx}}{c + eu_{xx}}$$

into the linear constant coefficient PDE

(ii)′ $$U_t = U_{XX}.$$

Proof. Set

$$dX = c\,dx + e\,dp$$

$$dP = a\,dx + b\,dp.$$

Then the 1-form

$$du - p\,dx + P\,dX$$

is closed since $bc - ae = 1$, and thus there is a function U such that

$$dU - P\,dX = du - p\,dx.$$

By inspection, X and P are linear in x, p while U is quadratic in x and p and linear in u, implying that $(x, u, p) \to (X, U, P)$ is a contact transformation defined on all of (x, u, p) space. By construction

$$\Omega = (dU - P\,dX) \wedge dt$$

$$\Upsilon = dU \wedge dX + dP \wedge dt$$

so that equation (i)′ is transformed into equation (ii)′.

This discussion illustrates the essential use of the full symmetry group in the theory of exterior differential systems, going beyond the changes of dependent and independent variable of classical PDE, and even beyond the more recently fashionable gauge transformations. Unless we use the full group of contact transformations, it looks as though we have many more classes of evolution equations admitting a conservation law than is actually the case.

6. Integrable extensions of parabolic systems. In this section, we will introduce and develop the beginnings of a structure theory for a wide generalization of the notion of conservation law, which we shall call *integrable extensions*. This generalization has been considered before in other contexts and is closely related to the notion of "pseudopotentials" as defined by Estabrook and Wahlquist [EW], a concept they developed in the course of their work on the $K\,dV$ equation.

In fact, the conservation laws we have been studying up to this point are often called *local* conservation laws in the literature, meaning that they are defined as 1-forms on some suitable space of jets of solutions of the equation.

If a given parabolic equation does admit a local conservation law, there is a process of "adjoining a potential" associated to this conservation law to create an extended system, which is also parabolic. This extended system may also admit conservation laws, even an infinite number of them. These new conservation laws are often called *nonlocal* conservation laws.

Before giving a general discussion, we shall illustrate this phenomenon for Burger's equation, recovering in the process the famous Hopf-Cole transformation.

Example 7 (continued): The Hopf-Cole transformation. We saw in §5 that the exterior differential system corresponding to Burger's equation

$$u_t = u_{xx} + 2uu_x = (u_x + u^2)_x$$

admits a single conservation law. Namely, in the 4-space with coordinates (x, t, u, p), the system $\mathscr{I}$ is generated by the 2-forms

$$\Omega = (du - p\,dx) \wedge dt$$

$$\Upsilon = du \wedge dx + d(p + u^2) \wedge dt,$$

and Υ is (up to constant multiples) the only closed 2-form which is a linear combination of Ω and Υ.

Of course, $\Upsilon = dv$ where

$$v = u\,dx + (p + u^2)\,dt.$$

The form v is closed on every integral manifold of our system, and this suggests that, corresponding to each solution u of Burger's equation, there should be a "potential" function v (uniquely determined up to an additive constant) so that

$$dv = u\,dx + (p + u^2)\,dt.$$

We are now going to see what happens if we regard v as a new coordinate and make the above relationship a part of our differential ideal: We will augment our given ideal $\mathscr{I}$ by adding the 1-form ψ where

$$\psi = dv - v\,dv - u\,dx - (p + u^2)\,dt.$$

Then the system $\mathscr{I}^+$ generated by $\mathscr{I}$ and ψ becomes the Monge-Ampere form of the exterior differential system for the equation

$$v_t = v_{xx} + (v_x)^2.$$

The system $\mathscr{I}^+$ has its invariants T and U equal to 0. Thus, we can deprolong it to a system $\mathscr{S}$ defined on $\mathbb{R}^4$ with coordinates (x, t, v, u) with generators

$$\Omega' = (dv - u\,dx) \wedge dt$$

$$\Upsilon' = dv \wedge dx + (du + u^2\,dx) \wedge dt.$$

Now, we can compute the conservation laws for $\mathscr{S}$. We find that the space of conservation laws is of infinite dimension.[1] Of course this implies by our structure theory that the equation $v_t = v_{xx} + (v_x)^2$ must be linearizable by a contact transformation.

[1] Here is how this computation goes: After applying the standard technique, the conservation laws are found to be represented by 2-forms of the form

$$\Phi = A\Upsilon' + B\Omega'$$

where A and B satisfy the condition that there exist functions A_2 and B_2 so that

$$dA = A_2\,dt + A\,dv + (Au - B)\,dx$$

$$dB = B_2\,dt + B\,dv + (A_2 + Au^2 - Bu)\,dx + A\,du.$$

Now, if we regard A and B as new variables and set $\alpha = dA - A\,dv - (Au - B)\,dx$ and $\beta = dB - B\,dv - (Au^2 - Bu)\,dx - A\,du$, then we see that the conservation laws correspond to the integral manifolds of the system

$$\mathscr{T} = \{\alpha \wedge dt, \alpha \wedge dx + \beta \wedge dt\}.$$

The system $\mathscr{T}$ is closed under exterior differentiation and it is easy to see that, with the independence condition $dx \wedge dt \neq 0$, this system is involutive with characters $(s_1', s_2', s_3', s_4') = (2, 0, 0, 0)$. Of course, this means that the space of integral manifolds is of infinite dimension.

Indeed, one can see by inspection that the above equation for v is equivalent to

$$(e^v)_t = (e^v)_{xx}.$$

It follows that $u = v_x = (\log(w))_x$ where w is a positive solution of the classical heat equation, i.e., $w_t = w_{xx}$.

Of course, this representation of the solutions of Burger's equation is well known. In fact, the introduction of the function v is known as the *Hopf-Cole* transformation in the literature on Burger's equation. More generally, in §2.1.4 of [MSS], it is pointed out that all of the "integrable" evolutionary parabolic equations (which the authors classify) can be linearized by adjoining potentials, possibly in sequence. Our point here is that the discovery of this linearizing transformation is a natural consequence of interpreting a conservation law as a potential, together with our general linearization result.

This idea of adjoining a potential has been considerably generalized. Perhaps the most famous example of this generalization is the so-called Bäckland transformation for the sine-Gordon equation $u_{xt} = \sin u$. Recall that, if $u(x, t)$ is any solution of this equation, then, for any constant $\lambda \neq 0$, the overdetermined system

$$v_x = u_x + (2\lambda^{-1}) \sin(v + u)/2$$

$$v_t = -u_t + (2\lambda) \sin(v - u)/2$$

is compatible and each of the solutions $v(x, t)$ is also a solution of the sine-Gordon equation.

Estabrook and Wahlquist [EW] interpreted this fact as follows: Consider, on $\mathbb{R}^5$ with coordinates (x, t, u, p, q), the differential system $\mathcal{I}$ generated by the 1-form $\theta = du - p\,dx - q\,dt$ and the 2-form $\Omega = (dp - \sin u\,dt) \wedge dx$. The integral manifolds of $(\mathcal{I}, dx \wedge dt)$ are then locally the graphs of solutions of the sine-Gordon equation. Now, the 1-form on $\mathbb{R}^6 = \mathbb{R}^5 \times \mathbb{R}$ defined by

$$\psi = dv - (p + (2/\lambda) \sin(v + u)/2)\,dx + (q - (2\lambda) \sin(v - u)/2)\,dt$$

has the (easily verified) property that $d\psi$ is in the algebraic ideal $\mathcal{I}^+$ generated by $\mathcal{I}$ and ψ. Thus, if $N^2 \subset \mathbb{R}^5$ is an integral manifold of $\mathcal{I}$, then $d\psi \equiv 0 \bmod \psi$ on $N \times \mathbb{R} \subset \mathbb{R}^6$. Hence, $\mathrm{N} \times \mathbb{R}$ is foliated by integral manifolds of $\mathcal{I}^+$.

Motivated by comparing how the function v was introduced in the example of Burger's equation and how v is introduced in the example of sine-Gordon, Estabrook and Wahlquist decided to call the v in the sine-Gordon example a "pseudopotential". Considering that we used a (local) conservation law to generate the potential in Burger's equation and that this led us to construct new, nonlocal conservation laws, we are tempted to regard "pseudopotentials" as a generalization of (local) conservation laws. From the point of view of generalized symmetries, this notion has, of course, already been developed extensively. In particular, the reader may want to compare the work of Bluman [Bl] and Krasilshchik and Vinogradov [KV] with our treatment.

For reasons stemming from certain similarities with (differential) Galois theory, we prefer to call constructions of this nature *integrable extensions*. The precise definition of the concept as we shall use it is as follows.

Definition. Given an exterior differential system with independence condition $(\mathscr{I}, \Omega)$ on a manifold M^m, an *integrable extension* $\mathscr{E} = (P, \sigma, \mathscr{T})$ of $(\mathscr{I}, \Omega)$ over M is a smooth manifold P^{m+r} together with a submersion $\sigma: P \to M$ and a differential ideal $\mathscr{T}$ on P with the properties that, first, $\mathscr{T}$ contains $\sigma^*(\mathscr{I})$ and that, second, there exists a Pfaffian system $\mathscr{V}$ of rank r on P which is transverse to the fibers of σ and so that, *algebraically*, $\mathscr{T}$ is generated by $\mathscr{V}$ and $\mathscr{I}$.

Locally, an integrable extension $\mathscr{T}$ of a system $\mathscr{I}$ can be described by adding to the generators for $\mathscr{I}$ a set of r 1-forms ψ^ρ $(1 \leqslant \rho \leqslant r)$ which are linearly independent on the fibers of σ and which satisfy the differential conditions

$$d\psi^\rho \equiv 0 \bmod \mathscr{I}, \psi^1, \ldots, \psi^r.$$

The reason that this is called an integrable extension is clear: If $N \subset M$ is an integral manifold of $\mathscr{I}$, then the system $\mathscr{T}$ restricted to $\sigma^{-1}(N)$ becomes a Frobenius system. There is an r-parameter family of integral manifolds of $\mathscr{T}$ lying over each integral manifold of $\mathscr{I}$.

We say that the extension $\mathscr{E}$ is *flat* if P can be covered by open sets on which the complement $\mathscr{V} \subset \mathscr{T}$ can be chosen to be an integrable (i.e., Frobenius) system. Clearly, all flat extensions of the same extension degree r (=dimension of the fibers of σ) are locally equivalent.

One way of generating local integrable extensions, directly generalizing what we did in the case of Burger's equation, is to take a differential system $\mathscr{I}$ which admits r conservation laws represented by closed 2-forms $\Upsilon^1, \ldots, \Upsilon^r$, consider r new variables p^ρ, and define $\psi^\rho = dp^\rho - v^\rho$, where v^ρ is a (locally defined) 1-form which satisfies $dv^\rho = \Upsilon^\rho$. We shall call this construction an *extension by conservation laws*.

The interesting question is whether this is essentially the only type of integrable extension. It is easy to see that the sine-Gordon example given above is not an extension by conservation laws, but the sine-Gordon example is hyperbolic and our main interest in this paper is the parabolic case.

The classification of integrable extensions. We will now specialize to the case of a non-Goursat parabolic system. Our main result in this section is the following one which shows that, except in the case of those Monge-Ampere systems with T and U equal to zero, the integrable extensions are easily classified.

THEOREM 1. *Let $\mathscr{I}$ be a non-Goursat parabolic system on M^7, and let $\mathscr{I}^{\langle\infty\rangle}$ be its infinite prolongation on $M^{\langle\infty\rangle}$. Let $\mathscr{E} = (P, \sigma, \mathscr{T})$ be an integrable extension of*

$\mathcal{I}^{\langle\infty\rangle}$ over $M^{\langle\infty\rangle}$. If $\mathcal{I}$ is not of Monge-Ampere type, then $\mathcal{E}$ is flat. If $\mathcal{I}$ is of Monge-Ampere type, then locally $\mathcal{E}$ can be regarded as the prolongation of an integrable extension over the underlying contact 5-manifold and the local structure equations of P can be written in the form

$$d\psi^\rho \equiv A^\rho(\omega^1 \wedge \theta_0 + \omega^2 \wedge \theta_1) + B^\rho\omega^2 \wedge \theta_0 \bmod \psi^1, \ldots, \psi^r.$$

If the Monge-Ampere invariant T of $\mathcal{I}$ is nonzero and $\mathcal{I}$ admits a conservation law, then locally there is a unique nonflat integrable extension $\mathcal{E}_1$ of $\mathcal{I}$ of fiber dimension 1 and all nonflat extensions of $\mathcal{I}$ of higher fiber dimension are flat extensions of $\mathcal{E}_1$. If the Monge-Ampere invariant T of $\mathcal{I}$ is nonzero and $\mathcal{I}$ does not admit a conservation law, or else if T vanishes but U does not, then $\mathcal{E}$ is flat.

Proof. First, choose an $\mathcal{I}^{\langle\infty\rangle}$-coframing $(\omega^1, \omega^2; \theta_0, \theta_1, \ldots)$ of $M^{\langle\infty\rangle}$ which satisfies the structure equations

$$d\theta_k \equiv -\theta_{k+1} \wedge \omega^1 - \theta_{k+2} \wedge \omega^2 \bmod \Theta_k\,(=\theta_0, \ldots, \theta_k).$$

(Such a coframing was constructed in §1.) By the integrable extension hypothesis, it is possible to choose a complement $\mathcal{V}$ to $\mathcal{I}$ in $\mathcal{T}$ with local generators ψ^ρ $(1 \leqslant \rho \leqslant r)$ so that, for some p and q sufficiently large, the structure equations of the following form are valid:

$$d\psi^\rho \equiv \sum_{j=0}^{p-1} a_j^\rho\omega^1 \wedge \theta_j + \sum_{j=0}^{p} b_j^\rho\omega^2 \wedge \theta_j + \sum_{0\leqslant i<j\leqslant q} c_{ij}^\rho\theta_i \wedge \theta_j \bmod \psi^1, \ldots, \psi^r.$$

More explicitly, there exist 1-forms φ_β^ρ (which are linear combinations of the ψ^σ, the ω^i, and the θ_k) so that

$$d\psi^\rho = -\varphi_\beta^\rho \wedge \psi^\beta + \sum_{j=0}^{p-1} a_j^\rho\omega^1 \wedge \theta_j + \sum_{j=0}^{p} b_j^\rho\omega^2 \wedge \theta_j + \sum_{0\leqslant i<j\leqslant q} c_{ij}^\rho\theta_i \wedge \theta_j.$$

Taking the exterior derivative of both sides and reducing modulo Θ_p, the ψ^ρ, and the "quadratic ideal" $(\Theta_\infty)^2$ yields the congruence

$$0 \equiv (a_{p-1}^\rho\omega^1 + b_{p-1}^\rho\omega^2) \wedge (\theta_{p+1} \wedge \omega^2) + (b_p^\rho\omega^2) \wedge (\theta_{p+1} \wedge \omega^1 + \theta_{p+2} \wedge \omega^2).$$

It follows that $a_{p-1}^\rho = b_p^\rho$. If $p = 1$, this says that, with a slight relabeling, the structure equations assume the form

$$d\psi^\rho = -\varphi_\beta^\rho \wedge \psi^\beta + B^\rho\omega^2 \wedge \theta_0 + A^\rho(\omega^1 \wedge \theta_0 + \omega^2 \wedge \theta_1) + \sum_{0\leqslant i<j\leqslant q} c_{ij}^\rho\theta_i \wedge \theta_j.$$

On the other hand, if $p \geqslant 2$, the relation $a_{p-1}^\rho = b_p^\rho$ shows that $d\psi^\rho$ can be rewritten in the form

$$d\psi^\rho \equiv \sum_{j=0}^{p-2} a_j^\rho \omega^1 \wedge \theta_j + \sum_{j=0}^{p-1} b_j^\rho \omega^2 \wedge \theta_j + \sum_{0 \leqslant i < j \leqslant q} c_{ij}^\rho \theta_i \wedge \theta_j$$

$$+ a_{p-1}^\rho(\omega^1 \wedge \theta_{p-1} + \omega^2 \wedge \theta_p).$$

Since $d\theta_{p-2} \equiv \omega^1 \wedge \theta_{p-1} + \omega^2 \wedge \theta_p \bmod \Theta_{p-2}$, it follows that, by modifying a_i^ρ and b_i^ρ for $0 \leqslant i \leqslant p-2$ and the appropriate c_{ij}^ρ as well and by possibly raising q, we may rewrite the structure equations in the form

$$d(\psi^\rho - a_{p-1}^\rho \theta_{p-2}) = -\varphi_\beta^\rho \wedge (\psi^\beta - a_{p-1}^\beta \theta_{p-2}) - \varphi_\beta^\rho \wedge (a_{p-1}^\beta \theta_{p-2}) + \sum_{j=0}^{p-2} a_j^\rho \omega^1 \wedge \theta_j$$

$$+ \sum_{j=0}^{p-1} b_j^\rho \omega^2 \wedge \theta_j + \sum_{0 \leqslant i < j \leqslant q} c_{ij}^\rho \theta_i \wedge \theta_j.$$

Thus, replacing each ψ^ρ by $\psi^\rho - a_{p-1}^\rho \theta_{p-2}$, we reduce p by one. (Some care is necessary here. The point is that the terms $\varphi_\beta^\rho \wedge \theta_{p-2}$, when expanded in terms of the new local coframing, consist of terms which are either in the ideal generated by the ψ^ρ, are quadratic in the θ_k, or else are of the form $\omega^i \wedge \theta_{p-2}$. Redistributing them as appropriate does not raise p.) Note that the operation of modifying the ψ^ρ affects the choice of the splitting $\mathscr{V}$ (which is spanned by the ψ^ρ), but of course it does not affect the ideal $\mathscr{T}$.

If $p - 1 \geqslant 2$, this construction can be repeated. In fact, by repeating this construction at most $p-2$ times, we reduce to the case where $p = 1$. Thus we may assume that we have chosen generators ψ^ρ so that the structure equations have the form

$$d\psi^\rho = -\varphi_\beta^\rho \wedge \psi^\beta + A^\rho(\omega^1 \wedge \theta_0 + \omega^2 \wedge \theta_1) + B^\rho \omega^2 \wedge \theta_0 + \sum_{0 \leqslant i < j \leqslant q} c_{ij}^\rho \theta_i \wedge \theta_j.$$

It is important to remark that the space $\mathscr{V}$ spanned by the ψ^ρ which satisfy these equations is a canonically defined complement to $\mathscr{I}$ in $\mathscr{T}$.

We will now derive further limitations on the c_{ij}^ρ. Let $s \geqslant 0$ be the largest integer so that there exists a nonzero c_{ij}^ρ so that $i + j = s$. For each $k \geqslant 0$ let Υ_k denote the *algebraic* ideal generated by $\psi^1, \ldots, \psi^r, \theta_0, \theta_1$, and all of the quadratic terms $\theta_i \wedge \theta_j$ where $i + j \leqslant k$. Expanding $d(d\psi^\rho) = 0$ and reducing modulo Υ_s yields the relations

$$\sum_{\substack{i+j=s-1\\ i<j}} c_{ij}^\rho[(-\theta_{i+2} \wedge \omega^2) \wedge \theta_j - \theta_i \wedge (-\theta_{j+2} \wedge \omega^2)] \equiv 0$$

$$\sum_{\substack{i+j=s\\ i<j}} c_{ij}^\rho[(-\theta_{i+1} \wedge \omega^1 - \theta_{i+2} \wedge \omega^2) \wedge \theta_j - \theta_i \wedge (-\theta_{j+1} \wedge \omega^1 - \theta_{j+2} \wedge \omega^2)] \equiv 0,$$

where the congruences are modulo θ_0 and θ_1. The second of these two relations uncouples into the relations

$$\sum_{\substack{i+j=s\\ i<j}} c_{ij}^{\rho}(\theta_{i+1} \wedge \theta_j + \theta_i \wedge \theta_{j+1}) \equiv \sum_{\substack{i+j=s\\ i<j}} c_{ij}^{\rho}(\theta_{i+2} \wedge \theta_j + \theta_i \wedge \theta_{j+2}) \equiv 0,$$

where, again, the congruences are taken modulo θ_0 and θ_1. For $s > 3$, these relations imply that $c_{ij}^{\rho} = 0$ for all $i + j = s$, contradicting our choice of s. Moreover, when $s = 3$ these relations imply that $c_{12}^{\rho} = c_{03}^{\rho}$. Now, writing C_i^{ρ} in place of c_{0i}^{ρ}, we have $d(\psi^{\rho})$ in the form

$$d\psi^{\rho} = -\varphi_{\beta}^{\rho} \wedge \psi^{\beta} + A^{\rho}(\omega^1 \wedge \theta_0 + \omega^2 \wedge \theta_1) + B^{\rho}\omega^2 \wedge \theta_0$$
$$+ C_1^{\rho}\theta_0 \wedge \theta_1 + C_2^{\rho}\theta_0 \wedge \theta_2 + C_3^{\rho}(\theta_0 \wedge \theta_3 + \theta_1 \wedge \theta_2).$$

Now, the terms C_3^{ρ} must all vanish. To see this, let us suppose that the forms $(\theta_0, \theta_1, \theta_2, \omega^2, \omega^1, \theta_3, \theta_4)$ satisfy the structure equations of a 1-adapted coframing as defined in §2. A short calculation shows that

$$d(d\psi^{\rho}) \equiv 2C_3^{\rho}\theta_1 \wedge \theta_3 \wedge \omega^1 \bmod \psi^1, \ldots, \psi^r, \theta_0, \omega^2, \theta_1 \wedge \theta_2.$$

Of course, this implies that $C_3^{\rho} = 0$. Now substituting this back into the calculation for $d(d\psi^{\rho})$, the formula simplifies to show that

$$d(d\psi^{\rho}) \equiv -(C_2^{\rho}\omega^1 - A^{\rho}\phi) \wedge \theta_2 \wedge \theta_1 \bmod \psi^1, \ldots, \psi^r, \theta_0, \omega^2.$$

Of course, this implies that $A^{\rho}\phi - C_2^{\rho}\omega^1 \equiv 0 \bmod \theta_0, \theta_1, \theta_2, \omega^2$. However, since $\phi = S_0\theta_3 + S_1\omega^1 + S_3\omega^2$, this clearly implies that $A^{\rho}S_0 = 0$ and $C_2^{\rho} = A^{\rho}S_1$.

Now suppose that we are on the open set where $S_0 \neq 0$. Then we must have $A^{\rho} = 0$ and hence $C_2^{\rho} = 0$. The formula for $d(d\psi^{\rho})$ now simplifies to

$$d(d\psi^{\rho}) \equiv (-C_1^{\rho}\theta_2 + B^{\rho}\omega^1) \wedge \omega^2 \wedge \theta_1 \bmod \psi^1, \ldots, \psi^r, \theta_0.$$

Of course, this implies that $B^{\rho} = C_1^{\rho} = 0$. But now we have

$$d(\psi^{\rho}) \equiv 0 \bmod \psi^1, \ldots, \psi^r,$$

so the extension $\mathcal{T}$ is flat. Thus, we have shown that, on the open set where $\mathcal{I}$ is not of Monge-Ampere type, any integral extension must be flat.

Let us restrict to the case of a Monge-Ampere system and suppose that the $\mathcal{I}$-coframing we chose was 3-adapted. The structure equations for the ψ^{ρ} now reduce to

$$d\psi^{\rho} = -\varphi_{\beta}^{\rho} \wedge \psi^{\beta} + A^{\rho}(\omega^1 \wedge \theta_0 + \omega^2 \wedge \theta_1) + B^{\rho}\omega^2 \wedge \theta_0$$
$$+ C_1^{\rho}\theta_0 \wedge \theta_1 + C_2^{\rho}\theta_0 \wedge \theta_2.$$

In particular, coupled with the structure equations of a 3-adapted coframe, these equations imply that the Pfaffian system $\mathscr{K}$ spanned by the $(r+5)$ 1-forms $\psi^1, \ldots, \psi^r, \theta_0, \theta_1, \theta_2, \omega^1, \omega^2$ form a Frobenius (i.e., completely integrable) system. It follows that locally there is a submersion of P onto a smooth manifold Q of dimension $(r+5)$ so that the system $\mathscr{K}$ is the pull-back of the 1-forms on Q. Moreover, from the formulas for $d\psi^\rho$, we see that the rank-r system $\mathscr{V}$ spanned by the ψ^ρ is well defined on Q. Moreover, Q has a natural smooth submersion to the underlying 5-manifold on which the Monge-Ampere system is defined. Thus, we may clearly disregard any of the extra prolongation variables and regard $\mathscr{T}$ on P as an integrable extension of a parabolic Monge-Ampere system on a 5-manifold.

Now, we easily compute that

$$d(d\psi^\rho) \equiv -C_2^\rho \omega^1 \wedge \theta_2 \wedge \theta_1 \bmod \psi^1, \ldots, \psi^r, \theta_0, \omega^2,$$

so we must have $C_2^\rho = 0$. Moreover, the formula for $d(d\psi^\rho)$ now simplifies further to show that

$$d(d\psi^\rho) \equiv (dA^\rho - A^\rho(\alpha + \rho) + A^\beta \varphi_\beta^\rho + B^\rho \omega^1 - C_1^\rho \theta_2) \wedge \omega^2 \wedge \theta_1$$
$$\times \bmod \psi^1, \ldots, \psi^r, \theta_0.$$

It follows that there are functions A_0^ρ, A_1^ρ, and A_2^ρ, so that

$$dA^\rho \equiv A^\rho(\alpha + \rho) - A^\beta \varphi_\beta^\rho + A_0^\rho \theta_0 + A_1^\rho \theta_1 + A_2^\rho \omega^2 - B^\rho \omega^1$$
$$+ C_1^\rho \theta_2 \bmod \psi^1, \ldots, \psi^r.$$

Substituting this back into the formula for $d(d\psi^\rho)$ yields the formula

$$d(d\psi^\rho) \equiv 2C_1^\rho \theta_0 \wedge \theta_2 \wedge \omega^1 \bmod \psi^1, \ldots, \psi^r, \omega^2, \theta_0 \wedge \theta_1.$$

Hence, C_1^ρ must also vanish.

Now, we deal with the case where the Monge-Ampere invariant T is nonzero. Substituting the relations $C_i^\rho = 0$ back into the structure equations, we compute that

$$d(d\psi^\rho) \equiv (A^\rho(P_1 - U) - 2B^\rho T)\theta_2 \wedge \theta_1 \wedge \theta_0$$
$$- (A_1^\rho + A^\rho H)\omega^1 \wedge \theta_1 \wedge \theta_0 \bmod \psi^1, \ldots, \psi^r, \omega^2.$$

In particular, it follows that we must have $A^\rho(P_1 - U) - 2B^\rho T = 0$. Thus, the structure equations for $d\psi^\rho$ must have the form

$$d\psi^\rho \equiv D^\rho[2T(\omega^1 \wedge \theta_0 + \omega^2 \wedge \theta_1) + (P_1 - U)\omega^2 \wedge \theta_0] \bmod \psi^1, \ldots, \psi^r.$$

If all the D^ρ are zero, then the extension is flat and we are done. Thus, we may suppose that at least one of the D^ρ is nonzero. For convenience, let us set

$$\Psi = 2T(\omega^1 \wedge \theta_0 + \omega^2 \wedge \theta_1) + (P_1 - U)\omega^2 \wedge \theta_0.$$

It is then possible to make a change of basis in the ψ^ρ so that

$$d\psi^\rho \equiv 0 \bmod \psi^1, \ldots, \psi^r \qquad \text{for } 1 \leqslant \rho < r$$

while $d\psi^r \equiv \Psi \bmod \psi^1, \ldots, \psi^r$. It follows that there exist 1-forms ξ^ρ for $1 \leqslant \rho < r$ so that

$$d\psi^\rho \equiv \xi^\rho \wedge \psi^r \bmod \psi^1, \ldots, \psi^{r-1} \qquad \text{for } 1 \leqslant \rho < r.$$

However, taking the exterior derivative of these equations and reducing modulo $\psi^1, \ldots, \psi^r$ yields

$$\xi^\rho \wedge \Psi \equiv 0 \bmod \psi^1, \ldots, \psi^r.$$

Since $\Psi \wedge \Psi \not\equiv 0 \bmod \psi^1, \ldots, \psi^r$, it follows that we must have $\xi^\rho \equiv 0 \bmod \psi^1, \ldots, \psi^r$. Of course, this implies that $d\psi^\rho \equiv 0 \bmod \psi^1, \ldots, \psi^{r-1}$ for $1 \leqslant \rho < r$. In other words, the system $\mathscr{V}'$ spanned by the first $(r-1)$ of the ψ^ρ is itself Frobenius. Thus, the dimension r extension of $\mathscr{I}$ is actually a dimension-1 extension of a flat dimension $(r-1)$ extension of $\mathscr{I}$.

Let us now restrict to a leaf L of $\mathscr{V}'$. Then all of the ψ^ρ for $\rho < r$ vanish. Writing ψ and φ for ψ^r and φ^r_r, respectively, we have the remaining structure equation

$$d\psi = -\varphi \wedge \psi + \Psi.$$

Differentiating this equation and reducing modulo ψ now yields $0 \equiv \varphi \wedge \Psi + d\Psi \bmod \psi$.

Now, Ψ and $d\Psi$ are defined on the manifold M and, by hypothesis, ψ is linearly independent from all of the forms on M. Thus, the relation $d\Psi \equiv -\varphi \wedge \Psi \bmod \psi$ cannot have a solution unless there exists a 1-form λ on M which satisfies $d\Psi = -\lambda \wedge \Psi$. Thus, we will have a contradiction unless such a λ exists. In particular, unless such a λ exists, every integrable extension of $\mathscr{I}$ must be flat. (Note, by the way, that if such a λ does not exist, then no multiple of Ψ can be closed and the space of conservation laws of $\mathscr{I}$ is trivial.)

Therefore, let us suppose that a 1-form λ exists on M so that $d\Psi = -\lambda \wedge \Psi$. Because Ψ is not decomposable, this λ is unique and hence is well defined on M. Substituting this back into our relation, we see that $(\lambda - \varphi) \wedge \Psi \equiv 0 \bmod \psi$. Again, since $\Psi \wedge \Psi \not\equiv 0 \bmod \psi$, it follows that $\varphi \equiv \lambda \bmod \psi$. Thus, the structure equation has become

$$d\psi = -\lambda \wedge \psi + \Psi.$$

Differentiating and using the relation $d\Psi = -\lambda \wedge \Psi$ now yields $d\lambda \wedge \psi = 0$. However, since ψ is linearly independent from the forms on M, this can only be true if $d\lambda = 0$. Thus, locally, we may set $\lambda = L^{-1}\,dL$ for some nonzero function L on M and rewrite the structure equation in the form $d(L\psi) = L\Psi$.

Of course this implies that $L\Psi$ is a conservation law for $\mathscr{I}$. In particular, we have shown that, unless $\mathscr{I}$ admits a conservation law, then any integrable extension of $\mathscr{I}$ is flat.

Let us now continue and replace Ψ by $L\Psi$, so that Ψ is a closed 2-form on M which represents the conservation law for $\mathscr{I}$ (which is unique up to constant multiples). Our arguments so far have shown that any nonflat extension has local structure equations of the form

$$\left.\begin{aligned} d\psi^\rho &\equiv 0 \\ d\psi^r &\equiv -\varphi \wedge \psi^r + \Psi \end{aligned}\right\} \bmod \psi^1, \ldots, \psi^{r-1}.$$

Differentiating this last equation yields $\varphi \wedge \Psi \equiv 0 \bmod \psi^1, \ldots, \psi^r$. Again, since $(\Psi \wedge \Psi) \not\equiv 0 \bmod \psi^1, \ldots, \psi^r$, it follows that $\varphi \equiv 0 \bmod \psi^1, \ldots, \psi^r$. Of course, this means that the equations above simplify to

$$\left.\begin{aligned} d\psi^\rho &\equiv 0 \\ d\psi^r &\equiv \Psi \end{aligned}\right\} \bmod \psi^1, \ldots, \psi^{r-1}.$$

Now let μ be a local 1-form on M satisfying $d\mu = \Psi$. The system $\{\psi^1, \ldots, \psi^{r-1}, \psi - \mu\}$ is then completely integrable, and it is easy to show that there are locally defined functions $p^1, \ldots, p^r$ so that the system $\mathscr{V}$ is generated by the 1-forms $dp^1, \ldots, dp^{r-1}, dp^r + \mu$ and the system $\mathscr{V}'$ is generated by the 1-forms $dp^1, \ldots, dp^{r-1}$. Of course, this is precisely the statement that the integrable extensions are flat extensions of a unique nontrivial extension of rank 1.

Finally, assume that T vanishes but U does not. In this case, the invariant P_1 vanishes and the relation that we derived above now shows that $A^\rho U = 0$. Of course, this implies that A^ρ is zero, so the structure equations now simplify to

$$d\psi^\rho \equiv B^\rho \omega^2 \wedge \theta_0 \bmod \psi^1, \ldots, \psi^r.$$

If all of the B^ρ are identically zero, then clearly the extension is flat, so there is nothing to prove. If at least one of the B^ρ is nonzero, it then follows that the Cartan system of $\mathscr{V}$ is the system $C(\mathscr{V}) = \{\psi^1, \ldots, \psi^r, \omega^2, \theta_0\}$. Since the Cartan system of any Pfaffian system is completely integrable, this would, in particular, imply that $d\theta_0 \equiv 0 \bmod C(\mathscr{V})$. However, the structure equation $d\theta_0 \equiv -\theta_1 \wedge \omega^1 \bmod \mathscr{I}$ clearly contradicts this. Thus, the proof of Theorem 1 is complete. □

In view of Theorem 1, it follows that the only remaining case in the problem of classifying the possible integrable extensions of $\mathscr{I}$ is the case where the invariants T and U vanish. The classification in this case is rather subtle, and we have not completed it. We will content ourselves with showing that, in this case, there can be nontrivial integrable extensions even when there are no local conservation laws.

One general method of constructing integrable extensions can be described as follows: Let $P \to M$ be a principal right G-bundle over a manifold M and let ψ be a connection on P, regarded as a $\mathfrak{g}$-valued 1-form on P. Let $\mathscr{I}$ be the differential ideal on M generated by the components of the corresponding curvature 2-form $\Psi = d\psi + (1/2)[\psi, \psi]$. Then clearly ψ defines an integrable extension of $\mathscr{I}$.

The case of extensions by conservation laws is covered by the case where G is an abelian group. If G is nonabelian, then it may well happen that $\mathscr{I}$ has no conservation laws.

It remains to be seen if one can construct such a connection for which the ideal $\mathscr{I}$ is a non-Goursat parabolic system. In the following example, we analyse the possibilities for the simple group SO(3). (We chose the Lie group SO(3) because of its simplicity. The group SL(2, $\mathbb{R}$) would have worked equally well and the results for this group are very similiar to those for SO(3), with obvious modifications made to take into account the slightly more complicated orbit structure of the adjoint representation of SL(2, $\mathbb{R}$).)

Example 9: SO(3)-*connection extensions.* We want to understand the SO(3)-connections α over a 4-manifold M such that the ideal $\mathscr{I}$ generated by the 2-form components of the curvature of α is a non-Goursat parabolic system. We also want to find examples for which this parabolic system does not have any conservation laws.

Let α be an SO(3)-valued 1-form on a 4-manifold M. For simplicity in the discussion below, we shall assume that M is simply connected.

We may write α in the form

$$\alpha = \begin{bmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{bmatrix},$$

where the α_i are ordinary 1-forms. The curvature of α is $\Phi = d\alpha + \alpha \wedge \alpha$. It is of the form

$$\Phi = \begin{bmatrix} 0 & -\Phi_3 & \Phi_2 \\ \Phi_3 & 0 & -\Phi_1 \\ -\Phi_2 & \Phi_1 & 0 \end{bmatrix},$$

where the 2-forms Φ_i are given by the formulas

$$\Phi_1 = d\alpha_1 + \alpha_2 \wedge \alpha_3$$

$$\Phi_2 = d\alpha_2 + \alpha_3 \wedge \alpha_1$$

$$\Phi_3 = d\alpha_3 + \alpha_1 \wedge \alpha_2 .$$

Let $\mathscr{I}$ denote the ideal in $\mathscr{A}^2(M)$ which is generated by the 2-forms Φ_i. We want to determine conditions on α so that this ideal is a non-Goursat parabolic system on M.

Now, the ideal $\mathscr{I}$ depends only on the gauge-equivalence class of α. Indeed, any gauge-equivalent connection will be of the form

$$\tilde{\alpha} = g^{-1}\, dg + g^{-1}\alpha g$$

for some smooth function $g\colon M \to \mathrm{SO}(3)$. The corresponding curvature will be $\tilde{\Phi} = g^{-1}\Phi g$, the components of which clearly generate the same ideal as the components of Φ.

Our assumption is that the ideal $\mathscr{I}$ be a non-Goursat parabolic system, and hence that it is generated locally by a pair of 2-forms. This implies that there is a unique linear relation among the Φ_i. Since, under the adjoint action, SO(3) acts transitively on lines in $\mathfrak{so}(3) = \mathbb{R}^3$, it follows that we may choose a gauge-equivalent connection $\tilde{\alpha}$ so that this relation takes the form $\tilde{\Phi}_3 = 0$. The remaining forms $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ are linearly independent, though they have a linear combination (unique up to multiples) which is decomposable.

Again appealing to our knowledge of the adjoint action of SO(3), we may arrange that $(\tilde{\Phi}_2)^2 = 0$. Of course, by the parabolic assumption, we must then have $\Phi_1 \wedge \Phi_1 \neq 0$, though the parabolic assumption will also imply that $\tilde{\Phi}_2 \wedge \tilde{\Phi}_1 = 0$. These normalizations determine $\tilde{\alpha}$ uniquely in the gauge equivalence class of α up to a finite group $\mathbb{Z}_2 \times \mathbb{Z}_2$. (The assumption that M be simply connected ensures that this normalization can be done globally on M.) Since we will be working with this connection for some time, we will simply rename it to be α.

According to these normalizations, it follows that one may locally choose a coframing $(\omega^2, \theta_0, \omega^1, \theta_1)$ of M^4 so that we have equations of the form

$$d\alpha_1 + \alpha_2 \wedge \alpha_3 = \Phi_1 = \theta_0 \wedge \omega^1 + \theta_1 \wedge \omega^2$$

$$d\alpha_2 + \alpha_3 \wedge \alpha_1 = \Phi_2 = \theta_0 \wedge \omega^2$$

$$d\alpha_3 + \alpha_1 \wedge \alpha_2 = \Phi_3 = 0.$$

First, we rule out the possibility that $\alpha_1 \wedge \alpha_2 = 0$ on any open set in M. Suppose, to the contrary, that this is the case. Then $\alpha_3 = df$ for some function f and so the connection $\tilde{\alpha}$ generated by the above formula with

$$g = \begin{bmatrix} \cos f & \sin f & 0 \\ -\sin f & \cos f & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

will satisfy $\tilde{\alpha}_3 = 0$ and $\tilde{\alpha}_1 \wedge \tilde{\alpha}_2 = 0$. Thus, our ideal $\mathscr{I}$ will be generated by $\tilde{\Phi}_1 = d\tilde{\alpha}_1$ and $\tilde{\Phi}_2 = d\tilde{\alpha}_2$. Now, by our hypothesis that these forms generate a parabolic system, we clearly cannot have either $\tilde{\alpha}_1$ or $\tilde{\alpha}_2$ vanishing on an open set. Let us then restrict our attention to the dense open set on which neither $\tilde{\alpha}_1$ nor $\tilde{\alpha}_2$ is zero. Then we may write $\tilde{\alpha}_2 = h\tilde{\alpha}_1$, where h is a nonzero function. Our ideal $\mathscr{I}$ is now generated in this open set by $d\tilde{\alpha}_1$ and $d\tilde{\alpha}_2 = d(h\tilde{\alpha}_1) = dh \wedge \tilde{\alpha}_1 + h\, d\tilde{\alpha}_1$. Of course, since the parabolic ideal $\mathscr{I} = \{d\tilde{\alpha}_1, dh \wedge \tilde{\alpha}_1 + h\, d\tilde{\alpha}_1\}$ can contain only one nonzero decomposable 2-form up to multiples, it follows that this must be $\Omega = dh \wedge \tilde{\alpha}_1$. Moreover, since Ω wedges with anything in the ideal to give zero, it follows that we must have $dh \wedge \tilde{\alpha}_1 \wedge d\tilde{\alpha}_1 = 0$. However, this implies that $d\Omega = -dh \wedge d\tilde{\alpha}_1$ satisfies $d\Omega \wedge \tilde{\alpha}_1 = 0$, and we clearly have $d\Omega \wedge dh = 0$. These combine to imply that $d\Omega$ is a multiple of $dh \wedge \tilde{\alpha}_1 = \Omega$. In particular, the form Ω is integrable, contrary to our assumption that the ideal $\mathscr{I}$ be a non-Goursat parabolic system.

Henceforth, we shall restrict our attention to the dense open set on which $\alpha_1 \wedge \alpha_2$ is nonzero.

Since $\alpha_1 \wedge \alpha_2$ is a nonzero, closed, decomposable 2-form, every point lies in a simply connected open set, say U, on which there exist independent functions x and y so that $\alpha_1 \wedge \alpha_2 = dx \wedge dy$. Moreover, there clearly must exist a function z on U so that $\alpha_3 = dz - x\, dy$, and there must also exist an $\mathrm{SL}(2, \mathbb{R})$-valued function A on U so that

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = A \begin{pmatrix} dx \\ dy \end{pmatrix}.$$

Now, by construction, $\Phi_1 = d\alpha_1 + \alpha_3 \wedge \alpha_2$ is a nondegenerate 2-form on U. Since α_1, α_2, and α_3 (and hence, Φ) can be expressed in terms of the functions x, y, z, and the components of A, it then follows that the map

$$(x, y, z, A)\colon U \to \mathbb{R}^3 \times \mathrm{SL}(2, \mathbb{R})$$

must be an immersion since the rank of its differential must be at least 4. This mapping is almost canonical; it depends on the choice of the functions x and y, which are determined up to a unimodular change of coordinates in two variables, and the choice of z, which is determined up to an additive constant. We are now going to show that the image of this mapping is an integral manifold of a certain natural exterior differential system on $X^6 = \mathbb{R}^3 \times \mathrm{SL}(2, \mathbb{R})$.

Indeed, let us regard x, y, and z as coordinates on the $\mathbb{R}^3$-factor of X^6 and $A: X^6 \to SL(2, \mathbb{R})$ as the projection on the second factor. Let α_1, α_2, and α_3 be defined as 1-forms on X^6 by the above formulas and set

$$dAA^{-1} = \begin{pmatrix} \pi_1 & \pi_3 \\ \pi_2 & -\pi_1 \end{pmatrix}.$$

Then the 1-forms α_i and π_i define a coframing of X^6 satisfying the structure equations

$$\begin{aligned} d\alpha_1 &= \pi_1 \wedge \alpha_1 + \pi_3 \wedge \alpha_2 & d\pi_1 &= \pi_3 \wedge \pi_2 \\ d\alpha_2 &= \pi_2 \wedge \alpha_1 - \pi_1 \wedge \alpha_2 & d\pi_2 &= 2\pi_2 \wedge \pi_1 \\ d\alpha_3 &= -\alpha_1 \wedge \alpha_2 & d\pi_3 &= 2\pi_1 \wedge \pi_3 . \end{aligned}$$

Let us define the 2-forms

$$\Phi_1 = d\alpha_1 + \alpha_2 \wedge \alpha_3 = \pi_1 \wedge \alpha_1 + (\pi_3 - \alpha_3) \wedge \alpha_2$$

$$\Phi_2 = d\alpha_2 + \alpha_3 \wedge \alpha_1 = (\pi_2 + \alpha_3) \wedge \alpha_1 - \pi_1 \wedge \alpha_2 .$$

Let $\mathscr{K}$ denote the differential ideal on X generated by the 4-forms

$$\Upsilon_1 = (\Phi_2)^2 = 2(\pi_2 + \alpha_3) \wedge \pi_1 \wedge \alpha_1 \wedge \alpha_2$$

$$\Upsilon_2 = \Phi_1 \wedge \Phi_2 = (\pi_2 + \alpha_3) \wedge (\pi_3 - \alpha_3) \wedge \alpha_1 \wedge \alpha_2$$

and consider the independence condition given by the 4-form $(\Phi_1)^2 = 2\pi_1 \wedge (\pi_3 - \alpha_3) \wedge \alpha_1 \wedge \alpha_2$.

The immersions (x, y, z, A): $U \to X^6$ constructed above are clearly integral manifolds of $(\mathscr{K}, \Phi_1 \wedge \Phi_1)$. Conversely, on any integral manifold of $(\mathscr{K}, \Phi_1 \wedge \Phi_1)$, the 2-forms Φ_1 and Φ_2 generate a parabolic system. (As we shall see below, the extra condition that this ideal be non-Goursat is a "higher" independence condition on integral manifolds of $(\mathscr{K}, \Phi_1 \wedge \Phi_1)$.)

Now, the independence condition implies that, on an integral manifold $N^4 \subset X$ of $(\mathscr{K}, \Phi_1^2)$, the 1-forms α_1, α_2, π_1, and $\pi_3 - \alpha_3$ are linearly independent. The vanishing of Υ_1 on N implies that $\pi_2 + \alpha_3$ is a linear combination of α_1, α_2, and π_1, and the vanishing of Υ_2 on N implies that $\pi_2 + \alpha_3$ is a linear combination of α_1, α_2, and $\pi_3 - \alpha_3$. Of course, these two conditions together imply that, on N, the 1-form $\pi_2 + \alpha_3$ must be a linear combination of α_1 and α_2. In other words, the 3-form $\Upsilon_0 = (\pi_2 + \alpha_3) \wedge \alpha_1 \wedge \alpha_2$ also vanishes on N. Note that Υ_1 and Υ_2 are multiples of Υ_0.

Thus, let $\mathscr{T}$ denote the exterior differential system generated by $\Upsilon_0 = (\pi_2 + \alpha_3) \wedge \alpha_1 \wedge \alpha_2$ and $d\Upsilon_0 = 2\pi_2 \wedge \pi_1 \wedge \alpha_1 \wedge \alpha_2$. We are then interested in the integral manifolds of $(\mathscr{T}, \Phi_1 \wedge \Phi_1)$. On every integral manifold of $(\mathscr{T}, \Phi_1^2)$, we have relations of the form

$$\pi_2 + \alpha_3 = p_1\alpha_1 + p_2\alpha_2$$

$$\pi_2 = p_3\alpha_1 + p_4\alpha_2 + p_5\pi_1,$$

and it easily follows that there is a 5-parameter family of integral elements of this system at every point of X. Moreover, the reduced characters are easily seen to be $(s'_0, s'_1, s'_2, s'_3, s'_4) = (0, 0, 1, 1, 0)$. It follows by Cartan's Test (see [BCG[3]]) that this system is involutive and that the integral manifolds depend on one function of three variables.

It remains to be explained how the condition of being non-Goursat is coded into this formulation. To see this, note that on an integral manifold on which the above relations hold, we will have

$$\Phi_2 = p_2\alpha_2 \wedge \alpha_1 - \pi_1 \wedge \alpha_2 = -(\pi_1 + p_2\alpha_1) \wedge \alpha_2,$$

while

$$d\Phi_2 = -\alpha_3 \wedge \Phi_1 = (p_4 - p_2)\pi_1 \wedge \alpha_1 \wedge \alpha_2 + \pi_3 \wedge (p_5\pi_1 + (p_1 - p_3)\alpha_1) \wedge \alpha_2.$$

It follows that $d\Phi_2$ is a multiple of Φ_2 if and only if we have $p_2p_5 - (p_1 - p_3) = 0$. Since $p_2p_5 - (p_1 - p_3) \neq 0$ is an open condition and the system is involutive, it follows that the generic integral manifold of $(\mathscr{T}, \Phi_1 \wedge \Phi_1)$ will determine a non-Goursat parabolic structure.

Thus, the space of gauge-diffeomorphism classes of local SO(3)-connections over open sets in $\mathbb{R}^4$ whose curvature generates a non-Goursat parabolic system $\mathscr{I}$ depends locally on one function of 3 variables. (It is possible to avoid the use of the Cartan-Kähler theorem in the above proof, but at the cost of some clarity.) Clearly, given such an α, the corresponding connection on the trivial principal SO(3)-bundle over M defines an integrable extension of $\mathscr{I}$ of rank 3 over M^4.

Note that, in the case of a non-Goursat integral, the common divisor of $\Phi_2 = \theta_0 \wedge \omega^2$ and $d\Phi_2$ is α_2. Of course, this implies that α_2 spans the first derived system of $\{\pi_1 + p_2\alpha_1, \alpha_2\} = \{\theta_0, \omega^2\}$. The induced non-Goursat parabolic system will be of quasi-evolutionary type if and only if this first derived system is completely integrable, i.e., if and only if $\alpha_2 \wedge d\alpha_2 = 0$. Since, on any integral we have

$$\alpha_2 \wedge d\alpha_2 = \pi_2 \wedge \alpha_1 \wedge \alpha_2 = p_5\pi_1 \wedge \alpha_1 \wedge \alpha_2,$$

it follows that the corresponding system represents a quasi-evolutionary system if and only if $p_5 = 0$. This represents an extra equation and it is not hard to carry out the Cartan-Kähler analysis to show that such a system can always be put in the normal form

$$\alpha_1 = z^{-1}\,dx + w\,dt$$

$$\alpha_2 = z\,dt$$

$$\alpha_3 = df - x\,dt,$$

where f is an arbitrary function of x and t satisfying $f_x > 0$. This normal form is unique up to choices involving functions of one variable. Thus, the quasi-evolutionary equations which can arise as curvature ideals of SO(3)-connections depend locally on one function of two variables.

Moreover, by applying the handy algorithm, it is easy to show that none of these systems ever have any conservation laws. Thus, we have constructed parabolic systems which have nontrivial integrable extensions, but no conservation laws, as desired.

Example 10. We shall study the modified reaction-diffusion equation

$$u_t = u^m(u_{xx} + u), \tag{1}$$

where $m \neq 0$ is a parameter. The standard reaction-dffusion equation is the case $m = 2$ studied in Example 8. We shall prove the following.

PROPOSITION. (i) *The equation (1) has exactly two independent conservation laws.*

(ii) *If we consider the integrable extension corresponding to these two conservation laws, then the new system has no conservation laws unless $m = 2$, and in that case it has exactly one conservation law.*

This is an example of a well-known phenomenon: Many interesting equations come with parameters, and the imposition of conservation laws will put (algebraic) conditions on those parameters.

Proof. The beginning of the argument is essentially the same as that for Example 8 in §5 (our equation reduces to (18) there when $m = 2$). We set $U = u^m$ and

$$\Omega = (du - p\,dx) \wedge dt$$

$$\Upsilon = du \wedge dx + U(dp + u\,dx) \wedge dt$$

$$\Phi = A\Upsilon + B\Omega,$$

and shall determine the conditions on A, B that $d\Phi = 0$. Now

$$d\Phi = dA \wedge du \wedge dx + U\,dA \wedge dp \wedge dt + \frac{mAU}{u}\,du \wedge dp \wedge dt + Uu\,dA \wedge dx \wedge dt + (mAU + AU)\,du \wedge dx \wedge dt + dB \wedge du \wedge dt - p\,dB \wedge dx \wedge dt - B\,dp \wedge dx \wedge dt.$$

Setting

$$dA = A_2\,dt + A_0(du - p\,dx) + A_1\,dx + A_3\,dp$$
$$dB = B_2\,dt + B_0(du - p\,dx) + B_1\,dx + B_3\,dp$$

and substituting into $d\Phi = 0$ gives

$$0 = (uUA_0 + mAU - B_3u)/u\,du \wedge dp \wedge dt + (A_2 + uUA_0 + mAU + AU - B_1)\,dt \wedge du \wedge dx + A_3\,dp \wedge du \wedge dx + (-UA_0p + UA_1 - uUA_3 + pB_3 + B)\,dx \wedge dp \wedge dt,$$

which implies

$$A_3 = 0$$
$$B_1 = A_2 + uUA_0 + mAU + AU$$
$$A_1 = \frac{1}{uU}(uB + pAUm)$$
$$B_3 = \frac{U}{u}(uA_0 + mA).$$

We now substitute these expressions into the formulas for dA, dB and impose the integrability condition $d^2A = 0$. This gives

$$0 = d^2A \equiv -2(A_0 + mA/u)\,dp \wedge dx \bmod (du - p\,dx), dt,$$

which implies $A_0 = -mA/u$. Substituting this into the expression for dA gives $0 = d^2A \equiv -B_0/U\,du \wedge dx \bmod dt$, which implies that $B_0 = 0$. This simplifies the formula for d^2A so that we have

$$0 = d^2A = dA_2 \wedge dt - (B_2/U)_t\,dt \wedge dx - (mA_2/u)_t\,dt \wedge du.$$

Thus, there is some function C so that

$$dA_2 = C\,dt - (mA_2/u)\,du - (B_2/U)\,dx.$$

We are now ready to turn to the consequences of $d^2B = 0$. We have

$$0 = d^2B = -(mA_2/u)\,du \wedge dx + (C + UA_2)\,dt \wedge dx + dB_2 \wedge dt.$$

Since we are assuming that we are not in the linear case, we know that $m \neq 0$, so $A_2 = 0$. By our formula for dA_2, this then gives $C = B_2 = 0$.

Collecting everything, we have

$$\begin{aligned} dA &= -(mA/u)\,du - (B/U)\,dx \\ dB &= AU\,dx, \end{aligned}$$

which is the same as the integrable system

$$\begin{aligned} d(AU) &= -B\,dx \\ dB &= AU\,dx. \end{aligned}$$

The solutions of this system are given by

$$\begin{aligned} A &= \frac{1}{U}(c_0 \cos x + c_1 \sin x) \\ B &= c_0 \sin x - c_1 \cos x, \end{aligned}$$

where c_0 and c_1 are arbitrary constants. This proves that (1) has exactly two independent conservation laws when $m \neq 0$. It follows that we may choose

$$\begin{aligned} \Phi_0 &= \frac{\cos x}{U}\Upsilon + \sin x\,\Omega \\ \Phi_1 &= \frac{\sin x}{U}\Upsilon - \cos x\,\Omega \end{aligned}$$

as a basis for the space of conservation laws.

We now turn to the integrable extension corresponding to Φ_0 and Φ_1. For this we first find 1-forms φ_0 and φ_1 with $d\varphi_0 = \Phi_0$ and $d\varphi_1 = \Phi_1$, and then introduce new variables z and w and consider the system generated by

$$\begin{aligned} \chi_0 &= dz - \varphi_0 \\ \chi_1 &= dw - \varphi_1. \end{aligned}$$

With $v = \int du/U$ we have

$$\varphi_0 = v \cos x\, dx + (p \cos x + u \sin x)\, dt$$

$$\varphi_1 = v \sin x\, dx + (p\ sin\ x - u \cos x)\, dt.$$

However, rather than use χ_0 and χ_1, it turns out to be more convenient to introduce a new variable r and consider the system generated by

$$\theta_0 = \sin x\, dz - \cos x\, dw - u\, dt$$

$$\theta_1 = \cos x\, dz + \sin x\, dw - v\, dx - p\, dt$$

$$\theta_2 = (1/U)(du - p\, dx) - (r + u)\, dt$$

$$\theta_3 = (1/U)(dp - r\, dx).$$

The linear combinations $\sin x\theta_0 + \cos x\theta_1$ and $-\cos x\theta_0 + \sin x\theta_1$ recover χ_0 and x_1, and clearly $\theta_2 = \theta_3 = 0$ defines the equation (1). The reason for choosing this basis is that if we now set

$$\omega^1 = dx$$

$$\omega^2 = U\, dt,$$

then we have

$$d\theta_0 \equiv -\theta_1 \wedge \omega^1 - \theta_2 \wedge \omega^2 \bmod \theta_0$$

$$d\theta_1 \equiv -\theta_2 \wedge \omega^1 - \theta_3 \wedge \omega^2 \bmod \theta_0, \theta_1.$$

Thus, we have the beginning of a 1-adapted basis for a parabolic system in the 7-dimensional space with variables (z, w, x, t, u, p, r), and we can seek to determine the conservation laws of this system.

To carry this out, we shall work directly with the general form

$$\Phi = A(\omega^1 \wedge \theta_0 + \omega^2 \wedge \theta_1) + B\omega^2 \wedge \theta_0 + C_1\theta_0 \wedge \theta_1 + C_2\theta_0 \wedge \theta_2$$

of the conservation laws, rather than trace through the coframe adaptations necessary in order to be able to apply the handy algorithm. Calculation gives

$$\theta_0 \wedge d\Phi \equiv (C_2/U)(\sin^2 x + \cos^2 x)\, du \wedge dz \wedge dx \wedge dw \bmod \omega^2$$

so that we must have $C_2 = 0$. Then we find that

$$d\Phi \equiv (dA + (mA/u)\, du + B\omega^1 - C_1\theta_2) \wedge \omega^2 \wedge \theta_1 \bmod \theta_0,$$

so that we must have

$$dA = -(mAu)\,du - B\omega^1 + C_1\theta_2 + B_0\theta_0 + B_1\theta_1 + B_2\omega^2$$

for some functions B_0, B_1, and B_2. Substituting this, we obtain, after a straightforward calculation, that

$$d\Phi \wedge \omega^2 \wedge \theta_1 = (1/u)(mAU - 2C_1 u)\,du \wedge dx \wedge dz \wedge dt \wedge dw$$

so that we must set

$$C_1 = \frac{mAU}{2u}.$$

Finally, we find that

$$d\Phi \equiv -\frac{mAU}{4u^2}(m-2)\,du \wedge dz \wedge dw \bmod dt, dx.$$

Thus, except when $m = 2$, the system has no conservation laws. In the case $m = 2$, we already know from Example 8 that there is exactly one conservation law, corresponding to the linear area contraction for the heat flow shrinking plane curves. □

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