Let $g_{\mathbb{Q}}$ be a split semisimple Lie algebra of linear transformations of the finite dimensional vector space $V_{\mathbb{Q}}$ over $\mathbb{Q}$. Let $h_{\mathbb{Q}}$ be a split Cartan subalgebra of $g_{\mathbb{Q}}$ and choose for each root $\alpha$ of $h_{\mathbb{Q}}$ a root vector $X_{\alpha}$ so that if $[X_{\alpha}, X_{-\alpha}] = H_{\alpha}$ then $\alpha(H_{\alpha}) = 2$ and so that there is an automorphism $\theta$ of $g_{\mathbb{Q}}$ with $\theta(X_{\alpha}) = -X_{-\alpha}$. Let $L$ be the set of weights of $h_{\mathbb{Q}}$ and if $\lambda \in L$ let $V_{\mathbb{Q}}(\lambda) = \{v \in V_{\mathbb{Q}} \mid Hv = \lambda(H)v \text{ for all } H \in h_{\mathbb{Q}}\}$; let $H_1, \ldots, H_p$ be a basis over $\mathbb{Z}$ of 

\[ \{H \mid \lambda(H) \in \mathbb{Z} \text{ if } V_{\mathbb{Q}}(\lambda) \neq 0\}. \]

As usual, there is associated to $g_{\mathbb{Q}}$ a connected algebraic group $G$ of linear transformations of $V_{\mathbb{C}} = V_{\mathbb{Q}} \otimes \mathbb{Q}\mathbb{C}$. If $H$ is some lattice in $V_{\mathbb{Q}}$ satisfying

1. $M = \sum_{\lambda \in L} M \cap V(\lambda)$,
2. $(X_{\alpha}^n/n!)M \subseteq M$ for all $\alpha$,

then we let $G_{\mathbb{Z}} = \{g \in G \mid gM = M\}$. Let $\omega$ be a left invariant form on $G_{\mathbb{R}}$ of highest degree which takes the value $\pm 1$ on $(\wedge_{i=1}^p H_1) \wedge (\wedge_{\alpha > 0} X_{\alpha})$ and let $[dg]$ be the Haar measure associated to $\omega$. Our purpose now is to show the following.

If $\xi(\cdot)$ is the Riemann zeta function, $\prod_{i=1}^p (t^{2a_i-1} + 1)$ is the Poincaré polynomial of $G_{\mathbb{C}}$, and $c$ is the order of the fundamental group of $G_{\mathbb{C}}$ then

\[ \int_{G_{\mathbb{Z}}/G_{\mathbb{R}}} [dg] = c \prod_{i=1}^p \xi(a_i). \]

The method to be used to find the volume of $G_{\mathbb{Z}} \backslash G_{\mathbb{R}}$ is not directly applicable to $[dg]$. So it is necessary to introduce another Haar measure on the group $G_{\mathbb{R}}$. Let $U$ be the connected subgroup of $G$ whose Lie algebra is spanned over $\mathbb{R}$ by \{\(X_\alpha - X_{-\alpha}, \ i(X_\alpha + X_{-\alpha}), \ iH_\alpha \mid \alpha \text{ a root}\}\} and let $K = G_{\mathbb{R}} \cap U$. Choose an order on the roots and let $N = N_{\mathbb{R}}$ be the set of real points on the connected algebraic subgroup of $G_{\mathbb{C}}$ with the Lie algebra $\sum_{\alpha > 0} \mathbb{C}X_{\alpha}$. Let $A_{\mathbb{R}}$ be the normalizer of $h_{\mathbb{C}}$ in $G_{\mathbb{R}}$. Let $dn$ be the Haar measure on $N$ defined by a form which
Volume of the fundamental domain

takes the value $\pm 1$ on $\cap_{i=1}^{P} H_i$. Let $dk$ be the Haar measure on $K$ such that the total volume of $K$ is one. Let $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$ and let $\xi_{2\rho}(a)$ be the character of $A_{\mathbb{C}}$ associated to $2\rho$. Finally let $dg$ be such that

$$
\int_{G_{\mathbb{R}}} \phi(g) dg = \int_{N \times A_{\mathbb{R}} \times K} |\xi_{2\rho}(a)|^{-1} \phi(nak) dn \ da \ dk.
$$

If $N^-$ is the set of real points on the group associated to $\sum_{\alpha < 0} C X_{\alpha}$ define $dn^-$ in the same way as we defined $dn$. It is easy to see that

$$
\int_{G} \phi(g) [dg] = \int_{N} dn \int_{A_{\mathbb{R}}} da \int_{N^-} dn^- |\xi_{2\rho}(a)|^{-1} \phi(na^-).
$$

Suppose $\phi(gk) = \phi(g)$ for all $g \in G_{\mathbb{R}}$ and all $k \in K$. Then

$$
\int_{G} \phi(g) dg = \int_{N \times A_{\mathbb{R}}} dn \ da |\xi_{2\rho}(a)|^{-1} \phi(na).
$$

On the other hand, if $n^- = n(n^-)a(n^-)k(n^-)$,

$$
\int_{G} \phi(g) [dg] = \int_{N^-} dn^- \left\{ \int_{A} da \int_{N} dn |\xi_{2\rho}(a)|^{-1} \phi(na^-a(n^-)k(n^-)) \right\}
$$

$$
= \left\{ \int_{A} da \int_{N} dn |\xi_{2\rho}(a)|^{-1} \phi(na) \right\} \left\{ \int_{N^-} |\xi_{2\rho}(a^-)| dn^- \right\}.
$$

It follows from a formula of Gindikin and Karpelevich that the second factor equals

$$
\prod_{\alpha > 0} \pi^{1 - \frac{b_{\alpha}}{2}} \Gamma(\rho(H_{\alpha})/2) = \prod_{\alpha > 0} \frac{\pi^{-\rho(H_{\alpha})/2} \Gamma(\rho(H_{\alpha})/2)}{\pi^{-(\rho(H_{\alpha})+1)/2} \Gamma((\rho(H_{\alpha})+1)/2)}
$$

$$
= \frac{\prod'_{\alpha > 0} \pi^{-\rho(H_{\alpha})/2} \Gamma(\rho(H_{\alpha})/2)}{\prod_{\alpha > 0} \pi^{-(\rho(H_{\alpha})+1)/2} \Gamma((\rho(H_{\alpha})+1)/2)},
$$

since when $\alpha$ is simple $\rho(H_{\alpha}) = 1$ and

$$
\pi^{-\frac{b_{\alpha}}{2}} \Gamma(\frac{1}{2}) = 1.
$$

The product in the numerator is taken over the positive roots which are not simple. By a well-known result, the numbers, with multiplicities, in the set

$$
\{\rho(H_{\alpha}) + 1 | \alpha > 0\}
$$
are just the numbers \( \rho(H_\alpha) \) with \( \alpha \) positive and not simple, together with the numbers \( a_1, \ldots, a_p \). So if
\[
\xi(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \xi(s),
\]
we have to show that
\[
\int_{G \setminus G_R} dg = \frac{c_{\alpha>0} \xi(\rho(H_\alpha) + 1)}{\Pi'_{\alpha>0} \xi(\rho(H_\alpha))}.
\]

By the way, it is well to keep in mind that \( \rho(H_\alpha) > 1 \) if \( \alpha \) is not simple.

Let \( A \) be the connected component of \( A_R \) and let \( M \) be the points of finite order in \( A_R \). Certainly \( A_R = AM \). Moreover, by Iwasawa, \( G = NAK \). If \( g = nak \) and \( a = \exp H \), we set \( H = H(g) \).

If \( \phi \) is an infinitely differentiable function with compact support on \( N \setminus G \) such that \( \phi(gk) = \phi(g) \) for all \( g \) in \( G \) and all \( k \) in \( K \) we can write \( \phi \) as a Fourier integral.
\[
\phi(g) = \frac{1}{(2\pi)^p} \int_{\text{Re}\lambda = \lambda_0} \exp(\lambda(H(g)) + \rho(G(g)) \Phi(\lambda)) d\lambda;
\]
\( \lambda \) is the symbol for an element of the dual of \( h_G \); \( \Phi(\lambda) \) is an entire complex-valued function of \( \lambda \); and \( d\lambda = dz_1 \wedge \ldots \wedge dz_p \) with \( z_i = \lambda(H_i) \). As in the lectures on Eisenstein series we can introduce
\[
\hat{\phi}(g) = \sum_{\gamma \in G \cap NM \setminus G} \phi(\gamma g).
\]

Our evaluation of the volume of \( G \setminus G_R \) will be based on the simple relation
\[
(\hat{\phi}, 1)(1, \hat{\psi}) = (1, 1)(\Pi \hat{\phi}, \Pi \hat{\psi}).
\]

The inner products are taken in \( L^2(G \setminus G_R) \) with respect to \( dg \) and \( \Pi \) is the orthogonal projection on the space of constant functions. Since
\[
(1, 1) = \int_{G \setminus G_R} dg,
\]
it is enough to find an explicit formula for the other three terms. Now
\[
(\hat{\phi}, 1) = \int_{G \setminus NM \setminus G} \phi(g) dg
\]
\[
= \mu(G \cap NM \setminus NM) \int_A |\xi_{2\rho}(a)|^{-1} \phi(a) da
\]
\[
= \Phi(\rho)
\]
since $\mu(G_{\mathbb{Z}} \cap NM \backslash NM) = 1$. To see the latter we have to observe that $M \subseteq G_{\mathbb{Z}}$ and that, as follows from results stated in Cartier’s talk, $\mu(G_{\mathbb{Z}} \cap N \backslash N) = 1$. It is also clear that $(1, \hat{\psi}) = \bar{\Psi}(\rho)$. The nontrivial step is to evaluate $(\Pi \hat{\phi}, \Pi \hat{\psi})$.

From the theory of Eisenstein series we know that

$$(\hat{\phi}, \hat{\psi}) = \frac{1}{(2\pi)^p} \int_{\text{Re}\lambda = \lambda_0} \sum_{s \in \Omega} M(s, \lambda) \Phi(\lambda) \bar{\Psi}(-s \lambda) |d\lambda|.$$ 

$\Omega$ is the Weyl group, $\lambda_0$ is any point such that $\lambda_0(H_\alpha) > 1$ for every simple root, and

$$M(s, \lambda) = \prod_{\alpha > 0} \frac{\xi(1 + s \lambda(H_\alpha))}{\xi(1 + \lambda(H_\alpha))} = \prod_{\alpha > 0; s\alpha < 0} \frac{\xi(\lambda(H_\alpha))}{\xi(1 + \lambda(H_\alpha))}.$$ 

In the lectures on Eisenstein series I introduced an unbounded self-adjoint operator $A$ on the closed subspace of $L^2(G_{\mathbb{Z}} \backslash G_{\mathbb{R}})$ generated by the functions $\hat{\phi}$ with $\phi$ of the form indicated above. Comparing the definition of $A$ with the formula for $(\hat{\phi}, 1)$ we see that

$$(A \hat{\phi}, 1) = (\rho, \rho)(\hat{\phi}, 1).$$

Since the constant functions are in this space $A1 = (\rho, \rho) \cdot 1$. As a consequence, if $E(x), -\infty < x < \infty$, is the spectral resolution of $A$ the constant functions are in the range of $E((\rho, \rho)) - E((\rho, \rho) - 0) = E$. We show that this range consists precisely of the constant functions and compute $(E\hat{\phi}, \hat{\psi}) = (\Pi \hat{\phi}, \Pi \hat{\psi})$.

Suppose $a > (\rho, \rho) > b$ and $a - b$ is small. According to a well-known formula,

$$\frac{1}{2} \left\{ (E(a)\hat{\phi}, \hat{\psi}) + (E(a - 0)\hat{\phi}, \hat{\psi}) \right\} - \frac{1}{2} \left\{ (E(b)\hat{\phi}, \hat{\psi}) + (E(b - 0)\hat{\phi}, \hat{\psi}) \right\}$$

is equal to

$$\lim_{\delta \to 0} \frac{1}{2\pi i} \int_{C(a,b,c,\delta)} (R(\mu, A)\hat{\phi}, \hat{\psi}) d\mu$$
if $C(a, b, c, \delta)$ is the following contour.

Recall that, if $\text{Re} \mu > (\lambda_0, \lambda_0)$,

$$
(R(\mu, A)\hat{\phi}, \hat{\psi}) = \sum_{s \in \Omega} \frac{1}{(2\pi i)^p} \int_{\text{Re} \lambda = \lambda_0} \frac{1}{\mu - (\lambda, \lambda)} M(s, \lambda) \Phi(\lambda) \bar{\Psi}(-s\bar{\lambda}) d\lambda.
$$

If $w = (w_1, \ldots, w_p)$ belongs to $\mathbb{C}^p$ let $\lambda(w)$ be such that $\lambda(H_{\alpha_i}) = w_i$, where $\alpha_1, \ldots, \alpha_p$ are the simple roots. Set

$$
\phi_p(w, s) = M(s, \lambda(w)) \Phi(\lambda(w)) \bar{\Psi}(-s\lambda\bar{w})),
Q_p(w) = (\lambda(w), \lambda(w)),
$$

then (a) is equal to

$$
\frac{1}{c} \sum_{s \in \Omega} \lim_{\delta \downarrow 0} \frac{1}{2\pi i} \int_{C(a, b, c, \delta)} d\mu \left\{ \frac{1}{(2\pi i)^p} \int_{\text{Re} w = w_0} \frac{1}{\mu - Q_p(w)} \phi_p(w, s) dw_1 \ldots dw_p \right\}
$$

provided each of these limits exist. The coordinates of $w_0$ must all be greater than one. We shall consider the limits individually.

Let $w^q = (w_1, \ldots, w_q)$ and define $\phi_q(w^q; s)$ inductively for $0 \leq q \leq p$ by

$$
\phi_q(q_1, \ldots, w_q; s) = \text{Residue}_{w_{q+1} = 1} \phi_{q+1}(w_1, \ldots, w_{q+1}; s).
$$

\footnote{The inner integral is defined for $\text{Re} \mu > Q_p(w_0)$. However, as can be seen from the discussion to follow, the function of $\mu$ it defines can be analytically continued to a region containing $C(a, b, c, \delta)$.}
It is easily seen that \( \phi_q(w^q; s) \) has no singularities in the region defined by the inequalities \( \Re w_i > 1, 1 \leq i \leq q \); that \( \phi_q(w^q; s) \) goes to zero very fast when the imaginary part of \( w^q \) goes to infinity and its real part remains in a compact subset of this region; and that there is a positive number \( \epsilon \) such that the only singularities of \( \phi_q(w^q; s) \) in

\[
\{(w_1, \ldots, w_q) \mid |\Re w_i - 1| < \epsilon, 1 \leq i \leq q\}
\]

lie on the hyperplanes \( w_i = 1 \) and are at most simple poles. \( \phi_0(s) \) is of course a constant. Set \( Q_q(w^q) = Q_p(w_1, \ldots, w_q, 1, \ldots, 1) \).

Let us show by induction that the given limit equals

\[
\lim_{\delta \to 0} \frac{1}{2\pi i} \int_{C(a,b,c,\delta)} d\mu \left\{ \frac{1}{2\pi i} \int_{\Re w^q = w_0^q} \frac{1}{\mu - Q_q(w^q)} \phi_q(w^q; s) dw_1 \ldots dw_q \right\}
\]

if \( w_0^q = (w_{0,1}, \ldots, w_{0,q}) \) with \( w_{0,i} > 1, 1 \leq i \leq q \). Of course, the above expression is independent of the choice of such a point \( w_0^q \). Take \( w_0^q = (1 + u, \ldots, 1 + u, 1 + v) \), with \( u \) and \( v \) positive but small, and \( w_0^{q-1} = (1 + u, \ldots, 1 + u) \). If \( \Lambda_1, \ldots, \Lambda_q \) are such that \( \Lambda_i(H_{\alpha j}) = \delta_{ij} \), then \( (\Lambda_i, \Lambda_j) \geq 0 \). As a consequence, if \( u \) is much smaller than \( v \), then

\[
Q_q(1 + u, \ldots, 1 + u, 1 - v) < (\rho, \rho).
\]

Choose (b) to be larger than the number on the left. Also

\[
\Re Q_q(w^q) = Q_q(\Re w^q) - Q_p(\Im w_1, \ldots, \Im w_q, 0, \ldots, 0).
\]

Thus there is a constant \( N \) such that if either \( \Re w_i = 1 + u, 1 \leq i \leq q - 1 \) and \( \Re w_q = 1 - v \) or \( \Re w_i = 1 + u, 1 \leq i \leq p \) and \( |\Re w_q - 1| \leq v \) and \( |\Im w_q| > N \), then

\[
\Re Q_q(w^q) < b - 1/N.
\]

In (b) we may perform the integrations in any order. Integrate first with respect to \( w_q \). If \( C \) is the indicated contour, the result is the sum of (b) with \( q \) replaced by \( q - 1 \) and

\[
\lim_{\delta \to 0} \frac{1}{(w\pi i)^q} \int_{\Re w^{q-1} = w_0^{q-1}} dw_1 \ldots dw_{q-1} \int_{C} dw_q \phi_q(w^q, s) \left\{ \frac{1}{2\pi i} \int_{C(a,b,c,\delta)} d\mu \right\} - \frac{1}{Q_q(w^q)} d\mu\}
\]
which is obviously zero.

The contour $C$

Taking $q = 0$ in (b) we get

$$
\lim_{\delta \to 0} \frac{\phi_0(s)}{2\pi i} \int_{C(a,b,c,\delta)} \frac{1}{\mu - (\rho, \rho)} d\mu = \phi_0(s).
$$

It is clear that $\phi_0(s)$ is zero unless $s$ sends every positive root to a negative root but that for the unique element of the Weyl group which does this

$$
\phi_0(s) = \frac{\Pi'_{\alpha > 0} \xi(\rho(H_\alpha)) \Phi(\rho) \Psi(\rho)}{\Pi_{\alpha > 0} \xi(\rho(H_\alpha) + 1)}
$$

since $s\rho = -\rho$. This is the result required.

Finally, I remark that although the method just described for computing the volume of $\Gamma \backslash G$ has obvious limitations, it can be applied to other groups. In particular it works for Chevalley groups over a numberfield.