The Volume of the Fundamental Domain for Some Arithmetical Subgroups of Chevalley Groups

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Let $\mathfrak{g}_{\mathbf{Q}}$ be a split semisimple Lie algebra of linear transformations of the finite dimensional vector space $V_{\mathbf{Q}}$ over $\mathbf{Q}$. Let $\mathfrak{h}_{\mathbf{Q}}$ be a split Cartan subalgebra of $\mathfrak{g}_{\mathbf{Q}}$ and choose for each root α of $\mathfrak{h}_{\mathbf{Q}}$ a root vector X_α so that if $[X_\alpha, X_{-\alpha}] = H_\alpha$ then $\alpha(H_\alpha) = 2$ and so that there is an automorphism θ of $\mathfrak{g}_{\mathbf{Q}}$ with $\theta(X_\alpha) = -X_{-\alpha}$. Let L be the set of weights of $\mathfrak{h}_{\mathbf{Q}}$ and if $\lambda \in L$ let

$$V_{\mathbf{Q}}(\lambda) = \{v \in V_{\mathbf{Q}} \mid Hv = \lambda(H)v \ \text{ for all } H \in \mathfrak{h}_{\mathbf{Q}}\};$$

let $H_1, \ldots, H_p$ be a basis over $\mathbf{Z}$ of

$$\{H \mid \lambda(H) \in \mathbf{Z} \text{ if } V_{\mathbf{Q}}(\lambda) \neq 0\}.$$

As usual, there is associated to $\mathfrak{g}_{\mathbf{Q}}$ a connected algebraic group G of linear transformations of $V_{\mathbf{C}} = V_{\mathbf{Q}} \otimes {}_{\mathbf{Q}}\mathbf{C}$. If H is some lattice in $V_{\mathbf{Q}}$ satisfying

(i) $M = \sum_{\lambda \in L} M \cap V(\lambda)$,

(ii) $(X_\alpha^n/n!)M \subseteq M$ for all α,

then we let $G_{\mathbf{Z}} = \{g \in G \mid gM = M\}$. Let ω be a left invariant form on $G_{\mathbf{R}}$ of highest degree which takes the value ± 1 on $(\wedge_{i=1}^p H_1) \wedge (\wedge_{\alpha>0} X_\alpha)$ and let $[dg]$ be the Haar measure associated to ω. Our purpose now is to show the following.

If $\xi(\cdot)$ is the Riemann zeta function, $\Pi_{i=1}^p (t^{2a_i - 1} + 1)$ is the Poincaré polynomial of $G_{\mathbf{C}}$, and c is the order of the fundamental group of $G_{\mathbf{C}}$ then

$$\int_{G_{\mathbf{Z}}/G_{\mathbf{R}}} [dg] = c\Pi_{i=1}^p \xi(a_i).$$

The method to be used to find the volume of $G_{\mathbf{Z}} \backslash G_{\mathbf{R}}$ is not directly applicable to $[dg]$. So it is necessary to introduce another Haar measure on the group $G_{\mathbf{R}}$. Let U be the connected subgroup of G whose Lie algebra is spanned over $\mathbf{R}$ by $\{X_\alpha - X_{-\alpha}, i(X_\alpha + X_{-\alpha}), iH_\alpha \mid \alpha \text{ a root}\}$ and let $K = G_{\mathbf{R}} \cap U$. Choose an order on the roots and let $N = N_{\mathbf{R}}$ be the set of real points on the connected algebraic subgroup of $G_{\mathbf{C}}$ with the Lie algebra $\sum_{\alpha>0} \mathbf{C}X_\alpha$. Let $A_{\mathbf{R}}$ be the normalizer of $\mathfrak{h}_{\mathbf{C}}$ in $G_{\mathbf{R}}$. Let dn be the Haar measure on N defined by a form which takes the value ± 1 on $\wedge_{i=1}^{p} H_i$. Let dk be the Haar measure on K such that the total volume of K is one. Let $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ and let $\xi_{2\rho}(a)$ be the character of $A_{\mathbf{C}}$ associated to 2ρ. Finally let dg be such that

$$\int_{G_{\mathbf{R}}} \phi(g)dg = \int_{N\times A_{\mathbf{R}}\times K} |\xi_{2\rho}(a)|^{-1}\phi(nak)dn\ da\ dk.$$

If N^- is the set of real points on the group associated to $\sum_{\alpha<0}\mathbf{C}X_\alpha$ define dn^- in the same way as we defined dn. It is easy to see that

$$\int_G \phi(g)[dg] = \int_N dn \int_{A_{\mathbf{R}}} da \int_{N^-} dn^- |\xi_{2\rho}(a)|^{-1}\phi(nan^-).$$

Suppose $\phi(gk) = \phi(g)$ for all $g \in G_{\mathbf{R}}$ and all $k \in K$. Then

$$\int_G \phi(g)dg = \int_{N\times A_{\mathbf{R}}} dn\ da|\xi_{2\rho}(a)|^{-1}\phi(na).$$

On the other hand, if $n^- = n(n^-)a(n^-)k(n^-)$,

$$\begin{aligned}\int \phi(g)[dg] &= \int_{N^-} dn^- \left\{\int_A da \int_N dn|\xi_{2\rho}(a)|^{-1}\phi(nan(n^-)a(n^-)k(n^-))\right\}\\ &= \left\{\int_A da \int_N dn|\xi_{2\rho}(a)|^{-1}\phi(na)\right\}\left\{\int_{N^-} |\xi_{2\rho}a(n^-)|dn^-\right\}.\end{aligned}$$

It follows from a formula of Gindikin and Karpelevich that the second factor equals

$$\begin{aligned}\prod_{\alpha>0}\frac{\pi^{+\frac{1}{2}^{b}}\Gamma(\rho(H_\alpha)/2)}{\Gamma((\rho(H_\alpha)+1)/2)} &= \prod_{\alpha>0}\frac{\pi^{-\rho(H_\alpha)/2}\Gamma(\rho(H_\alpha)/2)}{\pi^{-(\rho(H_\alpha)+1)/2}\Gamma((\rho(H_\alpha)+1)/2)}\\ &= \frac{\prod'_{\alpha>0}\pi^{-\rho(H_\alpha)/2}\Gamma(\rho(H_\alpha)/2)}{\prod_{\alpha>0}\pi^{-(\rho(H_\alpha)+1)/2}\Gamma((\rho(H_\alpha)+1)/2)},\end{aligned}$$

since when α is simple $\rho(H_\alpha) = 1$ and

$$\pi^{-\frac{1}{2}}\Gamma(\frac{1}{2}) = 1.$$

The product in the numerator is taken over the positive roots which are not simple. By a well-known result, the numbers, with multiplicities, in the set

$$\{\rho(H_\alpha) + 1 \mid \alpha > 0\}$$

are just the numbers $\rho(H_\alpha)$ with α positive and not simple, together with the numbers $a_1, \ldots, a_p$. So if

$$\xi(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\xi(s),$$

we have to show that

$$\int_{G_{\mathbf{Z}}/G_{\mathbf{R}}} dg = \frac{c\Pi_{\alpha>0}\xi(\rho(H_\alpha)+1)}{\Pi'_{\alpha>0}\xi(\rho(H_\alpha))}.$$

By the way, it is well to keep in mind that $\rho(H_\alpha) > 1$ if α is not simple.

Let A be the connected component of $A_{\mathbf{R}}$ and let M be the points of finite order in $A_{\mathbf{R}}$. Certainly $A_{\mathbf{R}} = AM$. Moreover, by Iwasawa, $G = NAK$. If $g = nak$ and $a = \exp H$, we set $H = H(g)$.

If ϕ is an infinitely differentiable function with compact support on $N\backslash G$ such that $\phi(gk) = \phi(g)$ for all g in G and all k in K we can write ϕ as a Fourier integral.

$$\phi(g) = \frac{1}{(2\pi)^p}\int_{\mathrm{Re}\lambda=\lambda_0} \exp(\lambda(H(g)) + \rho(G(g))\Phi(\lambda)|d\lambda|;$$

λ is the symbol for an element of the dual of $\mathfrak{h}_{\mathbf{C}}$; $\Phi(\lambda)$ is an entire complex-valued function of λ; and $d\lambda = dz_1 \wedge \ldots \wedge dz_p$ with $z_i = \lambda(H_i)$. As in the lectures on Eisenstein series we can introduce

$$\hat{\phi}(g) = \sum_{\gamma\in G_{\mathbf{Z}}\cap NM\backslash G_{\mathbf{Z}}} \phi(\gamma g).$$

Our evaluation of the volume of $G_{\mathbf{Z}}\backslash G_{\mathbf{R}}$ will be based on the simple relation

$$(\hat{\phi}, 1)(1, \hat{\psi}) = (1,1)(\Pi\hat{\phi}, \Pi\hat{\psi}).$$

The inner products are taken in $L^2(G_{\mathbf{Z}}\backslash G_{\mathbf{R}})$ with respect to dg and Π is the orthogonal projection on the space of constant functions. Since

$$(1,1) = \int_{G_{\mathbf{Z}}\backslash G_{\mathbf{R}}} dg,$$

it is enough to find an explicit formula for the other three terms. Now

$$\begin{aligned}(\hat{\phi}, 1) &= \int_{G_{\mathbf{Z}}\cap NM\backslash G_{\mathbf{R}}} \phi(g)dg \\ &= \mu(G_{\mathbf{Z}} \cap NM\backslash NM)\int_A |\xi_{2\rho}(a)|^{-1}\phi(a)da \\ &= \Phi(\rho)\end{aligned}$$

since $\mu(G_{\mathbf{Z}} \cap NM \backslash NM) = 1$. To see the latter we have to observe that $M \subseteq G_{\mathbf{Z}}$ and that, as follows from results stated in Cartier's talk, $\mu(G_{\mathbf{Z}} \cap N \backslash N) = 1$. It is also clear that $(1, \hat{\psi}) = \bar{\Psi}(\rho)$. The nontrivial step is to evaluate

$$(\Pi\hat{\phi}, \Pi\hat{\psi}).$$

From the theory of Eisenstein series we know that

$$(\hat{\phi}, \hat{\psi} = \frac{1}{(2\pi)^p} \int_{\mathrm{Re}\lambda=\lambda_0} \sum_{s\in\Omega} M(s,\lambda)\Phi(\lambda)\bar{\Psi}(-s\bar{\lambda})|d\lambda|.$$

Ω is the Weyl group, λ_0 is any point such that $\lambda_0(H_\alpha) > 1$ for every simple root, and

$$M(s,\lambda) = \Pi_{\alpha>0} \frac{\xi(1 + s\lambda(H_\alpha))}{\xi(1+\lambda(H_\alpha))} = \Pi_{\alpha>0;s\alpha<0} \frac{\xi(\lambda(H_\alpha))}{\xi(1+\lambda(H_\alpha))}.$$

In the lectures on Eisenstein series I introduced an unbounded self-adjoint operator A on the closed subspace of $L^2(G_{\mathbf{Z}} \backslash G_{\mathbf{R}})$ generated by the functions $\hat{\phi}$ with ϕ of the form indicated above. Comparing the definition of A with the formula for $(\hat{\phi}, 1)$ we see that

$$(A\hat{\phi}, 1) = (\rho,\rho)(\hat{\phi}, 1).$$

Since the constant functions are in this space $A1 = (\rho,\rho)\cdot 1$. As a consequence, if $E(x)$, $-\infty < x < \infty$, is the spectral resolution of A the constant functions are in the range of $E((\rho,\rho)) - E((\rho,\rho) - 0) = E$. We show that this range consists precisely of the constant functions and compute $(E\hat{\phi}, \hat{\psi}) = (\Pi\hat{\phi}, \Pi\hat{\psi})$.

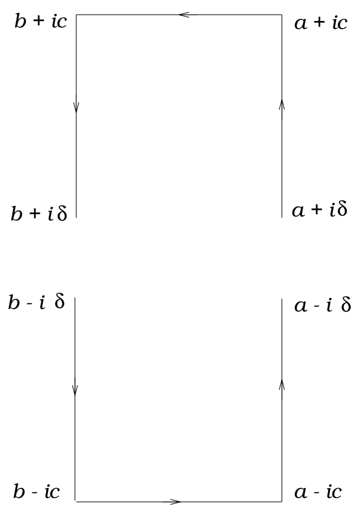
Suppose $a > (\rho,\rho) > b$ and $a - b$ is small. According to a well-known formula,

$$\frac{1}{2}\{(E(a)\hat{\phi},\hat{\psi}) + (E(a-0)\hat{\phi},\hat{\psi})\} - \frac{1}{2}\{(E(b)\hat{\phi},\hat{\psi}) + (E(b-0)\hat{\phi},\hat{\psi})\}$$

is equal to

$$(a) \qquad \lim_{\delta\downarrow 0} \frac{1}{2\pi i} \int_{C(a,b,c,\delta)} (R(\mu, A)\hat{\phi}, \hat{\psi}) d\mu$$

if $C(a, b, c, \delta)$ is the following contour.



Recall that, if $\mathrm{Re}\mu > (\lambda_0, \lambda_0)$,

$$(R(\mu, A)\hat{\phi}, \hat{\psi}) = \sum_{s\in\Omega} \frac{1}{(2\pi i)^p} \int_{\mathrm{Re}\lambda=\lambda_0} \frac{1}{\mu - (\lambda, \lambda)} M(s, \lambda)\Phi(\lambda)\bar{\Psi}(-s\bar{\lambda})d\lambda.$$

If $w = (w_1, \ldots, w_p)$ belongs to $\mathbf{C}^p$ let $\lambda(w)$ be such that $\lambda(H_{\alpha_i}) = w_i$, where $\alpha_1, \ldots, \alpha_p$ are the simple roots. Set

$$\begin{aligned} \phi_p(w, s) &= M(s, \lambda(w))\Phi(\lambda(w))\bar{\Psi}(-s\lambda\bar{w})), \\ Q_p(w) &= (\lambda(w), \lambda(w)), \end{aligned}$$

then (a) is equal to

$$\frac{1}{c} \sum_{s\in\Omega} \lim_{\delta\downarrow 0} \frac{1}{2\pi i} \int_{C(a,b,c,\delta)} d\mu \left\{ \frac{1}{(2\pi i)^p} \int_{\mathrm{Re}w=w_0} \frac{1}{\mu - Q_p(w)} \phi_p(w, s) dw_1 \ldots dw_p \right\}$$

provided each of these limits exist.[2] The coordinates of w_0 must all be greater than one. We shall consider the limits individually.

Let $w^q = (w_1, \ldots, w_q)$ and define $\phi_q(w^q; s)$ inductively for $0 \leqq q \leqq p$ by

$$\phi_q(q_1, \ldots, w_q; s) = \underset{w_{q+1}=1}{\mathrm{Residue}}\, \phi_{q+1}(w_1, \ldots, w_{q+1}; s).$$

[2] The inner integral is defined for $\mathrm{Re}\mu > Q_p(w_0)$. However, as can be seen from the discussion to follow, the function of μ it defines can be analytically continued to a region containing $C(a, b, c, \delta)$.

It is easily seen that $\phi_q(w^q; s)$ has no singularities in the region defined by the inequalities $\mathrm{Re}\ w_i > 1, 1 \leqq i \leqq q$; that $\phi_q(w^q; s)$ goes to zero very fast when the imaginary part of w^q goes to infinity and its real part remains in a compact subset of this region; and that there is a positive number ϵ such that the only singularities of $\phi_q(w^q; s)$ in

$$\{(w_1, \ldots, w_q) \bigm| |\mathrm{Re}\ w_i - 1| < \epsilon, 1 \leqq i \leqq q\}$$

lie on the hyperplanes $w_i = 1$ and are at most simple poles. $\phi_0(s)$ is of course a constant. Set $Q_q(w^q) = Q_p(w_1, \ldots, w_q, 1, \ldots, 1)$.

Let us show by induction that the given limit equals

$$(b) \qquad \lim_{\delta \downarrow 0} \frac{1}{2\pi i} \int_{C(a,b,c,\delta)} d\mu \left\{ \frac{1}{(2\pi i)^q} \int_{\mathrm{Re}\ w^q = w_0^q} \frac{1}{\mu - Q_q(w^q)} \phi_q(w^q; s) dw_1 \ldots dw_q \right\}$$

if $w_0^q = (w_{0,1}, \ldots, w_{0,q})$ with $w_{0,i} > 1, 1 \leqq i \leqq q$. Of course, the above expression is independent of the choice of such a point w_0^q. Take $w_0^q = (1+u, \ldots, 1+u, 1+v)$, with u and v positive but small, and $w_0^{q-1} = (1+u, \ldots, 1+u)$. If $\Lambda_1, \ldots, \Lambda_q$ are such that $\Lambda_i(H_{\alpha j}) = \delta_{ij}$, then $(\Lambda_i, \Lambda_j) \geqq 0$. As a consequence, if u is much smaller than v, then

$$Q_q(1+u, \ldots, 1+u, 1-v) < (\rho, \rho).$$

Choose (b) to be larger than the number on the left. Also

$$\mathrm{Re}\ Q_q(w^q) = Q_q(\mathrm{Re}\ w^q) - Q_p(\mathrm{Im} w_1, \ldots, \mathrm{Im} w_q, 0, \ldots, 0).$$

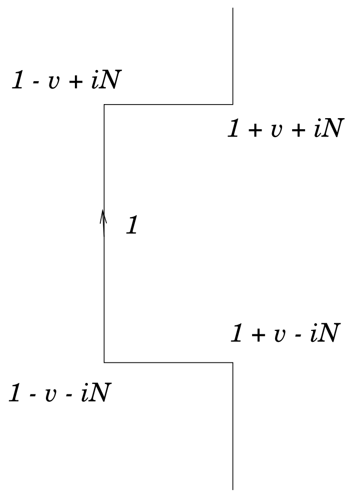
Thus there is a constant N such that if either $\mathrm{Re}\ w_i = 1+u, 1 \leqq i \leqq q-1$ and $\mathrm{Re}\ w_q = 1-v$ or $\mathrm{Re}\ w_i = 1+u, 1 \leqq i \leqq p$ and $|\mathrm{Re}\ w_q - 1| \leqq v$ and $|\mathrm{Im} w_q| > N$, then

$$\mathrm{Re}\ Q_q(w^q) < b - 1/N.$$

In (b) we may perform the integrations in any order. Integrate first with respect to w_q. If C is the indicated contour, the result is the sum of (b) with q replaced by $q-1$ and

$$\lim_{\delta \downarrow 0} \frac{1}{(w\pi i)^q} \int_{\mathrm{Re}\ w^{q-1} = w_0^{q-1}} dw_1 \ldots\ldots dw_{q-1} \int_C dw_q \phi_q(w^q, s) \left\{ \frac{1}{2\pi i} \int_{C(a,b,c,,\delta)} \frac{1}{\mu - Q_q(w^q)} d\mu \right\}$$

which is obviously zero.



The contour C

Taking $q = 0$ in (b) we get

$$\lim_{\delta \downarrow 0} \frac{\phi_0(s)}{2\pi i} \int_{C(a,b,c,\delta)} \frac{1}{\mu - (\rho, \rho)} d\mu = \phi_0(s).$$

It is clear that $\phi_0(s)$ is zero unless s sends every positive root to a negative root but that for the unique element of the Weyl group which does this

$$\phi_0(s) = \frac{\Pi'_{a>0} \xi(\rho(H_\alpha)) \Phi(\rho) \overline{\Psi(\rho)}}{\Pi_{\alpha>0} \xi(\rho(H_\alpha) + 1)}$$

since $s\rho = -\rho$. This is the result required.

Finally, I remark that although the method just described for computing the volume of $\Gamma\backslash G$ has obvious limitations, it can be applied to other groups. In particular it works for Chevalley groups over a numberfield.