ON THE CLASSIFICATION OF IRREDUCIBLE REPRESENTATIONS OF REAL ALGEBRAIC GROUPS

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Contents

1. Introduction

Suppose G is a connected reductive group over a global field F. Many of the problems of the theory of automorphic forms involve some aspect of study of the representation ρ of $G(\mathbf{A}(F))$ on the space of slowly increasing functions on the homogeneous space $G(F)\backslash G(\mathbf{A}(F))$. It is of particular interest to study the irreducible constituents of ρ. In a lecture [9], published some time ago, but unfortunately rendered difficult to read by a number of small errors and a general imprecision, reflections in part of a hastiness for which my excitement at the time may be to blame, I formulated some questions about these constituents which seemed to me then, as they do today, of some fascination. The questions have analogues when F is a local field; these concern the irreducible admissible representations of $G(F)$.

As I remarked in the lecture, there are cases in which the answers to the questions are implicit in existing theories. If G is abelian they are consequences of class field theory, especially of the Tate-Nakayama duality. This is verified in [10]. If F is the real or complex field, they are consequences of the results obtained by Harish-Chandra for representations of real reductive groups. This may not be obvious; my ostensible purpose in this note is to make it so. An incidental, but not unimportant, profit to be gained from this exercise is a better insight into the correct formulation of the questions.

Suppose the F is the real or complex field. Let $\Pi(G)$ be the set of infinitesimal equivalence classes of irreducible quasi-simple Banach space representations of $G(F)$ [16]. In the second section we shall recall the definition of the Weil group W_F of F as well as that of the associated or dual group $\widehat{G}$ of G and then introduce a collection $\Phi(G)$ of classes of homomorphisms of the Weil group of F into $\widehat{G}$. After reviewing in the same section some simple properties of the associate group we shall, in the third section, associate to each $\varphi \in \Phi(G)$ a nonempty finite set Π_φ in $\Pi(G)$. The remainder of the paper will be devoted to showing that these sets are disjoint and that they exhaust $\Pi(G)$. For reasons stemming from the study of

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L-functions associated to automorphic forms we say that two classes in the same Π_φ are L-indistinguishable.

Thus if $\widetilde{\Pi}(G)$ is the set of classes of L-indistinguishable representations of $G(F)$, then by definition the elements of $\widetilde{\Pi}(G)$ are parametrized by $\Phi(G)$. It will be seen that if G is quasi-split and G_1 is obtained from it by the inner twisting ψ then ψ defines an injection $\Phi(G_1) \hookrightarrow \Phi(G)$ and hence an injection $\widetilde{\Pi}(G_1) \hookrightarrow \widetilde{\Pi}(G)$. It will also be seen that for G quasi-split the set $\widetilde{\Pi}(G)$ is, in a sense to be made precise later, a covariant function of $\widehat{G}$. These properties of $\widetilde{\Pi}(G)$ provide answers to the questions of [9].

The classification of L-indistinguishable representation throws up more questions than it resolves, since we say nothing about the structure of the sets Π_φ themselves and hence do not really classify infinitesimal equivalence classes. None the less we do reduce the general problem to that of classifying the tempered representations. This is a considerable simplification. For example, Wallach [15] has proved that the unitary principal series are irreducible for complex groups. From this it follows that each Π_φ consists of a single class; so the classification is complete in this case. Since $\Phi(G)$ may, when F is complex, be easily identified with the orbits of the Weyl group in the set of quasi-characters of a Cartan subgroup $G(\mathbf{C})$, it is likely that the classification provided by this paper coincides with that of Zhelobenko. The set $\Pi(G)$ has been described by Hirai [7, 8] for $G = \mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$. It is a simple and worthwhile exercise to translate his classification into ours. In fact, the definitions of this paper were suggested by the study of his results. It would be interesting to know if each Π_φ consists of a single class when G is $\mathrm{GL}(n)$ and F is $\mathbf{R}$.

Important though these problems are, we do not try to decide which elements of which Π_φ are unitary or how the classes in a Π_φ are unitary or how the classes in a Π_φ decompose upon restriction to a maximal compact subgroup of $G(\mathbf{R})$.

The three main lemmas of this paper are Lemmas 3.13, 3.14, and 4.2. The first associates to each triplet consisting of a parabolic subgroup P over $\mathbf{R}$, a tempered representation of a Levi factor of $P(\mathbf{R})$, and a positive quasi-character of $P(\mathbf{R})$ whose parameter lies in the interior of a certain chamber defined by P, an irreducible quasi-simple representation of $G(\mathbf{R})$. The second lemma shows that these representations are not infinitesimally equivalent. The third shows that they exhaust the classes of irreducible quasi-simple representations.

As we observed above, the proofs are not very difficult. Unfortunately, they rely to some extent on unpublished results of Harish-Chandra. To prove that the sets Π_φ are disjoint we use results from [6], which includes no proofs. Moreover, and this is more serious, for the proof of Lemma 4.2 we use results from [5], which has only been partly reproduced in Appendix 3 of [16]. It contains theorems on differential equations which are used to study the asymptotic behavior of spherical functions not only in the interior of a positive Weyl chamber, as in [16], but also on the walls.

2. The associate group

We begin by recalling some of the constructions of [9]. If F is $\mathbf{C}$ the Weil group W_F is $\mathbf{C}^\times$. If F is $\mathbf{R}$ the Weil group W_F consists of pairs (z,τ), $z \in \mathbf{C}^\times$, $\tau \in \mathfrak{g}(\mathbf{C}/\mathbf{R}) = \{1,\sigma\}$ with multiplication defined by

$$(z_1,\tau_1)(z_2,\tau_2) = \big(z_1\tau_1(z_2)a_{\tau_1,\tau_2}, \tau_1\tau_2\big).$$

Here $a_{\tau_1,\tau_2} = 1$ if $\tau_1 = 1$ or $\tau_2 = 1$ and $a_{\tau_1,\tau_2} = -1$ if $\tau_1 = \tau_2 = \sigma$. For both fields we have an exact sequence

$$1 \longrightarrow \overline{F}^{\times} \longrightarrow W_F \longrightarrow \mathfrak{g}(\overline{F}/F) \longrightarrow 1 \quad .$$

Suppose $G^{\widehat{0}}$ is a connected reductive complex algebraic group, $B^{\widehat{0}}$ a Borel subgroup of $G^{\widehat{0}}$, and $T^{\widehat{0}}$ a Cartan subgroup of $G^{\widehat{0}}$ in $B^{\widehat{0}}$. For each root $\widehat{\alpha}$ of $T^{\widehat{0}}$ simple with respect to $B^{\widehat{0}}$ let $X_{\widehat{\alpha}} \neq 0$ in the Lie algebra $\widehat{\mathfrak{g}}$ of $G^{\widehat{0}}$ be such that

$$\operatorname{Ad} t(X_{\widehat{\alpha}}) = \alpha(t) X_{\widehat{\alpha}}, \qquad t \in T^{\widehat{0}}.$$

Let

$$A\Big(G^{\widehat{0}}, B^{\widehat{0}}, T^{\widehat{0}}, \{X_{\widehat{\alpha}}\}\Big)$$

be the group of complex analytic automorphisms ω of $G^{\widehat{0}}$ leaving $B^{\widehat{0}}$ and $T^{\widehat{0}}$ invariant and sending $X_{\widehat{\alpha}}$ to $X_{\omega\widehat{\alpha}}$, where $\omega\widehat{\alpha}$ is defined by

$$\omega\widehat{\alpha}(\omega t) = \widehat{\alpha}(t).$$

If instead of $B^{\widehat{0}}$, $T^{\widehat{0}}$, $\{X_{\widehat{\alpha}}\}$ we choose $\overline{B}^{\widehat{0}}$, $\overline{T}^{\widehat{0}}$, $\{\overline{X}_{\overline{\alpha}^{\wedge}}\}$ with the same properties there is a unique inner automorphism ψ such that

$$\overline{B}^{\widehat{0}} = \psi(B^{\widehat{0}}); \quad \overline{T}^{\widehat{0}} = \psi(T^{\widehat{0}}), \quad \overline{X}_{\psi\widehat{\alpha}} = \psi(X_{\widehat{\alpha}}).$$

Then

$$A\Big(G^{\widehat{0}}, \overline{B}^{\widehat{0}}, \overline{T}^{\widehat{0}}, \{\overline{X}_{\overline{\alpha}^{\wedge}}\}\Big) = \Big\{ \psi\omega\psi^{-1} \;\Big|\; \omega \in A\Big(G^{\widehat{0}}, B^{\widehat{0}}, T^{\widehat{0}}, \{X_{\widehat{\alpha}}\}\Big) \Big\}.$$

Suppose we have an extension

$$1 \longrightarrow G^{\widehat{0}} \longrightarrow \widehat{G} \longrightarrow W_F \longrightarrow 1$$

of topological groups. A splitting is a continuous homomorphism from W_F to $\widehat{G}$ for which the composition

$$W_F \longrightarrow \widehat{G} \longrightarrow W_F$$

is the identity. Each splitting defines a homomorphism of η of W_F into the group of automorphism of $\widehat{G}$. The splitting will be called admissible if, for each ω in W_F, $\eta(\omega)$ is complex analytic and the associated linear transformation of the Lie algebra of $G^{\widehat{0}}$ is semisimple. It will be called distinguished if there is a $B^{\widehat{0}}$, a $T^{\widehat{0}}$, and a collection $\{X_{\widehat{\alpha}}\}$ such that η factors through a homomorphism of $\mathfrak{g}(\overline{F}/F)$ into $A\Big(G^{\widehat{0}}, B^{\widehat{0}}, T^{\widehat{0}}, \{X_{\widehat{\alpha}}\}\Big)$. Two distinguished splittings will be called equivalent if they are conjugate under $G^{\widehat{0}}$.

We introduce a category $\widehat{\mathcal{G}}(F)$ whose objects are extensions of the above type, with $G^{\widehat{0}}$ a connected reductive complex algebraic group, together with an equivalence class of distinguished splittings. These we call special. A homomorphism

$$\varphi : \widehat{G}_1 \to \widehat{G}_2$$

of two objects in the category will be called an L-homomorphism if

$$\begin{array}{ccc} \widehat{G}_1 & \xrightarrow{\ \varphi\ } & \widehat{G}_2 \\ & \searrow \quad \swarrow & \\ & W_F & \end{array}$$

is commutative, if the restriction of φ to $G_1^{\widehat{0}}$ is complex analytic, and if φ preserves admissible splittings. Two L-homomorphisms will be called equivalent if there is a $g \in G_2^{\widehat{0}}$ such that

$$\varphi_2 = \operatorname{ad} g \circ \varphi_1.$$

An arrow in our category will be an equivalence class of L-homomorphisms. For simplicity, we do not distinguish in the notation between a homomorphism and its equivalence class.

For future reference we define a parabolic subgroup $\widehat{P}$ of $\widehat{G}$ to be a closed subgroup $\widehat{P}$ such that $P^{\widehat{0}} = \widehat{P} \cap G^{\widehat{0}}$ is a parabolic subgroup of $G^{\widehat{0}}$ and such that the projection $\widehat{P} \to W_F$ is surjective.

We also remark that $A\Big(G^{\widehat{0}}, B^{\widehat{0}}, T^{\widehat{0}}, \{X_{\widehat{\alpha}}\}\Big)$ contains no inner automorphisms. Thus if

$$\eta : \mathfrak{g}(\overline{F}/F) \to A\Big(G^{\widehat{0}}, B^{\widehat{0}}, T^{\widehat{0}}, \{X_{\widehat{\alpha}}\}\Big),$$
$$\overline{\eta} : \mathfrak{g}(\overline{F}/F) \to A\Big(G^{\widehat{0}}, \overline{B}^{\widehat{0}}, \overline{T}^{\widehat{0}}, \{\overline{X}_{\overline{\alpha}^{\wedge}}\}\Big)$$

are associated to two distinguished splittings of $\widehat{G}$ there is a $g \in G^{\widehat{0}}$, unique modulo the center, such that

$$\overline{\eta} = \operatorname{ad} g \circ \eta \circ \operatorname{ad} g^{-1}.$$

Suppose we are given a special distinguished splitting associated to the above map η. Let $\widehat{L}$ be the group of rational characters of $T^{\widehat{0}}$. If both variables on the right are treated as algebraic groups

$$\widehat{L} = \operatorname{Hom}(T^{\widehat{0}}, \mathbf{C}^{\times}).$$

Let conversely

$$L = \operatorname{Hom}(\mathbf{C}^{\times}, T^{\widehat{0}}).$$

Define a pairing

$$L \times \widehat{L} \to \mathbf{Z}$$

by

$$\widehat{\lambda}\big(\lambda(z)\big) = z^{\langle \lambda, \widehat{\lambda} \rangle}, \qquad z \in \mathbf{C}^{\times}.$$

This pairing identifies $\widehat{L}$ with $\operatorname{Hom}(L, \mathbf{Z})$. Associated to each root $\widehat{\alpha}$ of $T^{\widehat{0}}$ is a homomorphism of $\operatorname{SL}(2, \mathbf{C})$ into $G^{\widehat{0}}$. The composition

$$z \to \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \to G^{\widehat{0}}$$

factors through $T^{\widehat{0}}$ and defines an element α of L.

Let $\widehat{\Delta}$ be the set of roots simple with respect to $B^{\widehat{0}}$. Associated to $G^{\widehat{0}}$, $B^{\widehat{0}}$, $T^{\widehat{0}}$, $\{X_{\widehat{\alpha}}\}$ are a connected reductive group G^0 over F, a Borel subgroup B^0 of G^0, a Cartan subgroup T^0 in B^0, and isomorphisms η_α, $\widehat{\alpha} \in \widehat{\Delta}$, of the additive group with a subgroup of B^0 such that

$$L = \operatorname{Hom}\big(T, \operatorname{GL}(1)\big)$$

and

$$\Delta = \Big\{ \alpha \,\Big|\, \widehat{\alpha} \in \widehat{\Delta} \Big\}$$

is the set of simple roots of T^0 with respect to B^0. Moreover

$$\operatorname{ad} t\big(\eta_\alpha(x)\big) = \eta_\alpha\big(\alpha(t)x\big), \qquad x \in \overline{F},\ t \in T^0(\overline{F}).$$

The collection G^0, B^0, T^0, η_α is determined up to canonical isomorphism by these conditions. Any ω in $A\Big(\widehat{G^0}, \widehat{B^0}, \widehat{T^0}, \{X_{\widehat{\alpha}}\}\Big)$ acts on L and $\widehat{L}$. There is a unique way of letting ω act on G^0 so that

$$\omega\lambda(\omega t) = \lambda(t), \quad \lambda \in L,\ t \in T^0(\overline{F}),$$

and

$$\omega\eta_\alpha(x) = \eta_{\omega\alpha}(x), \qquad x \in \overline{F}.$$

The automorphism ω so obtained is defined over F. Thus

$$\eta : \mathfrak{g}(\overline{F}/F) \to A\Big(\widehat{G^0}, \widehat{B^0}, \widehat{T^0}, \{X_{\widehat{\alpha}}\}\Big)$$

defines an element of $H^1\Big(\mathfrak{g}(\overline{F}/F), \operatorname{Aut} G^0\Big)$ and hence an F-form G of G^0. In particular

$$G(F) = \Big\{\, g \in G^0(\overline{F}) \;\Big|\; \tau\eta(\tau)(g) = g\ \forall \tau \in \mathfrak{g}(\overline{F}/F) \,\Big\}.$$

Observe that the group G is quasi-split. Observe also that the data associated to two special distinguished splittings of $\widehat{G}$ are connected by a unique inner automorphism. It follows readily that the group G, together with B, T, $\{\eta_\alpha\}$, is determined up to canonical isomorphism by $\widehat{G}$.

Conversely, suppose we are given a quasi-split group G over F. Choose a Borel subgroup B and a Cartan subgroup T in B all defined over F. Interchanging the roles of L and $\widehat{L}$ and of Δ and $\widehat{\Delta}$, we pass from G, B, and T to $\widehat{G^0}$, $\widehat{B^0}$, $\widehat{T^0}$, and $\{X_\alpha\}$. The group $A\Big(\widehat{G^0}, \widehat{B^0}, \widehat{T^0}, \{X_{\widehat{\alpha}}\}\Big)$ may be identified with the group of automorphisms of L that leave the set Δ invariant. Define a homomorphism

$$\eta : \mathfrak{g}(\overline{F}/F) \to A\Big(\widehat{G^0}, \widehat{B^0}, \widehat{T^0}, \{X_{\widehat{\alpha}}\}\Big)$$

by

$$\eta(\tau)\lambda\big(\tau(t)\big) = \tau\big(\lambda(t)\big), \quad \lambda \in L,\ t \in T(\overline{F}).$$

This map allows us to define $\widehat{G}$, which again is determined up to canonical isomorphisms by G alone.

Suppose G_1 and G_2 are two quasi-split groups over F and $\psi : G_1 \to G_2$ is an isomorphism with $\psi^{-1}\tau(\psi)$ inner for each τ in $\mathfrak{g}(\overline{F}/F)$. Choose $g \in G_1(\overline{F})$ so that $\psi' = \psi \circ \operatorname{ad} g$ takes B_1 to B_2 and T_1 to T_2. Then ψ' determines a bijection $\Delta_1 \to \Delta_2$ as well as an isomorphism $\psi' : L_1 \to L_2$. These do not depend on the choice of g and determine an isomorphism $\widehat{\psi} : \widehat{G_1^0} \to \widehat{G_2^0}$. This isomorphism takes $\widehat{B_1^0}$ to $\widehat{B_2^0}$, $\widehat{T_1^0}$ to $\widehat{T_2^0}$, and $X_{\widehat{\alpha}_1}$ to $X_{\widehat{\alpha}_2}$ if α_1 and α_2 are corresponding elements in Δ_1 and Δ_2. Since $\psi'^{-1}\tau(\psi')$ takes T_1 to T_1, B_1 to B_1, and is inner it is the identity on T_1. It follows readily that

$$\eta_2(\tau)\psi'(\lambda_1) = \psi'\big(\eta_1(\tau)\lambda_1\big).$$

Thus $\widehat{\psi}$ may be extended to an isomorphism of $\widehat{G}_1$ with $\widehat{G}_2$ that preserves the splittings. It is determined uniquely by the conditions imposed upon it.

These are of course the considerations which allowed us to define $\widehat{G}$ in the first place. If $G_1 = G_2 = G$ then $\widehat{G}$ may be realized either as $\widehat{G_1^0} \times W_F$ or as $\widehat{G_2^0} \times W_F$ but these two groups are canonically isomorphic. There are occasions when a failure to distinguish between G and its realizations leads to serious confusion.

In general if G_1 is a connected reductive group over F we may choose an isomorphism ψ of G_1 with a quasi-split group G. The isomorphism ψ is to be defined over $\overline{F}$ and $\psi^{-1}\tau(\psi)$ is to be inner for $\tau \in \mathfrak{g}(\overline{F}/F)$. We may, taking into account the canonical isomorphisms above, define $\widehat{G}_1$ to be $\widehat{G}$. However, we should observe that the same difficulties are present here as in the definition of the fundamental group; the isomorphism ψ we write $\widehat{G}_1(\psi)$.

There are some further observations to be made before the task of this paper can be formulated. Let $\mathfrak{p}(G)$ and $\mathfrak{p}(G_1)$ be respectively the sets of conjugacy classes of parabolic subgroups of G and G_1 that are defined over F. Let $\mathfrak{p}(\widehat{G})$ be the classes of parabolic subgroups of $\widehat{G}$ with respect to conjugacy under $G^{\widehat{0}}$. We want to describe a bijection

$$\mathfrak{p}(G) \leftrightarrow \mathfrak{p}(\widehat{G})$$

and an injection

$$\mathfrak{p}(G_1) \hookrightarrow \mathfrak{p}(\widehat{G}).$$

For the first we recall that for a given T and B and the corresponding $T^{\widehat{0}}$, $B^{\widehat{0}}$ we have a bijection $\Delta \leftrightarrow \widehat{\Delta}$. It is well known that $\mathfrak{p}(G)$ is parametrized by the $\mathfrak{g}(\overline{F}/F)$-invariant subsets of Δ. The classes of parabolic subgroups of $G^{\widehat{0}}$ are parametrized by the subset of Δ. The normalizer of $P^{\widehat{0}}$ in $\widehat{G}$ is parabolic if and only if the associated subset of Δ is invariant under $\mathfrak{g}(\overline{F}/F)$. This yields the bijection.

The injection will now be defined by

$$\mathfrak{p}(G_1) \hookrightarrow \mathfrak{p}(G).$$

Suppose P_1 is a parabolic subgroup of G_1 defined over F. I claim that here there is a g in $G_1(\overline{F})$ such that if $\psi' = \psi \circ \operatorname{ad} g$ then $P = \psi'(P_1)$ is defined over F. The class of P depends only on ψ and the class of P_1. The required injection maps the latter class to the former. To prove that g exists we use the following lemma.

Lemma 2.1. *Let G' and G be connected reductive groups over F. Let G be quasi-split and let $\psi : G' \to G$ be an isomorphism defined over $\overline{F}$. Suppose $\psi^{-1}\tau(\psi)$ is inner for $\tau \in \mathfrak{g}(\overline{F}/F)$. If T' is a Cartan subgroup of G' defined over F there is a $g' \in G'(\overline{F})$ and a Cartan subgroup T in G defined over F such that $\psi' = \psi \circ \operatorname{ad} g'$ when restricted to T' yields an isomorphism of T' with T that is defined over F.*

Let G'_{der} be the derived group of G' and let G'_{sc} be its simply connected covering group. Define G_{der} and G_{sc} in the same way. Lift ψ to an isomorphism $\psi_{\text{sc}} : G'_{\text{sc}} \to G_{\text{sc}}$. Let T_{sc} be the inverse image of T' in G'_{sc}. Choose $\widetilde{t'} \in T'_{\text{sc}}(F)$ with image t' in $T'(F)$ so that T'_{sc} is the centralizer of $\widetilde{t'}$ and T' the centralizer of t'. Set $\widetilde{t}_1 = \psi_{\text{sc}}(\widetilde{t'})$. Since

$$\tau(\widetilde{t}_1) = \psi_{\text{sc}}\Big(\psi_{\text{sc}}^{-1}\tau(\psi_{\text{sc}})(\widetilde{t'})\Big), \quad \tau \in \mathfrak{g}(\overline{F}/F),$$

the conjugacy class of $\widetilde{t}_1$ is defined over F. By Theorem 1.7 of [14] there is a $\widetilde{g} \in G_{\text{sc}}(\overline{F})$ such that $\widetilde{t} = \operatorname{ad}\widetilde{g}(\widetilde{t}_1)$ lies in $G_{\text{sc}}(F)$. Let t be its projection in $G(F)$. The centralizer T of t is defined over F and if g' is the projection of $\widetilde{g}' = \psi_{\text{sc}}^{-1}(\widetilde{g})$ then $\psi' = \psi \circ \operatorname{ad} g$ maps t' to t and T' to T. Since both t' and t are rational over F the automorphism $\psi'^{-1}\tau(\psi')$ which is inner commutes with t' and hence with all of T'. It follows that $\psi' : T' \to T$ is defined over F.

We apply the lemma with G' equal to G_1 and with T' equal to a Cartan subgroup T_1 lying in P_1. Choose g so that if $\psi' = \psi \circ \operatorname{ad} g$ then $\psi'^{-1}\tau(\psi')$ lies in $T_1(\overline{F})$ for $\tau \in \mathfrak{g}(\overline{F}/F)$. Then if

$$P = \psi'(P_1)$$

we have

$$\tau(P) = \psi'\big(\psi'^{-1}\tau(\psi')(P_1)\big) = \psi(P_1) = P$$

and P is defined over F.

Let $\widehat{\mathfrak{p}}(G_1)$ be the image of $\mathfrak{p}(G_1)$ in $\mathfrak{p}(\widehat{G}_1)$.

Lemma 2.2. *If $\overline{P}_1^{\wedge} \supseteq \widehat{P}_1$ and the class of $\widehat{P}_1$ lies in $\widehat{\mathfrak{p}}(G_1)$ so does the class of $\overline{P}_1^{\wedge}$.*

Choose P_1 in G_1 that is defined over F. The parabolic subgroups of G_1 that are defined over F and contain P_1 belong to different classes. So do the parabolic subgroups of $\widehat{G}_1$ that contain $\widehat{P}_1$. We have only to verify that these sets contain the same number of elements. Choose T_1 in P_1 that is defined over F and choose an isomorphism ψ of G_1 with a quasi-split group G so that $\psi^{-1}\tau(\psi)$ is inner and commutes with T_1 for all $\tau \in \mathfrak{g}(\overline{F}/F)$. Let M_1 be a Levi factor of P_1 containing T_1 and let S_1 be a maximal torus in the center of M_1. Then $P = \psi(P_1)$, $M = \psi(M_1)$, and $S = \psi(S_1)$, as well as $\psi|_{S_1}$ are all defined over F. Thus the maximal split tori in S and S_1 have a common rank r and P and P_1 are both contained in 2^r parabolic subgroups defined over F. Since the number of parabolic subgroups of $\widehat{G}_1$ that contain $\widehat{P}_1$ is equal to the number of parabolic subgroups of G that are defined over F and contain P the required equality follows.

The group W_F lies in $\widehat{\mathcal{G}}(F)$. Let $\Phi(G_1)$ be the subset of

$$\operatorname{Hom}_{\widehat{\mathcal{G}}(F)}(W_F, \widehat{G}_1)$$

consisting of these φ such that the class of any parabolic subgroup $\widehat{P}$ containing $\varphi(W_F)$ lies in $\widehat{\mathfrak{p}}(G_1)$ under the above injection. In particular, for the quasi-split group G

$$\Phi(G) = \operatorname{Hom}_{\widehat{\mathcal{G}}(F)}(W_F, \widehat{G})$$

which is obviously a covariant functor of $\widehat{G}$.

We shall start in the next paragraph to relate $\Phi(G)$ to $\Pi(G)$. There are some simple properties of $\Phi(G)$ to establish first. The group $G(F)$ does not change on restriction of scalars and neither does $\Pi(G)$. We had best check that this is also true for $\Phi(G)$. Although there is, in the present circumstances, only one nontrivial way to restrict scalars, namely from $\mathbf{C}$ to $\mathbf{R}$, I would prefer not to take this explicitly into account.

Let E be a finite extension of F. We want first of all to define a faithful functor from $\widehat{\mathcal{G}}(E)$ to $\widehat{\mathcal{G}}(F)$. We imbed E in $\overline{F}$. Corresponding to this imbedding is an imbedding of W_E in W_F. Actually there is some arbitrariness in both imbeddings. Since, up to equivalence, it has no effect on the functor to be constructed, we ignore it. Suppose $\widetilde{G}^{\wedge}$ lies in $\widehat{\mathcal{G}}(E)$. Choose a distinguished splitting of $\widetilde{G}^{\wedge}$ and let $\widetilde{\eta}$ be the corresponding action of W_E on $\widetilde{G}^{\hat{0}}$. Let $G^{\hat{0}}$ be the set of functions h on W_F with values in $\widetilde{G}^{\hat{0}}$ satisfying

$$h(vw) = \widetilde{\eta}(v)\big(h(w)\big), \quad v \in W_E.$$

Let $\eta(v)$, $v \in W_F$, send h to h' with

$$h'(w) = h(wv).$$

With respect to this action form the semi-direct product

$$\widehat{G} = G^{\widehat{0}} \rtimes W_F.$$

It is easy to see that the given splitting of $\widehat{G}$ is distinguished and that $\widehat{G}$ lies in $\widehat{\mathcal{G}}(F)$. Observe also that there is an obvious bijection from $\mathfrak{p}(\widetilde{G}^\wedge)$ to $\mathfrak{p}(\widehat{G})$.

If we had chosen another distinguished splitting $\widetilde{\eta}'$ there would be a $g \in \widetilde{G}^{\widehat{0}}$ such that

$$\widetilde{\eta}'(w) = g\widetilde{\eta}(w)g^{-1}, \quad w \in W_E.$$

The map $h \to h'$ with $h'(w) = gh(w)g^{-1}$ together with the identity on W_F would yield an isomorphism between $\widehat{G}$ and the group constructed by means of $\widetilde{\eta}'$; so we need not worry about the arbitrariness of the distinguished splitting either.

Choose a set V of representatives v for $W_E\backslash W_F$. If $w \in W_F$ let

$$vw = d_v(w)v', \quad v, v' \in V.$$

If $\widetilde{\varphi}$ is an L-homomorphism from $\widetilde{G}_1^\wedge$ to $\widetilde{G}_2^\wedge$ let

$$\widetilde{\varphi}(1 \times w) = \widetilde{a}(w) \times w, \quad w \in W_E,$$

with respect to special distinguished splittings of $\widetilde{G}_1^\wedge$ and $\widetilde{G}_2^\wedge$. If $w \in W_F$ let $a(w)$ be the function in $G_2^{\widehat{0}}$ whose value at $v \in V$ is $\widetilde{a}\big(d_v(w)\big)$. If h is a function in $G_1^{\widehat{0}}$ let h' be the function in $G_2^{\widehat{0}}$ defined by

$$h'(v) = \widetilde{\varphi}\big(h(v)\big), \quad v \in V.$$

Define $\widehat{G}_1$ and $\widehat{G}_2$ as above and let φ be the homomorphism from the former to the latter defined by

$$\varphi(h \times w) = h'a(w) \times w.$$

A little calculation, which will be left to the reader, shows that φ is in fact an L-homomorphism and that its class is determined by that of $\widetilde{\varphi}$ alone and is independent of the auxiliary data. The reader will also easily verify that the class determined by $\widetilde{\varphi}_1\widetilde{\varphi}_2$ is $\varphi_1\varphi_2$.

Given φ we define $\widetilde{\varphi}$ as follows. If $\widetilde{h} \times w$, $w \in W_E$, belongs to $\widetilde{G}_1^\wedge$ we let h be a function in $G_1^{\widehat{0}}$ with $h(1) = \widetilde{h}$. If

$$\varphi(h \times w) = h' \times w$$

we set

$$\widetilde{\varphi}(\widetilde{h} \times w) = h'(1) \times w.$$

The class of $\widetilde{\varphi}$ depends only on that of φ. It is clear that this process inverts the operation of the previous section. A slight variant of Shapiro's lemma shows that the reciprocal is true. Starting from φ we construct $\widetilde{\varphi}$; from $\widetilde{\varphi}$ we pass to φ'. We have to show that φ and φ' lie in the same class. We may assume that the set of representatives V contains 1. Suppose

$$\varphi(w) = h_w \times w$$

and define h in $G_2^{\widehat{0}}$ by

$$h(v) = h_v(1).$$

It is easily verified that

$$\varphi(g) = h\varphi'(g)h^{-1}, \quad g \in \widehat{G}_1.$$

Thus our functor is fully faithful. The object of $\widehat{\mathcal{G}}(F)$ corresponding to W_E is W_F.

Suppose $\widetilde{G}$ is quasi-split over E and G over F is obtained from $\widetilde{G}$ by restriction of scalars. Then for any scheme Z over F

$$\mathrm{Hom}_F(Z,G) = \mathrm{Hom}_E(Z \otimes_F E, \widetilde{G})$$

because restrictions of scalars is the right adjoint of base change. In particular if a Borel subgroup $\widetilde{B}$ of $\widetilde{G}$ and a Cartan subgroup $\widetilde{T}$ of $\widetilde{B}$ are given, then restriction of scalars yields B and T in G; so G is quasi-split. We must verify that $\widehat{G}$ is obtained from $\widetilde{G}^\wedge$ by the functor introduced above.

Let L' be the group of functions λ' on $\mathfrak{g}(\overline{F}/F)$ with values in $\widetilde{L}^\wedge$ satisfying

$$\lambda'(\sigma\tau) = \sigma\lambda'(\tau), \quad \sigma \in \mathfrak{g}(\overline{F}/E),$$

and let Δ' be the set of λ' that are zero on all but one coset of $\mathfrak{g}(\overline{F}/E)$ on which they take values in $\widetilde{\Delta}$. All we have to do is show that L' is isomorphic to $\widehat{L}$ as a $\mathfrak{g}(\overline{F}/F)$ module in such a way that Δ' corresponds to Δ.

Since we have chosen an imbedding of E in F we may take $\overline{E}$ to be $\overline{F}$. Map $\overline{F} \otimes_F E$ to the ring R of $\overline{F}$-valued functions a on $\mathfrak{g}(\overline{F}/F)$ satisfying

$$a(\sigma\tau) = \sigma\big(a(\tau)\big), \quad \sigma \in \mathfrak{g}(\overline{F}/E),$$

by

$$\alpha \otimes \beta \longrightarrow a : \tau \longrightarrow \tau(\alpha)\beta \quad .$$

This is an isomorphism. Then

$$\begin{aligned}\widehat{L} &= \mathrm{Hom}_{\overline{F}}\Big(\mathrm{GL}(1) \otimes_F \overline{F}, T \otimes_F \overline{F}\Big) \\ &= \mathrm{Hom}_F\Big(\mathrm{GL}(1) \otimes_F, \overline{F}, T\Big) = \mathrm{Hom}_E\Big(\mathrm{GL}(1) \otimes_F R, \widetilde{T}\Big).\end{aligned}$$

Every $\tau \in \mathfrak{g}(\overline{F}/F)$ yields by evaluation a map $R \to \overline{F}$ and hence a map

$$\mathrm{Hom}_E\Big(\mathrm{GL}(1) \otimes_F R, \widetilde{T}\Big) \to \mathrm{Hom}_E\Big(\mathrm{GL}(1) \otimes_F \overline{F}, \widetilde{T}\Big) = \widetilde{L}^\wedge.$$

Thus every element of $\widehat{L}$ yields a function on $\mathfrak{g}(\overline{F}/F)$ with values in $\widetilde{L}^\wedge$. The function is easily seen to lie in L'. That the resulting map from $\widehat{L}$ to L' has the required properties is easy to see.

If we take L to be $\mathrm{Hom}(L, \mathbf{Z})$ we may identify L with the space of functions λ on $\mathfrak{g}(\overline{F}/F)$ with values in $\widetilde{L}$ satisfying

$$\lambda(\sigma\tau) = \sigma\big(\lambda(\tau)\big), \quad \sigma \in \mathfrak{g}(\overline{F}/E).$$

The pairing is

$$\langle \lambda, \widehat{\lambda} \rangle = \sum_{\mathfrak{g}(\overline{F}/E)\backslash\mathfrak{g}(\overline{F}/F)} \Big\langle \lambda(\tau), \widehat{\lambda}(\tau) \Big\rangle.$$

If z is an $\overline{F}$-valued point in $\mathrm{GL}(1)$ then

$$\lambda\Big(\widehat{\lambda}(z)\Big) = z^{\langle \lambda, \widehat{\lambda} \rangle} = z^{\sum \langle \lambda(z), \widehat{\lambda}(\tau) \rangle} = \prod \lambda(\tau)\Big(\widehat{\lambda}(\tau)(z)\Big) = \prod \tau^{-1}\Big\{\lambda(\tau)\Big(\widehat{\lambda}(\tau)(\tau z)\Big)\Big\}$$

because every rational character of $\mathrm{GL}(1)$ is defined over F. In general we have an isomorphism

$$T(\overline{F}) = \mathrm{Hom}_F(\mathrm{Spec}\,\overline{F}, T) = \mathrm{Hom}_E(\mathrm{Spec}\,R, \widetilde{T}) = \widetilde{T}(R).$$

Since each $\tau \in \mathfrak{g}(\overline{F}/F)$ yields a map $R \to \overline{F}$, we may associate to each $s \in T(\overline{F})$ a function $\tau \to s(\tau)$ on $\mathfrak{g}(\overline{F}/F)$ with values in $\widetilde{T}(\overline{F})$. If $s = \widehat{\lambda}(z)$ then $s(\tau) = \widehat{\lambda}(\tau)\big(\tau(z)\big)$. Since the points $\widehat{\lambda}(z)$ generate $T(\overline{F})$ we have

$$\lambda(s) = \prod \tau^{-1}\Big\{\lambda(\tau)\big(s(\tau)\big)\Big\}.$$

In particular if s lies in $T(F)$ then $s(\tau) = \widetilde{s}$ is independent of τ and lies in $\widetilde{T}(E)$.

It has already been pointed out that the definition of the associated group of an arbitrary connected reductive group G_1 depends on the choice of an isomorphism $\psi : G_1 \to G$ with G quasi-split. However, composing ψ with an inner automorphism has no effect on the construction. In particular, since $\psi^{-1}\tau(\psi)$, $\tau \in \mathfrak{g}(\overline{F}/F)$ is always supposed inner, ψ could be replaced by $\tau(\psi)$.

Lemma 2.3. *Suppose $\widetilde{G}_1$ and $\widetilde{G}$ are given over E with $\widetilde{G}$ quasi-split, together with an isomorphism $\widetilde{\psi} : \widetilde{G}_1 \to \widetilde{G}$ over $\overline{E}$. Let G_1 and G be obtained from $\widetilde{G}_1$ and $\widetilde{G}$ by restriction of scalars. There is associated to $\widetilde{\psi}$ an isomorphism $\psi : G_1 \to G$ over $\overline{F}$ defined up to composition with an inner automorphism and $\widehat{G}_1$ is obtained from $\widetilde{G}_1^{\wedge}(\widetilde{\psi})$ by the restriction of scalars functor from $\widehat{\mathcal{G}}(E)$ to $\widehat{\mathcal{G}}(F)$.*

Only the existence of ψ needs to be established. We imbed E in F and identify $\overline{E}$ with $\overline{F}$

$$\begin{aligned}\mathrm{Hom}_F(G_1 \otimes_F \overline{F}, G \otimes_F \overline{F}) &= \mathrm{Hom}_F(G_1 \otimes_F \overline{F}, G) \\ &= \mathrm{Hom}_E(G_1 \otimes_F R, \widetilde{G})\end{aligned}$$

and

$$\mathrm{Hom}_{\overline{F}}(G_1 \otimes_F \overline{F}, G_1 \otimes_F \overline{F}) = \mathrm{Hom}_E(G_1 \otimes_F R, \widetilde{G}_1).$$

Start from the identity morphism on the left to get a morphism from $G_1 \otimes_F R$ to $\widetilde{G}_1$. On the other hand, if we choose a set of representatives ρ for $\mathfrak{g}(\overline{F}/E)$ in $\mathfrak{g}(\overline{F}/F)$ we may imbed $\overline{F}$ in R by associating to $\alpha \in \overline{F}$ the function whose value at each ρ is α. This yields a morphism from Spec R to Spec $\overline{F}$ over E. The two morphisms together yield a morphism from $G_1 \otimes_F R$ to $\widetilde{G}_1 \otimes_E \overline{F}$. Composing with $\widetilde{\psi} : \widetilde{G}_1 \otimes_E \overline{F} \to \widetilde{G}$ we get a morphism from $G_1 \otimes_F R$ to $\widetilde{G}$ and hence $\psi : G_1 \otimes_F \overline{F} \to G$.

The invariance of $\Phi(G)$ under restriction of scalars is now clear. Suppose P is a parabolic subgroup of G over F. We may choose B and T in P. Now construct $\widehat{G}$, $B^{\widehat{0}}$, and $T^{\widehat{0}}$. Let $\widehat{P}$ be the parabolic subgroup of $\widehat{G}$ containing $B^{\widehat{0}}$ whose class corresponds to that of P. Let N be the unipotent radical of P, $\widehat{N}$ that of $P^{\widehat{0}}$, and let $M = P/N$, $\widehat{M} = \widehat{P}/\widehat{N}$. It is easily seen that $\widehat{M}$ belongs to $\widehat{\mathcal{G}}(F)$ and that $\widehat{M}$ is the associated group of M. If $\overline{P}^{\wedge}$ is another parabolic subgroup in the same class as $\widehat{P}$ there is a $g \in G^{\widehat{0}}$ such that $g\widehat{P}g^{-1} = \overline{P}^{\wedge}$. The induced map $\widehat{M} \to \overline{M}^{\wedge}$ is uniquely determined up to an inner automorphism by an element of $\overline{M}^{\widehat{0}}$. Thus if $\overline{P}^{\wedge}$ and P lie in corresponding classes in $\mathfrak{p}(\widehat{G})$ and $\mathfrak{p}(G)$ the associated group of M is canonically isomorphic, in the category $\widehat{\mathcal{G}}(F)$, $\overline{P}^{\wedge}/\overline{N}^{\wedge}$.

Suppose $\psi : G_1 \xrightarrow{\sim} G$ is such that $\psi^{-1}\tau(\psi)$ is inner for $\tau \in \mathfrak{g}(\overline{F}/F)$. If P_1 is a parabolic subgroup of G_1 over F we may always modify ψ by an inner automorphism so that $P = \psi(P_1)$ is defined over F. We readily deduce the following lemma.

Lemma 2.4. *Suppose P_1 is a parabolic subgroup of G_1 over F and $\widehat{P}_1$ is a parabolic subgroup of $\widehat{G}_1$ whose class corresponds to that of P_1. Then $\widehat{M_1} = \widehat{P}_1/\widehat{N}_1$ is canonically isomorphic in the category $\widehat{\mathcal{G}}(F)$ to the associate group of M_1.*

Choose a splitting $M_1 \to P_1$ defined over F and a splitting $\widehat{M_1} \to \widehat{P}_1$ that carries distinguished splittings of $\widehat{M_1}$ to distinguished splittings of $\widehat{G}_1$. The isomorphism between $\widehat{M_1}$ and the associated group of M_1 depends on the choice of P_1 and $\widehat{P}_1$ with M_1 and $\widehat{M_1}$ as Levi factors.

Lemma 2.5. *Suppose P_1 and $\widehat{P}_1$ are given as above with M_1 and $\widehat{M_1}$ as Levi factors. There is a bijection η between the parabolic subgroups of G_1 defined over F that contain M_1 as a Levi factor and the parabolic subgroups of $\widehat{G}_1$ that contain $\widehat{M_1}$ as Levi factor such that $\widehat{P}_1 = \eta(P_1)$, and such that the isomorphism between $\widehat{M_1}$ and the associated group of M_1 is the same for all pairs $\widehat{P}_1$, $\eta(\overline{P}_1)$.*

Take G quasi-split and let ψ be an isomorphism from G_1 to G with $\psi^{-1}\tau(\psi)$ inner for $\tau \in \mathfrak{g}(\overline{F}/F)$. We also suppose that there is a Cartan subgroup T_1 in M_1 defined over F such that each $\psi^{-1}\tau(\psi)$ commutes with the elements of T. Then $\psi(T_1)$, $M = \psi(M_1)$, and $P = \psi(P_1)$ are defined over F. In fact if $\overline{P}_1$ is any parabolic subgroup over F that contains M_1 then $\overline{P} = \psi(\overline{P}_1)$ is defined over F. The definitions are such that we may prove the assertions for G, M, P rather than G_1, M_1, P_1. Choose a Borel subgroup B over F that is contained in P and a Cartan subgroup T of B that is also defined over F. Then build $\widehat{G}$, $B^{\widehat{0}}$, $T^{\widehat{0}}$, and $\{X_{\widehat{\alpha}}\}$. We may replace $\widehat{G}_1$ by $\widehat{G}$ and suppose that $\widehat{P}$ contains $\widehat{B}$. Since any two Levi factors of $\widehat{P}$ are conjugate under $P^{\widehat{0}}$ (cf. [12], Theorem 7.1), we may also suppose that $\widehat{M}$ contains $\widehat{T}$.

Let $D(M)$ be the space of vectors in $L \otimes \mathbf{R}$ invariant under $\mathfrak{g}(\overline{F}/F)$ and orthogonal to the roots of $\widehat{M}$. By a chamber in $D(M)$ we mean a connected component of the complement of the union of the hyperplanes

$$\left\{\, a \in D(M) \;\middle|\; \langle a, \widehat{\alpha}\rangle = 0 \,\right\}$$

where $\widehat{\alpha}$ is a root of $T^{\widehat{0}}$ in $G^{\widehat{0}}$ but not in $M^{\widehat{0}}$. There is a bijection between chambers in $D(M)$ and parabolic subgroups $\overline{P}^{\wedge}$ of $\widehat{G}$ that contain $\widehat{M}$ as Levi factor. The subgroup $\overline{P}^{\wedge}$ corresponds to the chamber

$$C = \left\{\, a \in D(M) \;\middle|\; \langle a, \widehat{\alpha}\rangle > 0 \text{ if } X_{\widehat{\alpha}} \in \widehat{\mathfrak{p}},\, X_{\widehat{\alpha}} \notin \widehat{m} \,\right\}.$$

$\mathfrak{p}$ and $\widehat{\mathfrak{m}}$ are the Lie algebras of $\widehat{P}$ and $\widehat{M}$.

There is also a bijection between chambers of $D(M)$ and parabolic subgroups of G that are defined over F and contain M as Levi factor. If B is the Killing form, which may be degenerate, then C corresponds to $\overline{P}$ defined by the condition that it contain T and that a root α of T in G be a root in $\overline{P}$ if and only if $B(a, \alpha) \geqslant 0$ for all a in C. The bijection η is the composition of $\overline{P} \to C \to \overline{P}^{\wedge}$.

The Weyl groups $\widehat{\Omega}$ and Ω of $T^{\widehat{0}}$ in $G^{\widehat{0}}$ and of T in G are isomorphic in such a way that the reflections

$$\lambda \to \lambda - \langle \lambda, \widehat{\alpha}\rangle \alpha,$$
$$\widehat{\lambda} \to \widehat{\lambda} - \langle \alpha, \widehat{\lambda}\rangle \widehat{\alpha}$$

correspond. Suppose $\overline{P}^\wedge = \eta(\overline{P})$. There is an $\widehat{\omega}$ in $\widehat{\Omega}$ that takes every root of $T^{\widehat{0}}$ in $\overline{P}^{\widehat{0}}$ not in $M^{\widehat{0}}$ and every root in $M^{\widehat{0}} \cap B^{\widehat{0}}$ to a root of $T^{\widehat{0}}$ in $B^{\widehat{0}}$. Let h in the normalizer of $T^{\widehat{0}}$ in $G^{\widehat{0}}$ represent $\widehat{\omega}$ and let $\widehat{P}_0$ be $h\overline{P}^\wedge h^{-1}$, and $\widehat{M}_0$ be $h\widehat{M}h^{-1}$. We may suppose that

$$\operatorname{Ad} h(X_{\widehat{\alpha}}) = X_{\widehat{\omega}(\widehat{\alpha})}$$

if $\widehat{\alpha}$ is a root of $T^{\widehat{0}}$ in $M^{\widehat{0}} \cap B^{\widehat{0}}$. If g in the normalizer of T in $G(\overline{F})$ represents the element ω of Ω corresponding to $\widehat{\omega}$ then $P_0 = g\overline{P}g^{-1}$ contains B. It is clear that α is a root of T in P_0 if and only if $\widehat{\alpha}$ is a root of $T^{\widehat{0}}$ in $\widehat{P}_0$. Thus P_0 and $\widehat{P}_0$ and hence $\overline{P}$ and $\overline{P}^\wedge$ belong to corresponding classes in $\mathfrak{p}(G)$ and $\mathfrak{p}(\widehat{G})$.

If we build the associate group of M starting with M, $B \cap M$, and T we obtain $\widehat{M}$, $\widehat{B} \cap \widehat{M}$, $T^{\widehat{0}}$, and the collection $\{X_{\widehat{\alpha}}\}$ where $\widehat{\alpha}$ runs over the simple roots of $T^{\widehat{0}}$ in $M^{\widehat{0}}$ with respect to $B^{\widehat{0}} \cap M^{\widehat{0}}$. This gives the isomorphism of $\widehat{M}$ with the associate group of M defined by P and $\widehat{P}$. The isomorphism between $\widehat{M}$ and the associated group of M defined by $\overline{P}$ and $\overline{P}^\wedge$ is more complicated to obtain. This is not because of any intrinsic asymmetry but rather because of the simplifying assumption that $\widehat{P}$ contains $\widehat{B}$ and P contains B. We have to use g to establish an isomorphism between M and $M_0 = gMg^{-1}$ that we may assume is defined over F, then build the associate group of M_0 with respect to $B \cap M_0$ and T, obtaining thereby $\widehat{M}_0$, $\widehat{B} \cap \widehat{M}_0$, $T^{\widehat{0}}$, and $\{X_{\widehat{\alpha}}\}$, where $\widehat{\alpha}$ runs over the simple roots of $T^{\widehat{0}}$ in $M^{\widehat{0}}$ with respect to $B^{\widehat{0}} \cap M_0^{\widehat{0}}$, and finally we have to use the isomorphism between $\widehat{M}$ and $\widehat{M}_0$ given by h.

What has to be verified to prove the lemma is that, in the category $\widehat{\mathcal{G}}(F)$, the isomorphism between $\widehat{M}$ and $\widehat{M}_0$ given by h is equal to the isomorphism between them as two concrete realizations of the associate group of M. What is the latter isomorphism? The isomorphism $\operatorname{ad} g$ takes M to M_0, $B \cap M$ to $B \cap M_0$, T to T, and the root α of T in M to $\omega\alpha$. Then the isomorphism between $\widehat{M}$ and $\widehat{M}_0$ as realizations of the associate group takes $M^{\widehat{0}}$ to $M_0^{\widehat{0}}$, $B^{\widehat{0}} \cap M^{\widehat{0}}$ to $B^{\widehat{0}} \cap M_0^{\widehat{0}}$, $T^{\widehat{0}}$ to $T^{\widehat{0}}$, $X_{\widehat{\alpha}}$ to $X_{\widehat{\omega}\widehat{\alpha}}$, respects the splittings $\widehat{M} = M^{\widehat{0}} \times W_F$, $\widehat{M}_0 = M_0^{\widehat{0}} \times W_F$ built into the construction, and acts trivially on W_F. It is characterized by these properties. Since $(\omega\alpha)^\wedge = \widehat{\omega}\widehat{\alpha}$ the isomorphism given by h has all these properties except perhaps the last. To achieve the last we exploit the circumstance that we are not really working with isomorphisms but rather with classes of them to modify our initial choice of h.

The group W_F acts on $\widehat{L}$. Since in its action on $G^{\widehat{0}}$ it leaves $\overline{P}^{\widehat{0}}$, $M^{\widehat{0}}$, and $B^{\widehat{0}}$ invariant and since the normalizer of $T^{\widehat{0}}$ in $B^{\widehat{0}}$ is $T^{\widehat{0}}$, it is clear that on $\widehat{L}$

$$w\widehat{\omega} = \widehat{\omega}w, \quad w \in W_F.$$

That h can be modified in the fashion desired follows immediately from the next lemma.

Lemma 2.6. *Let $\widehat{G}$, $B^{\widehat{0}}$, $T^{\widehat{0}}$, and $\{X_{\widehat{\alpha}}\}$ be given. Suppose $\widehat{\omega} \in \widehat{\Omega}$ and that on $\widehat{L}$*

$$w\widehat{\omega} = \widehat{\omega}w, \quad w \in W_F.$$

Then $\widehat{\omega}$ is represented by an element h of the normalizer of $T^{\widehat{0}}$ in $G^{\widehat{0}}$ that commutes with w in W_F and satisfies

$$\operatorname{ad} h(X_{\widehat{\alpha}}) = X_{\widehat{\omega}\widehat{\alpha}}$$

if $\widehat{\alpha}$ is simple with respect to $B^{\widehat{0}}$.

We ignore for the moment the last condition and simply try to find an h that represents $\widehat{\omega}$ and is fixed by the action of W_F on $G^{\widehat{0}}$. The action of W_F on $G^{\widehat{0}}$ factors through $\mathfrak{g}(\overline{F}/F)$ and it is easier to forget about W_F and deal directly with $\mathfrak{g}(\overline{F}/F)$. Start off with any h that represents $\widehat{\omega}$. Then

$$\tau \to a_\tau(h) = \tau(h)h^{-1}$$

lies in $T^{\widehat{0}}$ and is a 1-cocycle of $\mathfrak{g}(\overline{F}/F)$ with values in $T^{\widehat{0}}$. If h is replaced by sh, $s \in T^{\widehat{0}}$, then $a_\tau(h)$ is replaced by $\tau(s)a_\tau(h)s^{-1}$; so our problem is to show that the class of the cocycle is trivial. Since

$$a_\tau(h_1h_2) = a_\tau(h_1)\widehat{\omega}_1\big(a_\tau(h_2)\big)$$

it will be enough to show this for a set of generators of the centralizer $\widehat{\Omega}_0$ of $\mathfrak{g}(\overline{F}/F)$ in $\widehat{\Omega}$.

Suppose A is the set of vectors in $L \otimes \mathbf{R}$ invariant under $\mathfrak{g}(\overline{F}/F)$. The group $\widehat{\Omega}_0$ acts faithfully on A and, as is easily seen, acts simply transitively on the chambers, that is, the connected components of the complement of the hyperplanes.

$$\big\{\, a \in A \;\big|\; \langle a, \widehat{\alpha}\rangle = 0 \,\big\}$$

where $\widehat{\alpha}$ is any root of $T^{\widehat{0}}$ in $G^{\widehat{0}}$. Each orbit O in $\widehat{\Delta}$ defines a reflection

$$S_O : a \to a - \frac{\langle a, \widehat{\alpha}_0\rangle}{|O|} \sum_{\widehat{\alpha}\in O} \alpha$$

where $\widehat{\alpha}_0$ is any element of O. These reflections are each given by an $\widehat{\omega}_0$ in $\widehat{\Omega}_0$ and the collection of $\widehat{\omega}_0$ generates $\widehat{\Omega}_0$. We have to show that each $\widehat{\omega}_0$ is represented by an element of $G^{\widehat{0}}$ that is fixed by $\mathfrak{g}(\overline{F}/F)$. Replacing $G^{\widehat{0}}$ by a subgroup if necessary, we may suppose that $O = \widehat{\Delta}$. Since the question only becomes more difficult if $G^{\widehat{0}}$ is replaced by a finite covering group, we may suppose $G^{\widehat{0}}$ is the product of a torus and a finite number of simple, simply connected groups. The torus may be discarded. Let

$$G^{\widehat{0}} = \prod_{i=1}^{r} G_i^{\widehat{0}}, \quad T^{\widehat{0}} = \prod_{i=1}^{r} T_i^{\widehat{0}}, \quad \widehat{\Omega} = \prod_{i=1}^{r} \widehat{\Omega}_i,$$

and

$$\widehat{\omega} = \widehat{\omega}_{\widehat{\Delta}} = \prod_{i=1}^{r} \widehat{\omega}_i.$$

If $\tau(G_i^{\widehat{0}}) = G_j^{\widehat{0}}$ then

$$\tau(\widehat{\omega}_i) = \widehat{\omega}_j.$$

Suppose $\mathfrak{g}(\overline{F}/E)$ is the stabilizer of $G_1^{\widehat{0}}$ in $\mathfrak{g}(\overline{F}/F)$. Then $\widehat{\omega}_1$ commutes with $\mathfrak{g}(\overline{F}/F)$. Suppose it is represented by h_1 in $G_1^{\widehat{0}}$ which is fixed by $\mathfrak{g}(\overline{F}/E)$. Set

$$h_j = \tau(h_1)$$

where τ is any element of $\mathfrak{g}(\overline{F}/F)$ that takes $G_1^{\widehat{0}}$ to $G_j^{\widehat{0}}$. Then h_j is well-defined and

$$h = \prod_{j=1}^{r} h_j$$

is fixed by $\mathfrak{g}(\overline{F}/F)$ and represents $\widehat{\omega}$.

We are now reduced to a situation in which $\widehat{G^0}$ is simple and simply connected and $\mathfrak{g}(\overline{F}/F)$ acts transitively on $\widehat{\Delta}$. There are two possibilities. The group $\widehat{G^0}$ is of type A_1 or A_2. In the first case $\mathfrak{g}(\overline{F}/F)$ acts trivially and there is nothing to prove. In the second we may take $\widehat{G^0}$ to be $\mathrm{SL}(3,\mathbf{C})$, $\widehat{T^0}$ to be the group of diagonal matrices, $\widehat{B^0}$ to be the group of upper triangular matrices, and the collection $\{X_{\widehat{\alpha}}\}$ to consist of

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $A\Big(\widehat{G^0}, \widehat{B^0}, \widehat{T^0}, \{X_{\widehat{\alpha}}\}\Big)$ consists of the trivial automorphism and the automorphism

$$H \to \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} {}^tH^{-1} \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We may take h to be

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Suppose $\widehat{\omega}$ is arbitrary in $\widehat{\Omega}_0$ and is represented by an h in $\widehat{G^0}$ that is fixed by $\mathfrak{g}(\overline{F}/F)$. In order to complete the proof of the lemma we have to show that there is an s in $\widehat{T^0}$ that is fixed by $\mathfrak{g}(\overline{F}/F)$ such that

$$\mathrm{ad}(hs)X_{\widehat{\alpha}} = X_{\widehat{\omega}\widehat{\alpha}}, \quad \widehat{\alpha} \in \widehat{\Delta}.$$

Let

$$\mathrm{ad}\, h(X_{\widehat{\alpha}}) = c(\widehat{\alpha})X_{\widehat{\omega}\widehat{\alpha}}.$$

Clearly $c(\tau\widehat{\alpha}) = c(\widehat{\alpha})$ for $\tau \in \mathfrak{g}(\overline{F}/F)$. We may choose $d(\widehat{\alpha})$, $\widehat{\alpha} \in \widehat{\Delta}$, such that $d(\tau\widehat{\alpha}) = d(\widehat{\alpha})$ and such that

$$d(\widehat{\alpha})^{|\mathfrak{g}(\overline{F}/F)|} = c(\widehat{\alpha}).$$

If t in $\widehat{T^0}$ satisfies

$$\widehat{\alpha}(t) = d(\widehat{\alpha})^{-1}, \quad \widehat{\alpha} \in \widehat{\Delta},$$

then

$$s = \prod_{\tau \in \mathfrak{g}(\overline{F}/F)} \tau(t)$$

is the required s.

Suppose φ_1 is an automorphism of G_1 such that $\varphi_1^{-1}\tau(\varphi_1)$ is inner for all $\tau \in \mathfrak{g}(\overline{F}/F)$. For example φ_1 could be defined over F. If ψ is an isomorphism of G_1 with a quasi-split group G, we define the automorphism φ of G by transport of structure. We have seen already that φ determines an automorphism $\widehat{\varphi}$ of $\widehat{G}$. By transport of structure again we obtain an automorphism $\widehat{\varphi}_1$ of $\widehat{G}_1$. It is easily seen that $\widehat{\varphi}_1$ depends only on φ_1 and not on ψ.

Lemma 2.7. *Suppose P_1 is a parabolic subgroup of G_1 over F and $\widehat{P}_1$ is a parabolic subgroup of $\widehat{G}$ whose class corresponds to that of P_1. Let M_1 be a Levi factor of P_1 over F and $\widehat{M_1}$, which we take as the associate group of M_1, a Levi factor of $\widehat{P}_1$. Suppose $g \in G_1(F)$ normalizes M_1. If φ_1 is the restriction of $\mathrm{Ad}\, g$ to M_1 and $\widehat{\varphi}_1$ the associated automorphism of*

$\widehat{M}_1$, there is an element h in the normalizer of $\widehat{M}_1$ in $G_1^{\hat{0}}$ such that $\widehat{\varphi}_1$ is the restriction of $\operatorname{Ad} h$ *to* $\widehat{M}_1$.

Suppose that g is only in $G_1(\overline{F})$ but that $g^{-1}\tau(g)$ lies in $M_1(\overline{F})$ for each τ. Then we can still define $\widehat{\varphi}_1$ and the lemma remains valid. We work with the weaker assumption. The advantage is that if ψ is an isomorphism of G_1 with a quasi-split group G such that $\psi^{-1}\tau(\psi) = \operatorname{ad} m_\tau$ with $m_\tau \in M(\overline{F})$ for each τ then $\psi(g)$ continues to satisfy the weaker assumption, for

$$\psi(g^{-1})\tau\big(\psi(g)\big) = \psi\big(g^{-1}m_\tau\tau(g)m_\tau^{-1}\big) \in M(\overline{F})$$

if $M = \psi(M_1)$. We prove the lemma for the group G. P_1 is replaced by $P = \psi(P_1)$ and M_1 by M. g is now in $G(\overline{F})$. We choose B and T such that $B \subseteq P$ and $T \subseteq M$.

We may compose g with any element of $M(\overline{F})$ and thus suppose that

$$g(B \cap M)g^{-1} = b \cap M, \quad gTg^{-1} = T.$$

Since g is determined by these conditions modulo T,

$$g\tau(g^{-1}) \in T, \quad \tau \in \mathfrak{g}(\overline{F}/F).$$

In particular g represents an element ω of Ω fixed by $\mathfrak{g}(\overline{F}/F)$. Let $\widehat{\omega}$ be the corresponding element of $\widehat{\Omega}_0$.

We construct $\widehat{G}$, $B^{\hat{0}}$, $T^{\hat{0}}$, and $\{X_{\widehat{\alpha}}\}$ corresponding to G, B, and T and realize $\widehat{\omega}$ by an h that satisfies the conditions of the preceding lemma. If we take $\widehat{P}$ to contain $B^{\hat{0}}$ it is clear that $\operatorname{Ad} h$ is equal to $\widehat{\varphi}_1$ on $\widehat{M}$.

For the next lemma we work in the category of tori over F. Suppose S is such a torus. Then $\widehat{S}$ admits by construction a special distinguished splitting. Also $\widehat{L}$ is a covariant functor of S and

$$S^{\hat{0}} = \operatorname{Hom}(\widehat{L}, \mathbf{C}^\times)$$

is a contravariant functor. So is $\widehat{S}$. $\Phi(S)$, which consists of classes of continuous homomorphisms of W_F into $\widehat{S}$, is also contravariant. We write a homomorphism φ as

$$\varphi(w) = a(w) \times w.$$

We compose φ_1 and φ_2 by setting

$$\varphi_1\varphi_2(w) = a_1(w)a_2(w) \times w.$$

This composition is actually defined for the classes and turns $\Phi(S)$ into an abelian group. $\Pi(S)$ is the group of continuous homomorphisms of $S(F)$ into $\mathbf{C}^\times$. Although the following lemma is valid over any local field, we prove it here only for the real and the complex field.

Lemma 2.8. *On the category of tori over F the group-valued functors Φ and Π are isomorphic.*

When F is $\mathbf{C}$ the lemma is particularly easy. Any homomorphism from the topological group $\mathbf{C}^\times$ to $\mathbf{C}^\times$ may be written as

$$z = e^x \to z^a\overline{z}^b = e^{az+b\overline{x}}$$

where a and b are two uniquely determined elements of $\mathbf{C}$ whose difference lies in $\mathbf{Z}$. If $\varphi \in \Phi(S)$ is a continuous homomorphism from $\mathbf{C}^\times$ to $\widehat{S} = S^{\hat{0}}$, let $\varphi(z) = a(z) \times z$, $z \in \mathbf{C}^\times$, and

$$\widehat{\lambda}\big(a(z)\big) = z^{\langle\mu,\widehat{\lambda}\rangle}\overline{z}^{\langle\nu,\widehat{\lambda}\rangle}$$

where μ and ν are uniquely determined elements of $L\otimes\mathbf{C}$ whose difference lies in L. Associate to φ the element of π of $\Pi(S)$ defined by

$$\pi : t \to \mu(t)\nu(\bar{t}) = e^{\langle\mu,H\rangle+\langle\nu,\overline{H}\rangle}$$

where $H \in \widehat{L}\otimes\mathbf{C}$ is defined by

$$\lambda(t) = e^{\langle\lambda,H\rangle}, \quad \lambda\in L.$$

That the map $\varphi\to\pi$ gives the required isomorphism of functors is easily seen.

Now let F be $\mathbf{R}$. Let φ be an honest homomorphism from $W_{\mathbf{R}}$ to $\widehat{S}$. Let $\varphi(w) = a(w)\times w$ and

$$\widehat{\lambda}\big(a(z)\big) = z^{\langle\mu,\widehat{\lambda}\rangle}\overline{z}^{\langle\nu,\widehat{\lambda}\rangle}, \quad z\in\mathbf{C}^\times$$

If σ is the nontrivial element of $\mathfrak{g}(\mathbf{C}/\mathbf{R})$ then $\nu = \sigma\mu$. Let

$$a(1\times\sigma) = \alpha, \quad \alpha\in S^{\widehat{0}}.$$

and let

$$\widehat{\lambda}(\alpha) = e^{2\pi i\langle\lambda_0,\widehat{\lambda}\rangle}, \quad \lambda_0\in L\otimes\mathbf{C}.$$

λ_0 is determined modulo L and

$$\lambda_0+\sigma\lambda_0 \equiv \frac{1}{2}(\mu-\nu) \qquad (\mathrm{mod}\ L).$$

μ and ν are determined by the class of φ alone but λ_0 is determined only modulo the sum of L and

$$\{\,\lambda-\sigma\lambda \mid \lambda\in L\otimes\mathbf{C}\,\}.$$

We write an element t in $S(\mathbf{C})$ as e^H where H in $\widehat{L}\otimes\mathbf{C}$ is defined by

$$\lambda(t) = e^{\langle\lambda,H\rangle,} \quad \lambda\in L.$$

t lies in $S(\mathbf{R})$ if and only if

$$H-\sigma\overline{H}\in 2\pi i\widehat{L}.$$

Define π by

$$\pi(t) = e^{\langle\lambda_0,H-\sigma\overline{H}\rangle+\langle\mu/2,H+\sigma\overline{H}\rangle}.$$

This is permissible, for if t is 1 then $H\in 2\pi i\widehat{L}$ and

$$\langle\lambda_0,H-\sigma\overline{H}\rangle+\left\langle\frac{\mu}{2},H+\sigma\overline{H}\right\rangle = \left\langle\lambda_0+\sigma\lambda_0+\frac{\mu}{2}-\frac{\sigma\mu}{2},H\right\rangle\in 2\pi i\mathbf{Z}.$$

On the other hand, if π is given extend it to a quasi-character π' of $S(\mathbf{C})$. Let

$$\pi'(t) = e^{\langle\mu_1,H\rangle+\langle\mu_2,\overline{H}\rangle.}$$

Define μ and λ_0 by

$$\mu_1 = \frac{\mu}{2}+\lambda_0, \quad \sigma\mu_2 = \frac{\mu}{2}-\lambda_0,$$

so that

$$\mu = \mu_1+\sigma\mu_2, \quad \lambda_0 = \frac{\mu_1}{2}-\frac{\sigma\mu_2}{2}.$$

Then

$$\lambda_0+\sigma\lambda_0 = \frac{1}{2}\{\mu_1+\sigma\mu_1-\mu_2-\sigma\mu_2\} \equiv \frac{1}{2}\{\mu_1+\sigma\mu_2-\sigma\mu_1-\mu_2\} \quad (\mathrm{mod}\ L)$$

and

$$\mu_1 + \sigma\mu_2 - \sigma\mu_1 - \mu_2 = \mu - \sigma\mu.$$

All we have to do is check that μ is determined by π alone and that λ_0 is determined modulo the sum of L and $\{\, \lambda - \sigma\lambda \mid \lambda \in L \otimes \mathbf{C} \,\}$ by π.

For this we may suppose that π is trivial. If $H \in \widehat{L} \otimes \mathbf{C}$ then

$$1 = \pi'(e^{H+\sigma\overline{H}}) = e^{\langle \mu, H\rangle + \langle \sigma\mu, \overline{H}\rangle}$$

and $\mu = 0$. If $\widehat{\lambda} \in \widehat{L}$ and $\sigma\widehat{\lambda} = \widehat{\lambda}$ there is an $H \in \widehat{L} \otimes \mathbf{C}$ such that

$$2\pi i\widehat{\lambda} = H - \sigma\overline{H}.$$

Thus

$$\langle \lambda_0, \widehat{\lambda}\rangle \in \mathbf{Z}.$$

It follows immediately that

$$\lambda_0 \in L + \{\, \lambda - \sigma\lambda \mid \lambda \in L \otimes \mathbf{C} \,\}.$$

There is one fact to be verified.

Lemma 2.9. *The functor from Φ to Π respects restriction of scalars.*

We consider restriction of scalars from $\mathbf{C}$ to $\mathbf{R}$. Let $\widetilde{S}$ be a torus over $\mathbf{C}$ and S the torus obtained by restriction of scalars. Then

$$S(\mathbf{R}) = \mathrm{Hom}_{\mathbf{R}}(\mathrm{Spec}\,\mathbf{R}, S) \simeq \mathrm{Hom}_{\mathbf{C}}(\mathrm{Spec}\,\mathbf{C}, \widetilde{S}) = \widetilde{S}(\mathbf{C}).$$

We denote corresponding elements in $S(\mathbf{R})$ and $\widetilde{S}(\mathbf{C})$ by s and $\widetilde{s}$. L is the group of functions on $\mathfrak{g}(\mathbf{C}/\mathbf{R})$ with values in $\widetilde{L}$ and $\mathfrak{g}(\mathbf{C}/\mathbf{R})$ operates by right translation. If $\lambda_1 = \lambda(1)$, $\lambda_2 = \lambda(\sigma)$ then

$$\lambda(s) = \lambda_1(\widetilde{s})\sigma\big(\lambda_2(\widetilde{s})\big).$$

If $\widetilde{s} = e^{H^\sim}$, $H^\sim \in \widetilde{L}^\wedge \otimes C$ then $s = e^H$ with $H = (H^\sim, \overline{H}^\sim)$ and

$$H + \sigma\overline{H} = 2(H^\sim, \overline{H}^\sim), \quad H - \sigma\overline{H} = 0,$$

and

$$e^{\langle \widetilde{\mu}, H^\sim\rangle + \langle \widetilde{\nu}, \overline{H}^\sim\rangle} = e^{\langle \lambda_0, H - \sigma\overline{H}\rangle + \langle \mu/2, H + \sigma\overline{H}\rangle}$$

if

$$\mu = (\widetilde{\mu}, \widetilde{\nu}), \quad \lambda_0 = \frac{1}{2}(\widetilde{\mu} - \widetilde{\nu}, 0).$$

Thus if the quasi-character $\widetilde{\pi}$ of $\widetilde{S}(\mathbf{C})$ is given by $\widetilde{\mu}$, $\widetilde{\nu}$, the associated quasi-character π of $S(\mathbf{R})$ is given by μ and λ_0.

On the other hand let $\widetilde{\varphi} : W_{\mathbf{C}} \to \widetilde{S}^\wedge$ be given by $\widetilde{\varphi}(z) = \widetilde{a}(z) \times z$ and let

$$\widehat{\lambda}\big(\widetilde{a}(z)\big) = z^{\langle \widetilde{\mu}, \widehat{\lambda}\rangle}\overline{z}^{\langle \widetilde{\nu}, \widehat{\lambda}\rangle}.$$

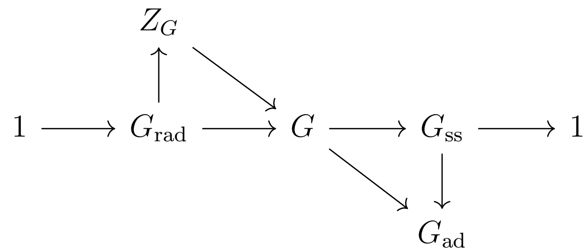
$\widehat{S^0}$ is the set of functions on $\mathfrak{g}(\mathbf{C}/\mathbf{R})$ with values in $\widehat{\widetilde{S}^0}$. If $\varphi : w \to a(w) \times w$ is obtained from $\widetilde{\varphi}$ by the restriction of scalars functor, then

$$a(z) = \big(\widetilde{a}(z), \widetilde{a}(\overline{z})\big), \quad a(1 \times \sigma) = \big(\widetilde{a}(-1), 1\big).$$

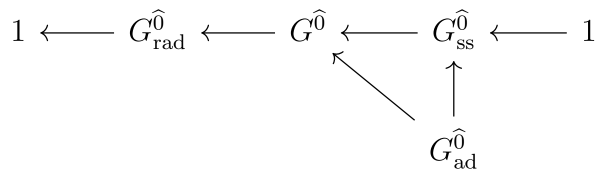
One calculates easily the corresponding μ and λ_0 and finds that they have the correct values.

Now take G connected and reductive. Let Z_G be its center. We want to use the previous lemma to associate to each element φ in $\Phi(G)$ a homomorphism X_φ of $Z_G(F)$ into $\mathbf{C}^\times$. Since

Z_G is not affected by an inner twisting, we could, but do not, suppose that G is quasi-split. Let G_{rad} be the maximal torus in Z and let G_{ss} be the quotient of G by G_{rad}. If G_{ad} is the adjoint group of G we have the following diagram



in which the horizontal line is exact. A pair B, T in G determines B_{ss}, T_{ss} and B_{ad}, T_{ad}. Using these to build the associate groups, we obtain



in which the horizontal line is exact.

In particular we have a map $\Phi(G) \to \Phi(G_{\mathrm{rad}})$, so that every element φ in $\Phi(G)$ determines a homomorphism of $G_{\mathrm{rad}}(F)$ into $\mathbf{C}^\times$. Thus when Z_G is connected we are able to define χ_φ. In general let

$$M = \mathrm{Hom}\Big(Z_G \otimes \overline{F}, \mathrm{GL}(1)\Big).$$

M is a $\mathfrak{g}(\overline{F}/F)$ module and there is surjection $\eta : L \to M$ whose kernel is the lattice generated by the roots. Let $\zeta : Q \to M$ be a surjective homomorphism of $\mathfrak{g}(\overline{F}/F)$-modules with Q free over Z. Let

$$\overline{L} = \left\{ (\lambda, p) \;\middle|\; \eta(\lambda) = \zeta(p) \right\}$$

and let

$$\overline{\Delta} = \left\{ (\alpha, 0) \;\middle|\; \alpha \in \Delta \right\}.$$

From $\overline{L}$ and $\overline{\Delta}$ and the cocycle defining G we construct $\overline{G}$. The surjection $\overline{L} \to L$ obtained by projection on the first factor yields an injection $G \to \overline{G}$ and a surjection $\overline{G}^\wedge \to \widehat{G}$ whose kernel is a torus over $\mathbf{C}$, namely

$$\mathrm{Hom}(\widehat{N}, \mathbf{C}^\times) = S^{\widehat{0}}$$

if N is the kernel of $\overline{L} \to L$ and S is the torus over F associated to the $\mathfrak{g}(\overline{F}/F)$-module N. Moreover $\overline{G}_{\mathrm{rad}} = Z_{\overline{G}}$ is the torus defined by Q.

There is an exact sequence

$$1 \longrightarrow \mathrm{Hom}\big(S(F), \mathbf{C}^\times\big) \longrightarrow \mathrm{Hom}\big(Z_{\overline{G}}(F), \mathbf{C}^\times\big) \longrightarrow \mathrm{Hom}\big(Z_G(F), \mathbf{C}^\times\big) \quad .$$

Every element of $\Phi(\overline{G})$ determines an element of the middle group and hence of the last. If $\overline{\varphi}_1$ and $\overline{\varphi}_2$ in $\Phi(\overline{G})$ have the same image in $\Phi(G)$ then, after an appropriate choice of representatives,

$$\overline{\varphi}_2(w) = a(w)\overline{\varphi}_1(w)$$

where $a(w) \in S^{\widehat{0}}$ and

$$\psi(w) = a(w) \times w$$

is an element of $\Phi(S)$. Thus the images of $\overline{\varphi}_1$ and $\overline{\varphi}_2$ in $\Phi(\overline{G}_{\rm rad})$ differ by an element in the image of $\Phi(S)$. By the functoriality of Lemma 2.8, they determine the same element of $\mathrm{Hom}\big(Z_G(F), \mathbf{C}^\times\big)$.

The next lemma will allow us to define χ_φ, $\varphi \in \Phi(G)$; it will remain, however, to verify that it is independent of the choice of Q.

Lemma 2.10. *Suppose $\overline{G}^\wedge$ and $\widehat{G}$ are objects in $\widehat{\mathcal{G}}(F)$ and $\widehat{\varphi} : \overline{G}^\wedge \to \widehat{G}$ is a surjective morphism. Suppose that the kernel of $\widehat{\varphi}$ is a torus $S^{\widehat{0}}$ in the center of $\overline{G}^{\widehat{0}}$. Then*

$$\mathrm{Hom}(W_F, \overline{G}^\wedge) \to \mathrm{Hom}(W_F, \widehat{G})$$

is surjective.

The assumption does not depend on the representative chosen. If $T^{\widehat{0}}$ is a Cartan subgroup of $G^{\widehat{0}}$ then

$$\overline{T}^{\widehat{0}} = \widehat{\varphi}^{-1}(T^{\widehat{0}})$$

is a Cartan subgroup of $\overline{G}^{\widehat{0}}$. If $\widehat{L}$ and $\overline{L}^\wedge$ are the lattices of rational characters of $T^{\widehat{0}}$ and $\overline{T}^{\widehat{0}}$ then $\widehat{L} \to \overline{L}^\wedge$ is injective and the quotient is torsion-free. Let

$$\psi \in \mathrm{Hom}(W_F, \widehat{G}).$$

We may assume that

$$\psi(\mathbf{C}^\times) \subseteq T^{\widehat{0}}.$$

Let

$$\widehat{\lambda}\big(\psi(z \times 1)\big) = z^{\langle\mu,\widehat{\lambda}\rangle}\overline{z}^{\langle\nu,\widehat{\lambda}\rangle}$$

where μ and ν lie in $L \otimes \mathbf{C}$ and $\mu - \nu \in L$. The map $\widehat{L} \to \overline{L}^\wedge$ leads to surjective maps $\overline{L} \to L$ and $\overline{L} \otimes \mathbf{C} \to \mathbf{C}$. Lift μ to $\widetilde{\mu}$ and ν to $\widetilde{\nu}$ in $\overline{L} \otimes \mathbf{C}$ so that $\widetilde{\mu} - \widetilde{\nu}$ lies in $\overline{L}$.

Define $\widetilde{\psi}(z \times 1)$ in $\overline{T}^{\widehat{0}}$ by

$$\widehat{\lambda}\Big(\widetilde{\psi}(z \times 1)\Big) = z^{\langle\widetilde{\mu},\widehat{\lambda}\rangle}\overline{z}^{\langle\widetilde{\nu},\widehat{\lambda}\rangle}, \quad \widehat{\lambda} \in \overline{L}^\wedge.$$

Lift $\psi(1 \times \sigma)$ arbitrarily to $\widetilde{\psi}(1 \times \sigma)$ and set, in general

$$\widetilde{\psi}(z \times \sigma) = \widetilde{\psi}(z \times 1)\widetilde{\psi}(1 \times \sigma).$$

Let

$$\widetilde{\psi}(w_1)\widetilde{\psi}(w_2) = a(w_1, w_2)\widetilde{\psi}(w_1 w_2),$$

where $a(w_1, w_2)$ is a continuous 2-cocycle on W_F with values in $S^{\widehat{0}}$. What we have to do is show that there is a continuous function $b(w)$ on W_F with values in $S^{\widehat{0}}$ such that

$$b(s_1)w_1\big(b(w_2)\big)a(w_1, w_2) = b(w_1 w_2).$$

What we do is introduce the extension K of topological groups defined by this cocycle and show that it splits continuously.

This is clear if $F = \mathbf{C}$; so take $F = \mathbf{R}$. Let $\widehat{N}$ be the lattice of rational characters of $S^{\widehat{0}}$. Consider the inverse image of $\mathbf{C}^\times$ in K. This extension of $\mathbf{C}^\times$ splits. Write an element in it as

$$s \times z = e^H \times e^x$$

with x in $\mathbf{C}$ and H in $N\otimes\mathbf{C}$. Let

$$\sigma(e^{H\times z})=e^{\sigma(H)+z\mu+\overline{z}\nu}\times e^{\overline{z}}$$

with μ and ν in $N\otimes\mathbf{C}$. Applying σ again we see that $\nu=-\sigma(\mu)$. Moreover $\mu+\sigma(\mu)$ must lie in N. In fact σ must fix the square of any lifting of $1\times\sigma$ to H. Since this square is of the form $s\times(-1)$,

$$e^{\pi i\left(\mu+\sigma(\mu)\right)}=1$$

and $\mu+\sigma(\mu)\in 2N$. Set

$$\gamma=\frac{\mu-\sigma(\mu)}{4},\quad \delta=\frac{\mu+\sigma(\mu)}{4}$$

and

$$\alpha=\gamma-\delta,\quad \beta=\gamma+\delta.$$

Then $\mu=2(\gamma+\delta)$, $\alpha-\beta=-2\delta$ lies in N and $\sigma(\alpha)+\mu=\beta$, $\sigma(\beta)-\sigma(\mu)=\alpha$. We replace the original splitting over $\mathbf{C}^\times$ by

$$e^z\to e^{z\alpha+\overline{z}\beta}\times e^z.$$

Since

$$\sigma\left(e^{z\alpha+\overline{z}\beta}\times e^z\right)=e^{z\left(\sigma(\alpha)+\mu\right)+\overline{z}\left(\sigma(\beta)-\sigma(\mu)\right)}\times e^{\overline{z}}=e^{z\beta+\overline{z}\alpha}\times e^{\overline{z}}$$

this new splitting is respected by the action of σ.

We have still to split the extension completely. Choose a representative h of $1\times\sigma$ in H. Let $h^2=s\times(-1)$. Let $S=e^H$ and $H=H_++H_-$, with $\sigma(H_+)=H_+$, $\sigma(H_-)=-H_-$. Replacing h by

$$(e^{-H_+/2}\times 1)h$$

if necessary, we may suppose that $H_+=0$. Since $\sigma(s)=s$, $2H$ lies in $2\pi iN$. Write

$$H=\pi i\left(\lambda-\sigma(\lambda)\right)$$

with $\lambda\in N\otimes\mathbf{C}$. We may modify the splitting over $\mathbf{C}^\times$ once again, replacing it by

$$e^z\to e^{z\lambda+\overline{z}\sigma(\lambda)}\times e^z.$$

In this new splitting over $\mathbf{C}^\times$, h^2 is given by

$$e^{H-\pi i\lambda+\pi i\sigma(\lambda)}\times -1=1\times -1.$$

We have now split the extension completely.

To show that χ_φ is independent of Q is easy. Suppose Q_1, Q_2 together with ζ_1, ζ_2 are two possible choices. Since we may replace the pair Q_1, Q_2 by Q_3, Q_1 or Q_3, Q_2 with

$$Q_3=\left\{\,(p_1,p_2)\;\middle|\;\zeta_1(p_1)=\zeta_2(p_2)\,\right\}$$

there is no harm in supposing that Q_1 is given by a surjective homomorphism $\xi:Q_1\to Q_2$. When this is so, Lemma 2.8 shows immediately that Q_1 and Q_2 give the same quasi-character χ_φ.

The following fact follows easily from the construction and Lemma 2.9.

Lemma 2.11. *The map $\varphi\to\chi_\varphi$ respects restriction of scalars.*

Let $\widehat{Z}$ be the center of $G^{\widehat{0}}$. The action of W_F on $\widehat{Z}$ is well-defined and so is the group $H^1(W_F, \widehat{Z})$, where it is understood that only continuous cocycles are to be considered. If $\varphi \in \Phi(G)$ and $\alpha \in H^1(W_F, \widehat{Z})$ define $\alpha\varphi$ by

$$\alpha\varphi(w) = \alpha(w)\varphi(w).$$

As is implicit in the notation and is easily verified the class of $\alpha\varphi$ depends only on that of α and φ. Thus the group $H^1(W_F, \widehat{Z})$ acts on $\Phi(G)$. We should also be able to make it act in $\Pi(G)$. To do this we associate to α a continuous homomorphism π_α of $G(F)$ into $C^\times$.

Let G_{der} be the derived group of G, G_{sc} the simply connected covering group of G_{der}, and G_{corad} the quotient of G by G_{der}. We have

$$\begin{array}{ccccccccc}
 & & G_{\mathrm{sc}} & & & & & & \\
 & & \downarrow & \searrow & & & & & \\
1 & \longrightarrow & G_{\mathrm{der}} & \longrightarrow & G & \longrightarrow & G_{\mathrm{corad}} & \longrightarrow & 1
\end{array}.$$

Passing to associate groups we have

$$\begin{array}{ccccccccc}
 & & G^{\widehat{0}}_{\mathrm{sc}} & & & & & & \\
 & & \uparrow & \nwarrow & & & & & \\
1 & \longleftarrow & G^{\widehat{0}}_{\mathrm{der}} & \longleftarrow & G^{\widehat{0}} & \longleftarrow & G^{\widehat{0}}_{\mathrm{corad}} & \longleftarrow & 1
\end{array}$$

Suppose we have a diagram

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & B & & & & \\
 & & & & \downarrow & \searrow & & & \\
1 & \longrightarrow & G_{\mathrm{sc}} & \longrightarrow & \widetilde{G} & \longrightarrow & D & \longrightarrow & 1 \\
 & & & \searrow & \downarrow & & & & \\
 & & & & G & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 1 & & & &
\end{array}$$

in which the vertical and horizontal lines are exact, B and D are tori, and $\widetilde{G}(F) \to G(F)$ is surjective. For example if R is the kernel of $G_{\mathrm{sc}} \to G$ and

$$K = \mathrm{Hom}\Big(R \otimes_F \overline{F}, \mathrm{GL}(1)\Big)$$

we could take a free $\mathfrak{g}(\overline{F}/F)$ module P that maps surjectively to K, set $\widetilde{L}$ equal to the group of pairs (λ, p), $\lambda \in L_{\mathrm{sc}}$, $p \in P$, with the same image in K, and $\widetilde{\Delta}$ equal to $\{\, (\alpha, 0) \mid \alpha \in \Delta \,\}$, and define G by means of $\widetilde{L}$, $\widetilde{\Delta}$, and the twisting defining G. Passing to associate groups yields

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & & \uparrow & & & & \\
 & & & & B^{\widehat{0}} & & & & \\
 & & & & \uparrow & \nwarrow & & & \\
1 & \longleftarrow & G_{\mathrm{sc}}^{\widehat{0}} & \longleftarrow & \widetilde{G}^{\widehat{0}} & \longleftarrow & D^{\widehat{0}} & \longleftarrow & 1 \\
 & & & \nwarrow & \uparrow & & & & \\
 & & & & G^{\widehat{0}} & \longleftarrow & \widehat{Z} & & \\
 & & & & \uparrow & & & & \\
 & & & & 1 & & & &
\end{array}
$$

This diagram gives $\widehat{Z}$ as the kernel of $D^{\widehat{0}} \to B^{\widehat{0}}$ and hence a map of $H^1(W_F, \widehat{Z})$ into the kernel of $H^1(W_F, D^{\widehat{0}}) \to H^1(W_F, B^{\widehat{0}})$. By Lemma 2.8 every element β of $H^1(W_F, D^{\widehat{0}}) = \Phi(D)$ yields a quasi-character of $D(F)$ and hence of $\widetilde{G}(F)$. It is trivial on $B(F)$ and hence gives a quasi-character of $G(F)$ if and only if β becomes zero in $H^1(W_F, B^{\widehat{0}})$. In particular every element α of $H^1(W_F, \widehat{Z})$ yields a quasi-character π_α of $G(F)$. If $\widetilde{G}_1$ and $\widetilde{G}_2$ are possible choices for $\widetilde{G}$ so is $\widetilde{G}_1 \times_G \widetilde{G}_2$. Using this, one shows easily that π_α does not depend on the choice of $\widetilde{G}$.

Lemma 2.12. *The map $\alpha \to \pi_\alpha$ respects restriction of scalars. If $\varphi' = \alpha\varphi$ then*

$$\chi'_\varphi(z) = \pi_\alpha(z)\chi_\varphi(z), \quad z \in Z(F).$$

The first assertion follows easily from Lemma 2.9 and the construction. Suppose G_1 is the group for which we are trying to prove the second assertion. Let G be quasi-split and let ψ be an isomorphism of G_1 and G such that $\psi^{-1}\tau(\psi)$ is inner for $\tau \in \mathfrak{g}(\overline{F}/F)$. We may so construct $\widetilde{G}_1$ and $\widetilde{G}$ that ψ lifts to $\widetilde{\psi} : \widetilde{G}_1 \to \widetilde{G}$. D_1 and D will be the same and

$$
\begin{array}{ccc}
\widetilde{G}_1 & \longrightarrow & D_1 \\
{\scriptstyle\widetilde{\psi}}\downarrow & & \Vert \\
\widetilde{G} & \longrightarrow & D
\end{array}
$$

will be commutative. Since ψ restricted to Z_{G_1} is defined over F and yields an isomorphism of Z_{G_1} with Z_{G_2} and since

$$\chi_\varphi\big(\psi(z)\big) = \chi_\varphi(z)$$

if $\varphi \in \Phi(G_1) = \Phi(G)$, we need only prove the lemma for G.

If $\widetilde{T}$ and T are corresponding Cartan subgroups of $\widetilde{G}$ and G, defined over F and lying in Borel subgroups over F, then

$$
\begin{array}{ccc}
\widetilde{T}^{\widehat{0}} & \longleftarrow & D^{\widehat{0}} \\
\uparrow & & \uparrow \\
T^{\widehat{0}} & \longleftarrow & \widehat{Z}
\end{array}
$$

is commutative. Thus on $T(F)$, π_α is the quasi-character defined by the image of α in

$$H^1(W_F, \widehat{T^0}) = \Phi(T).$$

Although we do not need to know it, it could be observed that $\widetilde{T}(F) \to D(F)$ is surjective because $H^1\Big(\mathfrak{g}(\overline{F}/F), T_{\mathrm{sc}}\Big) = 0$. Thus π_α is determined by its values on $T(F)$.

Now consider the objects used to define χ_φ. We had a surjection $\Phi(\overline{G}) \to \Phi(G)$. If φ is the image of $\overline{\varphi}$ then χ_φ is determined by the image of $\overline{\varphi}$ in $\Phi(\overline{G}_{\mathrm{rad}}^\wedge)$. But $\widehat{\overline{G}^0_{\mathrm{rad}}} = \widehat{\overline{T}^0_{\mathrm{rad}}}$ and, by Lemma 2.10, $\Phi(\overline{T}) \to \Phi(\overline{T}_{\mathrm{rad}})$ is surjective. Thus for $\varphi \in \Phi(G)$ there is an η that lies in the image of $\Phi(T) \to \Phi(G)$ and lifts to $\overline{\eta}$ in the image of $\Phi(\overline{T}) \to \Phi(\overline{G})$ such that $\overline{\eta}$ and $\overline{\varphi}$ have the same image in $\Phi(\overline{G}_{\mathrm{rad}})$. Then $\chi_\varphi = \chi_\eta$. If η is the image of β in $\Phi(T)$ then, by construction almost, χ_η is the restriction to $Z_G(F)$ of the quasi-character of $T(F)$ associated to β. Since $\overline{Z}^\wedge$, the center of $\widehat{\overline{G}^0}$, is the inverse image of $\widehat{Z}$ in $\overline{G}^0$ we may choose η' corresponding to φ' to be the image of $\gamma\beta$, if γ is the image of $\alpha \in H^1(W_F, \widehat{Z})$, in $H^1(W_F, \widehat{T^0}) = \Phi(T)$. The lemma now follows.

Notice that if $\psi : H \to G$ is defined over F and has an abelian kernel and an abelian cokernel then we can associate to it a homomorphism $\widehat{\psi} : \widehat{G} \to \widehat{H}$.

3. The definitions

The group $G(F)$ is a Lie group. Let $\mathfrak{g}$ be the tensor product of its Lie algebra with $\mathbf{C}$, let $\mathfrak{A}$ be the universal enveloping algebra of $\mathfrak{g}$ and let $\mathfrak{Z}$ be its center. A (continuous) representation π of $G(F)$ on a Banach space V will be called irreducible if V contains no nontrivial closed invariant subspaces; it will be called quasi-simple if the elements of $\mathfrak{Z}$ act on the infinitely differentiable vectors as scalars.

Let π be irreducible and quasi-simple. Let K be a maximal compact subgroup of $G(F)$ and let μ and ν be irreducible representations of K on the finite-dimensional spaces X and Y. Suppose we have K-homomorphisms ζ and η of X and Y into V and its dual V^* respectively. Suppose moreover that $\zeta(x)$ is infinitely differentiable for all $x \in X$. Let $\Psi = \Psi_{\zeta,\eta}$ be the function on $G(F)$ with values in $X^* \otimes Y^*$ defined by

$$\Psi(g) : (x, y) \to \big\langle \pi(g)\zeta(x), \eta(y) \big\rangle.$$

Then Ψ is a spherical function on $G(F)$ of type μ^*, ν^*, if μ^* and ν^* are contragredient to μ, ν. If we regard the elements of $\mathfrak{A}$ as left-invariant differential operators on $G(F)$ then

$$Z\Phi = \kappa(Z)\Psi, \quad Z \in \mathfrak{A},$$

if $\pi(Z) = \kappa(Z)I$. Because π is quasi-simple and irreducible, $\pi(z)$ is a scalar for $z \in Z_G(F)$ and

$$\Psi(gz) = \pi(z)\Psi(g).$$

If $G^0(F)$ is the connected component of $G(F)$ then

$$G(F) = KG^0(F) = KZ_G(F)G^0_{\mathrm{der}}(F)$$

(cf. [11]); so Ψ is determined by its restriction to $G^0_{\mathrm{der}}(F)$. Notice also that if $v \in V$, $v^* \in V^*$, and $\big\langle \pi(g)v, v^* \big\rangle = 0$ for all g then either v or v^* is zero.

It follows from these considerations and Proposition 9.1.3.1 of [16] that

$$\dim \mathrm{Hom}_K(X, V) < \infty,$$

so that any representation of K occurs with finite multiplicity in V. Let V_K be the space of K-finite vectors. Every vector in V_K is infinitely differentiable so that both $\mathfrak{A}$ and K operate on V_K. The representations π and π' on V and V' are said to be infinitesimally equivalent if the representations of the pair $\mathfrak{A}$, K on V_K' are algebraically equivalent. Since any two maximal compact subgroups of $G(F)$ are conjugate, this notion does not depend on the choice of K. $\Pi(G)$ will be the set of infinitesimal equivalence classes of irreducible quasi-simple representations of $G(F)$. We shall usually not distinguish between a representation and its class.

To every φ in $\Phi(G)$ we are going to associate a finite but nonempty set Π_φ in $\Pi(G)$ such that the following conditions are valid.

(i) If $\varphi \neq \varphi'$ then Π_φ and Π'_φ are disjoint.

(ii) If $\pi \in \Pi_\varphi$ then

$$\pi(z) = \chi_\varphi(z) I, \quad z \in Z_G(F).$$

(iii) If $\varphi' = \alpha\varphi$ with $\alpha \in H^1(W_F, \widehat{Z})$ then

$$\Pi'_\varphi = \left\{ \pi_\alpha \otimes \pi \;\middle|\; \pi \in \Pi_\varphi \right\}.$$

(iv) If $\eta : H \to G$ has abelian kernel and cokernel, if $\varphi \in \Phi(G)$ and $\varphi' = \widehat{\eta}(\varphi)$, then the pullback of any $\pi \in \Pi_\varphi$ to $H(F)$ is the direct sum of finitely many irreducible, quasi-simple representations, all of which lie in $\Pi_{\varphi'}$.

(v) If $\varphi \in \Phi(G)$ and one element of Π_φ is square integrable modulo $Z_G(F)$ then all elements are. This happens if and only if $\varphi(W_F)$ is contained in no proper parabolic subgroup of $\widehat{G}$.

We remark that the representation π is said to be square integrable modulo the center if $\pi = \zeta \otimes \pi'$ where ζ is one-dimensional and where π', which operates on V', is such that $\left| f'\big(\pi'(g)v'\big) \right|^2$ is an integrable function on $Z_G(F)\backslash G(F)$ for any K-finite $v' \in V'$ and any K-finite linear form f' on V'.

(vi) If $\varphi \in \Phi(G)$ is tempered then all elements are. With respect to a distinguished splitting, write $\varphi(w) = a(w) \times w$. The elements of Π_φ are tempered if and only if $\left\{ a(w) \;\middle|\; w \in W_F \right\}$ is relatively compact in $\widehat{G}$.

π, acting on V, is said to be tempered if $f\big(\pi(g)v\big)$ satisfies the weak inequality for any K-finite $v \in V$ and any K-finite linear form f on V.

Since we can always restrict scalars, we may as well take F to be $\mathbf{R}$. Let $\varphi \in \Phi(G)$. Let A be the Zariski-closure of the image of $W_{\mathbf{R}}$ under the composition of φ with the homomorphism of $\widehat{G}$ into the group of automorphism of $\widehat{\mathfrak{g}}$, the Lie algebra of $G^{\widehat{0}}$. Let B be the Zariski-closure of the image of $\mathbf{C}^\times$. Since the elements in the image of $\mathbf{C}^\times$ commute and are, by assumption, semisimple they can be simultaneously diagonalized. Thus every element of B is semisimple. The same is true for A, because $A^2 \subseteq B$. Since A is clearly supersolvable we may apply Theorem 5.16 of [13] to see that $\varphi(W_{\mathbf{R}})$ normalizes a Cartan subgroup $S^{\widehat{0}}$ in $G^{\widehat{0}}$. Since the group of automorphisms of $S^{\widehat{0}}$ is discrete, $\varphi(\mathbf{C}^\times)$ must centralize $S^{\widehat{0}}$. Consequently $\mathfrak{g}(\mathbf{C}/\mathbf{R}) = \mathbf{C}^\times\backslash W_{\mathbf{R}}$ acts on $S^{\widehat{0}}$, on $\widehat{M} = \mathrm{Hom}(S^{\widehat{0}}, \mathbf{C}^\times)$, on $M = \mathrm{Hom}(\widehat{M}, \mathbf{Z})$, and on $M \otimes \mathbf{R}$.

Suppose $\mathfrak{g}(\mathbf{C}/\mathbf{R})$ fixes a point λ in $M \otimes \mathbf{R}$. If $\widehat{P}$ is the parabolic subgroup of $\widehat{G}$ defined by the condition that $\widehat{\alpha}$ is a root of $S^{\widehat{0}}$ in $P^{\widehat{0}}$ if and only if $\langle \lambda, \widehat{\alpha} \rangle \geqslant 0$ then $\varphi(W_{\mathbf{R}})$ lies in $\widehat{P}$.

We shall first define Π_φ under the assumption that $\varphi(W_{\mathbf{R}})$ is contained in no proper parabolic subgroup of $\widehat{G}$. Then if λ is fixed by $\mathfrak{g}(\mathbf{C}/\mathbf{R})$

$$\langle \lambda, \widehat{\alpha} \rangle = 0$$

for all $\widehat{\alpha}$.

Lemma 3.1. *If $\Phi(G)$ contains a φ with the property that $\varphi(W_{\mathbf{R}})$ is contained in no proper parabolic subgroup of $\widehat{G}$ then G_{der} has a Cartan subgroup T_{der} with $T_{\mathrm{der}}(F)$ compact.*

We have a map $\widehat{G} \to \widehat{G}_{\mathrm{der}}$ that yields $\Phi(G) \to \Phi(G_{\mathrm{der}})$. Replacing φ by its image in $\Phi(G_{\mathrm{der}})$, we may suppose that $G = G_{\mathrm{der}}$. Then $\widehat{M} \otimes \mathbf{R}$ is spanned by the roots of $\widehat{\alpha}$. Consequently the nontrivial element σ in $\mathfrak{g}(\mathbf{C}/\mathbf{R})$ fixes no element of $M \otimes \mathbf{R}$ but 0 and acts as -1.

Let ψ be an isomorphism of G with a quasi-split group G'. Choose a Borel subgroup B' of G' over $\mathbf{R}$ and a Cartan subgroup T' of B'. Use G', B', and T' to build $\widehat{G}$, $B^{\widehat{0}}$ and $T^{\widehat{0}}$ as a concrete realization of the associate group. Replacing φ by another homomorphism in the same class, we may suppose that $S^{\widehat{0}} = T^{\widehat{0}}$, so that $\widehat{M} = \widehat{L}$. There are, however, two actions of σ on $\widehat{L}$, the one built into the construction of $\widehat{G}$, which we denote by $\widehat{\lambda} \to \sigma\widehat{\lambda}$, and the one defined by φ, which we denote by $\widehat{\lambda} \to \overline{\sigma}\widehat{\lambda}$. There is an $\widehat{\omega}$ in $\widehat{\Omega}$ such that

$$\overline{\sigma}\widehat{\lambda} = \widehat{\omega}\sigma\widehat{\lambda}.$$

Since $\overline{\sigma}$ acts as -1, σ commutes with $\overline{\sigma}$ and $\widehat{\omega}$. Let ω be the element of Ω', the Weyl group of T', corresponding to $\widehat{\omega}$.

Because any two quasi-split groups differing by an inner twisting are isomorphic, we may suppose that σ acts on $G'(\mathbf{C})$ in such a way that

$$\sigma\widehat{\lambda}(\sigma\tau) = \sigma\Big(\widehat{\lambda}(t)\Big), \quad t \in T'(\mathbf{C}),$$

and

$$\sigma(X'_\alpha) = X'_{\sigma\alpha}, \quad \alpha \in \Delta'.$$

Here the X'_α are appropriately chosen root vectors in the Lie algebra of G. Define $X_{-\alpha}$ so that if $H_\alpha = [X_\alpha, X_{-\alpha}]$ then $\alpha(H_\alpha) = 2$. The algebraic automorphism of G' defined by $t \to \omega(t)$, $X_\alpha \to X_{\omega(\alpha)}$ commutes with the action of σ and its square is 1. It is of course inner. We use the cocycle $a_1 = 1$, $\alpha_\sigma = \omega$ to twist G' and obtain G''. G'' contains the Cartan subgroup T'' obtained by twisting T'. Since

$$\lambda(\omega\sigma t) = \sigma\big(\lambda(t)\big)^{-1}$$

$T''(\mathbf{R})$ is compact.

There is an isomorphism η of G'' with G such that $\xi = \eta^{-1}\sigma(\eta)$ is inner. We may suppose that $\eta(T'') = T$ is a Cartan subgroup of G over $\mathbf{R}$ for which the compact part of $T(\mathbf{R})$ has maximal dimension. What we have to do to show that $T(\mathbf{R})$ is compact is to show that ξ, which normalizes T'', actually centralizes T'', for then $\eta : T'' \to T$ is defined over $\mathbf{R}$. We use an idea that can be found in many places. If $t \in T''(\mathbf{C})$ and λ is a rational character of T'' then $\lambda\big(\sigma(t)\big) = \sigma\big(\lambda(t)^{-1}\big)$. Thus

$$\lambda\big(\sigma\xi(t)\big) = \sigma\big(\lambda(\xi t)^{-1}\big) = \sigma\big(\xi^{-1}\lambda(t)^{-1}\big) = \xi^{-1}\lambda\big(\sigma(t)\big) = \lambda\big(\xi\sigma(t)\big)$$

and $\xi\sigma = \sigma\xi$. Since $\xi\sigma = 1$, $\xi^2 = 1$.

Suppose α is a root of T'' and $\xi\alpha = -\alpha$. Consider the subgroup H'' of G'' that is generated by $T''(\mathbf{C})$ and the one-parameter subgroups $\exp zX''_\alpha$, $\exp zX''_{-\alpha}$, $z \in \mathbf{C}$. H'' is invariant under

σ and ξ and $H = \eta(H'')$ is defined over $\mathbf{R}$. I claim that H_{sc} is isomorphic to SL(2) over $\mathbf{R}$ and that T_{sc}, the inverse image of T in H_{sc} is the Cartan subgroup whose set of real points is noncompact. This clearly contradicts the definition of T and shows that $\xi\alpha \neq -\alpha$ for all α. To prove the assertion about H_{sc}, we start from the observation that we may choose X''_α and $X''_{-\alpha}$ so that $[X''_\alpha, X''_{-\alpha}] = -H''_\alpha$ with $\alpha(H''_\alpha) = 2$ and so that $\sigma(X''_\alpha) = X''_{-\alpha}$. Then $\sigma(X''_{-\alpha}) = X''_\alpha$. Let $\xi(X''_\alpha) = aX''_{-\alpha}$, $\xi(X''_{-\alpha}) = bX''_\alpha$. Then

$$[aX''_{-\alpha}, bX''_\alpha] = -H''_\alpha;$$

so $ab = 1$. However the relation $\xi\sigma(\xi) = 1$ shows that $a\bar{a} = b\bar{b} = 1$. Recall that, on $\mathbf{C}$, σ is complex conjugation. Choose s in $T''(\mathbf{C})$ such that

$$\alpha(s)^{-1}\overline{\alpha(s)} = a.$$

Replacing η by $\eta \circ \operatorname{ad} s$, we suppose that $a = b = 1$. Set

$$H_\alpha = \eta(H''_\alpha), \quad X_\alpha = \eta(X''_\alpha), \quad X_{-\alpha} = -\eta(X''_\alpha).$$

Then

$$\sigma(X_\alpha) = \eta\big(\xi\sigma(X''_\alpha)\big) = \eta(X''_\alpha) = X_\alpha,$$
$$\sigma(X_{-\alpha}) = X_{-\alpha},$$
$$\sigma(H_\alpha) = H_\alpha.$$

Thus H_α, X_α, $X_{-\alpha}$ span a Lie algebra that, together with the action of σ on it, is isomorphic to the Lie algebra of SL(2). Since H_α lies in the Lie algebra of T, this gives the required assertion.

Let M'' be the lattice of rational characters of T''. Since $\alpha + \xi\alpha$ is different from 0 for all α there is a point H in the dual of $M'' \otimes \mathbf{R}$ such that

$$\langle \alpha + \xi\alpha, H\rangle = \langle \alpha, H + \xi H\rangle \neq 0$$

for all α. But ξ fixes $H + \xi H$ and therefore fixes the chamber in which it lies. Since ξ is inner this is possible only if ξ centralizes T''.

The lemma proved, we return to the original G and φ. Although it is not important, we choose for the sake of definiteness an isomorphism ψ of G with a quasi-split group G', with $\psi^{-1}\sigma(\psi)$ inner, choose B' and T', construct $G^{\widehat{0}}$, $B^{\widehat{0}}$, $T^{\widehat{0}}$ accordingly, and take the associate group to be $\widehat{G} = G^{\widehat{0}} \times W_{\mathbf{R}}$. We also suppose that $\varphi(W_{\mathbf{R}})$ normalizes $T^{\widehat{0}}$. Write $\varphi(w) = a(w) \times w$. If $z \in \mathbf{C}^\times$ then $a(z) \in T^{\widehat{0}}$. Let

$$\widehat{\lambda}\big(a(z)\big) = z^{\langle\mu,\widehat{\lambda}\rangle}\overline{z}^{\langle\nu,\widehat{\lambda}\rangle}.$$

If $\widehat{\lambda} \to \overline{\sigma}\widehat{\lambda}$ denotes the action of σ on $\widehat{L}$ defined by φ then $\nu = \overline{\sigma}\mu$, $\mu = \overline{\sigma}\nu$. Also if $a = a(1 \times \sigma)$,

$$\widehat{\lambda}\big(a\sigma(a)\big) = \widehat{\lambda}\big(a(-1)\big) = (-1)^{\langle\mu-\nu,\widehat{\lambda}\rangle}.$$

If $\langle\alpha, \widehat{\lambda}\rangle = 0$ for all roots α then $\widehat{\lambda}$ is a rational character of $G^{\widehat{0}}$ and we may define $\widehat{\lambda}(a)$. Notice in particular that $\langle\alpha, \widehat{\lambda}\rangle = 0$ for all roots α if $\widehat{\lambda} = \overline{\sigma}\widehat{\lambda}$. The next lemma is critical.

Lemma 3.2. *Suppose $h = a \times w$, with $w = 1 \times \sigma$, lies in $\widehat{G}$, normalizes $T^{\widehat{0}}$, and $h\widehat{\alpha} = -\widehat{\alpha}$ for every root $\widehat{\alpha}$. Then $a\sigma(a) \in T^{\widehat{0}}$ and, if δ is one-half the sum of the positive roots with respect to any order,*

$$\widehat{\lambda}\big(a\sigma(a)\big) = (-1)^{\langle 2\delta,\widehat{\lambda}\rangle}\widehat{\mu}(a) = (-1)^{\langle\delta-h\delta,\widehat{\lambda}\rangle}\widehat{\mu}(a)$$

if $\widehat{\mu} = \widehat{\lambda} + h\widehat{\lambda}$.

Of course an h satisfying the conditions of the lemma does not always exist. When it does a is any element of the normalizer of $\widehat{T^0}$ in $\widehat{G^0}$ that takes positive roots to negative roots.

That $a\sigma(a) \in \widehat{T^0}$ and that $2\delta = \delta - h\delta$ is clear. If $s \in \widehat{T^0}$ and h is replaced by sh then $a\sigma(a)$ is replaced by

$$sh(s)a\sigma(a)$$

where $h(s) = hsh^{-1}$. Since $\widehat{\mu}(a)$ becomes

$$\widehat{\mu}(sa) = \widehat{\lambda}(s)h\widehat{\lambda}(s)\widehat{\mu}(a)$$

and

$$\widehat{\lambda}\big(sh(s)\big) = \widehat{\lambda}(s)h\widehat{\lambda}(s)$$

we are free to replace h by sh. Thus we may suppose that $a \in \widehat{G^0_{\text{ss}}}$ or, more simply, that G, and hence $\widehat{G^0}$, is semisimple. Since it only makes the matter more difficult we may then replace G By G_{ad} and $\widehat{G^0}$ by $\widehat{G^0_{\text{ad}}}$, which is simply connected. Then the whole situation factors and we may finally assume that $\widehat{G^0}$ is simple and simply connected.

Suppose $\widehat{\beta}$ is the largest root with respect to the given order [2]. Then $\sigma(X_{\widehat{\beta}}) = \eta X_{\widehat{\beta}}$ with $\eta = \pm 1$. If σ acts trivially on $\widehat{G^0}$ then $\eta = 1$. In general I claim that if

$$\widehat{\beta} = \sum_{\widehat{\alpha}\in\widehat{\Delta}} n(\widehat{\alpha})\widehat{\alpha}$$

is the expression of $\widehat{\beta}$ as a sum of simple roots and if ℓ is one-half the sum of those $n(\widehat{\alpha})$ for which $\widehat{\alpha} \neq \sigma\widehat{\alpha}$ and $(\widehat{\alpha}, \sigma\widehat{\alpha}) \neq 0$ then $\eta = (-1)^\ell$. This statement is not true for $\widehat{\beta}$ alone but for any positive root

$$\widehat{\gamma} = \sum m(\widehat{\alpha})\widehat{\alpha}$$

fixed by σ. Of course $n(\widehat{\alpha})$ is to be replaced by $m(\widehat{\alpha})$ and η by $\eta(\widehat{\gamma})$, where

$$\sigma(X_{\widehat{\gamma}}) = \eta(\widehat{\gamma})X_{\widehat{\gamma}}.$$

We prove it by induction on $m = \sum m(\widehat{\alpha})$.

If $m = 1$ then $\ell = 0$; but by construction $\eta(\widehat{\gamma}) = 1$. Suppose $m > 1$, so that $\widehat{\gamma}$ is not simple. Choose a simple root $\widehat{\alpha}_1$ such that $(\widehat{\gamma}, \widehat{\alpha}_1) > 0$. If $\widehat{\alpha}_2 = \sigma\widehat{\alpha}_1$, then $(\widehat{\gamma}, \widehat{\alpha}_2) = (\widehat{\gamma}, \widehat{\alpha}_1)$. If $\widehat{\alpha}_1 = \widehat{\alpha}_2$ then $\overline{\overline{\widehat{\gamma}}} = \widehat{\gamma} - \widehat{\alpha}_1$ is also a root and

$$X_{\widehat{\gamma}} = [X_{\widehat{\alpha}_1}, X_{\overline{\overline{\widehat{\gamma}}}}];$$

so $\eta(\widehat{\gamma}) = \eta(\overline{\overline{\widehat{\gamma}}})$. Moreover $\ell(\widehat{\gamma}) = \ell(\overline{\overline{\widehat{\gamma}}})$. If $\widehat{\alpha}_1 \neq \widehat{\alpha}_2$ and $(\widehat{\alpha}_1, \widehat{\alpha}_2) = 0$ then $\overline{\overline{\widehat{\gamma}}} = \widehat{\gamma} - \widehat{\alpha}_1 - \widehat{\alpha}_2$ is a root. The integers $\ell(\widehat{\gamma})$ and $\ell(\overline{\overline{\widehat{\gamma}}})$ are equal. Since

$$X_{\widehat{\gamma}} = \Big[X_{\widehat{\alpha}_1}, [X_{\widehat{\alpha}_2}, X_{\overline{\overline{\widehat{\gamma}}}}]\Big] = \Big[X_{\widehat{\alpha}_2}, [X_{\widehat{\alpha}_1}, X_{\overline{\overline{\widehat{\gamma}}}}]\Big],$$

$\eta(\widehat{\gamma}) = \eta(\overline{\overline{\widehat{\gamma}}})$. If $(\widehat{\alpha}_1, \widehat{\alpha}_2) \neq 0$ then $\widehat{\alpha} = \widehat{\alpha}_1 + \widehat{\alpha}_2$ is a root and

$$X_{\widehat{\alpha}} = [X_{\widehat{\alpha}_1}, X_{\widehat{\alpha}_2}];$$

so $\eta(\widehat{\alpha}) = -1$. If $\widehat{\gamma} = \widehat{\alpha}$ we are done. Otherwise $\overline{\overline{\widehat{\gamma}}} = \widehat{\gamma} - \widehat{\alpha}$ is a root, $\ell(\widehat{\gamma}) = \ell(\overline{\overline{\widehat{\gamma}}}) + 1$, and $\eta(\widehat{\gamma}) = -\eta(\overline{\overline{\widehat{\gamma}}})$ because

$$X_{\widehat{\gamma}} = [X_{\widehat{\alpha}}, X_{\overline{\overline{\widehat{\gamma}}}}].$$

Since $(\widehat{\beta},\widehat{\alpha}) \geqslant 0$ for all positive roots, every root perpendicular to $\widehat{\beta}$ is a linear combination of simple roots perpendicular to it. Let $H^{\widehat{0}}$ be the connected subgroup of $G^{\widehat{0}}$ corresponding to the Lie algebra generated by $\left\{ X_{\widehat{\alpha}} \,\middle|\, (\widehat{\alpha},\widehat{\beta}) = 0 \right\}$. $\widehat{H} = H^{\widehat{0}} \times W_{\mathbf{R}} \subseteq \widehat{G}$ is also an associate group and we may assume the lemma has been proved for it. Let $J^{\widehat{0}}$ be the group corresponding to the Lie algebra generated by $X_{\widehat{\beta}}$, $X_{-\widehat{\beta}}$. $J^{\widehat{0}}$ is also invariant under $W_{\mathbf{R}}$. The groups $H^{\widehat{0}}$ and $J^{\widehat{0}}$ commute with each other. Let a_1 be an element of $H^{\widehat{0}}$ normalizing $T^{\widehat{0}}$ and taking positive roots in $H^{\widehat{0}}$ to negative roots. Let a_2 be an element of $J^{\widehat{0}}$ normalizing $T^{\widehat{0}}$ and taking $\widehat{\beta}$ to $-\widehat{\beta}$. a_1 fixes $\widehat{\beta}$. Thus if $\widehat{\alpha}$ is positive and $(\widehat{\alpha},\widehat{\beta}) > 0$, then $(a_1\widehat{\alpha},\widehat{\beta}) > 0$ and $a_1\widehat{\alpha}$ is positive. But $(a_2\widehat{\alpha},\widehat{\beta}) = -(\widehat{\alpha},\widehat{\beta}) < 0$ so $a_2\widehat{\alpha}$ is negative. The product a_1a_2 takes every positive root to a negative root and we may take $a = a_1a_2$. Since a_2 centralizes $H^{\widehat{0}}$, $a_1 \times (1 \times \sigma)$ in $\widehat{H}$ takes every positive root to its negative.

By induction
$$\widehat{\lambda}\big(a_1\sigma(a_1)\big) = (-1)^{\Sigma}\Gamma_0^{(\alpha,\widehat{\lambda})}$$
if $\Gamma_0 = \left\{ \alpha > 0 \,\middle|\, \langle \alpha,\widehat{\beta}\rangle = 0 \right\}$. $J^{\widehat{0}}$ is covered by $\mathrm{SL}(2,C)$. We may suppose that
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to X_{\widehat{\beta}}, \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \to X_{-\widehat{\beta}}.$$
Then the action of σ lifts to conjugation by
$$\begin{pmatrix} 1 & 0 \\ 0 & (-1)^{\ell} \end{pmatrix}.$$
Since we may take a_2 to be the image of
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$a_2\sigma(a_2)$ is the image of
$$\begin{pmatrix} (-1)^{\ell+1} & 0 \\ 0 & (-1)^{\ell+1} \end{pmatrix}.$$
Thus
$$\widehat{\lambda}\big(a_2\sigma(a_2)\big) = (-1)^{(\ell+1)\langle\beta,\widehat{\lambda}\rangle}.$$
To prove the lemma we have to show that
$$\ell\langle\beta,\widehat{\lambda}\rangle \equiv \sum_{\substack{\alpha>0 \\ \langle\alpha,\widehat{\beta}\rangle\neq 0 \\ \alpha\neq\beta}} \langle\alpha,\widehat{\lambda}\rangle \qquad (\mathrm{mod}\ 2).$$
$\alpha > 0$, $\langle\alpha,\widehat{\beta}\rangle \neq 0$, and if $\alpha \neq \beta$ then $\beta\langle\alpha,\widehat{\beta}\rangle - \alpha$ is also a positive root and is different from α. Thus the right side is $\ell'\langle\beta,\widehat{\lambda}\rangle$ if
$$\ell' = \frac{1}{2}\sum_{\substack{\alpha>0 \\ \alpha\neq\beta \\ \langle\alpha,\widehat{\beta}\rangle\neq 0}} \langle\alpha,\widehat{\beta}\rangle = \frac{1}{2}\left\{\sum_{\alpha>0}\langle\alpha,\widehat{\beta}\rangle\right\} - 1 = \langle\delta,\widehat{\beta}\rangle - 1 = \sum n(\widehat{\alpha}) - 1.$$
It would be enough to show that $\ell = \ell'$ (mod 2).

To finish up we make use of some standard facts [2]. The order h of a Coxeter element is $\ell'+1$. If σ acts trivially then $\ell = 0$. But if σ acts trivially then a itself must take every root to its negative. This forces $\ell'+2$ to be even ([2], p. 173). If σ does not act trivially the roots are all of the same length. There is an α in Δ such that $\langle \alpha, \widehat{\beta}\rangle = \langle \beta, \widehat{\alpha}\rangle = 1$ ([2], p. 165). Since $\langle 2\delta, \widehat{\alpha}\rangle = 2$

$$\widehat{\alpha}\big(a\sigma(a)\big) = (-1)^{\ell-\ell'}.$$

However h acts on the Lie algebra of $G^{\widehat{0}}$, and

$$h(X_{\widehat{\alpha}}) = cX_{-\widehat{\alpha}}, \quad h(X_{-\widehat{\alpha}}) = dX_{\widehat{\alpha}}, \quad h\big([X_{\widehat{\alpha}}, X_{-\widehat{\alpha}}]\big) = -[X_{\widehat{\alpha}}, X_{-\widehat{\alpha}}].$$

This forces cd to be 1; so

$$\widehat{\alpha}\big(a\sigma(a)\big)X_{\widehat{\alpha}} = h^2(X_{\widehat{\alpha}}) = X_{\widehat{\alpha}}$$

and $\ell - \ell'$ is even.

There is another lemma to be verified before we can define Π_φ.

Lemma 3.3. *Suppose $\varphi \in \Phi(G)$, $\varphi(W_R)$ is contained in no proper parabolic subgroup of $\widehat{G}$, $\varphi(W_{\mathbf{R}})$ normalizes $T^{\widehat{0}}$, and*

$$\widehat{\lambda}\big(\varphi(z)\big) = z^{\langle \mu, \widehat{\lambda}\rangle}\overline{z}^{\langle \nu, \widehat{\lambda}\rangle}, \quad z \in \mathbf{C}^\times.$$

Then $\langle \mu, \widehat{\alpha}\rangle$ is different from 0 for all roots $\widehat{\alpha}$. Moreover $\varphi(W_{\mathbf{R}})$ normalizes no other Cartan subgroup of $G^{\widehat{0}}$.

Suppose $\langle \mu, \widehat{\alpha}\rangle = 0$. $X_{\widehat{\alpha}}$ is fixed by $\varphi(z)$, $z \in \mathbf{C}^\times$. Let $h = \varphi(1 \times \sigma)$ and set

$$U = X_{\widehat{\alpha}} + h(X_{\widehat{\alpha}}).$$

Then $hU = U$. Let $\mathfrak{h}$ be the space of vectors H in the Lie algebra of $T^{\widehat{0}}$ for which $\widehat{\alpha}(H) = 0$. Then $\mathfrak{h} + \mathbf{C}U$ is the Lie algebra of a Cartan subgroup of $G^{\widehat{0}}$ normalized by $\varphi(W_{\mathbf{R}})$. It is however, clear that the action of σ on the roots does not take every root to its negative. This we know is incompatible with the assumption that $\varphi(W_{\mathbf{R}})$ is contained in no proper parabolic subgroup of $\widehat{G}$. Thus $\langle \mu, \widehat{\alpha}\rangle$ is never 0 and the centralizer of $\varphi(\mathbf{C}^\times)$ in the Lie algebra of $G^{\widehat{0}}$ is exactly the Lie algebra of $T^{\widehat{0}}$. The second assertion of the lemma follows.

If $n \in G^{\widehat{0}}$ and

$$\varphi' : w \to n\varphi(w)n^{-1}$$

satisfies the conditions of the lemma, with μ, ν replaced by μ', ν', then n must normalize $T^{\widehat{0}}$ and $\mu' = n\mu$, $\nu' = n\nu$. Consequently the orbit of μ under the Weyl group is determined by the class of φ alone. Since $\mu - \nu \in L$ and

$$\langle \mu - \nu, \widehat{\alpha}\rangle = \langle \mu - \overline{\sigma}\mu, \widehat{\alpha}\rangle = 2\langle \mu, \widehat{\alpha}\rangle$$

the number $\langle \mu, \widehat{\alpha}\rangle$ is real for all $\widehat{\alpha}$. Since it is different from 0, we may choose φ in its class so that $\langle \mu, \widehat{\alpha}\rangle > 0$ for all $\widehat{\alpha} \in \widehat{\Delta}$. This done, the only way we may modify φ is to replace it by

$$\varphi' : w \to s\varphi(w)s^{-1}$$

with $s \in T^{\widehat{0}}$.

We have observed that if $\langle \alpha, \widehat{\lambda}\rangle = 0$ for all roots α, we may define $\widehat{\lambda}(a)$, where $\varphi(1 \times \sigma) = a \times (1 \times \sigma)$. Choose $\lambda_0 \in L \otimes \mathbf{C}$ such that

$$\widehat{\lambda}(a) = e^{2\pi i \langle \lambda_0, \widehat{\lambda}\rangle}$$

for such $\widehat{\lambda}$. λ_0 is determined modulo

$$L + \sum_{\alpha\in\Delta} \mathbf{C}_\alpha + \{\, \lambda - \overline{\sigma}\lambda \mid \lambda \in L \otimes \mathbf{C} \,\}$$

or, since $\alpha = (\alpha - \overline{\sigma}\alpha)/2$,

$$L + \{\, \lambda - \overline{\sigma}\lambda \mid \lambda \in L \otimes \mathbf{C} \,\}.$$

We know that $\langle \mu, \widehat{\alpha} \rangle$ is real and different from 0 for all α. Let

$$\delta = \frac{1}{2} \sum_{\langle \mu, \widehat{\alpha}\rangle > 0} \alpha$$

and set $\mu_1 = \mu - \delta$, $\nu_1 = \overline{\sigma}\mu_1 = \nu + \delta$. Then $\mu_1 - \nu_1 = \mu - \nu - 2\delta \in L$, because $\mu - \nu \in L$. If $\widehat{\lambda} \in \widehat{L}$ then, by Lemma 3.2,

$$e^{\pi i \langle \mu - \nu, \widehat{\lambda} \rangle} = \widehat{\lambda}\big(a\sigma(a)\big) = e^{\pi i \langle 2\delta, \widehat{\lambda}\rangle + \langle \lambda_0 + \overline{\sigma}\lambda_0, \widehat{\lambda}\rangle};$$

so

$$\lambda_0 + \overline{\sigma}\lambda_0 \equiv \frac{\mu_1 - \nu}{2} \qquad (\mathrm{mod}\ L).$$

Thus

$$\begin{aligned} \langle \mu, \widehat{\alpha} \rangle = \frac{1}{2}\langle \mu - \overline{\sigma}\mu, \widehat{\alpha}\rangle &= \frac{1}{2}\langle \mu_1 - \nu_1, \widehat{\alpha}\rangle + \langle \delta, \widehat{\alpha}\rangle \\ &\equiv \langle \lambda_0 + \overline{\sigma}\lambda_0, \widehat{\alpha}\rangle + \langle \delta, \widehat{\alpha}\rangle = 1 \end{aligned}$$

modulo $\mathbf{Z}$ and $\langle \mu, \widehat{\alpha}\rangle$ is integral. If $\langle \mu, \widehat{\alpha}\rangle > 0$ then

$$\langle \mu_1, \widehat{\alpha}\rangle = \langle \mu, \widehat{\alpha}\rangle - \langle \delta, \widehat{\alpha}\rangle = \langle \mu, \widehat{\alpha}\rangle - 1 \geqslant 0.$$

Let S be a Cartan subgroup of G over F for which $S(\mathbf{R}) \cap G_{\mathrm{der}}(\mathbf{R})$ is compact. We may choose the isomorphism ψ of G with G' so that $\psi(S) = T'$. The isomorphism allows us to identify L, the lattice of rational characters of T', with the lattice of rational characters of S. Then the semi-direct product $\widehat{T^0} \times \mathfrak{g}(\mathbf{C}/\mathbf{R})$ with σ acting on $\widehat{T^0}$ as $\overline{\sigma}$ becomes the associate group of S. As in the proof of Lemma 2.8, μ_1, ν_1, and λ_0 then define a homomorphism χ of $S(\mathbf{R})$ into $\mathbf{C}^\times$. However since ψ is not uniquely determined, μ_1, ν_1 and λ_0 are only determined modulo the action of the Weyl group. Although the elements of the Weyl group of S may be represented by elements of $G(\mathbf{C})$ their action on S is defined over $\mathbf{R}$. If we replace μ_1, ν_1 and λ_0 by $\omega\mu_1$, $\omega\nu_1 = \overline{\sigma}\omega\mu_1$, and $\omega\lambda_0$, which is congruent to λ_0 modulo $\{\, \lambda - \overline{\sigma}\lambda \mid \lambda \in L \otimes \mathbf{C} \,\}$, then χ is replaced by $\chi' : t \to \chi\big(\omega^{-1}(t)\big)$. Let X_φ be the set of these quasi-characters χ. X_φ is determined by the class of φ alone.

To verify that the sets Π_φ, which we shall soon define, are disjoint and exhaust the classes of representations of $G(\mathbf{R})$ that are square-integrable modulo the center, we shall need the following lemma.

Lemma 3.4. *Suppose G has a Cartan subgroup S over $\mathbf{R}$ such that $S(\mathbf{R}) \cap G_{\mathrm{der}}(\mathbf{R})$ is compact and X is an orbit of the Weyl group in the set of quasi-characters of $S(\mathbf{R})$. Then there is a unique $\varphi \in \Phi(G)$ such that $\varphi(W_{\mathbf{R}})$ is contained in no proper parabolic subgroup of $\widehat{G}$ and such that $X = X_\varphi$.*

We have first to observe that an h in $\widehat{G}$ that satisfies the conditions of Lemma 3.2 exists. Let ψ be an isomorphism of G with a quasi-split group G' such that $\psi^{-1}\sigma(\psi)$ is inner. If B' and T' are chosen as usual, we may suppose that $\psi(S) = T'$. We have to show that there is an element $\widehat{\omega}$ of the Weyl group of $T^{\widehat{0}}$ such that $\widehat{\omega}\sigma\widehat{\alpha} = -\widehat{\alpha}$ for all $\widehat{\alpha}$. This is equivalent to showing that there is an element ω of the Weyl group of T' such that $\omega\sigma\alpha = -\alpha$ for all α. ω and $\widehat{\omega}$ will then be corresponding elements. Let $\psi\sigma(\psi^{-1}) = \operatorname{ad} n$, with n in the normalizer of T', then

$$-\psi^{-1}(\alpha) = \sigma\big(\psi^{-1}(\alpha)\big) = \psi^{-1}\big(\psi\sigma(\psi^{-1})\sigma\alpha\big) = \psi^{-1}(n\sigma\alpha)$$

and $n\sigma\alpha = -\alpha$. We take ω to be the element of the Weyl group represented by n.

Let $\overline{\sigma}$ be the action on $T^{\widehat{0}}$, $\widehat{L}$, and L determined by such an h. We regard $T^{\widehat{0}}$, with the action $\overline{\sigma}$, as the connected component of the associate group of S. If $\chi \in X$ choose μ_1, ν_1 and λ_0 so that

$$\chi(t) = e^{\langle \lambda_0, H - \overline{\sigma}\overline{H}\rangle + \langle \mu_1/2, H + \overline{\sigma}\overline{H}\rangle}$$

if $t = e^H$ lies in $S(\mathbf{R})$. Given X, μ_1, ν_1, and λ_0 are determined modulo the action of the Weyl group. Also $\mu_1 - \nu_1 \in L$ and, since $\overline{\sigma}\widehat{\alpha} = -\widehat{\alpha}$,

$$\langle \mu_1 - \nu_1, \widehat{\alpha}\rangle = \langle \mu_1 - \overline{\sigma}\mu_1, \widehat{\alpha}\rangle = 2\langle \mu_1, \widehat{\alpha}\rangle$$

is real. Choose an order on the roots so that $\langle \mu_1, \widehat{\alpha}\rangle \geqslant 0$ if $\widehat{\alpha}$ is positive; let δ be one-half the sum of the positive roots α with respect to this order, and set $\mu = \mu_1 + \delta$, $\nu = \overline{\sigma}\mu$. Since the δ which arise in this way differ by an element of the Weyl group that fixes μ_1, the orbit of μ, ν, and λ_0 under the Weyl group is determined by X_φ alone. The various μ are certainly nonsingular. To be definite choose the unique one that is positive with respect to $B^{\widehat{0}}$.

If φ is normalized in the way described earlier, it is clear that $X = X_\varphi$ only if

$$\widehat{\lambda}\big(\varphi(z)\big) = z^{\langle \mu, \widehat{\lambda}\rangle}\overline{z}^{\langle \nu, \widehat{\lambda}\rangle}, \quad z \in \mathbf{C}^\times.$$

Fix an $h = b \times (1 \times \sigma)$ satisfying the conditions of Lemma 3.2 and choose a representative λ_0. If $X = X_\varphi$ and φ gives rise to this particular λ_0 then $\varphi(1 \times \sigma) = a \times (1 \times \sigma)$ with $a = sb$, $s \in T^{\widehat{0}}$, and

$$e^{2\pi i \langle \lambda_0, \widehat{\lambda}\rangle} = \widehat{\lambda}(a) = \widehat{\lambda}(s)\widehat{\lambda}(b)$$

if $\langle \alpha, \widehat{\lambda}\rangle = 0$ for all α. An s in $T^{\widehat{0}}$ satisfying this condition always exists. We will be able to extend φ to $W_{\mathbf{R}}$ if

$$\widehat{\lambda}\big(a\sigma(a)\big) = e^{\pi i \langle \mu - \nu, \widehat{\lambda}\rangle}$$

for all $\widehat{\lambda}$. By Lemma 3.2 the left side is

$$e^{2\pi i \langle \delta, \widehat{\lambda}\rangle + 2\pi i \langle \lambda_0 + \overline{\sigma}\lambda_0, \widehat{\lambda}\rangle}.$$

Since

$$\frac{\mu_1 - \nu_1}{2} \equiv \lambda_0 + \overline{\sigma}\lambda_0 \qquad (\mathrm{mod}\ L)$$

it equals the right side. As for the uniqueness, if $\widehat{\lambda}(s) = 1$ whenever $\langle \alpha, \widehat{\lambda}\rangle = 0$ then, in particular, $\widehat{\lambda}(s) = 1$ when $\overline{\sigma}\widehat{\lambda} = \widehat{\lambda}$ and $s = t\overline{\sigma}(t^{-1})$ with $t \in T^{\widehat{0}}$. Then $sh = tht^{-1}$. Choosing a different representative for λ_0 forces the same kind of change in s; so the class of φ is determined uniquely by X.

Suppose $\varphi(W_{\mathbf{R}})$ is contained in a parabolic subgroup $\widehat{P}$ of $\widehat{G}$. Then $\varphi(W_{\mathbf{R}})$ is contained in a Levi factor $\widehat{M}$ of $\widehat{P}$ and normalizes a Cartan subgroup of $M^{\widehat{0}}$. But μ is regular; so $T^{\widehat{0}}$ is the only Cartan subgroup centralized by $\varphi(\mathbf{C}^\times)$ and therefore the only Cartan subgroup normalized by $\varphi(W_{\mathbf{R}})$. $\widehat{P}$ must then contain $T^{\widehat{0}}$. Since $\varphi(1 \times \sigma)$ takes each root to its negative, $\widehat{P}$ is $\widehat{G}$.

Suppose S and S' are two Cartan subgroups of G such that $S(\mathbf{R}) \cap G_{\mathrm{der}}(\mathbf{R})$ and $S'(\mathbf{R}) \cap G_{\mathrm{der}}(\mathbf{R})$ are compact. There is a $\mathfrak{g} \in G(\mathbf{C})$ such that $\operatorname{ad} g(S) = S'$. The restriction of $\operatorname{ad} g$ to S is defined over $\mathbf{R}$ and

$$X_\varphi = \left\{ \chi \circ \operatorname{ad} g \;\middle|\; \chi \in X'_\varphi \right\}.$$

If $g \in G(\mathbf{R})$ then $g = g_1 g_2$ where g_1 lies in $G^0_{\mathrm{der}}(\mathbf{R})$ and g_2 lies in the normalizer of S. If Ω is the Weyl group of S and Ω_1 consists of those elements in Ω with a representative in $G^0_{\mathrm{der}}(\mathbf{R})$, the connected component of $G_{\mathrm{der}}(\mathbf{R})$, and ω is the image of g_2 in Ω then $g \to \Omega_1 \omega$ is a well-defined map of $G(\mathbf{R})$ into Ω_1/Ω. The inverse image of Ω_1 in $G(\mathbf{R})$ is

$$G_0(\mathbf{R}) = S(\mathbf{R}) G^0_{\mathrm{der}}(\mathbf{R}) = S(\mathbf{R}) G^0(\mathbf{R}) = Z_G(\mathbf{R}) G^0_{\mathrm{der}}(\mathbf{R}).$$

If $\chi \in X_\varphi$ and μ_1 is defined above, choose an order on the roots so that $\langle \mu_1, \widehat{\alpha} \rangle \geqslant 0$ if α is positive and let δ be one-half the sum of the roots positive with respect to this order. By Harish-Chandra's theory of the discrete series there exists for each such pair χ, δ a unique irreducible representation $\pi_0(\chi, \delta)$ of $G_0(\mathbf{R})$, square-integrable modulo the center, whose character on the set of regular elements in $S(\mathbf{R})$ is

$$\epsilon(G) \sum_{\omega \in \Omega_1} \frac{\operatorname{sgn} \omega \chi(\omega s) e^{\delta(\omega H - H)}}{\Delta(s)}.$$

Here $s = e^H$ is a regular element in $S(\mathbf{R})$ and

$$\Delta(s) = \prod_{\langle \delta, \widehat{\alpha} \rangle > 0} (1 - e^{-\alpha(H)}).$$

$\epsilon(G)$, which is ± 1, depends on G alone.

It is clear that, if $\chi \in X_\varphi$, $\chi' \in X_{\varphi'}$, the representations $\pi_0(\chi, \delta)$ and $\pi_0(\chi', \delta')$ are equivalent if and only if $\varphi = \varphi'$ and there is an $\omega \in \Omega_1$ such that $\delta = \omega \delta'$ and

$$\chi'(s) = \chi(\omega s)$$

for all $s \in S(\mathbf{R})$. If $g \in G(\mathbf{R})$ has image $\Omega_1 \omega$ in $\Omega_1 \backslash \Omega$, $\pi_0 = \pi_0(\chi, \delta)$, and

$$\pi'_0(h) = \pi_0(g h g^{-1}), \quad h \in G_0(\mathbf{R}),$$

then $\pi'_0 = \pi_0(\chi', \delta')$ with $\omega \delta' = \delta$ and $\chi'(s) = \chi(\omega s)$. Thus the representations

$$\pi(\chi, \delta) = \operatorname{Ind}\bigl(G(\mathbf{R}), G_0(\mathbf{R}), \pi_0(\chi, \delta)\bigr)$$

are irreducible. We set

$$\Pi_\varphi = \left\{ \pi(\chi, \delta) \;\middle|\; \chi \in X_\varphi \right\}.$$

If the image of $G(\mathbf{R})$ in $\Omega_1 \backslash \Omega$ has e elements, then Π_φ contains

$$[\Omega : \Omega_1]/e$$

classes.

Before explaining why conditions (i) to (vi) are, insofar as they apply to the Π_φ already defined, fulfilled, we verify a simple lemma.

Lemma 3.5. *The restriction of an irreducible quasi-simple representation π of $G(\mathbf{R})$ to $G^\sigma_{\mathrm{der}}(\mathbf{R})$ is infinitesimally equivalent to the direct sum of finitely many irreducible representations of $G^0_{\mathrm{der}}(\mathbf{R})$.*

Let π act on V. Let K be a maximal compact subgroup of $G(\mathbf{R})$ and let K^0 be $K \cap G^0_{\mathrm{der}}(\mathbf{R})$. Since $\pi(z)$ is a scalar for $z \in Z_G(\mathbf{R})$ and since

$$\Big[K : K_0\big(K \cap Z_G(\mathbf{R})\big)\Big] < \infty$$

every irreducible representation of K_0 occurs with finite multiplicity in the restriction of π to K_0. Let ρ be an irreducible representation which actually occurs and $g_1, \ldots, g_r$ be a complete set of representatives for the cosets of $Z_G(\mathbf{R})G^0_{\mathrm{der}}(\mathbf{R})$ in $G(\mathbf{R})$. If ξ is the character of ρ let U_j be the range of the projection

$$E_j = \int_{K_0} \xi(k)\pi(g_j^{-1}kg_j)\,dk.$$

U_j is finite-dimensional. Let E be

$$\int_{K_0} \xi(k)\pi(k)\,dk.$$

If U is a closed nonzero subspace of V invariant under $G^0_{\mathrm{der}}(\mathbf{R})$ then $U \cap \Big(\sum_j U_j\Big)$ is not zero. If it were

$$0 = \pi(g_j)E_jU = E\pi(g_j)U$$

for each j. But V is contained in the closure of

$$\sum \pi(g_j)U$$

and $EV \neq 0$. As a consequence, among the closed nonzero subspaces of V invariant under $G^0_{\mathrm{der}}(\mathbf{R})$ there is at least one minimal one W. The representation of $G^0_{\mathrm{der}}(\mathbf{R})$ on W is irreducible.

Choose a maximal collection $h_1, \ldots, h_\ell$ from $\{g_1, \ldots, g_r\}$ such that

$$\sum_{i=1}^{\ell} \pi(h_i)W$$

is direct. Each $\pi(h_1)W$ is invariant and irreducible under $G^0_{\mathrm{der}}(\mathbf{R})$. Moreover

$$\bigoplus_{i=1}^{\ell} \pi(h_i)W$$

is dense in V and therefore contains all K-finite vectors. The lemma follows.

If $\Omega_2(G)$ is the group of elements in the Weyl group of S that can be realized in $G(\mathbf{R})$ then the character of $\pi(\chi, \delta)$, which certainly exists as a distribution, is given on the regular elements of $S(\mathbf{R})$ by the function

$$\epsilon(G) \sum_{\Omega_2(G)} \frac{\operatorname{sgn}\omega\chi(\omega s)e^{\delta(\omega H - H)}}{\Delta(s)}.$$

It follows that the sets Π_φ that have been defined so far are disjoint.

Suppose π is an irreducible quasi-simple representation of $G(\mathbf{R})$ on V that is square-integrable modulo the center. By the previous lemma the restriction of π to $G_0(\mathbf{R})$ is the direct sum of finitely many irreducible quasi-simple representations, each of which is clearly square-integrable modulo the center. Let π_0 be one of them and let π_0 act on $V_0 \subseteq V$. By the theory of the discrete series, G has a Cartan subgroup S over $\mathbf{R}$, so that $S(\mathbf{R}) \cap G_{\mathrm{der}}(\mathbf{R})$

is compact and there is a χ and a δ such that π_0 is $\pi_0(\chi,\delta)$. By Lemma 3.4 there is a φ such that $\chi \in X_\varphi$. If $g_1,\dots,g_r$ are a set of representatives for $G_0(\mathbf{R})\backslash G(\mathbf{R})$ then

$$V \supseteq \bigoplus_{i=1}^{r} \pi(g_i^{-1})V_0$$

because the representations $h \to \pi(g_i h g_i^{-1})$ are inequivalent. If $g_j g = h_j g_i$ and

$$v = \bigoplus \pi(g_i^{-1})v_i$$

then

$$\pi(g)v = \bigoplus \pi(g_j^{-1})v_j'$$

with $v_j' = \pi(h_j)v_i$. Mapping v to the function on $G(\mathbf{R})$ whose value at hg_i, $h \in G_0(\mathbf{R})$ is $\pi(h)v_i$ we obtain an infinitesimal equivalence of π with $\pi(\chi,\delta)$. This shows at least that (v) will be a consequence of (i) and that the union of the sets Π_φ will contain all classes that are square-integrable modulo the center.

Suppose $\eta : H \to G$ has abelian kernel and cokernel and $\varphi' = \widehat{\eta}(\varphi)$ where $\varphi \in \Phi(G)$ and $\varphi(W_{\mathbf{R}})$ is contained in no proper parabolic subgroup of $\widehat{G}$. Then $\varphi'(W_{\mathbf{R}})$ is contained in no proper parabolic subgroup of $\widehat{H}$. It follows from the preceding lemma that the restriction of any irreducible quasi-simple representations π of $G(\mathbf{R})$ to $H(\mathbf{R})$ is, infinitesimally, the direct sum $\bigoplus \pi_i$ of finitely many irreducible quasi-simple representations of $H(\mathbf{R})$, for the map $H^0_{\mathrm{der}}(\mathbf{R}) \to G^0_{\mathrm{der}}(\mathbf{R})$ is surjective. If π is square-integrable modulo the center so is each π_i. We consider the restriction of $\pi(\chi,\delta)$. $\pi(\chi,\delta)$ restricted to $G_0(\mathbf{R})$ is the sum

$$\bigoplus_{\Omega_1\backslash\Omega_2(G)} \pi_0(\omega^{-1}\chi,\omega^{-1}\delta)$$

and $\pi_0(\omega^{-1}\chi,\omega^{-1}\delta)$ restricted to $H_0(\mathbf{R})$ is irreducible. It is clearly equal to $\pi_0(\omega^{-1}\chi',\omega^{-1}\delta')$ if χ' is the quasi-character $s \to \chi\bigl(\eta(s)\bigr)$ on the inverse image of $S(\mathbf{R})$ in $H(\mathbf{R})$ and if δ' is the pullback of δ. It is also easy to see that $X_{\varphi'} = \left\{ \chi' \;\middle|\; \chi \in \chi_\varphi \right\}$. Thus $\pi(\chi,\delta)$ restricted to $H(\mathbf{R})$ is

$$\bigoplus_{\Omega_2(H)\backslash\Omega_2(G)} \pi(\omega^{-1}\chi',\omega^{-1}\delta')$$

with $\chi' \in X_{\varphi'}$ and condition (iv) is satisfied.

Condition (ii) is clear when the center of G is connected. In the general case it follows from (iv) and the definition of χ_φ. Condition (iii) is clear when G_{der} is simply connected. In the general case it follows from (iv) and the definition of π_α.

If the quasi-simple irreducible representation π is square-integrable modulo the center and if ζ is the quasi-character of $G_{\mathrm{rad}}(\mathbf{R})$ defined by

$$\pi(z) = \zeta(z)I, \quad z \in G_{\mathrm{rad}}(\mathbf{R}),$$

then π is tempered if and only if ζ is a character. This is so if and only if $\langle \mu, H\rangle$ is purely imaginary whenever $H \in \widehat{L} \otimes C$ satisfies $\sigma H = \overline{H}$ and $\langle \alpha, H\rangle = 0$ for all α. On the other hand, if $\varphi(w) = a(w) \times w$ then $\left\{ a(w) \;\middle|\; w \in W_{\mathbf{R}} \right\}$ is relatively compact if and only if $\left\{ a(z) \;\middle|\; z \in \mathbf{C}^\times \right\}$ is. This is so if and only if

$$z = e^x \to \widehat{\lambda}\bigl(a(z)\bigr) = z^{\langle \mu, \widehat{\lambda}\rangle} \overline{z}^{\langle \overline{\sigma}\mu, \widehat{\lambda}\rangle} = e^{\langle \mu, z\widehat{\lambda} + \overline{x}\,\overline{\sigma}\widehat{\lambda}\rangle}$$

is a character of $\mathbf{C}^\times$ for each $\widehat{\lambda}$, that is, if and only if $\langle \mu, H\rangle$ is purely imaginary when $H \in \widehat{L} \otimes C$ satisfies $\overline{\sigma} H = \overline{H}$. Any such H is a sum of terms of the form $x\widehat{\lambda} + \overline{x}\,\overline{\sigma}\widehat{\lambda}$ where $\widehat{\lambda}$ is either a root $\widehat{\alpha}$ or satisfies $\langle \alpha, \widehat{\lambda}\rangle = 0$ for all α. If $\widehat{\lambda} = \widehat{\alpha}$ then

$$x\widehat{\lambda} + \overline{x}\,\overline{\sigma}\widehat{\lambda} = (x - \overline{x})\widehat{\alpha}.$$

Since $\langle \mu, \widehat{\alpha}\rangle$ is real, $\langle \mu, x\widehat{\lambda} + \overline{x}\,\overline{\sigma}\widehat{\lambda}\rangle$ is purely imaginary. If $\langle \alpha, \widehat{\lambda}\rangle = 0$ for all α then $\overline{\sigma}\widehat{\lambda} = \sigma\widehat{\lambda}$. Condition (vi) is now clear.

Before completing the definition of the sets Π_φ, we remind ourselves of some properties of induced representations. Suppose P is a parabolic subgroup of G over $\mathbf{R}$, N its unipotent radical, and $M = P/N$. Suppose ρ is an irreducible quasi-simple representation of $M(\mathbf{R})$ on a Banach space V. Lifting, we may also treat ρ as a representation of $P(\mathbf{R})$. If $p \in P(\mathbf{R})$ let $\delta_P(p)$ be the square root of the absolute value of the determinant of the restriction of $\operatorname{Ad}\rho$ to the Lie algebra of N. Let $I(V)$ be the space of continuous functions on $G(\mathbf{R})$ with values in V that satisfy

$$\varphi(pg) = \delta_P(p)\rho(p)\varphi(g), \quad p \in P(\mathbf{R}).$$

$I(V)$ is a Banach space; let I_ρ be the representation of $G(\mathbf{R})$ on it by right translations.

There is a quasi-character ζ_ρ of $Z_G(\mathbf{R})$ such that $I_\rho(z)$ is the scalar $\zeta_\rho(z)$ when $z \in Z_G(\mathbf{R})$ and a unique positive real-valued quasi-character ξ_ρ of $G(\mathbf{R})$ such that $\left|\zeta_\rho(z)\right| = \xi_\rho(z)$ if $z \in Z_G(\mathbf{R})$. There is also a quasi-character ζ'_ρ of $Z_M(\mathbf{R})$ such that $\rho(z)$ is the scalar $\zeta'_\rho(z)$ if $z \in Z_M(\mathbf{R})$ and a positive quasi-character ξ'_ρ of $M(\mathbf{R})$ such that $\left|\zeta'_\rho(z)\right| = \xi'_\rho(z)$ if $z \in Z_G(\mathbf{R})$. ζ_ρ is the restriction of ζ'_ρ to $Z_G(\mathbf{R})$. If π is an irreducible quasi-simple representation of $G(\mathbf{R})$ we may also define ξ'_π and ζ'_π.

Suppose ρ^* is a quasi-simple irreducible representation of $M(\mathbf{R})$ on V^* and there is an $M(\mathbf{R})$-invariant bilinear form

$$(v, v^*) \to \langle v, v^*\rangle \in \mathbf{C}$$

on $V \times V^*$. We may introduce $I(V^*)$, $I_{\rho*}$, and the bilinear form

$$\langle \varphi, \psi\rangle = \int_K \bigl\langle \varphi(k), \psi(k)\bigr\rangle\, dk$$

on $I(V) \times I(V^*)$. It is known to be $G(\mathbf{R})$-invariant. Any K-finite continuous linear form on $I(V)$ is of the form $\varphi \to \langle \varphi, \psi\rangle$ for a suitable ψ in $I(V^*)$. We want to investigate the function $\bigl\langle I_\rho(g)\varphi, \psi\bigr\rangle$ for K-finite φ and ψ.

Let X be the lattice of rational characters of P and Y the lattice of rational characters of M_{rad}. There is an injection $X \to Y$ that leads to isomorphisms $X \otimes \mathbf{R} \xrightarrow{\sim} Y \otimes \mathbf{R}$, $X \otimes \mathbf{C} \xrightarrow{\sim} Y \otimes \mathbf{C}$. We identify these two spaces. If $D(P)$ is the set of invariant elements of $X \otimes \mathbf{R}$, then every $\lambda = \sum x_i\lambda_i$ in $D(P)$, $\lambda_i \in X$, $x_i \in \mathbf{R}$ defines a positive quasi-character π_λ of $M(\mathbf{R})$ by

$$\pi_\lambda(g) = \prod_i \bigl|\lambda_i(g)\bigr|^{x_i}, \quad g \in M(\mathbf{R}).$$

π_λ is a representation on C. If P is minimal over $\mathbf{R}$, we take π^*_λ to be π_λ^{-1} and φ and ψ to be identically 1 on K and set

$$\phi_\lambda(g) = \bigl\langle I_{\pi_\lambda}(g)\varphi, \psi\bigr\rangle.$$

As usual Ξ_G is the function ϕ_0.

Recall that if π is a quasi-simple, irreducible representation of $G(\mathbf{R})$ on W that is square-integrable modulo the center, and u and v are K-finite vectors in W and its dual, then $\langle \pi(g)u, v\rangle$ is bound by a constant times $\zeta'_\pi(g)\Xi_G(g)$.

We now prove an easy sequence of lemmas. G is a reductive linear group over $\mathbf{R}$. There is a hermitian form on the underlying real vector space that is invariant under K and with respect to which $G(\mathbf{R})$ is selfadjoint. Every $g \in G(\mathbf{R})$ is a product $g = kh$ where $k \in K$ and h is selfadjoint and positive with respect to the given form. Let $\ell(g)$ be the norm of $\log h$. We choose an abelian subgroup A of $G(\mathbf{R})$ every element of which is selfadjoint and positive and which is maximal with respect to this property. A is then connected and $G(\mathbf{R}) = KAK$. If P is a given parabolic subgroup over $\mathbf{R}$ we may, and do, take A in $P(\mathbf{R})$.

If we choose $\psi : G \xrightarrow{\sim} G'$ where G' is quasi-split and $\psi^{-1}\sigma(\psi)$ is inner, if B' and T' have the usual significance, and if $P' = \psi(P)$ contains B' and $\psi(A) \subseteq T'$ as we may suppose, then $X \otimes \mathbf{R}$ may be regarded as a subspace of $L \otimes \mathbf{R}$. Let $D^+(P)$ be the set of $\lambda \in D(P)$ such that $\langle \lambda, \widehat{\alpha}\rangle > 0$ if α is a root of T' in $N' = \psi(N)$. Let $\overline{D}^+(P)$ be its closure. If P is minimal let $A^+(P)$ be the set of a in A such that $\alpha(a) \geqslant 1$ if α is a root of T' in N'.

Lemma 3.6. *Let P be minimal over $\mathbf{R}$. There is an integer d and a constant c such that*

$$\phi_\lambda(a) \leqslant c\pi_\lambda(a)\delta_P^{-1}(a)\big(1 + \ell(a)\big)^d$$

if $a \in A^+(P)$, $\lambda \in \overline{D}^+(P)$.

The group $G(\mathbf{R}) = P(\mathbf{R})K$. Write $g = p(g)k(g)$. $p(g)$ is not uniquely determined but $\pi_\lambda\big(p(g)\big)$ and $\delta_P\big(p(g)\big)$ are and

$$\phi_\lambda(a) = \int_K \delta_P\big(p(ka)\big)\pi_\lambda\big(p(ka)\big)\, dk.$$

By Lemma 3.3.2.3 of [16]

$$\pi_\lambda\big(p(ka)\big) \leqslant \pi_\lambda(a).$$

Thus

$$\phi_\lambda(a) \leqslant \pi_\lambda(a)\phi_0(a) \leqslant c\pi_\lambda(a)\delta_P^{-1}(a)\big(1 + \ell(a)\big)^d.$$

The last inequality is a consequence of Theorem 8.3.7.4 of [16].

Lemma 3.7. *For each $\lambda \in \overline{D}^+(P)$ there is a positive constant $c(\lambda)$ such that*

$$\phi_\lambda(a) \geqslant c(\lambda)\pi_\lambda(a)\delta_P^{-1}(a)$$

for all $a \in A^+(P)$.

To prove this we remind ourselves of an integration formula (cf. [4]). Let P be for the moment any parabolic subgroup over $\mathbf{R}$. Let $\overline{N}$ be the unipotent radical of a parabolic subgroup over $\mathbf{R}$ opposed to P. $G(\mathbf{R})$ is again $P(\mathbf{R})K$ and we write $g = p(g)k(g)$. If f is any continuous function on $K \cap p(\mathbf{R})\backslash K$ then, with a suitable choice of the Haar measure on $\overline{N}(\mathbf{R})$,

$$\int_K f(k)\, dk = \int_{\overline{N}(\mathbf{R})} \delta_P^2\big(p(\overline{n})\big) f\big(k(\overline{n})\big)\, d\overline{n} \tag{3.1}$$

Take P to be minimal over $\mathbf{R}$, take

$$f(k) = \delta_P\big(p(ka)\big)\pi_\lambda\big(p(ka)\big),$$

and write $p(\overline{n}) = p_1$, $k(\overline{n}) = k_1$. Then

$$k_1 a = p_1^{-1} a a^{-1} \overline{n} a$$

and

$$\delta_P\big(p(k_1 a)\big)\pi_\lambda\big(p(k_1 a)\big)$$

equals

$$\big\{\delta_P^{-1}(p_1)\pi_\lambda^{-1}(p_1)\big\}\big\{\delta_P(a)\pi_\lambda(a)\big\}\Big\{\delta_P\big(p(a^{-1}\overline{n}a)\big)\pi_\lambda\big(p(a^{-1}\overline{n}a)\big)\Big\}.$$

Consequently

$$\phi_\lambda(a) = \delta_P(a)\pi_\lambda(a)\int_{\overline{N}(\mathbf{R})}\big\{\delta_P^{-1}(p_1)\pi_\lambda^{-1}(p_1)\big\}\Big\{\delta_P\big(p(a^{-1}\overline{n}a)\big)\pi_\lambda\big(p(a^{-1}\overline{n}a)\big)\Big\}\,d\overline{n}.$$

Substitute $a\overline{n}a^{-1}$ for $\overline{n}$ to obtain

$$\delta_P^{-1}(a)\pi_\lambda(a)\int_{\overline{N}(\mathbf{R})}\Big\{\delta_P^{-1}\big(p(a\overline{n}a^{-1})\big)\pi_\lambda^{-1}\big(p(a\overline{n}a^{-1})\big)\Big\}\Big\{\delta_P\big(p(\overline{n})\big)\pi_\lambda\big(p(\overline{n})\big)\pi_\lambda\big(p(\overline{n})\big)\Big\}\,d\overline{n}.$$

All we have to do is show that for a given $\lambda \in \overline{D}^+(P)$ the integral is bounded below by a positive constant as a varies over $A^+(P)$. If U is a relatively compact subset of $\overline{N}(\mathbf{R})$ so is

$$\bigcup_{a\in A^+(P)} aUa^{-1}.$$

Since the integrand is continuous and positive, the required estimate is certainly valid.

We shall eventually have to make use of a well-known result of Bhanu-Murty-Gindikin-Karpelevich. If P_1 is a parabolic subgroup of G over $\mathbf{R}$ that contains the minimal P we may set $P_1' = \psi(P_1)$, $N_1' = \psi(N_1)$. Suppose $\overline{P}_1$ is opposed to P and $\overline{N}_1$ is its unipotent radical. If $\langle\lambda, \widehat{\alpha}\rangle > 0$ whenever α is a root of T' in N_1' then

$$\int_{\overline{N}_1(\mathbf{R})}\delta_P\big(p(\overline{n})\big)\pi_\lambda\big(p(\overline{n})\big)\,d\overline{n} < \infty. \tag{3.2}$$

If P and P_0 are two parabolic subgroups of G over $\mathbf{R}$ and $P \supseteq P_0$ then $D(P) \subseteq D(P_0)$. Also if ξ is any positive quasi-character of $G(\mathbf{R})$ there is a λ in $D(G)$ such that $\xi = \pi_\lambda$.

Lemma 3.8. *Suppose P is a parabolic subgroup of G over $\mathbf{R}$. Suppose ρ and ρ^* are two irreducible quasi-simple representations of $M(\mathbf{R})$ on V and V^* respectively. Suppose that there is a nontrivial $M(\mathbf{R})$-invariant pairing $V \times V^* \to \mathbf{C}$. Let K' be the projection of $K \cap P(\mathbf{R})$ on $M(\mathbf{R})$ and suppose that for any two K'-finite vectors v and v^* there is a constant c such that*

$$\Big|\big\langle\rho(m)v, v^*\big\rangle\Big| \leqslant c\xi_\rho'(m)\Xi_M(m), \quad m \in M(\mathbf{R}).$$

Suppose $\xi_\rho' = \pi_\lambda$ with $\lambda \in \overline{D}^+(P)$. If P contains the minimal P_0 then $\lambda \in \overline{D}^+(P_0)$ and for any two K-finite φ and ψ in $I(V)$ and $I(V^*)$ there is a constant c such that

$$\Big|\big\langle I_\rho(g)\varphi, \psi\big\rangle\Big| \leqslant c\phi_\lambda(g), \quad g \in G(\mathbf{R}).$$

As usual we suppose that $\psi(P) = P'$ and $\psi(P_0')$ contain B'. If α is a root of T' in N_0' that is also a root in N' then $\langle\lambda, \widehat{\alpha}\rangle \geqslant 0$. If α is a root in N_0' but not in N' then $\langle\lambda, \widehat{\alpha}\rangle = 0$. Consequently $\lambda \in \overline{D}^+(P_0)$.

If $k \in K$, $g \in G(\mathbf{R})$ write $kg = pk_1,\ p \in P(\mathbf{R})$, $k_1 \in K$, and let m be the projection of p on $M(\mathbf{R})$. Then

$$\big\langle I_\rho(g)\varphi, \psi\big\rangle = \int_K \big\langle \varphi(kg), \psi(k)\big\rangle\, dk = \int_K \big\langle \delta_P(p)\rho(p)\varphi(k_1), \psi(k)\big\rangle\, dk.$$

There are functions φ_i in $I(V)$, functions φ_j in $I(V^*)$, and continuous functions a_i, b_j on K such that

$$\varphi(hk) = \sum a_i(k)\varphi_i(h), \quad \psi(hk) = \sum b_j(k)\psi_j(h)$$

for $h \in G(\mathbf{R})$ and such that $\varphi_i(1)$, $\psi_j(1)$ are K'-finite. Then

$$\big\langle \delta_P(p)\rho(p)\varphi(k_1), \psi(k_1), \psi(k)\big\rangle = \delta_P(p) \sum_{i,j} a_k(k_1)b_j(k)\big\langle \rho(m)\varphi_i(1), \psi_j(1)\big\rangle.$$

There is therefore a constant c such that

$$\Big|\big\langle \delta_P(p)\rho(p)\varphi(k_1), \psi(k)\big\rangle\Big| \leqslant c\delta_P(p)\pi_\lambda(m)\Xi_M(m).$$

We may lift M to a Levi factor of P, chosen so that $M(\mathbf{R})$ is selfadjoint with respect to the given hermitian form. Then $K \cap M(\mathbf{R})$ is a maximal compact subgroup of $M(\mathbf{R})$ and $K \cap M(\mathbf{R}) = K \cap P(\mathbf{R})$. The function $\varphi_\lambda(g)$ is given by

$$\int_K \delta_{P_0}\big(p_0(kg)\big)\pi_\lambda\big(p_0(kg)\big)\, dk = \int_K \left\{ \int_{K\cap M(\mathbf{R})} \delta_{P_0}\big(p_0(ukg)\big)\pi_\lambda\big(p_0(ukg)\big) du \right\} dk.$$

Set $p(kg) = p = nm$, $n \in N(\mathbf{R})$; then

$$ukg = unu^{-1}umk_1$$

and $unu^{-1} \in N(\mathbf{R}) \subseteq N_0(\mathbf{R})$. $P_0' = P_0 \cap M$ is a minimal parabolic subgroup of M over $\mathbf{R}$. Write $um = p_0k_0$, $p_0 \in P_0'(\mathbf{R})$, $k_0 \in K \cap M(\mathbf{R})$. Then

$$\delta_{P_0}\big(p_0(ukg)\big) = \delta_P(m)\delta_{P_0'}(p_0).$$

Because $\lambda \in D(P)$

$$\pi_\lambda\big(p_0(ukg)\big) = \pi_\lambda(m).$$

Thus

$$\phi_\lambda(g) = \int_K \delta_P(m)\pi_\lambda(m)\left\{ \int_{K\cap M(R)} \delta_{P_0'}(p_0) du \right\} dk.$$

The right side is

$$\int_K \delta_P(m)\pi_\lambda(m)\Xi_M(m)\, dk.$$

Since $\delta_P(p) = \delta_P(m)$, the lemma follows.

Corollary 3.9. *Assume in addition that $\lambda \in D(G) \subseteq D(P)$. Then*

$$\Big|\big\langle I_\rho(g)\varphi, \psi\big\rangle\Big| \leqslant c\pi_\lambda(g)\Xi_G(g).$$

One has only to observe that when $\lambda \in D(G)$

$$\phi_\lambda(g) = \pi_\lambda(g)\Xi_G(g).$$

We shall have to make use of some results from [6] that are considerably more serious than those of the preceeding lemmas. We recapitulate them in the form we require. Two conjugacy classes $\mathfrak{p}$ and $\mathfrak{p}'$ of parabolic subgroups of G over $\mathbf{R}$ are said to be associate if we can find $P \in \mathfrak{p}$, $P' \in \mathfrak{p}'$ such that P and P' have a common Levi factor. Given P and the Levi factor M, for which $M(\mathbf{R})$ is selfadjoint, we may, with no loss of generality, assume that it is the common factor, for we may replace P' by a conjugate.

If ρ is a quasi-simple irreducible representation of $M(\mathbf{R})$ we may define I_ρ with respect to P or to P'. To distinguish the two possibilities we write I_ρ^P, $I_\rho^{P'}$. To apply the results of [6] we take ρ to be square-integrable modulo the center. It then satisfies the conditions of Lemma 3.8. In fact, we may suppose, since it is only the infinitesimal equivalence class of ρ and I_ρ that interests us, that V is a Hilbert space and that $\pi_\lambda^{-1} \otimes \rho$ is unitary. Then we take V^* to be the dual space to V and ρ^* to be the representation contragredient to ρ. If (u, v) is the inner product on V then

$$(\varphi, \psi) = \int_K \big(\varphi(k), \psi(k)\big)\, dk$$

is an inner product on $I(V)$. If we assume in addition, as we must, that ρ satisfies the condition of the corollary, then $\pi_\lambda^{-1}(g) I_\rho(g)$ is unitary with respect to this inner product.

For a general quasi-simple irreducible ρ, the elements of $\mathfrak{z}$ operate on the infinitely differentiable vectors in $I(V)$ as scalars. Moreover the restriction of I_ρ^P to K contains any irreducible representation of K with finite multiplicity. Exploiting, for example, the fact that the characters of irreducible quasi-simple representations of $G^0(\mathbf{R})$ are functions, one sees that I_ρ^P admits a finite composition series. Our present stronger assumptions on ρ, which imply the existence of an inner product on $I(V)$ with respect to which the operators $I_\rho(g)$ differ by a scalar from a unitary matrix, allow us to conclude that I_ρ is infinitesimally equivalent to the direct sum of finitely many irreducible quasi-simple representations. From Lemma 8 and Theorem 3 of [6], we conclude that if I_ρ^P and $I_{\rho'}^{P'}$ have an irreducible constituent in common then the classes of P and P' are associate.

If P and P' have the common Levi factor M, then, by computing the character, one sees that I_ρ^P and $I_{\rho'}^{P'}$ are infinitesimally equivalent ([6], §11). By Lemma 12 of [6], the representations I_ρ^P and $I_{\rho'}^P$ are infinitesimally equivalent if and only if there is an h in the normalizer of M in $G(\mathbf{R})$ such that ρ' and $m \to \rho(hmh^{-1})$ are infinitesimally equivalent. What does not appear so clearly in [6] is that if I_ρ^P and $I_{\rho'}^P$ have an irreducible constituent in common, then there is an h in the normalizer of M in $G(\mathbf{R})$ such that ρ' and $m \to \rho(hmh^{-1})$ are infinitesimally equivalent.

This is an important point. We shall return to it after some considerations that are, unfortunately, only implicit in [6]. We take up once again the assumptions of Lemma 3.8. Suppose $\psi \in V^*$ is $K \cap M(\mathbf{R})$ finite. If $\varphi \in I(V)$ is K-finite then $\varphi(k) \in V$ is $K \cap M(\mathbf{R})$ finite for all $k \in K$ and $\{\, \varphi(k) \mid k \in K \,\}$ spans a finite-dimensional subspace of V. There is therefore a constant c such that

$$\Big| \big\langle \rho(m)\varphi(k), \psi \big\rangle \Big| \leqslant c\pi_\lambda(m)\Xi_M(m)$$

for all $m \in M(\mathbf{R})$ and all $k \in K$. Suppose $\varphi \in I(V)$, $\psi \in V^*$, and

$$\sup_{k,m} \frac{\Big| \big\langle \rho(m)\varphi(k), \psi \big\rangle \Big|}{\pi_\lambda(m)\Xi_M(m)} = \|\varphi\|_\psi < \infty. \tag{3.3}$$

If U is a compact subset of $G(\mathbf{R})$ and $p(kg) = n(kg)m(kg)$ then

$$\{\, m(kg) \mid k \in K, g \in U \,\}$$

is relatively compact. Set $m_1 = m(kg)$, $k_1 = k(kg)$, and let $m_2 \in M(\mathbf{R})$; then

$$\Big|\big\langle \rho(m)\varphi(kg), \rho(m_2)\psi \big\rangle\Big| = \delta_P(m_1)\Big|\big\langle \rho(m_2^{-1}mm_1)\varphi(k_1), \psi\big\rangle\Big|$$

which is bounded by

$$c\pi_\lambda(m_2^{-1}mm_1)\Xi_M(m_2^{-1}mm_1) \leqslant c'\pi_\lambda(m)\Xi_M(m).$$

For the last inequality see Proposition 8.3.7.2 of [16].

It follows easily from these considerations that if $f \in C_c^\infty\big(G(\mathbf{R})\big)$, if $\|\varphi'\|_\psi < \infty$, and

$$\varphi = I_\rho(f)\varphi' = \int_{G(\mathbf{R})} f(g) I_\rho(g)\varphi'\, dg$$

then

$$\lim_{g\to h}\big\|I_\rho(g)\varphi - I_\rho(h)\varphi\big\|_\psi = 0. \tag{3.4}$$

Notice in particular that if φ is K-finite there is an f such that

$$\varphi = I_\rho(f)\varphi.$$

If $\psi \in V^*$ let $I(V,\psi)$ be the set of all $\varphi \in I(V)$ satisfying (3.3) and (3.4). If v is a compactly supported measure on $M(\mathbf{R})$ and

$$\psi' = \int_{M(\mathbf{R})} \rho(m)\psi\, dv$$

then $I(V,\psi') \supseteq I(V,\psi)$. In particular $I(V,\psi)$ is the same for all nonzero $K \cap M(\mathbf{R})$ finite vectors ψ. If $I(V,\psi)$ contains the K-finite vectors then the restriction of I_ρ to $I(V,\psi)$ is infinitesimally equivalent to ψ.

Lemma 3.10. *Suppose ρ satisfies the conditions of Lemma 3.8 with $\lambda \in D(P)^+$. Suppose M is a Levi factor of P with $M(\mathbf{R})$ selfadjoint and $\overline{P} \cap P = M$. If $\overline{N}$ is the unipotent radical of $\overline{P}$, $\psi \in V^*$, and $\varphi \in I(V,\psi)$ then*

$$\int_{\overline{N}(\mathbf{R})} \big\langle \varphi(\overline{n}g), \psi \big\rangle\, d\overline{n}$$

is absolutely convergent.

We may take $g = 1$. Write $\overline{n} = nmk$, $n \in N(\mathbf{R})$, $m \in M(\mathbf{R})$, $k \in K$. Then

$$\Big|\big\langle \varphi(\overline{n}), \psi\big\rangle\Big| = \delta_P(m)\Big|\big\langle \rho(m)\varphi(k), \psi\big\rangle\Big| \leqslant c\delta_P(m)\pi_\lambda(m)\Xi_M(m).$$

We have seen that if $P_0 \subseteq P$ is minimal then

$$\int_{\overline{N}(\mathbf{R})} \delta_{P_0}\big(p_0(\overline{n})\big)\pi_\lambda\big(p_0(\overline{n})\big)\, d\overline{n}$$

is finite; it equals

$$\int_{\overline{N}(\mathbf{R})}\int_{K\cap M(\mathbf{R})} \delta_{P_0}\big(p_0(u\overline{n}u^{-1})\big)\pi_\lambda\big(p_0(u\overline{n}u^{-1})\big)\, d\overline{n}.$$

Since $p_0(u\overline{n}u^{-1}) = p_0(u\overline{n})$ we may proceed as in the proof of Lemma 3.8 to see that this double integral equals

$$\int_{\overline{N}(\mathbf{R})} \delta_P(m)\pi_\lambda(m)\Xi_M(m)\, d\overline{n}.$$

The lemma follows.

We set

$$\int_{\overline{N}(\mathbf{R})} \big\langle \varphi(\overline{n}g), \psi \big\rangle\, d\overline{n} = M(\varphi, \psi; g).$$

It is clear that

$$M(\varphi, \psi; gh) = M\big(I_\rho(h)\varphi, \psi; g\big)$$

and that

$$M\big(\varphi, \rho(m)\psi, g\big) = \delta_P^{-2}(m) M(\varphi, \psi; m^{-1}g).$$

Let V_0^* be the space of vectors ψ in V^* for which $I(V, \psi)$ contains the K-finite vectors. V_0^* is invariant under $M(\mathbf{R})$. If φ is K-finite

$$\psi \to M(\varphi, \psi, 1)$$

is a $K \cap M(\mathbf{R})$-finite linear form on V_0^*. There is therefore a $K \cap M(\mathbf{R})$-finite vector $M(\varphi)$ in V such that

$$M(\varphi, \psi, 1) = \big\langle M(\varphi), \psi \big\rangle$$

for $\psi \in V_0^*$.

The map $p \times \overline{n} \to p\overline{n}$ of $P(\mathbf{R}) \times \overline{N}(\mathbf{R})$ into $G(\mathbf{R})$ is a diffeomorphism of $P(\mathbf{R}) \times \overline{N}(\mathbf{R})$ with an open subset of $G(\mathbf{R})$. If f is an infinitely differentiable complex-valued function on $\overline{N}(\mathbf{R})$ with compact support X and $v \in V$ is K-finite, define $\varphi \in I(V)$ by

$$\varphi(p\overline{n}) = f(\overline{n})\delta_P(p)\rho(p)v.$$

The set

$$Y = \big\{\, m \in M(\mathbf{R}) \;\big|\; N(\mathbf{R})mK \cap X \neq \varnothing \,\big\}$$

is compact. If $m \in M(\mathbf{R})$ and $mk = n_1 m_1 \overline{n}$, $n_1 \in N(\mathbf{R})$, $m_1 \in M(\mathbf{R})$, $\overline{n} \in X$ then $m = m_1 m_2$ with $m_2 \in Y$. Thus if ψ is $K \cap M(\mathbf{R})$-finite

$$\Big|\big\langle \rho(m)\varphi(k), \psi \big\rangle\Big| \leqslant c\pi_\lambda(m_1)\Xi_M(m_1) \leqslant c'\pi_\lambda(m)\Xi_M(m).$$

Given ψ we can clearly choose v and f such that $M(\varphi, \psi, 1) \neq 0$. If d in $C_c^\infty\big(g(\mathbf{R})\big)$ is a sufficiently close approximation to the delta-function and $\varphi' = I_\rho(d)\varphi$ then $M(\varphi', \psi, 1)$ is also not zero. Since $\varphi' \in I(V, \psi)$ we have the following lemma.

Lemma 3.11. *If $\psi \in V^*$ is $K \cap M(\mathbf{R})$-finite there is a $\varphi \in I(V, \psi)$ such that*

$$M(\varphi, \psi, 1) \neq 0.$$

If $M(\mathbf{R})$ is selfadjoint and $A \subseteq P(\mathbf{R})$ then $A \subseteq M(\mathbf{R})$. Let $A(P)$ be the centralizer of $M(\mathbf{R})$ in A and let $A^+(P)$ consist of those a in $A(P)$ for which $\alpha(a) \geqslant 1$ if α is a root of T' in N'. We say that $a \to \infty$ in $A^+(P)$ if $\alpha(a) \to \infty$ for all such α.

Lemma 3.12. *Suppose that $\varphi \in I(V)$ and $\psi \in I(V^*)$ are K-finite. If $m \in M(\mathbf{R})$ is fixed, then*

$$\big\langle I_\rho(am)\varphi, \psi \big\rangle = \delta_P^{-1}(a)\Big\{\zeta'_\rho(a) M\big(\varphi, \psi(1), m\big) + o\big(\pi_\lambda(a)\big)\Big\}$$

as $a \to \infty$ in $A^+(P)$.

Since $\left|\zeta'_\rho(a)\right| = \pi_\lambda(a)$ the error term is smaller than the principal term if $M\bigl(\varphi,\psi(1),m\bigr) \neq 0$.

Replacing φ by $I_\rho(m)$, we suppose $m=1$. φ may no longer be K-finite.

$$\bigl\langle I_\rho(a)\varphi,\psi\bigr\rangle = \int_K \bigl\langle \varphi(ka),\psi(k)\bigr\rangle\,dk.$$

The integrand is clearly a function on $K\cap P(\mathbf{R})\backslash K$. Choose P and $\overline{N}$ as in the previous lemma and write $\overline{n} = nmk$, $n\in N(\mathbf{R})$, $m\in M(\mathbf{R})$, $k\in K$. Applying (3.1) we see that the integral is equal to

$$\int_{\overline{N}(\mathbf{R})} \delta_P^2(m)\bigl\langle\varphi(ka),\psi(k)\bigr\rangle\,d\overline{n}.$$

Let $a^{-1}\overline{n}a = n_1m_1k_1$. Since $ka = m^{-1}n^{-1}\overline{n}a = m^{-1}n^{-1}aa^{-1}\overline{n}a$ the intergrand is

$$\delta_P(m)\delta_P(a)\delta_P(m_1)\bigl\langle\rho(m^{-1}am_1)\varphi(k_1),\psi(k)\bigr\rangle.$$

If we substitute $a\overline{n}a^{-1}$ for $\overline{n}$ so that $\overline{n} = n_1m_1k_1$, $a\overline{n}a^{-1} = nmk$ the integral becomes

$$\delta_P^{-1}(a)\zeta'_\rho(a)\int_{\overline{N}(\mathbf{R})}\delta_P(m)\delta_P(m_1)\bigl\langle\rho(m^{-1}m_1)\varphi(k_1),\psi(k)\bigr\rangle\,d\overline{n}.$$

All we have to do is show that

$$\lim_{a\to\infty}\int_{\overline{N}(\mathbf{R})}\delta_P(m)\delta_P(m_1)\bigl\langle\rho(m^{-1}m_1)\varphi(k_1),\psi(k)\bigr\rangle\,d\overline{n} = M\bigl(\varphi,\psi(1),1\bigr).$$

In a moment we shall show that we may take the limit under the integral sign. Since $a\overline{n}a^{-1}\to 1$, we may suppose that $n\to 1$, $m\to 1$, $k\to 1$. The integrand approaches

$$\delta_P(m_1)\bigl\langle\rho(m_1)\varphi(k_1),\psi(1)\bigr\rangle = \bigl\langle\varphi(\overline{n}),\psi(1)\bigr\rangle$$

and the lemma follows.

The integral is dominated by a constant times

$$\delta_P(m)\delta_P(m_1)\pi_\lambda(m^{-1}m_1)\Xi_M(m^{-1}m_1).$$

Choose $P_0\subseteq P$ such that $P_0(R)\supseteq A$ and such that P_0 is minimal over $\mathbf{R}$. As usual let $P'_0 = P_0\cap M$. Since

$$\Xi_M(m^{-1}m_1) = \int_{K\cap M(\mathbf{R})}\delta_{P'_0}\bigl(p'_0(um^{-1}m_1)\bigr)du,$$

it is enough to show that the integral of

$$\delta_P(m)\delta_P(m_1)\pi_\lambda(m^{-1}m_1)\delta_{P'_0}\bigl(p'_0(um^{-1}m_1)\bigr)$$

is uniformly small on the complement of a large compact set in $\overline{N}(\mathbf{R})\times\bigl(K\cap M(\mathbf{R})\bigr)$.

Choose a $\overline{P}_0\subseteq\overline{P}$ that is opposed to P_0 so that

$$A\subseteq\overline{P}_0(\mathbf{R})\cap P_0(\mathbf{R}) = M_0(\mathbf{R})\subseteq M(\mathbf{R}).$$

Write an element of $\overline{N}_0(\mathbf{R})$ as $\overline{n}_0 = n_0m_0k_0$, $n_0\in N_0(\mathbf{R})$, $m_0\in M_0(\mathbf{R})$, $k_0\in K$. If f is a function on $K\cap M_0(\mathbf{R})\backslash K$ then

$$\int_K f(k)\,dk = \int_{\overline{N}_0(\mathbf{R})}\delta_{P_0}^2(m_0)f(k_0)\,d\overline{n}_0.$$

If $\overline{N}'_0 = \overline{N}_0 \cap M$ then $\overline{N}_0(\mathbf{R}) = \overline{N}'_0(\mathbf{R})\overline{N}(\mathbf{R})$. Let $\overline{n}_0 = \overline{n}'_0\overline{n}_1$; let $\overline{n}'_0 = n_2 m_2 v$, $n_2 \in N'_0(\mathbf{R})$, $m_2 \in M_0(\mathbf{R})$, $k \in K \cap N(\mathbf{R})$; let $v\overline{n}_1 v^{-1} = n_3 m_3 k_3$, $n_3 \in N_0(\mathbf{R})$, $m_3 \in M_0(\mathbf{R})$, $k_3 \in K$. We may suppose

$$m_0 = m_2 m_3, \quad k_0 = k_3 u.$$

Then the integral is equal to

$$\int_{\overline{N}'_0(\mathbf{R})} \int_{\overline{N}(\mathbf{R})} \delta^2_{P_0}(m_2 m_3) f(k_3 v) d\overline{n}_1 \, d\overline{n}'_0.$$

Replacing $\overline{n}_1$ by $a\overline{n}_1 a^{-1}$ so that $n_3 m_3 k_3 = va\overline{n}_1 a^{-1} v^{-1}$ we obtain

$$\delta_P^{-2}(a) \int_{\overline{N}'_0(\mathbf{R})} \int_{\overline{N}(\mathbf{R})} \delta^2_{P_0}(m_2 m_3) f(k_3 v) \, d\overline{n}_1 d\overline{n}'_0.$$

On the other hand, if $\overline{n} = nmk$

$$\int_K f(k) \, dk = \int_{\overline{N}(\mathbf{R})} \delta^2_P(m) \left\{ \int_{K \cap M(\mathbf{R})} f(uk) du \right\} d\overline{n}.$$

Replacing $\overline{n}$ by $a\overline{n}a^{-1}$, we obtain

$$\delta_P^{-2}(a) \int_{\overline{N}(\mathbf{R})} \delta^2_P(m) \left\{ \int_{K \cap M(\mathbf{R})} f(uk) du \right\} d\overline{n}$$

where nmk is now $a\overline{n}a^{-1}$.

Thus

$$\int_{\overline{N}(\mathbf{R})} \delta^2_P(m) \left\{ \int_{K \cap M(\mathbf{R})} f(uk) du \right\} d\overline{n} = \int_{\overline{N}'_0(\mathbf{R})} \int_{\overline{N}(\mathbf{R})} \delta^2_P(m_2 m_3) f(k_3 v) \, d\overline{n}_1 \, d\overline{n}'_0.$$

If X is a compact set in $\overline{N}'_0(\mathbf{R}) \times \overline{N}(\mathbf{R}) = \overline{N}_0(\mathbf{R})$ with complement $\mathcal{C}X$, then

$$Y = \left\{ \overline{n} \in \overline{N}(\mathbf{R}) \;\middle|\; \overline{n} \in P(\mathbf{R})X \right\}$$

is also compact. If f is positive

$$\int_{\mathcal{C}Y} \delta^2_P(m) \left\{ \int_{K \cap M(\mathbf{R})} f(uk) du \right\} d\overline{n} \leqslant \int_{\mathcal{C}X} \delta^2_{P_0}(m_2 m_3) f(k_3 v) \, d\overline{n}_1 \, d\overline{n}'_0.$$

Take f to be the function

$$\delta_P^{-1}(a)\pi_\lambda^{-1}(a)\pi_\lambda\big(p_0(ka)\big)\delta_{P_0}\big(p_0(ka)\big).$$

If $a\overline{n}a^{-1} = nmk$, then $uka = um^{-1}n^{-1}a\overline{n}$ and

$$\delta^2_P(m) f(uk) = \delta_P(m)\delta_P(m_1)\pi_\lambda(m^{-1}m_1)\delta_{P'_0}\big(p'_0(um^{-1}am_1)\big)$$

if $\overline{n} = n_1 m_1 k_1$. Observe that

$$\delta_P(a) = \delta_{P_0}(a)$$

and that, more generally,

$$\delta_{P_0}(m) = \delta_P(m)\delta_{P'_0}\big(p'_0(m)\big)$$

for all m in $M(\mathbf{R})$.

To finish the proof of the lemma, we show that

$$\delta^2_P(m_2 m_3) f(k_3 v)$$

is dominated on $\overline{N}_0(\mathbf{R})$ by an integrable function that is independent of a. Let $v\overline{n}_1v^{-1} = n_4m_4k_4$, $n_4 \in N_0(\mathbf{R})$, $m_4 \in M_0(\mathbf{R})$, $k_4 \in K$. Since $k_3va = m_0^{-1}n_0^{-1}a\overline{n}_0$ and $m_0 = m_2m_3$,

$$\delta^2_{P_0}(m_2m_3)f(k_3v) = \delta^2_{P_0'}(m_2)\delta_{P_0}(m_3m_4)\pi_\lambda(m_3^{-1}m_4)$$

m_2 does not depend on $\overline{n}_1$. Consider the function

$$\delta_{P_0}(m_3m_4)\pi_\lambda(m_3^{-1}m_4)$$

on $\overline{N}(\mathbf{R})$. Replacing $\overline{n}_1$ by $v^{-1}\overline{n}_1v$ we may, for the present purposes, suppose that $v = 1$. There is a δ in $D^+(P_0)$ such that

$$\delta_{P_0}(m) = \pi_\delta(m), \quad m \in M_0(\mathbf{R}).$$

There is a β, $1 > \beta > 0$, such that $\delta - \beta\lambda$ also lies in $D^+(P_0)$. Let $\alpha = 1 - \beta$. By Lemma 13 of [4],[1]

$$\pi_{\delta-\beta\lambda}(m_3) \leqslant 1, \quad \pi_{\alpha\lambda}(m_3^{-1}m_4) \leqslant 1.$$

By the formula of Bhanu-Murty-Gindikin-Karpelevich [3], the function

$$\pi_{\delta+\beta\lambda}(m_4)$$

is integrable on $\overline{N}(\mathbf{R})$. It does not depend on a. Since the transformation $\overline{n}_1 \to v^{-1}\overline{n}_1v$ is unimodular, it remains only to observe that, by the same formula,

$$\int_{\overline{N}_0'(\mathbf{R})} \delta^2_{P_0'}(m_2)\, d\overline{n}_0' < \infty.$$

We still suppose that $\lambda \in D(P^+)$. The space $I(V,\psi)$ is the same for all nonzero $K \cap M(\mathbf{R})$-finite ψ. Denote it by $I_0(V)$ and provide it with the norm

$$\|\varphi\|_\psi + \sup_{k\in K}\big\|\varphi(k)\big\|.$$

This norm depends in no essential way on ψ. The subspace

$$I_1(V) = \big\{\, \varphi \in I_0(V) \;\big|\; M(\varphi,\psi,g) = 0 \text{ for all } g \in G(\mathbf{R}) \,\big\}$$

is also independent of ψ. It is closed and $G(\mathbf{R})$-invariant. The quotient $J(V) = I_0(V)/I_1(V)$ is not zero. Let J_ρ be the representation of $G(\mathbf{R})$ on it. When we want to indicate the presence of P we write J_ρ^P instead of J_ρ.

Lemma 3.13. *The representation J_ρ is irreducible.*

Suppose $I_2(V)$ is a closed invariant subspace of $I_0(V)$ and $I_1(V) \subsetneq I_2(V) \subsetneq I_0(V)$. If $v^* \in V^*$ is $K \cap M(\mathbf{R})$-finite then the function

$$\varphi \to M(\varphi, v^*, 1)$$

on $I_0(V)$ is continuous. If it vanishes on all K-finite functions in $I_2(V)$ it vanishes identically on $I_2(V)$. Then

$$0 = M\big(I_\rho(g)\varphi, v^*, 1\big) = M(\varphi, v^*, g)$$

for all $\varphi \in I_2(V)$. This is impossible unless $v^* = 0$. On the other hand, there is a K-finite function ψ in $I(V^*)$ that is not zero but is orthogonal to $I_2(V)$. Then $\big\langle I_\rho(g)\varphi, I_{\rho^*}(k)\psi\big\rangle = 0$ for all $g \in G(\mathbf{R})$ and all $k \in K$. If φ in $I_2(V)$ is K-finite we may apply Lemma 3.12 to see that

$$M\big(\varphi, \psi(k), 1\big) = 0$$

[1](2018) I have been unable to make sense of this reference. (R.L.)

for all k. This in turn implies that $\psi(k) = 0$ for all k, which is a contradiction. The lemma follows.

Lemma 3.14.

(*a*) *Suppose ρ and ρ' satisfy the conditions of Lemma 3.8 with respect to P and P' respectively. Suppose $\lambda \in D^+(P)$ and $\lambda' \in D^+(P')$. If J_ρ^P and $J_{\rho'}^{P'}$ are infinitesimally equivalent there is an $h \in G(\mathbf{R})$ such that $P' = hPh^{-1}$, $M' = hMh^{-1}$, and such that $m \to p'(hmh^{-1})$ is infinitesimally equivalent to ρ.*

(*b*) *Suppose ρ satisfies the conditions of Lemma 3.8 with respect to P and $\lambda \in D^+(P)$. Suppose ρ' satisfies the conditions of Corollary 3.9 with respect to P'. If $P \neq G$, then J_ρ^P is infinitesimally equivalent to no constituent of $I_{\rho'}^{P'}$.*

We may certainly suppose that P and P' both contain P_0 minimal over $\mathbf{R}$, that $A \subseteq P_0(\mathbf{R})$, and that $M(\mathbf{R})$ and $M'(\mathbf{R})$ are selfadjoint.

If π and π' are two irreducible quasi-simple representations of $G(\mathbf{R})$ on W and W', respectively, then π and π' are infinitesimally equivalent if and only if for any K-finite vector $w \in W$ and any K-finite linear form f on W there are a K-finite $w' \in W'$ and a K-finite linear form f' on W' such that

$$f\big(\pi(g)w\big) = f'\big(\pi'(g)w'\big)$$

for all $g \in G(\mathbf{R})$. If $S(\pi)$ is the set of all $\nu \in D(P_0)$ such that for any K-finite f and w there is a constant c such that

$$\Big|f\big(\pi(a)w\big)\Big| \leqslant c\delta_{P_0}^{-1}(a)\pi_\nu(a)$$

on $A^+(P_0)$ then

$$S(\pi) = S(\pi').$$

$S(\pi)$ is a convex set. We introduce the Killing form $B(\mu,\nu)$ on $L \otimes \mathbf{R}$. It is positive semidefinite. If $S(\pi)$ is not empty there is a unique point $\nu(\pi)$ in its closure such that

$$B\big(\nu(\pi),\nu(\pi)\big) = \inf\big\{\, B(\nu,\nu) \;\big|\; \nu \in S(\pi)\,\big\}.$$

If π is J_ρ^P there is a K-finite φ in $I_0(V)$ and a K-finite ψ in $I(V^*)$ such that

$$f\big(\pi(g)w\big) = \big\langle I_\rho(g)\varphi,\psi\big\rangle$$

for all g. By Lemmas 3.6 and 3.8, the closure of $S(\pi)$ contains λ.

We may choose a K-finite ψ in $I(V^*)$ such that ψ is orthogonal to $I_1(V)$ and such that $\psi(1) \neq 0$. We may also choose a K-finite φ in $I_0(V)$ such that $M(\varphi,\psi,1) \neq 0$. Applying Lemma 3.12 to this pair we see that if $\nu = \lambda + \mu$ belongs to $S(\pi)$ then $\mu(a) \geqslant 1$ if $a \in A^+(P)$. Since $\lambda \in D^+(P)$, $B(\lambda,\mu)$ is then nonnegative and

$$B(\nu,\nu) = B(\lambda,\lambda) + 2B(\lambda,\mu) + B(\mu,\mu) \geqslant B(\lambda,\lambda).$$

Thus $\nu(\pi) = \lambda$.

If ρ' satisfies the conditions of Lemma 3.9 and π' is a constituent of $I_{\rho'}^{P'}$ then we can find φ' and ψ' such that

$$f'\big(\pi'(g)w'\big) = \big\langle I_{\rho'}(g)\varphi',\psi'\big\rangle.$$

It follows readily from Lemma 3.6 and Corollary 3.9 that λ' lies in the closure of $S(\pi')$. Since $\lambda' \in D(G) = D^+(G)$, $B(\lambda',\lambda') = 0$ and $\nu(\pi') = \lambda'$.

If P and P' contain P_0, $D^+(P)$ and $D^+(P')$ are disjoint unless $P = P'$. This gives the second part of the lemma and half of the first. We now suppose that $P = P'$, $M = M'$ and show that if J_ρ^P and $J_{\rho'}^{P'}$ are infinitesimally equivalent then so are ρ and ρ'.

Let ψ in $I(V^*)$, with $\psi(1) \neq 0$, be K-finite and orthogonal to $I_1(V)$. Let φ lie in $I_0(V)$ but not in $I_1(V)$ and be K-finite. Then there exist φ' and ψ' that satisfy analogous conditions with respect to ρ' such that

$$\big\langle I_\rho(g)\varphi, \psi\big\rangle = \big\langle I_{\rho'}(g)\varphi', \psi'\big\rangle.$$

Applying Lemma 3.12 we see that

$$\begin{aligned}\big\langle \rho(m)M(\varphi), \psi(1)\big\rangle &= \big\langle M(\varphi), \rho^*(m^{-1})\psi(1)\big\rangle \\ &= M\big(\varphi, \rho^*(m^{-1})\psi(1)1\big) = M\big(\varphi, \psi(1)m\big)\end{aligned}$$

is equal to

$$\big\langle \rho'(m)M(\varphi'), \psi'(1)\big\rangle.$$

Since $M(\varphi) \neq 0$, it follows that ρ and ρ' are infinitesimally equivalent.

There is a point mentioned earlier that remains to be settled. We have to show that if ρ and ρ' satisfy the conditions of Lemma 3.9 with respect to P and I_ρ^P and $I_{\rho'}^P$ contain infinitesimally equivalent irreducible constituents then there is an h in the normalizer of M in $G(\mathbf{R})$ such that ρ' and $m \to \rho(hmh^{-1})$ are infinitesimally equivalent.

We have defined the quasi-character π_ν on $M(\mathbf{R})$ for $\nu \in D(P)$. The same formula

$$\pi_\nu(g) = \prod_i \big|\lambda_i(g)\big|^{x_i}$$

if $x = \sum x_i\lambda_i$ serves to define it for $\nu \in D(P) \otimes \mathbf{C}$. Set $\rho_\nu = \pi_\nu \otimes \rho$. All the representations ρ_ν act on the same space. Take ν in $iD(P)$ and consider the functions

$$f^\nu(g) = \big\langle I_{P_\nu}(g)\varphi, \psi\big\rangle$$

where $\varphi \in I(V)$ and $\psi \in I(V^*)$ are K-finite. Let h run over a set of representatives for the normalizer of M in $G(\mathbf{R})$ modulo $M(\mathbf{R})$. If v does not lie in a certain finite set of hyperplanes the quasi-characters $\zeta_h^\nu : a \to \rho_\nu(hah^{-1})$ of $A(P)$ are distinct and if f_P is defined as in [6] then, by Theorem 5 of that paper

$$f_P^\nu(m) = \sum_h \theta_h^\nu(m)$$

where

$$\theta_h^\nu(am) = \zeta_h^\nu(a)\theta_h^\nu(m), \quad a \in A(P).$$

By Lemma 3.12 and by Lemma 9 of [6], $\theta_1^\nu(m)$ has the form

$$\theta_1^\nu(m) = \big\langle \rho_\nu(m)M_\nu(\varphi), \psi(1)\big\rangle$$

where $M_\nu(\varphi)$ lies in V. $M_\nu(\varphi)$ is a meromorphic function of ν in a neighborhood of $iD(P)$ and its singularity can be killed by a product of linear factors. Let ρ_ν^h be the representation $m \to \rho_\nu(hmh^{-1})$. Since I_{ρ_ν} and $I_{\rho_\nu^h}$ are infinitesimally equivalent there are a φ_ν^h and a ψ_ν^h such that

$$\big\langle I_{\rho_\nu}(g)\varphi, \psi\big\rangle = \Big\langle I_{\rho_\nu^h}(g)\varphi_\nu^h, \psi_\nu^h\Big\rangle.$$

It follows that θ_h^ν is of the form

$$\Big\langle \rho_\nu^h(m)M_\nu^h(\varphi), N_\nu^h(\psi)\Big\rangle$$

where $M_\nu^h(\varphi)$, $N_\nu^h(\psi)$ are $K \cap M(\mathbf{R})$-finite vectors in V and V^*. They too are meromorphic in a neighborhood $iD(P)$ and their singularities can be killed by a product of linear factors.

Let $\lambda_1, \dots, \lambda_r$ be a basis for $D(P)$. If $a = (a_1, \dots, a_r)$ is an r-tuple of non-negative integers set

$$\ell_a(m) = \prod_{i=1}^r \{\log \pi_{\lambda_i}(m)\}^{a_i}.$$

Let $\{h\}$ be a set of representatives for the normalizer of M in $G(\mathbf{R})$ modulo the group of those g in the normalizer for which $\rho^g : m \to \rho(gmg^{-1})$ is equivalent to ρ. If the reader is willing to admit[2] that, for a fixed m, $f_P^\nu(m)$ depends continuously on $\nu \in iD(P)$, he can conclude that

$$f_P^0(m) = \sum_h \sum_a \ell_a(m) \left\langle \rho^h(m)\varphi_a^h, \psi_a^h \right\rangle$$

where the sum on a is finite and φ_a^h and ψ_a^h are $K \cap M(\mathbf{R})$-finite vectors in V and V^*.

If M is also a Levi factor of P' then a similar result holds for $f_{P'}^0(m)$. Since not all of the functions $f_{P'}^0(m)$ can vanish identically, we conclude that from a nonzero matrix coefficient of any irreducible constituent of I_ρ^P we can retrieve at least one nonzero matrix coefficient of one of the representations ρ^h. This yields the required assertion.

We are now in a position to complete our definition of sets Π_φ. Let $\widehat{P}$ be minimal among the parabolic subgroups of $\widehat{G}$ containing $\varphi(W_{\mathbf{R}})$. The group $\widehat{G}$ may be represented as a semi-direct product $G^{\widehat{0}} \times W_{\mathbf{R}}$. Since $\mathbf{C}^\times \subseteq W_{\mathbf{R}}$ acts trivially we may divide out by it to obtain the algebraic group $G^{\widehat{0}} \times \mathfrak{g}(\mathbf{C}/\mathbf{R})$. The image of $\widehat{P}$ is also algebraic and we may use the theorem that any two maximal fully reducible subgroups in an algebraic group are conjugate to conclude that $\varphi(W_{\mathbf{R}})$ is contained in a Levi factor $\widehat{M}$ of $\widehat{P}$. By assumption the class of P lies in $\widehat{\mathfrak{p}}(G)$. Let P be a parabolic subgroup of G over $\mathbf{R}$ whose class corresponds to that of $\widehat{P}$. Since $\varphi(W_{\mathbf{R}})$ is contained in no proper parabolic subgroup of $\widehat{M}$ the earlier definition, together with Lemma 2.4, associates to φ a finite set $\Pi_\varphi(P, \widehat{P})$ in $\Pi(M)$.

Suppose $\varphi(W_{\mathbf{R}})$ is contained minimally in both $\widehat{P}$ and $\widehat{\overline{P}}$. Then

$$(\widehat{P} \cap \overline{P}^\wedge)\widehat{N} \supseteq (P^{\widehat{0}} \cap \overline{P}^{\widehat{0}})\widehat{N}$$

and the right side is a parabolic subgroup of $G^{\widehat{0}}$ ([1], Proposition 4.4). The right side is the connected component of the left. Since the left contains $\varphi(W_{\mathbf{R}})$ which projects onto $W_{\mathbf{R}}$, it is a parabolic subgroup of $\widehat{G}$. Since it is contained in $\widehat{P}$ and contains $\varphi(W_{\mathbf{R}})$ it is equal to $\widehat{P}$. By the same proposition $\overline{P}^{\widehat{0}}$ contains a maximal reductive subgroup of $P^{\widehat{0}}$. Reversing the roles of the two groups we see that $P^{\widehat{0}}$ contains a maximal reductive subgroup of $\overline{P}^{\widehat{0}}$. Since any two maximal reductive subgroups of $P^{\widehat{0}}$ or of $\overline{P}^{\widehat{0}}$ have the same dimension, $P^{\widehat{0}}$ and $\overline{P}^{\widehat{0}}$ have a common maximal reductive subgroup. As before, we can divide $\widehat{G}$ by $\mathbf{C}^\times \subseteq \widehat{G} = G^{\widehat{0}} \times W_{\mathbf{R}}$ to obtain an algebraic group. $\mathbf{C}^\times$ is contained in $\widehat{P} \cap \overline{P}^\wedge$ and the quotient of $\widehat{P} \cap \overline{P}^\wedge$ by $\mathbf{C}^\times$ is an algebraic group. Take a maximal reductive subgroup in it which contains the image of $\varphi(W_{\mathbf{R}})$. Let its inverse image in $\widehat{P} \cap \overline{P}^\wedge$ be $\widehat{M}$. $\widehat{M}$ contains $\varphi(W_{\mathbf{R}})$ and therefore projects onto $W_{\mathbf{R}}$. Since [12] any two maximal reductive subgroups of the quotient of $\widehat{P} \cap \overline{P}^\wedge$ by $\mathbf{C}^\times$

[2] The author leaves him to struggle with his conscience.

are conjugate, $\widehat{M}$ contains a Levi factor of $P^{\widehat{0}}$ and of $\overline{P}^{\widehat{0}}$. Thus $\widehat{M}$ itself is a Levi factor of $\widehat{P}$ and of $\overline{P}^{\wedge}$.

Since the set $\Pi_\varphi(P,\widehat{P})$ does not depend on $\widehat{M}$, we may, for our purposes, fix $\widehat{M}$ and let $\widehat{P}$, which does affect $\Pi_\varphi(P,\widehat{P})$, vary over the parabolic subgroups of $\widehat{G}$ with $\widehat{M}$ as Levi factor. Since the pair (M,P) together with the set $\Pi_\varphi(P,\widehat{P})$ is determined only up to conjugacy we may assume that M too is fixed. It will be supposed that $M(\mathbf{R})$ is selfadjoint, although this is not important.

If $D(M)$ is the space introduced in the proof of Lemma 2.5 and M is a Levi factor of P then $D(M)=D(P)$. Let $\lambda=\lambda_\varphi(P,\widehat{P})$ in $D(M)$ be defined by the condition that

$$\left|\zeta'_\rho(z)\right| = \pi_\lambda(z)$$

if $\rho\in\Pi_\varphi(P,\widehat{P})$ and $z\in Z_M(\mathbf{R})$. We observe next that P and $\widehat{P}$ may always be so chosen that $\lambda_\varphi(P,\widehat{P})\in\overline{D}^+(P)$. In fact, if we vary P and $\widehat{P}$ simultaneously as in Lemma 2.5 we may let P vary over all parabolic subgroups of G with M as Levi factor without affecting $\Pi_\varphi(P,\widehat{P})$ or $\lambda_\varphi(P,\widehat{P})$. For at least one such P, $\lambda_\varphi(P,\widehat{P})\in\overline{D}^+(P)$, the closure of the chamber corresponding to P. From now on we only consider pairs P, $\widehat{P}$ for which $\lambda_\varphi(P,\widehat{P})\in\overline{D}^+(P)$. There is, moreover, a unique parabolic subgroup P_1 of G over $\mathbf{R}$ such that $P_1\supseteq M$ and $\lambda_\varphi(P,\widehat{P})\in D^+(P_1)$. P_1 contains P and there is a unique $\widehat{P}_1$ containing $\widehat{P}$ such that P_1 and $\widehat{P}_1$ lie in corresponding classes.

We can characterize $\widehat{P}_1$ in terms of $\widehat{M}$ and $\varphi(W_\mathbf{R})$ alone, without reference to P and $\widehat{P}$. For this we shall have to take cognizance of the way $\widehat{M}$ is identified with the associate group of M. We recall that we choose an isomorphism ψ of G with a quasi-split group G', containing B' and T', so that $\psi^{-1}\sigma(\psi)$ is inner and so that $\psi(P)\supseteq B'$ and $M'=\psi(M)\supseteq T'$. We then use G', B', and T' to construct $G^{\widehat{0}}$, $B^{\widehat{0}}$, $T^{\widehat{0}}$ and $\widehat{G}=G^{\widehat{0}}\times W_\mathbf{R}$. Conjugating $\widehat{M}$ and $\widehat{P}$, and therefore also φ, but that does not matter, we arrange that $\widehat{P}\supseteq B^{\widehat{0}}$ and that $\widehat{M}\supseteq T^{\widehat{0}}$. The construction of $\widehat{G}$ is such that this $\widehat{M}$ can be trivially identified with the associate group of M. We may also suppose that $\varphi(\mathbf{C}^\times)\subseteq T^{\widehat{0}}\times\mathbf{C}^\times$.

Now that everything is explicit, let us recall how the restriction of ζ'_ρ, $\rho\in\Pi_\varphi(P,\widehat{P})$, to the connected component of $M_{\mathrm{rad}}(\mathbf{R})$ is determined by φ. We write $\varphi(w)=a(w)\times w$ with respect to the splitting $G^{\widehat{0}}\times W_\mathbf{R}$. As before let $\overline{\sigma}$ be the action of $\sigma\in\mathfrak{g}(\mathbf{C}/\mathbf{R})$ on $T^{\widehat{0}}$, L, and $\widehat{L}$ determined by φ. Choose μ and $\nu=\overline{\sigma}\mu$ in $L\otimes\mathbf{C}$ such that

$$\widehat{\lambda}\big(a(z)\big) = z^{\langle\mu,\widehat{\lambda}\rangle}\overline{z}^{\langle\nu,\widehat{\lambda}\rangle}$$

Any s in the connected component of $M_{\mathrm{rad}}(\mathbf{R})$ may be written as $s=e^H$, where $H=\overline{\sigma}\overline{H}$ lies in $\widehat{L}\otimes\mathbf{C}$ and $\langle\alpha,H\rangle=0$ if α is a root of T' in M'. Then

$$\zeta'_\rho(s) = e^{\langle\mu/2,H+\overline{\sigma}\overline{H}\rangle} = e^{\langle\mu,\overline{H}\rangle}.$$

Write $\mu=\mu_1+\mu_2$ where $\overline{\sigma}\mu_1=\mu_1$, $\overline{\sigma}\mu_2=-\mu_2$. Because $\varphi(W_\mathbf{R})$ is contained in no proper parabolic subgroup of $\widehat{M}$, $\langle\mu_1,\widehat{\alpha}\rangle=0$ if $\widehat{\alpha}$ is a root of $T^{\widehat{0}}$ in $M^{\widehat{0}}$. Since $\mu-\overline{\sigma}\mu=2\mu_2$ lies in L,

$$\overline{\langle\mu_2,H\rangle} = \langle\mu_2,\overline{H}\rangle = \langle\overline{\sigma}\mu_2,\overline{\sigma}\overline{H}\rangle = -\langle\mu_2,H\rangle$$

and $\langle\mu_2,H\rangle$ is purely imaginary. A similar calculation shows that

$$\mathrm{Re}\langle\mu_1,H\rangle = \langle\mathrm{Re}\,\mu_1,H\rangle.$$

Thus

$$\left|\zeta'_\rho(s)\right| = e^{\langle \mathrm{Re}\,\mu_1, H\rangle} = e^{\langle \mathrm{Re}\,\mu_1, H+\overline{H}\rangle/2} = \prod_i \left|\lambda_i(s)\right|^{x_i}$$

if $\mathrm{Re}\,\mu_i = \sum x_i\lambda_i$. It follows that if $\lambda = \lambda_\varphi(P, \widehat{P})$ then $\lambda = \mathrm{Re}\,\mu_1$. Thus P_1 is determined by the condition that $\widehat{\alpha}$ is a root of $T^{\widehat{0}}$ in $P_1^{\widehat{0}}$ if and only if $\langle \mathrm{Re}\,\mu_1, \widehat{\alpha}\rangle \geqslant 0$.

$\widehat{P}_1$ is therefore determined by φ alone. P_1 is then any parabolic subgroup of G over $\mathbf{R}$ containing M whose class corresponds to that of $\widehat{P}_1$. P and $\widehat{P}$ are then any pair with $P \subseteq P_1$, $\widehat{P} \subseteq \widehat{P}_1$ whose classes correspond. Choose a Levi factor M_1 of P_1 over $\mathbf{R}$ that contains M. $P \cap M_1 = P'$ is a parabolic subgroup of M_1. If $\rho \in \Pi_\varphi(P, \widehat{P})$ we may consider the irreducible constituents of $I_\rho^{P'}$, a representation of $M_1(\mathbf{R})$. Let Π'_φ be the set formed by the infinitesimal equivalence classes of these constituents as ρ varies over $\Pi_\varphi(P, \widehat{P})$. We have to observe that $\Pi'_\varphi \subseteq \Pi(M_1)$ is independent of the choice of P and $\widehat{P}$. Since Π'_φ is not affected if we simultaneously conjugate P' and $\Pi_\varphi(P, \widehat{P})$ with an element of $M_1(\mathbf{R})$, we need only check that Π'_φ is independent of $\widehat{P}$. But if we change $\widehat{P}$ then, by Lemma 2.5, we may change P and hence P' so that $\Pi_\varphi(P, \widehat{P})$ is not affected. Since $I_\rho^{P'}$ and $I_\rho^{\overline{P}'}$ are infinitesimally equivalent if P' and $\overline{P}'$ are two parabolic subgroups of M_1 with M as Levi factor, the set Π'_φ does depend only on φ.

We have normalized φ so that $\widehat{P}_1$ contains $B^{\widehat{0}}$. Suppose φ' is normalized in the same way and gives rise to the same $\widehat{P}_1$. We shall need to know that if Π'_φ and $\Pi'_{\varphi'}$ have an infinitesimal equivalence class in common then φ and φ' determine the same element of $\Phi(G)$. If Π'_φ and $\Pi'_{\varphi'}$ have an element in common then, as we can see from our review of the results of [6], the images $\varphi(W_{\mathbf{R}})$ and $\varphi'(W_{\mathbf{R}})$ may be supported to lie in the same $\widehat{M}$, no proper parabolic subgroup of which contains either of them. We may choose, tentatively, the same P and $\widehat{P}$ for both of them. Then there are a ρ in $\Pi_\varphi(P, \widehat{P})$ and a ρ' in $\Pi_{\varphi'}(P, \widehat{P})$ and a g in the normalizer of M in $M_1(\mathbf{R})$ such that ρ is equivalent to $m \to \rho'(gmg^{-1})$. If ψ_1 is the restriction of $\mathrm{ad}\,g$ to M and $\widehat{\psi}_1$ the associated automorphism of $\widehat{M}$ then, by condition (iv),

$$\Pi_{\widehat{\psi}_1(\varphi)}(P, \widehat{P}) = \left\{ m \to \rho(gmg^{-1}) \;\middle|\; \rho \in \Pi_\varphi(P, \widehat{P}) \right\}.$$

By Lemma 2.7, there is an h in the normalizer of $\widehat{M}$ in $M_1^{\widehat{0}}$ such that on $\widehat{M}$ the operator $\mathrm{ad}\,h$ is equal to $\widehat{\psi}_1$. We may replace φ by $\mathrm{ad}\,h \circ \varphi$. Then $\Pi_\varphi(P, \widehat{P})$ and $\Pi_{\varphi'}(P, \widehat{P})$ have an element in common; so φ and φ' belong to the same class in $\Phi(M_1)$ and hence in $\Phi(G)$.

We are now able to introduce Π_φ in general. Π_φ consists of the classes $J_\rho^{P_1}$, $\rho \in \Pi'_\varphi$. By Lemma 3.14, together with the preceding discussion, these sets are disjoint. The other conditions on the sets Π_φ are built into their definition.

4. Exhaustion

It remains to prove the following proposition.

Proposition 4.1. *The sets* Π_φ, $\varphi \in \Phi(G)$, *exhaust* $\Pi(G)$.

We agree to call an infinitesimal equivalence class essentially tempered if it is a constituent of some I_ρ where ρ satisfies the conditions of Corollary 3.9 and is square-integrable modulo the center. To prove the proposition, we have only to prove the following lemma.

Lemma 4.2. *If π is an irreducible quasi-simple representation of $G(\mathbf{R})$, there is a parabolic subgroup P of G over $\mathbf{R}$ and an essentially tempered representation ρ of $M(\mathbf{R})$, $M = P/N$, such that $\lambda \in D^+(P)$ and such that π is infinitesimally equivalent to J_ρ.*

λ has the same meaning as in Lemma 3.8. Notice that, by Lemma 3.14, ρ and P are uniquely determined by π. The lemma reduces the problem of classifying the classes of irreducible quasi-simple representations of $G(\mathbf{R})$ to that of classifying the classes of essentially tempered representations of the various $M(\mathbf{R})$.

Let π be given. The first, the easy, step is to find P. Let π act on V. If $g \in G(\mathbf{R})$ and v^* lies in the dual space of V define $\pi^*(g)v^*$ by

$$\big\langle v, \pi^*(g)v^*\big\rangle = \big\langle \pi(g^{-1})v, v^*\big\rangle.$$

If $f \in C_c^\infty(G)$ define $\pi^*(f)v^*$ by

$$\big\langle v, \pi^*(f)v^*\big\rangle = \int_{G(\mathbf{R})} f(g)\big\langle \pi(g^{-1})v, v^*\big\rangle\, dg.$$

It is clear that

$$\big\|\pi^*(f)v^*\big\| \leqslant \|v^*\| \int_{G(\mathbf{R})} \big|f(g)\big|\big\|\pi(g)\big\|\, dg.$$

It is also clear that every K-finite vector v^* is a finite linear combination

$$v^* = \sum_i \pi^*(f_i)v_i^*.$$

Let V^* be the set of all v^* for which

$$\lim_{g\to h}\big\|\pi^*(g)v^* - \pi^*(h)v^*\big\| = 0$$

for all h. Since V^* contains all vectors of the form $\pi^*(f)v^*$, it contains all K-finite vectors. The representation π^* on V^* is continuous and the pairing $(v, v^*) \to \langle v, v^*\rangle$ is $G(\mathbf{R})$-invariant.

Let $\mathfrak{X}$ be a finite collection of classes of irreducible representations of K. Let $V(\mathfrak{X})$ and $V^*(\mathfrak{X})$ be the direct sum of the subspaces of V and V^* transforming according to the representations in $\mathfrak{X}$. Consider the function Ψ from $G(\mathbf{R})$ to the dual $W(\mathfrak{X})$ of $V(\mathfrak{X}) \otimes V^*(\mathfrak{X})$ defined by

$$\Psi(g) : u \otimes v \to \big\langle \pi(g)u, v\big\rangle.$$

Choose a parabolic subgroup P_0 of G minimal among those defined over $\mathbf{R}$. We suppose that $A \subseteq P_0(\mathbf{R})$. $D(P_0)$ and the Lie algebra $\mathfrak{A}$ of A are in duality over $\mathbf{R}$ in such a way that

$$\pi_\lambda(\exp H) = e^{\langle\lambda, H\rangle}.$$

It will be convenient to shuck some of our earlier notation, which is not always appropriate to our present purposes. Write $D(P_0) = D_0 + D$, where D_0 is orthogonal to $\mathfrak{A} \cap \mathfrak{g}_{\mathrm{der}}$ and D is orthogonal to $\mathfrak{A} \cap \mathfrak{g}_{\mathrm{rad}}$. D has as basis the roots $\alpha_1, \ldots, \alpha_r$ of A simple with respect to $P_0(\mathbf{R})$. The Killing form $B(\mu, \nu)$ is nondegenerate on D and zero on D_0. Define $\beta_1, \ldots, \beta_r$ in D by

$$B(\alpha_i, \beta_j) = \delta_{ij}.$$

By Theorem 9.1.3.2 of [16] there is a countable subset $L(\pi, \mathfrak{X})$ in $D(P_0) \otimes \mathbf{C}$ such that in the interior of $A^+ = A^+(P_0)$ an expansion

$$\Psi(a) = e^{-\langle\delta, H\rangle} \sum_{\lambda \in L(\pi,\mathfrak{X})} p_\lambda(H) e^{\langle\lambda, H\rangle} \tag{4.1}$$

is valid. δ is one-half the sum of the roots of A positive with respect to P_0 and taken with multiplicity. $a = e^H$ and p_λ is a polynomial function of H with values in $W(\mathfrak{X})$ that does not vanish identically.

If λ and μ belong to $D(P_0) \otimes \mathbf{C}$ we write $\lambda \succ \mu$ if

$$\operatorname{Re}\lambda = \operatorname{Re}\mu + \sum_{i=1}^{r} x_i \alpha_i$$

with $x_i \geqslant 0$. Let $E(\pi, \mathfrak{X})$ be the set of λ maximal in $L(\pi, \mathfrak{X})$ with respect to this order. As in [16], $E(\pi, \mathfrak{X})$ is finite. There is a simple fact to be verified.

Lemma 4.3. *The set $E(\pi, \mathfrak{X})$ is the same for all $\mathfrak{X}$ for which $W(\mathfrak{X})$ is not zero.*

If $\mathfrak{X}$ and $\mathfrak{Y}$ are two finite collections of classes of irreducible representations of K we may also introduce a function Ψ with values in the dual $W(\mathfrak{X}, \mathfrak{Y})$ of $V(\mathfrak{X}) \otimes V^*(\mathfrak{Y})$ by

$$\Psi(g) : u \otimes v \to \langle \pi(g)u, v \rangle.$$

Thus Ψ also admits an expansion of the form (4.1). We introduce $E(\pi, \mathfrak{X}, \mathfrak{Y})$ and show that it is the same for all $\mathfrak{X}$ and $\mathfrak{Y}$ for which $V(\mathfrak{X}) \neq 0$ and $V^*(\mathfrak{Y}) \neq 0$.

It is clear that if $\mathfrak{X}_1 \subseteq \mathfrak{X}_2$ then every element of $E(\pi, \mathfrak{X}_1, \mathfrak{Y})$ is dominated by an element of $E(\pi, \mathfrak{X}_2, \mathfrak{Y})$. If $V(\mathfrak{X})$ is different from 0 then every K-finite vector in V is a finite sum

$$\sum \pi(X_i) u_i$$

with $u_i \in V(\mathfrak{X})$, $X_i \in \mathfrak{Y}$, the universal enveloping algebra of $\mathfrak{g}$. If

$$f(g) = \langle \pi(g)u, v \rangle$$

then

$$\langle \pi(g)\pi(X)u, v \rangle = Xf(g).$$

If $u \in V(\mathfrak{X})$, $v \in V^*(\mathfrak{Y})$ then $\langle \pi(g)\pi(X)u, v \rangle$ is a coordinate of the $W(\mathfrak{X}, \mathfrak{Y})$-valued function $X\Psi(g)$. It follows from Theorem 9.1.2.9 of [16] that every exponent in the expansion of $\langle \pi(g)\pi(X)u, v \rangle$ is dominated by an element of $E(\pi, \mathfrak{X}, \mathfrak{Y})$. This if $V(\mathfrak{X}_1) \neq 0$ and $V(\mathfrak{X}_2) \neq 0$ then every element of $E(\pi, \mathfrak{X}_1, \mathfrak{Y})$ is dominated by an element of $E(\pi, \mathfrak{X}_2, \mathfrak{Y})$ and conversely. The two sets are therefore the same.

We define a double action $w \to \tau_1(k_1) w \tau_2(k_2)$ of K on $W(\mathfrak{X}, \mathfrak{Y})$ by

$$\tau_1(k_1) w \tau_2(k_2) : u \otimes v \to w\big(\pi(k_1^{-1})u \otimes \pi^*(k_2)v\big).$$

We may interchange the roles of V and V^* and of $\mathfrak{X}$ and $\mathfrak{Y}$, replacing π by π^*. If ω, the element of the Weyl group of A that takes positive roots to negative roots, is represented by $k \in K$, then

$$\langle u, \pi^*(a)v \rangle = \langle \pi(a^{-1})u, v \rangle = \Big\langle \pi\big(\omega(a^{-1})\big)\pi(k)u, \pi^*(k)v \Big\rangle.$$

Thus $\Psi(a)$ is replaced by

$$\tau_1(k^{-1}) \Psi\big(\omega(a^{-1})\big) \tau_2(k)$$

and

$$E(\pi^*, \mathfrak{Y}, \mathfrak{X}) = \big\{ -\omega(\lambda) \bigm| \lambda \in E(\pi, \mathfrak{X}, \mathfrak{Y}) \big\}.$$

It follows that $E(\pi^*, \mathfrak{X}, \mathfrak{Y})$ is also independent of $\mathfrak{Y}$.

We all also need some simple geometric lemmas. We recall that $B(\alpha_i, \alpha_j) \leqslant 0$ if $i \neq j$ and that $B(\beta_i, \beta_j) \geqslant 0$ for all i and j. If F is a subset of $\{1, \ldots, r\}$ let D_F be the subspace of D

spanned by $\{\, \beta_i \mid i \in F \,\}$. If $i \in F$ let $\beta_i^F = \beta_i$; if $i \in F$ let β_i^F be the orthogonal projection of β_i on the orthogonal complement of D_F. Define α_i^F by

$$B(\alpha_i^F, \beta_j^F) = \delta_{ij}.$$

If $i \notin F$ then $\alpha_i^F = \alpha_i$. If $i \in F$ then

$$\alpha_i^F = \alpha_i + \sum_{k \notin F} c_{ik}\alpha_k.$$

If k is not in F then

$$0 = B(\alpha_i^F, \beta_k^F) = (\alpha_i, \beta_k^F) + c_{ik}.$$

$\left\{\, \alpha_k \;\middle|\; k \notin F \,\right\}$ is a basis for the orthogonal complement of D_F and $B(\alpha_k, \alpha_\ell) \leqslant 0$ if $k \neq \ell$. Since $\left\{\, \beta_k^F \;\middle|\; k \notin F \,\right\}$ is the dual basis, $B(\beta_k^F, \beta_\ell^F) \geqslant 0$. Therefore β_ℓ^F is a linear combination of the α_k with nonnegative coefficients. Since $B(\alpha_i, \alpha_k) \leqslant 0$, $B(\alpha_i, \beta_k^F) \leqslant 0$ for $k \notin F$ and $c_{ik} \geqslant 0$. Thus if i and j belong to F and $i \neq j$

$$B(\alpha_i^F, \alpha_j^F) = B(\alpha_i^F, \alpha_j) = B(\alpha_i, \alpha_j) + \sum_{k \notin F} c_{ik} B(\alpha_k, \alpha_j) \leqslant 0.$$

The inequality $B(\alpha_i^F, \alpha_j^F) \leqslant 0$ is also valid if one of i and j does not lie in F.

For each F let ϵ_F be the characteristic function of

$$\left\{\, \lambda \in D(P_0) \;\middle|\; B(\alpha_i^F, \lambda) > 0,\;\; i \in F,\;\; B(\beta_i^F, \lambda) \leqslant 0,\;\; i \notin F \,\right\}.$$

Lemma 4.4. *If* $\lambda \in D(P_0)$ *then*

$$\sum_F \epsilon_F(\lambda) = 1.$$

Suppose $B(\alpha_i, \lambda) > 0$ for all i. Then $B(\alpha_i^F, \lambda) > 0$ for all i and all F. Since the basis $\{\beta_i^F\}$ is dual to $\{\alpha_i^F\}$ and $B(\alpha_i^F, \alpha_j^F) \leqslant 0$ for $i \neq j$, β_i^F is a linear combination of the α_j^F with nonnegative coefficients and $B(\beta_i^F, \lambda) > 0$ for all i and F. Thus $\epsilon_F(\lambda) = 0$ unless $F = \{1, \ldots, r\}$ when $\epsilon_F(\lambda) = 1$. Thus all we have to do is show that ϵ_F is a constant.

A hyperplane defined by an equation $B(\alpha_i^F, \lambda) = 0$ or $B(\beta_i^F, \lambda) = 0$ for some i and F will be called special. If λ is any point in $D(P_0)$ and if $B(\alpha_i, \mu) > 0$ for all i, then for any sufficiently small positive real number a

$$\epsilon_F(\lambda) = \epsilon_F(\lambda - a\mu)$$

for all F. Moreover $\lambda - a\mu$ lies in no special hyperplane. To show that ϵ_F is a constant we have to show that it is constant on the complement of the special hyperplanes. For this we have only to verify that it is constant in a neighborhood of a point λ_0 lying in exactly one special hyperplane.

For this we may disregard all those F which lie neither in

$$S_1 = \left\{\, F \;\middle|\; B(\alpha_i^F, \lambda_0) = 0 \text{ for some } i \in F \,\right\}$$

nor in

$$S_2 = \left\{\, F \;\middle|\; B(\beta_i^F, \lambda_0) = 0 \text{ for some } i \notin F \,\right\}.$$

The sets S_1 and S_2 are disjoint. $F = \{1, \ldots, r\}$ does not belong to S_2. We can introduce a bijection between S_1 and S_2. If $F_1 \in S_1$ and α_i^F with $i \in F_1$ is orthogonal to λ_0 set $F_2 = F_1 - \{i\}$. $\alpha_i^{F_1}$ and $\beta_i^{F_2}$ both lie in the space spanned by $\left\{\, \beta_j \;\middle|\; j \in F_1 \,\right\}$ and are both orthogonal to $\{\, \beta_j \mid j \in F_2 \,\}$. Thus they are multiples of each other and $F_2 \in S_2$. It is clear that $F_1 \to F_2$ is a bijection. Since

$$1 = B(\alpha_i^{F_1}, \beta_i) = B(\alpha_i^{F_1}, \beta_i^{F_2}),$$

$\alpha_i^{F_1}$ is a positive multiple of $\beta_i^{F_2}$. We have relations

$$\alpha_j^{F_2} = a_j^{F_1} + c_j \alpha_i^{F_1}, \quad j \in F_2,$$
$$\beta_j^{F_2} = \beta_j^{F_1} + d_j \beta_i^{F_2}, \quad j \notin F_1.$$

Near λ_0

$$\operatorname{sign} B(\beta_j^{F_2}, \lambda) = \operatorname{sign} B(\beta_j^{F_1}, \lambda), \quad j \notin F_1,$$
$$\operatorname{sign} B(\alpha_j^{F_2}, \lambda) = \operatorname{sign} B(\alpha_j^{F_1}, \lambda), \quad j \in F_2.$$

Moreover, either $B(\alpha_i^{F_1}, \lambda) > 0$ or $B(\beta_i^{F_2}, \lambda) \leqslant 0$ but not both. Thus $\epsilon_{F_1} + \epsilon_{F_2}$ is constant near λ_0. The lemma follows.

If $\lambda \in D(P_0)$ let $F = F(\lambda)$ be the unique subset of $\{1, \ldots, r\}$ such that

$$B(\alpha_i^F, \lambda) > 0, \quad i \in F,$$
$$B(\beta_i^F, \lambda) \leqslant 0, \quad i \notin F.$$

Let λ^0 be the projection of λ on the sum of D_0 and D_F. Then $B(\alpha_i, \lambda^0) \geqslant 0$ for all i and $B(\alpha_i, \lambda^0) > 0$ if $i \in F$. This is clear because $B(\alpha_i, \lambda^0) = 0$ if $i \notin F$ and $B(\alpha_i, \lambda^0) = B(\alpha_i^F, \lambda^0) = B(\alpha_i^F, \lambda)$ if $i \in F$. Let $\lambda = \lambda^0 + \lambda^1$. Then

$$\lambda^1 = \sum_{i \notin F} b_i \alpha_i.$$

Notice that

$$b_i = B(\beta_i, \lambda^1) = B(\beta_i^F, \lambda^1) = B(\beta_i^F, \lambda) \leqslant 0.$$

Lemma 4.5. *Suppose λ and μ lie in $D(P_0)$ and*

$$\lambda^0 + \sum_{i=1}^{r} c_i \alpha_i = \mu^0 + \nu + \sum_{j \notin F(\mu)} d_j \alpha_j.$$

Suppose $c_i \leqslant 0$, $\nu \in D$, $B(\alpha_i, \nu) = 0$ if $i \notin F(\mu)$, and $B(\beta_i, \nu) \geqslant 0$ if $i \in F(\mu)$. Then $\lambda^0 \succ \mu^0$.

Certainly $\lambda^0 - \mu^0 \in D$. If $i \in F(\mu)$ then

$$B(\beta_i, \lambda^0 - \mu^0) = -c_i + B(\beta_i, \nu) \geqslant 0. \tag{4.2}$$

If $i \notin F(\mu)$ then

$$B(\alpha_i, \lambda^0 - \mu^0) = B(\alpha_i, \lambda^0) \geqslant 0.$$

If $i \notin F(\mu)$

$$\beta_i^F = \sum_{j \notin F(\mu)} e_j \alpha_j$$

with $e_j \geqslant 0$; so

$$B(\beta_i^F, \lambda^0 - \mu^0) \geqslant 0.$$

Moreover

$$\beta_i^F = \beta_i - \sum_{j \in F(\mu)} a_j \beta_j$$

and

$$a_j = B(\beta_i, \beta_j)/B(\beta_j, \beta_j) \geqslant 0.$$

For (4.2) we conclude that

$$B(\beta_i, \lambda^0 - \mu^0) \geqslant B(\beta_i^F, \lambda^0 - \mu^0) \geqslant 0.$$

The lemma follows.

Corollary 4.6. *If $\lambda \succ \mu$ then $\lambda^0 \succ \mu^0$.*

If $\lambda \succ \mu$ then

$$\lambda + \sum_{i=1}^{r} c_i \alpha_i = \mu$$

with $c_i \leqslant 0$. Since

$$\lambda = \lambda^0 + \sum_{i \notin F(\lambda)} b_i \alpha_i$$

with $B_1 \leqslant 0$ and

$$\mu = \mu^0 + \sum_{j \notin F(\mu)} d_j \alpha_j$$

the corollary follows.

Since the set $E(\pi, \mathfrak{X})$ is the same for all $\mathfrak{X}$ with $W(\mathfrak{X}) \neq 0$ we may denote it by $E(\pi)$. Consider

$$L^0(\pi, \mathfrak{X}) = \left\{ \lambda^0 \;\middle|\; \lambda = \operatorname{Re} \lambda', \lambda' \in L(\pi, \mathfrak{X}) \right\}$$

and

$$E^0(\pi) = \left\{ \lambda^0 \;\middle|\; \lambda = \operatorname{Re} \lambda', \lambda' \in E(\pi) \right\}.$$

Suppose μ^0 lies in $L^0(\pi, \mathfrak{X})$. There is a $\lambda' \in E(\pi)$ such that $\lambda' \succ \mu'$; then $\lambda = \operatorname{Re} \lambda' \succ \mu = \operatorname{Re} \mu'$ and $\lambda^0 \succ \mu^0$. Thus $L^0(\pi, \mathfrak{X})$ has a maximal element λ^0 and $\lambda^0 \in E^0(\pi)$. We fix such a λ^0 once and for all. Since λ^0 lies in the closure of $D^+(P_0)$ there is a unique P containing P_0 such that $\lambda^0 \in D^+(P)$. This will turn out to be the P which appears in Lemma 4.2.

To obtain the representation ρ we have to apply some results that appear in an unpublished manuscript of Harish-Chandra [5] but, to the best of my knowledge, nowhere else.

$D(P_0)$ is the sum of $D_1 = D(P)$ and its orthogonal complement D_2. A is a product $A_1 A_2$, where $A_1 = A(P) = \left\{ e^H \;\middle|\; H \perp D_2 \right\}$ and $A_2 = \left\{ e^H \;\middle|\; H \perp D_1 \right\}$. Let $L_1(\pi, \mathfrak{X})$ be the projection of $L(\pi, \mathfrak{X})$ on $D_1 \otimes \mathbf{C}$. The first result we need from [5] is that $\Psi(a) = \Psi(a_1, a_2) = \Psi(e^{H_1}, a_2)$ admits an expansion

$$e^{-\langle \delta, H_1 \rangle} \sum_{\lambda_1 \in L_1(\pi, \mathfrak{X})} \phi_{\lambda_1}(H_1, a_2) e^{\langle \lambda_1, H_1 \rangle} \tag{4.3}$$

valid for a_1 in the interior of $A_1^+ = A^+(P)$. $\phi_\lambda(H_1, a_2)$ is a polynomial function of H_1 whose coefficients are analytic functions of a_2. The degrees of these polynomials are bounded. If $a = e^H \in A^+ = A^+(P_0)$ then

$$\phi_\lambda(H_1, a_2) e^{\langle \lambda_1 - \delta, H_1 \rangle} = \sum p_\lambda(H) e^{\langle \lambda - \delta, H \rangle} \tag{4.4}$$

where the sum is taken over all $\lambda \in L(\pi, \mathfrak{X})$ whose projection on $D_1 \otimes \mathbf{C}$ is λ_1.

To exploit this expansion we have to generalize some considerations to be found in §9.1.2 of [16]. The generalization being quite formal, we shall be as sparing as possible with proofs.

Choose a Levi factor M of P over $\mathbf{R}$ such that $M(\mathbf{R})$ is selfadjoint. Let $\mathfrak{p}$, $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{k}$ in $\mathfrak{g}$ be the complexifications of the Lie algebras of $P(\mathbf{R})$, $M(\mathbf{R})$, $N(\mathbf{R})$, and K and let $\mathfrak{P}$, $\mathfrak{M}$, $\mathfrak{N}$, and $\mathfrak{K}$ be their universal enveloping algebras. Let $\mathfrak{q}$ be the orthogonal complement of $\mathfrak{m}$ in $\mathfrak{k} \cap \mathfrak{g}_{\mathrm{der}}$. As on p. 269 of [16], but with a different result, we define $\mathfrak{Q}$ to be the image of the symmetric algebra of $\mathfrak{q}$ in $\mathfrak{A}$.

Note that

$$\dim \mathfrak{q} = \dim \mathfrak{g} - \dim \mathfrak{p} = \dim \mathfrak{n}$$

and that

$$\dim \mathfrak{g} = 2 \dim \mathfrak{q} + \dim \mathfrak{m}.$$

Let $K_1 = K \cap M(\mathbf{R})$. It is a maximal compact subgroup of $M(\mathbf{R})$. Let U be a compact subset of $M(\mathbf{R})$ with $U = K_1 U = U K_1$. As m varies over U the eigenvalues of $\operatorname{ad} m$ in the orthogonal complement of $\mathfrak{m}$ in $\mathfrak{g}_{\mathrm{der}}$ lie in compact subset of $\mathbf{C}^\times$, say

$$\left\{ z \;\middle|\; \frac{1}{R} \leqslant |z| \leqslant R \right\}.$$

Let $A_1^+(R)$ be the set of all a in A_1 such that $\alpha(a) > R$ for every root of A_1 in $\mathfrak{n}$. If $m = m_1 a$, $m_1 \in U$, $a \in A_1^+(R)$ the centralizer of m in $\mathfrak{g}$ lies in m. Moreover

$$\mathfrak{g} = \operatorname{ad} m(\mathfrak{q}) \oplus \mathfrak{m} \oplus \mathfrak{q}. \tag{4.5}$$

To see this one has only to verify that

$$\operatorname{ad} m(\mathfrak{q}) \cap (\mathfrak{m} + \mathfrak{q}) = 0.$$

Since $\mathfrak{m}$ and $\mathfrak{q}$ are invariant under K_1 and $M(\mathbf{R}) = K_1 A K_1$ we may suppose that m_1, and hence m, lies in A. Suppose X lies in the above intersection. Let θ be the automorphism of $G(\mathbf{C})$ such that $\theta(g^{-1})$ is the conjugate transpose of g with respect to the hermitian form introduced earlier. θ is a Cartan involution. Let H lie in the Lie algebra of A_1 and set

$$X_H = (\operatorname{ad} H)^2 X.$$

Then $X_H \in \mathfrak{k}$ and $\operatorname{ad} m(X_H) \in \mathfrak{k}$. Consequently

$$\operatorname{ad} m(X_H) = \theta\big(\operatorname{ad} m(X_H)\big) = \operatorname{ad} m^{-1}(X_H)$$

and

$$\operatorname{ad} m^2(X_H) = X_H.$$

Since $\operatorname{ad} m$ has only positive eigenvalues and since its centralizer in $\mathfrak{g}$ is $\mathfrak{m}$, this equation implies that $X_H \in \mathfrak{m}$. Thus

$$(\operatorname{ad} H)^3 X = \operatorname{ad} H(X_H) = 0.$$

However, $\operatorname{ad} H$ is semisimple; so $X_H = 0$. Since H was arbitrary in the Lie algebra of A_1, X lies in $\mathfrak{m}$. Since both $\mathfrak{m} \cap \mathfrak{q}$ and $\mathfrak{m} \cap \operatorname{ad} m(\mathfrak{q})$ must be zero, X is zero.

The relation (4.5) yields an isomorphism

$$\mathfrak{A} \simeq \operatorname{ad} m(\mathfrak{Q}) \otimes \mathfrak{M} \otimes \mathfrak{Q} \simeq \mathfrak{Q} \otimes \mathfrak{M} \otimes \mathfrak{Q}.$$

If $X \in \mathfrak{A}$ we let X_m be the corresponding element on the right. The function Ψ restricted to $M(\mathbf{R})$ yields a function on $M(\mathbf{R})$ with values in $W(\mathfrak{X})$. If $X \in \mathfrak{M}$ we denote the result of applying X to this function at the point m by $\Psi(m, X)$. The actions τ_1 and τ_2 of K on $W(\mathfrak{X})$ yield actions of $\mathfrak{k}$. Let $X \to X^\sim$ be the involution of $\mathfrak{k}$ defined by $X^\sim = -X$, $X \in \mathfrak{k}$. If $X \in \mathfrak{A}$ and

$$X_m = \sum X_i \otimes Y_i \otimes Z_i$$

then

$$X\Psi(m) = \sum \tau_1(Z_i^\sim)\Psi(m, Y_i)\tau_2(Z_i^\sim).$$

Let $\overline{P} = \theta(P)$. Then $\overline{P}$ is defined over $\mathbf{R}$ and $\overline{P} \cap P = M$. Moreover

$$\mathfrak{g} = \bar{\mathfrak{n}} + \mathfrak{m} + \mathfrak{q}$$

and

$$\mathfrak{A} = \overline{\mathfrak{N}}\mathfrak{M}\mathfrak{Q}.$$

If $X = \sum Y_i Z_i$, $Y_i \in \mathfrak{M}$, $Z_i \in \mathfrak{Q}$ then

$$X_m = \sum 1 \otimes Y_i \otimes Z_i.$$

Suppose $\overline{X} \in \bar{\mathfrak{n}}$. Let $\overline{X} = \theta(X)$, $X \in \mathfrak{n}$. Then

$$Y = X + \overline{X}$$

lies in $\mathfrak{q}$. Let $X' = \operatorname{ad} m(X)$, $\overline{X}' = \theta(X')$ and

$$Y' = X' + \overline{X}' = X' + \theta\big(\operatorname{ad} m(X)\big).$$

Since

$$\operatorname{ad} m(Y) = X' + \operatorname{ad} m(\overline{X}) = X' + \theta\big(\operatorname{ad}\theta(m)(X)\big)$$

we have

$$\theta\Big(\big\{\operatorname{ad} m - \operatorname{ad}\theta(m)\big\}X\Big) = Y' - \operatorname{ad} mY.$$

We are still assuming that $m = m_1 a$, $m_1 \in U$, $a \in A_1^+(R)$; the restriction of $\operatorname{ad} m - \operatorname{ad}\theta(m)$ to $\mathfrak{n}$ is therefore invertible. Let $\mathfrak{F}$ be the ring of functions generated by the matrix coefficients of its inverse. $\mathfrak{F}$ does not contain 1. Replacing X by $\big\{\operatorname{ad} m - \operatorname{ad}\theta(m)\big\}^{-1}X$, we see that

$$\overline{X} = \sum f_i(m)\operatorname{ad} m(X_i) + \sum g_i(m) Z_i$$

with f_i, g_i in $\mathfrak{F}$ and X_i, Z_i in $\mathfrak{Q}$. Then

$$\overline{X}_m = \sum f_i(m) X_i \otimes 1 \otimes 1 + \sum g_i(m) 1 \otimes 1 \otimes Z_i.$$

One proves more generally by induction on the degree that

$$X_m = X_0 + \sum_i f_i(m) X_i \tag{4.6}$$

where $f_i \in \mathfrak{F}$, $X_i \in \mathfrak{Q} \otimes \mathfrak{M} \otimes \mathfrak{Q}$ and where $X_0 \in \mathfrak{M} \otimes \mathfrak{Q} \simeq \mathfrak{M}\mathfrak{Q}$ is uniquely defined by the condition that $X - X_0 \in \bar{\mathfrak{n}}\mathfrak{A}$.

Notice that as a function of $a \in A_1^+$ an element of $\mathfrak{F}$ is a linear combination of products of the functions $\big\{\alpha(a) - \alpha^{-1}(a)\big\}^{-1}$, α a root of A_1 in $\mathfrak{n}$ with coefficients that are analytic functions of m_1. Moreover $\big\{\alpha(a) - \alpha^{-1}(a)\big\}^{-1}$ admits an expansion.

$$\sum_{n=0}^{\infty} e^{-(2n+1)\langle\alpha, H\rangle} \tag{4.7}$$

for $a = e^H$ in $A_1^+(R)$.

If $X \in D$, the centralizer of K in $\mathfrak{A}$, and if $M(X)$ is the linear transformation of $W(\mathfrak{X})$ adjoint to the operator

$$u \otimes v \to \pi(X)u \otimes v$$

on $V(\mathfrak{X}) \otimes V^*(\mathfrak{X})$ then

$$X\Psi = M(X)\Psi.$$

$\lambda^0 \in D_1$ was fixed some time ago. There is at least one $\lambda_1^0 \in L_1(\pi, \mathfrak{X})$ with $\operatorname{Re} \lambda_1^0 = \lambda^0$. Fix such a λ_1^0. If $m \in M(\mathbf{R})$ we write $m = k_1 a k_2$ with k_1, k_2 in K_1 and a in A. We write $a = a_1 a_2$, $a_1 = e^{H_1}$, and set

$$\Phi(m) = e^{\langle \lambda_1^0 - \delta, H \rangle} \tau_1(k_2^{-1}) \phi_{\lambda_1^0}(H_1, a_2) \tau_2(k_1^{-1}).$$

Because of the uniqueness of the expansion (4.3), Φ is well-defined. The elements of $\mathfrak{K} \otimes \mathfrak{M} \otimes \mathfrak{K}$ act on Φ. $X \otimes Y \otimes Z$ sends Φ to Φ' with

$$\Phi'(m) = \tau_1(Z^\sim)\Phi(m, Y)\tau_2(X^\sim).$$

Let $X \in \mathfrak{D}$ and let X_0 be defined as in (4.6); then $X_0 \in \mathfrak{K} \otimes \mathfrak{M} \otimes \mathfrak{K}$ and

$$X_0 \Phi = M(X)\Phi. \tag{4.8}$$

To see this we start from the equation $X\Psi(m) = M(X)\Psi(m)$. If we set $m_2 = k_1 a_2 k_2$ and

$$\phi_{\lambda_1}(H_1, m_2) = \tau_1(k_2^{-1})\phi_{\lambda_1}(H_1, a_2)\tau_2(k_1^{-1})$$

the function $M(X)\Psi(m)$ has an expansion

$$\sum_{\lambda_1 \in L_1(\pi, X)} M(X)\phi_{\lambda_1}(H, m_2) e^{\langle \lambda_1 - \delta, H \rangle}.$$

The function $X\Psi(m)$ is equal to

$$X_0 \Psi(m) + \sum f_i(m) X_i \Psi(m).$$

X_0 and the X_i are acting as elements of $\mathfrak{K} \otimes \mathfrak{M} \otimes \mathfrak{K}$. Because of (4.7) each $f_i(m)$ has an expansion

$$\sum_{\mu_1} \epsilon_{\mu_1}(m_2) e^{-\langle \mu_1, H_1 \rangle}$$

valid for $m_2 \in U$, $a_1 \in A_1^+(R)$, where U is a compact set in $M(\mathbf{R})$ and $R = R(U)$ is chosen as before. μ_1 runs over the projections on D_1 of sums of positive roots of A in $\mathfrak{n}$. The sums are not empty and μ_1 is never zero. We may, for convergence offers no difficulty (cf. 16), apply X_i to Ψ term by term, expand the product $f_i(m) X_i \Psi(m)$ formally, add the results, and then compare coefficients of the exponentials $e^{\langle \lambda_1 - \delta, H_1 \rangle}$ on both sides of the equation.

We are interested in the terms corresponding to λ_1^0. If we incorporate the exponential, the term on the right is $M(X)\Phi(m)$. At first sight the term on the left seems more complicated. Suppose, however, that μ_1 is the projection on D_1 of a sum of positive roots of A in $\mathfrak{n}$, ν_1 lies in $L_1(\pi, \mathfrak{X})$, and $\nu_1 - \mu_1 = \lambda_1^0$. Let λ_1^0 be the projection of λ' in $L(\pi, \mathfrak{X})$ and let $\lambda = \operatorname{Re} \lambda'$; let ν_1 be the projection of ν' and let $\nu = \operatorname{Re} \nu'$. Then

$$\operatorname{Re} \lambda_1^0 = \lambda^0$$

and, if as before we define ν^0 to be the projection of ν on the sum of D_0 and $D_{F(\nu)}$, then

$$\operatorname{Re} \nu_1 = \nu + \sum_{j \notin F(\lambda)} c_j \alpha_j = \nu^0 + \sum_{i=1}^r b_i \alpha_i + \sum_{j \notin F(\lambda)} c_j \alpha_j$$

with $b_i \leqslant 0$. Also

$$\mu_1 = \sum_{i=1}^{r} d_i\alpha_i + \sum_{j\notin F(\lambda)} e_j\alpha_j$$

with $d_i \geqslant 0$. Moreover, at least one d_i, with $i \in F(\lambda)$, is positive. We have

$$\nu^0 + \sum_{i=1}^{r}(b_i - d_i)\alpha_i = \lambda^0 + \sum_{j\notin F(\lambda)} (e_j - c_j)\alpha_j.$$

It follows from Lemma 4.5 that $\nu^0 \succ \lambda^0$. By the very choice of λ^0, ν^0 is therefore equal to λ^0. However if $i \in F(\lambda)$ then

$$B(\beta_i, \nu^0 - \lambda^0) = d_i - b_i.$$

Since this is positive for at least one i, $\nu^0 \neq \lambda^0$. This is a contradiction. The term on the left in which we are interested is therefore $X_0\Phi(m)$. The relation (4.8) follows.

$\mathfrak{D}$ contains $\mathfrak{Z}$. As a linear space $\mathfrak{A}$ is a sum

$$\mathfrak{M} + \bar{\mathfrak{n}}\mathfrak{M} + \mathfrak{M}\mathfrak{n} + \bar{\mathfrak{n}}\mathfrak{M}\mathfrak{n}$$

and

$$\mathfrak{Z} \subseteq \mathfrak{M} + \bar{\mathfrak{n}}\mathfrak{M}\mathfrak{n}.$$

Thus if $X \in \mathfrak{Z} = \mathfrak{Z}_G$ then X_0 belongs to $\mathfrak{M}$ and in fact to $\mathfrak{Z}_M$. The map $X \to X_0$ is an injection of $\mathfrak{Z}_G$ into $\mathfrak{Z}_M$ and turns $\mathfrak{Z}_M$ into a finite $\mathfrak{Z}$-module. Notice also that $M(X)$ is a scalar $m(X)I$ if $X \in \mathfrak{Z}_G$.

According to (4.4) the restriction of Φ to A has an asymptotic expansion $\sum p_\lambda(H)e^{\langle\lambda-\delta,H\rangle}$ where λ runs over those elements of $L(\pi, \mathfrak{X})$ whose projection on $D_1 \otimes \mathbf{C}$ is λ_1^0. Suppose ν' is one of the indices for this sum. Let $\nu = \operatorname{Re}\nu'$ and define ν^0 as before. We can again apply Lemma 4.5 to see that $\nu^0 = \lambda^0$. Thus if $F = \left\{ i \,\middle|\, B(\alpha_i, \lambda^0) > 0 \right\}$ then $F = F(\nu)$ and

$$\operatorname{Re}\Big\{B(\beta_F^i, \nu)\Big\} \leqslant 0, \quad i \notin F. \tag{4.9}$$

In spite of the fact that Φ is not an eigenfunction of $\mathfrak{Z}_M$ but only of the image of $\mathfrak{Z}_G$ in $\mathfrak{Z}_M$ the considerations of §9.1.3 of [16], and hence those of its appendix as well as those of [5], may be applied to it. We do not want to apply them to obtain an asymptotic expansion, which we already have; we want to apply a further result (Theorem 4) of [5] that in conjunction with (4.9) and Lemma 3.7 easily implies the existence of a constant c and an integer d such that

$$\pi_{\delta_1}(m)\big\|\Phi(m)\big\| \leqslant c\big(1 + \ell(m)\big)^d \pi_{\lambda^0}(m)\Xi_M(m) \tag{4.10}$$

for all m. δ_1 is the projection of δ on D_1.

We had fixed $\mathfrak{X}$ but we may let it grow without changing λ^0. Thus

$$\Phi(m)(u \otimes v) = \Phi(m; u, v)$$

is defined for all K-finite $\mu \in V$, $v \in V^*$.

Lemma 4.7. *Suppose v in V^* is K-finite. If the function $\Phi\big(m; \pi(k)u, v\big)$ vanishes identically in m and k for some nonzero K-finite u in V then it vanishes identically for all such u.*

The function $\phi(m) = \big\langle \pi(m)u, v\big\rangle$, $m = a_1m_2$, $a_1 = e^{H_1} \in A_1$, $m_2 = k_1a_2k_2$, $a_2 \in A_2$, k_1, $k_2 \in K_1$, admits an asymptotic expansion $\sum a_{\lambda_1}(m; u, v)e^{\langle\lambda_1-\delta,H_1\rangle}$ with

$$a_{\lambda_1^0}(m; u, v)e^{\langle\lambda_1^0-\delta,H_1\rangle} = \Phi(m, u, v).$$

Suppose $X \in \mathfrak{A}$ and write

$$X_m = X_0 + \sum f_i(m) X_i.$$

For this we have to constrain m_2 to vary in some compact set U and a_1 to vary in $A_1^+(R)$, $R = R(U)$. Then

$$\big\langle \pi(m)\pi(X)u, v \big\rangle = X\phi(m) = X_0\phi(m) + \sum_i f_i(m) X_i \phi(m).$$

The symbol $X\phi(m)$ denotes the value of X applied to the function $\phi(g) = \big\langle \pi(g)u, v\big\rangle$ at the point m. X_0 and X_i are applied as elements of $\mathfrak{K} \otimes \mathfrak{M} \otimes \mathfrak{K}$.

The considerations used to prove the equality (4.8) show that if $u' = \pi(X)u$ then $a_{\lambda_1^0}(m; u', v)$ is the coefficient of $e^{\langle \lambda_1^0 - \delta, H_1\rangle}$ in the expansion of $X_0\phi(m)$.

$$X_0 = \sum 1 \otimes Y_j \otimes Z_j$$

with $Y_j \in \mathfrak{M}$, $Z_j \in \mathfrak{K}$. Applying Z_j we replace the coefficient $a_{\lambda_1^0}(m, u, v)$ by $a_{\lambda_1^0}\big(m, \pi(Z_j)u, v\big)$. If $\Phi\big(m, \pi(k)u, v\big) = 0$ for all m and k, this is zero. If the coefficient is zero before Y_j is applied it is zero after. Since every K-finite vector in V is of the form $\pi(X)u$, $X \in \mathfrak{A}$, the lemma follows.

There is certainly at least one K-finite v in V^*, which we fix once and for all, such that $\Phi(m; u, v)$ is not zero for all K-finite u.

Let $\mathfrak{T}$ be the Banach space of continuous functions θ on $M(\mathbf{R})$ for which

$$\|\theta\| = \sup \frac{\big|\theta(m)\big|}{\big(1 + \ell(m)\big)^{d-1} \pi_{\lambda^0}(m) \Xi_M(m)} < \infty.$$

If $m \in M(\mathbf{R})$ let $r(m)\theta$ be the function whose value at m_1 is

$$\theta(m_1 m)$$

Let $\mathfrak{W}$ be the space of all θ in $\mathfrak{T}$ for which

$$\lim_{m \to m_0} \big\| r(m)\theta - r(m_0)\theta \big\| = 0$$

for all m_0. If $u \in V$ is K-finite then

$$\theta_u : m \to \pi_{\delta_1}(m) \Phi(m; u, v)$$

lies in $\mathfrak{W}$ because of (4.10). Let $\mathfrak{V}$ be the closed subspace of $\mathfrak{W}$ generated by the functions $r(m)\theta_u$.

Lemma 4.8. *The representation r of $M(\mathbf{R})$ on $\mathfrak{V}$ admits a finite composition series.*

Let $\mathfrak{V}_0$ be the space of functions in $\mathfrak{V}$ of the form

$$\theta = r(f)\theta' = \int_{M(\mathbf{R})} f(m) r(m) \theta' dm$$

with $f \in C_c^\infty\big(m(\mathbf{R})\big)$. If $X \in \mathfrak{Z}_G \hookrightarrow \mathfrak{Z}_M$ and $\theta \in \mathfrak{V}_0$ then

$$X\theta = m'(X)\theta \tag{4.11}$$

if $m'(X) = m(X')$, where X' is the element of $\mathfrak{Z}_M$ defined by

$$X(\pi_{\delta_1}\phi)(m) = \pi_{\delta_1}(m) X' \phi(m).$$

If $K_1 = K \cap M(\mathbf{R})$, if φ_1 and φ_2 are two continuous functions on K_1, and if $\theta \in \mathfrak{V}_0$ let

$$\theta(m;\varphi_1,\varphi_2) = \int_{K_1}\int_{K_1} \varphi_1(k_1)\theta(k_1mk_2)\varphi_2(k_2)\,dk_1\,dk_2.$$

If $\varphi_1'(k) = \varphi_1(kk_1^{-1})$ and $\varphi_2'(k) = \varphi_2(k_2^{-1}k)$ then

$$\theta(k_1mk_2;\varphi_1,\varphi_2) = \theta(m,\varphi_1',\varphi_2').$$

If $\theta = r(m_1)\theta_u$ then

$$\begin{aligned}\theta(m,\varphi_1,\varphi_2) &= \pi_{\delta_1}(m)\iint \varphi_1(k_1)\Phi(k_1mm_1k_2;u,v)\varphi_2(k_2)\,dk_1\,dk_2\\ &= \pi_{\delta_1}(m)\iint \varphi_1(k_1)\Phi\big(mm_1,\pi(k_2)u,\pi^*(k_1^{-1})v\big)\varphi_2(k_2)\,dk_1\,dk_2\\ &= \pi_{\delta_1}(m)\Phi(mm_1,u',v')\end{aligned}$$

with

$$u' = \int \varphi_2(k_2)\pi(k_2)u\,dk_2, \quad v' = \int \varphi_1(k_1)\pi^*(k_1^{-1})v\,dk_1.$$

In particular, if $v' = 0$ then $\theta(m,\varphi_1,\varphi_2) = 0$ for $\theta = r(m_1)\theta_u$ and hence, by continuity, for any θ in $\mathfrak{V}_0$. There is a closed subspace of finite codimension in the space of continuous functions on K_1, invariant under left and right translations, such that $v' = 0$ whenever φ_1 lies in this subspace. Factoring out the subspace, we may regard φ_1 as varying over a finite-dimensional space. Let $\mathfrak{X}_1$ be a finite set of classes of irreducible representations of K_1. For other, more obvious, reasons, if θ is constrained to lie in the subspace $\mathfrak{V}(\mathfrak{X}_1)$ of $\mathfrak{V}_0$ spanned by vectors transforming according to one of the representations in $\mathfrak{X}_1$, then φ_2 may be regarded as varying over a finite-dimensional space. Using (4.11) and a simple variant of Proposition 9.1.3.1 of [16], we conclude that the space of functions $m \to \theta(m;\varphi_1,\varphi_2)$, where $\theta \in \mathfrak{V}(\mathfrak{X}_1)$ and φ_1 and φ_2 are continuous functions on K_1, is finite-dimensional. Since φ_1 and φ_2 may be allowed to approach the delta-function, we conclude that $\mathfrak{V}(\mathfrak{X}_1)$ itself lies in this space and is finite-dimensional. Since $\mathfrak{V}_0$ is dense in $\mathfrak{V}$ every irreducible representation of K_1 occurs with finite multiplicity in $\mathfrak{V}$.

To complete the proof we need a well-known fact, which we state as a lemma.

Lemma 4.9. *Let $X \to m'(X)$ be homomorphism of $\mathfrak{Z}_G \hookrightarrow \mathfrak{Z}_M$ into $\mathbf{C}$. There are only a finite number of infinitesimal equivalence classes of quasi-simple irreducible representations τ of $M(\mathbf{R})$ such that*

$$\tau(X) = m(X)I$$

for $X \in \mathfrak{Z}_G$.

Since there are only a finite number of ways of extending m to a homomorphism of $\mathfrak{Z}_M$ into $\mathbf{C}$, it is enough to prove the lemma for $G = M$; that is, we may assume that m' is already given on $\mathfrak{Z}_M$ and that

$$\tau(X) = m'(X)I$$

for all $X \in \mathfrak{Z}_M$.

Let τ act on W. We saw in Lemma 3.5 that the restriction of τ to the connected component $M^0(\mathbf{R})$ is the direct sum of finitely many irreducible representations. Let τ^0, acting on $W^0 \subseteq W$, be one of them. Because of Theorem 4.5.8.9 of [16], there are only finitely many possibilities for the class of τ^0.

Suppose W' is the space of all functions φ on $M(\mathbf{R})$ with values in W^0 satisfying

$$\varphi(m_0 m) = \tau^0(m_0)\varphi(m), \quad m_0 \in M^0(\mathbf{R}).$$

$M(\mathbf{R})$ acts on W' by right translations. There is an $M(\mathbf{R})$-invariant map from W' to W given by

$$\varphi \to \sum_{M^0(\mathbf{R})\backslash M(\mathbf{R})} \tau(g^{-1})\varphi(g).$$

We shall verify that W' admits a finite composition series

$$0 = W_0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_n = W'.$$

Then τ must be equivalent to the representation of $M(\mathbf{R})$ on one of the quotients $W_{i-1}\backslash W_i$. From this the lemma follows.

To show the existence of a finite composition series all we have to do is show that if

$$0 = W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_n = W'$$

is any chain of $M(\mathbf{R})$-invariant subspaces then $n \leqslant \left[M(\mathbf{R}) : M^0(\mathbf{R})\right]$. We could instead work with spaces of K-finite vectors invariant under the pair K, $\mathfrak{M}$. If $K^0 = K \cap M^0(\mathbf{R})$ then W' admits a composition series of length $\left[M(\mathbf{R}) : M^0(\mathbf{R})\right]$ with respect to the pair K^0, $\mathfrak{M}$. Any chain invariant with respect to this pair, and, *a fortiori,* any chain invariant with respect to K, $\mathfrak{M}$, has therefore length at most $\left[M(\mathbf{R}) : M^0(\mathbf{R})\right]$.

We return to the proof of Lemma 4.8. Let $\tau_1, \ldots, \tau_s$ be the classes corresponding to the given homomorphism m'. Choose for each i an irreducible representation σ_i occurring in the restriction of τ_i to K_1. Set $\mathfrak{X}_1 = \{\sigma_1, \ldots, \sigma_s\}$.

Suppose $\mathfrak{V}'' \subsetneq \mathfrak{V}'$ are closed $M(\mathbf{R})$-invariant subspaces of $\mathfrak{V}$. Let σ be a representation of K occurring in $\mathfrak{W}^0 = \mathfrak{V}''\backslash\mathfrak{V}'$. Let $\mathfrak{W}(\sigma)$ be the space of all vectors in $\mathfrak{W}^0$ transforming according to σ. $\mathfrak{W}(\sigma)$ is finite-dimensional. Among the nonzero subspaces of $\mathfrak{W}(\sigma)$ obtained by intersecting it with a closed $M(\mathbf{R})$ -invariant subspace of $\mathfrak{W}^0$, there is a minimal one $\mathfrak{W}'(\sigma)$. Let $\mathfrak{W}'$ be the intersection of all closed invariant subspaces of $\mathfrak{W}^0$ that contain $\mathfrak{W}'(\sigma)$. Let $\mathfrak{W}''$ be the closure of the sum of all closed invariant subspaces of $\mathfrak{W}'$ that do not contain $\mathfrak{W}'(\sigma)$. Then $\mathfrak{W}'' \subsetneq \mathfrak{W}'$ and the representation of $M(\mathbf{R})$ on $\mathfrak{W}''\backslash\mathfrak{W}'$ is irreducible. Since it must be one of $\tau_1, \ldots, \tau_s$, it contains one of $\sigma_1, \ldots, \sigma_s$.

Suppose we have a chain of closed $M(\mathbf{R})$-invariant subspaces

$$0 \subsetneq \mathfrak{V}_1 \subsetneq \cdots \subsetneq \mathfrak{V}_n = \mathfrak{V}.$$

Since one of $\sigma_1, \ldots, \sigma_s$ is contained in the representation of K_1 on the quotient of the successive subspaces, $n \leqslant \dim \mathfrak{V}(\mathfrak{X}_1)$. On the other hand, if these quotients are not irreducible the chain can be further refined. The lemma follows.

As before let $\overline{P} = \theta(P)$. Let $\mathfrak{U}$ be the space of continuous functions φ on $G(\mathbf{R})$ with values in $\mathfrak{V}$ which satisfy the following two conditions:

(i) If $\overline{n} \in \overline{N}(\mathbf{R})$ then $\varphi(\overline{n}g) = \varphi(g)$.

(ii) If $m \in M(\mathbf{R})$ then $\varphi(mg) = \pi_{\delta_1}^{-1}(m)r(m)\varphi(g)$.

The representation of $G(\mathbf{R})$ on $\mathfrak{U}$ by right translations is the induced representation $I_r^{\overline{P}}$. It is easily seen that every representation of K occurs with finite multiplicity in $I_r^{\overline{P}}$ and that

$$I_r^{\overline{P}}(X) = m(X)I, \quad I \in \mathfrak{Z}_G.$$

Thus $I_r^{\overline{P}}$ admits a finite composition series. We now show that π is infinitesimally equivalent to a subrepresentation of $I_r^{\overline{P}}$. For this we have only to define an injection of the K-finite vectors in V into $\mathfrak{U}$ which commutes with the action of K and $\mathfrak{U}$.

Recall that the vector v was fixed. Suppose u is K-finite. If $k_1 \in K_1$ then

$$\Phi\big(mk_1^{-1}, \pi(k_1k)u, v\big) = \Phi\big(m, \pi(k)u, v\big).$$

We define φ_u in $\mathfrak{U}$ by

$$\varphi_u(\overline{n}mk) : m_1 \to \pi_{\delta_1}(m_1)\Phi\big(m_1m, \pi(k)u, v\big). \tag{4.12}$$

The map $u \to \varphi_u$ is by our choice of v, an injection; it clearly commutes with the action of K. To verify that it commutes with the action of $\mathfrak{A}$ we have only to check that

$$\varphi_{\pi(X)u}(1) = \Big(I_r^{\overline{P}}(X)\varphi_u\Big)(1).$$

Set $\varphi_u = \varphi$ and $\varphi_{\pi(X)u} = \varphi'$. Then $\varphi(m) = \Phi(m; u, v)$ and $\varphi'(m) = \Phi\big(m, \pi(X)u, v\big)$. Recall that if X_0 is defined as in (4.6) and equals

$$\sum 1 \otimes Y_i \otimes Z_i$$

then

$$\varphi'(m) = \sum Y_i\varphi_i(m).$$

On the right Y_i is applied to a function of m and

$$\varphi_i(m) = \Phi\big(m, \pi(Z_i)u, v\big).$$

X_0 was so chosen that

$$X - \sum Y_iZ_i \in \overline{n}\mathfrak{A}$$

It is clear that if $Y \in \overline{\mathfrak{n}}$ and ψ is K-finite in $\mathfrak{U}$ then

$$I_r^{\overline{P}}(Y)\psi(1) = 0.$$

Thus

$$I_r^{\overline{P}}(X)\varphi(1) = \sum I_r^{\overline{P}}(Y_i)I_r^{\overline{P}}(Z_i)\varphi(1) = \sum I_r^{\overline{P}}(Y_i)\varphi_{\pi(Z_i)u}(1).$$

A close examination of the definition (4.12) shows that $I_r^{\overline{P}}(Y_i)\varphi_{\pi(Z_i)u}(1)$ is the function $m \to Y_i\varphi_i(m)$.

There must be an irreducible constituent ρ of the representation r on $\mathfrak{V}$ such that π is infinitesimally equivalent to a subrepresentation of $I_\rho^{\overline{P}}$. This ρ is the representation figuring in Lemma 4.2, which we are still in the process of proving. We must show that ρ is essentially tempered. Accepting this for the moment, we show that π is infinitesimally equivalent to the representation J_ρ^P.

An easy computation (for a special case, see Chapter 5 of [16]) shows that I_ρ^P and $I_\rho^{\overline{P}}$ have the same character and therefore the same irreducible constituents.

Let ρ act on W. J_ρ^P was introduced as the representation on the quotient $I_0(W)/I_1(W)$. All we have to do is verify that π cannot be a constituent of the restriction of I_ρ^P to $I_1(W)$.

The $\lambda = \lambda(\rho)$ that figures in Lemma 3.8 is λ^0. If π is a constituent of the restriction of I_ρ^P to $I_1(W)$ then, by Lemma 3.12,

$$\big\langle \pi(am)u, v\big\rangle = o\big(\delta_P^{-1}(a)\pi_{\lambda_0}(a)\big) \tag{4.13}$$

if m is fixed in $M(\mathbf{R})$ and $a \to \infty$ in $A^+(P)$. However, Theorem 3 of [5] assures us that the expansion (4.3) converges decently for fixed a_2 (cf. [16]), Appendix 3). We conclude from (4.13) and Lemma A.3.2.3 of [16] that the terms of (4.3) with $\operatorname{Re}\lambda_1 = \lambda^0$. This certainly contradicts the choice of λ^0.

We apply Lemma A.3.2.3 in the following manner. Choose $\lambda_1^0 \in L_1(\pi, \mathfrak{X})$ with $\operatorname{Re}\lambda_1^0 = \lambda^0$. Let a_2 be fixed. If $a_1 = e^{H_1}$ lies in A_1 then $\delta_P(a_1) = e^{\langle\delta, H_1\rangle}$. Thus

$$\sum_{\lambda_1 \in L_1(\pi,\mathfrak{X})} \phi_{\lambda_1}(H_1, a_2) e^{\langle \lambda_1 - \lambda_1^0, H_1\rangle} = o(1)$$

as $a_1 \to \infty$ in $A_1^+ = A^+(P)$. If $\epsilon > 0$ we can choose $R > 0$ and a finite subset S of $L_1(\pi, \mathfrak{X})$ so that if $\langle\alpha, H_1\rangle \geqslant R + \epsilon B(H_1, H_1)$ when α is a root of A_1 in $\mathfrak{n}$ then

$$\left| \sum_{\lambda_1 \in L_1(\pi,\mathfrak{X})} \phi_{\lambda_1}(H_1, a_2) e^{\langle \lambda_1 - \lambda_1^0, H_1\rangle} \right| \leqslant \epsilon$$

and

$$\left| \sum_{\lambda_1 \notin S} \phi_{\lambda_1}(H_1, a_2) e^{\langle \lambda_1 - \lambda_1^0, H_1\rangle} \right| \leqslant \epsilon.$$

Then

$$\left| \sum_{\lambda_1 \in S} \phi_{\lambda_1}(H_1, a_2) e^{\langle \lambda_1 - \lambda_1^0, H_1\rangle} \right| \leqslant 2\epsilon.$$

Lemma A.3.2.3 then implies that

$$\left| \phi_{\lambda_1^0}(H_1, a_2) \right| \leqslant 2\epsilon$$

for all H_1. Since ϵ is arbitrary $\varphi_{\lambda_1^0}(H_1, a_2) = 0$.

It remains to show that ρ is essentially tempered. Any K_1-finite linear form on $\mathfrak{V}$ is a linear combination of the functionals

$$\theta \to \theta(m_1, \varphi_1, \varphi_2)$$

where $m_1 \in M(\mathbf{R})$ and φ_1, φ_2 are continuous functions on K_1. Thus

$$\left| f\big(r(m)\theta\big) \right| \leqslant c\big(1 + \ell(m)\big)^{d-1} \pi_{\lambda_0}(m) \Xi_M(m).$$

A similar inequality is valid for the representation ρ. Set $\rho' = \pi_{\lambda^0}^{-1} \otimes \rho$. If $w \in W$ is K_1-finite and f is a K_1-finite linear form on W, an inequality

$$\left| f\big(\rho'(m)w\big) \right| \leqslant c\big(1 + \ell(m)\big)^{d-1} \Xi_M(m)$$

is satisfied.

To finish up we have only to prove the following lemma, in which we replace M by G and ρ by π in order to allow the symbols ρ, M, and P to take on a new meaning.

Lemma 4.10. *Suppose that π and π^* are quasi-simple irreducible representations of $G(\mathbf{R})$ on the Banach spaces V and V^* and that there is a nontrivial $G(\mathbf{R})$-invariant bilinear pairing*

$(u, v) \to \langle u, v \rangle$ of $V \times V^*$ into $\mathbf{C}$. Suppose there is an integer d such that for every K-finite u and v an inequality

$$\left|\langle \pi(g)u, v \rangle\right| \leqslant c\big(1 + \ell(g)\big)^d \Xi_G(g)$$

is satisfied. Then there is a parabolic subgroup P of G over $\mathbf{R}$ and a unitary representation ρ of $M(\mathbf{R})$, square-integrable modulo the center, such that π is a constituent of I_ρ^P.

We start from the expansion (4.1) and show that if $\lambda_0 \in L(\pi, \mathfrak{X})$ then $\operatorname{Re} B(\beta_i, \lambda_0) \leqslant 0$ for all i. If not, there is a linear combination $\beta = \sum b_i \beta_i$ with positive coefficients such that $\operatorname{Re} B(\beta, \lambda_0) > 0$. Choose H_0 in the Lie algebra of A so that $\langle \lambda, H_0 \rangle = B(\beta, \lambda)$ for all λ. Then e^{H_0} lies in the interior of A^+. Taking Lemma 3.6 and the assumption of the lemma into account, we see that for H in a small neighborhood of H_0

$$\sum_{\lambda \in L(\pi, \mathfrak{X})} p_\lambda(tH) e^{t\langle \lambda - \lambda_0, H \rangle} = o(1)$$

as $t \to \infty$. Applying Lemma A.3.2.3 as before we conclude that $p_{\lambda_0}(H) = 0$, a contradiction.

Let

$$E(\lambda) = \big\{\, i \;\big|\; \operatorname{Re} B(\beta_i, \lambda) = 0 \,\big\}.$$

Let E be maximal in the collection of $E(\lambda)$. P will be defined by demanding that $P \supseteq P_0$, a fixed parabolic subgroup minimal over $\mathbf{R}$, and that $D(P)$ be spanned by D_0 and $\{\, \beta_i \mid i \in E \,\}$.

This decided, we turn to the expansion (4.3). There is at least one λ_1^0 in $L_1(\pi, \mathfrak{X})$ with $\operatorname{Re} \lambda_1^0 = 0$. We fix it and define the function $\Phi(m)$ as before. If $\lambda_1 \in L_1(\pi, \mathfrak{X})$ then $\operatorname{Re} B(\beta_i, \lambda_1) \leqslant 0$ for $i \in E$. This allows us to argue as before and to show that the new Φ satisfies (4.8).

It satisfies a much improved form of (4.10). If $\lambda \in L(\pi, \mathfrak{X})$ has projection λ_1 in $L_1(\pi, \mathfrak{X})$ and $\operatorname{Re} \lambda_1 = 0$ then, by the maximality of E, $B(\beta_i, \lambda) < 0$ for $i \notin E$. Since the set $E(\pi)$ is finite there is a $\mu \in D(P_0)$ such that $B(\beta_i, \mu) = 0$ for $i \in E$ and $B(\beta_i, \mu) < 0$ for $i \notin E$ and such that

$$B(\beta_i, \mu) \geqslant \operatorname{Re} B(\beta_i, \lambda)$$

if $\lambda \in L(\pi, \mathfrak{X})$ and the real part of the projection of λ on $D(P) \otimes \mathbf{C}$ is zero. Theorem 4 of [5] implies that there are an integer d and a constant c such that

$$\pi_{\delta_1}(a) \big\|\Phi(a)\big\| \leqslant c\big(1 + \ell(a)\big)^d \Xi_M(a) e^{\langle u, H \rangle} \tag{4.14}$$

for $a = e^H$ in $A^+(P_0')$, where $P_0' = P_0 \cap M$. Using this inequality instead of (4.10) we proceed as before to define ρ. π is then a constituent of I_ρ^P. Since it follows easily from (4.14) that ρ is square-integrable modulo the center, the lemma is proved.

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