Comments on the previous text

I add here a few words, whose intention is not to rewrite the text in English for the benefit of those who cannot read Russian. It is rather to make clear in a brief space and in a language, my mother tongue English, unfortunately in an excessively widespread use at present – *auf die Straße gegangen* so to say – my notion of the final form the geometric theory for reductive groups over non-singular closed complex curves will take. The paper itself is concerned with $GL(2)$ over an elliptic curve, but it offers clear insights into the nature of the problem. I shall describe these as briefly and as clearly as I can because I think this problem is ripe for a general, although not immediate, solution.

Before we address the mathematics it is best to comment briefly on the unfortunate text *Is there an analytic theory of automorphic functions for complex algebraic curves,* cited as [FAF], of Frenkel. As is implicit in the reference to [F] (such references are to footnotes in my article Об аналитическом виде геометрической теории автоморфных форм, cited as [LGT]) I was introduced to the geometric theory by that paper, which I appreciated and which I read with some care. I was however not fully satisfied by the discussion of that paper. There was no proof of the existence of the Hecke *eigensheaves* defined there. Indeed, although there may be some value in the introduction of sheaves into a geometric theory, it is, I very strongly believe, first necessary to construct Hecke eigenfunctions. In the context of functional analysis and hermitian operators such eigenfunctions exist as a consequence of standard theories, such as that expounded by M. H. Stone in his book, *Linear transformations in Hilbert space and their applications to analysis* a book that I read with care as I was attempting to understand Eisenstein series many years ago. So existence is no problem. These theories are unfortunately often ignored by fans of sheaf cohomology, which certainly has its own place within mathematics, but perhaps not in the basic geometric theory of automorphic forms. I am not certain.

Frenkel was offered more than one possibility, namely two, one in Oslo, one in Minneapolis, to comment on my Russian article cited above. On neither occasion did he show any sign of having read it in a serious manner or understood its purposes. This is clear from the condescending introduction "Langlands envisioned ... certain Hecke operators, which he attempted to define" to the text cited above, the purpose of whose title, the English translation of my Russian title is difficult to fathom. I was reluctant to read further after this remark, a curious expression of disrespect not common in the Anglo-Saxon world, but, so far as I can ascertain, Frenkel's difficulty is that he cannot envision the possibility of two(!) geometric theories, a theory in the style of Hecke and a sheaf-theoretic theory. If, however, there are two, the problem arises of understanding their relation. I have not attempted to do so. This might very well be quite interesting. Frenkel could, if he had understood my paper, have attempted this in his Minneapolis lecture.

The point is that the theory I proposed and developed for elliptic curves is not the geometric theory of [F], thus the theory I refer to as the Russian theory. It is, rather, a theory in which Hecke operators are acting not on sheaves but on Hilbert spaces, so that their eigenvalues are defined. The Russian theory may very well have merits, but it is not a theory that can be regarded as an analogue of the arithmetic theory, or potential arithmetic theory, referred to with my name, thus it cannot be regarded as a 'geometric Langlands program'. There are no Hecke eigenvalues available. Thus even if the theory expounded in [F] has merits of its own, from a classical point of view thus from my point of view, it is not

what is wanted. I would be grateful if it could be referred to in a different manner, say 'A sheaf-theoretic geometric theory.' There will be no further reference to it below, although I would welcome a careful, conscientious study of the relation between the two different geometric theories, both of which are presently inadequate. I shall, rather, briefly indicate below what my paper, which is devoted to a specific group over a specific kind of curve, offers as a possible 'geometric Langlands theory.'

The general theory will be very difficult, but it is promising and it would be unfortunate if young mathematicians who might develop it were discouraged from developing it by ignorant but well-publicized remarks. The specific theory of my paper exploits two difficult texts: the first, the paper [A] of Atiyah for the group $GL(n)$ over an elliptic curve, is, so far as I know, very little read; the second, the paper [AB] of Atiyah-Bott, is I believe widely read. Atiyah's paper is a kind of geometric reduction theory, a description of all G-bundles, for a specific reductive group over a given closed non-singular curve. One task, a first task, of the general theory is the description of all such spaces, thus a general geometric reduction theory. If I were a good many years younger, I would find it an appealing task. It is relatively concrete. At my present age it would be fruitless. There is too much to be done.

The underlying space once understood, the second task is to understand the eigenfunctions and eigenvalues of the Hecke operators, thus in particular to define the Hecke operators. Their appropriate definition, as I repeated more than once, sometimes with great emphasis, in my Russian article, is brutal. It cannot be avoided and yet there is little reason - at first glance! - to expect it to be viable. In my article, I introduced the definition only with great reluctance and with little hope that it would be appropriate. It was, however, with great joy, expressed at the end of the account [LGT] that I began to see the justness of my crude and brutal determination of the theory (for $GL(2)$ over an elliptic curve) appearing. I tried twice to explain to Frenkel in correspondence that great attention was to be paid to the definitions of [LGT] but this advice fell on deaf ears, perhaps better, blind eyes.

The theory of [LGT], for – as I have stressed – the specific case of $GL(2)$ over an elliptic curve, refers constantly both to the paper [A] and to the paper [AB]. The paper [A] is perhaps the more difficult of the two but the paper [AB] is the richer. In particular, it makes the role of the L-group, only implicitly, and of the Yang-Mills equation in the geometric theory evident. Evident is perhaps the wrong word, but the burst of glee toward the end of [LGT] is the result of a long, careful examination in that paper of the theory of [AB] together with careful calculation of the Hecke eigenfunctions and eigenvalues, defined as in [LGT]. I repeat that these eigenfunctions and eigenvalues do not exist in the Russian theory, where we are dealing with sheaves. As I indicated above there is no reason, if one is so inclined, not to search for the relation between the Russian theory, in so far as it is more than supposition, and the theory developed in [LGT], although that is only for a specific group, but which, one hopes, can be developed in general. I, myself, would not expect any help from [FAF].

There was one surprising conclusion or conjecture resulting from [LGT] and this was that, such fine points as L-indistinguishability ignored, the automorphic forms associated to a group G and a base curve X are represented by the homomorphisms of a group, the galoisian group, into the L-group ${}^LG$. This is discussed in [LGT] but only for the specific group $GL(2)$ and only over an elliptic curve. It is otherwise conjectural. It is also closely related to the material in [AB].

I came to this conclusion when preparing, several months after [LGT] was finished, for a lecture *Otomorfik formların bağımsız bir teorisi var mı?* given in Istanbul much after the

paper [LGT] was written. The text for the lecture, posted on this site with this text, has to be rewritten. As posted it is the result of a hurried preparation in a foreign language of ideas that arose during the preparation. In any case, the conclusion reveals clearly the difference between the geometric theory and the arithmetic theory. The first seems to be described by the representations of a group, the galoisian group. The second by a Tannakian category, but here we are introducing matters beyond my ken, both the general notion and its specific instances. I hope to undertake a revision of the text just cited. As the lecture was prepared and given in Turkish, it will be accessible to an even smaller audience than the paper, but that is not important. What is? It is the principle that the Tannakian category – hypothetical – associated to automorphic forms is the category of representations of a group, the galoisian group, in the geometric theory, but not in the arithmetic theory.

The theory developed in the paper under discussion presumably has a general form. Before commenting on the efforts necessary, I repeat that there is no reason that there will not be two geometric theories: (i) a theory in the style of Hecke and in the style of my paper; (ii) a sheaf-theoretic theory, as discussed in [F]. My concern is with the first. I would also repeat that the discussion in [LGT] relies heavily on the concepts introduced or developed in [AB]. In particular, a form of the galoisian group appears there.

The development for general reductive groups over general algebraic curves is, in whole or in part, a challenging and attractive task for young mathematicians, but a difficult one and it would be a shame if they were discouraged by ignorant essays.

The first, very challenging task would be the development for general curves and general groups of the paper of Atiyah, perhaps one of his best. As I have already indicated, it is difficult and little read. It is also very concrete. It would be a shame to discourage youthful efforts to master the methods and complete, for general reductive groups and general compact algebraic curves, its very concrete and very important conclusions. It would also be a shame to discourage an understanding of the relations between the Yang-Mills equations of [AB], the parametrization of their solutions by the representations of a group, essentially a galoisian group, and its relation to the Hecke eigenfunctions, thus to the geometric theory in the form I have introduced it. A supplementary investigation could then be devoted to the relation with the geometric theory in the Russian sense.