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# COMPLEX DIFFERENTIAL AND INTEGRAL GEOMETRY AND CURVATURE INTEGRALS ASSOCIATED TO SINGULARITIES OF COMPLEX ANALYTIC VARIETIES

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## Introduction

This paper closely parallels the Gergen lectures which I was fortunate to have the opportunity to deliver at Duke University in March, 1978. My goal in those talks was to discuss the relationship between curvature and the singularities of complex-analytic varieties which was initiated in the recent work of Linda Ness[(1)], N. A'Campo, and Rémi Langevin[(2)], and which is somewhat broadened and extended here. Since these lectures were intended for a general audience, much of the first two sections of the paper is expository, covering known material but from a viewpoint intended to serve later needs. Indeed, the discussion of real differential and integral geometry is not logically necessary for the eventual results on curvature and singularities, but by thinking through this more intuitive and familiar material the extension to the complex case should appear natural, and moreover should permit deeper appreciation of the special features peculiar to Hermitian differential and integral geometry. In the remainder of this introduction a somewhat more detailed account of the contents of this paper will be given.

In the first section there is a discussion of Hermann Weyl's formula for the volume of the tube $\tau_r(M)$ of radius $r$ around an oriented $n$-manifold $M$ in $\mathbb{R}^N$. The result is

$$\text{vol } \tau_r(M) = \sum_{k=0}^{n} C(n, m, k)\, \mu_k(M) r^{m+k} \tag{0.1}$$

where $m = N - n$, the $C(n, m, k)$ are universal constants, and the

$$\mu_k(M) = C(k, n) \int_M I_k(R_M) dM$$

are integrals over $M$ of scalar invariants $I_k(R_M)$ constructed from the Riemann curvature tensor $R_M$. The point is that the right-hand side of (0.1) is an intrinsic invariant of the induced Riemannian metric and does not depend on the particular embedding $M \subset \mathbb{R}^N$. At one extreme $\mu_0(M) = \text{vol }(M)$ (as expected), while at the other extreme $I_n(R_M)$ is zero for $n$ odd and for $n$ even it is the integrand $K\, dM$ in the general Gauss-Bonnet theorem

$$C^{\underline{te}} \int_M K\, dM = \chi(M) \tag{0.2}$$

for a closed manifold.[(3)] In fact, it was by using tubes (their boundary, to be precise) that the general Gauss-Bonnet theorem was first deduced from the formula (0.2) for oriented hypersurfaces, where $K$ is then the Gauss-Kronecker curvature. We have recounted this development in §1b, partly because it is of historical interest and illustrates the observation that in geometry intrinsic formulas are frequently arrived at by extrinsic considerations, and partly because the Gauss-Bonnet is certainly the most important curvature integral. In §1c we show how, via the Gauss mapping, this extrinsic proof ties in with the more modern version using characteristic classes. The first section concludes with the observation that, among the coefficients $\mu_k(M)$, only the top one is topologically invariant.

Perhaps it should be remarked that the motivation for beginning these lectures with Weyl's formula is first of all due to the reproductive property (0.4) below of the coefficients $\mu_k(M)$ whose complex analogue will play an essential role in our study of singularities, and secondly because the Gauss-Bonnet integrand is the first one to be investigated near a singularity.

Next we turn to integral geometry for manifolds in $\mathbb{R}^N$. The starting point is Crofton's formula

$$\int n(C \cap L)dL = 2l(C) \tag{0.3}$$

expressing the length $l(C)$ of a (piecewise smooth) curve $C$ in $\mathbb{R}^2$ as the average of the number $n(C \cap L)$ of intersections of the curve with a line. The proof is given in a setting so that the generalizations will involve no essential new concept. Technically the argument is facilitated by the use of moving frames, both because they are geometrically natural and because by using them it is easy to recognize the invariant density $dL$ as being a constant expression in the differential coefficients of a moving frame.[4] In §2b we apply Crofton's formula to prove a result of Fenchel-Fary-Milnor concerning the total curvature of closed curves in $\mathbb{R}^3$. Because of the relationship between knots in $\mathbb{R}^3$ and isolated singularities of complex-analytic curves in $\mathbb{C}^2$ [5] this may be considered as our first result about curvature and singularities. Finally, in §2c we discuss a special case of Chern's kinematic formula[6]; this result states that the coefficients $\mu_k(M)$ in Weyl's tube formula have the reproductive property

$$\int \mu_k(M \cap L)dL = C^{\underline{te}}\, \mu_k(M) \tag{0.4}$$

where $L$ varies over the affine linear spaces of dimensions $(N + k - n)$ in $\mathbb{R}^N$. In case $M$ is a closed manifold we may use the Gauss-Bonnet formula (0.2) together with (0.4) to interpret the curvature integral

$$\mu_k(M) = \int_M I_k(R_M)dM$$

as (a constant times) the average Euler characteristic of intersections $M \cap L$. Alternatively, we may view (0.4) as expressing the average over linear spaces $L$ of the curvature integrals

$$\mu_k(M \cap L) = \int_{M \cap L} I_k(R_{M \cap L})$$

as an integral in the curvature of $M$ (the point is that it is *not* the case that $R_{M \cap L} = R_{M|M \cap L}$). It is this viewpoint will turn out to be most relevant to our study of singularities.

Next we turn to the complex analogue of Weyl's formula. Section 3 begins with a discussion of the Hermitian differential geometry of a complex manifold $M \subset \mathbb{C}^N$, with special emphasis being placed on the positivity and non-degeneracy of the basic Chern forms $c_k(\Omega_M)$ in the curvature $\Omega_M$ of $M$—cf. (3.7) and (3.11). Then there follows some remarks on integration over complex-analytic varieties. The point of this discussion is that we shall be interested in curvature integrals near singularities, and these are not automatically convergent as are those arising from the restriction of smooth forms in the ambient space, so it is perhaps worthwhile to get some preliminary feeling for singular integrals on singular varieties.

After these preliminaries we come to the volume of tube formula. The formula, which is derived as in the real case, has been in the literature for some time,[7] but it does not seem to have been noticed that the coefficients have the form

$$\mu_k(M) = C^{\underline{te}} \int_M c_k(\Omega_M) \wedge \phi^{n-k} \tag{0.5}$$

where $\phi$ is the Kähler form on $\mathbb{C}^N$. As proved in the reference given in footnote (7), a similar result holds for a compact complex manifold in $\mathbb{P}^N$, and in this case all the coefficients (0.5) are of a topological character, in sharp contrast to a closed real manifold in $\mathbb{R}^N$ or $S^N$. For $k = 0$, by the Wirtinger theorem

$$\mu_0(M) = C^{\underline{te}} \operatorname{vol}(M),$$

and so one way of interpreting (0.5) is an extension of the Wirtinger theorem to volumes of tubes. Additionally, the observation that the integrands $\phi^{n-k} \wedge c_k(\Omega_M)$ have definite signs and are pullbacks to $M$ of forms on $\mathbb{C}^N \times G(n, N)$ under the holomorphic Gauss mapping

$$z \to (z, T_z(M))$$

enables us to prove that the integrals (0.5) are absolutely convergent near a singularity of a complex-analytic variety and therefore to extend the tube formula to this case. Section 3 concludes with an observation concerning the growth properties of the $\mu_k(M)$ for an entire analytic set in $\mathbb{C}^N$.

Section 4 is devoted to complex integral geometry. Since intersection numbers of complex-analytic varieties meeting in isolated points are always positive, the complex analogue of Crofton's formula (0.3) has a topological character. Once this is properly formulated, the generalization (4.9) to intersections of an analytic variety in the Grassmanian with a variable Schubert cycle is formal. Applying this result to the Gaussian image of a complex manifold $M \subset \mathbb{C}^N$ gives the formula

$$(-1)^n C^{\underline{te}} \int_M K dM = \int_{\mathbb{P}^{N-1*}} n(M, H) dH \tag{0.6}$$

where $n(M, H)$ is the number of points (counted with multiplicities) $z \in M$ such that the tangent plane $T_z(M)$ lies in $z + H$ for a hyperplane $H$. This application of integral geometry to eliminate the boundary integral in the usual Gauss-Bonnet for open manifolds illustrates one of our guiding principles: The global enumerative or projective formulas of algebraic geometry remain true locally *on the average*. Thus, e.g., one complex analogue of (0.3) says that for an analytic curve $C$ in $\mathbb{P}^2$ the average number of intersections of $C$ with a line is equal to the area of $C$ in the Fubini-Study metric. Another illustration of this principle is the complex analogue

$$\int \left( \int_{M \cap L} c_{n-k}(\Omega_{M \cap L}) \right) dL = C^{\underline{te}} \int_M c_{n-k}(\Omega_M) \wedge \phi^k \tag{0.7}$$

of the kinematic formula (0.4). The reason for (0.7) is Chern's formula (0.4) together with the appearance (0.5) of the Chern forms in the volume of tube formula. However, the proof of (0.7) given in §4c uses Wirtinger-type principles peculiar to the complex case and is therefore simpler than the proof of (0.4). For later applications we need a variant of (0.7) when the linear spaces $L$ are constrained to pass through the origin, and this topic completes §4.

Finally, in section 5 we turn to the study of the curvature of complex-analytic varieties. The basic observations are (i) that if $V$ is an analytic variety, then even though the curvature (e.g., the Gauss-Kronecker curvature) at the smooth points may tend to $\pm\infty$ as $z$ tends to a singular point, suitable curvature integrals $\int_V P(\Omega_V)$ will converge; and (ii) that if $\{V_t\}$ $(0 < |t| < \delta)$ is a family of complex manifolds approaching an analytic variety $V_0$, then

$$\lim_{t \to 0} \int_{V_t} P(\Omega_{V_t}) \neq \int_{V_0} P(\Omega_{V_0}); \tag{0.8}$$

i.e., the limit of the curvature is not equal to the curvature of limit, in contrast to the behaviour of volume. Following some heurestic discussion in §5a, our main general result (5.11) computes the difference of the two sides of (0.8) as an intersection number of a subvariety $\Delta$ of the Grassmannian with a Schubert cycle. In §5b this cycle $\Delta$, which we call the *Plücker defect* associated to the family $\{V_t\}$, is shown to exist and the formula (5.11) derived.

It remains to interpret the intersection number in special cases, and for this we sketch a proof of a result (5.15) of Tessier about the Milnor numbers $\mu^{(i)}$ associated to an isolated hypersurface singularity. This enables us to use (5.11) when $P(\Omega) = c_n(\Omega) = (-1)^n C^{\underline{te}} KdA$ is (a constant times) the Gauss-Bonnet integrand, and then we deduce Langevin's formula (cf. footnote (2))

$$\lim_{\epsilon \to 0} \lim_{t \to 0} (-1)^n C^{\underline{te}} \int_{V_t[\epsilon]} KdA = \{\mu^{(n+1)} + \mu^{(n)}\} \tag{0.9}$$

where $V_t[\epsilon] = \{z \in V_t : \|z\| \leq \epsilon\}$, which is the result initially arousing our interest in the subject. In §5c we apply the kinematic formula (0.7) to extend (0.9) to higher codimension, thereby giving curvature formulas for all sums $\{\mu^{(k+1)} + \mu^{(k)}\}$. Adding these up with alternating signs enables us to isolate the top Milnor number $\mu^{(n+1)}(V_0)$, a topological invariant of the isolated singularity, and arrive at the formula

$$\text{(0.10)}\quad \mu^{(n+1)}(V_0) = \lim_{\epsilon \to 0} \lim_{t \to 0} \left[ \sum_{k=0}^{n} \frac{(-1)^k C(k, n)}{\epsilon^{2k}} \int_{V_t[\epsilon]} c_{n-k}(\Omega_{V_t}) \wedge \phi^k \right] + (-1)^{n-1}.$$

Now the general manner in which we derived (0.10) suggests many other numerical characters which may be associated either to singular varieties or to a family of complex manifolds tending to a singular limit. The numbers which are essentially new—i.e., are not expressible in terms of classical Milnor numbers—are associated to an isolated singularity in codimension $\geqq 2$, and arise using either the 1st order tangential structure as for an isolated surface singularity in $\mathbb{C}^4$, on the higher order osculating structure as for an isolated curve singularity in $\mathbb{C}^3$. We don't know if any among these has topological meaning, or if any may be used to detect the impossibility of smoothing, or indeed any futher non-trivial applications, but it seems pretty clear that additional work in the area is possible, and so in §5d we conclude with a few examples analogous to (0.9), general observations, and open questions.

**Footnotes**

1. L. Ness, *Curvature of algebraic plane curves*, I. Compositio Math., vol. 35 (1977), pp. 57–63.
2. R. Langevin, *Courbure et singularities complexes*, to appear in Comm. Math. Helv.
3. Throughout this paper $C^{\underline{te}}$ will denote a suitable positive constant depending only on dimensions and not on the manifold in question.
4. In fact, all computations are made in the setting of moving frames, so that the present work may be considered as a continuation of our previous expository paper in this journal (vol. 41 (1974), pp. 775–814). There we commented that the general theory of frames associated to higher order jets was complicated by the presence of singularities arising from "inflectionary behaviour". In this paper we will be eventually interested in singularities occurring already at the first order, so that its general character may be said to be the application of frames to the metric study of complex-analytic singularities.
5. J. Milnor, *Singular points of complex hypersurfaces*, Annals of Math Studies #61, Princeton Univ. Press, Chapter 10.
6. S. S. Chern, *On the kinematic formula in integral geometry*, J. of Math. and Mech., vol 16 (1966), pp. 101–118.

Our discussion of integral geometry has been much influenced by unpublished course notes by Chern from a class taught at Berkeley in the spring of 1965.

7. F. J. Flaherty, *The Volume of a Tube in Complex Projective Space*, Ill. Jour. Math., vol. 16 (1972), pp. 627–638.

## 1. Hermann Weyl's formula for the volume of tubes

(a) *Frames and derivation of the formula.* We will discuss a formula of H. Weyl[(1)] for the volume of the tube of radius $r$ around an $n$-dimensional manifold in $\mathbb{R}^N$. For this we shall use moving frames, and shall recall briefly their definition and structure equations (for more details cf. the paper quoted in footnote (4) of the introduction).

In $\mathbb{R}^N$ with orientation $dx_1 \wedge \cdots \wedge dx_N > 0$, an oriented frame is given by $\{x; e_1, \cdots, e_N\}$ where $x \in R^N$ is a position vector and $e_1, \cdots, e_N$ form an oriented orthonormal basis. The set of these frames constitutes the frame manifold $\mathscr{F}(\mathbb{R}^N)$, which upon choice of a reference frame may be identified with the group $E(N)$ of proper Euclidean motions with $x$ corresponding to the translation part and $e_1, \cdots, e_N$ to the rotation part of such a motion. Considering $x$ and the $e_i$ as maps from $\mathscr{F}(\mathbb{R}^N)$ to $\mathbb{R}^N$ and writing their differentials as

$$(1.1) \qquad \begin{aligned} dx &= \sum_i \omega_i e_i \\ de_i &= \sum_j \omega_{ij} e_j, \qquad \omega_{ij} + \omega_{ji} = 0 \end{aligned}$$

defines

$$N + \frac{N(N-1)}{2} = \frac{N(N+1)}{2}$$

independent $l$-forms $\omega_i$ and $\omega_{ij}$ on $\mathscr{F}(\mathbb{R}^N)$ which under the above identification are just the invariant Maurer-Cartan forms on $E(N)$. Taking exterior derivatives in (1.1) gives the Maurer-Cartan equations

$$(1.2) \qquad \begin{aligned} d\omega_i &= \sum_j \omega_j \wedge \omega_{ji} \\ d\omega_{ij} &= \sum_k \omega_{ik} \wedge \omega_{kj}. \end{aligned}$$

Now let $M_n \subset \mathbb{R}^N$ be a connected smooth manifold. Since we are primarily interested in local questions it is convenient to assume $M$ to be oriented and to have a smooth boundary $\partial M$. Letting $m = N - n$ be the codimension of $M$ we shall use throughout this paper the index ranges

$$1 \leq i, j, k \leq N; \; 1 \leq \alpha, \beta \leq n; \; n + 1 \leq \mu, \nu \leq N.$$

Associated to $M$ is the submanifold $\mathscr{F}(M) \subset \mathscr{F}(\mathbb{R}^N)$ of Darboux frames defined by the conditions: $x \in M$; $e_1, \cdots, e_n$ form an oriented basis for the tangent space $T_x(M)$; and $e_{n+1}, \cdots, e_N$ form an oriented basis for the normal space $N_x(M)$. Since $dx$ is tangent to $M$, the first equations in (1.1) and (1.2) become

$$(1.3) \qquad \begin{aligned} dx &= \sum_\alpha \omega_\alpha e_\alpha, \qquad \text{i.e., } \omega_\mu = 0 \\ d\omega_\alpha &= \sum_\alpha \omega_\beta \wedge \omega_{\beta\alpha}. \end{aligned}$$

It follows that

$$I = (dx, dx) = \sum_\alpha \omega_\alpha^2$$

is the (pullback to $\mathscr{F}(M)$ of the) first fundamental form of $M$, and that $\{\omega_{\alpha\beta}\}$ is the connection matrix for the associated Riemannian connection.

By the first relation in (1.3) we may think of $\mathscr{F}(M)$ as an integral manifold of the differential system[(2)]

$$\omega_\mu = 0, \qquad d\omega_\mu = 0$$

on $\mathscr{F}(\mathbb{R}^N)$. By (1.2), the second of these equations is

$$0 = \sum_\alpha \omega_\alpha \wedge \omega_{\alpha\mu},$$

which by the well-known Cartan lemma implies that

$$\omega_{\alpha\mu} = \sum_\beta h_{\alpha\beta\mu}\omega_\beta, \qquad h_{\alpha\beta\mu} = h_{\beta\alpha\mu}. \tag{1.4}$$

The linear system of quadratic forms

$$\mathrm{II} = \sum h_{\alpha\beta\mu}\omega_\alpha\omega_\beta \otimes e_\mu$$

is called the second fundamental form of $M$ in $\mathbb{R}^N$; for each unit normal $\xi = \sum_\mu \xi_\mu e_\mu$

$$(\mathrm{II}, \zeta) = \sum h_{\alpha\beta\mu}\omega_\alpha\omega_\beta\xi_\mu$$
$$= (d^2x, \xi)^{(3)}$$

gives the usual second fundamental form of the projection of $M$ into the $\mathbb{R}^{n+1}$ spanned by $T_x(M)$ and $\xi$. By the Cartan structure equation, the curvature matrix $\Omega_M = \{\Omega_{\alpha\beta}\}$ for $M$ is given by

$$\Omega_{\alpha\beta} = d\omega_{\alpha\beta} - \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta}$$

$$= \sum_\mu \omega_{\alpha\mu} \wedge \omega_{\mu\beta}$$

by (1.2)

$$= -\sum h_{\alpha\gamma\mu}h_{\beta\delta\mu}\omega_\gamma \wedge \omega_\delta$$

by (1.4). Setting

$$\Omega_{\alpha\beta} = \frac{1}{2} \sum R_{\alpha\beta\gamma\delta}\omega_\gamma \wedge \omega_\delta, \qquad R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma}$$

it follows that the components of the Riemann curvature tensor $R_M$ are given in terms of the 2nd fundamental form by

$$R_{\alpha\beta\gamma\delta} = -\sum_\mu (h_{\alpha\gamma\mu}h_{\beta\delta\mu} - h_{\alpha\delta\mu}h_{\beta\gamma\mu}). \tag{1.5}$$

In case $M \subset \mathbb{R}^{N+1}$ is an oriented hypersurface, we choose $e_{n+1}$ to be the outward normal and choose our tangent frame $e_1, \cdots, e_n$ to diagonalize the second fundamental form: thus

$$\mathrm{II} = \sum k_\alpha \omega_\alpha^2$$

where the $k_\alpha$ are the principal curvatures. The Riemann curvature tensor is

$$R_{\alpha\beta\gamma\epsilon} = k_\alpha k_\beta (\delta_\alpha^\epsilon \delta_\beta^\gamma - \delta_\alpha^\gamma \delta_\beta^\epsilon). \tag{1.6}$$

Letting $dM = \omega_1 \wedge \cdots \wedge \omega_n$ be the volume form on $M$, the Gauss-Kronecker curvature is defined by

$$\omega_{1,n+1} \wedge \cdots \wedge \omega_{n,n+1} = K\, dM, \tag{1.7}$$

which using $\omega_{\alpha,n+1} = k_\alpha \omega_\alpha$ gives

$$K = k_1 \cdots k_n. \tag{1.8}$$

We now denote by $\tau_r(M)$ the tube of radius $r$ around $M$. More precisely, if $N[r]$ is the tubular neighborhood of radius $r$ around the zero section in the normal bundle, then there is an obvious map (exponential map)

$$N[r] \to R^N$$

and $\tau_r(M)$ is its image, counted with whatever multiplicities arise from the focal behaviour of the normal geodesies. To explain Weyl's formula we shall use the following notation due to Flanders[4]: Given a vector space $E$, we set

$$\wedge^{k,l}(E) = \wedge^k(E) \otimes \wedge^l(E^*)$$

$$\wedge^{*,*}(E) = \bigoplus_{k,l} \wedge^{k,l}(E)$$

and make $\wedge^{*,*}(E)$ into an associative algebra by the rule

$$(a \otimes b) \cdot (c \otimes d) = (a \wedge c) \otimes (b \wedge d).$$

The diagonal $\oplus \wedge^{k,k}(E)$ is then a *commutative* subalgebra. Taking $E = T_x(M)$ we may consider the curvature

$$R_M = \sum R_{\alpha\beta\gamma\delta} e_\alpha \wedge e_\beta \otimes \omega_\gamma \wedge \omega_\delta \in \wedge^{2,2}(T_x(M)),$$

and define the scalar invariants $I_l(R_M)$ for $l = 2k$ an even integer by

$$I_l(R_M) = \text{Trace } (\wedge^k R_M) \qquad (l = 2k)$$

where $\wedge^k R_M \in \wedge^l(T_x(M)) \otimes \wedge^l(T_x(M))^*$. In components

$$I_l(R_M) = C^{te} \sum_{A,B} \epsilon_B^A R_{\alpha_1\alpha_2\beta_1\beta_2} \cdots R_{\alpha_{l-1}\alpha_l\beta_{l-1}\beta_l} \tag{1.9}$$

where $A = (\alpha_1, \cdots, \alpha_l)$ and $B = (\beta_1, \cdots, \beta_l)$ run over index sets selected from $(1, \cdots, n)$, and

$$\epsilon^A_B = \begin{cases} 0 \text{ is } A \text{ is not a rearrangement of } B \\ \pm 1 \text{ is the sign of permutation taking } \alpha_i \text{ to } \beta_i \text{ otherwise,} \end{cases}$$

and where $C^{\underline{te}} = C(n, l)$ is a suitable constant. Setting

$$\mu_l(M) = \int_M I_l(R_M)dM$$

the formula is

$$\text{vol } \tau_r(M) = \sum_{\substack{l=0 \\ l \equiv 0(2)}}^{n} C(l, n)\mu_l(M)r^{m+l}. \tag{1.10}$$

We shall sketch Weyl's proof, deferring the detailed argument until section 3c in which the complex case will be discussed. Points in $\tau_r(M)$ are

$$y = x + \sum_{\mu} t_\mu e_\mu.$$

It follows from the structure equations (1.3) and (1.4) that

$$dy = \sum (\omega_\alpha - t_\mu \omega_{\alpha\mu})e_\alpha + \sum (dt_\mu + t_\nu \omega_{\nu\mu})e_\mu$$

with repeated indices always being summed. Letting $dV$ denote the volume element in $\mathbb{R}^N$ and $dt = dt_{n+1} \wedge \cdots \wedge dt_N$, we infer that

$$\begin{aligned} dV &= \bigwedge_\alpha \left( \sum (\delta_{\alpha\beta} - h_{\alpha\beta\mu}t_\mu)\omega_\beta \right) \bigwedge_\mu dt_\mu \\ &= \sum_l P_l(t, h)dM\, dt \end{aligned}$$

where

$$P_l(t, k) = \frac{(-1)^l}{l!} \sum_{A,B} \epsilon^A_B h_{\alpha_1\beta_1\mu_1} \cdots h_{\alpha_l\beta_l\mu_l} t_{\mu_1} \cdots t_{\mu_l}. \tag{1.11}$$

By Fubini's theorem

$$\text{vol } \tau_r(M) = \int_M \left( \sum_l \int_{\|t\| \le r} P_l(t, h)dt \right) dM. \tag{1.12}$$

To evaluate the inner integral we utilize Weyl's notation $\langle f \rangle$ for the spherical average $\int_{\|t\|=1} f(t)dt$, and also his explicit evaluation (Weyl, loc. cit., page 465)

$$\langle t_{n+1}^{l_{n+1}} \cdots t_N^{l_N} \rangle = C(m; l_{n+1}, \cdots, l_N) \tag{1.13}$$

where the constant $C(m; l_{n+1}, \cdots, l_N)$ is clearly zero unless all exponents are even. Setting $l = 2k$ and

$$P_l(h) = \langle P_l(t, h)\rangle$$

we deduce that

$$P_l(h) = (-1)^l \sum \epsilon_B^A h_{\alpha_1\beta_1\mu_1} \cdots h_{\alpha_l\beta_l\mu_l} \langle t_{\mu_1} \cdots t_{\mu_l}\rangle$$

$$= C^{\underline{te}} \sum \sum_{\mu} \epsilon_B^A h_{\alpha_1\beta_1\mu_1} h_{\alpha_2\beta_2\mu_1} \cdots h_{\alpha_{2k-1}\beta_{2k-1}\mu_k} h_{\alpha_{2k}\beta_{2k}\mu_k}$$

$$= C^{\underline{te}} \sum \epsilon_B^A R_{\alpha_1\alpha_2\beta_1\beta_2} \cdots R_{\alpha_{l-1}\alpha_l\beta_{l-1}\beta_l}$$

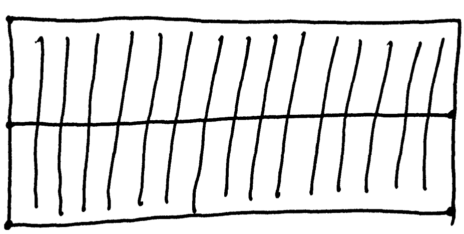
by (1.5) and a straightforward skew-symmetry argument

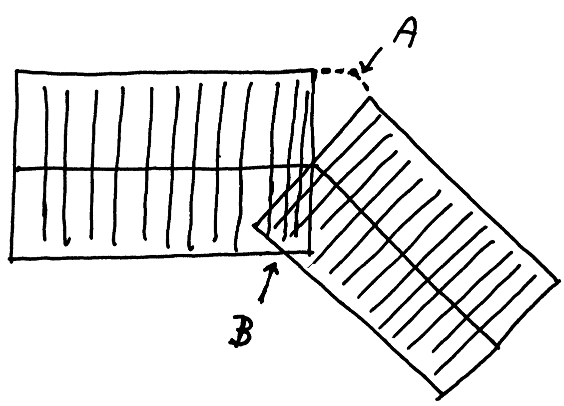
$$= C^{\underline{te}} I_l(R_M)$$

by (1.9). Aside from the explicit determination of constants (a non-trivial matter!), this implies Weyl's formula (1.10).

We remark that (1.10) may be extended to manifolds $M \subset S^N$ in spheres (Weyl, loc. cit.). We also note that the fact that $P_l(h)$ contains $h_{\alpha\beta\mu}$ only in the quantities $\sum_{\mu} h_{\alpha\beta\mu} h_{\alpha'\beta'\mu}$ may be deduced from the observation that $P_l(h)$ is invariant under substitutions $h_{\alpha\beta\mu} \to \sum_{\nu} h_{\alpha\beta\nu} g_{\nu\mu}$ for $(g_{\nu\mu})$ an arbitrary proper orthogonal matrix.

(b) *Tubes and the Gauss-Bonnet theorem*. We should like to make a few observations concerning (1.10). To begin with, as already remarked by Weyl, the first step (1.12) expressing the volume vol $\tau_r(M)$ in terms of the second fundamental form of $M$ in $R^N$ is "elementary calculus." The deeper and more interesting aspect, which we only outlined, is that the functions $P_l(h)$ are expressible in terms of the $R_{\alpha\beta\gamma\delta}$'s, and are therefore intrinsic invariants of the Riemannian metric. The simplest special case of this is that of an arc $C$ in the $(x, y)$-plane: it asserts the area of a strip of width $r$ about $C$ depends only on the length of $C$ and not on its curvature. This invariance under bending may be illustrated by considering the figures





The point is that the blank area $A$ is congruent to the doubly shaded region $B$.

A second remark concerns a pair of Riemannian manifolds $M_1$ and $M_2$ with $M_1 \times M_2$ being given the product metric. Denoting by

$$R_{M_1 + M_2} \in \wedge^{2,2}(T(M_1 \times M_2))$$

the curvature operator introduced in the formulation of (1.10), with the fairly obvious notation we have

$$R_{M_1 \times M_2} = R_{M_1} + R_{M_2}$$

which implies that

$$\wedge^l R_{M_1 \times M_2} = \sum_{p=0}^{l} \binom{l}{p} \wedge^p R_{M_1} + \wedge^{l-p} R_{M_2}.$$

Taking traces and integrating gives the functoriality property

$$\mu_l(M_1 \times M_2) = \sum_{p=0}^{l} \binom{l}{p} \mu_p(M_1) \cdot \mu_{l-p}(M_2). \tag{1.14}$$

This together with the reproductive property (0.4) possibly serve to characterize the curvature integrals $\mu_k(M)$.

The final observation is that, even if $M$ is a compact manifold without boundary, the $\mu_l(M)$ are metric but not in general topological invariants. For example, for small $r$ the dominant term in Weyl's formula is, as expected,

$$C^{\underline{te}} \text{ vol } (M) r^m.$$

On the other hand, the coefficient

$$\int_M I_n(R_M) dM, \qquad n \equiv 0(2),$$

of the highest power of $r$ does turn out to be topologically invariant. This realization was intimately connected with the discovery of the higher dimensional Gauss-Bonnet theorem, and we should like to briefly recount this development.[5] The starting point is the following theorem of H. Hopf: For $H \subset \mathbb{R}^N$ a compact oriented hypersurface, we consider the Gauss map

$$\gamma : H \to S^{N-1}$$

sending each point $y \in H$ to the outward unit normal $\nu(y)$. Hopf's result is that the degree of this map is a constant times the Euler-Poincaré characteristic $\chi(H)$.[6] Expressed in terms of integrals

$$\int_H d\sigma = C^{\underline{te}} \chi(H) \tag{1.15}$$

where $d\sigma$ is $\gamma^*$ (volume form on $S^{N-1}$).

For example, using Darboux frames $\{y; e_1, \cdots, e_{N-1}; e_N\}$ associated to $H$, we have

$$\nu(y) = e_N$$

so that

$$\begin{aligned} d\nu &= \sum_{\alpha} \omega_{N,\alpha}\, e_\alpha \\ &= - \sum_{\alpha} \omega_{\alpha,N}\, e_\alpha \\ &= - \sum_{\alpha,\beta} h_{\alpha\beta N}\, \omega_\beta\, e_\alpha \end{aligned}$$

where $\mathrm{II} = \sum h_{\alpha\beta N}\omega_\alpha\omega_\beta \otimes e_N$ is the second fundamental form of $H$. Thus

$$\begin{aligned} d\sigma &= \bigwedge_{\alpha} \omega_{N,\alpha} \\ &= C^{\underline{te}}\, KdH \end{aligned}$$

where $K$ is the Gauss-Kronecker curvature (1.8), and so by Hopf's theorem we infer that

$$C^{\underline{te}} \int_H KdH = \chi(H),$$

which is the Gauss-Bonnet theorem in this case.

Using tubes we may pass from a general $M_n \subset \mathbb{R}^N$ to a hypersurface. Namely, assuming that $M$ is oriented and compact without boundary, for sufficiently small $r_0$ the boundary of the tube $\tau_{r_0}(M)$ will be an oriented hypersurface $H$; for simplicity of notation let us assume that $r_0 = 1$. Expressing points $y \in H$ in the form

$$y = x + \sum_{\mu} t_\mu\, e_\mu, \qquad \|t\| = 1,$$

as in the proof of Weyl's formula, the Gauss mapping on $H$ is given by

$$\nu(y) = \sum_{\mu} t_\mu\, e_\mu.$$

By the structure equations

$$d\nu(y) = \sum_{\alpha} \left( \sum_{\mu} t_\mu\, \omega_{\mu\alpha} \right) e_\alpha + \sum_{\mu} (dt_\mu + t_\nu\, \omega_{\nu\mu})\, e_\mu. \tag{1.16}$$

The volume form on the unit normal sphere at $x$ is

$$\eta = C^{\underline{te}} \sum_{\mu} (-1)^{\mu-1}\, t_\mu\, dt_{n+1} \wedge \cdots \wedge \widehat{dt_\mu} \wedge \cdots \wedge dt_N.$$

Using $\sum_{\mu} t_\mu\, dt_\mu = 0$ on $H$, it follows from (1.16) that

$$\begin{aligned} d\sigma &= \bigwedge_{\alpha} (t_\mu \omega_{\mu\alpha}) \wedge \eta \\ &= P_n(h, t) dM \wedge \eta \end{aligned}$$

where $P_n(h, t)$ is given by (1.11). Consequently

$$\int_H d\sigma = C^{\underline{te}} \int_M I_n(R_M) dM.$$

In case the codimension $m$ is odd, we may use

$$\begin{aligned} \chi(H) &= \chi(M) \cdot \chi(S^{m-1}) \\ &= 2\,\chi(M) \end{aligned}$$

together with (1.15) to conclude that

$$C^{\underline{te}} \int_m I_n(R_M) dM = \chi(M). \tag{1.17}$$

In case $m$ is even we simply consider $M \subset \mathbb{R}^{N+1}$ by adding trivially an extra coordinate to $\mathbb{R}^N$, and then the argument still applies.

Setting $I_n(R_M)dM = KdM$, formula (1.17) is the famous Gauss-Bonnet theorem. As mentioned above, we have more or less retraced its original derivation, where it remains to show that the formula is valid for any Riemannian metric. One way, of course, is by quoting the Nash embedding theorem. For pedagogical purposes it is probably better to show directly that for a 1-parameter family of Riemannian metrics the variation of the integral

$$\frac{\partial}{\partial t}\left( \int_M I_n(R_{t,M}) dM \right) = 0,$$

by differentiation under the integral sign and using the Chern-Weil formalism to write

$$\frac{\partial}{\partial t} (I_n(R_{t,M}) dM) = d\eta_t$$

and then applying Stokes' theorem.[7]

A concluding remark is that, like many of the most beautiful formulas of geometry, the Gauss-Bonnet is an intrinsic relation which was however discovered and first proved by extrinsic methods. Chern's subsequence intrinsic proof[8] was based on the tangent sphere bundle rather than the extrinsic normal sphere bundle.

(c) *Gauss mapping and the Gauss-Bonnet theorem.* This last observation suggests that we consider directly the tangential Gauss mapping on $M$, which will be done following some observations on Grassmannians.

We denote by $G_R(n, N)$ the Grassmann manifold of oriented $n$-planes through the origin of $\mathbb{R}^N$. Note that $G_R(1, N)$ is the sphere $S^{N-1}$ and that $G_R(N - n, N)$

$\cong G_R(n, N)$. The manifold $\mathscr{F}_0(R^N)$ of all oriented orthonormal bases $\{e_1, \cdots, e_N\}$ for $\mathbb{R}^N$ is a fibre bundle over the Grassmannian by

$$\{e_1, \cdots, e_N\} \to \{n\text{-plane } T \text{ spanned by } e_1, \cdots, e_n\},$$

and it is convenient to "do calculus" for $G_R(n, N)$ up on $\mathscr{F}_0(R^N)$. For this we note that the unit multivector $e_1 \wedge \cdots \wedge e_n \in \wedge^n \mathbb{R}^N$ depends only on the $n$-plane $T$; this mapping induces the Plucker mapping $p$ in the following diagram

$$\begin{array}{ccc} \mathscr{F}_0(R^N) & \to & \mathscr{F}_0\left(R^{\binom{N}{n}}\right) \\ \downarrow & & \downarrow \\ G_R(n, N) & \xrightarrow{p} & S^{\binom{N}{n}-1}. \end{array} \tag{1.18}$$

(9)

Considering $\mathscr{F}_0(\mathbb{R}^N) \subset \mathscr{F}(\mathbb{R}^N)$ as a submanifold defined by $x = dx = 0$, the structure equations (1.1) and (1.2) are valid. If we note that in (1.18)

$$\begin{aligned} dp(T) &= d(e_1 \wedge \cdots \wedge e_n) \\ &= \sum_{\alpha,\mu} (-1)^{n-\alpha} \omega_{\alpha\mu} e_1 \wedge \cdots \wedge \hat{e}_\alpha \wedge \cdots \wedge e_n \wedge e_\mu, \end{aligned} \tag{1.19}$$

we deduce that the $n(N - n)$ forms $\{\omega_{\alpha\mu}\}$ are horizontal for the fibering $\pi$ and induce a basis for the cotangent space to the Grassmannian. Alternatively, the fibre $\pi^{-1}(T)$ consists of all frames obtained from a fixed one by proper rotations

$$e_\alpha \to \sum_\beta g_{\alpha\beta} e_\beta, \qquad e_\mu \to \sum_\nu g_{\mu\nu}\, e_\nu.$$

The subsequent change in the Maurer-Cartan matrix $\{\omega_{ij} = (de_i, e_j)\}$ is given by

$$\begin{aligned} \omega_{\alpha\beta} &\to \sum dg_{\alpha\gamma}(g^{-1})_{\gamma\beta} + \sum g_{\alpha\gamma}\omega_{\gamma\delta}(g^{-1})_{\delta\beta} \\ \omega_{\alpha\mu} &\to \sum g_{\alpha\beta}\omega_{\beta\nu}(g^{-1})_{\mu\nu} \end{aligned} \tag{1.20}$$

The second equation checks our remark about the horizontality of the $\omega_{\alpha\mu}$'s, while the first one has the following interpretation: Over the Grassmannian we consider the tautological or universal $n$-plane bundle

$$E \to G_R(n, N)$$

whose fibre over $T$ is just the $n$-plane $T$. Then $\{\omega_{\alpha\beta}\}$ gives a connection matrix, called the *universal connection*, for this bundle.(10) Its curvature matrix is

$$\begin{aligned} \Omega_{\alpha\beta} &= d\omega_{\alpha\beta} - \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} \\ &= -\sum_\mu \omega_{\alpha\mu} \wedge \omega_{\beta\mu}. \end{aligned}$$

We note that this connection is compatible with the metric, and as a consequence

$$\Omega_{\alpha\beta} + \Omega_{\beta\alpha} = 0.$$

Now when $n = 2k$ is even, an $n \times n$ skew-symmetric matrix $A$ has a scalar invariant called the Pfaffian $Pf(A)$, which has the property of being invariant under $A \to BA^tB$ for $B \in SO(n)$, and which satisfies

$$Pf(A)^2 = \det A.$$

Since multiplication of forms of even degree is commutative, we may take the Pfaffian of the curvature matrix $\{\Omega_{\alpha\beta}\}$, and define

$$Pf(\Omega_E) = C_n \sum_A \epsilon_I^A \Omega_{\alpha_1\alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{n-1}\alpha_n}. \tag{1.21}$$

This is a closed invariant $n$-form on the Grassmannian,[11] and (for a suitable constant $C_n$) its de Rham cohomology class in the Euler class of the universal bundle.

Returning to our even-dimensional oriented manifold $M_n \subset \mathbb{R}^N$, the Gauss mapping

$$\gamma : M \to G(n, N)$$

assigns to each $x \in M$ the tangent plane $T_x(M) \in G_R(n, N)$. Using Darboux frames, (1.4), and the Plücker embedding, the differential of the Gauss mapping is given by

$$\begin{aligned} d\gamma(x) &= d(e_1 \wedge \cdots \wedge e_n) \\ &= \sum (-1)^{n-\alpha}(h_{\alpha\beta\mu}\omega_\beta) e_1 \wedge \cdots \wedge \hat{e}_\alpha \wedge \cdots \wedge e_n \wedge e_\mu; \end{aligned}$$

i.e., the differential of the Gauss mapping is just the second fundamental form of $M$ in $\mathbb{R}^N$. Moreover, by definition

$$\gamma^* E = T(M),$$

and we infer that the Riemannian connection on $M$ is induced from the universal connection.[12] As a consequence, $\gamma^*(\Omega_E) = \Omega_M$ and the Pfaffian form pulls back to

$$\begin{aligned} & C_n \sum_A \epsilon_I^A \Omega_{\alpha_1\alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{n-1}\alpha_n} \\ &= C_n\left( \sum_{A,B} \epsilon_I^A \epsilon_I^B R_{\alpha_1\alpha_2\beta_1\beta_2} \cdots R_{\alpha_{n-1}\alpha_n\beta_{n-1}\beta_n}\right)\omega_1 \wedge \cdots \wedge \omega_n \\ &= n!\, C_n\left( \sum_{A,B} \epsilon_B^A R_{\alpha_1\alpha_2\beta_1\beta_2} \cdots R_{\alpha_{n-1}\alpha_n\beta_{n-1}\beta_n}\right) dM \\ &= C^{\underline{te}} I_n(R_M)\, dM \end{aligned}$$

by (1.9). In other words, the Gauss-Bonnet integrand is induced from the universally defined form $Pf(\Omega_E)$ under the Gauss mapping, and therefore in de Rham cohomology it represents the Euler class of the tangent bundle, which again implies the Gauss-Bonnet theorem for submanifolds of $\mathbb{R}^N$. One way to prove the Gauss-Bonnet theorem for a general compact Riemannian manifold $M$ is to observe that the preceding argument applies provided only that the Riemannian connection is induced from the universal connection by a map $f: M \to G(n, N)$ with $f^*E = T(M)$; that such a classifying map exists follows from the theorem of M. Narasimhan and Ramanan, Amer. J. Math., Vol. 83 (1961), pp. 563–572. An alternate proof which has worked well in teaching differential geometry is given at the very end of this paper.

In closing we remark that the other terms in Hermann Weyl's formula (1.10) are not pullbacks of invariant forms an the Grassmannian, and (as previously noted) are of a metric rather than a topological character. We are perhaps belaboring this point, because the complex case will be in sharp contrast in that all the terms turn out to be rigid.

**Footnotes**

1. H. Weyl, *On the volume of tubes*, Amer. J. of Math., vol. 61 (1939), pp. 461–472; cf. also H. Hotelling, *Tubes and spheres in n-spaces and a class of statistical problems*, American J. Math., vol. 61 (1939), pp. 440–460.

2. A *differential system* on a manifold $X$ is given by an ideal $\mathcal{I}$ in the algebra of differential forms which is closed under $d$; an *integral manifold* is a submanifold $Y \subset X$ such that $\phi|_Y = 0$ for all $\phi \in \mathcal{I}$.

3. Recall that

$$dx = \sum_{\alpha} \omega_\alpha e_\alpha \text{ and } de_\alpha = \sum_{\beta} \omega_{\alpha\beta}\, e_\beta + \sum_{\mu} \omega_{\alpha\mu} e_\mu,$$

so that the normal part of

$$d^2x \text{ is } \sum_{\alpha,\mu} \omega_\alpha \omega_{\alpha\mu} e_\mu.$$

4. H. Flanders, *Development of an extended differential calculus*, Trans. Amer. Math. Soc., vol. 57 (1951), pp. 311–326.

5. cf. C. B. Allendoerfer, *The Euler number of a Riemannian manifold*, Amer. J. Math., vol. 62 (1940), pp. 243–248; and C. B. Allendoerfer and A. Weil, *The Gauss-Bonnet theorem for Riemannian polyhedra*, Trans. Amer. Math. Soc., vol. 53 (1943), pp. 101–120. When $n = 2$ so that $M$ is a surface, we have $I_n(R_M) = C^{\underline{te}} R_{1212}$, which implies that $\mu_2(M)$ is a constant times the classical Gauss-Bonnet integral

$$\int_M K\, dA.$$

6. Recall that the degree is the number of points, counted with $\pm 1$ signs according to orientations, in $\gamma^{-1}(e)$ for a general value $e \in S^{N-1}$. This result is an immediate consequence of Hopf's theorem about the number of zeros of a vector field (cf. H. Hopf, Vektorfelder in Mannigfaltigkeiten, Math. Ann. vol. 96 (1927), pp. 225–250.)

7. Actually, this program will be discussed in some detail at the very end of the paper—cf. section 5d.

8. S. S. Chern, *A simple intrinsic proof of the Gauss-Bonnet theorem for Riemannian manifolds*, Ann. of Math., vol. 45 (1944), pp. 747-752.

9. The top mapping between frame manifolds is

$$\{\cdot\,\cdot, e_i, \cdot\,\cdot\} \to \{\cdot\,\cdot, e_I, \cdot\,\cdot\}$$

where $I = (i_1, \cdots, i_n)$ runs over increasing index sets and $e_I = e_{i_1} \wedge \cdots \wedge e_{i_n}$.

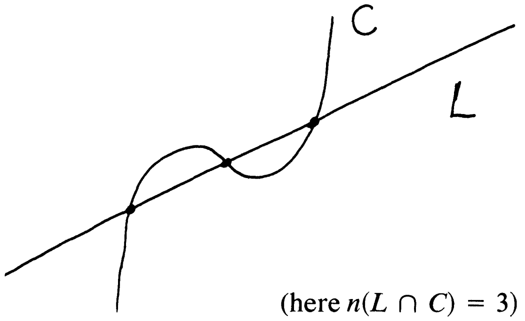
10. More precisely, in $\mathscr{F}_0(\mathbb{R}^N) \xrightarrow{\pi} G(n, N)$ the bundle $\pi^*E$ has the global frame $\{e_1, \cdots, e_n\}$ and is therefore trivial; $\{\omega_{\alpha\beta}\}$ is the connection matrix relative to this frame of the pullback to $\mathscr{F}_0(\mathbb{R}^N)$ of a connection in $E \to G(n, N)$. In the future we will generally omit such laborious descriptions of where various differential forms are defined.

11. A basic fact is that any invariant form $\phi$ on $G(n, N)$ is closed. Indeed, $\phi$ is an expression with constant coefficients in the $\{\omega_{\alpha\mu}\}$ and is invariant under $\omega_{\alpha\mu} \to \sum_\nu \omega_{\alpha\nu}\, g_{\nu\mu}$ for $(g_{\nu\mu}) \in SO(N-n)$. It follows that $\phi$ is a polynomial in the quantities $\sum_\mu \omega_{\alpha\mu} \wedge \omega_{\beta\mu}$, and hence has even degree. In particular $d\phi = 0$. Similar remarks will apply in the complex case.

12. Writing $ds^2 = \sum_\alpha \omega_\alpha^2$, the Riemannian connection matrix $\{\omega_{\alpha\beta}\}$ is uniquely characterized by $\omega_{\alpha\beta} + \omega_{\beta\alpha} = 0$ and $d\omega_\alpha = \sum_\beta \omega_\beta \wedge \omega_{\beta\alpha}$.

## 2. Integral geometry for manifolds in $\mathbb{R}^N$

(a) *Crofton's formula in the plane.* The first result in integral geometry[(1)] deals with the average of the number $n(L \cap C)$ of intersections of variable line $L$ with a piecewise smooth arc $C$ in the ordinary Euclidean plane




(here $n(L \cap C) = 3$)

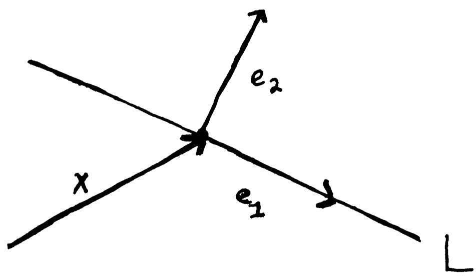
By average we mean with respect to the (suitably normalized) invariant measure $dL$ on the space $\tilde{G}_R(1, 2)$ of all lines in the plane, and the result is given by *Crofton's formula*

$$\int n(L \cap C)dL = 2l(C) \tag{2.1}$$

where $l(C)$ is the length of the arc $C$. To prove this we will find an explicit formula for $dL$ and then iterate the integral on the left in (2.1). Moving frames will facilitate the computation, since the $\omega_i$ and $\omega_{ij}$ are a basis for the invariant Maurer-Cartan forms on the Euclidean group and consequently $dL$ may be written as a constant expression in them.

Explicitly, we consider the manifold $\mathscr{F}(\mathbb{R}^2)$ of all frames $\{x; e_1, e_2\}$ as a bundle over $\tilde{G}_{\mathbb{R}}(1, 2)$ by the mapping

$$\{x; e_1, e_2\} \to \left\{ \begin{array}{c} \text{line } L \text{ through } x \text{ and} \\ \text{in the direction } e_1 \end{array} \right\}$$




The fibre consists of all frames $\{x^*; e_1^*, e_2^*\}$ where

$$x^* = x + \lambda e_1$$
$$e_1^* = \pm e_1, \qquad e_2^* = \pm e_2.$$

Taking the plus signs for a moment, it follows from the structure equations (1.1) and (1.3) that

$$\omega_1^* = \omega_1,$$
$$\omega_2^* = \omega_2 + \lambda\omega_{12},$$
$$\omega_{12}^* = \omega_{12}.$$

Consequently $\omega_2^* \wedge \omega_{12}^* = \omega_2 \wedge \omega_{12}$, and since the conditions that a line remain fixed are given by

$$\omega_2 = \omega_{12} = 0,$$

we may take our invariant measure to be

$$dL = \omega_2 \wedge \omega_{12}.$$

Here it is understood that the corresponding density $|\omega_2 \wedge \omega_{12}|$ is to be integrated; when this is done we may forget about the choice of $\pm$ sign above.

Now we consider the bundle $B \xrightarrow{\pi} C$ whose fibre over $x \in C$ consists of all lines passing through $x$. There is an obvious map

$$F : B \to \tilde{G}_{\mathbb{R}}(1, 2),$$

and the image $F(B)$ is the set of lines meeting $C$ and counted with multiplicities. It follows that

$$\int n(L \cap C)dL = \int_B F^*(\omega_2 \wedge \omega_{12}), \tag{2.2}$$

and we shall evaluate the right hand side by integration over the fibre. For this it is convenient to parametrize $C$ by arc length; i.e., to give $C$ by a vector-valued function $x(s)$ where $\|x'(s)\| = 1$. The Frénèt frame is defined by

$$dx = \omega_1^* e_1^*$$

$$de_1^* = \omega_{12}^* e_2^*$$

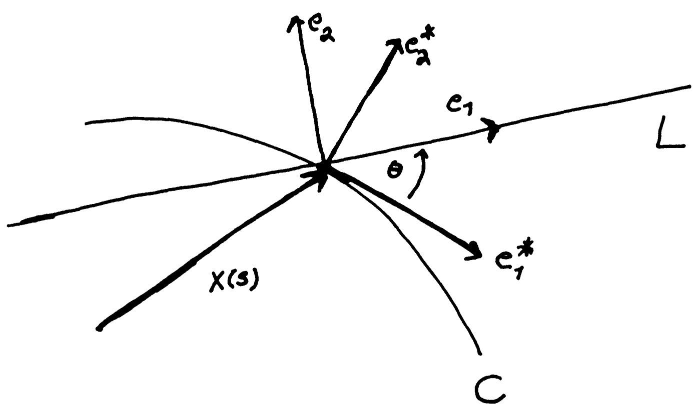
where $\omega_1^* = ds$ and $\omega_{12}^* = \kappa(s)ds$ with $\kappa(s)$ being the curvature.[2] The fibre $\pi^{-1}(x(s))$ consists of lines whose frames are $\{x; e_1, e_2\}$ where

$$x = x(s)$$

$$e_1 = \cos\theta\, e_1^* + \sin\theta\, e_2^*$$

$$e_2 = -\sin\theta\, e_1^* + \cos\theta\, e_2^*$$

as pictured by




Using $(\theta, s)$ as coordinates on $B$ (and dropping reference to $F^*$), we have

$$\begin{aligned}\omega_1 &= (dx, e_1) = \cos\theta\, ds,\\ \omega_{12} &= (de_1, e_2)\\ &\equiv (-\sin\theta\, e_1^* + \cos\theta\, e_2^*, e_2)d\theta \bmod ds\\ &\equiv d\theta \bmod ds.\end{aligned}$$

Thus $dL = \cos\theta\, d\theta \wedge ds$, and the right hand side of (2.2) is[3]

$$\begin{aligned}\int_C \left(\int_0^\pi |\cos\theta| d\theta\right) ds &= 2\int_C ds\\ &= 2l(C),\end{aligned}$$

completing the proof of Crofton's formula (2.1).

For future reference we note that the central geometric construction in the argument is the incidence correspondence $I \subset \mathbb{R}^2 \times \tilde{G}_{\mathbb{R}}(1, 2)$ defined by $\{(x, L) : x \in L\}$. There are two projections

$$\begin{array}{ccc} & I & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathbb{R}^2 & & \tilde{G}_{\mathbb{R}}(1, 2) \end{array}$$

and $B = \pi_1^{-1}(C)$. The basic integral in Crofton's formula is

$$\int_C (\pi_1)_*(\pi_2^* dL),$$

and as will emerge later the main ingredient which enables us to systematically evaluate such integrals is invariance under a suitably large group (cf. sections 4a and 4b).[4]

Finally we remark that on general grounds we may easily deduce that $\int n(L \cap C)dL$ is first of all additive in $C$, and then by passing to the limit that it is an integral

$$\int_C f(x, \kappa(s), \cdots, \kappa^{(n)}(s))ds$$

of some function of the curvature and its derivatives—indeed, *any* Euclidean invariant is of this general type. The main geometric point is that the average $\int n(L \cap C)dL$ is a bending invariant, and therefore does not involve the curvature and its derivatives.

(b) *Application of Crofton's formula to total curvature*. Of course there are Crofton formulas existing in great generality[5]; here we should like to observe that (2.1) remains valid in the elliptic non-Euclidean case. Explicitly, let $C$ be a curve lying in the unit 2-sphere $S^2$ and denote by $G_{\mathbb{R}}(2, 3)$ the great circles on $S^2$ parametrized by the planes $H$ through the origin $\mathbb{R}^3$. Then we claim that the relation

$$\pi \int n(H \cap C)dH = l(C) \tag{2.3}$$

is valid.

For the proof we consider the fibration

$$\mathcal{F}_0(\mathbb{R}^3) \to G_{\mathbb{R}}(2, 3)$$

given by

$$\{e_1, e_2, e_3\} \to e_1 \wedge e_2$$

where $e_1 \wedge e_2$ is the plane spanned by $e_1$ and $e_2$. We have encountered this map in section 1d above, and from the discussion there it follows that the invariant measure on $G_{\mathbb{R}}(2, 3)$ is

$$dH = \omega_{13} \wedge \omega_{23}.$$

If $C$ is given parametrically by the unit vector $e(t)$, then we attach to $C$ the Frénèt frame $\{e_1^*, e_2^*, e_3^*\}$ where

$$\begin{aligned} e_1^* &= e, \\ de_1^* &= \omega_{12}^* e_2^*; \qquad \text{i.e., } \omega_{13}^* = 0 \\ de_2^* &= -\omega_{12}^* e_1^* + \omega_{23}^* e_3^*. \end{aligned}$$

Then $\omega_{12}^* = \pm \|e'(t)\| dt$ is $\pm$ the element of arc length, and $\omega_{13}^* = \kappa^*(t)\omega_{12}^*$ where $\kappa^*$ is essentially the curvature. As before we have a diagram

$$\begin{array}{c} B \rightarrow G_{\mathbb{R}}(2, 3) \\ \downarrow \\ C \end{array}$$

where $B \subset C \times G_{\mathbb{R}}(2, 3)$ is the incidence manifold $\{(e, H) : e \in C \cap H\}$, and the left hand side of (2.3) is a constant times $\int_B |\omega_{13} \wedge \omega_{23}|$. To iterate the integral we parametrize all great circles passing through $e_1^*(t)$ by frames $\{e_1, e_2, e_3\}$ where

$$\begin{aligned} e_1 &= e_1^* \\ e_2 &= \cos\theta\, e_2^* + \sin\theta\, e_3^* \\ e_3 &= \sin\theta\, e_2^* + \cos\theta\, e_3^*. \end{aligned}$$

As before

$$|\omega_{13} \wedge \omega_{23}| = \|x'(t)\|\, |\cos\theta|\, d\theta \wedge dt,$$

and (2.3) follows, where the constant $\pi$ is determined by taking $C$ to be a great circle.

As an application of (2.3) we give what is the first relation between curvature and singularities. Recall that if $f(z_1, z_2) = 0$ defines a complex analytic curve $V$ passing through the origin in $\mathbb{C}^2$, then setting $S_\epsilon^3 = \{|z_1|^2 + |z_2|^2 = \epsilon^2\}$ the intersection $V_\epsilon = V \cap S_\epsilon^3$ defines a closed curve in the 3-sphere whose knot type reflects the topological structure of the isolated singularity which $V$ has at the origin[(6)]. We shall be concerned with the total curvature of a closed curve $C$ in $\mathbb{R}^3$ and shall prove the following results:[(7)] *The total curvature satisfies*

(2.4) $$\int |\kappa|\, ds \geq 2\pi$$

*with equality if and only if $C$ is a convex plane curve (Fenchel); and if $C$ is knotted then*

(2.5) $$\int |\kappa|\, ds \geq 4\pi$$

*(Fary-Milnor).*

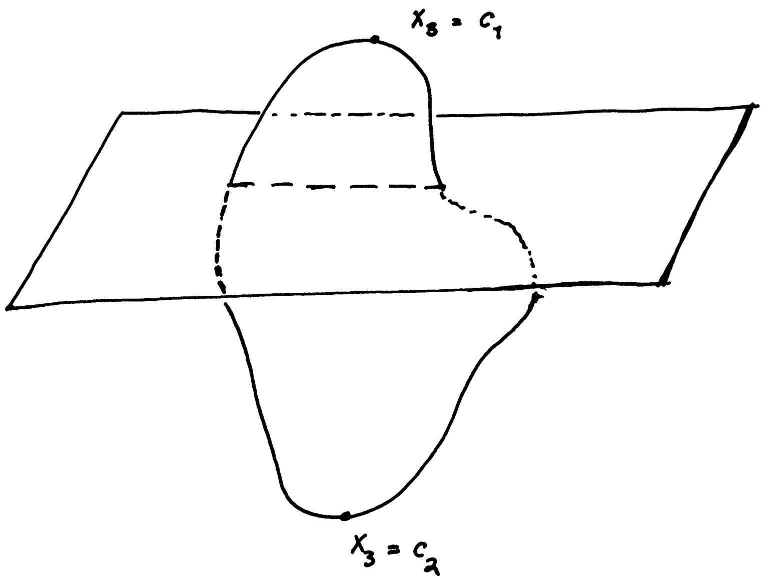
*Proof.* We give $C$ parametrically according to arc length by $x(s)$, and consider the Gauss map

$$\gamma : C \to S^2$$

defined by $\gamma(s) = x'(s)$. Now, and this is the main geometric point, for any great circle $H$, $n(H \cap \gamma(C))$ is just the number of times that the tangent line to $C$ lies in a 2-plane parallel to $H$. Equivalently, if we choose coordinates $(x_1, x_2, x_3)$ so that $H$ is the $(x_1, x_2)$-plane, then $n(H \cap \gamma(C))$ is just the number of critical values of the height function $x_3(s)$ on $C$. This is always an even number $\geq 2$, and if $n(H \cap \gamma(C))$ is equal to 2 on an open set of great circles then $C$ is unknotted. Indeed, for a generic choice of $H$ with $n(H \cap \gamma(C)) = 2$ and constants $c_1$ and $c_2$ corresponding to the maximum and minimum of the height function; the planes

$$x_3 = c, \qquad c_2 \leq c \leq c_1$$

will meet the curve in exactly two points, and $C$ is the boundary of the disc obtained by joining these points by a straight line




Now the length of the Gaussian image $\gamma(C)$ is just the total curvature, as follows from the Frénèt equations

$$\gamma(s) = e_1(s)$$

$$de_1(s) = \omega_{12}\, e_2(s),$$

$$\omega_{12} = \kappa(s)ds.$$

Then from (2.3)

$$\int |\kappa| ds = \pi \int n(H \cap \gamma(C)) dH,$$

and Fenchel's inequality is a consequence of $n(H \cap \gamma(C)) \geq 2$, while if $\int |\kappa| ds < 4\pi$ then $n(H \cap \pi(C)) = 2$ on an open set of great circles and we obtain the Fary-Milnor theorem.

It remains to examine the case of equality in (2.4). We shall first show that the $2^{\text{nd}}$ associated curve [(8)]

$$\gamma_2 : C \to G(2, 3)$$

which assigns to each point of $C$ its osculating 2-plane must be constant. Recall that either $x'(s) \wedge x''(s) \equiv 0$, in which case $C$ is a plane curve, or else in a neighborhood where $x'(s) \wedge x''(s) \neq 0$ we may define the Frénèt frame $\{x(s); e_1(s), e_2(s), e_3(s)\}$ so that the Frénèt equations

$$(2.6) \qquad \begin{cases} dx = e_1 \, ds \\ de_1 = \kappa \, e_2 \, ds \\ de_2 = -\kappa e_1 \, ds + \tau \, e_3 \, ds \\ de_3 = - \tau \, e_2 \, ds \end{cases}$$

are valid, where

$$\kappa(s) = \pm \frac{x' \wedge x''}{|x' \wedge x''|} \neq 0$$

is curvature and $\tau(s)$ is the torsion. The associated curve is given by

$$\gamma_2(s) = e_1 \wedge e_2,$$

and either this curve is a constant in which case $C$ lies a translate of the corresponding 2-plane, or else we have in a neighborhood of some point $s_0$

$$0 \neq \frac{d\gamma_2}{ds} = \tau \, e_1 \wedge e_3$$

by (2.6). Taking $x(s_0)$ to be the origin and our coordinate axes to be the $e_i(s_0)$, $C$ is given parametrically by

$$\left(s + \cdot \cdot, \kappa(s_0) \frac{s^2}{2} + \cdot \cdot, \tau(s_0) \frac{s^3}{3!} + \cdot \cdot\right), \qquad \kappa(s_0)\tau(s_0) \neq 0,$$

where the dots denote terms of order $\geq 4$. Setting

$$\begin{aligned} f_\epsilon(s) &= x_3 - \epsilon x_2 \\ &= \tau(s_0) \frac{s^3}{3!} - \epsilon\kappa(s_0) \frac{s^2}{2} + \cdot \cdot \end{aligned}$$

the function $f_\epsilon(s)$ has for sufficiently small $\epsilon \neq 0$ two critical values 0, $s_1$ corresponding to the two roots of

$$0 = f'_\epsilon(s) = s\left(\tau(s_0)\frac{s}{2} - \epsilon\kappa(s_0) + O(s^2)\right).$$

We note that $s_1$ is approximately equal to $2\epsilon\kappa(s_0)/\tau(s_0)$. Since

$$f''_\epsilon(s) = s\tau(s_0) - \epsilon\kappa(s_0) + O(s)$$

and

$$f''_\epsilon(0) = -\epsilon\kappa(s_0) \neq 0,$$

$$f''_\epsilon(2\epsilon\kappa(s_0)/\tau(s_0)) = \epsilon\kappa(s_0) \neq 0$$

we may assume that both these critical values are non-dengenerate. In summary, we have shown that in a neighborhood where $\kappa(s)\tau(s) \neq 0$, slight perturbations of the osculating 2-planes give height functions having two local non-degenerate critical values in addition to the absolute maximum and minimum. Thus $n(H \cap \gamma(C)) \geq 4$ on an open set and so equality cannot hold in (2.4).

Now we have proved that $C$ is a plane curve, and a similar argument shows that either $\kappa'(s) \equiv 0$, in which case $C$ is a circle, or $\kappa'(s) \neq 0$ in which case $C$ is a general convex plane curve. Q.E.D.

(c) *The kinematic formula.* We shall discuss briefly the reproductive property (0.4) of the curvature integrals in Weyl's tube formula (0.10). For this it is useful to pretend for a moment that we don't know Weyl's formula but do know the Gauss-Bonnet theorem (1.17), and pose the following question: Suppose that $M_n \subset \mathbb{R}^N$ is a compact manifold and $L \in \tilde{G}_{\mathbb{R}}(N - k, N)$ varies over the Grassmannian of affine $(N - k)$-planes in $\mathbb{R}^N$. Outside a set of measure zero in $\tilde{G}_{\mathbb{R}}(N - k, N)$ the intersection $M \cap L$ will be a smooth manifold of dimension $n - k$ in $L \cong \mathbb{R}^{N-k}$,[9] and we may ask for the average

$$\int \chi(M \cap L)dL$$

of the Euler characteristics of these intersections.

By the Gauss-Bonnet theorem this question is equivalent to determining the average

$$\int\left(\int_{M \cap L} I_{n-k}(R_{M \cap L})\right)dL \tag{2.6}$$

where $I_{n-k}(R_{M \cap L})$ is the Gauss-Bonnet integrand for $M \cap L$. According to two of our general principles, the expression for (2.6) should first of all be local—i.e., should be valid for a small piece of manifold as well as a global compact one—and secondly should be expressed as an integral over $M$ of a polynomial of degree $n - k$ in the entries of the curvature matrix $R_M$ of $M$. Recalling our discussions at the beginning of §1(b) and the end of §2(a), we may give an heuristic argument for the existence of a formula of the type

$$\int \left( \int_{M \cap L} I_{n-k}(R_{M \cap L}) \right) dL = C^{\underline{te}} \int_M I_{n-k}(R_M) \tag{2.7}$$

as follows: the left hand side of (2.7) may first of all be given as an integral over $M$ of an expression in the second fundamental form of $M$. Secondly, since the left hand side is invariant under isometric deformations of $M$ (i.e., bending), the same will be true of the expression in the second fundamental form, which must then be a polynomial in the curvature $R_M$ of $M$.

Of course this is not a proof, but it can be developed into one, as will be done in §4(c) for the complex case. The point we should like to emphasize is that the polynomials $I_{n-k}(R_M)$ are forced onto us in response to the question about expressing the average (2.6) as an integral over $M$, and there they turn out to be the same as those appearing in the formula for the volume of tubes. Unfortunately we do not know a direct way of establishing a link between these two.[10] A final remark is that the general reproductive property (0.4) may be deduced from the special case (2.7) when $n - k = \dim (M \cap L)$, this in the obvious way by expressing the left hand side of (0.4) as an interated integral of $\mu_k (M \cap L \cap L')$'s where $\dim (M \cap L \cap L') = k$.

The formula (2.7) is a special case of the general kinematic formula of Chern and Federer (cf. the references in footnote (6) of the introduction). This deals with the following question: Given a pair of manifolds $M_i$ ($i = 1, 2$) of dimensions $n_i$ with $n_i + n_2 \geq N$, as $g \in E(N)$ varies over the Euclidean group, express the average of the Euler characteristics

$$\int \chi(M_1 \cap gM_2) dg$$

in terms of invariants of the $M_i$. Here $dg$ is the kinematic density, i.e., the bi-invariant measure on $E(N)$. Formulas of this general sort originated with Blaschke (cf. the references cited in[5] of this section) and are among the deepest results in integral geometry. Their extension to the complex case in being done by Ted Shifrin in his Berkeley thesis.

A final remark is this: On $M_n \subset \mathbb{R}^N$ the Gauss-Bonnet integrand $I_n(R_M)dM$ is invariant under reversal of orientation. This is a reflection of the topological invariance of the Euler characteristic, and is proved by noting that both the Pfaffian $Pf(R_M)$ and volume form $dM$ change sign under reversal of orientation, so their product remains invariant. On the other hand the Pontrjagin forms change sign under reversal of orientation, reflecting the fact that the signature is an invariant of oriented manifolds. Consequently the simple-minded average[11]

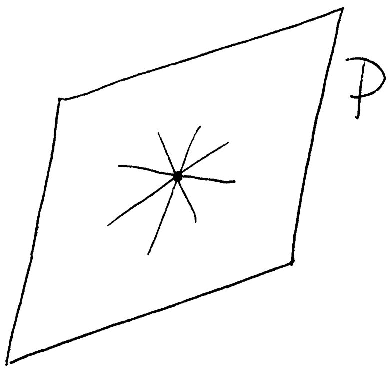
$$\int \int_{M \cap L} \mathrm{Pont}_{n-k}(R_{M \cap L}) dL = 0.$$

On the other hand, since the Pontrjagin numbers are cobordism invariants, in case $M$ is compact the absolute value

$$\left| \int_{M \cap L} \mathrm{Pont}_{n-k}(R_{M \cap L}) \right|$$

will be a constant independent of $L$.

*Proof.* A pencil $|L_t|$ of affine $(N-k)$-planes is given by choosing a fixed $(N-k-1)$-plane $\kappa$, a fixed $(N-k+1)$-plane $\kappa'$, and then $|L_t|$ is the family of $(N-k)$-planes containing $\kappa$ and contained in $\kappa'$. For example, a pencil of lines in $R^3$ is given by fixing a point $p$ and plane $P$ with $p \in P$, and then taking all lines in $P$ passing through $p$.




In any pencil $|L_t|$ the parameter space is the real projective line $S^1$ and decomposes into disjoint connected intervals $I[a, b] = \{a \leq t \leq b\}$ such that the $M \cap L_t$ are non-empty for $t \in I[a, b]$. A *chain will be given by a sequence of pencils* $L_t^{(l)}$ together with intervals $I[a_l, b_l]$ in their parameter space such that

$$M \cap L_{b_l}^{(l)} = M \cap L_{a_{l+1}}^{(l+1)}.$$

Since any two $L, L' \in \tilde{G}_R(N-k, N)$ for which $M \cap L$ and $M \cap L'$ are non-empty may be connected by a chain, it will suffice to establish the invariance of (2.8) when $L$ varies in such an interval $I[a, b]$ in a pencil.

Now we may assume that the axis $\kappa = \cap L_t$ of the pencil meets $M$ transversely and that the $L_t$ which are tangent are simply tangent away from $\kappa$. For $\epsilon > 0$ sufficiently small, the disjoint union

$$\tilde{M} = \bigcup_{a+\epsilon \leq t \leq b-\epsilon} (M \cap L_t)$$

will be a smooth manifold with boundary and the invariance of (2.8) results from the cobordism between $M \cap L_{a+\epsilon}$ and $M \cap L_{b-\epsilon}$. Q.E.D.

Now we may obviously ask whether, for a general manifold $M_n \subset R^N$, we may express the average

$$\int \left| \int_{M \cap L} \mathrm{Pont}_{k-1}(R_{M \cap L}) \right| dL$$

as a curvature integral over $M$, and which might allow us to prove by local methods the cobordism invariance of the Pontrjagin numbers?

**Footnotes**

1. The basic reference is the wonderful book, *Introduction to Integral Geometry*, Paris, Hermann, 1953, by L. A. Santaló.
2. The Gauss-mapping in this case is $\gamma(s) = e_1(s)$, and $\kappa(s)$ is the Jacobian of $\gamma$.
3. The angular integral goes from 0 to $\pi$, since we want to count each line only once.
4. cf. S. S. Chern, *On integral geometry in Klein spaces*, Ann. of Math., vol. 43 (1942), pp. 178–189. This paper isolated the group-theoretic notion of incidence which is ubiquitous in integral geometry.
5. cf. the recent book by L. Santaló, *Integral Geometry and Geometric Probability*, Addison-Wesley, Reading, Mass. (1976).
6. cf. Milnor's book referred to in footnote (5) of the Introduction.
7. In addition to the reference in footnote (5) of the Introduction, cf. S. S. Chern and R. Lashof, *On the total curvature of immersed manifolds*, Amer. J. Math., vol. 79 (1957), pp. 306–318.
8. The first associated curve is the Gauss mapping; it is always defined for smooth curves, whereas the 2$^{\text{nd}}$ associated curve is only defined under certain restrictions to be specified.
9. The measure zero set is the "dual variety" $M^*$ consisting of all $L$ which are tangent to $M$. The complement $\tilde{G}_R(N-k, N) - M^*$ will decompose into connected open sets on which $\chi(M \cap L)$ is constant.
10. Except, of course, that arising from considering orthogonally invariant polynomials in tensors having the symmetries of the Riemann curvature tensor.
11. Similarly, in Crofton's formula (2.1) the average of the intersection numbers of $L$ and $C$ is zero.

## 3. Hermitian differential geometry and volumes of tubes in the complex case

(a) *Frames and Chern forms for complex manifolds in* $\mathbb{C}^N$. We begin by discussing moving frames in Hermitian geometry. The complex frame manifold $\mathscr{F}(\mathbb{C}^N)$ consists of all frames $\{z; e_1, \cdots, e_N\}$ where $z \in \mathbb{C}^N$ is a position vector and the $\{e_i\}$ give a unitary basis for $\mathbb{C}^N$. The structure equations

$$
(3.1)\qquad \begin{aligned} dz &= \sum_i \omega_i\, e_i \\ de_i &= \sum_j \omega_{ij}\, e_j, \qquad \omega_{ij} + \bar{\omega}_{ji} = 0 \end{aligned}
$$

together with the integrability conditions

$$
(3.2)\qquad \begin{aligned} d\omega_i &= \sum_j \omega_j \wedge \omega_{ji} \\ d\omega_{ij} &= \sum_k \omega_{ik} \wedge \omega_{kj} \end{aligned}
$$

hold as in the real case. The forms $\omega_i$ are horizontal and of type (1, 0) for the fibering

$$\mathscr{F}(\mathbb{C}^N) \to \mathbb{C}^N$$

given by $\{z; e_1, \cdots, e_N\} \to z$, and

$$\phi = \frac{\sqrt{-1}}{2} \sum_i \phi_i \wedge \bar{\phi}_i$$

is (the pullback to $\mathscr{F}(\mathbb{C}^N)$ of) the standard Kähler form on $\mathbb{C}^N$. It will sometimes be convenient to write

$$\omega_i = (dz, e_i), \qquad \omega_{ij} = (de_i, e_j), \qquad \phi = \frac{\sqrt{-1}}{2} (dz, dz),$$

etc.

We shall also be interested in the submanifold $\mathscr{F}_0(\mathbb{C}^N) \subset \mathscr{F}(\mathbb{C}^N)$ of unitary frames $\{e_1, \cdots, e_N\}$ centered at the origin. The structure equations (3.1) and (3.2) are valid on $\mathscr{F}_0(\mathbb{C}^N)$ by setting $z = dz = 0$. The Grassmannian of complex $n$-planes through the origin in $\mathbb{C}^N$ will be denoted by $G(n, N)$; as in §1c there is a fibering

$$\mathscr{F}_0(\mathbb{C}^N) \xrightarrow{\pi} G(n, N)$$

given by

$$\pi(e_1, \cdots, e_N) = e_1 \wedge \cdots \wedge e_n$$

where $e_1 \wedge \cdots \wedge e_n$ denotes the $n$-plane spanned by $e_1, \cdots, e_n$. With the usual ranges of indices

$$1 \leq i, j \leq N; \qquad 1 \leq \alpha, \beta \leq n; \qquad n + 1 \leq \mu, \nu \leq N$$

the forms $\omega_{\alpha\mu}$ are horizontal for this fibering and of type (1, 0) for the usual complex structure on $G(n, N)$. Indeed, setting $\Delta = \{t: |t| < \epsilon\}$ the general holomorphic mapping $\Delta \to G(n, N)$ is given by the span of holomorphic vectors $f_\alpha(t)$ where $f_1(t) \wedge \cdots \wedge f_n(t) \neq 0$. If $\{e_\alpha(t); e_\mu(t)\}$ is any $C^\infty$ moving frame lying over this holomorphic curve in $G(n, N)$, then

$$e_\alpha(t) = \sum_\beta A_{\alpha\beta}(t) f_\beta(t),$$

and from $\bar{\partial} f_\beta(t) = 0$ we deduce that

$$\omega''_{\alpha\mu} = 0$$

(3.3)

$$\omega''_{\alpha\beta} = \sum_\gamma \bar{\partial} A_{\alpha\gamma} (A^{-1})_{\gamma\beta}$$

where $\psi = \psi' + \psi''$ denotes the type decomposition of a 1-form $\psi$ into the coefficients of $dt$ and $\overline{dt}$ respectively. The first equation in (3.3) checks our claim about the $\omega_{\alpha\mu}$ having type (1, 0), and the second has this interpretation: Over the Grassmannian we consider the universal $n$-plane bundle $E \to G(n, N)$; this is a holomorphic vector bundle having an Hermitian metric induced from that on $\mathbb{C}^N$, and hence there is a unique Hermitian connection $D$ with the properties

$$D'' = \bar{\partial}, \text{ and } D \text{ is compatible with the metric.}^{(1)}$$

The second property in (3.3) together with $\omega_{\alpha\beta} + \bar{\omega}_{\beta\alpha} = 0$ exactly imply that $\{\omega_{\alpha\beta}\}$ gives the connection matrix of the (pullback to $\mathscr{F}_0(\mathbb{C}^N)$ of) the universal bundle.

The curvature matrix $\Omega_E = \{\Omega_{\alpha\beta}\}$ is, by the Cartan structure equation,

$$\begin{aligned}\Omega_{\alpha\beta} &= d\omega_{\alpha\beta} - \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} \\ &= -\sum_\mu \omega_{\alpha\mu} \wedge \bar{\omega}_{\beta\mu}\end{aligned} \tag{3.4}$$

where we have used the second equation in (3.2). Setting

$$\det\left(\lambda I + \frac{\sqrt{-1}}{2\pi}\,\Omega_E\right) = \sum_{k=0}^{n} \lambda^{n-k}\, c_k(\Omega_E)$$

defines the basic Chern forms $c_k(\Omega_E)$, which are given explicitly by

$$\begin{aligned} c_k(\Omega_E) &= \frac{1}{k!}\left(\frac{\sqrt{-1}}{2\pi}\right)^k \sum_{A,B} \epsilon^A_B \Omega_{\alpha_1\beta_1} \wedge \cdots \wedge \Omega_{\alpha_k\beta_k} \\ &= C^{\underline{te}} \sum_{A,B,\mu} \epsilon^A_B \omega_{\alpha_1\mu_1} \wedge \cdots \wedge \omega_{\alpha_k\mu_k} \wedge \bar{\omega}_{\beta_1\mu_1} \wedge \cdots \wedge \bar{\omega}_{\beta_\kappa\mu_k}.\end{aligned} \tag{3.5}$$

These are closed forms on $G(n, N)$, and in de Rham cohomology they define the Chern classes $c_k(E) \in H^{2k}_{DR}(G(n, N))$. In particular the top Chern class $c_n(E)$ is represented by

$$c_n(\Omega_E) = \left(\frac{\sqrt{-1}}{2\pi}\right)^n \det(\Omega_{\alpha\beta}).$$

We remark that under the obvious embedding

$$j : G(n, N) \rightarrow G_{\mathbb{R}}(2n, 2N)$$

we have

$$j^* Pf(\Phi) = c_n(\Omega_E) \tag{3.6}$$

where $Pf(\Phi)$ is the Pfaffian in the curvature matrix $\Phi$ on $G_{\mathbb{R}}(2n, 2N)$ as defined by (1.15). This is straightforward to verify from the definitions.[(2)]

The formula (3.5) suggests that the Chern forms have sign properties in the holomorphic case. For example, suppose that

$$f : S \rightarrow G(2, n+2)$$

is a holomorphic mapping of a complex surface into the Grassmannian, and denote by $\Omega_E$ the curvature form of the bundle $E \to S$ induced from the universal 2-plane bundle over $G(2, n+2)$ (in other words, drop the $f^*$'s).

*Then there is the inequality*

$$c_2(\Omega_E) \geq 0, \tag{3.7}$$

*with equality holding if, and only if, either* i) *f maps S to a curve or* ii) *$E \to S$ has a trivial sub-line bundle (i.e., the planes $f(z) = \mathbb{C}_z^2$ have a fixed line in common).*

*Proof.* We have from (3.4)

$$\begin{aligned} c_2(\Omega_E) &= \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \Omega_{11} \wedge \Omega_{22} - \Omega_{12} \wedge \Omega_{21} \\ &= \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \sum_{\mu,\nu} (\omega_{1\mu} \wedge \bar{\omega}_{1\mu} \wedge \omega_{2\nu} \wedge \bar{\omega}_{2\nu} - \omega_{1\mu} \wedge \bar{\omega}_{2\mu} \wedge \omega_{2\nu} \wedge \bar{\omega}_{1\nu}) \\ &= -\frac{1}{2}\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \sum_{\mu,\nu} A_{\mu\nu} \wedge \bar{A}_{\mu\nu} \end{aligned}$$

where

$$A_{\mu\nu} = \omega_{1\mu} \wedge \omega_{2\nu} - \omega_{2\mu} \wedge \omega_{1\nu} = A_{\nu\mu}.$$

Since the $A_{\mu\nu}$ are (2, 0) forms it follows that $c_2(\Omega_E) \geq 0$ with equality holding if, and only if, $A_{\mu\nu} \equiv 0$ for all $\mu \leq \nu$.

Now recalling the basic isomorphism[3]

$$T(G(n, N)) \cong \text{Hom}\,(E, \mathbb{C}^N/E),$$

the differential of $f$ is

$$f_* : T_z(S) \to \text{Hom}\,(\mathbb{C}_z^2, \mathbb{C}^{N+2}/\mathbb{C}_z^2).$$

Indeed the matrix representing $f_*\left(\frac{\partial}{\partial z_\beta}\right)$ is

$$\left\{\omega_{\alpha\mu}\left(\frac{\partial}{\partial z_\beta}\right)\right\},$$

and writing

$$\omega_{\alpha\mu} = \sum_\beta A_{\alpha\beta\mu}\, dz_\beta$$

we deduce that the non-vanishing of

$$A_{\mu\nu} = (A_{11\mu}A_{22\nu} - A_{12\mu}A_{21\nu} - A_{21\mu}A_{12\nu} + A_{22\mu}A_{11\nu})dz_1 \wedge dz_2$$

is independent of the bases for $T_z(S)$, $\mathbb{C}_z^2$, and $\mathbb{C}^{n+2}/\mathbb{C}_z^2$. In particular we need not restrict ourselves to unitary bases for the last two vector spaces in order to investigate the meaning of the equations $A_{\mu\nu} = 0$ for all $\mu$, $\nu$.

For this reason, it will be convenient to use the manifold of all frames $\{v_1, v_2, \cdots, v_{n+2}\}$ for $\mathbb{C}^{n+2}$. This is a complex manifold fibering holomorphically over $G(2, n+2)$, and the structure equations

$$dv_i = \sum_j \theta_{ij} v_j$$

$$d\theta_{ij} = \sum_k \theta_{ik} \wedge \theta_{kj}$$

are valid as before, but where the $\{\theta_{ij}\}$ are now holomorphic differentials. In this setting our problem is to determine the meaning of the conditions

$$A_{\mu\nu} = \theta_{1\mu} \wedge \theta_{2\nu} - \theta_{2\mu} \wedge \theta_{1\nu} = 0$$

for all $\mu$, $\nu$.

As observed above, the tri-linear algebra data relevant to analyzing the differential $f_*$ is that of a linear map

$$A : \mathbb{C}^2 \to \operatorname{Hom}(\mathbb{C}^2, \mathbb{C}^{n+2}/\mathbb{C}^2).$$

Denote by $t = (t_1, t_2)$ coordinates on $\mathbb{C}^2$ and think of $A$ as being given by a pencil $A(t) = A_1 t_1 + A_2 t_2$ of $2 \times n$ matrices. We may consider the $2 \times 2$ minors of $A(t)$ as giving

$$\wedge^2 A(t) : \wedge^2 \mathbb{C}^2 \to \wedge^2 \mathbb{C}^n,$$

and we shall first assume that $\wedge^2 A(t) \not\equiv 0$. For example, supposing that the initial minor is non-zero we shall prove that $A_{34} \neq 0$. Effectively, we are then in the case $n = 2$ which we also assume in order to simplify notation.

Now $\wedge^2 A(t) : \wedge^2 \mathbb{C}^2 \to \wedge^2 \mathbb{C}^2$ is a quadratic function having two roots, which we may take to be $t_1 = 0$ and $t_2 = 0$ (the case of a double root must be treated separately). Then $A_1$ and $A_2$ are both $2 \times 2$ matrices of rank one, and if either

$$\ker A_1 \cap \ker A_2 \neq 0, \quad \text{or}$$

$$\operatorname{im} A_1 \cap \operatorname{im} A_2 \neq 0$$

we deduce that $\wedge^2 A(t) = 0$ for all $t$. It follows that we may choose a basis $v_1, v_2, v_3, v_4$ for $\mathbb{C}^4$ so that $v_1, v_2$ is a basis for $\mathbb{C}^2 \subset \mathbb{C}^4$ and

$$A_1 v_1 = v_3, \qquad A_1 v_2 = 0$$

$$A_2 v_2 = v_4, \qquad A_2 v_1 = 0.$$

Then

$$\theta_{13} = dz_1, \qquad \theta_{24} = dz_2, \qquad \theta_{23} = \theta_{14} = 0$$

and

$$A_{34} = dz_1 \wedge dz_2.$$

The case of a double root is similar to an argument we are about to give, and will therefore be omitted.

Now suppose that $\wedge^2 A(t) \equiv 0$. Then for all $t$, the linear transformation

$$A(t) : \mathbb{C}^2 \to \mathbb{C}^{n+2}/\mathbb{C}^2$$

has a kernel containing a non-zero vector $v(t)$. If the line $\mathbb{C}v(t)$ is not constant in $t$, then we may choose $v_1$, $v_2$ so that $A_1 v_2 = 0 = A_2 v_1$. Then the vectors $A_1 v_1$ and $A_2 v_2$ are proportional modulo $v_1$, $v_2$ since otherwise we would not have $\wedge^2 A(t) \equiv 0$. It follows that we may choose a frame $\{v_1, v_2, \cdots, v_{n+2}\}$ so that

$$A_1 v_1 = v_3, \qquad A_2 v_2 = v_3, \quad \text{and} \quad A_1 v_2 = A_2 v_1 = 0.$$

Then $\theta_{13} = dz_1$, $\theta_{23} = dz_2$, and all other $\theta_{\alpha\mu} = 0$; thus

$$A_{33} = 2\theta_{13} \wedge \theta_{23} = 2\, dz_1 \wedge dz_2$$

is non-zero.

In case $v(t) = v_1$ is constant in $t$, we will have the two cases:

$A_1 v_2$ and $A_2 v_2$ independent modulo $v_1$, $v_2$

$A_1 v_2$ proportional to $A_2 v_2$ modulo $v_1$, $v_2$.

In the first case we may choose a local frame so that

$$dv_1 = \theta_{11} v_1 + \theta_{12} v_2$$

$$dv_2 = \theta_{21} v_1 + \theta_{22} v_2 + \theta_{23} v_3 + \theta_{24} v_4$$

where $\theta_{23} \wedge \theta_{24} \neq 0$. From

$$0 = d\theta_{13} = \theta_{12} \wedge \theta_{23}, \qquad 0 = d\theta_{14} = \theta_{12} \wedge \theta_{24}$$

we deduce that $\theta_{12} = 0$. Then $dv_1 \equiv 0$ modulo $v_1$, which says exactly that the line $\mathbb{C}v_1 \subset \mathbb{C}^2_z$ is constant in $z$. Finally, in the second case we may choose a local frame so that

$$dv_1 = \theta_{11} v_1 + \theta_{12} v_2$$

$$dv_2 = \theta_{21} v_1 + \theta_{22} v_2 + \theta_{23} v_3,$$

from which it follows that the rank of $f_*$ is one and so $f(S)$ is a curve in $G(2, n+2)$. Q.E.D.

We will apply (3.7) to the holomorphic Gauss mapping. Suppose that $M \subset \mathbb{C}^N$ is a complex manifold and denote by $\mathscr{F}(M)$ the manifold of Darboux frames $\{z; e_1, \cdots, e_N\}$ defined by the conditions

$$\begin{cases} z \in M;\ e_1, \cdots, e_n \text{ give a unitary basis for } T_z(M); \text{ and } e_{n+1}, \\ \cdots, e_N \text{ give a unitary basis for the normal space } N_z(M). \end{cases}$$

We may think of $\mathscr{F}(M) \subset \mathscr{F}(\mathbb{C}^N)$ as an integral manifold of the differential system $\omega_\mu = 0$, and then (3.1) and (3.2) become

$$dz = \sum_{\alpha} \omega_\alpha \, e_\alpha$$

$$de_\alpha = \sum_{\beta} \omega_{\alpha\beta} \, e_\beta + \sum_{\mu} \omega_{\alpha\mu} \, e_\mu.$$

Setting $d\omega_\mu = 0$ in the first equation of (3.1) gives

$$\sum_{\alpha} \omega_\alpha \wedge \omega_{\alpha\mu} = 0,$$

which again by the Cartan lemma implies that

$$\omega_{\alpha\mu} = \sum_{\beta} h_{\alpha\beta\mu}\omega_\beta, \qquad h_{\alpha\beta\mu} = h_{\beta\alpha\mu}.$$

The first and second fundamental forms of $M \subset \mathbb{C}^N$ are defined by

$$\mathrm{I} = \sum \omega_\alpha \otimes \bar{\omega}_\alpha$$

$$\mathrm{II} = \sum h_{\alpha\beta\mu}\omega_\alpha\omega_\beta \otimes e_\mu.$$

The other part of the first equation in (3.1) is

$$d\omega_\alpha = \sum_{\beta} \omega_\beta \wedge \omega_{\beta\alpha}, \qquad \omega_{\alpha\beta} + \bar{\omega}_{\beta\alpha} = 0. \tag{3.8}$$

It is well known that given a Kähler metric I there is a unique matrix $\{\omega_{\alpha\beta}\}$ of 1-forms satisfying (3.8), which is then the connection matrix for the canonical Hermitian connection in the holomorphic tangent bundle $T(M)$. The curvature matrix $\Omega_M = \{\Omega_{\alpha\beta}\}$ is

$$\begin{aligned}\Omega_{\alpha\beta} &= d\omega_{\alpha\beta} - \sum_{\gamma} \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} \\ &= -\sum_{\mu} \omega_{\alpha\mu} \wedge \bar{\omega}_{\beta\mu}.\end{aligned}$$

Setting

$$\Omega_{\alpha\beta} = \sum_{\gamma,\delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}}\omega_\gamma \wedge \bar{\omega}_\delta$$

it follows that

$$\begin{aligned}R_{\alpha\bar{\beta}\gamma\bar{\delta}} &= -\sum_{\mu} h_{\alpha\gamma\mu}\bar{h}_{\beta\delta\mu} \\ &= R_{\gamma\bar{\beta}\alpha\bar{\delta}} = R_{\alpha\bar{\delta}\gamma\bar{\beta}}.\end{aligned} \tag{3.9}$$

As in the real case we may consider the holomorphic Gauss mapping

$$\gamma : M \to G(n, N)$$

with $\gamma^*E = T(M)$. The Hermitian connection and curvature are induced from those in $E \to G(n, N)$. There is a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\gamma_{\mathbb{R}}} & G_{\mathbb{R}}(2n, 2N) \\ & {\scriptstyle\gamma}\searrow \quad \nearrow{\scriptstyle j} & \\ & G(n, N) & \end{array}$$

where $\gamma_R$ is the usual real Gauss mapping, and we deduce from (3.6) and the discussion at the end of §1c that the Gauss Bonnet integrand is

$$\gamma^*_{\mathbb{R}} Pf(\Phi) = \gamma^* c_n(\Omega_E) = c_n(\Omega_M); \quad \text{i.e.,}$$

(3.10) $$C^{\underline{te}}\, KdM = c_n(\Omega_M).$$

Now a natural question is whether the Gaussian image $\gamma(M)$ has dimension $n$, and using (3.7) we shall prove:

*For $S \subset \mathbb{C}^{n+2}$ a complex-analytic surface*

(3.11) $$c_2(\Omega_S) \geq 0$$

*with equality holding only if* (i) *S is a plane,* (ii) *S is a developable ruled surface;*[4] *or* (iii) *S is a cone.*

*Proof.* We shall adhere to the notations in the proof of (3.7). If $c_2(\Omega_M) = 0$ and alternative ii) holds, then we may choose a local moving frame $\{e_1, e_2; e_3, \cdots, e_{n+2}\}$ so that

$$de_1 = \omega_{11} e_1, \qquad \text{and thus } \omega_{12} = \bar{\omega}_{21} = 0,$$
$$de_2 = \omega_{22} e_2 + \omega_{23} e_3 + \omega_{24} e_4.$$

From

$$\omega_{23} = h_{213}\omega_1 + h_{223}\omega_2 \text{ and}$$
$$h_{\alpha\beta\mu} = h_{\beta\alpha\mu}$$

we deduce that $\omega_{23} = h_{223}\omega_2$. Similarly $\omega_{24} = h_{224}\omega_2$, and it follows that the Gaussian image $\gamma(S)$ is a curve. The fibres of $\gamma : S \to \gamma(S)$ are defined by $\omega_2 = 0$; these are curves along which

$$dz = \omega_1 \, e_1, \qquad de_1 = \omega_{11} \, e_1.$$

We infer that these fibres are straight lines.

If alternative (ii) in (3.7) holds, then we may choose a moving frame so that

$$de_1 = \omega_{11}\, e_1 + \omega_{12}\, e_2$$

$$de_2 = \omega_{21}\, e_1 + \omega_{22}\, e_2 + \omega_{23}\, e_3.$$

As before, from $\omega_{23} = h_{213}\omega_1 + h_{223}\omega_2$ and $h_{213} = h_{123} = 0$ we have $\omega_{23} = \rho\,\omega_2$. From exterior differentiation of $\omega_{13} = 0$,

$$0 = d\omega_{13} = \omega_{12} \wedge \omega_{23}$$

which implies that $\omega_{12} = \sigma\omega_2$. The fibres of $\gamma$ are then defined by $\omega_2 = 0$, and along these curves $dz = \omega_1\, e_1$, $de_1 = \omega_{11}\, e_1$ as before.

Thus in both cases the fibres of the Gauss map are lines, and consequently $S$ is a ruled surface. As such, it may be given parametrically by

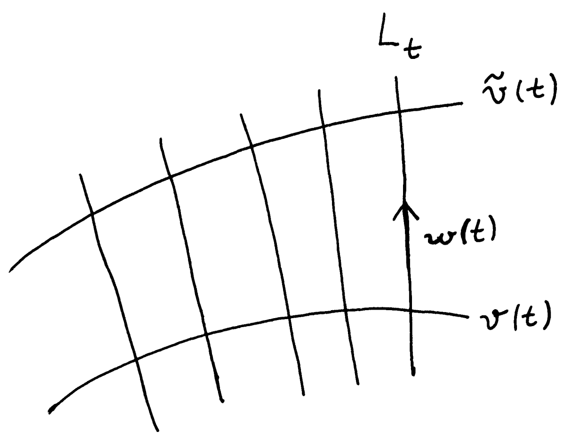
$$(t, \lambda) \rightarrow v(t) + \lambda w(t)$$

where $v(t)$, $w(t)$ are holomorphic vector-valued functions and $\lambda$ is a linear parameter on the line $L_t$. This representation is unique up to a substitution

$$\tilde{v}(t) = v(t) + \lambda(t)\, w(t)$$

$$\tilde{w}(t) = \zeta(t)\, w(t)$$

as depicted by




The Gauss map is

$$\gamma(t, \lambda) = (v' + \lambda w') \wedge w$$

and this map is degenerate exactly when $v' \wedge w$ and $w' \wedge w$ are proportional. If $w' \wedge w = 0$ so that $w' = \alpha w$, then $\tilde{w}' = \zeta' w + \zeta\alpha w$ and we may make $\tilde{w}' = 0$ by solving $\zeta' + \zeta\alpha = 0$. In this case $S$ is a cone (including the special case of a cylinder). If $w' \wedge w \neq 0$, then

$$v' \wedge w = \tau\, w' \wedge w.$$

Under the above substitution (and taking $\tilde{w} = w$),

$$\begin{aligned}\tilde{v}' \wedge \tilde{w} &= v' \wedge w + \lambda w' \wedge w \\ &= (\lambda + \tau) w' \wedge w,\end{aligned}$$

so that setting $\lambda + \tau = 0$ gives $v' \wedge w = 0$. Thus $w = \beta v'$ and taking $\zeta = \beta^{-1}$ we may assume $w = v'$. Then $S$ is given by

$$(t, \lambda) \to v(t) + \lambda v'(t),$$

and is a developable ruled surface. Q.E.D.

(b) *Remarks on integration over analytic varieties.* We now discuss some matters related to integration on analytic varieties. Let $V$ be an $n$-dimensional analytic variety defined in an open set $U \subset \mathbb{C}^N$. The set of singular points $V_s$ forms a proper analytic subvariety, and the complement $V^* = V - V_s$ is a complex manifold which is open and dense in $V$. We denote by $A_c^q(U)$ the $C^\infty$ differential forms of degree $q$ having compact support in $U$. The basic fact is that the linear function

$$T_V(\alpha) = \int_{V^*} \alpha, \qquad \alpha \in A_c^{2n}(U), \tag{3.12}$$

defines a closed, positive current of type $(n, n)$.[5] Essentially this means that the integral (3.12) is absolutely convergent, and that Stokes' theorem

$$\int_{V^*} d\alpha = 0, \qquad \alpha \in A_c^{2n-1}(U),$$

is valid. We shall also use the result that if $\{V_t\}$ is a family of varieties depending holomorphically on parameters (in a sense to be made precise in §5b when needed), then

$$\int_{V_t^*} \alpha$$

also depends holomorphically on $t$ (all we use is that it depends continuously).

In somewhat more detail, if

$$\phi = \frac{\sqrt{-1}}{2} \sum_i dz_i \wedge d\bar{z}_i$$

is the standard Kähler form on $\mathbb{C}^N$, and if for any index set $I = \{i_1, \cdots, i_n\}$ we denote by

$$\phi_I = \frac{\sqrt{-1}}{2}(dz_{i_1} \wedge d\bar{z}_{i_1}) \wedge \cdots \wedge \frac{\sqrt{-1}}{2}(dz_{i_n} \wedge d\bar{z}_{i_n})$$

the Euclidean measure on the corresponding $\mathbb{C}_I^n \subset \mathbb{C}^N$, then

$$\frac{\phi^n}{n!} = \sum_{\#I = n} \phi_I$$

and, for any affine $n$-plane $L$ in $\mathbb{C}^N$

$$\left.\frac{\phi^n}{n!}\right|_L = \text{Euclidean measure on } L \tag{3.13}$$

(to see this choose orthonormal coordinates so that $L$ is a translate of $\mathbb{C}^n$). Two important consequences of (3.13) are: (i) For any complex manifold $M$

$$\frac{1}{n!}\int_M \phi^n = \text{vol}\,(M) \tag{3.14}$$

is the Euclidean volume of $M$ (*Wirtinger theorem*), and (ii) for any $\alpha \in A_c^{2n}(U)$ there is an estimate

$$|\alpha| \leq C^{\underline{te}}\,\phi^n$$

in the sense that for all complex $n$-planes $L$

$$\alpha|_L \leq \xi \cdot \phi^n|_L$$

where the function $\xi$ on the Grassmannian is bounded. The fact that (3.12) defines a current is then equivalent to the finiteness of the volume

$$\frac{1}{n!}\int_{V^* \cap B[z,\epsilon]} \phi^n < \infty$$

of analytic varieties in the $\epsilon$-ball $B[z, \epsilon]$ around singular points $z \in V_s$. Similarly, Stokes' theorem follows from the usual version for manifolds together with the fact that the $(2n - 1)$-dimensional area of the boundary $\partial T_\epsilon(V_s)$ of the $\epsilon$-tube around the singular points tends to zero as $\epsilon \to 0$.

Later on we shall be examining more delicate integrals $\int_{V^*} \Phi$ where $\Phi$ is *not* the restriction of a form in $U$ (such as a curvature integral), or may be the restriction of a form but one having singularities on $V_s$. To obtain some feeling for these we shall examine one of the latter types here. With the notations

$$d^c = \frac{\sqrt{-1}}{4}\,(\bar{\partial} - \partial)$$

$$\omega = d\,d^c \log \|z\|^2$$

$$V[r] = \{z \in V : \|z\| \leq r\}$$

$$V[\rho, r] = V[r] - V[\rho], \qquad \rho \leq r,$$

we shall prove the formula[6]

$$\frac{1}{r^{2n}}\int_{V[r]} \phi^n - \frac{1}{\rho^{2n}}\int_{V[\rho]} \phi^n = \int_{V[\rho,r]} \omega^n. \tag{3.15}$$

Geometrically, if we consider the residual map

$$\mathbb{C}^N - \{0\} \to \mathbb{P}^{N-1}$$

defined by $z \to \overline{0z}$, then $\omega$ is the pullback of the standard Kähler form on $P^{N-1}$ and as such has a singularity at origin. To prove (3.15) we note that

$$d^c\|z\|^2 = \frac{\sqrt{-1}}{4}\{(z, dz) - (dz, z)\}$$

$$\phi = \frac{\sqrt{-1}}{2}(dz, dz) = dd^c\|z\|^2$$

$$d^c \log \|z\|^2 = \frac{\sqrt{-1}}{4}\left\{\frac{(z, dz) - (dz, z)}{\|z\|^2}\right\}$$

$$\omega = \frac{\sqrt{-1}}{2}\left\{\frac{(z, z)(dz, dz) - (dz, z)(z, dz)}{(z, z)^2}\right\}$$

If we define

(3.16) $$\eta = d^c \log \|z\|^2 \wedge \omega^{n-1}$$

then $d\eta = \omega^n$, and by Stokes' theorem the right hand side of (3.15) is $\int_{\partial V[r,\rho]} \eta$. Now, for any fixed $t$, on $\partial V[t]$ we have

$$0 = (dz, z) + (z, dz)$$

which implies that on $\partial V[t]$

$$\eta = \left(\frac{\sqrt{-1}}{2}\right)^n \frac{(z, dz)}{t^2} \wedge \left(\frac{(dz, dz)}{t^{2n-2}}\right)$$

$$= \frac{1}{t^{2n}} d^c \|z\|^2 \wedge \phi^{n-1}$$

Then

$$\int_{\partial V[t]} \eta = \frac{1}{t^{2n}} \int_{\partial V[t]} d^c \|z\|^2 \wedge \phi^{n-1}$$

$$= \frac{1}{t^{2n}} \int_{V[t]} \phi^n$$

proving (3.15).

As a consequence the function

$$\mu(V, r) = \frac{1}{r^{2n}} \int_{V[r]} \phi^n$$

$$= \frac{1}{r^{2n}} \text{vol } V[r]$$

is an increasing function of $r$, and the limit

$$\mu(V) = \lim_{r \to 0} \mu(V, r)$$

exists and is called the *Lelong number* of $V$ at the origin. We shall now prove that

$$\mu(V) = C^{\underline{te}}\, \mathrm{mult}_0\,(V) \tag{3.17}$$

is a (constant times) the multiplicity of $V$ at the origin.[7] For this we let

$$E \subset \mathbb{P}^{N-1} \times \mathbb{C}^N$$

be the closure of the incidence correspondence

$$\{(L, z) : L \in \mathbb{P}^{N-1},\ z \in \mathbb{C}^N - \{0\},\ z \in L\}.$$

Then $E \rightarrow \mathbb{C}^N$ is the blow-up of $\mathbb{C}^N$ at the origin, and $E \xrightarrow{\pi} \mathbb{P}^{N-1}$ is the total space of the universal Hopf line bundle. The closure of $V - \{0\}$ in $E$ gives the proper transform $\tilde{V}$ of $V$; since the fibre $\tilde{V} \cap \mathbb{P}^{N-1}$ of $\tilde{V} \rightarrow V$ over the origin is the limiting position of chords $\overline{0z}$ ($z \in V$), the intersection

$$T = \tilde{V} \cap \mathbb{P}^{N-1}$$

is the tangent cone to $V$ at the origin. We denote by $E[r] = \{(L, z) : \|z\| \leq r\}$ the tubular neighborhood of radius $r$ around the zero section and set

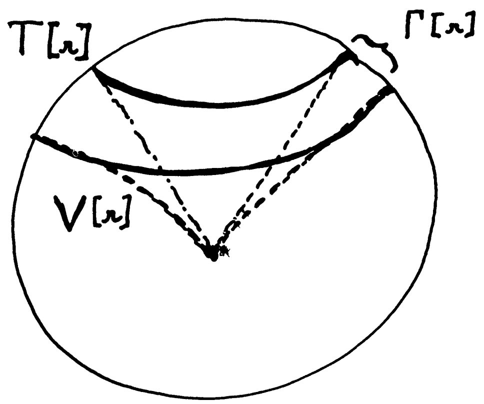
$$\tilde{V}[r] = \tilde{V} \cap E[r]$$

$$T[r] = (\pi^{-1}T) \cap E[r].$$

We note that $\partial \tilde{V}[r] = \partial V[r]$ and that the difference

$$\partial V[r] - \partial T[r] = \partial \Gamma[r]$$

where $\Gamma[r]$ is the locus where all lines $\overline{0z}$, for $0 < \|z\| < r$ and $z \in V$, meet the sphere of radius $r$




On $E$ we consider the $(2n - 1)$ form (3.16). Clearly $w$ is a smooth form on $E$, and $d^c \log \|z\|^2$ is a 1-form whose restriction to every complex line $\{re^{i\theta} \cdot z\}$ is $d\theta$. It follows then from the argument principle and Wirtinger theorem that

$$\int_{\partial T[r]} \eta = \int_T \omega^n$$
$$= C^{\underline{te}} \deg T$$
$$= C^{\underline{te}} \operatorname{mult}_0(V),$$

while by Stokes' theorem

$$\int_{\partial V[r]} \eta - \int_{\partial T[r]} \eta = \int_{\Gamma[r]} \omega^n$$

which tends to zero as $r \to 0$, thus proving (3.17).

We note the general principle, already familiar from the use of polar coordinates in elementary calculus, that blowing up frequently simplifies singular integrals.

In the next section we shall be considering varieties $V_t$ acquiring an isolated singularity at the origin and shall show that suitable curvature integrals

$$\lim_{t \to 0} \frac{1}{\epsilon^{2k}} \int_{V[\epsilon]} P_{n-k}(\Omega_{V_t}) \wedge \phi^k$$

exist and have limits as $\epsilon \to 0$, and shall eventually give geometric interpretations of these. For $k = n$ the above limit is $\frac{1}{\epsilon^{2n}}$ vol $V_0[\epsilon]$, which then tends to $\operatorname{mult}_0(V_0)$ as $\epsilon \to 0$.

(c) *Volume of tubes in the complex case*. We shall now derive the formula for the volume of the tube around a complex manifold $M_n \subset \mathbb{C}^N$. Proceeding as in the real case, we let $N[r]$ denote the ball of radius $r$ around $M$ embedded as zero cross-section in its normal bundle, and by $\tau_r(M)$ the image (counting multiplicities) of the map

$$N[r] \to \mathbb{C}^N.$$

Points in the image are

$$w = z + \sum_\mu t_\mu e_\mu, \qquad \|t\| \leq r,$$

and from the structure equations (3.1) and (3.2)

$$dw = \sum (\omega_\alpha + t_\mu \omega_{\mu\alpha}) e_\alpha + \sum (dt_\mu + t_\nu \omega_{\nu\mu}) e_\mu$$

with repeated indices being summed. Setting

$$\phi_\alpha = \omega_\alpha + t_\mu \omega_{\mu\alpha} = \omega_\alpha - t_\mu \bar{h}_{\alpha\delta\mu} \bar{\omega}_\delta$$
$$\phi_\mu = dt_\mu + t_\nu \omega_{\nu\mu}$$

(recall that $\mathrm{II} = \sum h_{\alpha\beta\mu}\omega_\alpha\omega_\beta \otimes e_\mu$ is the 2nd fundamental form of $M \subset \mathbb{C}^N$), the

Kähler form on $\mathbb{C}^N$ pulled back to $N[r]$ becomes

$$\phi = \frac{\sqrt{-1}}{2}\left(\sum_\alpha \phi_\alpha \wedge \bar{\phi}_\alpha + \sum_\mu \phi_\mu \wedge \bar{\phi}_\mu\right).$$

The volume form $\Phi$ on $\mathbb{C}^N$ is

$$\Phi = \frac{1}{N!}\phi^N$$

$$= C^{\underline{te}} \bigwedge_\alpha \phi_\alpha \bigwedge_\beta \bar{\phi}_\beta \bigwedge_\mu \phi_\mu \bigwedge_\nu \bar{\phi}_\nu.$$

We shall iterate the volume integral on $N[r]$; thus setting

$$\Psi(z, r) = \int_{\|t\| \leq r} \Phi(z, t)$$

we find

$$\text{vol}\, \tau_r(M) = \int_{z \in M} \Psi(z, r).$$

To evaluate the right hand side we fix $z$, assume as we may that our moving frame has been chosen so that $\omega_{\nu\mu}(z) = 0$ (this is convenient but not essential), and set

$$\boldsymbol{dt} = dt_{n+1} \wedge \cdots \wedge dt_N$$

$$\boldsymbol{\omega} = \omega_1 \wedge \cdots \wedge \omega_n$$

$$\bar{h}(t)_{\alpha\delta} = -\sum_\mu t_\mu \bar{h}_{\alpha\delta\mu}$$

$$\bar{h}(t)_{AD} = \frac{1}{k!}\det(\bar{h}(t)_{\alpha_i\delta_j})$$

where $A = (\alpha_1, \cdots, \alpha_k)$ and $D = (\delta_1, \cdots, \delta_k)$ are index sets selected from $(1, \cdots, n)$. Denoting by $A°$ the index set complementary to $A$,

$$\bigwedge_\alpha \phi_\alpha = \sum_{A,D} \pm \bar{h}(t)_{AD}\omega_{A°} \wedge \bar{\omega}_D$$

$$\bigwedge_\beta \bar{\phi}_\beta = \sum_{B,C} \pm h(\bar{t})_{BC}\bar{\omega}_{B°} \wedge \omega_C$$

where it is understood that the summation is over index sets having the same number of elements. It follows that

$$\Phi = C^{\underline{te}}\left(\sum \epsilon^A_{A'}\epsilon^B_{B'}\bar{h}(t)_{AB}h(t)_{B'A'}\right)\boldsymbol{\omega} \wedge \bar{\boldsymbol{\omega}} \wedge \boldsymbol{dt} \wedge \overline{\boldsymbol{dt}}.$$

For $\mu = (\mu_1, \cdots, \mu_k)$ with $\mu_1 \leq \cdots \leq \mu_k$ we set $t_\mu = t_{\mu_1} \cdots t_{\mu_k}$; then

$$\int_{\|k\| \leq r} t_\mu \bar{t}_\nu dt\, \overline{dt} = C(m, k) r^{2m + 2k} \delta^\mu_\nu$$

where $m = N - n$. Indeed, the unitary group $U(m)$ acts irreducibly on the space $\mathrm{Sym}^k(\mathbb{C}^{m*})$ of homogeneous polynomials of degree $k$ in $t_{n+1}, \cdots, t_N$; it leaves invariant the inner product

$$(t_\mu, t_\nu) = \int_{\|t\| = 1} t_\mu \bar{t}_\nu d\sigma$$

where $d\sigma$ is the invariant volume on the sphere, as well as the inner product induced from the standard one on $\mathbb{C}^m$. It follows from Schur's lemma that these two inner products are proportional. Setting

$$dM = \bigwedge_\alpha \frac{\sqrt{-1}}{2} (\omega_\alpha \wedge \bar{\omega}_\alpha)$$

we deduce that

$$\Psi = \sum_{k=0}^{n} P_k(\Omega_M) r^{2m + 2k}\, dM$$

where

$$\begin{aligned} P_k(\Omega_M) &= C^{\underline{te}} \sum \epsilon^{A'}_A \epsilon^{B'}_B \bar{h}_{\alpha_1\beta_1\mu_1} \cdots \bar{h}_{\alpha_k\beta_k\mu_k} h_{\beta'_1\alpha'_1\mu_1} \cdots h_{\beta'_k\alpha'_k\mu_k} \\ &= C^{\underline{te}} \sum \epsilon^A_{A'} \epsilon^B_{B'} h_{\alpha_1\beta_1\mu_1} \bar{h}_{\alpha'_1\beta'_1\mu_1} \cdots h_{\alpha_k\beta_k\mu_k} \bar{h}_{\alpha'_k\beta'_k\mu_k} \end{aligned}$$

since $h_{\alpha\beta\mu} = h_{\beta\alpha\mu}$

$$= C^{\underline{te}} \sum \epsilon^A_{A'} \epsilon^B_{B'}\, R_{\alpha_1\bar{\alpha}'_1\beta_1\bar{\beta}'_1} \cdots R_{\alpha_k\bar{\alpha}'_k\beta_k\bar{\beta}'_k}$$

by (3.9). Observe that this step is easier than in the real case where skew-symmetry conditions intervene. More importantly, using

$$\Omega_{\alpha\beta} = \sum_{\gamma,\delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}} \omega_\gamma \wedge \bar{\omega}_\delta$$

$$\phi = \frac{\sqrt{-1}}{2} \sum_\alpha \phi_\alpha \wedge \bar{\phi}_\alpha$$

we deduce that

$$P_k(\Omega_k) dM = c_k(\Omega_M) \wedge \phi^{n-k}$$

where

$$c_k(\Omega_M) = C^{\underline{te}} \sum \epsilon^A_{A'} \Omega_{\alpha_1\alpha'_1} \wedge \cdots \wedge \Omega_{\alpha_k\alpha'_k}$$

is, by (3.5), the $k$th basic Chern polynomial in the curvature matrix $\Omega_M$ of $M$.

Summarizing, we have arrived at our desired result:

$$\text{(3.18)} \qquad \operatorname{vol} \tau_r(M) = \sum_{k=0}^{n} C(k, n, m) r^{2(m+k)} \int_M c_k(\Omega_M) \wedge \phi^{n-k}.$$

We should like to make a few observations concerning this formula. The first is that an analogous result holds for a complex manifold $M_n \subset \mathbb{P}^N$ [8], more precisely, the formula is

$$\text{(3.19)} \qquad \operatorname{vol} \tau_r(M) = \sum_{k=0}^{n} C(k, n, m) r^{2(m+k)} \int_M P_k(\Omega_M, \phi) \wedge \phi^{n-k}$$

where

$$P_k(\Omega_M, \phi) = \sum_{l=0}^{k} C(l, k)\, c_l(\Omega_M) \wedge \phi^{k-l}.$$

In case $M$ is compact—i.e., is a projective algebraic variety—the integrals in (3.19) depend only on the tangential Chern classes $c_q(M) \in H^{2q}(M)$ and hyperplane class $\phi \in H^2(M)$. So we conclude that, just as the Wirtinger theorem (3.14) implies that the volume of $M$ is equal to (a constant times) its degree $\int_M \phi^n$, the formula (3.19) implies that the volume of the tube is again of a topological character—as noted above, this is in strong contrast to the real case.

The second remark concerns the volume of the tube near a singularity of an analytic variety $V_n \subset \mathbb{C}^N$. Setting $B[r] = \{\|z\| \leq r\}$, we suppose given an analytic variety in $B[1 + \epsilon]$ for some $\epsilon > 0$ and denote by $V$ that part of the variety in the unit ball. Then we claim that the integrals[9]

$$\int_{V^*} c_K(\Omega_{V^*}) \wedge \phi^{n-k}$$

converge, and consequently *the volume of the tube around a singular variety is finite*. To establish this, let

$$\Gamma \subset V \times G(n, N)$$

be the closure of the graph of the Gauss map

$$\gamma : V^* \to G(n, N).$$

It is easy to see that $\Gamma$ is an analytic variety of pure dimension $n$, and that the projection

$$\Gamma \xrightarrow{\pi} V$$

is an isomorphism on that part $\Gamma^*$ lying over $V^*$. In general there will be blowing down over the singular set of $V$, since for $z_0 \in V_s$ the fibre $\pi^{-1}(z_0) \subset G(n, N)$ is the limiting position of tangent planes $T_{z(t)}(V^*)$ along all analytic arcs $\{z(t)\} \subset V^*$ with $\lim_{t \to 0} z(t) = z_0$. Now the forms $\phi^{n-k} \wedge c_k(\Omega_E)$ are well-defined on $\mathbb{C}^N \times G(n, N)$, and hence by (3.12) are integrable over the smooth points of $\Gamma$. But then since

$$\gamma^*(\Omega_E) = \Omega_{V^*},$$

we conclude that

$$\int_{V^*} c_k(\Omega_{V^*}) \wedge \phi^{n-k} = \int_{\Gamma^*} c_k(\Omega_E) \wedge \phi^{n-k}$$

converges as asserted.

We remark that, upon approaching a singular point $z \in V_s$, as noted above the Gauss mapping may or may not extend according as to whether or not the tangent planes $T_{z(t)}(V^*)$ have a unique limiting position for all arcs $z(t)$ in $V^*$ which tend to $z$. Thus when $\dim V = 1$ it always extends, and in this case it may or may not happen that the Gaussian curvature $K(z(t)) \to -\infty$ as $z(t) \to z$. In fact the Gaussian curvature remains finite if, and only if, $z$ is an ordinary singularity of $V$.

Now we assume that $V$ is smooth except possibly at the origin. Then we will prove that the function

$$\mu_k(V, r) = \frac{(-1)^k}{r^{2n-2k}} \int_{V[r]} c_k(\Omega_V) \wedge \phi^{n-k} \tag{3.20}$$

is an increasing function of $r$[10]. For this we use the notations of section 3(b) above. By (3.6) the differential form $(-1)^k c_k(\Omega_V)$ is $C^\infty$, closed, and non-negative on $V^* = V - \{0\}$. Since $\omega = d\, d^c \log \|z\|^2$ is the pullback of the Kähler form on $\mathbb{P}^{N-1}$,

$$(-1)^k \int_{V[\rho,r]} c_k(\Omega_V) \wedge \omega^{n-k} \geq 0. \tag{3.21}$$

On the other hand, setting

$$\eta_k = d^c \log \|z\|^2 \wedge \omega^{n-k-1}$$

we have from $d\eta_k = \omega^{n-k}$ and Stokes' theorem that

$$(-1)^k \int_{V[\rho,r]} c_k(\Omega_V) \wedge \omega^{n-k} = (-1) \int_{\partial V[r]} c_k(\Omega_V) \wedge \eta_k - (-1)^k \int_{\partial V[\rho]} c_k(\Omega_V) \wedge \eta_k.$$

But on the sphere $\|z\| = t$

$$\eta_k = \frac{1}{t^{2n-2k}} d^c\|z\|^2 \wedge \phi^{n-k-1}$$

and therefore

$$\int_{\partial V[t]} c_k(\Omega_N) \wedge \eta_k = \frac{1}{t^{2n-2k}} \int_{\partial V[t]} c_k(\Omega_V) \wedge d^c\|z\|^2 \wedge \phi^{n-k-1} = \frac{1}{t^{2n-2k}} \int_{V[t]} c_k(\Omega_V) \wedge \phi^{n-k}.$$

Combining this with (3.21) and (3.22) gives our result.

Now suppose that $V \subset \mathbb{C}^N$ is an entire analytic set. It is by now well-established that the function $\mu_0(V, r)$ gives the basic analytic measure of the growth of $V$, playing to some extent a role analogous to the degree of an algebraic variety in projective space.(11) It may be that in more refined questions the function $\mu_k(V, r)$ should also be used as growth indicators, especially since they appear in the growth of the currents $T_\epsilon(V)$ obtained by the standard smoothing of the current $T_V$ defined by (3.12).

**Footnotes**

1. A good reference for general material on complex manifolds is S. S. Chern, *Complex Manifolds Without Potential Theory*, van Nostrand, 1968.
2. cf. the last section in S. S. Chern, *Characteristic classes of Hermitian manifolds*, Ann. of Math., vol. 47 (1946), pp. 85-121.
3. This is just another way of saying that the $n \times (N - n)$ matrix of forms $\{\omega_{\alpha\mu}\}$ gives a basis for the (1, 0) tangent space to $G(n, N)$.
4. A *ruled surface* is the locus of $\infty^1$ straight lines $L_t$ in $\mathbb{C}^{n+2}$; the tangent lines to a curve in $\mathbb{C}^{n+2}$ form a *developable ruled surface*; and finally a *cone* consists of $\infty^1$ lines through a fixed point, possibly at infinity. We note the following global corollary of (3.11), which was pointed out to us by Joe Harris: If $S \subset \mathbb{P}^N$ is a smooth algebraic surface which is not a plane, then the Gaussian image of $S$ is two-dimensional.
5. P. Lelong, *Fonctions plurisousharmoniques et formes differentielles positiv*, Paris, Gordon Breach, 1968.
6. This is the basic integral formula in holomorphic polar coordinates (these coordinates essentially amount to the standard Hopf bundle over $\mathbb{P}^{N-1}$).
7. P. Thie, *The Lelong number of points of a complex analytic set*, Math. Ann., vol. 172 (1967), pp. 269-312.
8. cf. the reference cited in footnote (7) of the introduction.
9. Recall that $V^* = V - V_s$ is the manifold of smooth points on $V$.
10. By closer examination of the behaviour near a singularity it seems likely that this remains true with no assumptions about the singularities of $V$.
11. cf. H. Skoda, *Sous-ensembles analytiques d'ordre fini ou infini dans* $\mathbb{C}^N$, Bull. Soc. Math. France, vol. 100 (1972), pp. 353-408.

## 4. Hermitian integral geometry

(a) *The elementary version of Crofton's formula.* We shall first take up the complex analogue of (2.1). Let $C$ be an analytic curve defined in some open set in $\mathbb{C}^2$. Denote by $\tilde{G}(, n)$ the Grassmannian of complex affine $k$-planes in $\mathbb{C}^n$, so that $\tilde{G}(1, 2)$ is the space of complex lines in the plane. For each line $L$ the analytic intersection number $\#(L, C)$ is defined and is a non-negative integer. By a basic fact in complex-analytic geometry, this is also the geometric number of intersections $n(L \cap C)$ of the line $L$ with the curve $C$. The point is that since complex manifolds are naturally oriented the geometric and topological intersection numbers coincide. We will prove that

$$\int n(L \cap C)dL = \text{vol}\,(C) \tag{4.1}$$

where $dL$ is a suitably normalized invariant measure on $\tilde{G}(1, 2)$.

In fact, we will give three quite different proofs of (4.1). The first will be a direct computation using frames analogous to the proof of (2.1). The remaining two will be based on general invariant-theoretic principles, and will generalize to give proofs of our two main integral-geometric formulas below.

*Proof #1.* Associated to $C$ is the manifold $\mathcal{F}(C)$ of Darboux frames $\{z^*; e_1^*, e_2^*\}$ where $z^* \in C$ and $e_1^* \in T_{z^*}(C)$ (cf. the discussion in §3(a)). The structure equations are

$$dz^* = \omega_1^* e_1^*$$

$$de_1^* = \omega_{11}^* e_1^* + \omega_{12}^* e_2^*$$

We observe that $\omega_1^*$ and $\omega_{12}^*$ are forms of type (1, 0), and that writing $\omega_{12}^* = h\,\omega_1^*$ the Kähler form (= volume form) and 1st Chern form in the tangent bundle are given respectively by

$$\phi = \frac{\sqrt{-1}}{2} \omega_1^* \wedge \bar{\omega}_1^*$$

$$c_1(\Omega_C) = \frac{\sqrt{-1}}{2\pi} \omega_{12}^* \wedge \bar{\omega}_{12}^*$$

$$= \frac{1}{4\pi} K \cdot \phi$$

where $K = -4|h|^2$ is the Gaussian curvature of the Riemann surface $C$.

Associated to a line $L \in \tilde{G}(1, 2)$ are the frames $\{z; e_1, e_2\}$ where $L$ is the line through $z$ in the direction $e_1$. Recalling the structure equations (3.1), we infer as in the real case that

$$dL = C^{\underline{te}}\, \omega_2 \wedge \bar{\omega}_2 \wedge \omega_{12} \wedge \bar{\omega}_{12}.$$

Proceeding as we did there, set

$$B = \{(z, L) : z \in L\} \subset \mathbb{C} \times \tilde{G}(1, 2)$$

so that we have a diagram

$$\begin{array}{ccc} B & \xrightarrow{F} & \tilde{G}(1,2) \\ \downarrow \pi & & \\ C & & \end{array}$$

The left hand side of (4.1) is $\int_B F^*\, dL$, and we shall evaluate this integral by integration over the fibres of $\pi$. Fixing $z \in C$, the lines through $z$ may be given parametrically by

$$z = z^*$$
$$e_1 = A_{11}e_1^* + A_{12}e_2^*$$
$$e_2 = A_{21}e_1^* + A_{22}e_2^*$$

where $A = (A_{ij}) \in U(2)$. From

$$\omega_2 = (dz_1, e_2) = A_{21}\omega_1^*$$

and

$$\begin{aligned}\omega_{12} &= (de_1, e_2) \\ &\equiv dA_{11}\bar{A}_{21} + dA_{12}\bar{A}_{22} \bmod \omega_1^* \quad \text{and } \bar{\omega}_1^*.\end{aligned}$$

we deduce that

$$F^*(dL) = \phi \wedge \psi(A, dA)$$

where $\psi(A, dA)$ is a 2-form on $U(2)$. The integral of $\psi$ over the fibre $\pi^{-1}(z)$ is a constant independent of $z$, and this implies (4.1).

*Proof #2.* This is based on the Wirtinger theorem (3.14), and the analytic curve $C$ will appear only at the very end.

We begin by defining the incidence correspondence

$$\mathrm{I} \subset \mathbb{C}^2 \times \tilde{G}(1, 2)$$

to be $\{(z, L) : z \in L\}$. The two projections

$$\begin{array}{ccccc} & & \mathrm{I} & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ \mathbb{C}^2 & & & & \tilde{G}(1, 2) \end{array}$$

have respective fibres $\pi_1^{-1}(z) = \{\text{lines through } z\}$, and $\pi_2^{-1}(L) = \{\text{points on the line } L\}$. Clearly I is a 3-dimensional complex manifold, and $\pi_2^* \, dL$ is a positive (2, 2) form on I. Integration over the fibres of the first projection gives

$$\hat{\phi} = (\pi_1)_*(\pi_2^* \, dL)$$

with the properties[1]: i) $\hat{\phi}$ is a (1, 1) form on $C^2$, and (ii) $\hat{\phi}$ is invariant under the group of unitary motions $z \to Az + b$, $A \in U(2)$. It follows easily that $\hat{\phi}$ is a constant multiple of the standard Kähler form $\phi$.[2]

Now setting $B = \pi_1^{-1}(C)$ and using (4.2) and $\pi_2^* \, dL|_B = F^* \, dL$

$$\begin{aligned}\int_B F^* \, dL &= \int_C (\pi_1)_* \, (F^* \, dL) \\ &= C^{te} \int_C \phi \\ &= C^{te} \operatorname{vol}(C)\end{aligned}$$

by the Wirtinger theorem. Q.E.D.

Note that this same argument fails in the real case due to

$$(\pi_1)_* (\pi_2^* \, dL) = 0,$$

expressing the fact that a real line has no natural orientation and so appears twice, with opposite orientations, in $\pi_1^{-1}(z)$. The point is that in the real case we must evaluate $\int n(L \cap C) \, dL$ as $\int_B |F^* \, dL|$, but because everything is oriented the absolute value signs disappear in the complex case.

*Proof #3*. In anticipation of later needs we shall prove an analogue of (4.1) for a complex analytic curve $C$ in the complex projective space $\mathbb{P}^n$, where $C$ is considered as the image of a Riemann surface with smooth boundary under a holomorphic mapping. Denoting by $\omega$ the Kähler form of the Fubini-Study metric on $\mathbb{P}^n$ and $\mathbb{P}^{n*}$ the space of hyperplanes $H$ in $\mathbb{P}^n$, the result we shall prove is

$$\frac{1}{\pi} \int_{P^{n*}} n(H \cap C) dH = \int_C \omega. \tag{4.3}$$

For the proof we denote points of $\mathbb{P}^n$ by $Z = [z_0, \cdots, z_n]$, hyperplanes $H \in P^{n*}$ by $H = [h_0, \cdots, h_n]$, and set

$$\begin{cases} \langle H, Z \rangle = \sum_i h_i z_i, & \text{and} \\ |H, Z| = |\langle H, Z \rangle|. \end{cases}$$

Then $\omega = dd^c \log \|Z\|^2$, and the 1-form

$$\eta_H = d^c \log \frac{\|Z\|^2 \cdot \|H\|^2}{|Z, H|^2}$$

is smooth in $\mathbb{P}^n - H$ and satisfies $d\eta_H = \omega$ there. Along $H$ it has a singularity, which in a normal disc $\{|t| < \epsilon\}$ looks up to a $C^\infty$ term like

$$\begin{aligned} d^c \log \frac{1}{|t|^2} &= \frac{\sqrt{-1}}{4} \left( \frac{dt}{t} - \frac{d\bar{t}}{\bar{t}} \right) \\ &= -\frac{1}{2} \, d\theta \end{aligned}$$

where $t = re^{i\theta}$. It follows that $\eta_H$ is integrable on $\mathbb{P}^n$ and satisfies the equation of currents[(3)]

$$\frac{1}{\pi} \, d\eta_H = \frac{1}{\pi} \omega - T_H \tag{4.4}$$

where $T_H$ is the current defined by integration over $H$. Assuming $H$ meets $C$ at a finite number of interior points, by (4.4) and Stokes' Theorem

$$\frac{1}{\pi}\int_C \omega = \#(H, C) + \frac{1}{\pi}\int_{\partial C}\eta_H. \tag{4.5}$$

Ignoring momentarily questions of convergence, we integrate (4.5) over $\mathbb{P}^{n*}$ and interchange an order of integration to obtain

$$\frac{1}{\pi}\int_C \omega = \int_{\mathbb{P}^{n*}} \#(H, C)dH + \int_{\partial C}\eta$$

where

$$\eta = \frac{1}{\pi}\int_{\mathbb{P}^{n}*} \pi_I \;\; t\,I$$

is the average of $\eta_H$. We claim that

$$\eta = 0, \tag{4.6}$$

which will certainly prove (4.3).

Now (4.6) can be proved by direct computation, but here is an alternative invariant-theoretic argument. We may consider $\eta$ as a current $T_\eta$ on $\mathbb{P}^n$ by

$$T_\eta(\alpha) = \frac{1}{\pi}\int_{\mathbb{P}^{n*}\times\mathbb{P}^n} \eta_H \wedge \alpha \; dH, \qquad \alpha \in A^{2n-1}(\mathbb{P}^n).$$

Since the unitary group $U(n+1)$ acts transitively on $\mathbb{P}^{n*}$ and satisfies

$$g^* \, dH = dH$$

$$g^*\eta_H = \eta_{gH}$$

for $g \in U(n+1)$, we deduce that $T_\eta$ is an invariant current. But it is well known (and easily proved) that the invariant currents on any compact symmetric space are just the harmonic forms,[4] and since the degree of $\eta$ is odd we conclude that $\eta = 0$.

Finally, the justification of the interchange of limits follows by a standard (and not particularly delicate) argument.

(b) *Crofton's formula for Schubert cycles*. One generalization of (4.1) deals with the intersection of Schubert cycles with an analytic subvariety $V$ in the Grassmannian $G(n, N)$. Recall that a flag $F$ in $\mathbb{C}^N$ is an increasing sequence of subspaces

$$(0) = W_0 \subset W_1 \subset \cdots \subset W_{N-1} \subset W_N = \mathbb{C}^N.$$

The unitary group $U(N)$ acts transitively on the manifold $F(N)$ of all flags. For each flag $F$ and sequence of integers $a = (a_1, \cdots, a_n)$ the Schubert cycle $\Sigma_a(F)$ is defined by

$$\Sigma_a(F) = \{T \in G(n, N): \dim T \cap W_{N-n+i-a_i} \geq i\}. \tag{4.7}$$

Thus $\Sigma_a(F)$ is the set of $n$-planes which fail to be in general position with respect to the flag $F$ in the amount specified by (4.7). The Schubert cycle forms an irreducible analytic variety of complex codimension $|a| = \sum_i a_i$ in the Grassmannian, and clearly

$$g\,\Sigma_a(F) = \Sigma_a(gF), \qquad g \in U(N).$$

Up to a constant there is a unique invariant measure on the space of Schubert cycles with fixed index sequence $a$; for simplicity of notation we shall write a general Schubert cycle as $\Sigma_a$ with $d\,\Sigma_a$ denoting the invariant measure.

We recall from section 3(a) the curvature matrix $\Omega_E = (\Omega_{\alpha\beta})$ of the universal $n$-plane bundle $E \to G(n, N)$. A fundamental result[5] is that the de Rham cohomology of the Grassmannian is represented by the invariant polynomials $P(\Omega_E)$ in the curvature matrix, and moreover these are just the polynomials in the basic Chern forms $c_k(\Omega_E)$ defined by (3.5). In particular, the fundamental class of each Schubert cycle $\Sigma_a(F)$ is represented by a polynomial $P_a(\Omega_E)$ which clearly does not depend on the particular choice of flag. For any closed algebraic subvariety $V \subset G(n, N)$ of dimension $|a|$ the intersection relation

$$\int_V P_a(\Omega_E) = \#\,(V, \Sigma_a) \tag{4.8}$$

is valid. We wish to prove that (4.8) is also true locally on the average. More precisely, we will prove that for $V \subset G(n, N)$ a piece of analytic variety of pure dimension $d = |a|$, we have *Crofton's formula (I)*:

$$\int_V P_a(\Omega_E) = \int \#\,(V, \Sigma_a)\,d\,\Sigma_a. \tag{4.9}$$

*Proof.* This is formally the same as our third proof of (4.1). For each flag $F \in F(N)$ there is a $(2d-1)$ form $\eta_{F,a}$ which satisfies,[6]

i) $\eta_{F,a}$ is integrable on $G(n, N)$ and $C^\infty$ on the complement of $\Sigma_a(F)$;
ii) the equation of currents

$$d\eta_{F,a} = P_a(\Omega_E) - T_{\Sigma_a(F)}$$

is valid; and
iii) for any unitary transformation $g \in U(N)$,

$$\eta_{gF,a} = g^*\eta_{F,a}.$$

The formula (4.9) is of a local character and so we may assume $V$ is a complex manifold with smooth boundary. Then for any $F \in F(N)$ such that $\Sigma_a(F)$ meets $V$ at a finite number of interior points, by property (ii) and Stokes' theorem

$$\int_V P_a(\Omega_E) = \#\,(V, \Sigma_a(F)) + \int_{\partial V} \eta_{F,a}.$$

Averaging this formula over $U(N)$, we are reduced to showing that the average

$$\eta = \int g^*\eta_{F,a}\,dg$$

is zero. But by the third property, for $g \in U(N)$

$$g^* \eta = \eta$$

and, as in the previous argument, $\eta$ is an invariant current of odd degree and is therefore identically zero. Q.E.D.

As an example, suppose we consider the sequence $(k, 0, \cdots, 0)$; the corresponding Schubert cycle will be denoted by $\Sigma_k$. For $k = n$ it is determined by a hyperplane $H \in \mathbb{P}^{N-1*}$, and is described by

$$\Sigma_n = \{T \in G(n, N); T \subset H\}. \tag{4.10}$$

In general, the polynomial corresponding to the Schubert cycle $\Sigma_k$ is $(-1)^k c_k(\Omega_E)$ where $c_k(\Omega_E)$ is the $k$th basic Chern form given explicitly by (3.5).

As an application of (4.9), we consider a complex manifold $M \subset \mathbb{C}^N$ and consider its image under the holomorphic Gauss mapping[7]

$$\gamma: M \to G(n, N).$$

We recall from section 3(a) that

$$\begin{cases} T(M) = \gamma^* E; \quad \text{and} \\ \Omega_M = \gamma^* \Omega_E, \end{cases}$$

and therefore (4.9) implies that

$$\int_M P_a(\Omega_M) = \int \#\,(\gamma(M), \Sigma_a)\, d\,\Sigma_a \tag{4.11}$$

for any codimension $n$ Schubert cycle $\Sigma_a$. In particular, taking $\Sigma_a = \Sigma_n(H)$ we infer from (4.10)

$$\#\,(\gamma(M), \Sigma_n(H)) = \begin{cases} \text{number of points } z \in M \text{ where the tangent} \\ \text{plane } T_z(M) \text{ lies in } z + H \end{cases} \tag{4.12}$$

$$= n(M, H)$$

where the last equality is a definition. Since by (3.10)

$$(-1)^n c_n(\Omega_M) = C^{\underline{te}} K\, dM$$

is the Gauss-Bonnet integrand, we have arrived at what might be called the *average Gauss-Bonnet theorem*

$$(-1)^n C^{\underline{te}} \int_M K dM = \int_{\mathbb{P}^{N-1*}} n(M, H)\, dH \tag{4.13}$$

where the integrand on the right is given by (4.12).

To interpret this equation we recall the usual form

$$C^{\underline{te}} \int_M K dM = \chi(M) + \int_{\partial M} k_g \tag{4.14}$$

of the Gauss-Bonnet theorem for a manifold with boundary. Here, $\chi(M)$ is the topological Euler characteristic and $k_g$ is the generalized "geodesic curvature". In case $M$ is compact the boundary integral is out and by Hopf's theorem $\chi(M)$ is $(-1)^n$ times the number of zeroes of a generic section of $T^*(M)$. Now each linear function

$$H(z) = \sum_i h_i z_i$$

gives on $\mathbb{C}^N$ the 1-form

$$\psi_H = \sum_i h_i \, dz_i,$$

and by (4.12) $n(M, H)$ is just the number of zeroes of $\psi_H$. Consequently we may think of the right hand side of (4.13) as the average number of zeroes of 1-forms $\psi_H$.

(c) *The second Crofton formula.* Considering still a complex manifold $M \subset \mathbb{C}^N$ we now wish to know about the intersections of its Gaussian image with Schubert cycles $\Sigma_a$ of codimension $n - k < n$. For a general affine $(N - k)$ plane $L \in \tilde{G}(N - k, N)$ the intersection $M \cap L$ will be a complex manifold of dimension $n - k$, and (4.11) applies to give

$$\int_{M \cap L} P_a(\Omega_{M \cap L}) = \int \#\,(\gamma(M \cap L), \Sigma_a)\, d\,\Sigma_a. \tag{4.15}$$

Here, $\gamma(M \cap L)$ is the Gaussian image of the intersection $M \cap L$ (*not* the Gaussian image of $M$ along $M \cap L$), and $\Omega_{M \cap L}$ is the curvature of $M \cap L$ (*not* the curvature of $M$ restricted to $M \cap L$). Averaging both sides of (4.15) over $\tilde{G}(N - k, N)$ gives

$$\int \left( \int_{M \cap L} P_a(\Omega_{M \cap L}) \right) dL = \iint \#\,(\gamma(M \cap L), \Sigma_a)\, d\,\Sigma_a\, dL. \tag{4.16}$$

It is clearly desirable to express the left hand side of (4.16) as a curvature integral on all of $M$, and for the basic Chern forms this is accomplished by the *Crofton formula (II)*:

$$\int \left( \int_{M \cap L} c_{n-k}(\Omega_{M \cap L}) \right) dL = \int_M c_{n-k}(\Omega_M) \wedge \phi^k. \tag{4.17}$$

Before embarking on the proof we remark that (4.17) should be considered deeper than (4.9) in that the curvature of a variable intersection $M \cap L$ is being related to the curvature of $M$. Moreover, as it stands (4.17) does not appear to have an obvious real analogue in terms of Pontryagin classes.

The reason why (4.17) holds is related to Chern's general kinematic formula discussed in §2(c), especially the reproductive property which we recall is the following: For $M \subset \mathbb{R}^N$ an oriented real manifold and $\tilde{G} = \tilde{G}_R(N - k, N)$ the

Grassmannian of oriented affine $(N - k)$ planes,

$$\int_{\tilde{G}} \left( \int_{M \cap L} I_l(R_{M \cap L}) \right) dL = C^{\underline{te}} \int_M I_l(R_M) \tag{4.18}$$

where the $I_l(R)$ are the curvature polynomials (1.9) appearing in Hermann Weyl's tube formula (1.10). In the complex case we have seen in (3.18) that the integrals in the tube formula are just those on the right hand side of (4.17), and consequently this result is the complex analogue of Chern's formula (4.18).

Now presumably the real proof could be adapted to give (4.17). However, it is convenient to take advantage of properties peculiar to the complex case, and so we shall give an argument along the lines of the second proof of (4.1). The following notations and ranges of indices will be used:

$G(l, N)$ = Grassmann manifold of $\mathbb{C}^l$'s through the origin in $\mathbb{C}^N$;

$\tilde{G}(l, N)$ = Grassmannian of affine $l$-planes in $\mathbb{C}^N$;

$\hat{G}(l, N)$ = flag variety of pairs $\{z, L\} \in \mathbb{C}^N \times \tilde{G}(l, N)$ where $z \in L$; note that

$$\hat{G}(l, N) \simeq \mathbb{C}^N \times G(l, N)$$

where $\{z, L\}$ maps to $(z, L_{-z})$ with $L_{-z}$ denoting the translate of $L$ by $-z$;

$\hat{G}(l, m, N)$ = set of triples $\{z, S, L\} \in \mathbb{C}^N \times \tilde{G}(l, N) \times \tilde{G}(m, N)$ where

$$z \in S \subset L;$$

$U \to \hat{G}(l, m, N)$ is the universal vector bundle with fibre $S_{-z}$ over $\{z, S, L\}$; and

$$\begin{gathered} 1 \le a, b, \le n - k;\ 1 \le \alpha, \beta \le n;\ 1 \le A, B \le N - k;\ 1 \le i, j, k \le N \\ n - k + 1 \le r, s \le N - k;\ n + 1 \le \mu, \nu \le N;\ N - k + 1 \le \rho, \sigma \le N. \end{gathered} \tag{4.19}$$

To each frame $\{z; e_1, \cdots, e_N\} \in \mathscr{F}(\mathbb{C}^N)$ we associate the point $\{z, S, L\} \in \hat{G}(n - k, N - k, N)$ where

$$S_{-z} = e_1 \wedge \cdots \wedge e_{n-k}, \quad L_{-z} = e_1 \wedge \cdots \wedge e_{N-k}.$$

Setting

$$\Omega_{ab} = - \sum_r \omega_{ar} \wedge \bar{\omega}_{br}$$

we have from (3.4) and (3.5)

$$\begin{aligned} c_{n-k}(\Omega_U) &\equiv C^{\underline{te}} \det \Omega_{ab} \bmod \omega_{a\rho}, \bar{\omega}_{a\rho} \\ dL &= C^{\underline{te}} \bigwedge_\rho \omega_\rho \wedge \bar{\omega}_\rho \bigwedge_{A,\rho} \omega_{A\rho} \wedge \bar{\omega}_{A\rho} \end{aligned} \tag{4.20}$$

Then

$$\Psi = c_{n-k}(\Omega_U) \wedge dL$$

is a well-defined differential form of type $(p, p)$ on $\hat{G}(n - k, N - k, N)$ where

$$p = n + k(N - k).$$

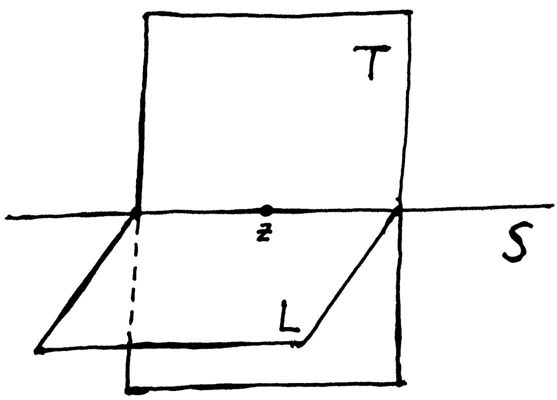
We now denote points in $\hat{G}(n, N)$ by $\{z^*, T\}$ and consider the incidence correspondence

$$I \subset \hat{G}(n, N) \times \hat{G}(n - k, N - k, N)$$

defined to be $\{z^*; T; z, S, L\}$ where

$$z = z^*, \qquad S \subset T$$

Recall that $T$, $S$, $L$ are affine $n$, $n - k$, and $N - k$ planes all passing through $z$, and so in general $\dim (T \cap L) = n - k$. The picture for $N = 3, n = 2, k = 1$ is




Consider now the projection

$$\pi : I \to \hat{G}(n, N)$$

onto the first factor. The typical fibre $F$ may be identified as follows: In $\mathbb{C}^N$ we fix a $\mathbb{C}^n$ and consider all flags $S \subset L$ where $\dim S = n - k$, $\dim L = N - k$, and $S \subset \mathbb{C}^n$. This is $F$, which is itself fibered according to

$$G(N - n, N - n + k) \to F \to G(n - k, n)$$

by $\{S, L\} \to S$. It follows that $\dim F = k(N - k)$, and consequently the fibre integral

$$\psi = \pi_* \Psi$$

is a form of type $(n, n)$ on $\hat{G}(n, N) \simeq \mathbb{C}^N \times G(n, N)$. It seems possible that, for $dL$ suitably normalized,

(4.21) $$\psi = \phi^k \wedge c_{n-k}(\Omega_E)$$

where $\Omega_E$ is the curvature in the universal bundle $E \to G(n, N)$, but we are unable to establish (4.21). As will seen below, (4.17) would be an immediate consequence of this result.

What we will do is prove enough of (4.21) for our purposes. Namely, on $\mathbb{C}^N \times G(n, N)$ there is an intrinsic differential ideal [8] $\mathcal{I}$ defined as follows: Over $\{z, T\}$ we consider frames $\{z; e_1, \cdots, e_N\}$ where $T_{-z} = e_1 \wedge \cdots \wedge e_n$. Then the forms

$$\omega_\mu, \; d\omega_\mu \; (\mu = n + 1, \cdots, N)$$

generate an intrinsic differential ideal $\mathcal{I}$, and we shall prove that

$$\psi \equiv \phi^k \wedge c_{n-k}(\Omega_E) \text{ modulo } \mathcal{I}. \tag{4.22}$$

Assuming this result for a moment, we will complete the proof of (4.17). For this consider $M$ as embedded in $\hat{G}(n, N) \cong \mathbb{C}^n \times G(n, N)$ by the refined Gauss mapping

$$z \to \{z, T_z(M)\}.$$

Using Darboux frames we see that $M$ is an integral manifold of the differential system $\mathcal{I}$, so that by (4.22) and (4.2) (cf. footnote[2])

$$\begin{aligned}\int_M \phi^k \wedge c_{n-k}(\Omega_M) &= \int_M \psi \\ &= \int_{\pi^{-1}(M)} \Psi \\ &= \int_{\hat{G}(N-k,N)} \left( \int_{M \cap L} c_{n-k}(\Omega_U) \right) dL.\end{aligned}$$

Now $M \cap L$ is mapped into $\hat{G}(n - k, N - k, N)$ by $z \to \{z, T_z(M \cap L), L\}$ for generic $L$, and so

$$\int_{M \cap L} c_{n-k}(\Omega_U) = \int_{M \cap L} c_{n-k}(\Omega_{M \cap L}).$$

Combining with the previous step gives (4.17).

Turning to the proof of (4.22), we first note that the form $\psi$ on $\mathbb{C}^N \times G(n, N)$ is invariant under the group of transformations

$$\{z, T\} \to \{gz + b, gT\}$$

where $g \in U(N)$ and $b \in \mathbb{C}^N$. It follows that $\psi$ is determined by its value at one point, say $\{0, \mathbb{C}^n\}$. If we write the (1, 0) cotangent space at this point as $V \oplus W$,[9] then

$$\begin{aligned}\psi &\in \wedge^n(V \oplus W) \otimes \wedge^n(\bar{V} \oplus \bar{W}) \\ &\simeq \bigoplus_{p,q} (\wedge^p V \otimes \wedge^q \bar{V}) \otimes (\wedge^{n-p} W \otimes \wedge^{n-q} \bar{W}).\end{aligned} \tag{4.23}$$

We will first show that under the decomposition (4.23),

$$\psi \in (\wedge^k V \otimes \wedge^k \bar{V}) \otimes (\wedge^{n-k} W \otimes \wedge^{n-k} \bar{W}). \tag{4.24}$$

To verify this apply the automorphism $z^* = \rho z$ to $\mathbb{C}^N$. The Maurer-Cartan forms $\omega_{ij} = (de_i, e_j)$ and volume form on $G(N-k, N)$ are preserved, while $\omega_i^* = (dz^*, e_i) = \rho\omega_i$. From (4.20) we infer that

$$\Psi^* = |\rho|^{2k}\Psi,$$

which implies

$$\psi^* = |\rho|^{2k}\psi.$$

On the other hand, if $\psi = \sum \psi_{p,q}$ is the decomposition (4.23) of $\psi$,

$$\psi^*_{p,q} = \rho^p \bar{\rho}^q \psi_{p,q},$$

from which we conclude that $\psi = \psi_{k,k}$, thus establishing (4.24).

In terms of a frame $\{0; e_1, \cdots, e_N\}$ lying over the fixed point and using the index range (4.19), $\psi$ is a sum of terms

$$P(\omega_\alpha \wedge \bar{\omega}_\beta, \omega_\alpha \wedge \bar{\omega}_\mu, \omega_\mu \wedge \bar{\omega}_\beta, \omega_\mu \wedge \bar{\omega}_\nu)\; Q(\omega_{\alpha\mu} \wedge \bar{\omega}_{b\nu})$$

where $P$ is a homogeneous polynomial of degree $k$ and $Q$ is one of degree $n-k$. When the frame undergoes a rotation

$$e_\alpha \to \sum_\beta g_{\alpha\beta} e_\beta, \qquad e_\mu \to \sum_\nu h_{\mu\nu} e_\nu$$

for unitary matrices $g$ and $h$,

$$\begin{cases} \omega_\alpha \to \sum_\beta g_{\alpha\beta}\omega_\beta,\ \omega_\mu \to \sum_\nu h_{\mu\nu}\omega_\nu \\ \omega_{\alpha\mu} \to \sum_{\beta,\nu} g_{\alpha\beta}\omega_{\beta\nu}\bar{h}_{\mu\nu} \end{cases}$$

From the theory of unitary invariants[10] it follows that $\psi$ is expressible in terms of the quantities

$$\phi_1 = \sum_\alpha \omega_\alpha \wedge \bar{\omega}_\alpha,\ \phi_2 = \sum_\mu \omega_\mu \wedge \bar{\omega}_\mu,\ \Omega_{\alpha\beta} = -\sum_\mu \omega_{\alpha\mu} \wedge \bar{\omega}_{\beta\mu}$$

$$\eta = \sum_{\alpha,\mu} \omega_\alpha \wedge \omega_{\alpha\mu} \wedge \bar{\omega}_\mu,\ \bar{\eta},\ \xi = \sum_{\alpha,\beta} \omega_\alpha \wedge \Omega_{\alpha\beta} \wedge \bar{\omega}_\beta, \tag{4.25}$$

$$\sum_{\alpha,\beta,\mu} \omega_\alpha \wedge \omega_{\alpha\mu} \wedge \bar{\omega}_{\beta\mu} \wedge \bar{\omega}_\beta, \quad \text{and} \quad \sum_{\alpha,\mu,\nu} \bar{\omega}_\mu \wedge \omega_{\alpha\mu} \wedge \bar{\omega}_{\alpha\nu} \wedge \omega_\nu.$$

Of course we would like to show that $\psi$ is a constant multiple of $(\phi_1 + \phi_2)^k \wedge \det \Omega_{\alpha\beta}$, which would prove (4.21). To establish (4.22) we observe that modulo the differential ideal $\mathcal{I} = \{\omega_\mu, d\omega_\mu\}$ the list (4.25) reduces to the quantities.

$$\phi_1,\ \Omega_{\alpha\beta}$$

$\left(\text{we use here that } d\omega_\mu \equiv \sum_\alpha \omega_\alpha \wedge \omega_{\alpha\mu} \bmod \omega_\nu\right)$. It follows that

$$\psi \equiv C^{\underline{te}}\phi_1^k \wedge P_{n-k}(\Omega_{\alpha\beta}) \bmod \mathscr{I} \tag{4.26}$$

where $P_{n-k}(\Omega_{\alpha\beta})$ is homogeneous of degree $n - k$ and invariant under

$$\Omega_{\alpha\beta} \to \sum_{\gamma,\delta} g_{\alpha\gamma}\bar{g}_{\beta\delta}\Omega_{\gamma\delta}.$$

Appealing again to the theory of unitary invariants (cf. footnote (5)) we deduce that $P_{n-k}(\Omega_{\alpha\beta})$ is a polynomial in $c_1(\Omega_{\alpha\beta}), \cdots, c_{n-k}(\Omega_{\alpha\beta})$.

To determine which polynomial we may argue as follows: In $G(n, N) \times G(n-k, N-k, N)$ we consider that part $I_0$ of the incidence correspondence lying over $z = 0$. Denoting by $\zeta$ the pullback to $G(n-k, N-k, N)$ of the fundamental class of $G(n-k, N)$ and by $c_{n-k}(U)$ the Chern class of $U \to G(n-k, N-k, N)$,

$$c_{n-k}(U) \wedge \zeta \in H^{2(n-k+k(N-k))}(I_0).$$

Under the projection $I_0 \xrightarrow{\pi} G(n, N)$, the Gysin image

$$\pi_*(c_{n-k}(U) \wedge \zeta) \in H^{2(n-k)}(G(n, N))$$

is expressible as a polynomial in $c_1(E), \cdots, c_{n-k}(E)$, and this polynomial is just $P_{n-k}$ whose determination is consequently a topological question. Since our argument that $P_{n-k} = C^{\underline{te}}c_{n-k}(E)$ is messy and we have no need for the explicit form of the result the argument will be omitted.

We observe that since any polynomial in the quantities (4.25) gives an invariant differential form on $\tilde{G}(n, N)$, an invariant-theoretic proof of (4.21) will require use of the additional property $d\psi = 0$. For example, let us examine the terms $\phi_1^l \wedge \phi_2^{k-l}$ which might appear in $\psi$. Since by the structure equations (3.2)

$$d\phi_1 = \eta + \bar{\eta} = -d\phi_2,$$

we deduce that

$$\begin{cases} d(\phi_1^l \wedge \phi_2^{k-l}) = (\eta + \bar{\eta}) \wedge (\phi_1^{l-1} \wedge \phi_2^{k-l-1})\,(l\phi_2 - (k-l)\phi_1) \\ d(Q(\Omega_{\alpha\beta})) = 0 \end{cases} \tag{4.27}$$

Writing

$$\psi = \left(\sum_{l=0}^{k} c_l\phi_1^l \wedge \phi_2^{k-l} \wedge Q_l(\Omega_{\alpha\beta})\right) + (\text{other terms})$$

it follows recursively from (4.27) and $d\psi = 0$ that

$$c_l = C^{\underline{te}}\binom{k}{l},$$

and therefore

$$\psi = C^{\underline{te}}\phi^k \wedge P(\Omega_{\alpha\beta}) + (\text{other terms}).$$

It seems possible that further additional argument might eliminate the "other terms" thereby leading to a proof of (4.21).

(d) *The third Crofton formula*. For applications to the study of isolated singularities we shall require a variant of (4.17) where the linear spaces are constrained to pass through the origin. Letting $M_n \subset \mathbb{C}^N$ be a complex manifold not passing through the origin and setting

$$\omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \|z\|^2,$$

the formula to be established is the *Crofton formula* III

$$(4.28) \qquad \int\int_{M \cap L_0} c_{n-k}(\Omega_{M \cap L_0})\, dL_0 = C^{\underline{te}} \int_M c_{n-k}(\Omega_M) \wedge \omega^k,$$

where $L_0$ varies over the Grassmannian $G(N-k, N)$ of $\mathbb{C}^{N-k}$'s through the origin on $\mathbb{C}^N$.

The proof of this result is a little long and will be given in several steps. We note that the result is easy when $n - k = 0$; the left hand side is the average of the number $\#(M \cap L_0)$ of intersection points of $M$ with linear spaces through the origin of complementary dimension, and the right hand side is the volume of the image of $M$ under the residual mapping

$$\pi : M \to \mathbb{P}^{N-1}.$$

The equality is then a consequence of the usual Crofton's formula for $\bar{M} = \pi(M)$. In general the idea will be to deduce (4.28) from the analogue of (4.17) for $\bar{M} \subset \mathbb{P}^{N-1}$. The difficulty is that the two metrics in the tangent bundle of $M$ induced from $M \to \mathbb{C}^N$ and $M \to \mathbb{P}^{N-1}$ will not coincide, and it is a priori possible that an additional invariant, such as the angle between the position vector $z \in M$ and tangent $n$-plane $T_z(M)$, could enter into the right hand side of (4.28).

(i) Over $P^{N-1}$ we consider the manifold $\mathcal{F}_0(\mathbb{C}^N)$ of unitary frames $\{Z_0, \cdots, Z_{N-1}\}$ for $\mathbb{C}^N$ where $Z_0$ lies over the point $\bar{Z}_0 = \pi(Z_0) \in P^{N-1}$. On $\mathcal{F}_0(\mathbb{C}^N)$ the structure equations

$$\begin{cases} dZ_i = \sum_{jk=0}^{N-1} \theta_{ij} Z_j, \qquad \theta_{ij} + \bar{\theta}_{ji} = 0 \\ d\theta_{ij} = \sum_{k=0}^{N-1} \theta_{ik} \wedge \theta_{kj} \end{cases}$$

are valid, and

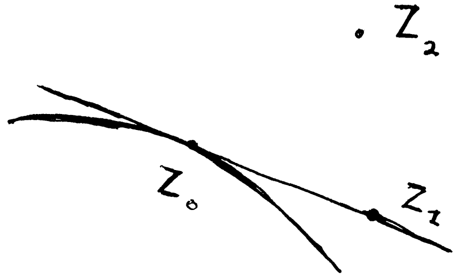
$$\omega = \frac{\sqrt{-1}}{2\pi} \left( \sum_{j=1}^{N-1} \theta_{0j} \wedge \overline{\theta_{0j}} \right)$$

is the pullback to $\mathscr{F}_0(\mathbb{C}^N)$ of the standard form $\omega = \frac{\sqrt{-1}}{2}\partial\bar\partial \log \|z\|^2$ on $\mathbb{P}^{N-1}$—for a proof cf. the reference given in footnote(4) of the introduction.

Now let $\bar M \subset \mathbb{P}^{N-1}$ be a complex manifold. For a point $Z_0 \in \bar M$ we recall that the projective tangent space to $\bar M$ at $\bar Z_0$ is the $P^n_{\bar Z_0}$ obtained as the limiting position of chords $\bar Z_0 \bar Z$ as $\bar Z \in M$ tends to $\bar Z_0$. We define $\mathscr{F}(\bar M) \subset \mathscr{F}_0(\mathbb{C}^N)$ to be the frames $\{Z_0, \cdots, Z_{N-1}\}$ where

$$\begin{cases} \bar Z_0 \in \bar M, \quad \text{and} \\ \bar Z_0, \cdots, \bar Z_n \text{ span } P^n_{\bar Z_0}. \end{cases}$$

The picture for an analytic curve in $\mathbb{P}^2$ is




We shall use the range of indices

$$\begin{cases} 1 \leqq \alpha, \beta \leq n; & 0 \leqq i, j \leqq N-1 \\ 0 \leqq a, b \leq n; & n-1 \leqq \rho, \sigma \leqq N-1 \end{cases}$$

Since on $\mathscr{F}(\bar M)$

$$dZ_0 \equiv \{Z_0, \cdots, Z_n\},$$

it follows that $\mathscr{F}(\bar M) \subset \mathscr{F}_0(\mathbb{C}^N)$ is an integral manifold of the differential system

$$\theta_{0\rho} = 0.$$

As a consequence we have

$$\omega = \frac{\sqrt{-1}}{2\pi} \sum_\alpha \theta_{0\alpha} \wedge \bar\theta_{0\alpha}, \quad \text{and}$$

$$0 = d\theta_{0\rho} = \sum_\alpha \theta_{0\alpha} \wedge \theta_{\alpha\rho}$$

which by the Cartan lemma implies

(4.29) $$\theta_{\alpha\rho} = \sum k_{\alpha\beta\rho}\theta_{0\beta}, \qquad k_{\alpha\beta\rho} = k_{\beta\alpha\rho}.$$

The second fundamental form of $\bar M \subset \mathbb{P}^{N-1}$ is defined to be

$$\mathrm{II} = \sum k_{\alpha\beta\rho}\theta_{0\alpha}\theta_{0\beta} \otimes Z_\rho.$$

From the structure equations

$$d\theta_{0\alpha} = \theta_{00} \wedge \theta_{0\alpha} + \sum_\beta \theta_{0\beta} \wedge \theta_{\beta\alpha}$$

$$= \sum_\beta \theta_{0\beta} \wedge \theta_{\beta\alpha} - \delta_{\beta\alpha}\theta_{00},$$

and we infer that

$$\phi_{\alpha\beta} = \theta_{\alpha\beta} - \delta_{\alpha\beta}\theta_{00} = -\bar{\phi}_{\beta\alpha}$$

is the connection matrix for the Hermitian connection associated to the metric

$$ds^2 = \sum_\alpha \theta_{0\alpha}\bar{\theta}_{0\alpha}$$

on $\bar{M}$. The curvature matrix is

$$\Phi_{\alpha\beta} = d\phi_{\alpha\beta} - \sum_\gamma \phi_{\alpha\gamma} \wedge \phi_{\gamma\beta}$$

$$= \theta_{\alpha 0} \wedge \theta_{0\beta} + \sum_\rho \theta_{\alpha\rho} \wedge \theta_{\rho\beta} - \delta_{\alpha\beta}\left(\sum_\gamma \theta_{0\gamma} \wedge \theta_{\gamma 0}\right)$$

which, upon setting

$$\theta = \sum_\gamma \theta_{0\gamma} \wedge \bar{\theta}_{0\gamma} = \frac{2\pi}{\sqrt{-1}}\,\omega$$

and using (4.29), gives

$$\Phi_{\alpha\beta} = \theta_{0\beta} \wedge \overline{\theta_{0\alpha}} + \delta_{\alpha\beta} \cdot \theta - \sum k_{\alpha\gamma\rho}\bar{k}_{\beta\delta\rho}\theta_{0\gamma} \wedge \bar{\theta}_{0\delta}. \tag{4.30}$$

As a check on signs and constants, the holomorphic sectional curvature in the direction $Z_1$ is the coefficient of $\theta_{01} \wedge \bar{\theta}_{01}$ in $\Phi_{11}$, which by (4.30) is

$$2 - \sum |k_{11\rho}|^2,$$

as it should be.

The Chern forms of $\bar{M} \subset \mathbb{P}^{N-1}$ are as usual defined by

$$\det\left(\lambda\delta_{\alpha\beta} + \frac{\sqrt{-1}}{2\pi}\,\Phi_{\alpha\beta}\right) = \sum_{k=0}^{n} c_k(\Phi_{\bar{M}})\lambda^{n-k}.$$

Although we shall not need it, the quantities

$$\int_{\bar{M}} c_{n-k}(\Phi_{\bar{M}}) \wedge \omega^k$$

are the coefficients in the expansion of the volume of the tube of radius $r$ around $\bar{M}$ in $P^{N-1}$. Also, by exact analogy with (4.17) it may be proved that (cf. §3(c))

$$\text{(4.31)} \qquad \int\!\!\int_{\bar{M}\cap\bar{L}} c_{n-k}(\Phi_{\bar{M}\cap\bar{L}})\, d\bar{L} = C^{\underline{te}} \int_{\bar{M}} c_{n-k}(\Phi_{\bar{M}}) \wedge \omega^k$$

where $\bar{L} = \pi(L_0)$ varies over the Grassmannian of $\mathbb{P}^{N-k-1}$'s in $\mathbb{P}^{N-1}$. We may view (4.31) as a local averaged version of the usual *adjunction formulas* in algebraic geometry.

(ii) Retaining the preceding notations, we consider the vector bundle $F \to \bar{M}$ whose fibre over $\bar{Z}_0$ is the $(n+1)$-plane spanned by $Z_0, \cdots, Z_n$. The associated projective bundle is the bundle of tangent projective spaces to $\bar{M} \subset \mathbb{P}^{N-1}$, and we have the *Euler sequence*

$$0 \to H^* \to F \to T(\bar{M}) \otimes H^* \to 0$$

where $H^* \to \bar{M}$ is the universal line bundle with fibres $\mathbb{C} \cdot Z_0$, and $T(\bar{M})$ is the holomorphic tangent bundle. The connection matrix for the Hermitian connection in $F$ is

$$\theta_{ab} = (dZ_a, Z_b),$$

and the curvature matrix is

$$\Theta_{ab} = d\theta_{ab} - \sum_c \theta_{ac} \wedge \theta_{cb}$$

$$= \sum_\rho \theta_{a\rho} \wedge \theta_{\rho b}.$$

From $\theta_{0\rho} = 0$ it follows that

$$\begin{cases} \Theta_{00} = 0 = \Theta_{0\alpha}, \quad \text{and} \\ \Theta_{\alpha\beta} = -\sum_\rho \theta_{\alpha\rho} \wedge \overline{\theta_{\beta\rho}}. \end{cases}$$

Using (4.31) we will prove that

$$\text{(4.32)} \qquad \int\!\!\int_{\bar{M}\cap\bar{L}} c_{n-k}(\Theta_{\bar{M}\cap\bar{L}})\, d\bar{L} = \sum_{l=0}^{n-k} C_l \int_M c_l(\Theta_M) \wedge \omega^{n-l}.$$

*Proof.* According to (4.30) the relation between the curvature matrices $\Phi = \Phi_{\bar{M}}$ and $\Theta = \Theta_{\bar{M}}$ is

$$\text{(4.33)} \qquad \begin{cases} \Phi_{\alpha\beta} = \theta_{0\beta} \wedge \bar{\theta}_{0\alpha} + \delta_{\alpha\beta} \cdot \theta + \Theta_{\alpha\beta} \\ \Theta_{\alpha\beta} = -\sum k_{\alpha\gamma\rho} \bar{k}_{\beta\delta\rho} \theta_{0\gamma} \wedge \bar{\theta}_{0\delta}. \end{cases}$$

We will show that

$$\text{(4.34)} \qquad c_k(\Phi) \wedge \omega^{n-k} = \sum_{l=0}^{k} C_l (c_l(\Theta) \wedge \omega^{n-l}), \qquad C_k = 1,$$

which when combined with (4.31) will establish (4.32).

For $k = 1$ we have (4.33)

$$c_1(\Phi) = (n + 1)\,\theta + c_1(\Theta)$$

which trivially implies (4.34). For $k = 2$, which is the crucial case, we have from (4.33)

$$c_2(\Phi) \wedge \omega^{n-2} = c_2(\Theta) \wedge \omega^{n-2} + C^{\underline{te}}\omega^n + L(\Theta)\omega^n$$

where $L(\Theta)$ is linear in the entries of $\Theta$. From (4.30) we deduce that $L(\Theta)$ is a linear combination of the expressions

$$k_{\alpha\gamma\rho}\bar{k}_{\beta\delta\rho},$$

where repeated indices will now be summed. From the theory of unitary invariants (cf. the reference in footnote(10) of §4) we infer that $L(\Theta)$ is a linear combination of the two expressions

$$k_{\alpha\gamma\rho}\bar{k}_{\alpha\gamma\rho},\ k_{\alpha\gamma\rho}\bar{k}_{\gamma\alpha\rho}.$$

By the symmetry $k_{\alpha\gamma\rho} = k_{\alpha\gamma\rho}$ these are both equal, and since

$$c_1(\Theta) = \frac{\sqrt{-1}}{2\pi}\left(\sum\Theta_{\alpha\alpha}\right) = -\frac{\sqrt{-1}}{2\pi}\sum\left(k_{\alpha\gamma\rho}\bar{k}_{\alpha\delta\rho}\theta_{0\gamma} \wedge \bar{\theta}_{0\delta}\right).$$

$$L(\Theta)\omega^n = C^{\underline{te}}c_1(\Theta) \wedge \omega^{n-1}.$$

This establishes (4.34) for $k = 2$, and the general argument is similar. We note again the essential role played by the symmetry of the second fundamental form.

We also remark that, by the discussion in §3(c), the formula (4.32) extends to the case where $\bar{M} \subset \mathbb{P}^{N-1}$ may have singularities; the point is that both sides are defined as the corresponding integrals over smooth points and these integrals are absolutely convergent.

(iii) Now let $M \subset \mathbb{C}^N - \{0\}$ be a complex manifold with residual image $\bar{M}$ in $\mathbb{P}^{N-1}$. Over $M$ we consider the usual Darboux frames $\{z, e_1, \cdots, e_N\}$, and over $\bar{M}$ we have the frames $\{Z_0, \cdots, Z_{N-1}\} \in \mathscr{F}(\bar{M})$ where $Z_0 = e^{i\theta}z/\|z\|$. Over smooth points of $\bar{M}$ we have $z \wedge e_1 \wedge \cdots \wedge e_n \neq 0$ and the two sets of vectors $\{z, e_1, \cdots, e_n\}$ and $\{Z_0, \cdots, Z_n\}$ both span the fibre of the bundle $F$ at $Z$. The pair of exact sequences

$$(4.35)\qquad \begin{cases} 0 \to H^* \to F \to T(\bar{M}) \otimes H^* \to 0 \\ 0 \to T(M) \to F \to Q \to 0 \end{cases}$$

contains the relationship between the bundles $T(M)$ and $T(\bar{M})$, both of which are complex-analytically isomorphic to the holomorphic tangent bundle of $M$ but which have quite different metrics. By (4.32) we have averaging formulas for the Chern forms $c_k(\Theta)$ of $F$, and we want to use these to deduce averaging formulas for the Chern forms $c_k(\Omega)$ of $T(M)$.

For this we decompose

$$z = z_t + z_n$$

into its tangential part

$$z_t = \sum_\alpha a_\alpha e_\alpha$$

and normal part

$$z_n = \sum_\mu z_\mu e_\mu .$$

We may consider $z_n$ as a holomorphic section of the line bundle $Q$ in (4.35), and the curvature matrix for this line bundle is the (1, 1) form

$$\Psi = -\partial\bar{\partial} \log \|z_n\|^2 . \tag{4.36}$$

Setting $z_* = \frac{1}{\|z_n\|} \cdot z_n$ we obtain a unitary frame $\{e_1, \cdots, e_n; z_*\}$ for $F$. The Chern form $c_k(\Theta)$ may be computed using this frame, as well as the previous one $\{Z_0, \cdots, Z_n\}$. The curvature matrix of $F$ in the first frame will be denoted by

$$\begin{pmatrix} \Omega_{\alpha\beta} & \Omega_{\alpha,*} \\ \Omega_{*,\alpha} & \Omega_{*,*} \end{pmatrix}$$

To compute it we use the second fundamental form of $T(M)$ in $F$—the point is that we know the curvature matrix $\Omega_{\alpha\beta}$ of $T(M)$ and $\Psi$ of $Q$, and want to determine from these and the second fundamental form the curvature matrix of $F$. Summing repeated indices the second fundamental form of $T(M) \subset F$ is given by the vector of (1, 0) forms

$$\begin{aligned} \omega_{\alpha,*} &= (de_\alpha, z_\alpha) \\ &= \frac{1}{\|z_n\|} \omega_{\alpha\mu}\bar{z}_\mu . \end{aligned}$$

It follows that

$$\left\{ \begin{aligned} \Omega^*_{\alpha\beta} &= \Omega_{\alpha\beta} + \omega_{\alpha,*} \wedge \bar{\omega}_{\beta,*} \\ \Omega_{**} &= \Psi - \sum_\alpha \omega_{\alpha,*} \wedge \bar{\omega}_{\alpha,*} \end{aligned} \right\} \tag{4.37}$$

(to check signs, recall that curvatures decrease on sub-bundles and increase on quotient bundles). There are similar formulas for $\Omega_{\alpha,*}$ and $\Omega_{*,\alpha}$. From (4.37) we infer that

$$c_1(\Theta_M) = c_1(\Omega_M) + \Psi_M . \tag{4.38}$$

Using this we will prove (4.28) when $n - k = 1$; i.e., when the intersections $M \cap L_0$ have complex dimension one.

Now (4.38) is valid for any complex manifold in $\mathbb{C}^N - \{0\}$, and applying it to $M \cap L_0$ and integrating gives

$$(4.39) \qquad \int_{M \cap L_0} c_1(\Omega_{M \cap L_0}) = \int_{M \cap L_0} c_1(\Theta_{M \cap L_0}) - \int_{M \cap L_0} \Psi_{M \cap L_0}.$$

We now average both sides of (4.39) over $L_0 \in G(N - n + 1, N)$ and use (4.32) to obtain

$$(4.40) \qquad \int \left( \int_{M \cap L_0} c_1(\Omega_{M \cap L_0}) \right) dL_0 = \int_M c_1(\Theta_M) \wedge \omega^{n-1} + C^{\underline{te}} \int_M \omega^n - \int \left( \int_{M \cap L_0} \Psi_{M \cap L_0} \right) dL_0.$$

We must examine the term on the far right.

In $\mathbb{C}^N - \{0\} \times G(N - n + 1, N)$ we consider the incidence correspondence

$$I = \{z, L_0) : z \in L_0\}$$

(actually, $I$ should be considered in $\mathbb{P}^{N-1} \times G(N - n + 1, N)$). The fibre of $\pi: I \to \mathbb{C}^N - \{0\}$ is $I_z \cong G(N - n, N - 1)$, and so $\pi_*(dL_0)$ is an $(n - 1, n - 1)$ form on $C^N - \{0\}$. Since this form is the pullback of a form on $P^{N-1}$ and is unitarily invariant, it is a multiple of $\omega^{n-1}$. We may then write

$$(4.41) \qquad dL_0 = \omega^{n-1} \wedge \Psi$$

where $\Psi$ is a form on $I$ which restricts to the fundamental class on each fibre $I_z$. Here we are using that the cohomology of $I$ is additively isomorphic to $H^*(\mathbb{P}^{N-1}) \otimes H^*(G(N - n, N - 1))$, and in this decomposition the cohomology class of $dL_0$ appears in $H^{2(n-1)}(\mathbb{P}^{N-1}) \otimes H^*(G(N - n, N - 1))$. Now we denote by $z_n(L_0)$ the normal vector for $M \cap L_0 \subset L_0$ at $z$. Since

$$T_z(M \cap L_0) = T_z(M) \cap L_0$$

it follows by an easy invariant-theoretic argument that

$$\int_{I_z} \log \|z_n(L_0)\| \Psi = C^{\underline{te}} \log \|z_n\|.$$

Combining this with (4.41) and (4.40) gives

$$\begin{aligned} \int \left( \int_{M \cap L_0} \Psi_{M \cap L_0} \right) dL_0 &= \int \left( \int_{M \cap L_0} - \partial\bar{\partial} \log \|z_n(L_0)\|^2 \right) \Psi \wedge \omega^{n-1} \\ &= \int \left\{ \int_{M \cap L_0} \left( -\partial\bar{\partial} \int_{I_z} \log \|z_n(L_0)\|^2 \Psi \right) \right\} \omega^{n-1} \\ &= C^{\underline{te}} \int_M -\partial\bar{\partial} \log \|z_n\|^2 \wedge \omega^{n-1} \end{aligned}$$

$$= C^{\underline{te}} \int_M \Psi_M \wedge \omega^{n-1}.$$

By computing a standard example we may check that the constant is one. Combining with (4.40) we conclude that

$$\int \left( \int_{M \cap L_0} c_1(\Omega_{M \cap L_0}) \right) dL_0 = \int_M (c_1(\Theta_M) + \Psi_M) \wedge \omega^{n-1} + C^{\underline{te}} \int_M \omega^n$$
$$= \int_M c_1(\Omega_M) \wedge \omega^{n-1} + C^{\underline{te}} \int_M \omega^n,$$

which establishes (4.28) when $n - k = 1$.

The argument for general $k$ is similar using all the equations (4.32) and (4.34) and will not be given in full detail now.

*Appendix to sections 2 and 4. Some general observations on integral geometry.* Upon scanning sections 2 and 4 on integral geometry the reader may suspect that the various Crofton formulas are different manifestations of the same basic phenomenon, and we want to explain that now. Given a connected Lie group $G$ and closed subgroups $H$ and $K$ we denote left cosets by $\xi = gH$ and $x = g'K$. We assume given an incidence correspondence

$$I \subset G/H \times G/K$$

which is invariant under the action of $G$. In practice $I$ will be the union of $G$-orbits but in general will not be acted on transitively by $G$.[11] Denoting by $\pi_1$ and $\pi_2$ the respective projections of $I$ onto $G/H$ and $G/K$ the basic operation in integral geometry is

$$\Phi \to (\pi_1)_*(\pi_2^*\Phi) \tag{4.42}$$

where $\Phi$ is a differential form on $G/K$. If we denote the right side of (4.42) by $I(\Phi)$, then $\Phi \to I(\Phi)$ takes invariant forms to invariant forms and all of our integral-geometric formulas arise by evaluating $I(\Phi)$ over suitable submanifolds of $G/H$. Here are some illustrations.

*Example 1.* Suppose that $G = E(n)$ is the real Euclidean group and

$$\begin{cases} G/H = \mathbb{R}^n \text{ is Euclidean space} \\ G/K = \mathbb{R}^{n*} \text{ is the space of affine hyperplanes.} \end{cases}$$

The incidence correspondence is

$$I = \{(x, \xi) : x \in \xi\}.$$

Given a curve $C$ in $\mathbb{R}^n$ and taking $\Phi = |d\xi|$ to be the invariant volume on $\mathbb{R}^{n*}$, Crofton's formula (2.1) is just the evaluation of

$$\int_C |I(\Phi)|. \tag{4.43}$$

Note the integration of densities, since without absolute values the result would be zero.

Here is a proof of (2.1) which illustrates the general principles of integral geometry: By the $G$-invariance of $\Phi$ and the incidence correspondence $I$, the integral in (4.43) is a $G$-invariant density on $C$. According to the theory of moving frames[12] this density is expressible in terms of the basic invariants which describe the position of $C$ in $G/H$ (in this case arclength, curvature, torsion, . . .). Moreover, and this is the main point, it is visibly clear that if two curves $C$ and $C$; osculate to *first* order at some $x_0 \in \mathbb{R}^{n}$[13], then the densities $|I(\Phi)|$ on $C$ and $C'$ agree at this point. It follows that $|I(\Phi)|$ is a constant multiple of arclength $ds$, which implies (2.1).

In general the order of contact necessary to determine $I(\Phi)$ at some point gives a first idea of what sort of formula we will obtain.

*Example 2.* The Crofton formulas (2.3) and (4.1) may be similarly formulated. In fact (4.1) is easier, since in the complex case there is no problem with orientations and we may use the following argument: taking $G$ to be the group of complex Euclidean motions and

$$\begin{cases} G/K = \mathbb{C}^n, \\ G/H = \mathbb{C}^{n*}, \end{cases}$$

and taking $\phi$ and $\phi^*$ to be the respective Kähler forms on $\mathbb{C}^n$ and $\mathbb{C}^{n*}$ and $\Phi = (1/n!)\,\phi^{*n}$ the volume form, we deduce that

$$I(\Phi) = (\pi_1)_*(\pi_2^*\Phi)$$

is a $G$-invariant (1, 1) form on $\mathbb{C}^n$. It follows trivially from Schur's lemma that $I(\Phi)$ is a multiple of $\phi$. For a holomorphic curve $C$ in $\mathbb{C}^n$, by the Wirtinger theorem (3.19)

$$\int_C \phi = \mathrm{vol}\,(C)$$

is the area of $C$, and by evaluating a constant we have proved (4.1).

*Example 3.* This gives the method for establishing the reproductive property (0.4). Again $G$ is the Euclidean group acting now on $\mathbb{R}^N$, and we first take

$$G/K = \hat{G}_{\mathbb{R}}(n - k, N - k, N)$$

to be the triples $(x, S, L)$ where

$$\begin{cases} x \in \mathbb{R}^N,\ S \text{ is an } (n-k)\text{-plane, } L \text{ is} \\ \text{an } (N-k)\text{-plane, and } x \in S \subset L; \end{cases}$$

this is a flag manifold. Secondly, we take

$$G/H = \hat{G}_{\mathbb{R}}(n, N)$$

to be the pairs $(x^*, T)$ where

$$\begin{cases} x^* \in \mathbb{R}^N,\ T \text{ is an } n\text{-plane,} \\ \text{and } x^* \in T. \end{cases}$$

The incidence correspondence is (cf. Figure 8 in section 4c)

$$I = \{((x^*, T), (x, S, L)) \colon x = x^*,\ S \subset T\}. \tag{4.44}$$

We note that $I$ is not a single $G$-orbit since the planes $T$ and $L$ will have an invariant, namely their "angle."

Now we assume that $n - k$ is even and denote by $U \to G_R(n - k, N - k, N)$ the universal bundle with fibre $S_{-x}$, and by

$$\pi \colon \hat{G}_{\mathbb{R}}(n - k, N - k, N) \to \tilde{G}_{\mathbb{R}}(N - k, N)$$

the projection $(x, S, L) \to L$ onto the Grassmannian of affine $(N - k)$-planes in $\mathbb{R}^N$. For $\Phi$ we take

$$\Phi = |Pf(U) \wedge \pi^* dL| \tag{4.45}$$

where $Pf(U)$ is the Pfaffian (1.21) of $U$ and $dL$ is the invariant volume on $\tilde{G}_{\mathbb{R}}(n - k, N)$. For $M_n \subset \mathbb{R}^N$ a manifold we let

$$M = \{(x, T) \colon x \in M \quad \text{and} \quad T = T_x(M)\}$$

be its Gaussian image in $\hat{G}(n, N)$ and

$$\hat{M} = \{(x, T); (x, S, L) \colon x \in M,\ T = T_x(M),\ S \subset T\}$$

the inverse image of $M$ in the incidence correspondence (4.44). Then by integration over the fibre

$$\int_M |I(\Phi)| = \int_M \Phi, \tag{4.46}$$

and we shall outline how (4.46) implies (0.4).

At a general point of $\hat{M}$ we have

$$\dim (T_x(M) \cap L) = n + (N - k) - N = n - k = \dim S$$

so that

$$S = T_x(M) \cap L = T_x(M \cap L).$$

By iteration of the integral on the right hand side of (4.46),

$$\int_M \Phi = \int \left( \int_{\hat{M} \cap L} Pf(U) \right) dL$$
$$= \int \left( \int_{M \cap L} Pf(\Omega_{M \cap L}) \right) dL$$

where $Pf(\Omega_{M \cap L})$ is the Pfaffian in the curvature matrix $\Omega_{M \cap L}$, and is therefore the Gauss-Bonnet integrand for $M \cap L$. It is important to note that $Pf(\Omega_{M \cap L})$ is invariant under reversal of orientation of $M \cap L$, and so the absolute value sign in (4.45) only pertains to $dL$.

We have now evaluated $\int_{\hat{M}} \Phi$ as being the left hand side of (0.4). As for $\int_M |I(\Phi)|$, it is clearly a second order invariant of $M \subset \mathbb{R}^N$, and hence is expressed as the integral in some universal polynomial of degree $2(n - k)$ in the second fundamental form of $M$. The main step is to show, by an argument using Meusnier's theorem, that this polynomial has the same invariance properties as those in Weyl's tube formula (1.10); the details are given in the reference cited in footnote 6 of the introduction.

*Example 4.* The same procedure as in example 3 may be used in the complex case, only the argument is simpler since we may take

$$\Phi = c_{n-k}(\Omega_U) \wedge dL$$

without absolute value signs, and then

$$I(\Phi) = (\pi_1)_*(\pi_2^*\Phi)$$

is a $G$-invariant closed $(n, n)$ form on $\hat{G}(n, N)$, which we may seek to determine without reference to any complex manifold $M \subset \mathbb{C}^N$. This is the procedure followed in our proof of (4.17).

*Example 5.* Here we take $G = U(N)$, $G/K$ to be the manifold $F(N)$ of all flags $W_0 \subset W_1 \subset - \subset W_N = \mathbb{C}^N$, and $G/H$ the Grassmannian $G(n, N)$. Then, for a sequence $a = (a_1, \cdots, a_n)$ of integers as in section 4(a), we let

$$I \subset G(n, N) \times F(N)$$

be defined by the Schubert conditions

$$I = \{(T, F)\colon T \in \Sigma_a(F)\}.$$

Taking $\Phi = dF$ the invariant volume on $F(N)$ we have that

$$I(\Phi) = (\pi_1)_*(\pi_2^* dF)$$

is an invariant form on $G(n, N)$.

The determination of $I(\Phi)$ brings out one of the salient features of Hermitian integral geometry. Namely, since the invariant differential forms on $G(n, N)$ are isomorphic to the cohomology, the form $I(\Phi)$ is uniquely specified by its cohomology class, and consequently the determination of the mapping $\Phi \to I(\Phi)$ is a purely topological question. In the case at hand

$$I(\Phi) = P_\alpha(\Omega_E),$$

and this implies the formula (4.9).

**Footnotes**

1. We recall that for $V \xrightarrow{\pi} W$ a surjective proper mapping of complex manifolds with generic fibre of dimension $k$, integration over the fibre gives a map

$$\pi_*: A_c^{(p+k,q+k)}(V) \to A_c^{(p,q)}(W)$$

with the defining property

$$\int_V \psi \wedge \pi^*\eta = \int_W \pi_*\psi \wedge \eta, \qquad \eta \in A^*(W). \tag{4.2}$$

It follows that $\pi_*$ commutes with $\partial$ and $\bar{\partial}$, and also preserves positive forms. We shall also have occasion to use integration over the fibre when $V$ is singular, but the extension to these cases will be obvious.

2. The point is that, by Schur's lemma, any Hermitian form on $\mathbb{C}^N$ which is invariant under the unitary group must be a multiple of the standard one. As much more sophisticated use of invariant theory occurs in section 4(c) below—cf. the discussion pertaining to the reference cited in footnote 10.

3. cf. the reference given in footnote 5 of section 3.

4. cf. footnote 11 in section 1 for an argument which covers all the cases we shall use in this paper.

5. Once we know that, by averaging, the deRham cohomology of $G(n, N)$ is isomorphic to the cohomology computed from the complex of invariant forms, the argument goes as follows: The invariant forms are exterior polynomials in the quantities $\omega_{\alpha\mu}$, $\bar{\omega}_{\beta\nu}$ which are invariant under $\omega_{\alpha\mu} \to \sum_{\beta,\nu} g_{\alpha\beta}\omega_{\beta\nu}\bar{h}_{\nu\mu}$ where $g$ and $h$ are unitary matrices. Using the $h$'s, we deduce that only the expressions

$$\Omega_{\alpha\beta} = \sum_{\mu} \omega_{\alpha\mu} \wedge \bar{\omega}_{\beta\mu}$$

appear; in particular, the invariant forms are all of even degree. Using the $g$'s, any invariant form is a polynomial in the elementary symmetric functions of $(\Omega_{\alpha\beta})$. By direct integration one finds that $c_k(\Omega_E)$ is Poincaré dual to $\Sigma_k$.

6. cf. W. Stoll, *Invariant Forms on Grassmann Manifolds*, Annals of Math. Studies no. 89, Princeton University Press, (1977). The form $\eta_{F,a}$ may also be constructed by applying standard potential theory to the Grassmannian to solve the equation of currents (ii).

7. As we saw in (3.11), aside from a few special cases the Gaussian image $\gamma(M)$ may be expected to have dimension $n$.

8. cf. footnote 2 in section 1.

9. We note that $V = T_0(\mathbb{C}^N)^* \simeq \mathbb{C}^{N*}$ and $W \simeq T^*_{\mathbb{C}^n}(G(n, N)) \cong \mathbb{C}^n \otimes \mathbb{C}^{N-n*}$; cf. footnote 3 in section 3.

10. H. Weyl, *The Classical Groups*, Princeton University Press, 1939.

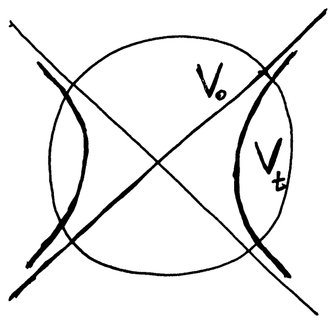
11. In the reference cited in footnote[4] of section 2, cosets $\xi$ and $x$ are defined to be incident if $gH$ and $g'K$ have in common a left coset relative to $H \cap K$. This notion is not sufficiently broad to cover the examples we have in mind.

12. cf. the recent works, Gary Jensen, *Higher Order Contact of Submanifolds of Homogeneous Spaces*, Lecture Notes in Math, no. 610, Springer-Verlag (1977); Mark Green, *The moving frame, differential invariants, and rigidity theorems for homogeneous spaces*, to appear in Duke Math. J.

13. This means that $C$ and $C'$ pass through $x_0$ and have the same tangent line there.

## 5. Curvature and Plücker Defects

(a) *Gauss-Bonnet and the Plücker paradox*. To obtain an heuristic idea of what is to come, let us consider a family of complex-analytic curves $\{V_t\}$ defined in some open set $U \subset \mathbb{C}^2$ and parametrized by the disc $B = \{t: |t| < 1\}$. For example, we may think of $V_t$ as defined by $f(x, y; t) = 0$ where $f$ is holomorphic in $U \times B$. We will also assume that $V_t$ is smooth for $t \neq 0$ while $V_0$ has one ordinary double point at the origin, as illustrated by the following figure




$f(x, y; t) = x^2 - y^2 - t$

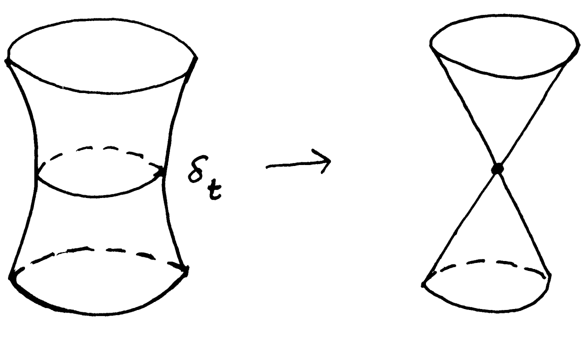
By shrinking $U$ slightly if necessary, we may consider all of the $V_t$ as Riemann surfaces with boundary; this is clear for $t \neq 0$, while $V_0$ is the image of a smooth Riemann surface $\tilde{V}_0$ (possibly disconnected as in the above example) under a holomorphic immersion which identifies two points. Note that the boundaries $\partial V_t \to \partial V_0$ as $t \to 0$, and that the Euclidean metric on $\mathbb{C}^2$ induces metrics on all the $V_t$ including $\tilde{V}_0$. By the usual Gauss-Bonnet theorem (cf. (4.14))

$$\frac{1}{2\pi} \int_{V_t} KdA = \chi(V_t) + \int_{\partial V_t} k_g ds \tag{5.1}$$

where $k_g$ is the geodesic curvature. This formula holds for any metric on a Riemann surface with boundary and is therefore valid for $t \neq 0$ and for $t = 0$ with $\tilde{V}_0$ in place of $V_0$. Since there are no singularities near the boundary

$$\lim_{t \to 0} \int_{\partial V_t} k_g ds = \int_{\partial V_0} k_g ds.$$

On the other hand, since the topological picture near the origin is



where the vanishing cycle $\delta_t \in H_1(V_t)$ shrinks to a point as $t \to 0$, we have

$$\chi(V_t) = \chi(\tilde{V}_0) - 2, \qquad t \neq 0.$$

It follows that

$$\lim_{t \to 0} \frac{1}{2\pi} \int_{V_t} KdA = \frac{1}{2\pi} \int_{V_0} KdA - 2. \tag{5.2}$$

On the other hand, it is clearly the case that for $W \subset U$ a compact region not containing the double point

$$\lim_{t \to 0} \int_{V_t \cap W} KdA = \int_{V_0 \cap W} KdA.$$

Setting $V_t[\epsilon] = V_t \cap B[0, \epsilon]$ where $B[0, \epsilon]$ is the $\epsilon$-ball around the origin, we deduce a special case of the theorem of Langevin[(1)]

$$\lim_{\epsilon \to 0} \lim_{t \to 0} \frac{1}{2\pi} \int_{V_t[\epsilon]} KdA = -2. \tag{5.3}$$

This result establishes in principle the basic link between curvature and singularities.

We note that (5.3) has the following consequence due to Linda Ness[(2)]: Since, by the discussion in section 4(b) (which in this case is quite obvious), the areas

$$\int_{V_t[\epsilon]} dA = 0(\epsilon) \tag{5.4}$$

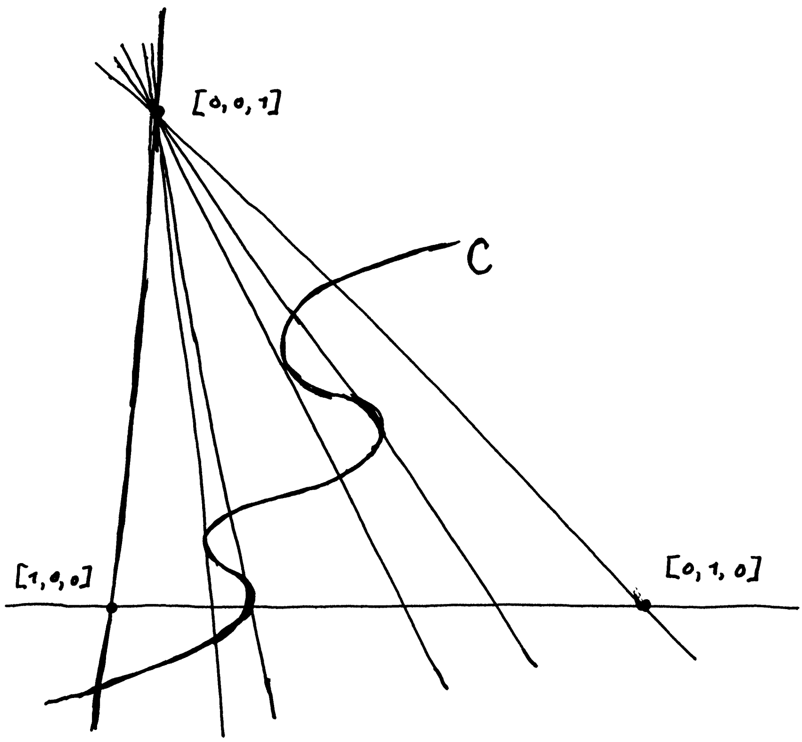
tend uniformly to zero, we deduce that there must be points $p_t \in V_t$ where the Gaussian curvature $K(p_t) \to -\infty$. Moreover, because $K \leq 0$ we may use (5.4) to estimate the size of the region where $K \leq -C^{\underline{te}} < 0$. It will be more convenient for us to do this indirectly using the theory of currents.

These considerations are closely related to the Plücker paradox, which arose in the very early days of algebraic geometry and brought into clear focus the necessity for exercising caution in treating singularities. Suppose that $C \subset \mathbb{P}^2$ is an algebraic plane curve given in affine coordinates by $f(x, y) = 0$ where $f$ is a polynomial of degree $d$. The dual curve $C^* \subset \mathbb{P}^{2*}$ is defined to be the set of tangent lines to $C$; i.e., the image under the Gauss mapping

$$\gamma: C \to \mathbb{P}^{2*}.$$

It is again an algebraic curve of some degree $d^*$, called the *class* of $C$, and the dual of $C^*$ is again $C$[(3)].

To compute $d^*$ we recall that the degree of any algebraic curve is the number of its intersections with a general line. By projective duality a line in $\mathbb{P}^{2*}$ is given by the pencil $P^1(p)$ of lines $L$ through a fixed point $p$ in $\mathbb{P}^2$. We may choose our coordinates so that $p = [0, 0, 1]$ is the point at infinity along the $y$-axis and such that the line at infinity is not tangent to $C$. The pencil $P^1(p)$ then consists of the vertical lines in the affine plane $\mathbb{C}^2$, and the class $d^*$ is just the




number of vertical tangents to the finite curve $f(x, y) = 0$. Finally, again by general position we may assume that each such vertical tangent is simple (i.e., is not a flex-tangent). Now the tangent to $C$ at a smooth point $(x, y)$ is vertical if

$$\frac{\partial f}{\partial y}(x, y) = 0,$$

and so it would appear that $d^*$ is just the number of solutions to the simultaneous polynomial equations

$$f(x, y) = 0, \qquad \frac{\partial f}{\partial y}(x, y) = 0, \tag{5.5}$$

which according to Bezout's theorem would give

$$d^* = d(d - 1).$$

If this were correct, then by double duality

$$\begin{aligned} d &= d^*(d^* - 1) \\ &= d(d - 1)(d^2 - d - 1) \end{aligned}$$

which is a contradiction if $d \geq 3$.

Of course the point is that our computation of $d^*$ is correct only when $C$ is smooth. Put differently, the number of solutions to the equations (5.5) gives the degree of the *algebraic* dual $\hat{C}$, defined as the set of lines which intersect $C$ multiply in less than $d$ distinct points. In case $p_1, \cdots, p_m$ are the singular points of $C$, then $\hat{C}$ is reducible and in fact

$$\hat{C} = C^* + \mu_1 P^1(p_1) + \cdots + \mu_m P^1(p_m)$$

where the $\mu_i$ are positive integers and

$$d(d-1) = d^* + \mu_1 + \cdots + \mu_m.$$

Now suppose that $\{C_t\}$ is a family of curves parametrized by the disc and with $C_t$ smooth for $t \neq 0$ while $C_0$ has singular points $p_1, \cdots, p_m$. Then $\hat{C}_t = C_t^*$ for $t \neq 0$ while clearly $\lim_{t \to 0} \hat{C}_t = \hat{C}_0$. It follows that

$$\lim_{t \to 0} C_t^* = C_0^* + \mu_1 P^1(p_1) + \cdots + \mu_m P^1(p_m); \tag{5.6}$$

i.e., when the curves $C_t$ acquire a singularity their Gaussian images are not continuous in $t$ and the discrepancy

$$\lim_{t \to 0} \gamma(C_t) - \gamma(C_0) = \sum_i \mu_i P^1(p_i)$$

may be called the Plücker defect associated to the situation. It is this Plücker defect which explains the phenomenon (5.2) and which will be systematized in the following discussion.

(b) *The Plücker defect and Langevin's formula*. Suppose now that $\{V_t\}_{t \in B}$ is a family of $n$-dimensional analytic varieties defined in a neighborhood of an open set $U \subset \mathbb{C}^N$ with $V_t$ smooth for $t \neq 0$. More precisely, we should be given a neighborhood $W$ of $U$ and an analytic subvariety

$$X \subset B \times W$$

such that for $t \neq 0$ the intersection $X \cdot (\{t\} \times W)$ is smooth. We define $V_t$ by

$$X \cdot (\{t\} \times U) = \{t\} \times V_t$$

and will assume that the boundaries $\partial V_t = V_t \cap \partial U$ are smooth for $t \neq 0$. Denote by $V_0^*$ the smooth points of $V_0$ and consider the graph

$$\Gamma^* = \{(p, \gamma_t(p)) \in X \times G(n, N)\}$$

of the Gauss mappings

$$\gamma_t \colon V_t \to G(n, N)$$

for $t \neq 0$ together with

$$\gamma_0 \colon V_0^* \to G(n, N).$$

The closure $\Gamma$ of $\Gamma^*$ is an analytic subvariety of $X \times G(n, N)$, and the projection

$$\pi : \Gamma \to B$$

has fibres $\pi^{-1}(t) = \Gamma_t$ the Gaussian images $\gamma_t(V_t)$ for $t \neq 0$, while the fibre $\Gamma_0$ is generally *not* the closure $\gamma_0(V_0^*)$. Writing

$$\Gamma_0 = \overline{\gamma(V_0^*)} + \Delta \tag{5.7}$$

defines $\Delta$ as an analytic subvariety of $G(n, N)$, one possibly having a boundary corresponding to the points of $V_s \cap \partial U$, and one whose irreducible components generally have multiplicities. We shall call $\Delta$ the *Plücker defect*[4] associated to the family $\{V_t\}_{t \in B}$.

It is also possible to define $\Delta$ as a current $T_\Delta$ by the formula

$$T_\Delta(\alpha) = \lim_{t \to 0} \int_{V_t} \gamma_t^*(\alpha) - \int_{V_0^*} \gamma_0^*(\alpha) \tag{5.8}$$

where $\alpha$ is a $C^\infty$ form on $G(n, N)$. Clearly this is just the current associated to the variety defined by (5.7). With this definition one may prove directly that $T_\Delta$ is a positive current of type $(n, n)$, which is closed in case $V_0$ has isolated interior singularities. Moreover, by taking a smaller class of "test forms" $\alpha$ we may refine the data of the Plücker defect.

Another possibility is to define $\Delta$ by using resolution of singularities.[5] By successively blowing up $X$ beginning along the singular locus of $V_0$ we arrive at

$$\tilde{X} \xrightarrow{\tilde{\omega}} B$$

where $\pi^{-1}(t) = V_t$ for $t \neq 0$ and where $\pi^{-1}(0)$ is a divisor with normal crossings. We may even assume that the Gauss mappings

$$\gamma_t \colon \tilde{\omega}^{-1}(t) \to G(n, N)$$

are defined for all $t$ including $t = 0$. Writing

$$\pi^{-1}(0) = \tilde{V}_0 + \mu_1 D_1 + \cdots + \mu_m D_m$$

where $\tilde{V}_0$ is the proper transform of $V_0$ (so that $\tilde{V}_0 \to V_0$ is a desingularization), it follows that

$$\begin{cases} \gamma_0(\tilde{V}_0) = \overline{\gamma_0(V_0^*)}, \text{ and} \\ \Delta = \mu_1\gamma_0(D_1) + \cdots + \mu_m\gamma_0(D_m). \end{cases} \tag{5.9}$$

Each of the characterizations (5.7)–(5.9) of the Plücker defect turns out to be of use.

Now let $a = (a_1, \cdots, a_n)$ be any sequence of integers with

$$|a| = \sum_\alpha a_\alpha = n$$

representing a codimension $n$ Schubert condition on $G(n, N)$ and denote by $P_a(\Omega_E)$ the corresponding polynomial in the curvature representing the cohomology class dual to the fundamental cycle of $\Sigma_a$. Then, from §3(b)

$$\lim_{t \to 0} \int_{\Gamma_t} P_a(\Omega_E) = \int_{\Gamma_0} P_a(\Omega_E).$$

On the other hand, for $t \neq 0$

$$\int_{\Gamma_t} P_a(\Omega_E) = \int_{V_t} P_a(\Omega_{V_t})$$

where $\Omega_{V_t}$ is the curvature matrix in the tangent bundle of $V_t$, while by the discussion of §3(c) for $t = 0$ the integral

$$\int_{V_0^*} P_a(\Omega_{V_0})$$

converges and is equal to

$$\int_{\gamma_0(V_0^*)} P_a(\Omega_E).$$

From this together with (5.8) we infer that

$$\lim_{t \to 0} \int_{V_t} P_a(\Omega_{V_t}) = \int_{V_0} P_a(\Omega_{V_0}) + \int_{\Delta} P_a(\Omega_E).$$

We may evaluate the term on the far right by the first Crofton formula (4.9) and obtain

$$\text{(5.10)} \qquad \lim_{t \to 0} \int_{V_t} P_a(\Omega_{V_t}) = \int_{V_0} P_a(\Omega_{V_0}) + \int \#\,(\Delta, \Sigma_a)\, d\Sigma_a$$

This is our main general result expressing the difference between the limit of the curvature and the curvature of the limit. It is clear that there is an analogous result for a family of either analytic or algebraic varieties in $\mathbb{P}^N$.

In case $V_0$ has only isolated interior singularities, $\Delta$ is an algebraic variety in $G(n, N)$, the intersection number $\#\,(\Delta, \Sigma_a)$ is constant for all Schubert cycles in the family, and (5.10) becomes our

*Main Formula (I). For a family of complex-analytic varieties $V_t$ acquiring an isolated singularity*

$$\text{(5.11)} \qquad \lim_{t \to 0} \int_{V_t} P_a(\Omega_{V_t}) = \int_{V_0} P_a(\Omega_{V_0}) + \#\,(\Delta, \Sigma_a)$$

*where $\Delta$ is the Plücker defect.* Note that if we combine (5.11) with the Crofton formula (4.9) we obtain

$$\text{(5.12)} \qquad \lim_{t \to 0} \int \#(\gamma(V_t), \Sigma_a)\, d\Sigma_a = \int \#\,(\gamma(V_0), \Sigma_a)\, d\Sigma_a + \#\,(\Delta, \Sigma_a)$$

For example, suppose that we consider the basic Schubert cycle $\Sigma_n$; then

$$P_n(\Omega_{v_t}) = c_n(\Omega_{V_t}) = (-1)^n C^{\underline{te}} KdA$$

is the Gauss-Bonnet integrand (we use $dA$ instead of $dV_t$), and (5.11) is

$$\lim_{t \to 0} C^{\underline{te}} \int_{V_t} KdA = C^{\underline{te}} \int_{V_0} KdA + (-1)^n \# (\Delta, \Sigma_n). \tag{5.13}$$

If we assume that the origin is the only isolated singularity of $V_0$ and set $V_t[\epsilon] = \{z \in V_t: \|z\| \leq \epsilon\}$ then by the discussion in section 3(c)

$$\lim_{\epsilon \to 0} \int_{V_0[\epsilon]} KdA = 0$$

which, when combined with (5.13) gives

$$\lim_{\epsilon \to 0} \lim_{t \to 0} C^{\underline{te}} \int_{V_t[\epsilon]} KdA = (-1)^n \# (\Delta, \Sigma_n). \tag{5.14}$$

Now suppose that $V_0 \subset \mathbb{C}^{n+1}$ is a hypersurface with an isolated singularity at the origin. If $V_0$ is given by an analytic equation $f(z_1, \cdots, z_{n+1}) = 0$, then setting $V_t = \{f(z) = t\}$ embeds $V_0$ in a family $\{V_t\}$ with $V_t$ smooth for $t \neq 0$. By (4.10) the Schubert cycles $\Sigma_n$ are in one-to-one correspondence with the hyperplanes $H$ through the origin, and by (4.12)

$$\#(\gamma(V_t), H) = \{\text{number of times the tangent plane to } V_t \text{ is parallel to } H\}.$$

We shall sketch the proof of the following result of Tessier:[6]

*For t sufficiently small and H generic,*

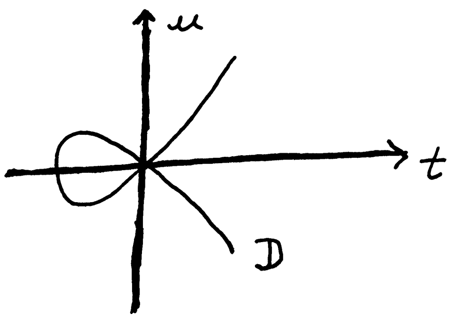
$$\# (\gamma(V_t), H) = \# (\gamma(V_0), H) + \{\mu^{(n+1)} + \mu^{(n)}\} \tag{5.15}$$

*where* $\mu^{(i)}$ *is the* $i^{\text{th}}$ *Milnor number of* $V_0$.

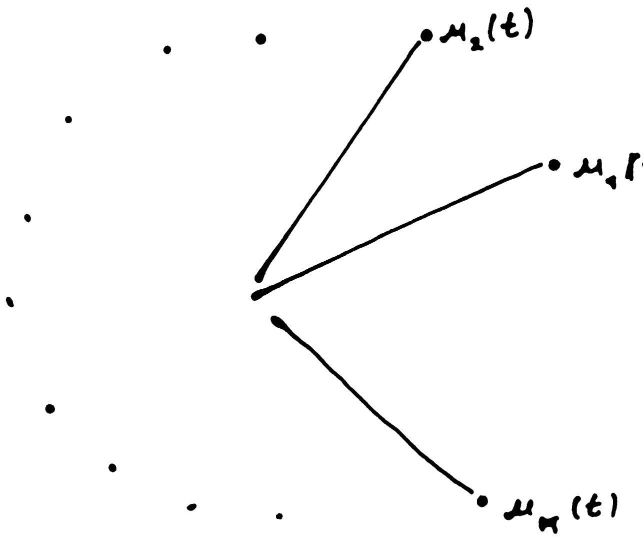
*Proof.* We recall[7] that for $\epsilon$, $t$ sufficiently small and $n \geq 2$, $V_t[\epsilon]$ has the homotopy type of a wedge of $n$-spheres; the number of these is the top Milnor number $\mu^{(n+1)}$. The remaining Milnor numbers are defined by: $\mu^{(n-k+1)}$ = top Milnor number of $L \cap V_0$ where $L$ is a generic $\mathbb{C}^{N-k}$ through the origin in $\mathbb{C}^N$, and where we agree to set $\mu^{(1)} = \text{mult}_0(V_0) - 1$ and $\mu^{(0)} = 1$. Choose a generic linear coordinate system $(z_1, \cdots, z_n, u)$ such that the hyperplanes parallel to $H$ are given by $u$ = constant. The projection

$$u : V_t \to \mathbb{C}$$

fibres $V_t$ by varieties $V_{t,u}$ of dimension $n - 1$, and the critical values $u_1, \cdots, u_\kappa$ correspond exactly to tangent hyperplanes $H_{u_\lambda}$ parallel to $H$. We may assume that each $H_{u_\lambda}$ is simply tangent, and thus by Lefschetz theory[8] as $u \to u_\lambda$ the $V_{t,u}$ acquire an ordinary double point with there being a single vanishing cycle $\delta \in H_{n-1}(V_{t,0})$. The picture in the (real) $(t, u)$ plane is something like




where $D$ is the discriminant curve of points $(t, u)$ such that $V_{t,u}$ is singular. The picture in the complex $u$-plane is




where the $u_\lambda(t)$ are the intersections of $D$ with the vertical line $t$ = constant. It follows that the Euler characteristics are related by

$$\chi(V_t) = \chi(V_{t,0}) + (-1)^{n-1}\kappa. \tag{5.16}$$

On the other hand, $V_t - V_t[\epsilon]$ and $V_{t,0} - V_{t,0}[\epsilon]$ may be assumed homotopic for $\epsilon$, $t$ sufficiently small while

$$\chi(V_t[\epsilon]) = 1 + (-1)^n \mu^{(n+1)}$$
$$\chi(V_{t,0}[\epsilon]) = 1 + (-1)^{n-1} \mu^{(n)}.$$

Together with (5.16) this implies that

$$\kappa = \mu^{(n+1)} + \mu^{(n)},$$

which proves (5.15).

As a consequence of (5.15),

$$\lim_{t \to 0} \int \#(\gamma(V_t), H)dH = \int \#(\gamma(V_0), H)dH + \mu^{(n+1)} + \mu^{(n)},$$

which when combined with (5.12) and (5.14) gives *Langevin's theorem* (0.9)

$$\lim_{\epsilon \to 0} \lim_{t \to 0} C^{te} \int_{V_t} KdA = (-1)^n\{\mu^{(n+1)} + \mu^{(n)}\}. \tag{5.17}$$

(c) *Extension to higher codimension and isolation of the top Milnor number.* Even though our main formula (5.11) is fairly general in scope, it is clearly of the same character as the special case (5.17). The extension to higher codimension is perhaps more novel, relying as it does on the formula (4.28) instead of the simpler Crofton formula (4.9).

As above we assume given $\{V_t\}$ where $V_t$ is smooth for $t \neq 0$ while $V_0$ has an isolated singularity at the origin. For a generic linear space $L \in G(N-k, N)$, the intersections $V_t \cap L$ will be transverse and therefore smooth for $t \neq 0$ while $V_0 \cap L \subset L$ will have an isolated singularity at the origin. We may define the corresponding Plücker defect $\Delta_L$, which is then an analytic subvariety of $G(n-k, L) \cong G(n-k, N-k)$. By (5.8) and (5.15)

$$\lim_{t \to 0} \int_{V_t \cap L} c_{n-k}(\Omega_{V_t \cap L}) = \int_{V_0 \cap L} c_{n-k}(\Omega_{V_0 \cap L}) + \{\mu^{(n-k+1)} + \mu^{(n-k)}\}. \tag{5.18}$$

For $t \neq 0$ the Crofton's formula (III) (4.28) gives

$$\int \left( \int_{V_t \cap L} c_{n-k}(\Omega_{V_t \cap L}) \right) dL = C^{te} \int_{V_t} c_{n-k}(\Omega_{V_t}) \wedge \omega^k. \tag{5.19}$$

To evaluate the right hand side we use the

LEMMA: *For $t \neq 0$ and $\psi$ a closed $(n-k, n-k)$ form on $V_t$,*

$$\int_{V_t[\epsilon]} \psi \wedge \omega^k = \frac{1}{\epsilon^{2k}} \int_{V_t[\epsilon]} \psi \wedge \phi^k \tag{5.20}$$

*Proof.* Referring to the notation in the proof of (3.15), we set

$$\sigma = \psi \wedge d^c \log \|z\|^2 \wedge \omega^{k-1}.$$

Then $d\sigma = \psi \wedge \omega^k$, while on the boundary $\partial V_t[\epsilon]$

$$d^c \log \|z\|^2 \wedge \omega^{k-1} = \frac{1}{\epsilon^{2k}} d^c \|z\|^2 \wedge \phi^{k-1}$$

(cf. the computation following (3.16)). By Stokes' theorem, which may be applied since $V_t$ is smooth for $t \neq 0$,

$$\int_{V_t[\epsilon]} \psi \wedge \omega^k = \int_{\partial V_t[\epsilon]} \sigma$$

$$= \frac{1}{\epsilon^{2k}} \int_{\partial V_t[\epsilon]} \psi \wedge d^c \|z\|^2 \wedge \phi^{k-1}$$

$$= \frac{1}{\epsilon^{2k}} \int_{V_t[\epsilon]} \psi \wedge \phi^k$$

since $d\, d^c \|z\|^2 = \phi$. Q.E.D.

Combining (5.19) and (5.20) gives

$$\int \left( \int_{V_t[\epsilon] \cap L} c_{n-k}(\Omega_{V_t \cap L}) \right) dL = \frac{C^{te}}{\epsilon^{2k}} \int_{V_t[\epsilon]} c_{n-k}(\Omega_{V_t}) \wedge \phi^k \tag{5.21}$$

On the other hand it is not difficult to show that

$$\lim_{\epsilon \to 0} \int_{V_0[\epsilon] \cap L} c_{n-k}(\Omega_{V_0 \cap L}) = 0$$

uniformly in $L$, so that combining (5.18)–(5.21) we obtain the extension of (5.11) to arbitrary codimension

$$\lim_{\epsilon \to 0} \lim_{t \to 0} \frac{(-1)^{n-k} C^{te}}{\epsilon^{2k}} \int_{V_t[\epsilon]} c_{n-k}(\Omega_{V_t}) \wedge \phi^k = \{\mu^{(n-k+1)} + \mu^{(n-k)}\}. \tag{5.22}$$

Note that for $k = n$ the left hand side is

$$\lim_{\epsilon \to 0} \lim_{t \to 0} \frac{C^{te}}{\epsilon^{2k}} \int_{V_t[\epsilon]} \phi^n = \mathrm{mult}_0(V_0)$$

by (3.17), while $\mu^{(1)} + \mu^{(0)}$ is also equal to $\mathrm{mult}_0(V_0)$ by our conventions. Adding up the formulas (5.22) with alternating signs telescopes the right hand side and gives for the top Milnor number $\mu^{(n+1)} = \mu^{(n+1)}(V_0)$ the formula

$$\mu^{(n+1)}(V_0) = \lim_{\epsilon \to 0} \lim_{t \to 0} \left[ \sum_{k=0}^{n} (-1)^k \frac{C(k, n)}{\epsilon^{2k}} \int_{V_t[\epsilon]} c_{n-k}(\Omega_{V_t}) \wedge \phi^k \right] + (-1)^{n-1} \tag{5.23}$$

where the $C(k, n)$ are suitable positive constants.[9]

(d) *Further generalizations and open questions*. Due to the rather general approach we have taken in discussing curvature and singularities (e.g., (5.11)), in addition to the Milnor numbers (5.23) several other numerical characters are suggested which one may associate to a family of complex manifolds $V_t \subset \mathbb{C}^N$ acquiring an isolated singularity. Those which are essentially new arise only in higher codimension, and may or may not be of higher order depending on whether $\dim V_t \leq [N/2]$. We shall illustrate some of the possibilities by discussing the two simplest cases.

(i) Suppose that $\{S_t\}$ is a family of surfaces given in some open set in $\mathbb{C}^4$ with $S_t$ being smooth for $t \neq 0$ while $S_0$ has an isolated singularity at the origin. The

Plücker defect $\Delta$ is then an algebraic surface[10] in the Grassmannian $G(2, 4)$. Recall that $G(2, 4)$ has dimension four, and that there are two families of Schubert cycles in the middle dimension which may be described as follows:

$$\begin{cases} \text{for a hyperplane } H, \Sigma_2(H) \text{ is the set of} \\ \text{2-planes } T \text{ such that } T \subset H \text{ (cf. (4.10));} \end{cases}$$

$$\begin{cases} \text{for a line } L, \Sigma_{1,1}(L) \text{ is the set of} \\ \text{2-planes } T \text{ such that } L \subset T. \end{cases}$$

The first intersection number $\#(\Delta, H)$ describes the limit of the number $\delta$ of critical values in a pencil of sections

$$C_{\lambda,t} = S_t \cap (\lambda + H);$$

as such it has to do with the number of vanishing cycles in the pencil $|C_{\lambda,t}|_\lambda$ and therefore with the Milnor number of $S_0$ (cf. the proof of (5.15)). The formula

$$\text{(5.24)} \qquad \lim_{\epsilon \to 0} \lim_{t \to 0} C^{\underline{te}} \int_{S_t[\epsilon]} c_2(\Omega_{S_t}) = \delta$$

is a consequence of (5.14) and is of the same character as (5.17).

Consideration of the other Schubert cycle $\Sigma_{1,1}(L)$ leads to the following geometric interpretation: Under a generic projection $\mathbb{C}^4 \to \mathbb{C}^3$, $S$ goes to a surface $S'$ having a finite number $\delta'$ of isolated singularities, and for generic $L$ this number is $\#\ (\Delta, \Sigma_{1,1}(L))$. In a manner similar to (5.24) we infer from (5.14) that

$$\text{(5.25)} \qquad \lim_{\epsilon \to 0} \lim_{t \to 0} C^{\underline{te}} \int_{S_t[\epsilon]} \{c_1(\Omega_{S_t})^2 - c_2(\Omega_{S_t})\} = \delta'$$

Based on the following analogy it seems possible that some variant of $\delta'$ will have topological meaning: For a smooth algebraic surface $W \subset \mathbb{P}^3$ the Chern numbers $c_1^2$ and $c_2$ may both be calculated from the degree, so that e.g., $c_1^2 - c_2$ is determined by $c_2$. But for a non-degenerate smooth surface $W \subset \mathbb{P}^4$ these two numbers are independent in the sense that neither one determines the other. Thus it seems reasonable that, at least for those surfaces $S_0 \subset \mathbb{C}^4$ which are limits of smooth surfaces, the independent numbers $c_2$ and $c_1^2 - c_2$ can be localized to yield two distinct invariants.

(ii) Suppose now that $\{C_t\}$ is a family of curves in an open set in $\mathbb{C}^3$ tending to a limit curve $C_0$ having an isolated singularity at the origin. The Gaussian images $\gamma_t(C_t) \subset G(1, 3) \cong \mathbb{P}^2$ give a family of analytic curves in the projective plane with

$$\lim_{t \to 0} \gamma_t(C_t) = \Delta + \gamma_0(C_0)$$

where the algebraic curve $\Delta$ is the Plücker defect. Recalling that $C^{\underline{te}}K\,dA$ is the pullback under $\gamma_t$ of the standard Kähler form on $\mathbb{P}^2$, we infer as in (5.14) that

$$\lim_{\epsilon \to 0} \lim_{t \to 0} C^{\underline{te}} \int_{C_t[\epsilon]} KdA = \#(\Delta, H), \tag{5.26}$$

and this number $\delta$ has the following interpretation: A generic pencil of plane sections $D_{t,\lambda} = C_t \cap (\lambda + H)$ represents $C_t$ as a $d = \text{mult}_0(C_0)$-sheeted covering of a disc in the $\lambda$-plane having $\delta$ branch points.

Now, assuming that $C_0$ is non-dengenerate, we may consider the second order Gauss map

$$\gamma_t^* : C_t \to \mathbb{P}_2^*$$

which assigns to each $z \in C_t$ the osculating 2-plane.[11] As before we may define the 2nd order Plücker defect $\Delta^*$ by

$$\lim_{t \to 0} \gamma_t^*(C_t) = \gamma_0^*(C_0) + \Delta^*,$$

and the letting $C^{\underline{te}} K^*dA$ denote the pullback under $\gamma_t^*$ of the Kähler form on $\mathbb{P}^{2*}$ we infer the formula

$$\lim_{\epsilon \to 0} \lim_{t \to 0} C^{\underline{te}} \int_{C_t[\epsilon]} K^*dA = \#(\Delta^*, L^*), \tag{5.27}$$

where $L^* \subset \mathbb{P}^{2*}$ is a generic line. The number $\delta^*$ on the right hand side of (5.27) has the following geometric interpretation: Under a generic linear projection $\mathbb{C}^3 \to \mathbb{C}^2$ the curve $C_t$ projects onto a plane curve $C'_t$, and $\delta^*$ is the number of flexes of $C'_t[\epsilon]$ as $\epsilon$, $t \to 0$.

Perhaps the general thrust of the discussion may be summarized as follows: In a family of global smooth algebraic varieties $V_t \subset \mathbb{P}^N$ tending to a singular variety $V_0$, some of the projective characters may jump in the limit at $t \to 0$[12]. This jump is measured by the intersection number of Schubert cycles with the Plücker defect $\Delta$ (and its higher order analogues $\Delta^*$, etc.), and by the analytic Plücker formulas[13] may be expressed as a difference

$$\lim_{t \to 0} \int_{V_t} \{\text{curvature form}\} - \int_{V_0} \{\text{curvature form}\}.$$

Moreover, in case $V_0$ has an isolated singularity this whole process may be localized around the singular point, so that the correction factor which must be subtracted from the Plücker formulas for $V_t$ to obtain those for $V_0$ is[14]

$$\lim_{\epsilon \to 0} \lim_{t \to 0} \int_{V_t[\epsilon]} \{\text{curvature form}\}.$$

Finally, and most importantly, certain combinations of these expressions will have intrinsic meaning for $V_0$, and may even yield topological invariants of the singularity as was the case in (5.23).

(iii) We will conclude with some observations about curvature integrals in the real and complex cases. In real differential geometry one generally encounters two types of curvature integrals, illustrated for a Riemannian surface by the two expressions

$$\int_S KdA, \qquad \int_S |K|dA \tag{5.28}$$

In general the first type includes characteristic classes and Weyl's coefficients (1.10), and reflects intrinsic metric properties of the manifold which may even be of a topological nature. The second type usually describes extrinsic properties of the manifold in Euclidean space[59]. For a complex manifold $M_n \subset \mathbb{C}^N$, because of sign properties such as

$$(-1)^k c_k(\Omega_M) \wedge \phi^{n-k} \geq 0$$

the distinction between the two types of curvature integrals illustrated in (5.28) seems to disappear, and the situation may be said either to be simpler or more rigid, depending on one's viewpoint.

Finally, along similar lines it would seem interesting to try and draw conclusions on the curvature of real algebraic curves acquiring a singularity. For example, one might examine the real curves $C_{t,\mathbb{R}} = \mathbb{R}^2 \cap C_t$ where $C_t \subset \mathbb{C}^2$ is defined by $f(x, y) = t$ with $f(x, y)$ being a weighted homogeneous polynomial having real coefficients. Superficial considerations suggest that the curvature of $C_{t,\mathbb{R}}$ in $\mathbb{R}^2$ tends to $\infty$, while on the Riemann surface $C_t$ the geodesic curvature of the (purely imaginary) vanishing cycles remains bounded.

(iv) This paragraph is an afterthought. Upon reviewing the preceeding discussion about "topologically invariant curvature integrals associated to a singularity" it seems to me that some clarification is desirable.

To begin with let us consider the pedogogical question of how one might best prove the Gauss-Bonnet theorem

$$C^{\underline{te}} \int_M KdM = \chi(M) \tag{5.29}$$

in a course on differential geometry. Chern's intrinsic proof (cf. footnote (8) in section 1) is probably the quickest but in my experience leaves some mystery as to the origin of his formulas. The Allendoerfer-Weil proof (cf. footnote (5) in section 1) is intuitively appealing and explains the origin of the formula, but relies on either the Nash embedding theorem or an unpleasant construction to show that the left hand side of (5.29) is independent of the metric. Finally, the proof by characteristic classes, interpreting the Gauss-Bonnet integrand as a de Rham representative of the Euler class of the tangent bundle, is conceptually satisfying but involves establishing a fairly elaborate machine. The following proof, in four steps, represents a compromise:

a) Prove (5.29) for oriented real hypersurfaces using the Hopf theorem, as was done in section 1(b), where $KdM$ is the Gauss-Kronecker curvature.

b) Using a) and the tube construction, deduce (5.29) for submanifolds $M_n \subset \mathbb{R}^N$ as was also done in section 1(b), this step explains the origin of the Gauss-Bonnet integrand.

c) Following the discussion in §1(c), use the Gauss mapping

$$\gamma : M \to G_{\mathbb{R}}(n, N)$$

to show that $C^{\underline{te}}\ KdM = \gamma^* Pf(\Omega_E)$ is induced from a closed form on the Grassmannian.

d) Finally, to show that the left hand side of (5.29) is independent of the Riemannian metric $g$ on $M$, we proceed as follows: Denote by $I = \{0 \le s \le 1\}$ the interval and suppose that $\{M_s\}$ is a 1-parameter family of manifolds in $\mathbb{R}^N$ given by a mapping

$$f\colon M \times I \to \mathbb{R}^N$$

where $f\colon M \times \{s\} \to \mathbb{R}^N$ is a smooth embedding with image $M_s$. Define

$$\gamma\colon M \times I \to G_{\mathbb{R}}(n\ N)$$

by $\gamma(x, s) = T_{f(x,s)}(M_s)$ and set

$$\gamma^* Pf(\Omega) = \Phi - \Psi \wedge ds$$

where $\Phi$ and $\Psi$ do not involve $ds$. Then $\Phi | M \times \{s\}$ is the Gauss-Bonnet integrand $C^{\underline{te}}\ K_s dM_s$ for the metric $g_s$ on $M$, and, using $(x, s)$ as product coordinates on $M \times I$, from

$$\begin{aligned} 0 &= \gamma^* Pf(\Omega_E) \\ &= \frac{\partial \Phi}{\partial s} \wedge ds - d_x \Psi \wedge ds \end{aligned}$$

we deduce that, with the obvious notation,

$$\frac{\partial \Phi}{\partial s} = d_x \Psi \tag{5.30}$$

It follows that

$$\begin{aligned} \frac{\partial}{\partial s}\left( C^{\underline{te}} \int_M K_s dM_s \right) &= \int_M \frac{\partial \Phi}{\partial s} \\ &= \int_M d_x \Psi \\ &= 0 \end{aligned}$$

by Stokes' theorem. Now, and this is the point, a simple local calculation shows that $\Psi$ depends only on the family of metrics $g_s$, and consequently the formula (5.30) can be established for a general 1-parameter family of metrics which need not come from embeddings in $\mathbb{R}^N$. Since any metric can be connected by a linear homotopy to one coming from such an embedding, we have thus established (5.29) for any metric. The point is that, just as the expressions in Weyl's tube formula turn out to be intrinsic invariants of the induced metric and therefore have meaning for any Riemannian manifold, the same will be true for the variation (5.30) of the Gauss-Bonnet integrand.

An interesting question is whether we can use a similar argument to establish the topological invariance of the top Milnor number $\mu^{n+1}$ of a hypersurface $V_0 \subset \mathbb{C}^{n+1}$ having an isolated singularity at the origin. This is especially intriguing since the example of Briancon and Speder (cf. Tessier, loc. cit.) shows that the lower Milnor numbers may not be topologically invariant. Suppose then that $\{V_{t,s}\}$ is a family of complex-analytic hypersurfaces parametrized by $\Delta \times I$ where $\Delta$ is the disc $\{|t| < 1\}$ and $I$ is the real interval $\{0 \leq s \leq 1\}$, and where $V_{t,s}$ is smooth for $t \neq 0$ while $V_{0,s}$ has an isolated singularity at the origin. We assume that, for fixed $s$, the $V_{t,s}$ fill out a neighborhood $U$ of the origin, and setting $U^* = U - \{0\}$ define

$$\gamma\colon U^* \times I \to \mathbb{C}^{n+1} \times G(n, n+1)$$

by $\gamma(z, s) = (z, T_z(V_{t,s}))$ where $z \in V_{t,s}$ $(t \neq 0)$. For any invariant curvature polynomial $P_k(\Omega_E)$ on $G(n, N)$ we set

$$\gamma^*(\phi^{n-k} \wedge P_k(\Omega_E)) = \Phi - \Psi \wedge ds$$

on $U^* \times I$, so that

$$\Phi | V_{t,s} = \phi^{n-k} \wedge P_k(\Omega_{V_{t,s}}).$$

Since this form is closed, as in (5.30) we deduce that

$$\frac{\partial \Phi}{\Phi s} = d_z \Psi$$

so that for $t \neq 0$

$$\frac{\partial}{\partial s}\left(\frac{1}{\epsilon^{2(n-k)}}\right)\int_{V_{t,s}[\epsilon]} P_k(\Omega_{V_{t,s}}) \wedge \phi^{n-k} = \frac{1}{\epsilon^{2(n-k)}} \int_{\partial V_{t,s}[\epsilon]} \Psi. \tag{5.31}$$

Observe that the right hand side of (5.31) is defined also for $t = 0$.

If we now let $\Psi_k$ be the form $\Psi$ arising from $c_k(\Omega_E)$, then (5.23) together with the topological invariance of the Milnor numbers suggests that

$$\lim_{\epsilon \to 0}\left(\sum_{k=0}^{n} \frac{(-1)^k}{\epsilon^{2(n-k)}} \int_{\partial V_{0,s}[\epsilon]} \Psi_k\right) = 0. \tag{5.32}$$

On the other hand, since the lower Milnor numbers are not topologically invariant, we will *not* have

$$\lim_{\epsilon \to 0}\left(\frac{1}{\epsilon^{2(n-k)}} \int_{\partial V_{0,s}[\epsilon]} \Psi_k\right) = 0. \tag{5.33}$$

One way in which this makes sense is for the individual forms $\Psi_k$ to have a singularity at the origin of just the right order to prevent (5.33) from holding (cf. the proof of (3.17) in §3(b)), but in the alternating sum

$$\sum_k \frac{(-1)^k}{\epsilon^{2(n-k)}} \Psi_k$$

cancellation occurs in the highest order term. If this were the case, then there would be a procedure for looking into the topological invariance of other curvature integrals. We note in closing that the question is purely one on the singular varieties $V_{0,s}$ and might be examined by blowing up the origin in a fashion similar to how (3.17) was established.

**Footnotes**

1. cf. the references cited in footnote 2 of the introduction.
2. cf. the reference cited in footnote 1 of the introduction.
3. There is a discussion of the Plücker formulas in the paper cited in footnote 4 of the introduction.
4. The motivation for this terminology is given by the discussion at the end of the preceeding section 5(a).
5. H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Annals of Math., vol. 79 (1964).
6. B. Tessier, *Introduction to equisingularity problems*, Proc. Symposia in Pure Math., vol. 29 (1975), pp. 593–632; this paper contains an extensive bibliography on singularities.
7. Milnor numbers were introduced in Milnor's book cited in footnote 5 in the introduction. There is an extensive bibliography on material pertaining to Milnor numbers given in Tessier's article referred to in footnote 6 above.
8. Of course the classic source is S. Lefschetz, *L'Analysis situs et la géométrie algébrique*, Gauthier-Villars, Paris, (1924). A recent paper which in fact is pertinent to our present study is A. Landman, *On the Picard-Lefschetz transformations*, Trans. Amer. Math. Soc., vol. 181 (1973), pp. 89–126.
9. As noted in §3(a) (cf. (3.7)) the differential forms $c_{n-k}(\Omega_{V_t})$ have the positivity property

$$(-1)^{n-k} c_{n-k}(\Omega_{V_t}) \wedge \phi^k \geq 0,$$

which is consistent with the positivity of the right hand side of (5.22). We may ask if the more subtle positivity

$$\lim_{\epsilon \to 0} \lim_{t \to 0} \sum \frac{(-1)^k C(k, n)}{\epsilon^{2k}} c_{n-k}(\Omega_{V_t}) \wedge \phi^k \geq 0$$

is valid?

10. Of course it may happen that $\Delta$ is degenerate, but in view of (3.11) it would seem that this should be considered an exceptional occurrence.
11. In this discussion, $\gamma_t^*$ does *not* denote the pullback of forms.
12. e.g., the degree will not, but the class will.
13. cf. the discussion in the reference cited in footnote 4 of the introduction.
14. G. Laumon, *Degré de la variété duale d'une hypersurface à singularitiés isolés*, Bull. Soc. Math. France (1976), pp. 51–63.
15. cf. The Fenchel-Fary-Milnor theorem in §2(b) the reference cited in Footnote 7 there.

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