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Correspondence and Cycle Spaces: A result comparing their cohomologies

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Outline.

1. Introduction

Let G be a reductive, semi-simple Lie group, $B \subset G$ a Borel subgroup and $X = G/B$ the corresponding *flag manifold.* Let G_0 be a connected real form that contains a compact maximal torus T; this means in particular that the complexified Lie algebra $\mathfrak{t}\otimes_{\mathbb{R}}\mathbb{C} =: \mathfrak{h}$ is a Cartan subalgebra corresponding to a Cartan subgroup $H \subset G$. By a *flag domain* D we mean an open G_0-orbit $G_0(x_0)$ of G acting on X whose isotropy group is compact. Flag manifolds and flag domains have over the years played a central role in representation theory, both finite and infinite dimensional ([**Sch2**], [**Sch3**], [**BE**], [**FHW**] and the references cited therein). Recently they, together with more general homogeneous complex manifolds G_0/L where $L \supset T$ and L is the compact centralizer of a circle $S^1 \subset T$, have appeared in Hodge theory in the form of Mumford-Tate domains [**GGK1**]. For the case $G_0 = \mathcal{U}(2,1)$, the corresponding Mumford-Tate domains have also appeared in very interesting recent work on arithmetic automorphic representation theory ([**C1**], [**C2**], [**C3**]). In the recent exposition [**GGK2**] of aspects of that work, together with extensions of it, certain constructions concerning the complex geometry of flag domains arose. These constructions play a central role in the use of *Penrose transforms* ([**BE**], [**EGW**], [**C2**]). In the exposition [**GGK2**], in special cases they were used under the term *correspondence spaces.* In that work the general construction and properties of these spaces, together with their relation to the *cycle spaces* [**FHW**] that have been in use since the mid-to-late 1960's ([**Sch1**] and [**GS**]), were discussed. The primary purpose of the present paper is to give the formal definition and some properties of the correspondence space $\mathcal{W}$ and to state and prove a result relating the complex geometry of $\mathcal{W}$ to that of the cycle space $\mathcal{U}$.

To give the informal statement of the result we first comment that both $\mathcal{W}$ and $\mathcal{U}$ are used to relate the cohomology $H^*(D, L_\mu)$ of homogeneous line bundles

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$L_\mu \to D$ to global, holomorphic objects. In the case of the correspondence space $\mathcal{W}$ the object is a holomorphic de Rham cohomology group, and it is therefore a quotient of spaces of global holomorphic sections of vector bundles. In this case there is the isomorphism [**EGW**]

$$H^*(D, L_\mu) \cong H^*\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi(L_\mu)); d_\pi\big). \tag{1.1}$$

In the examples considered so far, there are canonical "harmonic" representatives for classes on the RHS, so that in particular cohomology classes on the LHS can be "evaluated" at points of $\mathcal{W}$. For D a Mumford-Tate domain, $\mathcal{W}$ has an *arithmetic structure* (think of CM points in a Shimura variety), and the main result of [**GGK2**] concerns classes in $H^*(D, L_\mu)$ that take arithmetically defined values at arithmetic points of $\mathcal{W}$.

The "correspondence space" arises from the following consideration: The equivalence classes of homogeneous complex structures on G_0/T are indexed by W/W_K where W, W_K are the Weyl groups of G, G_0 respectively. We denote these by D_w, $w \in W/W_K$. The universality property of $\mathcal{W}$ gives diagrams

$$\begin{array}{ccc} & \mathcal{W} & \\ {}^{\pi}\swarrow & & \searrow^{\pi'} \\ D_w & & D_{w'}. \end{array} \tag{1.2}$$

Using (1.1) applied to D_w and $D_{w'}$ the existence of certain canonical classes on $\mathcal{W}$ gives multiplication mappings

$$H^q\big(\Gamma(\Omega^\bullet_\pi(L_\mu)); d_\pi\big) \to H^{q'}\big(\Gamma(\Omega^\bullet_{\pi'}(L'_{\mu'})); d_{\pi'}\big)$$

which lead to Penrose transforms

$$H^q(D_w, L_\mu) \to H^{q'}(D_{w'}, L'_{\mu'}).$$

One may think of this as an analogue of the maps on ordinary cohomology in classical algebraic geometry induced by a cycle on $\mathcal{W}$ in a diagram (1.2) where the objects are algebraic varieties.

In the case of the cycle space $\mathcal{U}$ there is always a map

$$H^q(D, L_\mu) \to H^0(\mathcal{U}, F^{0,q}_\mu)$$

where the $F^{p,q}_\mu \to \mathcal{U}$ are holomorphic vector bundles whose rank is $h^q(Z, \Lambda^p N_{Z/D}(L_\mu))$. There are conditions under which this map is injective and a description of its image (cf. [**FHW**], Theorem (3.4) and Corollary (3.5) below).

The main result of this paper is to relate the two global holomorphic objects which realize $H^*(D, L_\mu)$. The result applies only in the case that D is *non-classical*, meaning that it does not fibre holomorphically or anti-holomorphically over an Hermitian symmetric domain. This is the primary case of interest in [**C1**], [**C2**], [**C3**], [**GGK2**] as it is the situation where new geometric and arithmetic phenomena occur. The result is the

THEOREM 1.1. *In case D is non-classical there is a spectral sequence with*

$$\begin{cases} E_1^{p,q} = H^0(\mathcal{U}, F^{p,q}_\mu) \\ E_\infty^{p,q} = \mathrm{Gr}^p H^{p+q}\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi(L_\mu)); d_\pi\big). \end{cases}$$

For D non-classical we have $\mathcal{U} \subset G/K$ and there are maximal compact subvarieties $Z_u \subset D$, $u \in \mathcal{U}$ given by the translates by $g \in G$ of $Z = K/T$ where $gZ \subset D$. Denoting by $N_{Z/D}$ the normal bundle of Z in D, the proof of the theorem will yield the

COROLLARY 1.2. *If* $H^k(Z, \Lambda^{q-k+1} N_{Z/D}(L_\mu)) = 0$ *for* $0 \leqq k \leqq q-1$, *then*

$$H^q\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi(L_\mu)); d_\pi\big) \cong \ker\{\, H^0(D, F^{0,q}_\mu) \xrightarrow{d_1} H^0(D, F^{1,q}_\mu)\,\}.$$

When the isomorphism (1.1) is used on the LHS this corollary is closely related to the result in [**WZ**].

The original motivation for much of the work that this paper is drawing from was concerned not with the $H^q(D, L_\mu)$ but rather with the *automorphic cohomology groups* $H^q(\Gamma\backslash D, L_\mu)$ where $\Gamma \subset G_0$ is a discrete, co-compact and neat subgroup. In section 4.1 we will show that the equation (1.1), and the results (1.1) and (1.2) remain valid as stated when the spaces are factored by Γ. The main new ingredient used here is a result from [**BHH**] which shows that the quotient space $\Gamma\backslash\mathcal{U}$ is Stein.

In section 4.2 we shall show that the de Rham cohomology

$$H^*\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi(L_\mu)); d_\pi\big)$$

may be written as $\mathfrak{n}$-cohomology $H^*(\mathfrak{n}, \mathcal{O}G_{\mathcal{W}})_{-\mu}$ for a G_0-module $\mathcal{O}G_{\mathcal{W}}$. The spectral sequence (1.1) may then be interpreted as the Hochschild-Serre spectral sequence for $\mathfrak{n}_c \subset \mathfrak{n}$ where $\mathfrak{n}_c = \mathfrak{n} \cap \mathfrak{k}$ for $\mathfrak{k}$ the complexification of Lie algebra of the maximal compact subgroup K_0 of G_0.[1]

The differentials

$$d_r \colon E^{p,q}_r \to E^{p+r,q-r+1}_r$$

are linear differential operators of degree r, and we shall give a result (Theorem (4.4)) defining and computing their *symbols*. We shall also describe the *characteristic varieties* in our two examples.

In section 4.3 we shall analyze the spectral sequence in the special case of the two examples discussed in [**GGK2**]. This analysis will include determination of the symbol sequence and characteristic varieties for the linear PDE systems whose solutions are the Harish-Chandra module $V_{\mu+\rho}$ with infinitesimal character $\chi_{\mu+\rho}$ where $\mu + \rho$ is in the closure of the anti-dominant Weyl chamber. Of particular interest here are the cases where $\mu + \rho$ is singular, in which case $V_{\mu+\rho}$ is a limit of discrete series. The PDE systems have quite a different character than when $\mu + \rho$ is regular, as is not surprising due to the much greater intricacy of $\mathfrak{n}$-cohomology in these cases.

This paper is a companion work to [**GGK2**], one which completes a definition promised there and which relates the global holomorphic realization of cohomology that was an essential ingredient in [**GGK2**] to the other one that has appeared in the literature. The general context for this work is the relation between representation theory and the geometry of complex homogeneous manifolds. This is a vast and rich subject and we have chosen to refer to the references in some of our primary sources, specifically [**Sch2**], [**Sch3**], [**BE**] and [**FHW**], for excellent expositions of the general theory and for guides to the literature. We also note [**Sch1**], where much of the connection between homogeneous complex manifolds and representation theory had its origin.

[1]The subscript "c" for $\mathfrak{n}_c$ refers to "compact," as $\mathfrak{n}_c$ is the direct sum of the negative root spaces corresponding to the compact roots.

This paper is dedicated to Joe Harris on the occasion of his 60th birthday. The talk that the second author of this paper gave at Joe's 60th conference had the theme that understanding in depth "elementary" examples that have a rich geometry is both interesting in its own right and serves to suggest interesting general structures. The examples presented in that talk were based on [**GGK2**] and are recalled briefly in this paper. We feel that the theme mentioned above is very harmonious with Joe's approach to mathematics.

Notations.

- G is a reductive, semi-simple Lie group with Lie algebra $\mathfrak{g}$;
- $H \subset G$ is a Cartan subgroup with Lie algebra $\mathfrak{h}$;
- $B \subset G$ is a Borel subgroup with associated flag manifold $X = G/B$;
- G_0, with Lie algebra $\mathfrak{g}_0$, is a real form of G;
- we assume that the real form H_0 of H is a compact maximal torus $T \subset G_0$; we shall use the notations H_0 and T interchangeably;
- $K_0 \subset G_0$ is a maximal compact subgroup with $T \subset K_0$ and complexification $K \subset G$;
- $\Phi, \Phi^+, \Phi_c, \Phi_n$ are respectively the roots, positive roots, compact roots and non-compact roots of $(\mathfrak{g}, \mathfrak{h})$;
- W, W_K are the Weyl groups of (G, H), (K_0, T) respectively;
- it is known that the homogeneous complex structures D_w on G_0/T are parametrized by $w \in W/W_K$; they are the open orbits of G_0 acting on X;
- we shall denote by D one of the D_w and by $Z \cong K_0/T \cong K/B_K$, where $B_K = K \cap B$, is a maximal compact subvariety of D;
- the root space decomposition of $\mathfrak{g}$ is denoted
$$\mathfrak{g} = \mathfrak{h} \oplus \left(\underset{\alpha\in\Phi}{\oplus} \mathfrak{g}^{\alpha} \right)$$
where $\mathfrak{g}^{\alpha}$ is the α-root space;
- we have
$$\begin{cases} \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \\ \mathfrak{n} = \underset{\alpha\in\Phi^+}{\oplus} \mathfrak{g}^{-\alpha}; \end{cases}$$
- the Cartan decomposition of $\mathfrak{g}$ is
$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p};$$
we have
$$\begin{cases} \mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^- \\ \mathfrak{n} = \mathfrak{n}_c \oplus \mathfrak{p}^- \end{cases}$$
where $\mathfrak{p}^+ = \underset{\alpha\in\Phi^+_{nc}}{\oplus} \mathfrak{g}^{\alpha}$, $\mathfrak{p}^- = \overline{\mathfrak{p}^+}$ and $\mathfrak{n}_c = \underset{\alpha\in\Phi^+_c}{\oplus} \mathfrak{g}^{-\alpha}$;
- the compact maximal torus is
$$T = \mathfrak{t}/L,$$
where L is a lattice, and we denote by
$$\Gamma \subset i\check{\mathfrak{t}}$$
the weight lattice, which up to a factor of 2π is identified with $\mathrm{Hom}(L, \mathbb{Z})$;
- given a weight μ there is a corresponding character χ_μ of T which induces a homogeneous line bundle $L_\mu \to G_0/T$;

- $L_\mu \to G_0/T$ is made into a holomorphic line bundle over D in the usual way; i.e., by extending it to a holomorphic character $\chi_\mu \colon H \to \mathbb{C}^*$ and then extending it to B via the map $B \to H$.

2. Correspondence and cycle spaces

Cycle spaces and correspondence spaces first arose from special cases of flag domains D, cycle spaces initially in the context of Hodge theory and then representation theory, correspondence spaces in the context of integral geometry and Penrose-type transforms. Both the cycle spaces $\mathcal{U}$ and correspondence spaces $\mathcal{W}$ considered here are open subsets in G-homogeneous projective algebraic varieties, and the basic diagram relating $D, \mathcal{W}$ and $\mathcal{U}$ is an open subset of a diagram of G-homogeneous algebraic varieties. For later reference we now record this diagram:

(2.1)
$$\begin{array}{ccccc} & & G/H & & \\ & & \downarrow & & \\ & & G/B_K & & \\ & \swarrow & & \searrow & \\ G/B & & & & G/K. \end{array}$$

The space G/H is sometimes called the *enhanced flag variety.* Double homogeneous space fibrations in the lower part of the diagram are classical in integral geometry [**Ch**]. We note the following general properties:

(2.2a) *The fibres of $G/B_K \to G/B$ and of $G/H \to G/B_K$ are contractible affine algebraic varieties;*

(2.2b) *The fibres of $G/B_K \to G/K$ are projective algebraic varieties.*

Discussion. (see [**FHW**] for detailed proofs): From

$$\mathfrak{b} = \mathfrak{b}_K \oplus \mathfrak{p}^-$$

where $\mathfrak{b}_K = \mathfrak{b} \cap \mathfrak{k}$ one may show that

$$\exp \colon \mathfrak{p}^+ \xrightarrow{\sim} B/B_K$$

is a bi-holomorphic map. A similar argument works for

$$\mathfrak{b}_K = \mathfrak{h} \oplus \mathfrak{n}^+.$$

Finally, K/B_K is the flag variety for K.

The definition of the correspondence space derives from *Matsuki duality* between G_0-orbits $\mathcal{O}_{G_o}$ and K-orbits $\mathcal{O}_K$ in the flag variety X ([**FHW**] and [**Sch3**]). We recall that the pair $(\mathcal{O}_{G_0}, \mathcal{O}_K)$ are *Matsuki dual* if the intersection $\mathcal{O}_{G_0} \cap \mathcal{O}_K$ consists of exactly one K_0-orbit. The relation "contained in the closure of" partially orders the set of K-orbits as well as the set of G_0-orbits, and the duality

$$\{G_0\text{-orbits in } X\} \longleftrightarrow \{K\text{-orbits in } X\}$$

reverses the closure relationships. For $x_0 \in X$ such that $G_0(x_0) = \mathcal{O}_{G_0}$ is open, or equivalently $K(x_0) = \mathcal{O}_K$ is compact, we set

$$\mathcal{W}_G = \{\, g \in G : g\mathcal{O}_K \cap \mathcal{O}_{G_0} \neq \emptyset \text{ and is compact} \,\}^0/H.$$

Here, $\{\ \}^0$ denotes the connected component of the identity. Since $H \subset K$ this definition makes sense and $\mathcal{W}_G \subset G/H$ is an open set.

THEOREM-DEFINITION 2.1. *For an open G_0-orbit $\mathcal{W}_G$ is independent of x_0. We denote it by $\mathcal{W}$ and define it to be the* **correspondence space** *associated to (G_0, H).*

We recall here our blanket assumption that D is non-classical; i.e., it does not fibre holomorphically or anti-holomorphically over an Hermitian symmetric domain.

The property in the theorem is called *universality.* We will infer it from a similar property for the cycle space $\mathcal{U}$, to which we now turn. Set

$$\mathcal{U}_G = \{\, g \in G : g\mathcal{O}_K \subset \mathcal{O}_{G_0} \,\}^0/K.$$

We note that the term $\{\quad\}^0$ is the same as for $\mathcal{W}_G$; we have given this description because the cycle spaces associated to $D = G_0/T$ initially arose as

$$\{\, gZ : g \in G \text{ and } gZ \subset D \,\},$$

which is the set of translates gZ of the maximal compact subvariety $Z = K_0/T$ by $g \in G$ such that gZ remains in D (cf. [**Sch1**] and [**GS**]). The universality of $\mathcal{W}$ is a consequence of

THEOREM (universality, [**FHW**]). *For open G_0-orbits, $\mathcal{U}_G$ is independent of x_0.*

The proof of this theorem is based on Matsuki duality.

Before turning to the basic diagram and the properties of $\mathcal{W}$ and $\mathcal{U}$, we have previously noted that the open G_0-orbits D_w are indexed by the elements $w \in W/W_K$. Equivalently, a complex structure D_w on G_0/T is given by a choice Φ_w^+ of positive roots, and two such G_0-homogeneous complex structures $D_w, D_{w'}$ are equivalent if $w \equiv w' \bmod W_K$. Each D_w has a distinguished point $x_w \in D_w$ as follows:

If $x_0 \in G/B$ is the identity coset, then

$$x_w = w x_0 w^{-1} \in D_w.$$

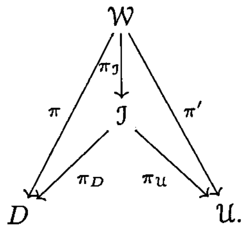
It follows that $D_w = G_0(x_w)$ and the compact K-orbit $Z_w \subset D_w$ given by the duality theorem is $Kx_w = wZw^{-1}$ where $Z = K_0/T \subset G_0/T$.

We will now describe the basic diagram for D. By the remark just given there will be a corresponding basic diagram for each D_w. Letting $\{Z_u, u \in \mathcal{U}\}$ be the family of maximal compact subvarieties $Z_u \subset D$ parametrized by $\mathcal{U}$, we define the *incidence correspondence* $\mathcal{I} \subset D \times \mathcal{U}$ by

$$\mathcal{I} = \{(x, u) : x \in Z_u\}.$$

DEFINITION 2.2. The *basic diagram* is

(2.3)



The maps are those induced by the maps in (2.1), where we note the inclusion

$$\mathcal{J} \subset G/B_K.$$

THEOREM 2.3.

(1) *$\mathcal{W}$ is a Stein manifold;*
(2) *the fibres of $\mathcal{W} \to D$ are contractible;*
(3) *the fibres of $\mathcal{W} \to \mathcal{J}$ are contractible.*

PROOF. For G_0 of Hermitian type, and recalling our assumption that D is non-classical, this result largely follows from the results in [**FHW**]. Specifically, we have that:

- $\mathcal{U}$ is Stein ([**FHW**], Corollary 6.3.3); and
- the fibres of $\pi' \colon \mathcal{W} \to \mathcal{U}$ are affine algebraic varieties.[2]

The latter statement follows by observing that (2.3) is an open subset of (2.1), and $\mathcal{W} \subset G/H$ is the inverse image of $\mathcal{U} \subset G/K$. A similar argument applies to $\mathcal{W} \to \mathcal{J}$, where a typical fibre is B/B_K. From [**FHW**], (6.23), in case G_0 is of Hermitian type the fibres $\mathcal{J} \to D$ are contractible. This case covers the two examples discussed below. The general case when G_0 is not of Hermitian type is more complex and will be discussed elsewhere. □

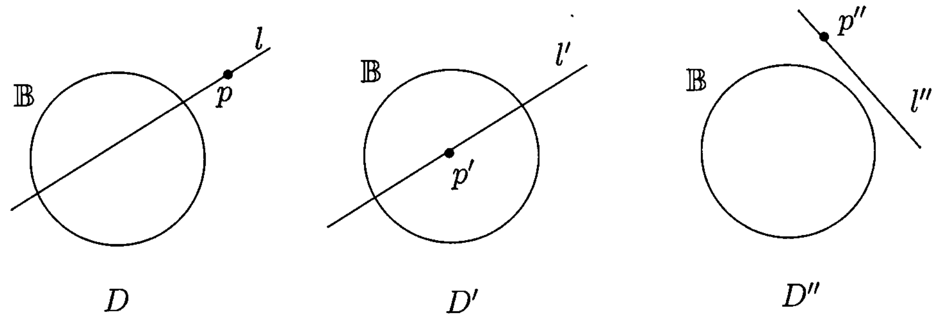
EXAMPLE 2.4. $\mathcal{U}(2,1)^3$ ([**EGW**], [**C1**], [**C2**] and [**GGK2**]).

- $\mathbb{H}$ is the standard Hermitian form on $\mathbb{C}^3$ with matrix diag$(1,1,-1)$ and
$$\mathcal{U}(2,1) = \{\, g \in \mathrm{GL}(3,\mathbb{C}) : {}^t\bar{g}\mathbb{H}g = \mathbb{H}\,\};$$
- points $p \in \mathbb{P}^2$ are given by homogeneous column vectors $p = {}^t[p_1, p_2, p_3]$ and lines $l \in \check{\mathbb{P}}^2$ by homogeneous row vectors $l = [l_1, l_2, l_3]$;
- the unit ball $\mathbb{B} \subset \mathbb{P}^2$ is given by $\{p : {}^t\bar{p}\mathbb{H}p < 0\}$; $\mathbb{B}^c = \mathbb{P}^2 \backslash \mathcal{C}l(\mathbb{B})$ is the complement of the closed ball;
- the flag variety is described as the standard incidence correspondence $X = \{(p,l) : \langle l, p\rangle = 0\}$ in $\mathbb{P}^2 \times \check{\mathbb{P}}^2$.

[2]The fibre $\pi'^{-1}(u_0) \cong K/H$ is the enhanced flag variety of $Z = Z_{u_0}$. It is a general result [**Bo**] that the quotient by the Cartan subgroup H of the affine variety K is again an affine algebraic variety. We note that K is reductive with center contained in H, so the quotient is the same as one of a semi-simple complex linear group by a Cartan subgroup.

[3]We use $\mathcal{U}(2,1)$ rather than $S\mathcal{U}(2,1)$ because $\mathcal{U}(2,1)$ is the Mumford-Tate group of a generic polarized Hodge structure with Hodge numbers $h^{3,0} = 1$, $h^{2,1} = 2$ and having an action by $\mathbb{Q}(\sqrt{-d})$ (cf. [**GGK1**]).

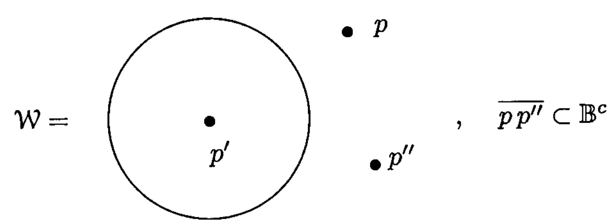
There are three flag domains given as open orbits of $\mathcal{U}(2,1)$ acting on X and which may be pictured as follows:



Here, D is non-classical and D', D'' are classical. For example, $(p', l') \to p'$ fibres D' over the ball with $\mathbb{P}^1$ fibres.

The enhanced flag variety $\mathrm{GL}(3, \mathbb{C})/H$ is given by the set of *projective frames*, defined as triples of points $p, p', p'' \in \mathbb{P}^2$ where $p \wedge p' \wedge p'' \neq 0$.

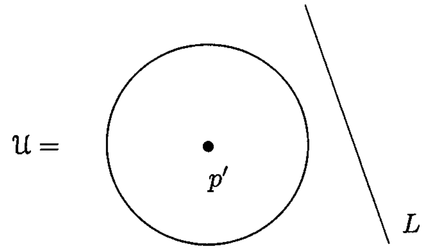
The correspondence space is pictured by



where $\overline{p\,p''}$ is the line joining p and p''. The maps $\mathcal{W} \to D, D', D''$ are given by

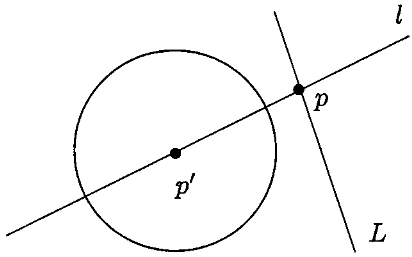
$$p, p', p'' \to \begin{cases} (p, \overline{p\,p'}) \in D \\ (p', \overline{p'\,p}) \in D' \\ (p, \overline{p\,p''}) \in D''. \end{cases}$$

The cycle space $\mathcal{U}$ is pictured by



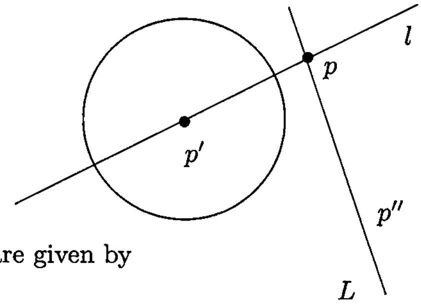
From the picture we see that $\mathcal{U} \cong \mathbb{B} \times \overline{\mathbb{B}}$ where $\overline{\mathbb{B}}$ is the conjugate complex structure on $\mathbb{B}$ and is isomorphic to the set of lines not meeting the closure of $\mathbb{B}$. The corresponding compact subvariety $Z(p', L) \cong \mathbb{P}^1$ is given by $\{(p, l)\} \subset D$ in the picture

$Z(p', L) =$

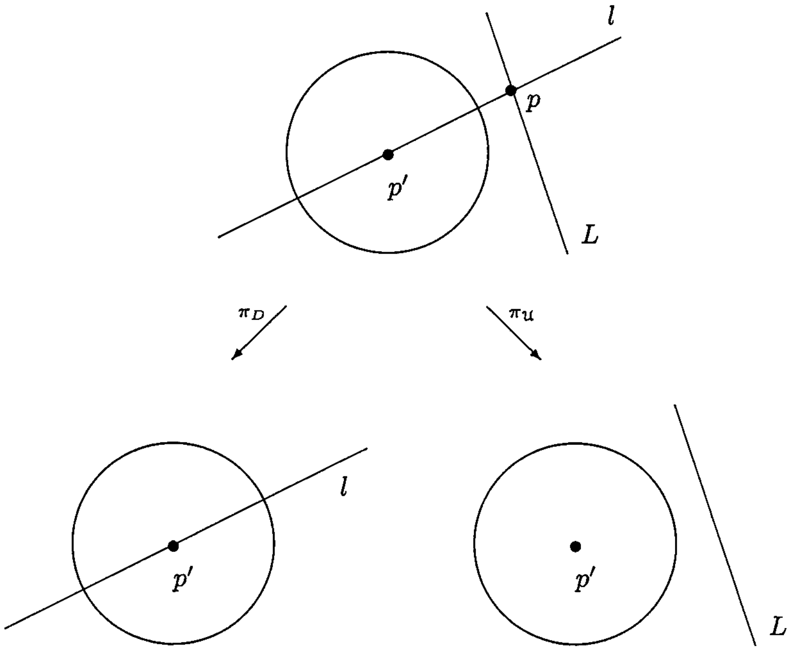


The incidence correspondence $\mathcal{I} = \{(p, l), (p', L)\} \subset D \times \mathcal{U}$ is as pictured in the above figure. The map $\mathcal{W} \to \mathcal{I}$ is given by

$p, p', p'' \longrightarrow$



The maps $\mathcal{I} \to D$, $\mathcal{I} \to \mathcal{U}$ are given by



All of the properties in the basic diagram (2.3) may be readily verified from the above pictures. The standard root diagram for $\mathcal{U}(2,1)$ is where the compact roots are labelled $\boxed{\cdot}$ and the Weyl chambers C, C', C'' correspond to the complex structures D, D', D''. Here, C is non-classical and C', C'' are classical.

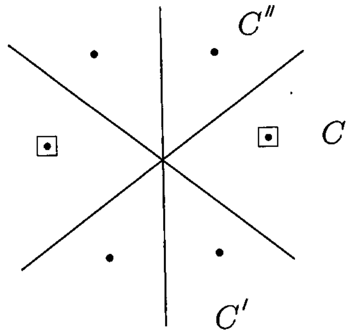
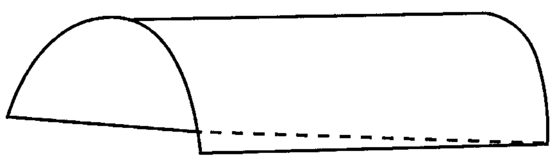


FIGURE 1

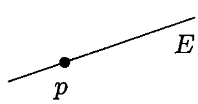
EXAMPLE 2.5. Sp(4) ([**GGK2**]).

- Q is the alternating form on $V = \mathbb{C}^4$ with matrix $\begin{pmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$;
- $\sigma\colon V \to V$ is a conjugation defined in the standard basis v_1, v_2, v_3, v_4 by $\sigma v_1 = iv_4$, $\sigma v_2 = iv_3$ (and then $\sigma v_3 = iv_2$, $\sigma v_4 = iv_1$);
- $\mathbb{H}$ is the Hermitian form defined by $\mathbb{H}(u, v) = iQ(u, \sigma v)$;
- $\mathbb{H}(v, \sigma v) = 0$ defines a real quadric hypersurface $Q_{\mathbb{H}} \subset \mathbb{P}^3$ which we picture as



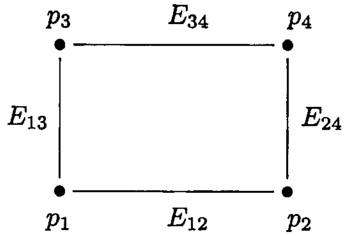
;

- $\mathrm{Sp}(4) = \mathrm{Aut}_\sigma(V, Q)$ is a real form of $\mathrm{Aut}(V, Q)$;
- a *Lagrange flag* is a flag $(0) \subset F_1 \subset F_2 \subset F_3 \subset V$ where $\dim F_j = j$ and with $F_2 = F_2^\perp$, $F_3 = F_1^\perp$, the $\perp$ being with respect to Q;
- a Lagrange flag is given projectively by a pair (p, E) where $p \in \mathbb{P}^3$ and $E \subset \mathbb{P}^3$ is a Lagrange line



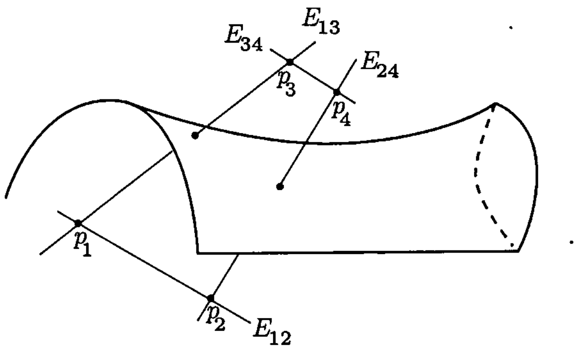
(think of $p = [F_1]$, $E = [F_2]$);
- a *Lagrange frame* is a basis f_1, f_2, f_3, f_4 for V for which $Q(f_i, f_j)$ is the above matrix Q;
- a *Lagrange quadrilateral* is a projective frame p_1, p_2, p_3, p_4 for $\mathbb{P}^3$ where $p_i = [f_i]$ for a Lagrange frame f_1, f_2, f_3, f_4.

The flag variety X is the set of Lagrange flags. The enhanced flag variety is the set of Lagrange quadrilaterals. We may picture a Lagrange quadrilateral as



where the depicted lines $E_{ij} = \overline{p_i p_j}$ are Lagrangian lines in $\mathbb{P}^3$. The diagonal lines are not Lagrangian.

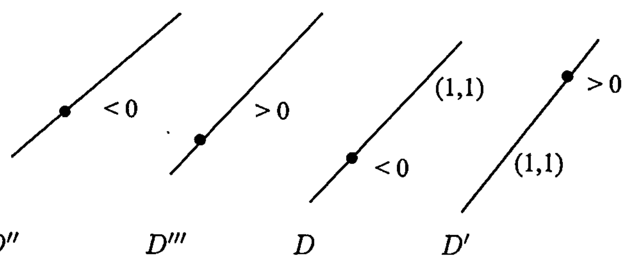
The correspondence space $\mathcal{W}$ is the set of Lagrange quadrilaterals positioned relative to the real hyperquadric $Q_{\mathbb{H}}$ as in the picture

$\mathcal{W} =$ 

The pictured Lagrangian lines E_{ij} are of three types

- E_{12} lies "inside" $Q_{\mathbb{H}}$, meaning that $\mathbb{H} < 0$ on the corresponding Lagrangian 2-plane $\tilde{E}_{12}$ in V;
- E_{13} meets $Q_{\mathbb{H}}$ in a real circle; as a consequence $\mathbb{H}$ has signature $(1,1)$ on $\tilde{E}_{13}$; E_{24} has a similar property;
- E_{34} lies "outside" $Q_{\mathbb{H}}$, meaning that $\mathbb{H} > 0$ on $\tilde{E}_{34}$.

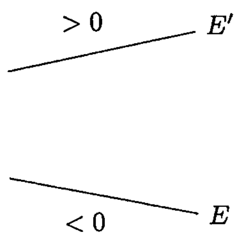
There are eight orbits of the four Lagrange flags in the above picture; thus we have (p_1, E_{12}) and (p_2, E_{12}) associated to E_{12}. These orbits give eight complex structures on $G_{\mathbb{R}}/T$, of which four pairs are equivalent under the action of W_K. The four types may be pictured as the orbits of

D_w's $=$ 

The notations mean that $\mathbb{H} < 0$, $\mathbb{H} > 0$ on the first two, $\mathbb{H}$ has signature $(1,1)$ on the second two, and on the two where $\mathbb{H}$ has signature $(1,1)$ we have indicated the sign of $\mathbb{H}$ on the marked points. Here, D and D' are non-classical and D'', D''' are classical.

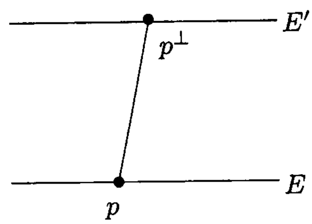
The cycle space is pictured as

$\mathfrak{U} =$



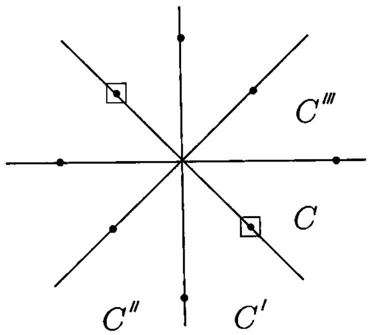
Here E, E' are Lagrangian lines on which $\mathbb{H}\big|_E < 0$, $\mathbb{H}\big|_{E'} > 0$ respectively (think of E as "inside" $Q_{\mathbb{H}}$ and E' as "outside"). The corresponding cycle $Z(E,E')$ in D is $\{(p, \overline{p\,p^\perp})\}$

$Z(E,E') =$



where $p \in E$ and $p^\perp \in E'$ is the unique point in E' with $Q(p,p^\perp) = 0$.

The standard root diagram for Sp(4) is



where the compact roots are marked ⊡ and the Weyl chambers corresponding to the complex structures are as indicated. Here, C and C' are non-classical and C'', C''' are classical.

3. The comparison theorem

Let $D = G_0/T$ be a flag domain and

$$L_\mu \to D$$

a holomorphic line bundle defined by a weight $\mu \in \Lambda$. As will now be explained, over each of the correspondence and cycle spaces there are global holomorphic objects to which the sheaf cohomology groups $H^q(D, L_\mu)$ map. In the case of $\mathcal{W}$ the mapping is an isomorphism and the holomorphic object is a quotient of global holomorphic sections of a holomorphic vector bundle. In the examples of interest to us there will usually be distinguished representatives of equivalence classes in the quotient space. In the case of the cycle space there are conditions under which the mapping is injective and the image can be identified; the global holomorphic object is sections of a bundle. The objective of this section is to relate these two ways of realizing $H^q(D, L_\mu)$.

We begin by recalling the result from [**EGW**]. Let M, N be a complex manifolds and

$$\pi \colon M \to N$$

a holomorphic submersion. We identify holomorphic vector bundles and their sheaves of sections. For $F \to N$ a holomorphic vector bundle we let

- $\pi^{-1}F$ be the pullback to M of the sheaf F;
- π^*F be the pullback to M of the bundle F.

We may think of $\pi^{-1}F \subset \pi^*F$ as the sections of π^*F that are constant along the fibres of $M \to N$.

Next we let Ω^q_π be the sheaf over M of relative holomorphic q-forms. We have

$$0 \to \pi^*\Omega^1_N \to \Omega^1_M \to \Omega^1_\pi \to 0,$$

and this defines a filtration $F^m\Omega^q_M$ with

$$\Omega^q_\pi \cong \Omega^q_M / F^q\Omega^q_M.$$

In local coordinates (x^i, y^α) on M such that $\pi(x^i, y^\alpha) = (y^\alpha)$, $F^m\Omega^q_M$ are the holomorphic differentials generated over Ω^{q-m}_M by terms $dy^{\alpha_1} \wedge \cdots \wedge dy^{\alpha_m}$. Thus $F^m\Omega^q_M = \operatorname{image}\{\pi^*\Omega^m_N \otimes \Omega^{q-m}_M \to \Omega^q_M\}$. From this description we see that we have

$$d \colon F^m\Omega^q_M \to F^m\Omega^{q+1}_M,$$

and consequently there is an induced relative differential

$$d_\pi \colon \Omega^q_\pi \to \Omega^{q+1}_\pi.$$

Setting $\Omega^q_\pi(F) = \Omega^q_\pi \otimes_{\mathcal{O}_M} \pi^*F$, since the transition functions of π^*F may be taken to involve only the y^α's, we may define

$$d_\pi \colon \Omega^q_\pi(F) \to \Omega^{q+1}_\pi(F)$$

to obtain the complex $(\Omega^\bullet_\pi(F); d_\pi)$. Using the holomorphic Poincaré lemma with holomorphic dependence on parameters one has the resolution

$$0 \to \pi^{-1}F \to \Omega^0_\pi(F) \xrightarrow{d_\pi} \Omega^1_\pi(F) \xrightarrow{d_\pi} \Omega^2_\pi(F) \to \cdots . \tag{3.1}$$

Denoting by $\mathbb{H}^*(M, \Omega^\bullet_\pi(F))$ the hypercohomology of the complex $(\Omega^\bullet_\pi(F), d_\pi)$, from (3.1) we have

$$H^*(M, \pi^{-1}F) \cong \mathbb{H}^*(M, \Omega^\bullet_\pi(F)). \tag{3.2}$$

We denote by

$$H^*\big(\Gamma(M, \Omega^\bullet_\pi(F)); d_\pi\big)$$

the de Rham cohomology groups arising by taking the global holomorphic sections of the complex $(\Omega^\bullet_\pi(F); d_\pi)$.

THEOREM 3.1. *Assume that M is Stein and the the fibres of $M \to N$ are contractible. Then*

$$H^*(N,F) \cong H^*\big(\Gamma(M,\Omega^\bullet_\pi(F)); d_\pi\big).$$

DISCUSSION. Using the spectral sequence

$$E_2^{p,q} = H^q_{d_\pi}\big(H^p(M,\Omega^\bullet_\pi(F))\big) \Rightarrow \mathbb{H}^{p+q}(M,\Omega^\bullet_\pi(F))$$

and the assumption that M is Stein to have $H^p(M,\Omega^\bullet_\pi(F)) = 0$ for $p > 0$ gives

(3.3) $$H^*(M,\pi^{-1}F) \cong H^*\big(\Gamma(M,\Omega^\bullet_\pi(F)); d_\pi\big).$$

Next, in the situations with which we shall be concerned, the submersion $M \to N$ will be locally over N a topological product. Then by the contractibility of the fibres the direct image sheaves

$$R^q_\pi(\pi^{-1}F) = 0 \text{ for } q > 0.$$

The Leray spectral sequence thus gives

(3.4) $$H^q(N,F) \cong H^q(M,\pi^{-1}F);$$

here the LHS is $H^q(N,R^0_\pi(\pi^{-1}F)) = H^q(N,F)$. Combining (3.3) and (3.4) gives the theorem.

NOTE 3.2. The second part of this argument is due to Buchdahl; cf. (14.2.3) in [**FHW**].

Using Theorem (2.3) we now apply this result to $\mathcal{W} \xrightarrow{\pi} D$ and $F = L_\mu$ to have the

COROLLARY 3.3. $H^q(D,L_\mu) \cong H^q\big(\Gamma(\mathcal{W},\Omega^\bullet_\pi(L_\mu)); d_\pi\big)$.

In this way, the coherent cohomology $H^q(D,L_\mu)$ is realized by global, holomorphic data. As noted above, in examples considered in [**GGK2**] there are canonical "harmonic" representatives of classes in the RHS of the corollary.

To state our main result we first define bundles

$$F^{p,q}_\mu \to \mathcal{U}$$

as follows: For $u \in \mathcal{U}$ let $Z_u \subset D$ be the corresponding maximal compact subvariety. Let $F^{p,q}_\mu = R^q_{\pi_\mathcal{U}}(\Omega^p_{\pi_D}(L_\mu))$. Then the fibre

$$F^{p,q}_{\mu,u} = H^q(Z_u, \Lambda^p N_{Z_u\backslash D}(L_\mu)).$$

THEOREM 3.4. *There exists a spectral sequence with*

$$\begin{cases} E_1^{p,q} = H^0(\mathcal{U},F^{p,q}_\mu), \quad \textit{and} \\ E_\infty^{p,q} = \mathrm{Gr}^p H^{p+q}\big(\Gamma(\mathcal{W},\Omega^\bullet_\pi(L_\mu)); d_\pi\big). \end{cases}$$

Using (3.3) we have the following result that is implicit in [**WZ**].

COROLLARY 3.5. *There exists a spectral sequence with*

$$\begin{cases} E_1^{p,q} = H^0(\mathcal{U},F^{p,q}_\mu) \\ E_\infty^{p,q} = \mathrm{Gr}^p H^{p+q}(D,L_\mu). \end{cases}$$

If $H^0(Z,\Lambda^{q+1}N_{Z/D}(L_\mu)) = \cdots = H^{q-1}(Z,\Lambda^2 N_{Z/D}(L_\mu)) = 0$, then

(3.5) $$H^q(D,L_\mu) \cong \ker\{\, H^0(\mathcal{U},F^{0,q}_\mu) \xrightarrow{d_1} H^0(\mathcal{U},F^{1,q}_\mu)\,\}.$$

Thus under the vanishing condition in the corollary, the coherent cohomology $H^q(D, L_\mu)$ is, in a different way from (3.3), realized as a global, holomorphic object.

The differential d_1 is a linear, first order differential operator whose symbol will be identified below following the proof of Theorem (3.4).

PROOF OF THEOREM (3.4). Referring to the basic diagram (3.3) we have on $\mathcal{W}$ the exact sequence of relative differentials

$$0 \to \pi_{\mathcal{J}}^*\Omega^1_{\pi_D} \to \Omega^1_\pi \to \Omega^1_{\pi_{\mathcal{J}}} \to 0. \tag{3.6}$$

This induces a filtration on $\Omega^\bullet_\pi$, and hence one on the complex

$$\Gamma(\mathcal{W}, \Omega^\bullet_\pi(L_\mu); d_\pi).$$

This filtration then leads to a spectral sequence abutting to

$$H^*(\Gamma(\mathcal{W}, \Omega^\bullet_\pi(L_\mu)); d_\pi).$$

We will identify the E_1-term with that given in the statement of the theorem.

The first observation is that in this spectral sequence we have

$$\begin{cases} E_0^{p,q} \cong \Gamma(\mathcal{W}, \Omega^q_{\pi_{\mathcal{J}}} \otimes \pi_{\mathcal{J}}^*\Omega^p_{\pi_D}(L_\mu)) \\ d_0 \;\; = d_{\pi_{\mathcal{J}}}. \end{cases}$$

Thus

$$\begin{cases} E_1^{p,q} \cong H^q(\Gamma(\mathcal{W}, \Omega^\bullet_{\pi_{\mathcal{J}}} \otimes \pi_{\mathcal{J}}^*\Omega^p_{\pi_D}(L_\mu)); d_{\pi_{\mathcal{J}}}) \\ d_1 \text{ is induced by } d_\pi. \end{cases}$$

By [**EGW**] applied to $\mathcal{W} \xrightarrow{\pi_{\mathcal{J}}} \mathcal{J}$ we have

$$\begin{cases} E_1^{p,q} \cong H^q(\mathcal{J}, \Omega^p_{\pi_D}(L_\mu)) \\ d_1 \text{ is induced by } d_\pi. \end{cases}$$

Since $\mathcal{U}$ is Stein, and the sheaves $R^q_{\pi_{\mathcal{U}}}\Omega^p_{\pi_D}(L_\mu)$ are coherent, the Leray spectral sequence applied to $\mathcal{J} \xrightarrow{\pi_{\mathcal{U}}} \mathcal{U}$ and $\Omega^p_{\pi_D}(L_\mu)$ gives

$$\begin{cases} E_1^{p,q} \cong H^0(\mathcal{U}, R^q_{\pi_{\mathcal{U}}}\Omega^p_{\pi_D}(L_\mu)) \\ d_1 \text{ is induced by } d_\pi. \end{cases}$$

It remains to establish the identification

$$R^q_{\pi_{\mathcal{U}}}\Omega^p_{\pi_D}(L_\mu) \cong F^{p,q}_\mu. \tag{3.7}$$

This will be done by identifying the various tangent spaces at the reference point $(x_0, u_0) \in \mathcal{J}$. For this we will identify locally free sheaves F with vector bundles and denote by F_p the fibre at the point p. We then have the identifications

- $T_{x_0}D = \mathfrak{n}^+$;
- $T_{x_0}Z = \mathfrak{n}^+_c$;
- $N_{Z/D,x_0} = \mathfrak{p}^+$;
- $T_{u_0}\mathcal{U} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$;
- $T_{(x_0,u_0)}\mathcal{J} = \mathfrak{n}^+_c \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$

and $T_{(x_0,u_0)}\mathcal{J}$ maps to $T_{x_0}D = \mathfrak{n}^+ = \mathfrak{n}^+_c \oplus \mathfrak{p}^+$ and $T_{u_0}\mathcal{U} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ by the evident projections.

It follows that

- $\Omega^1_{\pi_D,(x_0,u_0)} = \mathfrak{p}^{-*} \cong \mathfrak{p}^+ = N_{Z/D,x_0}$

where the isomorphism is via the Cartan-Killing form. □

The proof also allows us to identify the symbol $\sigma(d_1)$ of the differential operator d_1, as follows: Recall that

$$\sigma(d_1)\colon F^{0,q}_{u_0} \otimes T^*_{u_0}\mathcal{U} \to F^{1,q}_{\mu,u_0},$$

or using the definition of the $F^{p,q}_{\mu}$

$$\sigma(d_1)\colon H^q(Z, L_\mu) \otimes T^*_{u_0}\mathcal{U} \to H^q(Z, N_{Z/D}(L_\mu)). \tag{3.8}$$

Using the identification $T^*_{u_0}\mathcal{U} \cong \mathfrak{p}^* \cong \mathfrak{p}$ we have the inclusion

$$\mathfrak{p} \hookrightarrow H^0(Z, N_{Z/D}) \tag{3.9}$$

given geometrically by considering $X \in \mathfrak{p} \subset \mathfrak{g}$ as a holomorphic vector field along Z and then taking the normal part of X. Combining this with the evident map

$$H^q(Z, L_\mu) \otimes H^0(Z, N_{Z/D}) \to H^q(Z, N_{Z/D}(L_\mu))$$

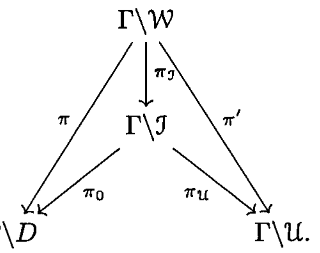
gives the symbol map (3.8).

This assertion will be proved when we revisit the symbol issue in section 4.2 (cf. Theorem 4.4).

4. Variants and applications

4.1. Quotienting by a discrete group. Let $\Gamma \subset G_0$ be a discrete, co-compact and neat subgroup. A principal motivation for [**GGK2**] was to understand the geometric and arithmetic properties of the *automorphic cohomology groups* $H^q(\Gamma\backslash D, L_\mu)$, objects that had arisen many years ago but whose above mentioned properties had to us remained largely mysterious until the works [**C1**], [**C2**], and [**C3**]. In studying the automorphic cohomology groups it is important to be able to take the quotient of the basic diagram (2.3) by Γ, which is then

(4.1)



Here we note that the group G_0 acts equivariantly on the diagram (2.3), and so the above quotient diagram is well-defined. The basic result concerning it is

THEOREM 4.1. *$\Gamma\backslash\mathcal{W}$ is Stein, and the fibres of π, π_D and $\pi_{\mathcal{J}}$ are contractible.*

PROOF. We first note that because Γ is assumed neat, any γ of finite order is the identity. Therefore, no $\gamma \in \Gamma$, $\gamma \neq e$, has a fixed point acting on D or on $\mathcal{U}$. For D this is because the isotropy subgroup of G_0 fixing any point $x \in D$ is compact. For $u \in \mathcal{U}$, if γ fixes u then it maps the compact subvariety $Z_u \subset D$ to itself, so again γ is of finite order. It follows that the fibres in (4.1) are biholomorphic to those in the basic diagram (2.3).

The next, and crucial, step is the result in [**BHH**] (cf. also 6.3.3 in [**FHW**]) that there exist strictly plurisubharmonic functions on $\mathcal{U}$ that are exhaustion functions modulo G_0. As in the proof in loc. cit., this induces a strictly plurisubharmonic exhaustion function of $\Gamma\backslash\mathcal{U}$, which is therefore a Stein manifold. Then $\Gamma\backslash\mathcal{W} \to \Gamma\backslash\mathcal{U}$

is a fibration over a Stein manifold with affine algebraic varieties as fibres, which implies that $\Gamma\backslash\mathcal{W}$ is itself Stein. □

The proof of Theorem (3.4) then applies verbatim to give

$$H^*(\Gamma\backslash D, L_\mu) \cong H^*\big(\Gamma(\Gamma\backslash\mathcal{W}, \Omega^\bullet_\pi(L_\mu)); d_\pi\big). \tag{4.2}$$

The double appearance of the notation Γ in the RHS is unfortunate, but we hope that the meaning is clear. We also have a spectral sequence with

$$\begin{cases} E_1^{p,q} = H^0(\Gamma\backslash\mathcal{U}, F_\mu^{p,q}) \\ E_\infty^{p,q} = \mathrm{Gr}^p H^{p+q}(\Gamma\backslash D, L_\mu). \end{cases} \tag{4.3}$$

4.2. $\mathfrak{n}$-cohomology interpretation. A familiar theme in the study of cohomology of homogeneous spaces and their quotients is to represent that cohomology by Lie algebra cohomology. For flag domains one considers $\mathfrak{n}$-cohomology where $\mathfrak{n}$ is the direct sum of the negative root spaces. Even though $\mathcal{W}$ is not a homogeneous space for G_0, we will show that the global de Rham cohomology groups $H^*\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi(L_\mu)); d_\pi\big)$ can be realized as $\mathfrak{n}$-cohomology for a certain G_0-module $\mathcal{O}G_\mathcal{W}$. Using this interpretation we will then observe that our spectral sequence is just the familiar Hochschild-Serre spectral sequence.

The definition of $\mathcal{O}G_\mathcal{W}$ is as follows: From the basic diagrams (2.1), (2.3) we obtain

$$\begin{array}{ccc} G_\mathcal{W} & \subset & G \\ \downarrow & & \downarrow f \\ \mathcal{W} & \subset & G/H \\ \downarrow \pi_\mathcal{J} & & \downarrow \\ \mathcal{J} & \subset & G/B_K \\ \downarrow \pi_D & & \downarrow \\ D & \subset & G/B. \end{array} \qquad \pi = \pi_D \circ \pi_\mathcal{J} : \mathcal{W} \to D \tag{4.4}$$

Definition 4.2. $G_\mathcal{W} = f^{-1}(\mathcal{W})$ is the open subset of G lying over $\mathcal{W}$ in the diagram (4.4), and

$$\mathcal{O}G_\mathcal{W} = \Gamma(G_\mathcal{W}, \mathcal{O}_{G_\mathcal{W}})$$

is the algebra of holomorphic functions on $G_\mathcal{W}$.

As we shall discuss below, $\mathcal{O}G_\mathcal{W}$ is a somewhat strange object but it is not as intractable as the definition might suggest. Since $G_\mathcal{W} \subset G$ is G_0-invariant, $\mathcal{O}G_\mathcal{W}$ is a G_0-module and therefore $\mathfrak{n}$-cohomology with coefficients in $\mathcal{O}G_\mathcal{W}$ is well-defined.

In fact, since

$$D = G_0(x_0) \subset G/B$$

and

$$\mathcal{W} = \{\, g \in G : gK(x_0) \subseteq D \,\}/H$$

we have

$$G_0\mathcal{W} \subseteq \mathcal{W}, \qquad \mathcal{W}K \subseteq \mathcal{W}$$

where K is acting on $\mathcal{W}$ on the right. Thus, G_0 and K act on $\mathcal{O}G_{\mathcal{W}}$ by

$$\begin{cases} (gh)(w) = h(gw) & g \in G_0, h \in \mathcal{O}G_{\mathcal{W}}, w \in G_{\mathcal{W}} \\ (hk)(w) = h(wk) & k \in K. \end{cases}$$

Because $G_{\mathcal{W}} \subset G$ is an open set; in fact it is $\{g \in G : gK(x_0) \subseteq D\}$, the Lie algebra $\mathfrak{g}$, viewed as left invariant vector fields on G, acts on $\mathcal{O}G_{\mathcal{W}}$ on the left. When $\mathfrak{g}$ is viewed as right invariant vector fields it acts on $\mathcal{O}G_{\mathcal{W}}$ on the right. These two actions commute, and we will use the right action of $\mathfrak{n}$ to define $H^*(\mathfrak{n}, \mathcal{O}G_{\mathcal{W}})$. These groups then have an action of G_0 on the left and an action of H on the right.

THEOREM 4.3.

(1) *There is the natural identification*

$$H^*\big(\Gamma(\mathcal{W}, \Omega^\bullet_\pi(L_\mu)); d_\pi\big) \cong H^*(\mathfrak{n}, \mathcal{O}G_{\mathcal{W}})_{-\mu}.$$

(2) *The Hochschild-Serre spectral sequence associated to the sub-algebra $\mathfrak{n}_c \subset \mathfrak{n}$ coincides with the spectral sequence given in Theorem* (3.4).

PROOF. The notation $(\ )_{-\mu}$ on the RHS of the isomorphism above means the following: The Cartan subgroup H acts on the right on $G_{\mathcal{W}}$ and therefore acts on the complex $(\Lambda^\bullet \mathfrak{n}^* \otimes \mathcal{O}G_{\mathcal{W}}, \delta)$ that computes Lie algebra cohomology. Then $H^*(\mathfrak{n}, \mathcal{O}G_{\mathcal{W}})_{-\mu}$ is that part of $H^*(\mathfrak{n}, \mathcal{O}G_{\mathcal{W}})$ that transforms by the character χ_μ^{-1} of H corresponding to the weight $-\mu$. This enters the picture because holomorphic sections of $\pi^* L_\mu \to \mathcal{W}$ are given by holomorphic functions on $G_{\mathcal{W}}$ that transform by χ_μ under the right action of H.

The proof of (1) in the above theorem is basically the observation from the proof of Theorem (3.4), and using the identification (3.6), that we have the natural identification of complexes

$$\Gamma\big(\mathcal{W}, \Omega^\bullet_\pi(L_\mu); d_\pi\big) \cong (\Lambda^\bullet \mathfrak{n}^* \otimes \mathcal{O}G_{\mathcal{W}}; \delta)_{-\mu}. \tag{4.5}$$

Here "natural" means that the action of G_0 on the LHS in (4.5) is given by the G_0-module structure of $\mathcal{O}G_{\mathcal{W}}$.

Turning to (2) in the theorem, here the basic observation is that when pulled back to $G_{\mathcal{W}}$, the exact sequence (3.6) is the dual to the restriction to $G_{\mathcal{W}} \subset G$ of the exact sequence of homogeneous vector bundles over G/H given by the exact sequence of H-modules

$$0 \to \mathfrak{n}_c \to \mathfrak{n} \to \mathfrak{p}^- \to 0.$$

From this we may infer (2) in the theorem. □

For later use we note that using the above identifications and $\mathfrak{p}^{-*} \cong \mathfrak{p}^+$ via the Cartan-Killing form,

$$E_1^{p,q} = H^q(\mathfrak{n}_c, \Lambda^p \mathfrak{p}^+ \otimes \mathcal{O}G_{\mathcal{W}})_{-\mu}. \tag{4.6}$$

Using this interpretation we shall now compute the symbol $\sigma(d_1)$ of

$$d_1 \colon H^0\big(\mathcal{U}, R^q_{\pi_{\mathcal{U}}} \Omega^p_{\pi_D}(L_\mu)\big) \to H^0\big(\mathcal{U}, R^q_{\pi_{\mathcal{U}}} \Omega^{p+1}_{\pi_D}(L_\mu)\big).$$

Following the notation from section 3 and the identification there of the fibre of the vector bundle $F^{p,q}_{\mu,u_0} \to \mathcal{U}$ and tangent space $T_{u_0}\mathcal{U}$ at the reference point, and identifying Z_{u_0} with Z to simplify the notation, the symbol $\sigma(d_1)$ of the 1st-order linear differential operator is a map

$$\sigma(d_1) \colon H^q\big(Z, \Lambda^p N_{Z/D}(L_\mu)\big) \otimes \mathfrak{p}^* \to H^q\big(Z, \Lambda^{p+1} N_{Z/D}(L_\mu)\big).$$

THEOREM 4.4. *With the identifications* $\mathfrak{p}^* \cong \mathfrak{p}$ *given by the Cartan-Killing form and inclusion* $\mathfrak{p} \hookrightarrow H^0(Z, N_{Z/D})$ *the symbol is given by*

$$\sigma(d_1)\varphi \otimes X = \varphi \wedge X.$$

Here, on the LHS we have $X \in \mathfrak{p}$ and $\varphi \in H^q(Z, \Lambda^p N_{Z/D}(L_\mu))$, and on the RHS X is the corresponding normal vector field in $H^0(Z, N_{Z/D})$. The map is $H^q(Z, \Lambda^p N_{Z/D}(L_\mu)) \otimes H^0(Z, N_{Z/D}) \to H^q(Z, \Lambda^{p+1} N_{Z/D}(L_\mu))$ induced by $\Lambda^p N_{Z/D} \otimes N_{Z/D} \to \Lambda^{p+1} N_{Z/D}$.

PROOF. To compute the symbol on $\varphi \otimes X$, we take a section f of $F^{p,q}$ defined near u_0 with $f(u_0) = 0$ and whose linear part is $\varphi \otimes X$. Then by definition

$$\sigma(d_1)\varphi \otimes X = (d_1 f)(u_0).$$

We shall give the computation when $p = 0$, $q = 1$ as this will indicate how the general case goes. Pulled back to $G_{\mathcal{W}}$ we may write

$$f = \sum_{\alpha \in \Phi_c^+} f_\alpha \omega^{-\alpha}$$

where the f_α are holomorphic functions that vanish along the inverse image of Z_{u_0}. Then

$$d_1 f = \sum_{\begin{cases}\alpha \in \Phi_c^+ \\ \beta \in \Phi_{nc}^+\end{cases}} (f_\alpha X_{-\beta}) \omega^{-\beta} \wedge \omega^{-\alpha} + \sum_{\alpha \in \Phi_c^+} f_\alpha d_\pi \omega^{-\alpha}.$$

The first term is the right action on f_α by the left invariant vector field $X_{-\beta}$. The second term vanishes along the inverse image of Z_{u_0}. As for the first term, under the pairing

$$\begin{pmatrix}\text{normal vector fields} \\ \text{to } Z_{u_0}\end{pmatrix} \otimes \begin{pmatrix}\text{holomorphic functions} \\ \text{vanishing along } Z_{u_0}\end{pmatrix} \to \mathcal{O}_{Z_0}$$

when evaluated along Z_{μ_0} the first term is the value along Z_{u_0} of

$$\sum_{\begin{cases}\alpha \in \Phi_c^+ \\ \beta \in \Phi_{nc}^+\end{cases}} f_\alpha X_{-\beta\alpha} X_\beta \otimes \omega^{-\alpha}$$

where $X_\beta \otimes \omega^{-\alpha} \in \mathfrak{p}^+ \otimes \mathfrak{n}^*$ and $f_\alpha X_{-\beta}\big|_{Z_0} \in \mathcal{O}_{Z_0}$. □

DISCUSSION. The G_0-module $\mathcal{O}G_{\mathcal{W}}$ is certainly not a Harish-Chandra, or HC, module, but it does have an interesting structure, reflecting the fact that $\mathcal{W}$ is a mixed algebro-geometric/complex analytic object, as we now explain.

The fibres of

$$\begin{array}{ccc} \mathcal{W} & \subset & G/H \\ \big\downarrow{\scriptstyle \pi'} & & \big\downarrow \\ \mathcal{U} & \subset & G/K \end{array}$$

are affine algebraic varieties isomorphic to the enhanced flag variety K/H. We may smoothly and equivariantly compactify G/H so that each fibre $g^{-1}(u)$, $u \in \mathcal{U}$, is the complement of a divisor with normal crossings. Then we may consider the G_0-invariant sub-algebra $\mathcal{O}G_{\mathcal{W}}^{\text{alg}} \subset \mathcal{O}G_{\mathcal{W}}$ of functions that are rational along each fibre, and by truncating Laurent series we may write $\mathcal{O}G_{\mathcal{W}}^{\text{alg}}$ as the union of G_0-submodules that are fibrewise K-finite acting on the right. Thus as a G_0-module over the G_0-module $\mathcal{O}(\mathcal{U}) = \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}})$ we see that $\mathcal{O}G_{\mathcal{W}}$ has a reasonable structure.

As for the G_0-module $\mathcal{O}(\mathcal{U})$, from **[FHW]** we see that $\mathcal{U}$ has the function-theoretic characteristics of a bounded domain of holomorphy (contractible, Stein, Kobayashi hyperbolic). In fact, for G_0 of Hermitian type, $\mathcal{U} \cong \Omega \times \overline{\Omega}$ where Ω is an Hermitian symmetric domain and where G_0 acts diagonally. Again, $\mathcal{O}(\mathcal{U})$ is not a HC-module but it seems to be a reasonable object to study. It will be further discussed in a future work. Here we shall illustrate it in the case of $S\mathcal{U}(2,1)$.

EXAMPLE 4.5. We represent elements of $G = \mathrm{SL}(3,\mathbb{C})$ as

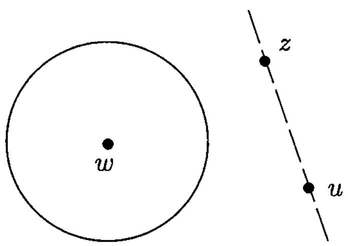
$$g = \begin{pmatrix} z_1 & w_1 & u_1 \\ z_2 & w_2 & u_2 \\ z_3 & w_3 & u_3 \end{pmatrix} = (z, w, u).$$

Taking as Hermitian form $\mathbb{H} = \mathrm{diag}(1,1,-1)$, $G_{\mathcal{W}} \subset G$ is defined by the conditions

$$\begin{cases} \mathbb{H}(w) < 0 \\ \mathbb{H}(z \wedge w) > 0. \end{cases}$$

The map $G_{\mathcal{W}} \to \mathcal{W}$ is given by

$(z, w, u) \to$



the dashed line indicating that the line $\overline{zu}$ lies in $\mathbb{B}^c$. The space $\mathcal{O}G_{\mathcal{W}}$ is spanned by the functions

$$w_1^i w_2^j w_3^k (z_2u_3 - z_3u_2)^l (z_3u_1 - z_1u_3)^m (z_1u_2 - z_2u_1)^n z_1^p z_2^q z_3^r u_1^a u_2^b u_3^c$$

where

$$i, j, i+j+k, l, m, l+m+n, p, q, r, a, b, c \geq 0.$$

There are relations among the generators, such as

$$\left(\frac{z_2u_3 - z_3u_2}{z_1u_2 - z_2u_1}\right)(z_1u_2 - z_2u_1) = z_2u_3 - z_3u_2.$$

4.3. Symbol maps for the two examples. In this section we shall discuss the *symbol sequence* and *characteristic variety* for each of our two examples. Before doing this we shall briefly explain the italicized terms.

In general, over a complex manifold M suppose we are given holomorphic vector bundles $E_i \to M$ and linear, 1st order differential operators $P_i \colon E_i \to E_{i+1}$ that form a complex

$$\begin{cases} E_1 \xrightarrow{P_1} E_2 \xrightarrow{P_2} E_3 \to \cdots \to E_m \to 0, \\ \qquad P_{i+1} \circ P_i = 0. \end{cases} \tag{4.7}$$

This general framework was the object of study of an extensive and rich theory developed by Spencer and his collaborators in the 1960's (cf. [**BCG**[3]], especially Chapters IX and X). In that theory, one assumes that $E_1 \xrightarrow{P_1} E_2$ is involutive with solution sheaf Θ, and then one seeks to construct the remaining terms in the above sequence that gives an exact sequence of sheaves which then provides a "resolution" of Θ. For each $x \in M$ and $\xi \in T^*_xM$, the pointwise symbol maps $\sigma(P_i)\colon E_{i,x} \otimes T^*_xM \to E_{i+1,x}$ give a complex, called the symbol sequence,

$$E_{1,x} \xrightarrow{\sigma(P_1)(\xi)} E_{2,x} \xrightarrow{\sigma(P_2)\xi} E_{3,x} \to \cdots \to E_{m,x}$$

whose cohomology is an important invariant of the situation (4.7). Also central to the theory is the characteristic variety $\Xi \subset \mathbb{P}T^*_xM$ defined by

$$\Xi = \{\, [\xi] \in \mathbb{P}T^*M : \ker \sigma(P_1)(\xi) \neq 0 \,\}.$$

Roughly speaking one has

- $\Xi = \mathbb{P}T^*M$ means that the PDE system defining Θ is *underdetermined*;
- $\operatorname{codim} \Xi = 1$ means that the PDE system is *determined*;
- $\operatorname{codim} \Xi > 1$ means that it is *overdetermined*;
- $\Xi = \emptyset$ means that the PDE is maximally overdetermined or *holonomic*.

In this latter case the sections of Θ over M are a *finite dimensional* vector space.

We observe that

$$R^q_{\pi_{\mathcal{U}}}\pi^*_D L_\mu \xrightarrow{d_1} R^q_{\pi_{\mathcal{U}}}\Omega^1_{\pi_D}(L_\mu) \xrightarrow{d_1} R^q_{\pi_{\mathcal{U}}}\Omega^2_{\pi_D}(L_\mu) \to \cdots$$

is a complex of the type (4.7) whose symbol sequence and characteristic variety are naturally associated to the spectral sequence (3.4). Although we do not know if the first d_1 is involutive or what the characteristic variety is in general, we shall now discuss the latter for our two examples.

In fact, to a general complex (4.7) there is naturally associated a spectral sequence leading to a definition of "secondary characteristic varieties." We suspect this construction may appear in the literature; we shall give and illustrate it in the situation studied here.

We will omit reference to the character μ and denote by

$$\begin{aligned} \mathcal{F}^{p,q} &= \text{sheaf of holomorphic sections of } F^{p,q} \to \mathcal{U} \\ &= R^q_{\pi_{\mathcal{U}}}\Omega^p_{\pi_D}(L) \end{aligned}$$

with stalks $\mathcal{F}^{p,q}_u$ for $u \in \mathcal{U}$. For $\mathfrak{m}_u \subset \mathcal{O}_{\mathcal{U},u}$ the maximal ideal, we define

$$\operatorname{Gr}^k \mathcal{F}^{p,q}_u = \frac{\mathfrak{m}^k_u \otimes \mathcal{O}_{\mathcal{U},u}\mathcal{F}^{p,q}_u}{\mathfrak{m}^{k+1} \otimes_{\mathcal{O}_{\mathcal{U},u}} F^{p,q}_u}.$$

This is a locally free coherent sheaf over $\mathcal{U}$ whose typical fibre is

$$S^k\mathfrak{p}^* \otimes H^q(Z, \wedge^p N_{Z/D}(L)).$$

Combining the above identification $\mathfrak{p}^* \cong \mathfrak{p}$ with the inclusion

$$\mathfrak{p} \hookrightarrow H^0(Z, N_{Z/D}),$$

the cup-products give maps

$$S^k\mathfrak{p} \otimes H^q(Z, \wedge^p_{N_{Z/D}}(L)) \to S^{k-1}\mathfrak{p} \otimes H^q(Z, \wedge^{p+1}N_{Z/D}(L)).$$

The composition of these maps for k followed by $k-1$ is zero, and thus we have for each k a complex of sheaves

$$\mathrm{Gr}^k\mathcal{F}^{0,q} \to \mathrm{Gr}^{k-1}\mathcal{F}^{1,q} \to \cdots \to \mathrm{Gr}^{k-r}\mathcal{F}^{r,q}$$

where $r = \min(k, \operatorname{codim}_D Z)$ and where the maps are $\mathcal{O}_{\mathcal{U}}$-linear. For $k=1$ this is just the symbol map. For $k=2$ it is

$$\begin{array}{ccccc} \mathrm{Gr}^2\mathcal{F}^{0,q} & \xrightarrow{d_1} & \mathrm{Gr}^1\mathcal{F}^{1,q} & \xrightarrow{d_1} & \mathrm{Gr}^0\mathcal{F}^{2,q} \\ & & \overset{d_2}{\searrow} & & \\ \mathrm{Gr}^2\mathcal{F}^{0,q-1} & \longrightarrow & \mathrm{Gr}^1\mathcal{F}^{1,q-1} & \longrightarrow & \mathrm{Gr}^0\mathcal{F}^{2,q-1}. \end{array}$$

Here we continue to denote by $\sigma_1(\xi)$ the natural maps induced by the usual symbol $\sigma_1(\xi)$ when $k=1$.

The maps in the fibres at u depend only on $\xi \in T_u^*\mathcal{U}$. In a typical fibre we have

$$k=1 \quad \xi \otimes H^q(Z,L) \xrightarrow{\sigma_1(\xi)} H^q(Z, N_{Z/D}(L))$$

$$\begin{array}{lccccc} k=2 & \xi^{(2)}H^q(Z,L) & \xrightarrow{\sigma_1(\xi)} & \xi\otimes H^q(Z,N_{Z/D}(L)) & \xrightarrow{\sigma_1(\xi)} & H^q(Z,\wedge^2 N_{Z/D}(L)) \\ & & & \overset{\sigma_2(\xi)}{\searrow} & & \\ & \xi^{(2)}\otimes H^{q-1}(Z,L) & \longrightarrow & \xi\otimes H^q(Z,N_{Z/D}(L)) & \longrightarrow & H^{q-1}(Z,\wedge^2 N_{Z/D}(L)). \end{array}$$

Definition 4.6. The *secondary symbol* $\sigma_2(\xi)$ is defined by the dotted arrow above. The *secondary characteristic variety* Ξ_2 is defined by

$$\Xi_2 = \{\xi : \sigma_1(\xi) = 0, \sigma_2(\xi) = 0\}.$$

The definition of $\sigma_2(\xi)$ is clearly related to the differential d_2 in our spectral sequence. Recall that d_2 is a linear differential operator of degree $\leqq 2$ defined on $\ker d_1$. We are not aware of how one may define the symbol of such an operator; the above is one possible construction defined on decomposable elements $\xi^{(2)}$ where $\sigma_1(\xi) = 0$; i.e., $\xi \in \Xi$. In the discussion below of $\mathrm{Sp}(4)$ we shall abuse notation and denote by $\sigma(d_2)$ the above construction extended in the special case (i) there to not necessarily decomposable elements in $\mathfrak{p}^{(2)}$. This discussion is not meant to be rigorous or definitive, but rather our interest is to illustrate interesting behavior of Harish-Chandra modules associated to degenerate, or close to being degenerate in the sense that $\mu+\rho$ is near to a wall, discrete series and limits of such.

$\underline{S\mathcal{U}(2,1)}$: As complex manifolds, we have $Z = \mathcal{U}(2)/T \cong S\mathcal{U}(2)/T_S$ where $T_S = \overline{S\mathcal{U}(2)\cap T}$. As homogeneous complex manifolds they are distinct and have different sets of homogeneous vector bundles (cf. section II.A in [**GGK2**] for a discussion and illustration of this point). Here, for simplicity we shall use $Z = S\mathcal{U}(2)/T_S$,[4] and we denote by $W \cong \mathbb{C}^2$ the standard representation of $S\mathcal{U}(2)$ with $W^{(n)}$ being the n^{th} symmetric product. We then have

$$W = H^0(\mathcal{O}_Z(1)).$$

[4]In the case of $\mathrm{Sp}(4)$ discussed below, in order to be able to use weight considerations we shall need to use the homogeneous complex manifold $Z = \mathcal{U}(2)/T$.

From [**GGK2**], (A.IV.F.6) we have the identification of holomorphic vector bundles over Z

$$N_{Z/D} \cong \mathcal{O}_Z(1) \oplus \mathcal{O}_Z(1).$$

Setting

$$\deg L_\mu\big|_Z = k,$$

we then have the following tables of the fibres of $R^q_{\pi_{\mathcal{U}}}\Omega^p_{\pi_D}(L_\mu) \to \mathcal{U}$ (here q is the y-axis and p is the x-axis, and $W^{(j)} = 0$ for $j < 0$):

$k = -l-2, l > 0$

$W^{(l)^*}$	$\overset{2}{\oplus} W^{(l-1)^*}$	$W^{(l-2)^*}$
0	0	0

$k = -2$

$W^{(0)}$	0	0
0	0	$W^{(0)}$

$k = -1$

0	0	0
0	0	0

$k \geqq 0$

0	0	0
$W^{(k)}$	$\overset{2}{\oplus} W^{(k+1)}$	$W^{(k+2)}$

Using the identification

$$\begin{cases} \mathfrak{p} = H^0(Z, N_{Z/D}) = W \oplus W \\ \mathfrak{p}^* \cong \mathfrak{p}, \text{ as in Theorem (4.4),} \end{cases}$$

we shall analyze the various cases.

$\underline{k = -l-2}$: The symbols are then

$$\begin{cases} \text{(i)} \quad W^{(l)^*} \otimes \left(\overset{2}{\oplus} W\right) \to \overset{2}{\oplus} W^{(l-1)^*} \\ \text{(ii)} \quad \left(\overset{2}{\oplus} W^{(l-1)^*}\right) \otimes \left(\overset{2}{\oplus} W\right) \to W^{(l-2)^*}. \end{cases}$$

These may be identified as follows:

(i) $P \otimes (w \oplus w') \longrightarrow P\lfloor w \oplus P\lfloor w'$ $\qquad P \in W^{(l)^*}; w, w' \in W$

(ii) $(P \oplus P') \otimes (w \oplus w') \longrightarrow P\lfloor w' - P'\lfloor w$ $\qquad P, P' \in W^{(l-1)^*}$.

It follows that (i) is injective unless w, w' are linearly dependent, and by a Koszul-type argument except in this case we have image (i) = kernel (ii). This gives the

CONCLUSION 4.7. For $k \leqq -3$ the characteristic variety Ξ is a quadric in $\mathbb{P}^3$; hence $\operatorname{codim} \Xi = 1$. For ξ non-characteristic the symbol sequence is exact.

$\underline{k \geqq 0}$: The symbols are then maps

(i) $W^{(k)} \otimes \left(\overset{2}{\oplus} W\right) \to \overset{2}{\oplus} W^{(k+1)}$;

(ii) $\left(\overset{2}{\oplus} W^{(k+1)}\right) \otimes \left(\overset{2}{\oplus} W\right) \to W^{(k+2)}$.

They may be identified as follows:

$$\text{(i)} \qquad P \otimes (w \oplus w') \to Pw \oplus Pw' \quad P \in W^{(k)}; w, w' \in W$$
$$\text{(ii)} \quad (P \oplus P') \otimes (w \oplus w') \to Pw' - P'w \quad P, P' \in W^{(k+1)}; w, w' \in W.$$

It follows that (i) is injective, unless of course $w = w' = 0$, and then the symbol sequence is exact.

CONCLUSION 4.8. For $k \geqq 0$ the characteristic variety Ξ is empty, and the symbol sequence is exact.

We observe that by (3.5) in the cases $k \leqq -2$ and $k \geqq 0$ the maps

$$H^1(D, L_\mu) \hookrightarrow H^0(\mathcal{U}, R^1_{\pi_\mathcal{U}} \pi_D^* L_\mu) \qquad k \leqq -2$$
$$-H^0(D, L_\mu) \hookrightarrow H^0(\mathcal{U}, R^0_{\pi_\mathcal{U}} \pi_D^* L_\mu) \qquad k \geqq 0$$

are injective. For $k \leqq -3$ and $k \geqq 0$ the image is just

$$\ker\{d_1 : H^0(\mathcal{U} R^q_{\pi_\mathcal{U}} \pi_D^* L_\mu) \to H^0(\mathcal{U}, R^q_{\pi_\mathcal{U}} \Omega^1_{\pi_D}(L_\mu))\}.$$

For $k = -2$ a very interesting special circumstance, to be discussed below, arises.

We also note that from the above conclusion we have that when $k \geqq 0$

$$\dim H^0(D, L_\mu) < \infty.$$

REMARK 4.9. For a general $D = G_0/T$ we will have

$$\text{(4.8)} \qquad \dim H^q(D, L_\mu) < \infty \qquad 0 \leqq q < d = \dim K_0/T$$

provided that for $Z = K_0/T$ and non-zero $\xi \in \mathfrak{p}$ the map

$$\text{(4.9)} \qquad H^q(Z, L_\mu) \xrightarrow{\xi} H^q(Z, N_{Z/D}(L_\mu))$$

is injective, where we are using the inclusion $\mathfrak{p} \hookrightarrow H^0(Z, N_{Z/D})$. We note that (4.8) is not true for D classical and $q = 0$ (take any μ such that $\mu + \rho$ is dominant), and it is not true for D non-classical and $q = d$ (take any μ such that $\mu + \rho$ is anti-dominant). In general, the LHS is known by the Borel-Weil-Bott theorem. For the RHS there is a composition series for $N_{Z/D}$ whose line bundle factors are the L_β where $\beta \in \Phi^+_{nc}$ is a positive non-compact root. Thus, at least in principal, one might hope to analyze the map (4.9).[5] We are not aware of any case of a non-classical D where it fails to be injective for non-zero ξ.

We note finally that

LEMMA 4.10. *If μ is in the anti-dominant Weyl chamber and $N_{Z/D} \to Z$ is ample, then there is a filtration $F^p H^q(D, L_\mu)$ such that for $q < d$, the associated graded has* $\dim \operatorname{Gr}^\bullet H^q(D, L_\mu) < \infty$.

PROOF. The filtration

$$F^p L_\mu = \mathcal{I}^p_Z \underset{\mathcal{O}_Z}{\otimes} L_\mu$$

of L_μ leads to a spectral sequence abutting to $H^*(D, L_\mu)$ and, using that $F^p L_\mu / F^{p+1} L_\mu \cong \operatorname{Sym}^p N^*_{Z/D}(L_\mu)$, with E_1-terms given by

$$E_1^{p,q} = H^{p+q}(Z, \operatorname{Gr}^p L_\mu) = H^{p+q}(Z, \operatorname{Sym}^p N^*_{Z/D}(L_\mu)).$$

[5] As is evident from the works of Schmid (cf. the references in [**Sch2**]) and from part IV of [**FHW**] the combinatorics of the extension data in the composition series are quite intricate.

By the ampleness assumption

$$E_1^{p,q} = 0 \qquad \text{for} \qquad 0 \leqq q < d,\ p \geqq p_0(\mu),$$

which gives the conclusion. □

The condition of ampleness is rare but does occur, especially in low dimensional examples including the two discussed in this paper. We suspect that in fact (4.8) is valid, but to be able to conclude this one needs $\cap_p F^p H^q(D, L_\mu) = (0)$.

Returning to the general discussion of the symbol map for $S\mathcal{U}(2,1)$, in many ways the most interesting is the case $\underline{k=-2}$: Then we have for the symbol a map

$$\sigma(d_2) \cdot W^{(0)} \otimes p^{*(2)} \to W^{(0)}. \tag{4.10}$$

Calculations that are in progress for a separate work indicate that

The symbol map (4.8) *is given by*

$$\sigma(d_2)P = \tfrac{1}{2}\langle P, \Omega\rangle, \qquad P \in \mathfrak{p}^{*(2)}$$

where $\Omega \in \mathfrak{g}^{(2)}$ *is the Casimir operator.*

Here we are writing $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ and thinking of $\mathfrak{p}^{*(2)} \subset \mathfrak{g}^{*(2)}$.

Representation-theoretic interpretation:[6] Referring to the root diagram in Figure 1 where C is the positive Weyl chamber for the non-classical complex structure on $\mathcal{U}(2,1)/T$, for weights μ such that for $\rho = \frac{1}{2}(\sum_{\alpha\in\Phi^*}\alpha)$ we have

$$\mu + \rho \in -C; \tag{4.11}$$

i.e., $\mu + \rho$ is *anti-dominant*, Schmid has shown that $H^1(D, L_\mu)$ is the HC-module $V_{\mu+\rho}$ with infinitesimal character $\chi_{\mu+\rho}$. Since

$$(4.11) \Rightarrow \deg L_\mu \leqq -3$$

from the discussion above and the results of [**Sch2**] we have

LEMMA 4.11. *For a weight* μ *satisfying* (4.11), *the HC-module associated to the discrete series representation with Harish-Chandra character* $\Theta_{\mu+\rho}$ *is realized as the kernel of the linear,* 1^{st} *order differential operator above whose characteristic variety is a quadric in* $\mathbb{P}^3$.

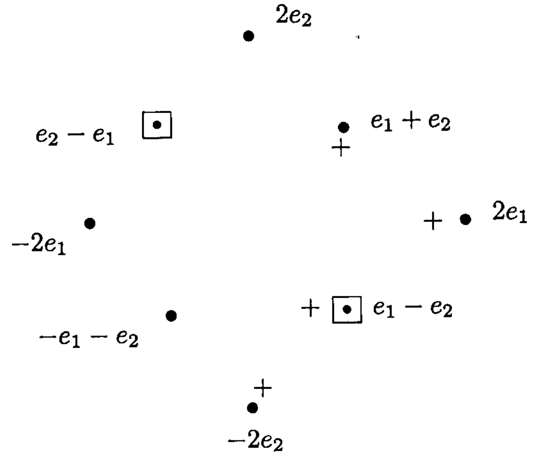
For the weight $\mu = -\rho$, so that $L_\mu\big|_Z = \omega_Z$ is the canonical bundle, $H^1(D, L_{-\rho})$ is the HC-module associated to the *totally degenerate limit of discrete series* (TDLDS) $(0, C)$ with infinitesimal character $\chi_0 = 0$ and corresponding to the non-classical Weyl chamber C. We expect to then have the conclusion

CONCLUSION 4.12. The HC-module associated to the TDLDS V_0 is the kernel of the scalar, linear 2^{nd} order PDE given above whose symbol is $(1/2)\Omega$ where Ω is the Casimir operator.

<u>Sp(4)</u>: In discussing $S\mathcal{U}(2,1)$ we have been treating $Z = \mathcal{U}(2)/T$ as the homogeneous space $S\mathcal{U}(2)/T_S$ where $T_S = S\mathcal{U}(2) \cap T$. For Sp(4) weight considerations

[6]These will be more extensively discussed in joint work in preparation with Matt Kerr which is a sequel to [**GGK2**].

require that we use the full $\mathcal{U}(2)$ symmetry group. In the weight diagram



we have labelled the positive roots for the Weyl chamber C corresponding to our non-classical complex structure D by $+$, and the compact roots by $\boxdot$. We denote by $L_{k_1e_1+k_2e_2} \to D$ the $\mathcal{U}(2)$-homogeneous line bundle given by the character of T corresponding to the weight $k_1e_1 + k_2e_2$. We then set

- $W = \mathcal{U}(2)$-module $H^0(Z, L_{e_1})$;
- $\delta = \mathcal{U}(2)$-module $\Lambda^2 W$ given by the character of $\mathcal{U}(2)$ with weight $e_1 + e_2$;
- $W_k^{(n)} = \mathcal{U}(2)$-module $\operatorname{Sym}^n W \otimes \delta^k$.

Then we have as $\mathcal{U}(2)$-modules

(4.12)
- $H^0(Z, L_{k_1e_1+k_2e_2}) = W_{k_2}^{(k_1-k_2)} \qquad (= 0 \text{ if } k_1 > k_2)$;
- $H^1(Z, L_{k_1e_1+k_2e_2}) = W_{k_1+1}^{(k_2-k_1-2)} \qquad (= 0 \text{ if } k_2 > k_1 + 2)$;
- $W_k^{(n)*} = W_{-n-k}^{(n)}$;
- $W_k^{(n)} \otimes W_l^{(m)} = \bigoplus_{i \geqq 0} W_{i+k+l}^{(n+m-2i)} \qquad \text{if } m \leq n.$

From the root diagram we may infer that for the normal bundle $N_{Z/D} \to Z$ we have as $\mathcal{U}(2)$-homogeneous vector bundles

$$\text{(4.13)} \qquad \begin{cases} N_{Z/D} = L_{-2e_2} \oplus N' \\ 0 \to L_{e_1+e_2} \to N' \to L_{2e_1} \to 0. \end{cases}$$

Using this we see that as $\mathcal{U}(2)$-modules

$$\text{(4.14)} \qquad H^0(Z, N_{Z/D}) = \underbrace{W_0^{(2)} \oplus W_{-2}^{(2)}}_{\mathfrak{p}} \oplus W_1^{(0)}.$$

Here, we have the inclusion $\mathfrak{p} \hookrightarrow H^0(Z, N_{Z/D})$ given by the terms over the bracket, and there is one "extra" deformation of Z; i.e., not coming from moving Z by G, corresponding to $W_1^{(0)}$. Since $H^1(Z, N_{Z/D}) = 0$, this extra infinitesimal deformation of $Z \subset D$ is unobstructed (cf. Part IV in [**FHW**] for a general discussion of this point).

For the line bundle $L_\mu = L_{k_1e_1+k_2e_2}$ we have

$$\begin{cases} \deg L_\mu = k_1 - k_2 \\ \omega_Z = L_{e_2-e_1} \qquad (\Rightarrow \deg \omega_Z = -2). \end{cases}$$

The following tables show where the non-zero groups $H^q(Z, \Lambda^p N_{Z/D}(L_\mu))$ occur. The specific $\mathcal{U}(2)$-modules can be identified using (4.12) and (4.13), and this will be done in two cases of particular interest.

$\underline{k \leqq -5}$:

*	*	*	*
0	0	0	0

Here, the term in the upper right-hand position is zero if $k = -5$. It may be checked that

$$\mu + \rho \text{ anti-dominant } \Rightarrow k \leqq -5.$$

Thus, the HC module associated to the discrete series have the above picture. The spectral sequence degenerates at E_2, which is a general phenomenon.

$\underline{k = -4}$:

*	*	*	0
0	0	0	*

$\underline{k = -3}$:

*	*	0	0
0	0	*	*

$\underline{k = -2}$:

*	0	0	0
0	*	*	*

$\underline{k = -1}$:

0	0	0	0
0	*	*	*

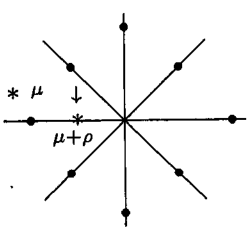
$\underline{k \geq 0}$:

0	0	0	0
*	*	*	*

The cases $k \leqq -5$, which include the discrete series, and $k \geqq 0$ where the characteristic variety $\Xi = \emptyset$ and $\dim H^0(D, L_\mu) < \infty$, are similar to the $S\mathcal{U}(2,1)$ example discussed above. Here we only analyze two particularly interesting cases:

(1) $L_{-\rho} = L_{-2e_1+e_2}$ corresponding to a TDLDS;
(2) $L_{-3e_1+e_2}$ corresponding to a non-degenerate limit of discrete series (NDLDS).

In case (2) the picture for $\mu = -3e_1 + e_2$ and $\mu + \rho$ is



The arrow means that the NDLDS is associated to a non-classical, anti-dominant Weyl chamber; that is it is a non-holomorphic NDLDS.

Case (1): The picture here is

$W_{-1}^{(1)}$	$W_0^{(1)}$	0	0
0	0	$W_{-1}^{(1)}$	$W_0^{(1)}$

We have as above

$$\mathfrak{p} = W_0^{(2)} \oplus W_{-2}^{(2)} \cong \mathfrak{p}^*.$$

Dualizing the symbol map

$$W_{-1}^{(1)} \otimes \mathfrak{p}^* \to W_0^{(1)}$$

for $d_1 \colon E_1^{0,1} \to E_1^{0,2}$ at our reference point gives, using $\mathfrak{p} \cong \mathfrak{p}^*$,

$$W_{-1}^{(1)} \otimes W_{-1}^{(1)} \to \mathfrak{p} = W_{-2}^{(2)} \oplus W_0^{(2)}.$$

By consideration of weights we end up in the $W_{-2}^{(2)}$-factor. This map is thus

$$H^0(Z, \mathcal{O}(1)) \otimes H^0(Z, \mathcal{O}(1)) \to H^0(Z, \mathcal{O}(2)).$$

Unwinding the dualities, we see that the non-zero part of the symbol is a map

$$\begin{array}{ccc} V \otimes S^2V^* & \longrightarrow & V^* \\ \Uparrow & & \Uparrow \\ u \otimes q & \longrightarrow & u \rfloor q \end{array}$$

where V is a 2-dimensional vector space. This map is an isomorphism if, and only if, $q \in S^2V^*$ is a non-singular quadric. This gives the

CONCLUSION 4.13. The characteristic variety $\Xi \subset \mathbb{P}(W_0^{(2)} \oplus W_{-2}^{(2)})$ is the projectivization of (non-singular quadric in $W_0^{(2)}) \oplus W_{-2}^{(2)}$.

This is a singular quadric in $\mathbb{P}\mathfrak{p}^*$. Since $d_1 \colon E_1^{1,0} \to E_1^{1,1}$ is a determined linear PDE system it is consistent that $\operatorname{codim} \Xi = 1$.

Turning to d_2, since for a 1st order determined linear PDE $P \colon E \to E'$ whose characteristic variety is a hypersurface, the solutions that vanish to 2nd order at a point define a linear subspace

$$F_x \subset E_x \otimes S^2T_x^*M$$

that projects *onto* E_x in the sense that the natural map $F_x \otimes S^2T_xM \to E_x$ is surjective as explained above, we may consider the symbol of d_2 as a map

$$\sigma(d_2) \colon W_{-1}^{(1)} \otimes S^2\mathfrak{p} \to W_{-1}^{(1)}.$$

Now by (4.12)

$$S^2\mathfrak{p} = S^2W_0^{(2)} \oplus \left(W_0^{(2)} \otimes W_{-2}^{(2)}\right) \oplus S^2W_{-2}^{(2)}.$$

By weight considerations, only $W_0^{(2)} \otimes W_{-2}^{(2)}$, which contains the map $\mathrm{id}_{W_0^{(2)}} \colon W_0^{(2)} \to W_0^{(2)}$, is going to map $W_{-1}^{(1)}$ to $W_{-1}^{(1)}$. Then by (4.12)

$$(4.15) \qquad \begin{cases} \operatorname{Hom}(W_{-1}^{(1)}, W_{-1}^{(1)}) = W_{-1}^{(2)} \oplus W_0^{(0)} \\ W_{-2}^{(2)} \otimes W_0^{(2)} = W_{-2}^{(4)} \oplus \underbrace{W_{-1}^{(2)} \oplus W_0^{(0)}}. \end{cases}$$

Thus the only potentially non-zero piece of $\sigma(d_2)$ arises from the term over the brackets. In fact, since

$$\mathfrak{p} = W_0^{(2)} \oplus W_{-2}^{(2)}$$

and

$$q \in W_0^{(2)} \cong S^2V^*$$

has rank one, taking for example $q = z_1^{*2}$ and $z_2 \in \ker d_1 \subset V$ then

$$q \oplus p \in S^2V \oplus S^2V^*$$

maps to $\mathrm{Hom}(W_{-1}^{(1)}, W_{-1}^{(1)})$ by

$$p = az_1^2 + bz_1z_2 + cz_2^2$$
$$q \otimes p \to 2az_1^* \otimes z_1 + bz_1^* \otimes z_2$$

and $z_2 \to 0$ under $q \otimes p$. Thus $z_2 \in \ker d_2$ so that we are led to the

CONCLUSION 4.14. Viewing the symbol as a map

$$\mathrm{Hom}\left(W_{-1}^{(1)}, W_{-1}^{(1)}\right) \to S^2\mathfrak{p},$$

from (4.15), the only non-zero part is a map

$$W_{-1}^{(2)} \oplus W_0^{(0)} \to W_{-1}^{(2)} \oplus W_1^{(0)}.$$

This map is a constant c times the identity.

We suspect, but have not proved, that $c \neq 0$; i.e., the characteristic variety of d_2 is non-trivial. The relation, if any, between the symbol $\sigma(d_2)$ and the Casimir operator is not yet understood by the authors. The case where $c = 0$ will be commented on at the end of this section.

Case (2): Here the picture is

$W_{-2}^{(2)}$	$W_{-2}^{(0)} \oplus W_{-1}^{(2)} \oplus W_0^{(0)}$	$W_{-1}^{(0)}$	0
0	0	0	$W_0^{(0)}$

This is derived from (4.12) and (4.13), and for the $E_1^{2,0}$ and $E_1^{2,1}$ terms uses that in the cohomology sequence

$$0 \to L_{-2e_1} \to N' \otimes L_{-2e_2} \otimes L_{-2e_1+e_2} \to L_{-e_1-e_2} \to 0$$

we have

$$H^0(Z, L_{-e_1-e_2}) \xrightarrow{\sim} H^1(Z, L_{-2e_1}).$$

Using $\mathfrak{p} \cong \mathfrak{p}^*$, for the symbol

$$\sigma(d_1)\colon E_1^{0,1} \otimes \mathfrak{p} \to E_1^{1,1} \tag{4.16}$$

we have

$$\begin{aligned} W_{-2}^{(2)} \otimes \mathfrak{p} &= \left(W_{-2}^{(2)} \otimes W_0^{(2)}\right) \oplus \left(W_{-2}^{(2)} \otimes W_{-2}^{(2)}\right) \\ &\cong \left(W_{-2}^{(4)} \oplus \underbrace{W_{-1}^{(2)} \oplus W_0^{(0)}}\right) \oplus \left(W_{-4}^{(4)} \oplus W_{-3}^{(2)} \oplus \underbrace{W_{-2}^{(0)}}\right). \end{aligned}$$

By weight considerations, only the terms over the brackets can map to something non-zero under d_1. Thus the symbol map is

$$\underbrace{\left(W_{-2}^{(2)} \otimes W_0^{(2)}\right)} \oplus \underbrace{\left(W_{-2}^{(2)} \otimes W_{-2}^{(2)}\right)} \to \underbrace{\left(W_{-1}^{(2)} \oplus W_0^{(0)}\right)} \oplus \underbrace{W_{-2}^{(0)}}$$

where the terms over the single and double brackets correspond under the symbol map and may be seen to be surjective. In fact, using from (4.12) that

$$W_0^{(2)} \cong W_{-2}^{(2)*}$$

the map over the double brackets is just contraction of $Q_1 \in W_{-2}^{(2)}$ with $Q_2 \in W_0^{(2)}$ twisted by δ^2.

The map over the single brackets is of the general form

$$S^2V \otimes S^2V \to (V \otimes V) \otimes (V \otimes V) \to V \otimes \Lambda^2 V \otimes V \to \Lambda^2 V \otimes S^2 V$$

together with

$$S^2V \otimes S^2V \to \Lambda^2 V \otimes S^2 V$$

where V is a 2-dimensional vector space. Together these two maps give

$$S^2V \otimes S^2V \to V \otimes \Lambda^2 V \otimes V \to \Lambda^2 V \otimes V \otimes V.$$

In coordinates and taking the above duality into account, the map is

$$\Big(\sum_{i,j} a_{ij} z_i^* z_j^*\Big) \otimes \Big(\sum_{k,l} b_{kl} z_k z_l\Big) \to \sum_{i,k}\Big(\sum_j a_{ij} b_{jk}\Big) z_i^* z_k.$$

There are three cases depending on the rank of $Q_2 = \sum_{k,l} b_{kl} z_k z_l$.

Rank $Q_2 = 2$: Taking $Q_2 = z_1 z_2$, the above map on $Q_1 = \sum_{i,j} z_i^* z_j^*$ is

$$\begin{cases} z_1^{*2} \to z_1^* \otimes z_2 \\ z_2^{*2} \to z_2^* \otimes z_1 \\ z_1^* z_2^* \to z_1^* \otimes z_2 + z_2^* \otimes z_1. \end{cases}$$

In this case there is no kernel contracting with Q_2.

Rank $Q_2 = 1$: Taking $Q_2 = z_1^2$ we have

$$\begin{cases} z_1^{*2} \to 2z_1^* \otimes z_1 \\ z_1^* z_2^* \to 2z_2^* \otimes z_1 \\ z_2^{*2} \to 0. \end{cases}$$

Now

$$\mathfrak{p} = W_0^{(2)} \oplus W_{-2}^{(2)}$$

where $Q_2 \in W_0^{(2)}$ and $Q_1 \in W_{-2}^{(2)}$. Then

- $Q_2 = z_1 z_2$ mapping to the kernel of the $\smile$ part of d_1 is zero;
- $Q_2 = z_1^2$ mapping to the kernel of the $\smile$ part of d_1 is z_2^{*2};
- $Q_2 = 0$ mapping to the kernel of the $\smile$ part of d_1 is all of $W_{-2}^{(2)}$.

If now $Q_2 = z_1^2$, then for $Q_1 = az_1^{*2} + bz_1^* z_2^* + cz_2^{*2}$ the kernel of the $\stackrel{\smile}{\smile}$ part of d_1 takes z_2^{*2} to $2cz_2$. So there is one further condition on Q_1, namely $c = 0$, to have a non-trivial $\ker d_1$. If $Q_2 = 0$, then Q_1 contracts to zero with a codimension $\geqq 2$ subspace of $W_{-2}^{(2)}$ for any Q_1. Thus the characteristic variety has codimension 2.

Rank $Q_2 = 0$: Then contracting with Q_1 we always get a rank 2 kernel. But $Q_2 = 0$ is a codimension 3 condition.

CONCLUSION 4.15. The characteristic variety Ξ of the symbol map (4.16) has $\operatorname{codim} \Xi = 2$.

This is consistent with $d_1 = E_1^{0,1} \to E_1^{1,1}$ being an overdetermined PDE system.

Remarks concerning degenerate symbols: We begin with the general

OBSERVATION 4.16. Schmid's result [**Sch2**] that $H^r(D, L_\mu) = 0$ for $r > d = \dim Z$ implies conditions on the differentials in the spectral sequence in Theorem (3.4).

Specifically, no terms in $E_1^{p,q}$ can survive to $E_\infty^{p,q}$ if $p + q > d$.

As we shall now discuss, this has implications for the symbol maps. For this we make the following

CONVENTION. If $P\colon E \to F$ is a differential operator of order $\leqq k$ whose symbol mapping $E \otimes S^k T^* M \to F$ is zero, then P has order $\leqq k-1$. We define *the* symbol of $P\colon E \to F$ to be the first non-zero map $E \otimes S^l T^* M \to F$.

Referring to the discussion below (4.14), if $c = 0$ then d_2 is a differential operator of order $\leqq 1$. If it is truly of order 1, then the symbol is a map

$$W_{-1}^{(2)} \oplus W_0^{(0)} \to W_{-2}^{(2)} \oplus W_0^{(2)},$$

which by weight considerations must be zero. Thus, d_2 is a scalar operator

$$W_{-1}^{(1)} \to W_{-1}^{(1)}$$

which must be a multiple of the identity. We again suspect, but have not proved, that if this situation does occur then the multiple is non-zero.

When we turn to case (ii), from (4.16) the mapping

$$d_2\colon \ker d_1 \cap E_1^{1,1} \to E_1^{3,0}$$

must be an isomorphism. By our convention above, the symbol $\sigma(d_2)$ must be non-zero. The various cases where d_2 is of actual order $2, 1, 0$ can be analyzed using weight considerations, but we shall not do so here.

We conclude with a remark about the case $k \leqq -5$, where the picture is

*	*	*	*
0	0	0	0

In this case we have an exact sequence

$$0 \to H^0(D, L_\mu) \to E_1^{0,1} \xrightarrow{d_1} E_1^{1,1} \xrightarrow{d_1} E_1^{2,1} \to E_1^{3,1} \to 0. \tag{4.17}$$

Now $E_1^{p,q} = H^0\big(\mathfrak{U}, R^q_{\pi_{\mathfrak{U}}} \Omega^p_{\pi_D}(L_\mu)\big)$, and since $\mathfrak{U}$ is a Stein manifold we believe it follows that setting $\Theta_\mu = \ker\{\, d_1 : R^1_{\pi_{\mathfrak{U}}} \pi_D^* L_\mu \to R^1_{\pi_{\mathfrak{U}}} \Omega^1(L_\mu)\,\}$ we have over $\mathfrak{U}$ the exact *sheaf* sequence

$$\begin{aligned} 0 \to \Theta_\mu \to R^1_{\pi_{\mathfrak{U}}} \pi_D^* L_\mu &\xrightarrow{d_1} R^1_{\pi_{\mathfrak{U}}} \Omega^1(L_\mu) \\ &\xrightarrow{d_1} R^1_{\pi_{\mathfrak{U}}} \Omega^2_{\pi_D}(L_\mu) \xrightarrow{d_1} R^1_{\pi_{\mathfrak{U}}} \Omega^3_{\pi_D}(L_\mu) \to 0. \end{aligned}$$

Although we have not tried to analyze this, it seems interesting and reasonable that this should be a Spencer resolution as in [**BCG**[3]], Chapter X. In fact, this could be a general phenomenon for the discrete series.

References

[BE] R. Baston and M. Eastwood, *The Penrose Transform; its Interaction with Representation Theory*, Oxford University Press, Oxford, 1989. MR1038279 (92j:32112)

[Bo] A. Borel, *Linear Algebraic Groups*, Springer-Verlag, New York, 1991. MR1102012 (92d:20001)

[BCG^3] R. Bryant, S. S. Chern, R. Gardner, H. Goldschmidt, and P. Griffiths, *Exterior Differential Systems*, *Math. Sci. Res. Inst. Publ.* **18**, Springer-Verlag, New York, 1991, viii+475pp. MR1083148 (92h:58007)

[BHH] D. Burns, S. Halverscheid, and R. Hind, The geoemtry of Grauert tubes and complexification of symmetric spaces *Duke J. Math.* **118** (2003), 465–491. MR1983038 (2004g:32025)

[C1] H. Carayol, Limites dégénérées de séries discrètes, formes automorphes et variétés de Griffiths-Schmid: le case du groupe $U(2,1)$, *Compos. Math.* **111** (1998), 51–88. MR1611063 (99k:22027)

[C2] ———, Quelques relations entre les cohomologies des variétés de Shimura et celles de Griffiths-Schmid (cas du group SU(2,1)), *Compos. Math.* **121** (2000), 305–335. MR1761629 (2001e:11056)

[C3] ———, Cohomologie automorphe et compactifications partielles de certaines variétés de Griffiths-Schmid, *Compos. Math.* **141** (2005), 1081–1102. MR2157130 (2006e:11065)

[Ch] S. S. Chern, On integral geometry in Klein spaces, *Ann. of Math.* **43** (1942), 178–189. MR0006075 (3:253h)

[EGW] M. Eastwood, S. Gindikin, and H. Wong, Holomorphic realizations of $\bar{\partial}$-cohomology and constructions of representations, *J. Geom. Phys.* **17**(3) (1995), 231–244. MR1358737 (96m:22021)

[FHW] G. Fels, A. Huckelberry, and J. A. Wolf, *Cycle Spaces of Flag Domains, A Complex Geometric Viewpoint*, *Progr. Math.* **245**, (H. Bass, J. Oesterlé, and A. Weinstein, eds.), Birkhäuser, Boston, 2006. MR2188135 (2006h:32018)

[GGK1] M. Green, P. Griffiths, and M. Kerr, Mumford-Tate Groups and Domains: Their Geometry and Arithmetic, *Annals of Math. Studies* **183**, Princeton University Press, Princeton, NJ, 2012. MR2918237

[GGK2] ———, Special values of automorphic cohomology classes, available at http://www.math.wustl.edu/~matkerr/.

[GS] P. Griffiths and W. Schmid, Locally homogeneous complex manifolds, *Acta. Math.* **123** (1969), 253–302. MR0259958 (41:4587)

[Sch1] W. Schmid, Homogeneous complex manifolds and representations of semisimple Lie groups, Ph.D. dissertation, Univ. California, Berkeley (1967); reprinted in *Representation Theory and Harmonic Analysis on Semisimple Lie Groups*, *Math. Surveys and Monogr.* **31**, Amer. Math. Soc. (1989), 223–286. MR1011899 (90i:22025)

[Sch2] ———, Discrete series, *Proc. Symp. Pure Math.* **61** (1997), 83–113. MR1476494 (98k:22051)

[Sch3] ———, Geometric methods in representation theory (lecture notes taken by Matvei Libine), in *Poisson Geometry Deformation Quantisation and Group Representations*, 273–323, *London Math. Soc. Lecture Note Ser.* **323**, Cambridge Univ. Press, Cambridge, 2005. MR2166454

[WZ] J. A. Wolf and R. Zierau, Holomorphic double fibration transforms, *Proc. Symp. Pure Math.* **68** (2000), 527–551. MR1767909 (2001f:32034)

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