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Chapter II

CURVATURE PROPERTIES OF THE HODGE BUNDLES

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We consider a polarized variation of Hodge structure $\phi : S \to \Gamma\backslash D$, which we think of locally as a variable polarized Hodge decomposition on a fixed vector space:

$$\begin{cases} H = \bigoplus_{p+q=n} H_s^{p,q} \\ F_s^p = H_s^{n,0} \oplus \cdots \oplus H_s^{p,n-p}, \end{cases}$$

where s varies over the variety S. (To be strictly correct, s should be in the universal covering $\tilde{S}$, for otherwise it may not be possible to have the fixed vector space H. Locally the description just given is fine.) We have

$$\frac{\partial H_s^{p,q}}{\partial \bar{s}} \subseteq H_s^{p+1,q-1} \oplus H_s^{p,q}$$

and by conjugation,

$$\frac{\partial H_s^{p,q}}{\partial s} \subseteq H_s^{p,q} \oplus H_s^{p-1,q+1};$$

or in terms of filtrations,

$$\frac{\partial F_s^p}{\partial \bar{s}} \subseteq F_s^p$$

and

$$\frac{\partial F_s^p}{\partial s} \subseteq F_s^{p-1} .$$

The differential ϕ_* of the period map assumes values in the horizontal subspace

$$\phi_* : T_s(S) \to \bigoplus_p \mathrm{Hom}(H_s^{p,q}, H_s^{p-1,q+1}) .$$

If $v \in T_s(S)$ is a tangent vector, we write

$$\phi_*(v) = \bigoplus_p v_p ,$$

where

$$v_p \in \mathrm{Hom}(H_s^{p,q}, H_s^{p-1,q+1}) ,$$

and we write

$$v_p^* \in \mathrm{Hom}(H_s^{p-1,q+1}, H_s^{p,q})$$

for the adjoint of v_p.

Since the variation of Hodge structure is polarized by the second bilinear relation, on the Hodge bundle $\mathcal{H}^{p,q}$ there is a Hermitian metric given by

$$\langle \psi, \eta \rangle = (\sqrt{-1})^{p-q} Q(\psi, \overline{\eta}) ,$$

making the Hodge bundle $\mathcal{H}^{p,q}$ into a Hermitian vector bundle.

§1. *Connections and curvature*

We recall here the curvature formulas for subbundles and quotient bundles of a Hermitian vector bundle. These are worked out in Griffiths and Harris [6, pp. 71-79]; however, because we use a different subscript convention, we will sketch the computations below.

If E is a Hermitian vector bundle with connection $D : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ and $e_1, \cdots, e_n$ is a unitary frame for E, then the connection matrix θ

relative to this frame is the matrix of 1-forms given by

$$De_j = \Sigma \theta_{ij} e_i .$$

The curvature matrix Θ is the matrix of (1,1)-forms given by

$$D^2 e_j = \Sigma \Theta_{ij} e_i .$$

Since

$$\begin{aligned} D^2 e_j &= \Sigma\, d\theta_{ij} e_i - \Sigma\, \theta_{ij} \wedge \theta_{\ell i} e_\ell \\ &= \Sigma\, d\theta_{ij} e_i + \Sigma\, \theta_{\ell i} \wedge \theta_{ij} e_\ell \\ &= \Sigma\, (d\theta_{ij} + \theta_{ik} \wedge \theta_{kj}) e_i \quad [\text{replacing } \ell \text{ by } i \text{ and } i \text{ by } k] , \end{aligned}$$

we have

$$\Theta = d\theta + \theta \wedge \theta .$$

Given an exact sequence

$$0 \to S \to E \to Q \to 0 ,$$

the *second fundamental form*

$$\sigma : \mathcal{A}^0(S) \to \mathcal{A}^1(Q)$$

is the composition of $D|_S$ followed by the projection to the quotient bundle Q. Using the Hermitian metric we may identify the quotient bundle Q with $(S)^\perp$ and write $E = S \oplus Q$. Let $e_1, \cdots, e_r$ be a unitary frame for S and $e_{r+1}, \cdots, e_n$ a unitary frame for Q. Then the connection matrix for E is

$$\theta = \begin{bmatrix} \theta_S & -{}^t\overline{\sigma} \\ \sigma & \theta_Q \end{bmatrix}$$

and the curvature matrix is

$$\Theta = d\theta + \theta\wedge\theta = \begin{bmatrix} d\theta_S + \theta_S\wedge\theta_S - {}^t\bar{\sigma}\wedge\sigma & * \\ * & d\theta_Q + \theta_Q\wedge\theta_Q - \sigma\wedge{}^t\bar{\sigma} \end{bmatrix}.$$

Therefore, if Θ_S and Θ_Q are the curvature matrices of S and Q respectively, then

(1) $$\Theta_S = \Theta|_S + {}^t\bar{\sigma}\wedge\sigma$$

and

(2) $$\Theta_Q = \Theta|_Q + \sigma\wedge{}^t\bar{\sigma}.$$

Denote the metric on E by $\langle\ ,\ \rangle$. A connection D on E is the *metric connection* if

(a) $d\langle v,w\rangle = \langle Dv,w\rangle + \langle v,Dw\rangle$, and

(b) the (0,1)-part of D is $\bar{\partial}$.

If $D = D' + D''$ is the decomposition of D into its (1,0)-part and (0,1)-part, then the metric connections on the subbundle S and the quotient bundle Q are

$$D_S = (D'|_S - \sigma) + \bar{\partial}$$

and

$$D_Q = D'|_Q + (\bar{\partial} + {}^t\bar{\sigma}).$$

PROPOSITION 3. *Let $D = D' + D''$ be the metric connection on a Hermitian vector bundle E. If e is a holomorphic section of E, then*

$$\partial\bar{\partial}\langle e,e\rangle = \langle D'e,D'e\rangle - \langle\Theta e,e\rangle.$$

Proof. By comparing the types in Condition (a) of a metric connection, we have

$$\partial\langle v,w\rangle = \langle D'v,w\rangle + \langle v,D''w\rangle,$$

$$\bar{\partial}\langle v,w\rangle = \langle D''v,w\rangle + \langle v,D'w\rangle.$$

Therefore,

$$\begin{aligned}\partial\bar{\partial}\langle e,e\rangle &= \partial\langle e,D'e\rangle \\ &= \langle D'e,D'e\rangle + \langle e,D''D'e\rangle \\ &= \langle D'e,D'e\rangle + \langle e,\Theta e\rangle && \text{because } \Theta = D''D' \text{ for holomorphic sections} \\ &= \langle D'e,D'e\rangle - \langle\Theta e,e\rangle && \text{because } {}^t\bar{\Theta} = -\Theta.\end{aligned}$$

q.e.d.

§2. *The curvature of Hodge bundles*

There are two metrics on the cohomology bundle $\mathcal{H}$: the Hodge metric $\langle\ ,\ \rangle$ and the nondegenerate indefinite Hermitian form $(-\sqrt{-1})^n Q(\ ,\bar{\ })$, which we will call the *indefinite metric* and denote by $(\ ,\ )_i$. On the Hodge bundle $\mathcal{H}^{p,q}$ these two metrics differ by a sign

$$\langle\ ,\ \rangle = (-1)^p(\ ,\ )_i.$$

So the curvature of $\mathcal{H}^{p,q}$ is the same relative to either metric.

Let $\psi_p : \mathcal{F}^p \to \mathcal{H}/\mathcal{F}^p$ be the second fundamental form of $\mathcal{F}^p$ in $\mathcal{H}$ relative to the indefinite metric. By the horizontality of the Gauss-Manin connection, ψ_p induces a map: $\mathcal{F}^p/\mathcal{F}^{p+1} \to \mathcal{F}^{p-1}/\mathcal{F}^p$, which we also denote by ψ_p. Thus the second fundamental form may be viewed as a map of Hodge bundles

$$\psi_p : \mathcal{H}^{p,q} \to \mathcal{H}^{p-1,q+1}.$$

A section e of the Hodge bundle $\mathcal{H}^{p,q}$ is said to be *quasi-horizontal* if $\psi_p e = 0$.

PROPOSITION 4. *Let $\langle\ ,\ \rangle$ be the Hodge metric and e and e' two C^∞ sections of the Hodge bundle $\mathcal{H}^{p,q}$. The curvature form Θ of $\mathcal{H}^{p,q}$ is given by*

$$\langle\Theta e,e'\rangle = \langle\psi_p e,\psi_p e'\rangle + \langle{}^t\bar{\psi}_{p+1}e,{}^t\bar{\psi}_{p+1}e'\rangle.$$

Proof. Since ψ_p is defined relative to the indefinite metric, we will compute $\Theta_{\mathcal{H}^{p,q}}$ relative to the indefinite metric. Consider the defining exact sequences for the Hodge bundles:

$$0 \to \mathcal{F}^p \to \mathcal{H} \to \mathcal{H}/\mathcal{F}^p \to 0$$

$$0 \to \mathcal{F}^{p+1} \to \mathcal{F}^p \to \mathcal{H}^{p,q} \to 0 .$$

By the curvature formulas for subbundles and quotient bundles ((1) and (2)),

$$\Theta_{\mathcal{F}^p} = {}^t\overline{\psi}_p \wedge \psi_p \quad \text{[because } \mathcal{H} \text{ is flat]}$$

$$\Theta_{\mathcal{H}^{p,q}} = {}^t\overline{\psi}_p \wedge \psi_p + \psi_{p+1} \wedge {}^t\overline{\psi}_{p+1} .$$

Therefore,

$$(\Theta_{\mathcal{H}^{p,q}} e,e)_i = -(\psi_p e, \psi_p e)_i - ({}^t\overline{\psi}_{p+1} e, {}^t\overline{\psi}_{p+1} e)_i .$$

In terms of the Hodge metric this formula is

$$(-1)^p \langle \Theta e,e \rangle = -(-1)^{p-1} \langle \psi_p e, \psi_p e \rangle - (-1)^{p+1} \langle {}^t\overline{\psi}_{p+1} e, {}^t\overline{\psi}_{p+1} e \rangle .$$

q.e.d.

A (1,1)-form

$$\omega = \frac{\sqrt{-1}}{2} \Sigma h_{ij}(z)\, dz_i \wedge d\overline{z}_j$$

is *positive* if (h_{ij}) is a positive definite Hermitian matrix. A matrix A of (1,1)-forms is *positive* if $\langle Ae,e \rangle$ is a positive (1,1)-form for every vector e. The matrix A is *negative* if $-A$ is positive. The notion of positive or negative semidefiniteness is defined analogously. A Hermitian vector bundle E is positive or negative according as $i\Theta_E$ is positive or negative.

COROLLARY 5. *The Hodge bundles* $\mathcal{H}^{0,n}$ *are negative semidefinite.*

Proof. By the proposition

$$i \langle \Theta_{\mathcal{H}^{0,n}} e,e \rangle = i \langle {}^t\overline{\psi}_1 e, {}^t\overline{\psi}_1 e \rangle ,$$

which is a negative semidefinite (1,1)-form. q.e.d.

For the proof of the next proposition recall that a real-valued function f is *plurisubharmonic* if its *Levi form* $i\partial\overline{\partial} f$ is positive semidefinite. By the maximum principle the only plurisubharmonic functions on a compact manifold are the constants.

PROPOSITION 6. *Over a compact base a holomorphic quasihorizontal section of* $\mathcal{H}^{p,q}$ *is holomorphic and flat as a section of* $\mathcal{H}$.

Proof. Combining the Levi form formula (Proposition 3) with the curvature formula (Proposition 4), we get

$$\partial\overline{\partial} \langle e,e \rangle = \langle D'_p e, D'_p e \rangle - \langle \psi_p e, \psi_p e \rangle - \langle {}^t\overline{\psi}_{p+1} e, {}^t\overline{\psi}_{p+1} e \rangle .$$

Because e is quasihorizontal,

$$(*) \qquad i\partial\overline{\partial} \langle e,e \rangle = i \langle D'_p e, D'_p e \rangle - i \langle {}^t\overline{\psi}_{p+1} e, {}^t\overline{\psi}_{p+1} e \rangle .$$

This is positive semidefinite, since D'_p has type (1,0) and ${}^t\overline{\psi}_{p+1}$ has type (0,1). So $\langle e,e \rangle$ is plurisubharmonic and hence constant. It then follows from (*) that $D'_p e = 0$ and ${}^t\overline{\psi}_{p+1} e = 0$. Since

$$\nabla e = D'_p e + \psi_p e = 0$$

and

$$\overline{\partial} e = D''_p e \pm {}^t\overline{\psi}_{p+1} e = 0 ,$$

e is flat and holomorphic as a section of $\mathcal{H}$. q.e.d.

APPLICATION 7 (Theorem of the Fixed Part). *Let* $\{\mathcal{H}_Z, \mathcal{H}^{p,q}, \nabla, Q, S\}$ *be a polarized variation of Hodge structure over a compact base. If* e *is a*

global holomorphic flat section of $\mathcal{H}$, *then the* (p,q)-*components of* e *are also holomorphic and flat as sections of* $\mathcal{H}$.

Proof. Let $e = e_r + e_{r+1} + \cdots + e_n$ be the decomposition of e with $e_p \in \mathcal{H}^{p,q}$. Since e is flat,

$$0 = \nabla e = (D'_r + \psi_r)e_r + (D'_{r+1} + \psi_{r+1})e_{r+1} + \cdots .$$

By comparing types

$$\psi_r e_r = 0 .$$

Similarly, since e is holomorphic,

$$0 = D''e = (D''_r \pm {}^t\overline{\psi}_{r+1})e_r + \cdots$$

so that

$$D''_r e_r = 0 .$$

Therefore, e_r is a holomorphic flat section of $\mathcal{H}^{r,n-r}$. By Proposition 6 it is holomorphic and flat as a section of $\mathcal{H}$. Next apply this argument to $e - e_r$, which we now know to be a global holomorphic flat section of $\mathcal{H}$. By induction all the (p,q)-components of e are holomorphic and flat as sections of $\mathcal{H}$. q.e.d.

For the next application, we note that if $\mathcal{V}$ and $\mathcal{V}'$ are two variations of Hodge structures over the same base S, then $\pi_1(S)$ acts on $\mathrm{Hom}(H_C, H'_C)$ by

$$(g, \sigma) \mapsto g^{-1}\sigma g$$

for $g \in \pi_1(S)$ and $\sigma \in \mathrm{Hom}(H_C, H'_C)$.

APPLICATION 8 (Rigidity Theorem). *Let* $\mathcal{V}$ *and* $\mathcal{V}'$ *be two variations of Hodge structures over a compact base space* S. *Suppose* $\mathcal{V}$ *and* $\mathcal{V}'$ *agree at one point* s_0 *via an isomorphism* $\mu_0 : H_C \to H'_C$ *and suppose the monodromy action is equivariant:*

$$\mu_0(ge) = g\mu_0(e) .$$

Then $\mathcal{V}$ *and* $\mathcal{V}'$ *are isomorphic everywhere.*

Proof. By parallel translation μ_0 extends to a flat, possibly multivalued section μ of $\mathrm{Hom}(\mathcal{V}, \mathcal{V}')$, which is an isomorphism of vector spaces everywhere. The equivariance of the monodromy says precisely that this section μ is single-valued. By the theorem of the fixed part μ has type (0,0) everywhere, so it preserves the Hodge filtrations everywhere. q.e.d.

APPLICATION 9 (Semi-simplicity). Deligne has proved that the monodromy representation of a variation of Hodge structure over a quasiprojective base is completely reducible over $\mathbf{Q}$ (Deligne [4, 4.2.6)]). To keep the presentation simple we will content ourselves with the following weaker statement.

THEOREM 9.1. *Let* $\mathcal{V} = \{\mathcal{H}_Z, \mathcal{H}^{p,q}, \nabla, Q, S\}$ *be a variation of Hodge structure over a compact base,* Γ *the monodromy group, and* H^Γ *the space of invariant cohomology classes. Then the restriction of the bilinear form* Q *to* H^Γ *is nondegenerate. It follows that if* $(H^\Gamma)^\perp$ *is the orthogonal complement of* H^Γ *with respect to* Q, *then*

$$H = H^\Gamma \oplus (H^\Gamma)^\perp$$

as Γ-*modules.*

Proof. Let $C : H \to H$ be the Weil operator. We claim that if $e = \Sigma e_p$ is an invariant cohomology class, then so are each Hodge component e_p and Ce. Let e(s) be the flat global section of the cohomology bundle $\mathcal{H}$ obtained by parallel translating e and let $e(s) = \Sigma e_p(s)$ be its Hodge decomposition. By the theorem of the fixed part, each $e_p(s)$ is also flat. Therefore, $e_p(s)$ can be obtained from e_p by parallel translation. Since $e_p(s)$ is a single-valued flat section, e_p is an invariant cohomology class. If $\gamma \in \Gamma$, then

$$\gamma Ce = \gamma\Sigma(\sqrt{-1})^{p-q} e_p$$
$$= \Sigma(\sqrt{-1})^{p-q} e_p$$
$$= Ce .$$

This proves the claim. Since $Q(C\bar{e},e) > 0$ for $e \neq 0$, the bilinear form Q is nondegenerate on H^Γ.

Because Γ preserves Q, the orthogonal complement $(H^\Gamma)^\perp$ is also invariant under Γ. Let $e \in H^\Gamma \cap (H^\Gamma)^\perp$. Then $Q(C\bar{e},e) = 0$. So $e = 0$ and $H^\Gamma \cap (H^\Gamma)^\perp = \{0\}$. Since H^Γ and $(H^\Gamma)^\perp$ have complementary dimensions in H, we get the direct sum decomposition

$$H = H^\Gamma \oplus (H^\Gamma)^\perp . \qquad \text{q.e.d.}$$

§3. *Curvature of the Classifying Space*

Let X be a Hermitian manifold. Denote by $T_x(X)$ the $(1,0)$-tangent space at x in X. For a holomorphic tangent vector ξ in $T_x(X)$ we define the *holomorphic sectional curvature* $K(\xi)$ as follows.

For the metric $ds^2 = h(z)dzd\bar{z}$ on the unit disk Δ, the Gaussian curvature is defined to be

$$K = -\frac{1}{\pi}\frac{1}{h}\frac{\partial^2 \log h}{\partial z \partial \bar{z}} .$$

In general, if X is a Hermitian manifold with metric ds_X^2 and $f:\Delta \to X$ is a holomorphic map such that

$$(*) \qquad f(0) = x \quad \text{and} \quad f_*(\partial/\partial z)_0 = \xi ,$$

then we set

$$K_f = \text{Gaussian curvature of } f^* ds_X^2 \text{ at the origin in } \Delta .$$

The *holomorphic sectional curvature* $K(\xi)$ is defined to be

$$K(\xi) = \sup_f K_f ,$$

where f ranges over all holomorphic maps $f:\Delta \to X$ satisfying the initial conditions $(*)$.

FACT 10 (Wu [12]). *If* $(R_{i\bar{j}k\bar{\ell}})$ *is the curvature tensor relative to an orthonormal frame near* x *and* $\xi = (\xi^j)$ *is a unit vector relative to the same frame, then*

$$K(\xi) = \Sigma R_{i\bar{j}k\bar{\ell}}\xi^i\bar{\xi}^j\xi^k\bar{\xi}^\ell .$$

EXAMPLE 11. On the unit disk Δ the Poincaré metric

$$ds_\Delta^2 = \frac{2}{\pi}\frac{dz\,d\bar{z}}{(1-|z|^2)^2}$$

has curvature $K \equiv -1$. On the upper half-plane $\mathfrak{h} = \{w = u+iv \mid v > 0\}$, which is conformally equivalent to Δ, the Poincaré metric is given by

$$ds_{\mathfrak{h}}^2 = \frac{1}{2\pi}\frac{dw\,d\bar{w}}{v^2} .$$

THE AHLFORS LEMMA. *Let* X *be a Hermitian manifold and* $T_h \subset T(X)$ *a holomorphic subbundle of the tangent bundle such that the holomorphic sectional curvature* $K(\xi) \leq -1$ *for all nonzero vectors* ξ *in* T_h. *Then for any holomorphic map* $f:\Delta \to X$ *such that* $f_*(\partial/\partial z) \in (T_h)_{f(z)}$, *we have*

$$f^*(ds_X^2) \leq ds_\Delta^2 .$$

REMARK 12. The map f in the Ahlfors Lemma is an integral manifold of T_h. The assertion is that such a map is *distance-decreasing*.

Recall that for the classifying space D, we have

$$\begin{array}{l} T(D) \subset \bigoplus_p \bigoplus_{r\geq 0} \mathrm{Hom}(\mathcal{H}^{p,q}, \mathcal{H}^{p-r,q+r}) \\ \cup \\ T_h(D) \subset \bigoplus_p \mathrm{Hom}(\mathcal{H}^{p,q}, \mathcal{H}^{p-1,q+1}) . \end{array}$$

Since each Hodge bundle $\mathcal{H}^{p,q}$ has a Hermitian metric, the tangent bundle $T(D)$ inherits a Hermitian metric by functoriality.

BASIC COMPUTATION 13 ([3] and [7]). The holomorphic sectional curvature of the horizontal subbundle $T_h(D)$ is bounded above by a negative constant:

$$K(\xi) \le -A < 0 \quad \text{for all} \quad \xi \in T_h(D) .$$

REMARK 14. This follows (nontrivially) from the curvature formula for the Hodge bundles (Proposition 4). It is convenient to normalize so that $A = 1$.

As a consequence of this computation and the Ahlfors Lemma, the classifying space D has all the function-theoretic properties of a bounded domain *relative to horizontal maps.*

A special case of the distance-decreasing property of the period map is the following.

PROPOSITION 15. *Let* $\phi : \Delta^* \to \Gamma\backslash D$ *be a period map and* $\phi_* : T\Delta^* \to TD = \oplus \mathrm{Hom}(\mathcal{F}^p, \mathcal{H}/\mathcal{F}^p)$ *its differential. Write* $\phi_* = \oplus(\phi_*)_p$. *If* e *is a section of* $\mathcal{F}^p$, *then*

$$\left\|(\phi_*)_p\left(\frac{\partial}{\partial r}\right)e\right\| \le \frac{C\|e\|}{r \log \frac{1}{r}}$$

for some constant C.

Proof. Recall that the Poincaré metric on Δ^* is

$$\frac{2}{\pi}\,\frac{dr \otimes dr + r^2 d\theta \otimes d\theta}{\left(r \log \frac{1}{r}\right)^2} .$$

Hence,

$$\left\|\frac{\partial}{\partial r}\right\|_{\Delta^*} = \sqrt{\frac{2}{\pi}}\,\frac{1}{r \log \frac{1}{r}} .$$

By the distance-decreasing property of the period map,

$$\left\|\phi_* \frac{\partial}{\partial r}\right\| \le \sqrt{\frac{2}{\pi}}\,\frac{1}{r \log(1/r)} .$$

Therefore,

$$\|(\phi_*)_p(\partial/\partial r)e\| \le \|(\phi_*)_p(\partial/\partial r)\|\,\|e\| \le \|\phi_*\,\partial/\partial r\|\,\|e\|$$

$$\le \sqrt{\frac{2}{\pi}}\,\frac{\|e\|}{r \log(1/r)} . \qquad \text{q.e.d.}$$

APPLICATION 16 (Removable Singularity Theorem). *Let* Z *be a subvariety of the algebraic variety* S *and* $\phi : S - Z \to \Gamma\backslash D$ *a variation of Hodge structure such that the monodromy is locally finite around* Z. *Then* ϕ *extends to give an extended variation of Hodge structure* $\phi : S \to \Gamma\backslash D$.

For a proof see Griffiths [5, p. 156].

APPLICATION 17 (Monodromy Theorem). *Let* $\phi : \Delta^* \to \{T^n\}\backslash D$ *be a variation of Hodge structure over the punctured disk with*

$$\phi_*(\textit{generator of } \pi_1(\Delta^*)) = T .$$

Then all the eigenvalues of T *are roots of unity.*

The proof that follows is due to Borel. We will need to quote the theorem of Kronecker that an algebraic integer all of whose conjugates have absolute value 1 is a root of unity.

Proof. Since T can be represented as an integer matrix, the eigenvalues of T are algebraic integers. By Kronecker's theorem, it suffices to show that their conjugates are all of absolute value 1.

Because the upper half-plane $\mathfrak{h}$ is simply connected, the map ϕ can be lifted as in the following commutative diagram

$$\begin{tikzcd} \mathfrak{h} \arrow[r, dashed, "\tilde{\phi}"] \arrow[d, "e^{2\pi i z}"'] & D \arrow[d] \\ \Delta^* \arrow[r, "\phi"] & \{T^n\}\backslash D . \end{tikzcd}$$

Note that $\tilde{\phi}(z+1) = T\tilde{\phi}(z)$. Let $\{z_n\}$ be a sequence of points in $\mathfrak{h}$ and p_0 the base point in D. Denote by $\rho_X(x,y)$ the distance between the points x and y on a Hermitian manifold X. Then

$$\begin{aligned}
&\rho_D(\tilde{\phi}(z_n+1), \tilde{\phi}(z_n)) \\
&\quad = \rho_D(T\tilde{\phi}(z_n), \tilde{\phi}(z_n)) \\
&\quad = \rho_D(Tg_n p_0, g_n p_0) \quad \text{for some } g_n \in G_{\mathbf{R}}, \text{ because } G_{\mathbf{R}} \text{ acts transitively on } D \\
&\quad = \rho_D(g_n^{-1} T g_n p_0, p_0) \quad \text{by the } G_{\mathbf{R}}\text{-invariance of the metric on } D.
\end{aligned}$$

On the other hand, by the distance-decreasing property of horizontal maps,

$$\rho_D(\tilde{\phi}(z_n+1), \tilde{\phi}(z_n)) \le \rho_{\mathfrak{h}}(z_n, z_n+1) = \frac{1}{\operatorname{Im} z_n}.$$

So if $\{z_n\}$ is chosen so that $\operatorname{Im} z_n \to 0$, then

$$\rho_D((g_n^{-1} T g_n)p_0, p_0) \to 0 \quad \text{as} \quad n \to \infty.$$

Because D is a metric space,

$$\lim_{n\to\infty} (g_n^{-1} T g_n) p_0 = p_0.$$

It follows that $g = \lim_{n\to\infty} g_n^{-1} T g_n$ is in the stabilizer H of p_0. Because H is compact, the eigenvalues of g and hence of T all have absolute value 1. q.e.d.

REMARK 18. A matrix T satisfying

$$(T^N - I)^{k+1} = 0$$

for some integers N and k is said to be *quasi-unipotent*. The least such k is called the *index* of quasi-unipotency. An equivalent formulation of the monodromy theorem is that T is quasi-unipotent. Schmid has shown that for a variation of Hodge structure of weight n over the punctured disk Δ^*, the index of quasi-unipotency is at most n. See Schmid [11].

REMARK 19. In the geometric case, the monodromy theorem is due to Landman [10] and Katz [9].

§4. *Algebraization and regular singular points*

Let $\mathcal{V} = \{\mathcal{H}_{\mathbf{Z}}, \mathcal{H}^{p,q}, \nabla, Q, S\}$ be a variation of Hodge structure over an algebraic base S. A priori the Hodge bundles are only holomorphic bundles. Of course, if the base is a smooth compact algebraic variety, then by Serre's GAGA principle, the Hodge bundles are also algebraic bundles. It turns out that even when the base is noncompact, the Hodge bundles can be given an algebraic structure which is uniquely characterized by a growth condition on its sections. The proof of this algebraization theorem is based on a curvature computation which we will briefly explain.

First, some notations and terminologies. Let S be an algebraic variety, not necessarily compact. A *smooth compactification* of S is a smooth compact variety $\bar{S}$ such that $D = \bar{S} - S$ is a divisor with normal crossings. Thus the neighborhoods at infinity are punctured polycylinders $P^* = (\Delta^*)^k \times \Delta^{n-k}$. Such a smooth compactification exists by Hironaka. We let η_{P^*} be the Poincaré metric on P^*. Recall that for a polarized Hodge structure $\{H_{\mathbf{Z}}, H^{p,q}, Q\}$ a positive definite metric can be defined on $H^{p,q}$ by setting

$$\langle \psi, \eta \rangle_p = (\sqrt{-1})^{p-q} Q(\psi, \bar{\eta}).$$

The *Hodge length* of v in H, $v = \sum_{p=1}^{n} v^{p,q}$, is then

$$\|v\| = \Big(\sum_{p=1}^{n} \langle v^{p,q}, v^{p,q} \rangle_p\Big)^{1/2}.$$

The algebraicity of the Hodge bundles is a consequence of the following general theorem.

THEOREM 20. *With the notations above, let* $E \to S$ *be a Hermitian vector bundle whose curvature satisfies*

$$-C\eta_{P^*} \leq \frac{\langle \theta e, e\rangle}{\langle e, e\rangle} \leq C\eta_{P^*}$$

locally at infinity. Then there exists a unique algebraic structure on E *whose algebraic sections* $e(s)$ *are characterized by moderate growth, that is,*

$$\|e(s)\| = 0(|s|^{-a})$$

for a local parameter s *on* S *and for some positive integer* a.

The point is that by the Ahlfors Lemma the curvatures of the Hodge bundles satisfy the inequality of the theorem and so over an algebraic base, $\mathcal{H}^{p,q}$ has the prescribed algebraic structure. To get an algebraic structure with moderate growth on $\mathcal{F}^p$ one first shows that the extension class of

$$0 \to \mathcal{F}^{p+1} \to \mathcal{F}^p \to \mathcal{H}^{p,q} \to 0$$

is algebraic. By induction we may assume $\mathcal{F}^{p+1}$ algebraic. The algebraicity of $\mathcal{F}^p$ then follows from that of $\mathcal{H}^{p,q}$, $\mathcal{F}^{p+1}$, and the extension class.

REMARK 21. When the variation of Hodge structure comes from geometry, there is an a priori algebraic structure on the Hodge bundles $\mathcal{H}^{p,q}$. For if $\pi: \mathcal{X} \to S$ is a family of polarized algebraic varieties, then $\mathcal{H}^{p,q}$ can be identified with the direct image sheaf

$$\mathcal{H}^{p,q} \simeq \mathcal{R}^p \pi_* \Omega^p_{\mathcal{X}/S},$$

where $\Omega^p_{\mathcal{X}/S} = \Lambda^p \Omega^1_{\mathcal{X}/S}$ is the sheaf of algebraic p-forms along the fiber. However, it can be shown that this a priori algebraic structure agrees with the algebraic structure with moderate growth given by the theorem (see Cornalba and Griffiths [2]).

Regular singular points

To explain intuitively the theorem on regular singular points, we consider a family of smooth projective varieties $f: \mathcal{X} \to \Delta^*$. Let $\omega(s) \in H^k(X_s)$ be a smoothly varying collection of C^∞ k-forms on the fibers of the family and $c(s_0)$ a k-cycle in X_{s_0}. Horizontally displace $c(s_0)$ to obtain k-cycles $c(s) \in H_k(X_s, Z)$. Although $c(s)$ is in general a multivalued section of $\bigcup_s H_k(X_s, Z)$, over an angular sector Δ^* it has single-valued branches. The assertion of the theorem is that *if the Hodge length of* $\omega(s)$ *has moderate growth, i.e.,*

$$\|\omega(s)\| = 0(|s|^{-a}) \text{ for some positive integer } a,$$

then over any angular sector the period of $\omega(s)$ *relative to a single-valued branch of* $c(s)$ *also has moderate growth:*

$$\left| \int_{c(s)} \omega(s) \right| = 0(|s|^{-a}).$$

To formalize this, we first make a few definitions.

DEFINITION 22. Let $\pi: E \to \Delta^*$ be a Hermitian vector bundle. A holomorphic section e of E is said to be *meromorphic at the origin* if

$$\|e(s)\| = 0(|s|^{-a}) \text{ for some positive integer } a.$$

Such a section is also called a *meromorphic section* of E.

DEFINITION 23. Let $\pi: E \to \Delta^*$ be a holomorphic vector bundle with a connection ∇, and let $\{v_1, \cdots, v_n\}$ be a (possibly multivalued) flat frame for E. The connection is said to have *regular singular point* at the origin (relative to the flat frame $\{v_1, \cdots, v_n\}$) if on any angular sector the coeffi-

cients of any meromorphic section e relative to the flat frame $\{v_1,\cdots,v_n\}$ have at most poles at the origin; in other words, if $e(s) = \Sigma b_i(s) v_i$, then the $b_i(s)$ are multi-valued meromorphic functions at the origin.

THEOREM 24. *The Gauss-Manin connection of the cohomology bundle $\pi : \mathcal{H} \to \Delta^*$ of a polarized variation of Hodge structure $\mathcal{V} = \{\mathcal{H}_{\mathbf{Z}}, \mathcal{H}^{p,q}, \nabla, Q, \Delta^*\}$ has a regular singular point at the origin.*

The proof of this theorem depends on an estimate for the length of a holomorphic flat section of $\pi : \mathcal{H} \to \Delta^*$. Here by a flat section we mean a flat section with respect to the Gauss-Manin connection ∇. If D is the metric connection on $\mathcal{H}$ relative to the indefinite metric $(-\sqrt{-1})^n Q(\ ,\bar{\ })$. Then the Gauss-Manin connection is the (1,0)-part D' of D. Thus a holomorphic flat section is also flat with respect to D (but not with respect to the metric connection of the Hodge metric on $\mathcal{H}$).

Recall that the metric connection $D_p = D'_p + D''_p$ on the Hodge bundle $\mathcal{H}^{p,q}$ is related to the metric connection $D = D' + D''$ on $\mathcal{H}$ by

$$D'_p + \psi_p = D'|_{\mathcal{H}^{p,q}}$$

$$D''_p \pm {}^t\psi_{p+1} = D''|_{\mathcal{H}^{p,q}} .$$

PROPOSITION 25. *On any angular sector of Δ^* the Hodge length of a holomorphic flat section e of $\mathcal{H} \to \Delta^*$ satisfies*

$$C_1\left(\log \frac{1}{|s|}\right)^{-k} \le \|e(s)\|^2 \le C_2\left(\log \frac{1}{|s|}\right)^{k}$$

where C_1, C_2, k are positive constants.

Proof. Let $e = \Sigma e_p$ be the Hodge decomposition of e, where e_p is the (p,n–p)-component. Because e is flat,

$$D'_p e_p = -\psi_{p+1} e_{p+1} .$$

Because e is holomorphic,

$$D''_p e_p = \pm {}^t\psi_{p-1} e_{p-1} .$$

Thus the radial derivative of the Hodge length is

$$\begin{aligned}\frac{\partial}{\partial r} \langle e,e\rangle &= \Sigma \frac{\partial}{\partial r} \langle e_p, e_p\rangle \\ &= 2 \operatorname{Re} \Sigma \langle D\left(\frac{\partial}{\partial r}\right) e_p, e_p\rangle \\ &= -2 \operatorname{Re} \Sigma \langle \psi_{p+1}\left(\frac{\partial}{\partial r}\right) e_{p+1}, e_p\rangle \pm \langle {}^t\psi_{p-1}\left(\frac{\partial}{\partial r}\right) e_{p-1}, e_p\rangle .\end{aligned}$$

By the Schwarz inequality,

$$|\langle \psi_{p+1}\left(\frac{\partial}{\partial r}\right) e_{p+1}, e_p\rangle| \le \|\psi_{p+1}\left(\frac{\partial}{\partial r}\right) e_{p+1}\| \, \|e_p\| .$$

By the distance-decreasing property of the period map,

$$\left\|\psi_{p+1}\left(\frac{\partial}{\partial r}\right) e_{p+1}\right\| \le \frac{C\|e_{p+1}\|}{r \log \frac{1}{r}} .$$

Therefore,

$$\left|\langle \psi_{p+1}\left(\frac{\partial}{\partial r}\right) e_{p+1}, e_p\rangle\right| \le \frac{C\|e_{p+1}\| \, \|e_p\|}{r \log \frac{1}{r}} \le \frac{C\|e\|^2}{r \log \frac{1}{r}}$$

So the radial derivative satisfies

$$\frac{\left|\frac{\partial}{\partial r} \langle e,e\rangle\right|}{\langle e,e\rangle} \le \frac{k}{r \log \frac{1}{r}}$$

for some constant k. Integrating this inequality with respect to r yields the desired estimate. q.e.d.

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REMARK ADDED IN PROOF: There is increasing evidence that the curvature properties of the Hodge bundles play an essential role in the classification theory of algebraic varieties. We give here a short bibliography to this interesting development:

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