A defect relation for equidimensional holomorphic mappings between algebraic varieties

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Introduction

(a) *Statement of defect relations.* The purpose of this paper is to prove versions of the first and second main theorems of Nevanlinna theory [8], together with the corresponding defect relations, in the case of a non-degenerate holomorphic mapping

$$f\colon \mathbf{C}^n \longrightarrow V \tag{0.1}$$

where V is a smooth, n-dimensional, projective algebraic variety. To state the defect relations, we introduce the following notation: For cohomology classes $\alpha, \beta \in H^2(V, \mathbf{R})$ and a real number λ, we write

$$\lambda > \left[\frac{\alpha}{\beta}\right]$$

to mean that $\lambda\beta - \alpha$ is represented by a positive definite closed (1, 1) form on V. Similarly, we may define $\lambda \leqq [\alpha/\beta]$. Let $\mathbf{L} \to V$ be a positive line bundle and $D_1, \cdots, D_k$ divisors in $|\mathbf{L}|$ whose irreducible components are smooth. Suppose that $D_1 + \cdots + D_k$ has normal crossings and the following inequality of Chern classes holds:

$$k > \left[\frac{c_1(\mathbf{K}_V^*)}{c_1(\mathbf{L})}\right]. \tag{0.2}$$

Nevanlinna defects $\delta(D)$ ([8], page 266) may be defined, for every divisor $D \in |\mathbf{L}|$, with the properties that $0 \leqq \delta(D) \leqq 1$ and $\delta(D) = 1$ if $f(\mathbf{C}^n)$ does not meet D. Our principal defect relation is the inequality

$$\sum_{j=1}^{k} \delta(D_j) \leqq \left[\frac{c_1(\mathbf{K}_V^*)}{c_1(\mathbf{L})}\right]. \tag{0.3}$$

Taking $V = \mathbf{P}_n$ and the $D_j = H_j$ to be $(n + 2)$ hyperplanes in general position, we obtain

$$\sum_{j=1}^{n+2} \delta(H_j) \leqq n + 1 . \tag{0.4}$$

For $n = 1$, (0.4) reduces to the classical Nevanlinna defect relation given in [8], page 266.

Our proof of (0.3) is, in principle, analogous to the differential geometric proof in the 1-variable case given by F. Nevanlinna ([8], pages 247–256). There are, however, some differences. In the first place F. Nevanlinna used the uniformization theorem to produce a metric with constant negative curvature on $\mathbf{P}_1 - \{a_1, \cdots, a_k\}$ $(k \geqq 3)$, whereas we give a direct algebro-geometric argument to construct a volume form Ψ on $V - (D_1 + \cdots + D_k)$ which satisfies $\operatorname{Ric}\Psi > 0$ and $(\operatorname{Ric}\Psi)^n \geqq \Psi$, this being the analogue of curvature $\leqq -1$. That the construction of such a Ψ should be possible was suggested by considering the branched coverings constructed in the first author's thesis [1]. In the second place, our derivation of the defect relations from the second main theorem seems more straightforward than that in [8] because, by keeping all of the expressions intrinsic, we are able to avoid a lot of extraneous remainder terms in the estimates. Finally, we obtain both the first and second main theorems quite easily using the language of currents [7] and the Poincaré equation (cf. (1.13) below).

$$\frac{\sqrt{-1}}{\pi}\partial\bar{\partial} \log |f(z)| = D_f \tag{0.5}$$

where $f(z)$ is a holomorphic function, D_f is the divisor of zeroes of f, and $\log |f|$ is a locally L^1 function considered as a current. In conclusion then, we feel that by isolating the essential algebro-geometric content of the problem, it has been possible to find a general result whose proof, even in the 1-variable case, is simpler and more intrinsic than those at hand.

(b) *Acknowledgments and references to other works.* Our first main theorem is similar to that of Stoll ("Value distribution of holomorphic maps," Several complex variables I, Springer-Verlag lecture notes #155 (1970), pages 165–190). In both cases the essential point is to use a "special exhaustion function" to get rid of the intractible remainder term which appears in the

general first main theorem (Stoll, loc. cit.). Our second main theorem is obtained by twice integrating the equation of currents (3.5), which is the globalized and intrinsic version of (0.5). This method appears implicitly in [8] and has been used recently by Kodaira [6] to derive an inequality in the spirit of Nevanlinna theory. Finally, from (0.4) we obtain an affirmative answer to problem V on Chern's list [2], as it follows that a holomorphic mapping

$$f\colon \mathbf{C}^n \longrightarrow \mathbf{P}_n - (H_1 \cup \cdots \cup H_{n+2})$$

must be degenerate. In [1] this degeneracy theorem was proved for $n+3$ hyperplanes. Subsequently, by a very direct and entirely different method, Mark Green (cf. [3]) obtained the sharpest possible degeneracy statement for maps

$$f\colon \mathbf{C}^m \longrightarrow \mathbf{P}_n - (H_1 \cup \cdots \cup H_l) .$$

Theorem (*Green*): Let $f\colon \mathbf{C}^m \longrightarrow \mathbf{P}_n$ be a holomorphic map that omits $n+k$ hyperplanes in general position, $k \geqq 1$. Then the image of f is contained in a projective linear subspace of dimension $\leqq [n/k]$, where the brackets mean greatest integer. Furthermore, this bound is sharp.

Our results were obtained after his work had been done.

In [9] Stoll has proved a defect relation of the form $\sum_{\mu=1}^{q} \delta(H_\mu) \leqq n+1$ for holomorphic maps $f\colon M_m \to \mathbf{P}_n$ where M belongs to a class of complex manifolds and where f satisfies certain additional hypotheses. Stoll has written us that these restrictive assumptions simplify considerably in the case $M = \mathbf{C}^m$ to where "the worst one is that f be not rational." We hope at a later time to investigate the relationship of his theory to the defect relation (0.3).

Finally, it is possible to derive defect relations for non-degenerate holomorphic maps $f\colon A \to V$ where A is an arbitrary smooth affine variety and $\dim_{\mathbf{C}} A = \dim_{\mathbf{C}} V$. The main inequality is the same as (0.3), except that now a non-negative term κ appears on the R.H.S. of (0.3) with the properties that (a) $\kappa = 0$ if $A = \mathbf{C}^n$; and (b) $\kappa = 0$ if f is transcendental. Thus, e.g., the degeneracy statement now says that a non-degenerate holomorphic map

$$f\colon A \longrightarrow \mathbf{P}_n - (H_1 + \cdots + H_{n+2})$$

is necessarily rational. Details of this result will appear at a later time.

1. Notations, terminology, and sign conventions

(a) *Line bundles and Chern classes.* Let M be a complex manifold. Relative to a suitable open covering $\{U_\alpha\}$ of M, a holomorphic line bundle is given by transition functions $f_{\alpha\beta} \in \mathcal{O}^*_{U_\alpha \cap U_\beta}$ satisfying the usual cocycle condition. The holomorphic sections σ of $\mathbf{L} \to M$ are given by holomorphic functions $\sigma_\alpha \in \mathcal{O}(U_\alpha)$ which satisfy $\sigma_\alpha = f_{\alpha\beta}\sigma_\beta$ in $U_\alpha \cap U_\beta$. A metric in $\mathbf{L}$ is given by positive C^∞ functions a_α in U_α such that the length

$$|\sigma|^2 = \frac{|\sigma_\alpha|^2}{a_\alpha}$$

is well-defined. Thus $a_\alpha = |f_{\alpha\beta}|^2 a_\beta$ in $U_\alpha \cap U_\beta$.

In particular, if the U_α are coordinate neighborhoods with holomorphic coordinates $z_\alpha = (z_\alpha^1, \cdots, z_\alpha^n)$ in U_α, then the *canonical bundle* $\mathbf{K}_V$ is defined by the transition functions

$$\kappa_{\alpha\beta} = \det\left(\frac{\partial z_\beta^i}{\partial z_\alpha^j}\right).$$

Thus, if $\sigma = \{\sigma_\alpha\}$ is a holomorphic section of $\mathbf{K}_V$, then the holomorphic n-forms $\sigma_\alpha dz_\alpha^1 \wedge \cdots \wedge dz_\alpha^n$ patch together to give a holomorphic n-form on M. A *volume form* Ψ is defined to be a positive C^∞ section of $\mathbf{K}_V \otimes \overline{\mathbf{K}}_V$ which is C^∞ outside an analytic subset of M. Locally

$$\Psi = \xi_\alpha \prod_{j=1}^n \left(\frac{\sqrt{-1}}{2} dz_\alpha^j \wedge d\bar{z}_\alpha^j\right)$$

where $\xi_\alpha \geqq 0$. In $U_\alpha \cap U_\beta$ we have

$$\xi_\alpha = |\kappa_{\alpha\beta}|^2 \xi_\beta ,$$

so that if Ψ is C^∞ everywhere then the ξ_α give a metric in $\mathbf{K}_V$.

We shall use differential forms and deRham cohomology, denoted by $H^*(M, \mathbf{R})$. As usual on complex manifolds, the exterior derivative may be written as $d = \partial + \bar{\partial}$, and we define

$$d^c = \frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial) . \tag{1.1}$$

From (1.1) it follows that

$$dd^c = \left(\frac{\sqrt{-1}}{2\pi}\right)\partial\bar{\partial} . \tag{1.2}$$

The factor π is put in (1.1) so that we will have the useful equation

$$\int_{|z|=1} d^c \log |z|^2 = 1 . \tag{1.3}$$

If φ is a C^∞ real form of type $(1, 1)$, then locally

$$\varphi = \sqrt{-1}\left(\sum_{i,j} \varphi_{ij} dz_i \wedge d\bar{z}_j\right) \qquad (\varphi_{ij} = \bar{\varphi}_{ji}) .$$

We shall say that φ is positive, written $\varphi > 0$, if (φ_{ij}) is a positive definite Hermitian matrix. The notation $\varphi \geqq 0$ has the obvious meaning. If u is a

C^∞ function, then $dd^c u \geqq 0$ is the same as the usual condition $(\partial^2 u/\partial z_i \partial \bar{z}_j) \geqq 0$ for plurisubharmonicity.

Definition. Let $\mathbf{L} \to M$ be a holomorphic line bundle with transition functions $\{f_{\alpha\beta}\}$ and metric $\{a_\beta\}$. Then the *Chern class* $c_1(\mathbf{L}) \in H^2(M, \mathbf{R})$ is represented by the real $(1, 1)$ form ω given in U_α by

$$\omega = dd^c \log a_\alpha \,. \tag{1.4}$$

Remarks. If ω and ω' arise from two metrics $\{a_\alpha\}$ and $\{a'_\alpha\}$, then

$$\omega - \omega' = dd^c b$$

where b is a positive C^∞ function on M. We shall check our signs and constants by computing $c_1(\mathbf{L})$ for the standard positive line bundle on $\mathbf{P}_1$. Cover $\mathbf{P}_1$ by open sets U_1, U_2 with coordinates z_1, z_2 such that $z_1 z_2 = 1$ in $U_1 \cap U_2$. Taking $f_{12} = z_1$, we obtain $\mathbf{L} \to \mathbf{P}_1$ having a section σ given by $\sigma_1 = z_1$, $\sigma_2 = 1$. The functions $a_j = (1 + |z_j|^2)$ $(j = 1, 2)$ give a metric in $\mathbf{L}$ since

$$\frac{|z_1|^2}{1 + |z_1|^2} = \frac{1}{1 + |z_2|^2}$$

in $U_1 \cap U_2$. The form ω representing $c_1(\mathbf{L})$ is given by

$$\omega = \frac{\sqrt{-1}}{2\pi}\left\{\frac{dz_1 \wedge d\bar{z}_1}{(1 + |z_1|^2)^2}\right\}$$

in U_1. Using polar coordinates $z_1 = re^{i\theta}$, we have

$$\int_{\mathbf{P}_1} \omega = \frac{\sqrt{-1}}{2\pi}\iint \frac{dz_1 d\bar{z}_1}{(1 + |z_1|^2)^2} = \frac{1}{\pi}\int_0^\infty \int_0^{2\pi} \frac{r d\theta dr}{(1 + r^2)^2} = 1 \,.$$

This verifies our signs and constants.

Definition. Let Ψ be volume form on M. Then the *Ricci form* $\operatorname{Ric}\Psi$ is the real $(1, 1)$ form defined where Ψ is positive and given locally by

$$\operatorname{Ric}\Psi = dd^c \log \xi_\alpha \,. \tag{1.5}$$

Remark. If Ψ is everywhere C^∞ then $\operatorname{Ric}\Psi = c_1(\mathbf{K}_V)$ in $H^2(M, \mathbf{R})$.

Definition. If $\mathbf{L} \to M$ is a holomorphic line bundle, then we will write $c_1(\mathbf{L}) > 0$ to mean that there is a positive $(1, 1)$ form ω representing $c_1(\mathbf{L})$ in $H^2(M, \mathbf{R})$.

Remark. In case M is a compact Kähler manifold, any positive form ω representing $c_1(\mathbf{L})$ is of the form (1.4) for some metric in $\mathbf{L}$ (c.f. Kodaira [5]). The notation

$$c_1(\mathbf{L}) + c_1(\mathbf{K}_V) > 0 \tag{1.6}$$

is the same as $c_1(\mathbf{L} \otimes \mathbf{K}_V) > 0$. If M is compact Kähler, then (1.6) implies that there are metrics in **L** and $\mathbf{K}_V$ such that (1.6) holds for the forms (1.4) representing the Chern classes of the bundles in question.

(1.7) LEMMA. *Let* $\mathbf{L} \to M$ *be a holomorphic line bundle having a metric. If* σ *is a holomorphic section of* $\mathbf{L} \to M$, *then outside the divisor of* σ *the differential form representing the Chern class is given by the formula*

$$c_1(\mathbf{L}) = -dd^c \log |\sigma|^2 .$$

Proof. Locally $|\sigma|^2 = |\sigma_\alpha|^2/a_\alpha$ so that $dd^c \log|\sigma|^2 = dd^c \log(1/a_\alpha)$. Now use (1.4).

Definition. $|\mathbf{L}|$ denotes the complete linear system of effective divisors D on M given by the zeroes of a holomorphic section of $\mathbf{L} \to M$.

(b) *Currents and forms in* $\mathbf{C}^n$. We shall work extensively on $\mathbf{C}^n$, and for this we use the notations:

$$(1.8)\quad \begin{cases} ||z||^2 = |z_1|^2 + \cdots + |z_n|^2 & \text{for } z = (z_1, \cdots, z_n) \in \mathbf{C}^n; \\ \varphi = dd^c||z||^2 = \dfrac{\sqrt{-1}}{2\pi}\{\sum_{j=}^n dz_j \wedge d\bar{z}_j\} ; & \\ \varphi_k = \underbrace{\varphi \wedge \cdots \wedge \varphi}_{k} ; & \\ \Phi = \varphi_n = \text{volume form on } \mathbf{C}^n ; & \\ \psi = dd^c \log ||z||^2 ; & \\ \psi_k = \underbrace{\psi \wedge \cdots \wedge \psi}_{k} ; & \\ \sigma = d^c \log ||z||^2 \wedge \psi_{n-1} ; & \\ B[r] = \{z \in \mathbf{C}^n : ||z|| \leqq r\} = \text{ball of radius } r \text{ in } \mathbf{C}^n ; & \\ S[r] = S \cap B[r] & \text{for any subset } S \subset \mathbf{C}^n . \end{cases}$$

(1.9) LEMMA. $\psi_n = 0$, $d\sigma = 0$, *and for any* $r > 0$

$$\int_{\partial B[r]} \sigma = 1 .$$

Proof. We consider the *Hopf fibration*

$$\pi: \partial B[r] \longrightarrow \mathbf{P}_{n-1}$$

of the $(2n-1)$ sphere $\partial B[r]$ over the projective space of lines through the origin in $\mathbf{C}^n$. It is easy to see that

$$\psi = \pi^*(\omega)$$

for a closed, C^∞ (1, 1) form ω on $\mathbf{P}_{n-1}$. This implies that $\psi_n = 0$, which in turn gives $d\sigma = 0$. By Stokes' theorem, the integral in (1.9) is independent of r.

To see explicitly what ω is, we observe first that all of the forms φ, ψ,

σ, ω are invariant under the unitary group acting on $\mathbf{C}^n$. Secondly, the computation given following (1.4) shows that the restriction of ω to the $\mathbf{P}_1$ given by $z_3 = \cdots = z_n = 0$ is the Chern class of the standard positive line bundle on $\mathbf{P}_1$. It follows that $\omega = c_1(\mathbf{H})$ for the standard positive line bundle $\mathbf{H} \to \mathbf{P}_{n-1}$. Thus

$$\int_{P_{n-1}} \omega_{n-1} = 1 \ .$$

To complete the proof of (1.9), we iterate the integral in question to obtain

$$\int_{\partial B[r]} \sigma = \int_{\xi \in \mathbf{P}_{n-1}} \left\{ \int_{\partial \xi[r]} d^c \log ||z||^2 \right\} \omega_{n-1}(\xi) \ .$$

By invariance under the unitary group, $\int_{\partial \xi[r]} d^c \log ||z||^2$ is a constant independent of ξ. To evaluate this constant, we take the line ξ_0 given by $z_2 = \cdots = z_n = 0$. Then

$$\int_{\partial \xi_0[r]} d^c \log ||z||^2 = \int_{|z_1|=r} d^c \log |z_1|^2 = 1$$

by (1.3). Q.E.D.

We shall use the language of *currents* (cf. Lelong [7]) on the complex manifold M. Let $A_c^{p,q}(M)$ be the C^∞ (p, q) forms on M having compact support. The space $\mathcal{C}^{p,q}(M)$ of currents of type (p, q) is the space of linear functions on $A_c^{n-p,n-q}(M)$ having the usual continuity properties. The operators ∂ and $\bar{\partial}$ may be defined on the space of currents by the formulae

$$\begin{cases} \partial T(\alpha) = -T(\partial \alpha) \\ \bar{\partial} T(\alpha) = -T(\bar{\partial} \alpha) \end{cases}$$

where $T \in \mathcal{C}^{p,q}(M)$ and $\alpha \in A_c^{n-p,n-q}(M)$. The main examples of currents we shall use are:

(i) Each $C^\infty(p, q)$ form η on M gives a current η in $\mathcal{C}^{p,q}(M)$ by the formula

$$\eta(\alpha) = \int_M \eta \wedge \alpha \qquad (\alpha \in A_c^{n-p,n-q}(M)) \ . \tag{1.10}$$

It follows from Stokes' theorem that ∂ in the sense of currents agrees with ∂ in the usual sense.

(ii) If θ is a differential form on M whose coefficients are locally L^1, then θ defines a current by the same formula (1.10). In most cases of interest, θ is C^∞ outside a certain subset S of M, and care must be taken to distinguish $\partial\theta$ in the sense of currents from $\partial\theta$ in the usual sense around S. We shall always use the notation $\partial\theta$ in the sense of currents. Thus

$$\partial\bar{\partial}|z|^{2\lambda} = \frac{\lambda^2 dz \wedge d\bar{z}}{|z|^{2-2\lambda}} \tag{1.11}$$

if $\lambda \geqq 0$, but not otherwise. On the other hand, for the forms ψ_k in (1.8), we have

$$\partial\psi_k = 0 = \bar{\partial}\psi_k$$

in both senses because the singularities of ψ_k are not too bad.

(iii) If D is a divisor on M, then D defines a current in $\mathcal{C}^{1,1}(M)$ by

$$D(\alpha) = \int_D \alpha$$

for $\alpha \in A_c^{n-1,n-1}(M)$ (cf. Lelong [7]). In this formula, multiplicities are computed using the rule $\int_{D_1+D_2} = \int_{D_1} + \int_{D_2}$.

(1.12) LEMMA. *Let* $\mathbf{L} \rightarrow M$ *be a complex line bundle having a metric, and let* δ *be a holomorphic section of* $\mathbf{L} \rightarrow M$ *having divisor* D. *Then* $\log|\delta|^2$ *is a locally* L^1 *function on* M *and defines a current which satisfies*

$$dd^c \log|\delta|^2 = D - c_1(\mathbf{L}) .$$

Proof. On $M - D$ our lemma reduces to Lemma (1.7). Thus we must show that, if f is holomorphic on an open set U in $\mathbf{C}^n$, then

$$dd^c \log|f|^2 = D \tag{1.13}$$

where D is the divisor $f = 0$.

It will suffice to prove (1.13) when f is irreducible. Then it is well known that we have an equation (1.13) up to signs and constants. To check these, we may verify the equation on $\mathbf{C}$

$$dd^c \log|z|^2 = \{0\}$$

by using (1.3). Q.E.D.

To state our final lemma, we suppose given an open set $U \subset \mathbf{C}^n$ and holomorphic functions $f, g \in \mathcal{O}(U)$ and a positive C^∞ function $a(z)$. Consider the volume form (with singularities)

$$\Psi = \frac{|f|^2\Phi}{|h|^2(\log|h|^2)^2}$$

where $h = \sqrt{a}\cdot g$ and Φ is the Euclidean volume form given in (1.8).

(1.14) LEMMA. *The Ricci form* Ric Φ *is locally* L^1 *in* U, *and we have the equation of currents*

$$dd^c \log\xi = D_f - D_g + \operatorname{Ric}\Phi .$$

Proof. Since $\log\xi = \log|f|^2 - \log|h|^2 - \log(\log|h|^2)^2$, it follows from

(1.12) that $\log \xi$ is L^1 and

$$dd^c \log \xi = D_f - D_g - dd^c \log a - dd^c \log (\log |h|^2)^2 .$$

From this it will suffice to show that $\partial\bar\partial \log (\log |h|^2)^2$ in the sense of currents is the same as $\partial\bar\partial \log (\log |h|^2)^2$ in the sense of differential forms. This is a question involving singularities of differential forms, and it is easily checked that it will suffice to verify the necessary equations in case $a = 1$. Then, in the sense of differential forms,

$$\text{(1.15)} \qquad \begin{aligned} \partial\bar\partial \log (\log |g|^2) &= d\left(\frac{1}{\log |g|^2} \frac{\overline{dg}}{g}\right) \\ &= -\frac{1}{|g|^2 \log |g|^2} dg \wedge \overline{dg} . \end{aligned}$$

Using (1.15) and Stokes' theorem, it will suffice to show that

$$\text{(1.16)} \qquad \lim_{\varepsilon\to 0} \int_{|g|=\varepsilon} \frac{\alpha \wedge \overline{dg}}{g \log |g|^2} = 0$$

for any compactly supported C^∞ form α. For this we may easily reduce to the case where g is irreducible. Then the Cauchy residue formula gives

$$\lim_{\varepsilon\to 0} \int_{|g|=\varepsilon} \alpha \wedge d(\arg g) = \int_{D_g} \alpha .$$

From this it follows that

$$\lim_{\varepsilon\to 0} \frac{1}{2\pi\sqrt{-1}} \int_{|g|=\varepsilon} \frac{\alpha \wedge \overline{dg}}{g \log |g|^2} = \lim_{\varepsilon\to 0} \frac{1}{\log \varepsilon^2} \int_{|g|=\varepsilon} \alpha \wedge d(\arg g) = 0 .$$

Finally, we shall use the following

Definition. Let M be a complex manifold and D a divisor on M. Then we shall say that D has *normal crossings* if locally D is the union of finitely many non-singular components given by the zeroes of distinct coordinate functions. If in addition each irreducible component of D is nonsingular, we say that D has *simple normal crossings.*

2. Construction of a volume form

Let V be a smooth, projective variety and $\mathbf{L} \to V$ a holomorphic line bundle.

(2.1) PROPOSITION. *Suppose that* $c_1(\mathbf{L}) + c_1(\mathbf{K}_V) > 0$ *and that* $D \in |\mathbf{L}|$ *is the union of nonsingular components* D_j, $j = 1, \cdots, l$, *such that* $D = D_1 + \cdots + D_l$ *has normal crossings. Then there is a volume form* Ψ *on* $V - D$ *such that*

$$(2.2)\qquad \begin{cases} \text{(i)} & \operatorname{Ric}\Psi > 0 \\ \text{(ii)} & \Psi \leqq (\operatorname{Ric}\Psi)^n \\ \text{(iii)} & \int_{V-D} (\operatorname{Ric}\Psi)^n < \infty \, . \end{cases}$$

Proof. Let Ω be a C^∞ volume form on V such that $\operatorname{Ric}(\Omega) = c_1(\mathbf{K}_V)$. Let $\delta_i \in \Gamma(V, [D_i])$ be a section defining D_i, so that $\delta = \delta_1 \cdots \delta_l \in \Gamma(V, \mathbf{L})$ defines D. Hence we have

$$(2.3)\qquad -dd^c \log|\delta|^2 = c_1(\mathbf{L}) \, .$$

Multiplying δ_j by a suitable constant, we may assume that $|\delta_j| < \varepsilon$ for any given $\varepsilon > 0$. We will show that for ε sufficiently small, the volume form

$$(2.4)\qquad \Psi = \frac{\Omega}{\prod_{j=1}^{l} |\delta_j|^2 (\log|\delta_j|^2)^2}$$

satisfies the requirements of the proposition. From (2.3) and (2.4) we have

$$(2.5)\qquad \operatorname{Ric}\Psi = c_1(\mathbf{L}) + c_1(\mathbf{K}_V) - \sum_{j=1}^{l} dd^c \log(\log|\delta_j|^2)^2 \, .$$

By hypothesis, the C^∞ (1, 1) form

$$\omega_1 = c_1(\mathbf{L}) + c_1(\mathbf{K}_V)$$

is positive on all of V. Now

$$(2.6)\qquad -dd^c \log(\log|\delta_j|^2) = -\frac{dd^c \log|\delta_j|^2}{\log|\delta_j|^2} + \frac{\sqrt{-1}}{2\pi} \frac{\partial \log|\delta_j|^2 \wedge \bar{\partial} \log|\delta_j|^2}{(\log|\delta_j|^2)^2} \, .$$

Since $dd^c \log|\delta_j|^2$ is C^∞ on V, we make ε very small and absorb $-(dd^c \log|\delta_j|^2/\log|\delta_j|^2)$ into ω_1 to obtain

$$(2.7)\qquad \operatorname{Ric}\Psi = \omega_2 + \frac{\sqrt{-1}}{2\pi} \sum_{j=1}^{l} \frac{\partial \log|\delta_j|^2 \wedge \bar{\partial} \log|\delta_j|^2}{(\log|\delta_j|^2)^2}$$

where ω_2 is positive and C^∞ on V. We now localize around $z_0 \in D$. Because D is assumed to have normal crossings, we may choose local holomorphic coordinates $z_1, \cdots, z_n$ such that D is given by

$$\zeta = z_1 \cdots z_k = 0 \, .$$

Thus we have the inequality

$$(2.8)\qquad \operatorname{Ric}\Psi \geqq \frac{\alpha\sqrt{-1}}{2\pi} \left\{ \sum_{j=1}^{n} dz_j \wedge \overline{dz_j} + \frac{\sqrt{-1}}{2\pi} \sum_{j=1}^{k} \frac{\partial \log|\delta_j|^2 \wedge \bar{\partial} \log|\delta_j|^2}{(\log|\delta_j|^2)^2} \right\}$$

for some $\alpha > 0$. By a change of scale we may assume $\alpha = 1$. Now $|\delta_j|^2 = a_j |z_j|^2$ for some positive C^∞ function a_j. We set

$$\begin{cases} \varphi_j = \partial \log a_j \\ \psi_j = \partial \log|z_j|^2 \, . \end{cases}$$

Thus the second term in the R.H.S. of (2.8) is

$$(2.9)\quad \frac{\sqrt{-1}}{2\pi}\left\{\sum_{j=1}^{k}\frac{\varphi_j\wedge\bar{\varphi}_j}{(\log|\delta_j|^2)^2}+\sum_{j=1}^{k}\frac{\varphi_j\wedge\bar{\psi}_j+\psi_j\wedge\bar{\varphi}_j}{(\log|\delta_j|^2)^2}+\sum_{j=1}^{k}\frac{\psi_j\wedge\bar{\psi}_j}{(\log|\delta_j|^2)^2}\right\}.$$

We absorb the first term of (2.9) as before to get

$$(2.10)\quad \operatorname{Ric}\Psi \geqq \frac{\sqrt{-1}}{2\pi}\left\{\sum_{j=1}^{n}dz_j\wedge\overline{dz}_j+\sum_{j=1}^{k}\frac{\varphi_j\wedge\bar{\psi}_j+\psi_j\wedge\bar{\varphi}_j}{(\log|\delta_j|^2)^2}+\sum_{j=1}^{k}\frac{\psi_j\wedge\bar{\psi}_j}{(\log|\delta_j|^2)^2}\right\}.$$

By the Cauchy-Schwarz inequality we have

$$\sqrt{-1}\frac{\varphi_j\wedge\bar{\psi}_j+\psi_j\wedge\bar{\varphi}_j}{(\log|\delta_j|^2)^2}\leqq\sqrt{-1}\left(\frac{\varphi_j\wedge\bar{\varphi}_j}{(\log|\delta_j|^2)^{2\beta}}+\frac{\psi_j\wedge\bar{\psi}_j}{(\log|\delta_j|^2)^{4-2\beta}}\right)$$

where $\beta>0$ is some small number. Choosing ε still smaller then gives

$$(2.11)\quad \operatorname{Ric}\Psi\geqq\gamma\frac{\sqrt{-1}}{2\pi}\left\{\sum_{j=1}^{n}dz_j\wedge d\bar{z}_j+\sum_{j=1}^{k}\frac{dz_j\wedge d\bar{z}_j}{|z_j|^2(\log|z_i|^2)^2}\right\},\quad \gamma>0.$$

From (2.11) we find

$$(2.12)\quad (\operatorname{Ric}\Psi)^n\leqq\frac{\gamma^n}{\prod_{j=1}^{k}|z_j|^2(\log|z_j|^2)^2}\left\{\prod_{j=1}^{n}\frac{\sqrt{-1}}{2\pi}dz_j\wedge\overline{dz}_j\right\}.$$

This estimate, combined with the compactness of V, shows that there is a positive constant c such that $(\operatorname{Ric}\Psi)^n\geqq c\Psi$. Since $\operatorname{Ric}(c\Psi)=\operatorname{Ric}\Psi$, we may make a change of scale to get $(\operatorname{Ric}\Psi)^n\geqq\operatorname{Ric}\Psi$.

To verify that integral of $(\operatorname{Ric}\Psi)^n$ is finite, we have locally the easy upper bound

$$(2.13)\quad (\operatorname{Ric}\Psi)^n\leqq c'\prod_{j=1}^{k}\left[\frac{\sqrt{-1}}{2}\frac{dz_j\wedge\overline{dz}_j}{|z_j|^2(\log|z_j|^2)^2}\right]\prod_{j=k+1}^{n}\left[\frac{\sqrt{-1}}{2}dz_j\wedge\overline{dz}_j\right]$$

for some $c'>0$. Finiteness now follows from the convergence of

$$\int_0^{1/2}\frac{dr}{r(\log r^2)^2}<\infty.\qquad\text{Q.E.D.}$$

Remark. Recall that the Poincaré volume

$$\Phi=\gamma\frac{\prod_{j=1}^{n}\frac{\sqrt{-1}}{2}(dz_j\wedge\overline{dz}_j)}{\prod_{j=1}^{n}(1-|z_j|^2)^2}$$

on the polycylinder $P=\{z\mid |z_j|<1\}$ satisfies $(\operatorname{Ric}\Phi)^n=\Phi$. Now Φ descends to the punctured polycylinder $P^*=\{z\mid 0<|z_1|<1,\ |z_j|<1\}$ to give a volume form

$$\Psi_0 = \gamma \frac{\frac{\sqrt{-1}}{2}(dz_1 \wedge \overline{dz}_1)}{(\log |z_1|^2)^2 |z_1|^2} \cdot \frac{\prod_{j=2}^n \frac{\sqrt{-1}}{2}(dz_j \wedge \overline{dz}_j)}{\prod_{j=2}^n (1 - |z_j|^2)}$$

such that $(\mathrm{Ric}\,\Psi_0)^n = \Psi_0$. The volume form Ψ constructed above is a global version of the Poincaré volume on P^*.

For $\dim V = 1$ our proposition reduces to

COROLLARY. *Let V be a compact Riemann surface of genus g. Then, if $N > 2 - 2g$, there exists on $V - \{x_1, \cdots, x_N\}$ a complete metric ds^2 whose Gaussian curvature is everywhere $\leqq -1$.*

Of course, this corollary was given classically as a consequence of the uniformization theorem. However, the above proof is an elementary algebro-geometric character.

COROLLARY. *Let $D_1, \cdots, D_N$ be smooth divisors in $\mathbf{P}_n$ such that $D = D_1 + \cdots + D_N$ has simple normal crossings and $\deg(D) \geqq n + 2$. Then there is a volume form Ψ on $\mathbf{P}_n - D$ such that*

(i) $\mathrm{Ric}\,\Psi > 0$

(ii) $\Psi \leqq (\mathrm{Ric}\,\Psi)^n$

(iii) $\int_{\mathbf{P}_n - D} (\mathrm{Ric}\,\Psi)^n < \infty$.

We remark that if the D_i are hyperplanes, then the requirement that D have simple normal crossings is the same as requiring the D_i to be in general position.

3. A second main theorem for non-degenerate holomorphic maps

Let V be a smooth projective variety, $\mathbf{L} \to V$ a holomorphic line bundle such that

$$c_1(\mathbf{L}) + c_1(\mathbf{K}_V) > 0 ,$$

and $D \in |\mathbf{L}|$ a divisor with simple normal crossings. Let Ψ be the volume form constructed in Proposition 2.1 and given explicitly by (2.3).

We consider a non-degenerate holomorphic mapping $f: B[R] \to V$. The notation $(*)_f$ will be used for the pullback under f of an object $(*)$ on V. Thus $D_f = f^{-1}(D)$ and $\Psi_f = f^*(\Psi)$. We remark that $\mathrm{Ric}\,\Psi_f = (\mathrm{Ric}\,\Psi)_f$. We write

$$\Psi_f = \xi \cdot \Phi$$

where Φ is the Euclidean volume form and

(a) $\xi \geqq 0$

(b) $\xi \in L^1_{\mathrm{loc}}(\mathbf{C}^n)$

(c) ξ is C^∞ on $\mathbf{C}^n - D_f$

(d) $\xi = 0$ on R_f .

Applying Lemma (1.14) to Ψ_f, we see that Ric Ψ_f is locally L^1 and that the following equation of currents holds

$$dd^c \log \xi = R - D_f + \operatorname{Ric} \Psi_f \,. \tag{3.2}$$

To state our second main theorem (S.M.T.), we shall use the following notations (cf. (1.8)):

$$\left\{\begin{aligned} T_1(r) &= \int_0^r \left(\int_{B[t]} (\operatorname{Ric} \Psi)_f \wedge \varphi_{n-1}\right)\frac{dt}{t^{2n-1}} \\ N(D, r) &= \int_0^r \left(\int_{D_f[t]} \psi_{n-1}\right)\frac{dt}{t} \quad (\textit{counting function}) \\ N_1(r) &= \int_0^r \left(\int_{R[t]} \psi_{n-1}\right)\frac{dt}{t} \quad (\textit{ramification term}) \\ \mu(r) &= \int_{\partial B[r]} \log \xi \cdot \sigma \,. \end{aligned}\right. \tag{3.3}$$

(3.4) PROPOSITION. *The following equality holds*:

$$T_1(r) + N_1(r) = N(D, r) + \mu(r) + O(1) \tag{S.M.T.}$$

where $O(1)$ *is a bounded term depending on* D *but not on* r.

Proof. We may assume that D_f and R_f do not pass through the origin. The L^1 function $\log \xi$ defines a current by the usual formula

$$\log \xi(\alpha) = \int_{B[R]} \log \xi \cdot \alpha \cdot \Phi$$

where α is a compactly supported C^∞ function on $B[R]$. From (3.2) we have

$$dd^c \log \xi = R - D_f + \operatorname{Ric} \Psi_f \,. \tag{3.5}$$

The (S.M.T.) (3.4) will be the twice integrated form of (3.5).

The currents R, D_f, and Ric Ψ_f are all closed and positive. As usual when one has positive linear functionals on the C^∞ functions, their domain of definition may be extended to suitable L^1-forms. In particular, we may apply the right-hand side of (3.5) to the form ψ_{n-1} (cf. (1.8)), and using $d\psi_{n-1} = 0$ together with Stokes' theorem we obtain the equation

$$\int_{\partial B[t]} d^c \log \xi \wedge \psi_{n-1} = \int_{R[t]} \psi_{n-1} - \int_{D_f[t]} \psi_{n-1} + \int \operatorname{Ric} \Psi_f \wedge \psi_{n-1} \,. \tag{3.6}$$

Standard estimates show that $\operatorname{Ric} \Psi_f \wedge \psi_{n-1}$ is L^1. In all cases below convergence of improper integrals is guaranteed by the integrable nature of the singularities of $dx/(\log x)^2 x$ at $x = 0$. Convergence estimates of this type are treated in Stoll's work on value distribution theory (see [10]).

In order to integrate (3.6) once more, we shall use the following two lemmas:

(3.7) LEMMA. *Referring to the notations* (1.10), *we have*

$$d^c \log \xi \wedge \psi_{n-1}\Big|_{\partial B[t]} = t\frac{\partial \log \xi \cdot \sigma}{\partial t}\Big|_{\partial B[t]},$$

where the meaning of the operator "$t(\partial/\partial t)$" will be explained in the proof.

(3.8) LEMMA. *Let η be a closed C^∞ (1, 1) form on $B[R]$. Then we have*

$$\frac{1}{t^{2n-2}}\int_{B[t]}\eta \wedge \varphi_{n-1} = \int_{B[t]}\eta \wedge \psi_{n-1}\,.$$

Assuming these lemmas, we may rewrite (3.6) in the form

$$\text{(3.9)}\qquad \frac{1}{t^{2n-2}}\int_{B[t]}\operatorname{Ric}\Psi_f \wedge \varphi_{n-1} + \int_{R[t]}\psi_{n-1} = \int_{D_f[t]}\psi_{n-1} + t\frac{\partial}{\partial t}\int_{\partial B[t]}\log \xi\cdot\sigma$$

for almost all $t \in \mathbf{R}^+$. Integration of (3.9) with respect to $(dt)/t$ gives

$$\text{(3.10)}\qquad T_1(r) + N_1(r) = N(D, r) + \mu(r)\,.$$

Proof of (3.7). First we must explain what the notation "$t(\partial\alpha/\partial t)$" means for a C^∞ function α. Let θ be the exterior normal vector of unit length to the spheres $\partial B[t]$. Then θ is a C^∞ vector field on $\mathbf{C}^n - \{0\}$ and $t(\partial\alpha/\partial t) = \theta\cdot\alpha$.

Now consider the Hopf fibration $\pi\colon \partial B[t] \to \mathbf{P}_{n-1}$ (cf. the proof of Lemma (1.9)). Since $\sigma = d^c \log ||z|| \wedge \psi_{n-1}$ and $\psi_{n-1} = \pi^*(\omega_{n-1})$ where ω is the standard (1, 1) form on $\mathbf{P}_{n-1}$, it will suffice to check that the restrictions of $\theta\cdot\alpha d^c \log ||z||$ and $d^c\alpha$ to each line λ through the origin are the same on $\lambda \cap \partial B[t] = \partial\lambda[t]$. Using polar coordinates $re^{i\theta}$ on the line λ, we must show that

$$\text{(3.11)}\qquad \left(r\frac{\partial\alpha}{\partial r}\right)d^c \log r \equiv d^c\alpha \ \ (dr)\,.$$

This equation is straightforward to verify. Q.E.D.

Proof of (3.8). Writing $\eta = d\xi$ for a C^∞ 1-form ξ and using Stokes' theorem, it will suffice to verify that

$$\text{(3.12)}\qquad \frac{1}{t^{2n-2}}\varphi_{n-1}|_{\partial B[t]} = \psi_{n-1}|_{\partial B[t]}\,.$$

From (1.8) we see that (3.12) will follow by taking exterior powers of the congruence

$$\text{(3.13)}\qquad \varphi \equiv ||z||^2\psi \text{ modulo } (d||z||)\,.$$

Since $\varphi = dd^c||z||^2$ and $\psi = dd^c \log ||z||$ we immediately obtain (3.13).

Q.E.D.

4. The defect relation (preliminary form)

Throughout this section V will be a smooth, projective variety, $\mathbf{L} \to V$ a holomorphic line bundle, and $D \in |\mathbf{L}|$ a divisor with simple normal crossings. We shall also use the notations (3.3). We now derive a basic estimate which, combined with the first main theorem in the next section, yields the desired defect relations.

(4.1) PROPOSITION. *Let* $f: \mathbf{C}^n \to V$ *be a non-degenerate holomorphic mapping and assume that* $c_1(\mathbf{L}) + c_1(\mathbf{K}_V) > 0$. *Then there exists a positive function* $T_1(r)$ *with* $\lim_{r\to\infty} T_1(r) = \infty$ *such that*

$$\limsup_{r\to\infty}\left[\frac{N(D, r)}{T_1(r)}\right] \geqq 1 . \tag{4.2}$$

Proof. We define the function

$$T^\sharp(r) = \int_0^r\left(\int_{B[t]} n\xi^{1/n}\Phi\right)\frac{dt}{t^{2n-1}} \tag{4.3}$$

where ξ is given by (3.1). From the Hadamard inequality

$$n(\det H)^{1/n} \leqq \operatorname{Trace} H$$

for an Hermitian positive (semi-definite) matrix, we obtain from $(\mathrm{Ric}\Psi)^n \geqq \Psi$ that

$$T^\sharp(r) \leqq T_1(r) . \tag{4.4}$$

(4.5) LEMMA. *Referring to the* (S.M.T.) (3.4), *we have*

$$\mu(r) + n(4n-2)\log r \leqq \log\left\{\frac{d^2T^\sharp(r)}{ds^2}\right\}$$

where $d/ds = r^{2n-1}d/dr$.

Proof. Using $\log\xi = n\log\xi^{1/n}$, Lemma (1.9), and the concavity of the logarithm ([8], page 251), we have

$$\mu(r) \leqq n\log\left(\int_{\partial B[r]}\xi^{1/n}\sigma\right) . \tag{4.6}$$

Now we use the equations

$$\begin{cases} \displaystyle\int_{\partial B[r]} n\xi^{1/n}\cdot\sigma = \frac{1}{t^{2n-1}}\frac{d}{dr}\left(\int_{B[r]} n\xi^{1/n}\Phi\right) ; \\ \displaystyle\int_{B[r]} n\xi^{1/n}\Phi = r^{2n-1}\frac{dT^\sharp(r)}{dr} \end{cases}$$

together with (4.6) to obtain

$$\mu(r) \leqq -n(2n-1)\log r + \log\left(\frac{d}{dr}\left\{r^{2n-1}\frac{dT^\sharp(r)}{dr}\right\}\right) . \tag{4.7}$$

Our lemma follows from (4.7).

Combining (4.5) with the (S.M.T.) (3.4) gives

$$(4.8)\qquad T_1(r) + N_1(r) + n(4n-2)\log r \leqq N(D, r) + \log\left\{\frac{d^2T^\sharp(r)}{ds^2}\right\} + O(1)\ .$$

In order to convert the inequality (4.8) into a good form, we must eliminate the derivatives in front of $T^\sharp(r)$. For this we use the following lemma from [8], page 253:

(4.9) LEMMA. *Suppose that $f(r)$, $g(r)$, $\alpha(r)$ are positive increasing functions of r where $g'(r)$ is continuous and $f'(r)$ is piecewise continuous. Suppose moreover that $\int^\infty dr/\alpha(r) < \infty$. Then*

$$(4.10)\qquad f'(r) \leqq g'(r)\alpha(f(r))$$

holds except on an open set $I \subset R^+$ such that

$$\int_I dg \leqq \int^\infty \frac{dr}{\alpha(r)}\ .$$

Remark. Following Weyl we shall use the notation

$$a(r) \leqq b(r) \qquad \|_g$$

to mean that the inequality holds except on a set I such that $\int_I dg < \infty$.

Taking $f(r) = T^\sharp(r)$, $g(r) = r^\mu/\mu$, $\alpha(r) = r^\nu$ with $\mu > 1$, $\nu > 1$ we obtain from (4.10) that

$$(4.11)\qquad \frac{dT^\sharp}{dr} \leqq r^{\mu-1}(T^\sharp)^\nu\ . \qquad \|_g$$

Keeping the same g and α and taking $f = r^{2n-1}dT^\sharp/dr$ we find

$$(4.12)\qquad \frac{d}{dr}\left(r^{2n-1}\frac{dT^\sharp}{dr}\right) \leqq r^{\mu-1}r^{(2n-1)\nu}\left(\frac{dT^\sharp}{dr}\right)^\nu\ . \qquad \|_g$$

Combining (4.11) and (4.12) we obtain

$$(4.13)\qquad \frac{d^2T^\sharp}{ds^2} \leqq r^{4n-2+\varepsilon}(T^\sharp)^{2+\delta} \qquad \|_g$$

where $\delta > 0$ and $\varepsilon > 0$ can be made arbitrarily small by choosing ν and μ close to 1. Putting (4.13) and (4.8) together gives

$$(4.14)\qquad T_1 + N_1 \leqq N + \varepsilon\log r + (2+\delta)\log T^\sharp + O(1)\ . \qquad \|_g$$

(4.15) LEMMA. *$T_1(r) \geqq c\log r$ for some $c > 0$.*

Proof.

$$\begin{aligned} T_1(r) &= \int_0^r\left(\int_{B[t]}\mathrm{Ric}\,\Psi_f \wedge \varphi_{n-1}\right)\frac{dt}{t^{2n-1}} \\ &= \int_0^r\left(\int_{B[t]}\mathrm{Ric}\,\Psi_f \wedge \psi_{n-1}\right)\frac{dt}{t} \\ &\geqq c\log r \end{aligned}$$

where $c = \int_{B[1]} \mathrm{Ric}\, \Psi_f \wedge \psi_{n-1} > 0$. Q.E.D.

Dividing (4.14) by $T_1(r)$ and using (4.4) and (4.15) gives the inequality

$$1 + \frac{N_1}{T_1} \leqq \left(\frac{N}{T_1}\right) + \frac{n\varepsilon}{c} + (2 + \delta)\frac{\log T_1}{T_1} + \frac{O(1)}{T_1} . \qquad \|_g$$

Letting: $r \to \infty$ we obtain

$$1 + \liminf_{r\to\infty}\left[\frac{N_1(r)}{T_1(r)}\right] \leqq \limsup_{r\to\infty}\left[\frac{N(D, r)}{T_1(r)}\right] + \frac{n\varepsilon}{c} .$$

Finally letting $\varepsilon \to 0$ we have

$$(4.16) \qquad 1 + \liminf_{r\to\infty}\left[\frac{N_1(r)}{T_1(r)}\right] \leqq \limsup_{r\to\infty}\left[\frac{N(D, r)}{T_1(r)}\right] .$$

Our proposition follows from this. Q.E.D.

(4.17) COROLLARY. *Let V and D be as above. Then any holomorphic mapping $f: \mathbf{C}^n \to V - D$ is degenerate.*

Proof. $N(D, r) \equiv 0$ if $f(\mathbf{C}^n)$ does not meet D.

(4.18) COROLLARY. *Let $H_1, \cdots, H_{n+2}$ be hyperplanes in general position in $\mathbf{P}_n$. Then any mapping $f: \mathbf{C}^n \to \mathbf{P}_n - (H_1 \cup \cdots \cup H_{n+2})$ is degenerate.*

Remark. If one is only interested in these corollaries, then direct arguments may be obtained either from Proposition (2.1) or from (4.8). These will be given in section 6 below.

5. The first main theorem and defect relations

Let V be a smooth projective variety, $\mathbf{L} \to V$ a holomorphic line bundle, and ω a (1, 1) form which represents $c_1(\mathbf{L})$ in the sense of section 1. If ω' is another such form, then from the remarks below (1.4)

$$(5.1) \qquad \omega - \omega' = dd^c\rho$$

for some C^∞ function ρ on V.

Let $f: \mathbf{C}^n \to V$ be a holomorphic mapping. For a while it is not necessary that f be non-degenerate.

Definition. The *order function* $T(\mathbf{L}, r)$ for f relative to the line bundle $\mathbf{L} \to V$ is given by

$$(5.2) \qquad T(\mathbf{L}, r) = \int_0^r \left(\int_{B[t]} \omega_f \wedge \varphi_{n-1}\right)\frac{dt}{t^{2n-1}} .$$

(5.3) LEMMA. *$T(\mathbf{L}, r)$ is intrinsically defined up to a constant $O(1)$.*

Proof. Referring to (5.1) and (5.2), it will suffice to show that

$$(5.4) \qquad \int_0^r \left(\int_{B[t]} dd^c \rho_f \wedge \varphi_{n-1} \right) \frac{dt}{t^{2n-1}} = O(1) .$$

Applying Stokes' theorem, the integral is

$$\begin{aligned} &\int_0^r \left(\int_{\partial B[t]} d^c \rho_f \wedge \varphi_{n-1} \right) \frac{dt}{t^{2n-1}} \\ &= \int_0^r \left(\int_{\partial B[t]} d^c \rho_f \wedge \psi_{n-1} \right) \frac{dt}{t} && \text{(by (3.8))} \\ &= \int_0^r \left(\frac{\partial}{\partial t} \int_{\partial B[t]} \rho_f \sigma \right) dt && \text{(by (3.7))} \\ &= \int_{\partial B[r]} \rho_f \cdot \sigma = O(1) . && \text{Q.E.D.} \end{aligned}$$

Remark. The same proof applies when ρ in (5.1) is C^∞ on $V - D$ and where ρ, $d^c\rho$, and $dd^c\rho$ are also L^1 on V. Referring to (1.14), (2.3), and (3.3) we have

$$(5.5) \qquad T_1(r) = T(\mathbf{L}, r) + T(\mathbf{K}_V, r) + \int_{\partial B[r]} \log (\log |\delta|^2)^2 \cdot \sigma .$$

To give the first main theorem (F.M.T.), we let $\delta \in H^0(V, \mathcal{O}(\mathbf{L}))$ and define the proximity function

$$m(\delta, r) = \int_{\partial B[r]} \log \frac{1}{|\delta_f|^2} \cdot \sigma .$$

Observe that if the section δ has no zeroes, then $m(\delta, r) = O(1)$. Moreover, multiplying δ by a non-zero constant changes $m(\delta, r)$ by a constant, so that the proximity function essentially depends only on the divisor D of δ. Assuming that $|\delta| \leqq 1$, we set

$$(5.6) \qquad m(D, r) = \int_{\partial B[r]} \log \frac{1}{|\delta_f|^2} \cdot \sigma \geqq 0 \qquad (\textit{proximity function}) .$$

(5.7) PROPOSITION. *Let $D \in |\mathbf{L}|$ and $f: \mathbf{C}^n \to V$ be a holomorphic mapping such that all components of D_f are divisors. Then we have*

$$(\text{F.M.T.}) \qquad N(D, r) + m(D, r) = T(\mathbf{L}, r) + O(1) .$$

Proof. Referring to (1.7) and (1.8), we have the equation of currents

$$(5.8) \qquad dd^c \log |\delta_f|^2 = D_f - \omega_f .$$

Following the proof of (3.4), we assume that D_f does not pass through the origin and then integrate (5.8) once to obtain

$$(5.9) \qquad \int_{B[t]} d^c \log |\delta_f|^2 \wedge \psi_{n-1} = \int_{D_f[t]} \psi_{n-1} - \int_{B[t]} \omega_f \wedge \psi_{n-1} .$$

Using (3.7) and (3.8) we may integrate (5.9) once more to obtain

$$\int_{\partial B[r]} \log |\delta_f|^2 \sigma = N(D, r) - T(\mathbf{L}, r) + O(1) .$$

Our proposition now follows from this. Q.E.D.

(5.10) COROLLARY. *Assume* $c_1(\mathbf{L}) > 0$. *Then we have the Nevanlinna inequality* (*cf.* [8], *page* 175)

$$N(D, r) < T(\mathbf{L}, r) + O(1)$$

for all $D \in |\mathbf{L}|$.

Remark. This remarkable inequality says that the growth of the area of $f^{-1}(D)$ is bounded for *every* $D \in |\mathbf{L}|$ by the order function $T(\mathbf{L}, r)$. To explain this a little further, we assume that the complete linear system $|\mathbf{L}|$ has no base points and therefore gives a holomorphic mapping

$$\chi_L : V \longrightarrow \mathbf{P}_N \tag{5.11}$$

such that $\chi_L^*(\mathbf{H}) = \mathbf{L}$ where $\mathbf{H} \to \mathbf{P}_N$ is the standard positive line bundle. The divisors $D \in |\mathbf{L}|$ are all hyperplane sections, and therefore we may consider D as a point of the dual projective space $\mathbf{P}_N^*$. Taking $\omega = \chi_L^*(\omega_{P_N})$ where ω_{P_N} is the usual Kähler form on $\mathbf{P}_N$, and denoting by $d\mu(D)$ the standard measure of total volume 1 on $\mathbf{P}_N^*$, we have the equality (cf. [8], page 131)

$$T(\mathbf{L}, r) = \int_{D \in P^*_N} N(D, r) d\mu(D) . \tag{5.12}$$

In order words, $T(\mathbf{L}, r)$ is the *average growth* of the area of $D_f[r]$, and the Nevanlinna inequality says that $N(D, r)$ is bounded for all D by its average value.

Definition. For $D \in |\mathbf{L}|$ we define the *deficiency*

$$\delta(D) = 1 - \limsup_{r\to\infty} \left[\frac{N(D, r)}{T(\mathbf{L}, r)}\right] . \tag{5.13}$$

Remarks. It follows from (5.10) that

$$0 \leqq \delta(D) \leqq 1 ,$$

and $\delta(D) = 1$ if $f(\mathbf{C}^n)$ does not meet the divisor D. From (5.12) we have

$$0 = \int_{D \in P^*_N} \delta(D) d\mu(D) , \tag{5.14}$$

so that $\delta(D) = 0$ for almost all $D \in |\mathbf{L}|$ (Liouville theorem). The defect relation which we shall prove is a very strong quantitative version of this observation.

Before giving the defect relations, we need an auxiliary estimate similar to Lemma (5.3).

(5.15) LEMMA. *Let V be a smooth projective variety, $\mathbf{L} \to V$ a positive line bundle and $D_1, \cdots, D_k \in |\mathbf{L}|$ divisors such that $D = D_1 + \cdots + D_k \in |\mathbf{L}^k|$ has simple normal crossings. Then there are constants c_1, c_2, c_3 such that*

$$\begin{aligned} 0 &\leqq [kT(\mathbf{L}, r) + T(\mathbf{K}_V, r)] - T_1(r) \\ &\leqq c_1 \log \bigl(kT(\mathbf{L}, r) + c_2\bigr) + c_3 . \end{aligned}$$

Proof. Writing each D_j as a union of smooth components E_i, we have $D = E_1 + \cdots + E_p$. Thus each E_i is defined by a section δ_i of the line bundle $[E_i]$. Using (2.5) we see that

$$(5.16) \quad [T(\mathbf{L}^k, r) + T(\mathbf{K}_V, r)] - T_1(r) = \sum_{i=1}^p \int_0^r \Bigl(\int_{B[t]} dd^c \log (\log |\delta_i|^2)^2 \psi_{n-1}\Bigr)\frac{dt}{t} .$$

Following the same reasoning as in the proof of (5.3), we have

$$(5.17) \qquad [T(\mathbf{L}^k, r) + T(\mathbf{K}_V, r)] - T_1(r) = \sum_{i=1}^p \int_{\partial B[r]} \log (\log |\delta_i|^2)^2 \sigma .$$

We may choose $|\delta_i|^2 < \varepsilon$, where ε is so small that $\log (\log |\delta_i|^2)^2 \geqq 0$. This gives the first inequality above. To obtain the second, we apply concavity of the logarithm to get

$$\sum_{i=1}^n \int_{\partial B[r]} \log (\log |\delta_i|^2)^2 \sigma \leqq \sum_{i=1}^p \log \int_{\partial B[s]} - \log |\delta_i|^2 \sigma = \sum_{i=}^p 2 \log m_i(r) .$$

From the first main theorem (5.7) we conclude that

$$(5.18) \qquad \sum_{i=1}^p \int_{\partial B[r]} \log (\log |\delta_i|^2)^2 \sigma \leqq 2 \sum_{i=1}^p \log \bigl(T([E_i], r) + O(1)\bigr) .$$

Using (5.3) we find

$$(5.19) \qquad \begin{aligned} T\bigl([E_i], r\bigr) &\leqq T(\mathbf{L}^k, r) + O(1) \\ kT(\mathbf{L}, r) &= T(\mathbf{L}^k, r) + O(1) . \end{aligned}$$

Applying (5.19) to (5.17) and (5.18) yields the desired inequality. Q.E.D.

(5.20) MAIN THEOREM (*defect relation*). *Let V be a smooth projective variety, $\mathbf{L} \to V$ a positive line bundle, and $D_1, \cdots, D_k \in |\mathbf{L}|$ divisors such that $D = D_1 + \cdots + D_k \in |\mathbf{L}^k|$ has simple normal crossings and the following inequality of Chern classes holds:*

$$k > \Bigl[\frac{c_1(\mathbf{K}_V^*)}{c_1(\mathbf{L})}\Bigr] .$$

Then we have

$$(\text{D.R.}) \qquad \sum_{j=1}^k \delta(D_j) \leqq \Bigl[\frac{c_1(\mathbf{K}_V^*)}{c_1(\mathbf{L})}\Bigr] .$$

Proof. Let $l > 0$ be such that

$$(*) \qquad (k - l)c_1(\mathbf{L}) + c_1(\mathbf{K}_V) \geqq 0 \ .$$

Thus $k - l \geqq [c_1(\mathbf{K}_V^*)/c_1(\mathbf{L})]$. Using the notation $\overline{\lim}$ for lim sup, we have

$$\begin{aligned}
\sum_{j=1}^{k} \delta(D_j) &= \sum_{j=1}^{k} \left\{1 - \overline{\lim}_{r\to\infty} \frac{N(D_j, r)}{T(\mathbf{L}, r)}\right\} \\
&\leqq k - \overline{\lim}_{r\to\infty} \left\{\frac{N(D, r)}{T(\mathbf{L}, r)}\right\} \\
&\leqq k - l\cdot\overline{\lim}_{r\to\infty} \left\{\frac{N(D, r)}{lT(\mathbf{L}, r)}\right\} \\
&\leqq k - l\,\overline{\lim}_{r\to\infty} \left\{\frac{N(D, r)}{T(\mathbf{L}^k, r) + T(\mathbf{K}_V)}\right\} && \text{(using } (*)) \\
&\leqq k - l\,\overline{\lim}_{r\to\infty} \left\{\frac{N(D, r)}{T_1(r)}\right\} && \text{(by (5.15))} \\
&\leqq k - l && \text{(by (4.2))} \ .
\end{aligned}$$

Since l is any positive number subject only to $(*)$, we have in conclusion that

$$\sum_{j=1}^{k} \delta(D_j) \leqq \left[\frac{c_1(\mathbf{K}_V^*)}{c_1(\mathbf{L})}\right] . \qquad \text{Q.E.D.}$$

(5.21) COROLLARY. *Let $f: \mathbf{C}^n \to \mathbf{P}_n$ be a non-degenerate holomorphic mapping and let $H_1, \cdots, H_{n+m}$ be hyperplanes in general position in $\mathbf{P}_n$. Then*

$$\sum \delta(D_j) \leqq n + 1 \ .$$

For $n = 1$ this is the famous defect relation of R. Nevanlinna [8], page 246.

(5.22) COROLLARY. *Let V be a smooth projective variety, $\mathbf{L} \to V$ a positive line bundle such that $\mathbf{K}_V = \mathbf{L}^{-m}$ for some m, and $D_1, \cdots, D_k \in |\mathbf{L}|$ divisors such that $D_1 + \cdots + D_k$ has simple normal crossings. Then if $k > m$ we have*

$$\sum \delta(D_j) \leqq m \ .$$

For example, if V_n is a smooth hypersurface of degree d in $\mathbf{P}_{n+1}$ and $\mathbf{L}$ is the hyperplane bundle, then $m = (n + 2) - d$.

Remark. If $m > 0$ so that the canonical bundle of V is negative (e.g., $V = \mathbf{P}_n$), then this defect relation has the same quantitative effect as (5.21). If $m = 0$, then we obtain $\delta(D_j) = 0$, so that no divisor is deficient. Finally, if $m < 0$ so that $\mathbf{K}_V$ is positive, then we find that f must be degenerate (cf. (4.17) above). Thus, as $\mathbf{K}_V$ becomes less negative, the mapping f becomes increasingly more rigid. As an example, let V be a smooth hypersurface of degree 4 in $\mathbf{P}_3$, i.e., a $K - 3$ surface. Let $\mathbf{L} \to V$ be the hyperplane bundle and let $D \in |\mathbf{L}|^k$ be a divisor with simple normal crossings. Then D is never deficient.

6. Variants and applications

(a) *Schottky-Landau theorems.* Let V be a smooth, projective variety, $\mathbf{L} \to V$ a holomorphic line bundle such that $c_1(\mathbf{L}) + c_1(\mathbf{K}_V) > 0$, and $D \in |\mathbf{L}|$ a divisor with simple normal crossings. Fixing a point $x_0 \in V - D$ and a metric in the tangent space $T_{x_0}(V - D)$, we let $\mathcal{F}$ be the set of all holomorphic mappings

$$f\colon B[R] \longrightarrow V - D \tag{6.1}$$

such that $f(0) = x_0$ and $|\det\{f(x_0)\}| \geqq 1$. For each $f \in \mathcal{F}$ we denote by $R(f)$ the maximal radius such that f is defined on the ball of radius R for all $R < R(f)$. Our Schottky-Landau theorem is

(6.2) PROPOSITION. *For each $f \in \mathcal{F}$ we have*

$$R(f) \leqq R_0$$

where R_0 depends only on V, D, and x_0.

First Proof. In this proof we shall only use (ii) of Proposition (2.1), together with the Schwarz Lemma from §2 of [4]. We consider the Poincaré-Bergmann metric $ds^2_{B[R]}$ on $B[R]$. If $\Theta[R]$ is the associated volume form, then it is well known that

$$(\operatorname{Ric}\Theta[R])^n = c_R\Theta[R] \tag{6.3}$$

for some positive constant c_R. We normalize $\Theta[R]$ so that $c_R = 1$ in (6.3) for all R. When this is done we have

$$\Theta[R](0) = \frac{c'}{R^{2n}}\prod_{j=1}^{n}\left\{\frac{\sqrt{-1}}{2\pi}dz_j \wedge d\bar{z}_j\right\} \qquad (c' > 0)\,. \tag{6.4}$$

(This is easily verified by considering the biholomorphic mapping $B[1] \to B[R]$ given by $z \to Rz$.)

Now let $f\colon B[R] \to V - D$ be normalized as above. From (2.1), (6.3), and Proposition 2.7 in [4] it follows that

$$f^*\Psi \leqq \Theta[R]\,. \tag{6.5}$$

On the other hand, since f is normalized, there is an inequality

$$c'' \prod_{j=1}^{n}\left\{\frac{\sqrt{-1}}{2\pi}dz_j \wedge d\bar{z}_j\right\} \leqq f^*\Psi(0) \tag{6.6}$$

for some $c'' > 0$. Combining (6.4), (6.5), and (6.6) gives

$$R^{2n} \leqq c'/c''\,,$$

from which we obtain $R \leqq \sqrt[2n]{c'/c''}$. Q.E.D.

Second proof. This follows the arguments given in Carlson [1] and Kodaira [6]. Using (4.3), (4.4), (4.5), and (3.4) we have

$$T^{\#}(r) + (4n - 2)\log r \leqq \log\left\{\frac{d^2 T^{\#}(r)}{ds^2}\right\}. \tag{6.7}$$

From the normalization conditions we obtain, as in the proof of (4.15), that

$$T^{\#}(r) \geqq c \log r \tag{6.8}$$

for some constant $c > 0$. Combining (6.7) and (4.13) we find

$$T^{\#}(r) \leqq \varepsilon \log r + c_1 \log\{T^{\#}(r)\}. \qquad \|_g \tag{6.9}$$

Putting together (6.8) and (6.9) gives

$$\begin{gathered}\frac{\log\{T^{\#}(r)\}}{T^{\#}(r)} \geqq c_1 > 0 \qquad \|_g \\ T^{\#}(r) \geqq c \log r.\end{gathered} \tag{6.10}$$

If we assume that the calculus lemma introduces no exceptional intervals so that (6.9) holds everywhere, then the two equations in (6.10) yield

$$\log r \leqq c_2,$$

from which it follows that $R(f) \leqq e^{c_2}$.

In general there will be exceptional intervals and a more refined estimate is needed (see [1], [6]). Q.E.D.

(6.11) COROLLARY. *Under the above assumptions on V and D, any holomorphic mapping $f\colon \mathbf{C}^n \to V - D$ is degenerate.*

Remarks. For $V = \mathbf{P}_1$ and $D = \{0, 1, \infty\}$, we recover the usual Schottky-Landau theorem ([8], page 61). In case $c_1(\mathbf{K}_V) > 0$, we may take D to be empty and Proposition 6.2 reduces to the one given in [4] and [6].

The corollary is the same as (4.17), so that we have given two quite different proofs of this result.

(b) *Remarks on the case* $c_1(\mathbf{L}) + c_1(\mathbf{K}_V) \geqq 0$. Thus far we have chosen our line bundle $\mathbf{L}$ so as to have $c_1(\mathbf{L}) + c_1(\mathbf{K}_V) > 0$. When the inequality is not strict, there is a volume form Ψ on $V - D$ such that

$$\operatorname{Ric}\Psi \geqq 0.$$

For example, we may set

$$\Psi = \frac{\Omega}{|\delta|^2}$$

where Ω is the volume form given by a metric in $\mathbf{K}_V$ and δ is a section of $\mathbf{L}$

defining D. Observe that $\int_{V-D}\Psi = +\infty$, so that we shall only consider holomorphic mappings

$$f\colon \mathbf{C}^n \longrightarrow V - D \tag{6.12}$$

where $D \in |\mathbf{L}|$ may have *arbitrary* singularities. The proof of Proposition 3.4 may be repeated to yield the inequality

$$0 \leqq N_1(r) \leqq \mu(r) + O(1)\ . \tag{6.13}$$

(6.14) PROPOSITION. *For any f as in* (6.12), *we have*

$$\int_{B[R]} \Psi_f \geqq cR^{2n} \qquad (c > 0)\ .$$

Proof. Writing $\Psi_f = \xi\cdot\Phi$, from (6.13) we obtain

$$\int_{\partial B[r]} \log \xi\cdot\sigma \geqq c_1 \tag{6.15}$$

for some (possibly negative) constant c_1. Using the concavity of the logarithm, we have from (6.15) the inequality

$$\log\left\{\frac{1}{r^{2n-1}}\frac{d}{dr}\int_{B[r]}\xi\cdot\Phi\right\} \geqq c_1\ .$$

It follows that

$$\frac{d}{dr}\int_{B[r]}\Psi_f \geqq cr^{2n-1} \qquad (c = e^{c_1} > 0)\ ,$$

and our proposition follows by integrating this inequality.

(6.16) COROLLARY [6]. *Let V be a smooth projective variety with $c_1(\mathbf{K}_V) \geqq 0$. Let Ω be any volume form on V and $f\colon \mathbf{C}^n \to V$ a non-degenerate holomorphic mapping. Then*

$$\int_{B[R]} f^*(\Omega) \geqq cR^{2n} \qquad (c > 0)\ .$$

(6.17) COROLLARY. *Let V be a smooth projective variety with $c_1(\mathbf{K}_V) \geqq 0$, and let $D \in |\mathbf{L}|$ be a divisor with simple normal crossings where $c_1(\mathbf{L}) > 0$. Then the deficiency $\delta(D) = 0$. In particular, the image $f(\mathbf{C}^n)$ meets D.*

Proof. Since $c_1(\mathbf{K}_V) \geqq 0$ and $\mathbf{L}$ is positive, we have

$$\left[\frac{c_1(\mathbf{K}_V^*)}{c_1(\mathbf{L})}\right] = 0\ .$$

Our result now follows immediately from the defect relation for $k = 1$.

(6.18) COROLLARY. *Let V, $\mathbf{L}$, f be as in the previous corollaries. Then the ramification term satisfies*

$$\liminf_{r\to\infty}\left[\frac{N_1(r)}{T(\mathbf{L}, r)}\right] = 0 .$$

Proof. From Lemma (5.15) and $\mathbf{K}_V \geqq 0$ we conclude

$$T(\mathbf{L}, r) \leqq T_1(r) + c_1 \log\big(T(\mathbf{L}, r) + c_2\big) + c_3 .$$

Referring to estimate (4.16), we obtain

$$1 + \underline{\lim}_{r\to\infty}\left[\frac{N_1(r)}{T_1(r)}\right] \leqq \overline{\lim}_{r\to\infty}\left[\frac{N(D, r)}{T(\mathbf{L}, r)}\right] \leqq 1$$

where the last inequality comes from the first main theorem (5.7). Thus we get

$$\underline{\lim}_{r\to\infty}\left[\frac{N_1(r)}{T_1(r)}\right] = 0 .$$

Using Lemma (5.15) and the fact that $\mathbf{L}$ is positive, we find

$$T_1(r) \leqq T(\mathbf{L}_V, r) + T(\mathbf{K}_V, r) \leqq mT(\mathbf{L}, r) + \text{const.}$$

for m sufficiently large. This yields

$$0 = \underline{\lim}_{r\to\infty}\left[\frac{N_1(r)}{T_1(r)}\right] \geqq \underline{\lim}_{r\to\infty}\frac{1}{m}\left[\frac{N_1(r)}{T(\mathbf{L}, r)}\right] . \qquad \text{Q.E.D.}$$

Remark. The previous corollaries show that, if V is a smooth projective variety with $c_1(\mathbf{K}_V) \geqq 0$, then every non-degenerate holomorphic mapping $f: \mathbf{C}^n \to V$ must have the following properties: (i) f is transcendental (cf. Proposition 6.22); (ii) the image $f(\mathbf{C}^n)$ meets every positive divisor (with simple normal crossings) on V; and (iii) the size of the ramification divisor of f is relatively small compared to that of $f^{-1}(D)$ for D a divisor on V.

(c) *Holomorphic mappings with growth conditions*. Let V be a smooth projective variety, $\mathbf{L} \to V$ a positive line bundle, and $D_1, \cdots, D_k \in |\mathbf{L}|$ divisors such that $D^\# = D_1 + \cdots + D_k$ has simple normal crossings and $kc_1(\mathbf{L}) + c_1(\mathbf{K}_V) > 0$. From Corollary (4.17) we find that any holomorphic mapping $f: \mathbf{C}^n \to V - D^\#$ is degenerate. It shall now be proved more generally that the growth of $f^{-1}(D)$ for any $D \in |\mathbf{L}|$ is determined by the growth of the finitely many divisors $f^{-1}(D_j)$ $(j = 1, \cdots, k)$.

(6.19) PROPOSITION. *Let $\tau(r)$ be a continuous increasing function of r such that $N(D_j, r) = O(\tau(r))$ for $j = 1, \cdots, k$. Then we have $N(D, r) = O(\tau(r))$ for all $D \in |\mathbf{L}|$.*

Proof. It follows from (4.16) and (5.15) that

$$\overline{\lim}_{r\to\infty}\frac{N(D^\#, r)}{T(\mathbf{L}, r)} \geqq c > 0 .$$

Since $T(\mathbf{L}, r)$ is a continuous increasing function of r, we obtain from $N(D^{\#}, r) = O(\tau(r))$ that

$$T(\mathbf{L}, r) = O(\tau(r)) .$$

Our proposition now follows from this together with the Nevanlinna inequality (5.10). Q.E.D.

To give an application, we first prove

(6.20) PROPOSITION. *Let V be a smooth, projective variety, $\mathbf{L} \to V$ a positive line bundle, and $f: \mathbf{C}^n \to V$ a (possibly degenerate) holomorphic mapping. Then f is rational if, and only if, the order function $T(\mathbf{L}, r)$ satisfies*

$$T(\mathbf{L}, r) = O(\log r) .$$

Proof. Replacing $\mathbf{L}$ by $\mathbf{L}^N$ for large N, we may assume that $\mathbf{L} \to V$ is very ample; i.e., the complete linear system $|\mathbf{L}|$ induces a projective embedding $V \subset \mathbf{P}_m$ $(m = \dim |\mathbf{L}|)$.

If f is algebraic, then the divisors D_f $(D \in |\mathbf{L}|)$ are all algebraic hypersurfaces of degree d in $\mathbf{C}^n$. It follows from [7] that

$$\int_{D_f[r]} \psi_{n-1} \leqq d$$

for all r, and thus $N(D, r) \leqq d \log r + O(1)$. Using (5.12) we have $T(\mathbf{L}, r) \leqq d \log r + O(1)$.

Conversely, suppose that $T(\mathbf{L}, r) \leqq d \log r + O(1)$. It follows from the Nevanlinna inequality (5.10) that $N(D, r) \leqq d \log r + O(1)$ for all $D \in |\mathbf{L}|$. Using Lelong's theorem [7] it now follows that *all* D_f are algebraic hypersurfaces of degree d in $\mathbf{C}^n$. From this we easily conclude that f is rational. Q.E.D.

Combining Propositions (6.19) and (6.20) we have

(6.21) COROLLARY. *Let V be a smooth projective variety, $\mathbf{L} \to V$ a positive line bundle such that $kc_1(\mathbf{L}) + c_1(\mathbf{K}_V) > 0$, and let $D_1, \cdots, D_k \in |\mathbf{L}|$ divisors such that $D_1 + \cdots + D_k$ has simple normal crossings. Then f is a rational mapping if, and only if, the divisors $f^{-1}(D_j)$ $(j = 1, \cdots, k)$ are algebraic.*

Remark. The same corollary (and proof) holds for holomorphic mappings of finite order.

Along similar lines, we have

(6.22) PROPOSITION. *Let V be a smooth, projective variety, $\mathbf{L} \to V$ a line bundle such that $c_1(\mathbf{L}) + c_1(\mathbf{K}_V) \geqq 0$, and $D \in |\mathbf{L}|$ a divisor with normal crossings. Then any non-degenerate holomorphic mapping $f: \mathbf{C}^n \to V - D$*

is transcendental.

Proof. Suppose that $f: \mathbf{C}^n \to V - D$ is rational. Then by desingularizing the graph of $f: \mathbf{P}_n \to V$ we may find a smooth, projective variety W and a diagram of rational holomorphic mappings

$$\begin{array}{ccccc} & & W & & \\ & {\scriptstyle\pi_1}\swarrow & & \searrow{\scriptstyle\pi_2} & \\ \mathbf{P}_n & & & & V \end{array} \tag{6.23}$$

together with a divisor $E \subset W$ satisfying $\pi_1(E) = \mathbf{P}_n - \mathbf{C}^n$, $\pi_2(W - E) \subset V - D$, and such that (6.23) restricts to

$$\begin{array}{ccccc} & & W - E & & \\ & {\scriptstyle\pi_1}\swarrow & & \searrow{\scriptstyle\pi_2} & \\ \mathbf{C}^n & & \xrightarrow{\quad f \quad} & & V - D \end{array}$$

where $f = \pi_2 \circ \pi_1^{-1}$. We set $W[r] = \pi_1^{-1}(B[r])$. Let Ψ be the volume form on $V - D$ given by $\Psi = \Omega/|\delta|^2$. From Proposition (6.14) we have $\int_{B[R]} f^*\Psi \geqq cR^{2n}$. On the other hand,

$$\int_{B[R]} f^*\Psi = \int_{W[R]} \pi_2^*\Psi = \int_{\pi_2(W[R])} \Psi \, .$$

Comparing (2.2) with the local form of π_2 we find that

$$\int_{\pi_2(W[R])} \Psi = O(\log R) \, .$$

This is a contradiction. Q.E.D.

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(Received August 30, 1971)