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Definition of Families of L-functions

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The zoo of automorphic cusp forms π on $GL(n)$ (n a natural number) over $\mathbb{Q}$ correspond bijectively to their standard $L-$functions $L(s,\pi)$, and they constitute a countable set containing species of different types.[1] For example, there are self-dual forms, ones corresponding to finite Galois representations, ones corresponding to Hasse-Weil zeta functions of varieties defined over $\mathbb{Q}$, etc. From a number of points of view (including the nontrivial problem of effectively isolating special forms), one is led to study such $L(s,\pi)$'s in families in which the π's have a similar flavor. Some applications actually demand the understanding of the behavior of the L-functions as π varies over such a family. Other applications involve questions about an individual L-function. In practice a "family" is investigated as it arises. For example, the density Theorems of Bombieri -Vinogradov (see [B],[V]) are concerned with showing that in a suitable sense most Dirichlet L-functions have few violations of the Riemann Hypothesis and, as such, this is a powerful substitute for the latter. Other examples are the $GL(2)$ subconvexity results which can all be proved by deforming the given form in a family (see [I-S] and [M-V] for an account). In the analogous function field setting the notion of a family of zeta functions is well defined, coming from the notion of a family of varieties defined over some base. Here, too, the power of deforming in a family in order to understand individual members is amply demonstrated in the work of Deligne [D]. In the number field setting, there is no formal definition of a family $\Im$ of L-functions. My purpose here is to give a working definition, and these pages can be viewed as an appendix to [K-S] and [I-S]. In [K-S] a symmetry type "$G(\Im)$" is associated to various sets of $L-$functions that are special cases of what we define below. The expectation is that all families have a symmetry associated with them and that these determine, for example, the distributions low-lying zeros for the L-functions in the family. We also give a universal algorithm as to how to compute the symmetry.

For the purpose of giving a general definition, we assume freely any standard conjectures when convenient. While many of these conjectures are well out of reach, there are numerous cases where they are known and where the discussion below is at least partially provable.

[1] For simplicity we suppress the t-dependence identifying π with $\pi\otimes|\ \ |^{it}$.

First some notation.

1) $N(\pi)$ is the conductor of π, it is the positive integer defined as the product to appropriate powers of the primes v at which π_v is ramified (here $\pi \cong \otimes \pi_v$). It is the integer appearing in the functional equation of the standard L-function $L(s, \pi)$.

2) $c(\pi)$ is the analytic conductor of π as defined in [I-S]. It is product of conductor $N(\pi)$ with an archmidean factor depending on π_∞. The analytic conductor measures the "complexity" of an automorphic form much like a height function for rational points in diophantine analysis. As in that setting, the set $S(X) = \{$cuspidal π's on $GL(n)$ with $c(\pi) < X\}$ is finite (see Brumley [Br]). There is no doubt that a "Weyl-Shanuel" type asymptotic formula exists for $S(X)$ and deriving such, starting with $n = 2$ would be of interest.

3) $e(\pi)$ is the root number of π that is the "sign" of functional equation.

4) For an automorphic cusp form π on $G = GL(n)$ let H_π be the smallest subgroup of $GL(n, \mathbb{C})$ associated with π such that the semisimple conjugacy classes corresponding to almost all of the π_v's lie in H (and in which these become equidistributed with respect to Haar measure on H intersect the unitary group, i.e., the "Sato -Tate" conjecture for π). This group is defined in Langlands [L], it is only defined up to conjugacy, and it is closely connected to the poles at $s = 1$ of $L(s, \pi, \rho)$ for an irreducible representation ρ of ${}^L G$. For example H_π should be finite iff π corresponds to a Galois representation.

While it is problematic at the present time to give a geometric definition of a family of automorphic forms, the philosophy of what should be true is based in part on the geometric analogues. Hence, it is desirable that our definition capture these families, too. Let R be a geometric family of smooth projective varieties V defined over a base space space T, all being defined over $\mathbb{Q}$. So for each t in T outside a Zariski closed set, V_t is smooth and projective over $\mathbb{Q}$. Fix the piece $L(s, V_{t,j})$ of the Hasse-Weil zeta function corresponding to the cohomology in dimension j. Each $L(s, V_{t,j})$ is expected to have analytic continuation to the complex plane and to satisfy a functional equation (which we normalize automorphically, s into $1 - s$) and, hence, it should be of the form: $L(s, \pi(t))$ for an automorphic form $\pi(t)$ on $GL(m)$ (with m the dimension of the corresponding cohomology group). We call such a family of automorphic forms $\pi(t)$, t in T, a geometric family.

Another family is that of automorphic forms that arise from finite Galois representations, that is Artin L-functions. Functoriality asserts that each such L-function is an L-function of an automorphic form on the corresponding $GL(n)$. Choosing a set of such Galois representations which are defined by some algebraic number theoretic constraint (such as the Dedekind zeta functions of extensions of a fixed degree) leads to families of Artin L-functions.

The universal family of L-functions (of degree n) consists of all $L(s,\pi)$ with π automorphic and cuspidal on $GL(n)$. In studying this family (or any family below), we always use the analytic conductor $c(\pi)$ to order (or to collect in finite sets) the members of the family.

We first define families of automorphic cusp forms on $GL(n)$ and then use these to define families of L-functions. One expects that all "L-functions" are finite products of these standard L-functions of cusp forms on $GL(n)$ and so restricting to subsets of these should not lose anything. Our definition of a family is made to include geometric families, families defined through local harmonic analysis and also functoriality.

Definition 1: (Family of automorphic forms) A family $\Im$ of automorphic forms is a subset of the universal family gotten by imposing one or more of the following conditions.

I) (The degenerate case) $\Im$ is π, a single such form.

II) (root numbers and central characters) Fix the central character χ, or restrict to self dual forms and then further to forms with a given sign $e(\pi)$ of their root number.

III) (given H type) Fix H a semisimple algebraic subgroup of $GL(n,C)$ and restrict to π's for which $H_\pi = H$.

IV)(harmonic analysis constraints)
(i) Let S be a finite set of places of $\mathbb{Q}$ (which may include ∞) and for $v \in S$ let B_v be an open (nice) subset of $GL(n,\mathbb{Q}_v)^{temp}$, the tempered dual of $GL(n,\mathbb{Q}_v)$. . We restrict to π's for which $\pi_v \in B_v$ for $v \in S$ and outside of S, π_v is unramified.

(ii) As in (i) except that outside of S there are no constraints on π_v.

V)(geometric families) Fix a geometric family R of varieties V_t defined over $\mathbb{Q}$ and restrict π to be a $\pi(t)$ corresponding to $V_{t,j}$, for t in T, as described above.

VI)(Functorial transfers) If H is a reductive algebraic group defined over $\mathbb{Q}$ (or a number field) and ρ a homomorphism of ${}^LH \to^L G$, with $G = GL(n)$, then functorial image of the automorphic forms on H give a family of forms in $GL(n)$. This includes the algebraically defined families of Artin L-functions.

Definition 2: (Family of L-functions) Fix $n_1, n_2, \ldots, n_k$ positive integers and irreducible finite dimensional representations $\rho_1, \rho_2, \ldots, \rho_k$ of $GL(n_j)$ for $j = 1, \ldots k$ and families of automorphic forms $\Im_j$ on $GL(n_1), \dots, GL(n_k)$. Define the family $\Im(\Im_1, \dots, \Im_k, \rho_1, \dots, \rho_k)$ of L-functions to be the tensor product L-functions:

$$L(s, \pi_1, \pi_2, \dots, \pi_k, \rho_1, \dots, \rho_k) := L(s, \rho_1(\pi_1) \times \rho_2(\pi_2) \times \dots \times \rho_k(\pi_k))$$

as π_j varies in each $\Im_j$ respectively.

This gives a family of L-functions of degree n equal to $(dim\ \rho_1) \times \dots \times (dim\ \rho_k)$ and these L-functions are conjecturally standard L-functions of automorphic forms on $GL(n)$. In general they need not correspond to cusp forms, but for simplicity in what follows I assume that this is the case (or at least that if not that any common factor in all the L-functions in the family is fixed, e.g., something like the Riemann Zeta function). Also, in the definition of a family of L-functions, we also assume that $\Im$ is infinite.

Discussion: At the very least, in studying such a family one needs to be able to understand asymptotically the cardinalities of the sets

(1) $S(\Im, X) = \{\pi \text{ in } \Im : c(\pi) < X\}$, as X goes to infinity.

Question 1:

Is $| S(\Im, X) |$ asymptotic to aX^b for some positive constants a and b depending on $\Im$? (or perhaps in some singular cases one should allow powers of $\log X$).

This is so in all cases I know (some are mentioned below), and I expect it is true in general. If so, one would naturally call b the dimension of the family.

More generally for each $m > 0$ there should be a number $t(m, \Im)$ such that as X goes to infinity,

$$\sum_{\substack{\pi \in \Im \\ c(\pi) < X}} a_\pi(m) \sim | S(\Im, X) | \, t(m, \Im), \tag{2}$$

here $a_\pi(m)$ is the m-th coefficient of the Dirichlet series defining $L(s, \pi)$. (Note this is normalized so that $a_\pi(1) = 1$).

The point of our definition is that the trace formula, and especially recent analytic developments around it, allow us to compute the asymptotics in (1) and (2), for families defined by conditions of type IV and some of type VI. As to those of type V the asymptotics in (2) can be computed using arithmetic geometric methods (that is the monodromy group of the family (see Katz [K] for an exposition). The families which are most problematic in terms of present technology, for determining the asymptotics in (1) and (2) are those associated with number fields and Artin's Galois representations. For example, counting the family of Dedekind zeta functions of number fields of a fixed degree is problematic if the degree is bigger than 5 (see below).

Symmetry type: A family $\Im$ of L-functions as defined above has a symmetry type $G(\Im)$. It can be determined from the function field analogue when such exists. There is also an algorithmic calculation which leads to a determination of the symmetry of the family. That is for families as above one can determine the densities of low-lying zeros for test functions whose support of their fourier transforms are small and these densities distinguish between the expected universal symmetry types. In more detail, assume that one has (2) above in the stronger form with a remainder of $O(X^{-b}m^r)$ with $b > 0$ and also the asymptotics of

$$\frac{1}{x}\sum_{p\leq x} t(p^e, \Im)\log p, \text{ as } x \to \infty \qquad \text{for } e = 1 \text{ or } 2.$$

Then one can determine the density of low zeros of $L(s,\pi)$ for π in $\Im$ in restricted ranges (as in [K-S]). As in that paper and for the irreducible families defined above, we expect that there are only five types of symmetries $G(\Im)$, these being SP, SO, SO (even), SO(odd) and U. The densities computed above, even in the restricted ranges, suffice to distinguish these five symmetries. Thus, as long as we can compute a little more than (1) and (2) above, we have a predicted symmetry. It in turn predicts the fluctuations of many other quantities associated with the family – for example, the moments (see [Ke-Sn]). All of these predictions can be tested numerically as they have been in many special cases ([R]), and certainly further such checking is needed if we are to believe in general that $G(\Im)$ is restricted universally as suggested above. It would complement nicely the high zero universality for a given $L(s,\pi)$ found in [R-S].

Some examples of where the above has or is being carried out are the following:

A) The first and most basic case is that of Artin L-functions corresponding to quadratic extensions of $\mathbb{Q}$. This can be realized by (III), (IV) or (VI) above. It has been investigated from all points of view (see [K-S] and [R]) and the symmetry is symplectic. The next case of this type is that of Dedekind zeta functions of cubic extensions. After division by $\zeta(s)$ this gives a family of $GL(2)$ L-functions. Very recently A. Yang (thesis PU 2009) has determined that its symmetry is also symplectic. Using the recent work of Bhargava on counting number fields of degree 4 and 5, the corresponding families of Dedekind zeta functions should also be approachable.

B) As far as families of type (IV) go, the paper [I-L-S] examines this for a number of such families in $GL(2)$. For example, using condition (II) to fix the central character to be trivial and (IV)(i) with S simply ∞ and the open set all discrete series, yields the family of holomorphic forms of any weight for $PSL(2, Z)$. This has an SO symmetry and if we further fix the epsilon factor we get an SO(even) and SO(odd) symmetry respectively. The recent work Lapid and Mueller [L-M] when extended as in [Sa] will lead to the (1) and (2) above for various large familes in $GL(n)$ for any n. For example the symmetry type and corresponding analysis for the universal unramified family – that is all cusp forms f on $GL(n)$ which are everywhere unramified – should emerge from these analytic developments of the trace formula.

C) As for geometric families (V), one of the few cases where the corresponding automorphic forms are known to exist is that of families of elliptic curves over $\mathbb{Q}$ (due to the work of Wiles, et al.). The symmetry of the universal such family of Hasse-Weil L-functions, as well as numerous special families of elliptic curves are determined in the thesis of S. J. Miller (see [M]) and also M. Young (see [Y]). They all have SO symmetries or SO(even), SO(odd) symmetries if one restricts the epsilon factor. This should even be true with a simple modification for families of elliptic curves over $\mathbb{Q}(t)$ which have positive rank r, over $\mathbb{Q}(t)$. In this case, the generic member of the family has rank at least r and the modification in question is that the low-lying zeros beyond these "forced" r zeros obey the same laws as described by the SO symmetry, though the numerics in this case are less convincing [M2].

D) An example of a family that is formed out of condition (IV) and then the more general combination as in Definition 2 via generalized tensor product L-functions is the following: π varies over holomorphic forms weight k of full level for $SL(2, Z)$ and $\pi\prime$ is a fixed every unramified form on $GL(2)$. Form the family of degree 6 L-functions $L(s, sym^2\pi \times \pi\prime)$. This family arises in the computation of the variance of quantum fluctuations on the modular surface ([L-S]). It is a self-dual family with root number equal to 1. However, as has been shown in [D-M], it has, perhaps somewhat surprisingly, an SO(even) symmetry rather than an SP symmetry. X. Li's [L] recent proof of the first case of subconvexity for a $GL(3)$ form, that is for $L(s, sym^2\pi)$ in the t aspect, uses a deformation in the family $L(s, sym^2\pi \times \pi\prime)$, where this time π is fixed and $\pi\prime$ varies over unramified $GL(2)$ forms and also unitary Eisenstein series. The symmetry type for this family can be computed easily.

One may speculate about an intrinsic definition of the symmetry $G(\Im)$. That is in a spectral interpretation of the zeros of $L(s, \pi)$. The operators corresponding to each $\pi \in \Im$ should preserve a bilinear form defining the symmetry. Moreover, as we vary over $\pi \in \Im$ ordered by their analytic conductors, these operators should become equidistributed in the corresponding space of operators which enjoy the symmetry. For the (Paul) Cohen, Connes, Meyer spectral interpretation, this is discussed in [P].

References

[B] E. Bombieri. *Mathematatika* 12 (1965): 201-225.

[V] A. I. Vinogradov. Izv Akad Nauk. *SSSR Ser. Math* 29 (1965): 903-924.

[I-S] H. Iwaniec and P Sarnak. *GAFA* special volume 2000, 705-741.

[M-V] P. Michel and A. Venkatesh. *Proceedings ICM*, Madrid. (2006): 421-457

[K-S] N. Katz and P. Sarnak. *BAMS* 36 (1999): 1-26.

[D] P. Deligne. *IHES* 43 (1972): 206-226.

[Br] F. Brumley. *American Journal of Math* 128 (2006): 1455-1474.

[L] R. Langlands. "Beyond Endoscopy," in *Contributions to automorphic forms, geometry and number theory.* Johns Hopkins Press (2004).

[K] N. Katz. *BAMS* 23 (1990): 269-309.

[Ke-Sn] J. Keating and N. Snaith. *Comm. Math Phys.* 214 (2000): 91-110.

[R-S] Z. Rudnick and P. Sarnak. *Duke Math Journal* 81 (1996): 269-322.

[I-L-S] H. Iwaniec, W. Luo and P. Sarnak. *IHES* 91 (2001): 55-131.

[L-M] E. Lapid and W. Mueller. "Spectral asymptotics for arithmetic quotients of $SL(n, R)/SO(n)$." *math arXiv* 19 (November 2007).

[Sa] P. Sarnak. "Statistical properties of eigenvalues of the Hecke Operators." *Analytic Number Theory and Diophantine problems, Prog in Math* 70 Birhhauser (1987): 321-331.

[M] S. J. Miller. *Compostio Math* 140 (2004): 952-992.

[Y] M. Young. *J. of Amer. Math Soc.* 19 (2006): 205-250.

[M2] S. J. Miller. *Exp. Math* 15, no.3 (2006):257-279.

[L-S] W. Luo and P. Sarnak. *Ann. Sci. École Norm. Sup.* 37 (2004): 769-799.

[D-M] E. Duenez and S. J. Miller. *Compositio Math* 142 (2006): 1403-1425.

[Li] X. Li. "Bounds for $GL(3) \times GL(2)$ and $GL(3)L$-functions," to appear in the *Annals of Math.*

[C-K-R-S] B. Conrey, J. Keating, M. Rubinstein and N. Snaith. *Proc. London Math Soc.* 91 (2005): 33-104.

[R] M. Rubinstein. Ph.D. Thesis, Princeton University, 1998.

[P] F. Paugan. "Spectral symmetries of zeta functions." *math arXiv* 0803.0199.
www.ma.utexas.edu/users/miker/thesis/thesis.html