PHILLIP A. GRIFFITHS
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Deformations of G-Structures

Part A: General Theory of Deformations

By

PHILLIP A. GRIFFITHS in Berkeley

This paper is concerned with deformations of structures on manifolds. It is divided into two parts: Part A treats the question of defining a general theory of deformations which will generalize the theory in, e.g. [1], [7], and [10], while at the same time retaining some geometric or analytical significance. The second section, Part B, investigates in more detail the implications of our general theory on the "classical" structures in differential geometry.

Let $G \subseteqq GL(n, R)$ be a linear Lie group, and let X be an n-dimensional manifold. A *G-Structure* on X is a reduction of the structure group of the tangent bundle of X from $GL(n, R)$ to G; geometrically, a G-structure gives a principal fibre bundle $G \to B_G \to X$ where B_G consists of all *G-frames* on X. The G-structure is *integrable* if X has a coordinate covering such that the coordinate frames are G-frames. A deformation theory of integrable G-structures has been given in [11], and, in case $n = 2k$ and $G \subseteqq GL(k, \mathbf{C}) \subset GL(2k, \mathbf{R})$, some considerable progress has been made towards obtaining general results generalizing the well known variation of complex analytic structure. In [1] a deformation theory of Riemannian manifolds of constant curvature was proposed; a variant of this was used in [13], although the problem in these cases was specifically to prove the "rigidity" of a structure, rather than to discuss the geometric significance of the existence of deformations.

After some preliminaries in § I, we shall, in § II, give a general definition of deformations of G-structures generalizing the theories described above. Our definition may be verbally stated as follows: A 1-parameter deformation of a G-structure $G \to B_G \to X$ is given by a 1-parameter family of G-structures $G \to B_G(t) \to X$, $B_G(0) = B_G$, such that the deformed structures have precisely the same *local* properties as the original G-structure. In other words, we shall assume the local triviality of our deformations, and then seek the global implications of this hypothesis. Clearly, such a theory generalizes the special cases given above.

In § III we discuss the relationship of our theory with the theory of sheaves; the possibility of such a relationship was one of the motivating factors in our definition. Paragraph IV is devoted to the higher order theory of deformations; as was indicated, for complex structures, in [2], the main synthesis here is gained by systematically introducing the enveloping algebra sheaf of the sheaf of germs of infinitessimal automorphisms of the G-structure. As a new application, we discuss conditions when global infinitesimal automorphisms are "stable" under deformations.

The class of G-structures may be partitioned into two subclasses, those whose local automorphism groups are Lie groups (structures of *finite type*), and those whose local groups are not locally compact (structures of *infinite type*). We call the former *geometric structures*, and Part B is devoted to the deformations of these structures. The most significant fact which turns up here is that, for a wide class of manifolds with geometric structures, the study of their deformations may be reduced to problems in Lie groups. The general program of Part B is to make this reduction (§§ V, VI, and VII), and then examine what geometric significance this has for deformations (§§ VII and VIII). In § IX, some examples of deformations of geometric structures are constructed, along with the parameter variety of the deformation. It is seen that these varieties can be, at best, only locally real analytic sets, as happens for complex structures.

An interesting fact which arises is that, at least with our method of approach, the global analysis of geometric structures does not turn out to be any easier than the global analysis of infinite structures as far as deriving theorems in deformation theory is concerned. Thus, in § VIII, after having worked "formally" throughout the preceeding part of the paper, the methods of partial differential equations are introduced, as in [8], to prove the existence of deformations of structures.

The author would like to acknowledge many valuable conversations with his collegues on the topics discussed below; in particular, S. S. Chern suggested the general problem of defining a deformation theory, and, also, much of the material in § IV has been influenced by written communications from D. C. Spencer.

Contents

I. G-Structures: Basic Definitions and Preliminaries

1. *Basic Definitions.* Let X be any (differentiable) manifold and let $GL(n, \mathbf{R}) \longrightarrow B \xrightarrow{\pi} X$ be the principal bundle of the tangent bundle $T(X)$ of X;

we shall sometimes consider B geometrically as the bundle of all tangent frames on X. If $G \subset GL(N, \mathbf{R})$ is a closed linear subgroup of $GL(N, \mathbf{R})$, then there is a fibre bundle diagram

$$\begin{tikzcd} B \arrow[rr, "\tau_G"] \arrow[dr, "\pi"'] & & B/G \arrow[dl, "\pi_G"] \\ & X & \end{tikzcd} ;$$

the bundle B/G is the bundle $BX_G GL(n, \mathbf{R})$ over X and the fibre is the coset space $GL(N, \mathbf{R})/G$. We describe τ_G in geometrical terms. If $x \in X$, then the fibre $B = \pi^{-1}(x)$ may be thought of as all invertible linear mappings $f : \mathbf{R}^n \to \to T_x(X)$ (= tangent space to X at x). Then $\tau_G(f)$ is the set of linear mappings $f \circ g : \mathbf{R}^n \to T_x(X)$ as g runs through $G \subset GL(n, \mathbf{R})$. Equivalently, if $f = (e_1, \ldots, e_n)$ is any frame at $x \in X$, then $\tau_G(f)$ consists of all frames $f \circ g = \left(\sum_j g_1^j e_j, \ldots, \sum_j g_n^j e_j\right)$ where $g = (g_j^i) \in G$.

Definition 1.1: A G-structure on X is given by a cross section $\sigma : X \to B/G$.

The mapping σ picks out a distinguished set B_G of frames over X as follows: If $x \in X$, then in a neighborhood U of $\sigma(x) \in B/G$, we may find a local cross section $i : U \to B|\tau_G^{-1}(U)$. Then, for $x^1 \in \pi_G(U)$, $(i \circ \sigma)(x^1)$ is a frame $(e_1(x^1), \ldots, e_n(x^1)) = f(x^1)) \in B_{x^1}$, and $B_G|\pi^{-1}(\pi_G(U))$ is the set of frames $f(x^1)g$ as x^1 varies in $\pi_G(U)$ and g ranges over G.

This totality of all such "admissable" frames gives a principal bundle.

$$(1.1) \qquad G \longrightarrow B_G \xrightarrow{\pi} X ,$$

and we shall frequently use (1.1) to signify the giving of a G-structure on X.

Suppose now that we are given $\sigma : X \to B/G$, and let $\mathbf{U} = \{U_j\}$ be an open covering of X by coordinate neighborhoods U_j. Over U_j we may determine a local section $\hat{\sigma}_j : U_j \to B|\pi^{-1}(U_j)$ such that $\tau_G(\hat{\sigma}_j) = \sigma$. If $\{f_{ij}\}$ is the system of transition functions of X relative to the covering $\mathbf{U}$, then the system $\{J(f_{ij})\}$ gives the transition functions of the bundle B, where, for a mapping $h : \mathbf{R}^n \to \mathbf{R}^n$, $J(h)$ is its Jacobian matrix. Since $\tau_G(\hat{\sigma}_i) = \tau_G(\hat{\sigma}_j)$ in $U_i \cap U_j$, we have

$$(1.2) \qquad J(f_{ij})\hat{\sigma}_j = \hat{\sigma}_i g_{ij}$$

where $g_{ij} : U_i \cap U_j \to G$ is a smooth mapping. If $x_i = (x_i^1, \ldots, x_i^n)$ are the coordinates in U_i, we set $dx_i = (dx_i^1, \ldots, dx_i^n)$ and $w_i = dx_i \hat{\sigma}_i^*$ where, for any $A \in GL(n, \mathbf{R})$, $A^* = {}^tA^{-1}$. Then, in $U_i \cap U_j$, we have from (1.2) $w_j = dx_j \hat{\sigma}_j^* = dx_j^t J(f_{ij})\hat{\sigma}_i^* g_{ij}^* = w_i g_{ij}^*$. The cocycle condition $g_{ij}^* g_{jk}^* = g_{ik}^*$ (in $U_i \cap U_j \cap U_k$) follows from the cocycle condition on $\{J(f_{ij})\}$. Thus, associated to $\{\sigma_i\}$, we have a collection $\{w_i\}$ of 1-forms of maximal rank satisfying

$$(1.3) \qquad w_j = w_i g_{ij}^* ; \quad g_{ij}^* g_{jk}^* = g_{ik}^* .$$

Conversely, a system $\{w_j\}$ satisfying (1.3) determines a section $\sigma : X \to B/G$.

Definition 1.2: A G-structure $\sigma : X \to B/G$ is *integrable* if we may choose $\{U_j\}$, $\{\hat{\sigma}_j\}$ above so that $\hat{\sigma}_j : U_j \to G$.

Remark: Equivalent to integrability are the statements:

$$J(f_{ij}) \in G, \quad \text{or} \left(\frac{\partial}{\partial x_i^1}, \ldots, \frac{\partial}{\partial x_i^n}\right) \in B_G .$$

2. *Local Automorphisms of G-Structures.* Given a local mapping $f: X \to X$, we always have its linearization $f_*: T_x(X) \to T_{f(x)}(X)$. If f_* is injective, we may lift f to a bundle mapping

$$\begin{array}{ccc} f_*: B & \longrightarrow & B \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ f: X & \longrightarrow & X . \end{array}$$

Given a local mapping f of maximal rank, there is an induced mapping $f_*: B/G \to B/G$, and $f_*(\sigma)$ is a local section of $B/G \xrightarrow{\pi_G} X$.

Definition 1.3: $\underline{\Gamma_G}$ is the sheaf of non-abelian groups composed of germs of local bi-mappings of X such that $f_*(\sigma) = \sigma \circ f$. Germs in $\underline{\Gamma_G}$ are called *local bi-G-mappings*.

As it stands, $\underline{\Gamma_G}$ is a sheaf over $X \times X$; a more manageable sheaf is the following:

Definition 1.4: $\Gamma_G[t]$ is the sheaf of germs of local bi-G-mappings of X which depend parametrically upon t varying in a neighborhood of $\mathbf{o} \in \mathbf{R}^1$ and which reduce to the identity at $\mathbf{o}$.

Thus, a germ of $\Gamma_G[t]$ over an open set $U \subset X$ is given by open sets W, W^1 with $W \cap W^1 \supseteq U$, an open set $V \supset \{o\}$ in $\mathbf{R}^1$, and a family of bi-G-mappings $f_t: W \to W^1 (t \in V)$ with $f_0 =$ identity.

Proposition 1.1: *For a local bi-map f of X, the following are equivalent:* (i) $f_*(\sigma) = \sigma$, (ii) $f_*(B_G) \subseteq B_G$, (iii) $f_*(\hat{\sigma}_i) = \hat{\sigma}_i g_i$ *for some* $g_i: U_i \to G$, *and* (iv) $f_*(w_i) = w_i g_i^*$.

Given a family of local vector fields $\theta(t)$ on X, we may define a family $f(t)$ of local bi-maps as follows: (i) $f(0) =$ Identity, and (ii) $\frac{df(t)}{dt} = \theta(t) \circ f(t)$ (where we consider $\theta(t)$ as a family of maps of X). We set $f(t) = \exp(\theta(t))$. (C.f. § IV.1 below.)

Definition 1.5: We define a sheaf of germs of vector fields $\underline{\Theta_G}$ an X by letting $\underline{\Theta_G}$ be these germs of vector fields θ such that $\exp(t\theta) \in \Gamma_G[t]$.

Definition 1.6: We define $\underline{\Theta_G[t]}$ to be these germs of vector fields $\theta(t)$, depending on t, such that $\exp(\theta(t)) \in \Gamma_G[t]$.

Letting $\mathbf{g}$ be the real Lie algebra of G, we have

Proposition 1.2: *Let θ be a germ of a vector field, and let L_θ denote the Lie derivative along θ. Then the following are equivalent:* (i) $\theta \in \Theta_G$, (ii) $L_\theta(\hat{\sigma}_i) = \hat{\sigma}_i g_i$ *where* $g_i: U_i \to \mathbf{g}$, *and* (iii) $L_\theta(w_i) = -w_i^t g_i$.

Proof: (iii): Let $\theta \in \Theta_G$. Then

$$L_\theta\Big(w_i\Big| = \frac{d}{dt}\exp(t\theta)^* w_i\Big]_{t=0} = \frac{d}{dt} w^i g_i(t)\Big]_{t=0} = w^i \frac{dg_i(t)}{dt}\Big]_{t=0} \text{and} \frac{dg_i(t)}{dt}\Big]_{t=0} : U_i \to \mathbf{g}.$$

If, conversely, θ satisfies (iii), then we easily see that $\theta \in \Theta_G$.

Corollary: *Θ_G is a sheaf of Lie algebras.*

Proof: $L_{[\theta,\theta^1]} = [L_\theta, L_{\theta^1}]$ and $\mathfrak{g}$ is a linear Lie algebra.

Remark: If the G-structure $\sigma : X \to B/G$ is integrable, then, in U_i, all germs $\frac{\partial}{\partial x_i^\alpha} = (\alpha = 1, \ldots, n) \in \Theta_G$.

Examples: (i) If $G = O(n)$, then a G-structure is just a Riemannian structure, a germ in Γ_G is a local isometry, and an element $\theta \in \Theta_G$ is a germ of a Killing vector field. (ii) If $G = I$, then a G-structure is just a parallelism given by global 1-forms $w_1, \ldots, w_n$; these Pfaffians may be thought of as giving an affine connexion in the tangent bundle of X. This connexion has zero curvature, and its torsion is given by $n^2\left(\frac{n-1}{2}\right)$ functions $C^i_{jk}(x)$ an X where $dw_i = C^i_{jk} w_j \wedge w_k$. The elements $\theta \in \Theta_G$ are then just the infinitesimal affine motions of this linear connexion. (iii) If $G = GL(n/2, C)$ (n even), and if we have an integrable G-structure, then X is a complex manifold, and Θ_G may be thought of as the sheaf of holomorphic vector fields.

We have the sheaf of rings $\mathscr{R}$ of C^∞ functions an X, and given a G-structure $\sigma : X \to B/G$, we define

Definition 1.7: $\mathscr{R}_G$ = largest sheaf of subrings of $\mathscr{R}$ such that, for any $f \in \mathscr{R}_G$, $\theta \in \Theta_G$, $f \cdot \theta \in \Theta_G$.

There is a mapping $p_x : \Theta_{Gx} \to T_x(X)$ defined by $p_x(\theta) = \theta(x) \in T_x(X)$ for any germ θ in the stalk $(\Theta_G)_x$. We set $d(x) = \dim(\operatorname{im} p_x)$.

Definition 1.8: The G-structure $G \to B_G \to X$ is *normal* if $d(x)$ is constant on X. The structure is *transitive* if it is normal and $d(x) = \dim X$.

Examples: (i) If the G-structure is integrable, then it is transitive. If this is the case, and if $G = GL(m/2, C)$, then $\mathscr{R}_G$ is the sheaf of germs of holomorphic functions. (ii) Let $G = GL(n, \mathbf{R})$ (So that we have a parallelism) and suppose that $d(x) \geqq 1$ ($x \in X$). Then $\mathscr{R}_G$ is the constant sheaf $\mathbf{R}$ of real numbers.

Proof: A vector field $\theta \in \Theta_G$ if, and only if, $L_\theta(w_i) = 0$ $(i = 1, \ldots, n)$. Thus, if e_i is dual to w_i, $\theta \in \Theta_G$ if, and only if, $[\theta, e_i] = 0$ $(i = 1, \ldots, n)$. But, for a function f, $[f\theta, e_i] = f[\theta, e_i] - (e_i f)\theta$; thus $[f\theta, e_i] = 0$ $(i = 1, \ldots, n)$ if, and only if, $(e_i f) = 0$ $(i = 1, \ldots, n)$ which happens if and only if f is a constant.

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Remark: Example (ii) leads us to make the following convention: If $d(x) = 0$, then we set $(\mathscr{R}_G)_x = 0$; if $d(x) = 0$, then $(\mathscr{R}_G)_x$ is as given in Definition 1.7.

3. *Flat G-Structures on* $\mathbf{R}^n$ (see [9]). Let $\mathbf{R}^n$ be real Euclidean n-space with global linear coordinates $(x^1, \ldots, x^n)$, let $G \subset GL(n, \mathbf{R})$ be as in § I.1 and denote by $\tilde{G}$ the inhomogeneous linear group with homogeneous part G. Let f be the frame $\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right)$ at the origin $\mathbf{o}$, and let $G \to B_G \to \mathbf{R}^n$ be the bundle of G-frames where B_G = orbit of f under $\tilde{G}$. (Remark: we have an exact sequence of groups $G \longrightarrow \tilde{G} \xrightarrow{\pi} \mathbf{R}^n$ and this may be identified with the fibration $G \to B_G \to \mathbf{R}^n$.) We shall describe Θ_G for this integrable real-analytic G-structure on $\mathbf{R}^n$; it will clearly suffice to describe the stalk $(\Theta_G)_o$ at the origin. Letting

$e_i = \left.\frac{\partial}{\partial x^i}\right]_o$, any germ $\xi(x)$ of an analytic vector field at o may be written

$$\xi(x) = \sum_{i, j_1, \ldots, j_q} (\xi^i_{j_1, \ldots, j_q} x^{j_1} \ldots x^{j_q}) e_i \tag{1.4}$$

where $\xi^i_{j_1, \ldots, j_q}$ is symmetric in the indices $j_\alpha (\alpha = 1, \ldots, q)$.

Proposition 1.3: *$(\Theta_G)_o$ consists precisely of those convergent series* (1.4) *satisfying the following condition: For each q and for any $q - 1$ vectors $\lambda_1 = (\lambda_1^1, \ldots, \lambda_1^n), \ldots, \lambda_{q-1} = (\lambda_{q-1}^1, \ldots, \lambda_{q-1}^n)$*

$$L^i_j = \sum_{j_1, \ldots, j_{q-1}} \xi^i_{j_1, \ldots, j_q} \lambda_1^{j_1} \ldots \lambda_{q-1}^{j_{q-1}} \in \mathfrak{g} . \tag{1.5}$$

Remark: Let $S^q = q^{th}$ symmetric product; then (1.5) states that $\xi^i = (\xi^i_{j_1, \ldots, j_q}) \in \mathrm{Hom}(S^q(\mathbf{R}^n), \mathbf{R}^n)$ and, under the pairing

$$\mathrm{Hom}(S^q(\mathbf{R}^n), \mathbf{R}^n) \otimes S^{q-1}(\mathbf{R}^n) \to \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n) = gl(n, \mathbf{R}), \ \xi^i \otimes S^{q-1}(\mathbf{R}^n) \to \xi \in gl(n, \mathbf{R}) .$$

Definition 1.9: We let $\mathfrak{g}^q \subsetneqq \mathrm{Hom}(S^q(\mathbf{R}^n), \mathbf{R}^n)$ be those tensors ξ^i satisfying (1.5).

Remark: $\mathfrak{g}^0 \cong \mathbf{R}^n$ and $\mathfrak{g}^1 \cong \mathfrak{g}$.

Definition 1.10: G is said to be of *finite type* if $\mathfrak{g}^q = 0$ for some $q > 0$. If G is of finite type, then the least q such that $\mathfrak{g}^q = 0$ is the *order of G*.

Examples: (i) If G is discrete, then it is of finite type of order 1, (ii) if $G = O(n)$, then $\mathfrak{g}^2 =$ tensors ξ^i_{jk} satisfying $\xi^i_{jk} = -\xi^j_{ik} = \xi^i_{kj}$; thus $\mathfrak{g}^2 = 0$ and G has order 2, (iii) if $G = GL(n)$ or $Sp(n)$, then G is of infinite type; this is also true if $G = SL(n)$, (iv) if $G = C(n)$ is the conformal group in n-variables, then G is infinite if $n = 2$; otherwise, it is of finite type of order 3, (v) if G = non-singular matrices of the form $\begin{pmatrix} A & B \\ O & C \end{pmatrix}$, then G is of infinite type, (vi) the question of what irreducible linear groups are of finite type is taken up in [9]; also, a general class of linear groups of finite type is given in § VI.3 below.

Let G be connected. We shall construct a sequence $G^\mu (G^0 = G)$ of linear groups by giving the linear Lie algebra $\mathfrak{g}_k$ of G^k. Let $\dim G = d$; then $\mathfrak{g}_1 \subset \subset gl(n + d, \mathbf{R})$ is the abelian Lie algebra consisting of all matrices of the form $\xi^2 = \begin{pmatrix} 0 & \xi^2 \\ 0 & 0 \end{pmatrix}$ where $\xi^2 \in \mathfrak{g}^2$. (This makes sense since $\xi^2 \in \mathrm{Hom}(\mathbf{R}^n, \mathfrak{g})$.) In general, $\mathfrak{g}_k (k > 0)$ is abelian and is defined by $\mathfrak{g}_k = (\mathfrak{g}_{k-1})_1$. In this way we get a sequence of linear groups $G = G^0, G^1, G^2, \ldots$; G is of finite type of order q if, and only if, $G^q = I$, $G^{q-1} \neq 1$.

II. Deformations of G-Structures

1. *Geometric Definition.* Let D be a manifold with parameter t and distinguished point $t_0 \in D$, and let $\sigma : X \to B/G$ be a G-structure on X.

Intuitively, a deformation of σ with parameter space D should be given by a family of cross-sections $\sigma(t) : X \to B/G$ $(t \in D)$ with $\sigma(t_0) = \sigma$ which is smooth in t. However, this definition is too broad, as it clearly leads to an infinite dimensional variation space. Example: if g is any Riemannian metric an X, the, for any symmetric tensor ξ, $g + t\xi$ gives a deformation of g for t

small. We shall give a definition which will generalize the complex analytic case and which, we hope, will be of a geometric interest. We first give an example.

Example: Let $G = O(n)$ and let $g \in \mathrm{Hom}(S^2 T(X), \mathbf{R})$ be the Riemannian metric of the Levi-Civita connection. Then we shall define a deformation of g to be given by a family $g_t (g_{t_0} = g)$ of *locally isometric* Riemannian structures on X. For example, if X is a real 2-torus with angular parameters θ_1, θ_2, and if $g = d\theta_1^2 + d\theta_2^2$, then such a family g_t is given by $g_t = d\theta_1^2 + (1-t)\, d\theta_2^2 (-\varepsilon < < t < \varepsilon)$. Or we may set $g_t = d\theta_1^2 + 2t\, d\theta_1\, d\theta_2 + d\theta_2^2$; both of these families give locally isometric structures which are, however, globally inequivalent. Or, if $g = e^{\mathrm{Sin}\,\theta_2} d\theta_1^2 + d\theta_2^2$, then $g_t = e^{\mathrm{Sin}\,\theta_2} d\theta_1^2 + (1-t)\, d\theta_2^2$ gives a deformation, but $g_t = e^{\mathrm{Sin}\,\theta_2} d\theta_1^2 + 2t\, d\theta_1\, d\theta_2 + d\theta_2^2$ does *not*.

We now give a formal definition. Let D be a neighborhood of $\mathbf{o}$ in $\mathbf{R}^m$ and let $t = (t^1, \ldots, t^m)$ be a parameter in D. Then a deformation of the G-structure $\sigma : X \to B/G$ is given by a family of G-structures $\sigma(t) : X \to B/G (\sigma(0) = \sigma$, $t \in D)$, depending smoothly on t, and satisfying the following condition:

(2.1) *Condition A:* There exists a covering $\mathbf{U} = \{U_i\}$ of X and local sections $\hat{\sigma}_i(t) : U_i \to B|\pi^{-1}(U_i)$ with $\tau_G(\hat{\sigma}_i(t)) = \sigma(t)$ (as described in § I), and, for each i, there exist a family of local bi-mappings $\varphi_i(t) : U_i \to U_i (t \in D)$ and mappings $g_i(t) : U_i \to G$ such that

$$\varphi_i(t)_* \hat{\sigma}_i(0) = \hat{\sigma}_i(t) g_i(t)\ . \tag{2.2}$$

Remarks: Equivalent to (2.2) is the equation $\varphi_i(t)^* \sigma(0) = \sigma(t)$. If the G-structure given by σ is integrable, transitive, or normal, then so are the structures $\sigma(t)$. Also, if $G = 0(n)$ and the Riemannian structure is locally symmetric, then the deformed structures are also. Finally, if $G = I$ and if the structure functions $C^i_{jk}(x)$ are constant, then they are constant for any deformed structure.

Let D^1 be a neighborhood of $\mathbf{o}$ in $\mathbf{R}^m$ with parameter t^1, and let $\sigma(t^1) : X \to \to B/G(\sigma(0) = \sigma, t^1 \in D^1)$ be another deformation of σ. Then $\sigma(t^1)$ is *equivalent* to $\sigma(t)$ if there exists a bi-map $\psi : D \to D^1$ and a family of bi-maps $\boldsymbol{\psi}(t) : X \to \to X (t \in D)$ such that $\boldsymbol{\psi}(t)_* \sigma(t) = \sigma(\psi(t))$. With this definition we may define: (i) trivial families of deformations, (ii) germs of deformations, and (iii) universal germs of deformations in an obvious fashion.

The above definition is simple enough, but is ill suited for analytical purposes, so we shall give another equivalent definition.

2. *Basic Sheaves; Coordinates of the First Kind.* Let $\sigma(t)$ $(t \in D)$ be a deformation of $\sigma(0) = \sigma : X \to B/G$ as described above. We define a differentiable manifold $\mathscr{V}$ with projection $\tilde{\omega} : \mathscr{V} \to D$ by setting $\mathscr{V} = X \times D$ and $\tilde{\omega}(x, t) = t$. We define now a G-structure along the fibres of $\mathscr{V}$. Let T_F be the sub-bundle of $T(\mathscr{V})$ given by vertical vectors (i.e. tangent vectors ξ with $\tilde{\omega}_*(\xi) = 0$); then T_F has structure group $GL(n, \mathbf{R})$; let $GL(n, \mathbf{R}) \to \mathscr{B} \to \mathscr{V}$ be the principal bundle.

We then have a diagram

$$\begin{tikzcd} \mathscr{B} \arrow[rr] \arrow[dr] & & \mathscr{B}/G \arrow[dl] \\ & \mathscr{V} & \end{tikzcd}$$

, and the deformation $\sigma(t)$ gives us a section $\Sigma : \mathscr{V} \to \mathscr{B}/G$ defined by $\Sigma | X \times \{t\} = \sigma(t)$. We now investigate what effect Condition A ((2.1)) has on this fibre space picture.

Let $\mathscr{T}$ = sheaf of germs of vector fields on $\mathscr{V}$; $\mathscr{T}_F$ = germs of vector fields along the fibres of $\mathscr{V} \xrightarrow{\tilde{\omega}} D$, and $\mathscr{T}_D$ = sheaf of germs of vector fields on D. Then we have

$$0 \longrightarrow \mathscr{T}_F \longrightarrow \mathscr{T} \xrightarrow{\tilde{\omega}_*} \mathscr{T}_D \longrightarrow 0 . \tag{2.3}$$

We now let $\hat{\Xi}_G$ = sheaf of germs of vector fields ξ on $\mathscr{V}$ such that $(\exp t\xi)_* \Sigma = \Sigma$; set $\Psi_G = \hat{\Xi}_G \cap \mathscr{T}_F$, and then we have

$$0 \longrightarrow \Psi_G \longrightarrow \hat{\Xi}_G \xrightarrow{\varphi} Q \longrightarrow 0 \quad \text{for some quotient sheaf } Q . \tag{2.4}$$

Recall that for $\xi \in \mathscr{T}_x$ (= stalk at $x \in \mathscr{V}$), we defined $p_x(\xi) = \xi(x)$. Then we have:

Proposition 2.1: *Condition A implies that*

$$\varphi \circ p_x((\hat{\Xi}_G)_x) = p_x(\tilde{\omega}^{-1}(\mathscr{T}_D))_x .$$

Remark: Intuitively, we have a local infinitessimal G-motion in each *horizontal* direction of $\mathscr{V} \xrightarrow{\tilde{\omega}} D$.

Proof: For simplicity, we suppose that $\dim D = 1$. We choose a coordinate neighborhood U_i on X such that $U_i \times D$ is a neighborhood of $x = (x^0, t_0) \in \mathscr{V}$. A germ in $\tilde{\omega}^{-1}(\mathscr{T}_D)$ is given by a vector field $f(x_i, t)\frac{\partial}{\partial t}$ (where $f(x_i, t)$ is a C^∞ function), and $p_x\left(f(x_i, t)\frac{\partial}{\partial t}\right) = f(x_i^0, t^0)\frac{\partial}{\partial t}\Big]_{t=t^0} \in T_{(x_i^0, t^0)}(\mathscr{V})$; thus $p_x(\tilde{\omega}^{-1}(\mathscr{T}_D))_x$ is given by the vectors $\lambda\frac{\partial}{\partial t}\Big]_{t=t_0} \in T_{(x_i, t_0)}(\mathscr{V})$ ($\lambda \in \mathbf{R}$); we must produce a germ $\xi(x_i, t) = \sum_\alpha \xi^\alpha(x_i, t)\frac{\partial}{\partial x_i^\alpha} + \lambda(x_i, t)\frac{\partial}{\partial t}$ in $(\hat{\Xi}_G)_x$ such that $\lambda(x_i^0, t_0) \neq 0$, since $\varphi \circ p_x(\xi(x_i, t_0)) = \lambda(x_i^0, t)\frac{\partial}{\partial t}\Big]_{t=t_0}$.

By (2.2) we have a 1-parameter family of local bi-maps $\varphi_i(t) : U_i \to U_i (t \in D)$ such that $\varphi_i(t)_* \sigma(0) = \sigma(t)$. We define a 1-parameter family of local bi-maps $\psi_s : U_i \times D \to U_i \times D$, ψ_0 = Identity, by $\psi_s(x_i, t) = (\varphi_i(t+s)\,\varphi_i(t)^{-1}x_i, t+s)$ (D being shrunk if necessary). Then $\psi_s(x_i, t)_* \Sigma(x_i, t) = \Sigma(x_i^1, t+s)$ where $x_i^1 = \varphi_i(t+s)\,\varphi_i(t)^{-1}x_i$, and thus ψ_s is a 1-parameter family of local bi-G-mappings on $\mathscr{V}$, and $\frac{d\psi_s}{ds}\Big]_{s=0}$ is a germ in $\hat{\Xi}_G$.

But $\frac{d\psi_s}{ds}\Big]_{s=0} = \Sigma\, \xi^\alpha(x_i, t)\frac{\partial}{\partial x_i^\alpha} + \frac{\partial}{\partial t}$, and this proves the Proposition.

Corollary: *Let $T_D \subset \mathcal{T}_D$ be the germs of vector fields $\sum_{i-1}^{m} \lambda^i \frac{\partial}{\partial t^i}$ with constant coefficients. Then we have a natural diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Psi_G & \longrightarrow & \Xi_G & \xrightarrow{\omega} & Q & \longrightarrow & 0 \\ & & & & & & \uparrow & & \\ & & & & & & T_D & & \\ & & & & & & \uparrow & & \\ & & & & & & 0 . & & \end{array}$$

Definition 2.1: We set $\Xi_G = \varphi^{-1}(T_D)$ so that we have

$$0 \longrightarrow \Psi_G \longrightarrow \Xi_G \xrightarrow{\tilde{\omega}} T_D \longrightarrow 0 , \tag{2.5}$$

and we call (2.5) the *basic sheaf sequence* of the deformation $\mathscr{V} \to D$.

Remark: Although we shall not make use of it, it is true that the hypothesis of Proposition 2.1 implies Condition A, and this gives an alternate definition of our deformations. A modified version of this will be given below.

Let $\{U_i\}$ be a coordinate covering of X with coordinates $x_i = (x_i^1, \cdots, x_i^n)$ in U_i and transition functions $x_i = f_{ij}(x_j)$. Then, relative to the product covering $\{U_i \times D\}$ of $\mathscr{V}$, a deformation as in § II.1 is given by a system $(\{x_i\}, \{f_{ij}\}, \{\hat{\sigma}_i(t)\}, \{\varphi_i(t)\}, \{g_i(x_i, t)\})$ satisfying (2.2). For brevity, we write this system as $(\{x_i\}, \{f_{ij}\}, \{\varphi_i(t)\})$.

Definition 2.2: The system $(\{x_i\}, \{f_{ij}\}, \{\varphi_i(t)\})$ is called a *coordinate system of the first kind* an $\mathscr{V}$.

Example: If X has an integrable complex structure, then we let $\mathbf{U} = \{U_i\}$ be a coordinate covering with holomorphic coordinates $Z_i = (Z_i^1, \ldots, Z_i^n)$. Then, since the deformed structures are integrable, a coordinate system of the first kind is given by giving C^∞ functions. $\mathcal{T}_i(Z_i, t) = (\mathcal{T}_i^1(Z_i, t), \ldots, \mathcal{T}_i^n(Z_i, t))$ such that $\mathcal{T}_i(Z_i, 0) = Z_i$ and such that $\mathcal{T}_i(Z_i, t)$, for fixed t, gives holomorphic coordinates on the manifold $X_t = \hat{\omega}^{-1}(t)$ corresponding to $\sigma(t)$: $X \to \mathscr{B}/GL(n, C)$.

As another example, if $G = O(n)$, then the G-structure is given, in U_i, by a metric $ds^2 = g_{ij}(x)\, dx^i dx^j$. Then $\sigma(t) \mid U_i$ is given by $ds_t^2 = g_{ij}(x, t) dx^i dx^j$ and a family of local bi-maps $\varphi_i(t): U_i \to U_i$ such that $\varphi_i(t)^* \, ds_t^2 = ds^2$.

3. *Coordinates of the Second kind.* Continuing on with the example of complex structure of the end of § II.2, we recall that, in [7], a deformation of a complex manifold was given by keeping the *same* coordinates $Z_i = (Z_i^1, \ldots, Z_i^n)$ locally, but then varying the transition functions. Thus, $X_t = \tilde{\omega}^{-1}(t)$ has a coordinate covering $\mathbf{U} = \{U_i\}$ with coordinates Z_i in U_i, and, in $U_i \cap U_j$, $Z_i = f_{ij}(Z_j, t)$ where $f_{ij}(Z_i, 0) = f_{ij}(Z_j)$ are the transition functions of $X \cong X_0$. The generalization of this to G-structures is what we shall call *coordinates of the second kind.* Before making the formal definition, we see what these coordinates will mean for the Riemannian structure at the end of § II.2. We shall have a fibre-space $\mathscr{V} \xrightarrow{\tilde{\omega}} D$ where $\tilde{\omega}^{-1}(t) = X_t$ will have a Riemannian

structure g_t; coordinates of the second kind will be given by a coordinate covering $\{W_i\}$ of $\mathscr{V}$, such that $\tilde{\omega}(W_i) = D$, with coordinates (y_i, t) in W_i, and transition functions $y_i = f_{ij}(y_j, t)$, $t = t$. Finally, the family of Riemannian metrics along the fibres of $\mathscr{V} \xrightarrow{\tilde{\omega}} D$, when restricted to W_i, will be given by $ds^2 = g_{\alpha\beta}(y)\, dy_i^\alpha\, dy_j^\beta + h_\alpha(y, t)\, dy_1^\alpha\, dt + p(y, t)\, dt^2$. Thus, $ds^2 | W_i \cap X_t$ is the same for all t.

To make all this precise, we assume that we are given a manifold X and a G-structure $\sigma : X \to B/G$. A deformation of σ will be given, first of all, by a fibre space $\mathscr{V} \xrightarrow{\tilde{\omega}} D$ such that $\tilde{\omega}^{-1}(t)$ is a manifold. We then require a section $\Sigma : \mathscr{V} \to \mathscr{B}/G$, where, setting $\Sigma | X_t = \sigma(t) : X_t \to B_t/G$, we assume that $X_0 = \tilde{\omega}^{-1}(0)$ is G-isomorphic to X. So far we have no restriction that the deformation of σ should locally preserve structure. This is given by

Condition B: We assume that we have a covering $\boldsymbol{W} = \{W_i\}$ of $\mathscr{V}$, $\tilde{\omega}(W_i) = D$, with coordinates (y_i, t) in W_i and transition functions $y_i = f_{ij}(y_j, t)$, $t = t$ in $W_i \cap W_j$. For each i, $\Sigma | W_i$ is given by $\hat{\sigma}_i(y_i, t) : W_i \to GL(n, \mathbf{R})$, and we require that there exist a mappings $g_i(y_i, t) : W_i \to G$ such that, for each *fixed* t,

$$\hat{\sigma}_i(y_i, t) = \hat{\sigma}_i(y_i, 0)\, g_i(y_i, t)\,. \tag{2.6}$$

We call such a coordinatization *coordinates of the second kind.*

There is a consistency relation which must be satisfied in order that (2.6) be invariant under coordinate changes. This requirement is that there exist $g_{ij} : W_i \cap W_j \to G$ such that

$$\mathscr{J}_y(f_{ij}(y_{ij}, t))\, \hat{\sigma}_j(y_j, t) = \hat{\sigma}_i(y_i, t)\, g_{ij}(y_i, t) \tag{2.7}$$

where $\mathscr{J}_y(f_{ij}(y_j, t))$ is the Jacobian of $f_{ij}(y_j, t)$ with respect to the y-variables. We write (2.7) more briefly as

$$\mathscr{J}_y(f_{ij}(t))\, \hat{\sigma}_j(t) = \hat{\sigma}_i(t)\, g_{ij}(t)\,. \tag{2.8}$$

Let $\sigma : X \to B/G$ and suppose that we are given a differentiable family $\sigma(t) : X \to B/G$ ($\sigma(0) = \sigma$, $t \in D$) of G-structures. Then we may form $\mathscr{V} \xrightarrow{\tilde{\omega}} D$ and we have $\Sigma : \mathscr{V} \to B/G$.

Theorem 2.1: *On the family $\mathscr{V} \xrightarrow{\tilde{\omega}} D$, Condition A is completely equivalent to Condition B.*

Proof: Suppose that, relative to covering $\{U_i \times D\}$ of $\mathscr{V}$, $(\{x_i\}, \{f_{ij}\}, \{\hat{\sigma}_i(t)\}, \{\varphi_i(t)\})$ gives a coordinate system of the first kind (Condition A); then $\varphi_i(t)_* \hat{\sigma}_i(0) = \hat{\sigma}_i(t)\, g_i(t)$. We define new coordinates (y_i, t) in $W_i \cong U_i \times D$ by setting $y_i = \varphi_i(t)^{-1} x_i$, $t = t$, and we write $(y_i, t) = \tau_i(x_i, t)$. For fixed t, we have $\hat{\sigma}_i(y_i, t) = \tau_i(x_i, t)_* \hat{\sigma}_i(x_i, t) = \varphi_i(t, x_i)_*^{-1} \hat{\sigma}_i(x_i, t)$

$$= \hat{\sigma}_i(x_i, 0)\, g_i(t)^{-1}$$
$$= \hat{\sigma}_i(y_i, 0)\, g_i(t)^{-1} \text{ (Since } \varphi_i(0) = \text{Identity)}\,.$$

Thus equation (2.6) is satisfied, we must verify (2.8). The change of coordinates from (y_j, t) to (y_i, t) is given by: $t = t$, and $y_i = \varphi_i(t)^{-1}(x_i) = \varphi_i(t)^{-1} f_{ij}(x_j) = \varphi_i(t)^{-1} \circ f_{ij} \circ \varphi_j(t)\,(y_j) = f_{ij}(y_j, t)$. Thus, for fixed t, we have

$$\begin{aligned}
&\mathscr{I}_y\big(f_{ij}(y_{ij}, t)\big)\, \sigma_j(y_j, t) \\
&\quad = f_{ij}(y_j, t)_*\, \hat{\sigma}_j(y_j, t) \\
&\quad = \varphi_i(t)_*^{-1} (f_{ij})_*\, \varphi_j(t)_*\, \varphi_j(t)_*^{-1}\, \hat{\sigma}_j(x_j, t) \\
&\quad = \varphi_i(t)_*^{-1}\, \hat{\sigma}_i(x_i, t)\, g_i(t) \\
&\quad = \hat{\sigma}(y_i, t)\, g_i(t).\ \ \text{Thus Condition A implies Condition B}.
\end{aligned}$$

The converse is proven by reversing the above calculations. Q. E. D.

Definition 2.3: Let X be a manifold, and let $\sigma : X \to B/G$ be a G-structure on X. Then we define a *deformation*, with parameter space D, of σ by: (i) a family $\sigma(t) : X \to B/G$ ($\sigma(0) = \sigma$, $t \in D$) of G-structures satisfying Condition A; as (ii) a fibre space $\mathscr{V} \xrightarrow{\tilde{\omega}} D$ with $\Sigma : \mathscr{V} \to \mathscr{B}/G$ such that $\Sigma | \tilde{\omega}^{-1}(0) = \Sigma | X_0 : X_0 \to B/G$ is G-isomorphic to X and such that Condition B is satisfied.

Remark: We give now a rephrasing, due to CHERN, of the definition of deformation of G-structure. Let $G \to B_G \to X$ be a G-structure on X, I the interval $[-1, 1] \subset \mathbf{R}$, and G^* the linear group of $(n+1) \times (n+1)$ matrices $g^* = \begin{pmatrix} g & * \\ 0 & 1 \end{pmatrix}$ ($g \in G$, $* \in \mathbf{R}$). Set $\mathscr{V} = X \times I$, and let $\tilde{w} : \mathscr{V} \to I$ be the projection. Then a deformation, with parameter space I, of the G-structure on X is given by:

(i) A G^*-structure $G^* \to B_{G^*} \to \mathscr{V}$ on $\mathscr{V}$

(ii) This G^*-structure is required to admit *G-local cross-sections* in the fibering $\mathscr{V} \xrightarrow{\tilde{\omega}} I$ in the following sense: For each point $v_0 = (x_0, t_0) \in \mathscr{V}$, there should exist neighborhoods U of x_0 in X, and W of t_0 in I such that the induced G^*-structure on $U \times W \subset \mathscr{V}$ is G^* isomorphic to a G'-structure where G' = all matrices $\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$ ($g \in G$).

(iii) The G^*-structure on $\mathscr{V}$, when restricted to $\tilde{w}^{-1}(0)$, gives $G \to B_G \to X$.

We shall see below that the group $H^1(X, \Theta_G)$ may be interpreted as the *obstruction* to reducing the structure group of any such G^*-structure on $\mathscr{V}$ down to the group G'.

III. The Relation with Sheaf Theory

1. The Infinitesimal Deformation. Let $\sigma : X \to B/G$ give rise to a G-structure $G \longrightarrow B_G \xrightarrow{\pi} X$, and let $\mathscr{V} \xrightarrow{\tilde{\omega}} D$ ($\tilde{\omega}^{-1}(0) \cong X$) be a deformation of this structure. We assume, for simplicity, that $\dim D = 1$. Thus we have a 1-parameter family $\sigma(t) : X \to B/G$ of G-structures, with $\sigma(0) = \sigma$, and such that each $\sigma(t)$ is locally isomorphic to $\sigma(0)$. From the basic sheaf sequence (2.5), we get the exact cohomology sequence

$$\text{(3.1)} \qquad \cdots \longrightarrow H^0(\mathscr{V}, \Xi_G) \xrightarrow{\tilde{\omega}} H^0(\mathscr{V}, T_D) \xrightarrow{\delta} H^1(\mathscr{V}, \Psi_G) \longrightarrow \cdots.$$

Proposition 3.1: *If X is compact, and if $\delta = 0$ in (3.1), then the deformation $\mathscr{V} \xrightarrow{\tilde{\omega}} D$ is trivial; i.e. there exists a $D^1 \subset D$ such that $\{o\} \subset D^1$, and a 1-parameter family of bi-maps $f(t^1): X_0 \to X_{t^1}$, ($t^1 \in D^1$) such that $f(t^1)_* \sigma(0) = \sigma(t^1)$.*

Proof: The proof is not hard, and goes as follows: Let $\{W_i\}$, $\{y_i\}$, $\{f_{ij}(y_j, t)\}$ be a coordinate system of the second kind on $\mathscr{V}$. Then we may consider, in W_i, the vector field $\frac{\partial}{\partial t}$ as a section of $T_D | W_i$. If $W_i \cap W_j \neq 0$, then $y_i^\alpha = f_{ij}^\alpha(y_j, t)$ ($\alpha = 1, \ldots, n$) and $t = t$, thus $\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \sum_{\alpha=1}^{n} \frac{\partial f_{ij}^\alpha}{\partial t}(y_j, t) \frac{\partial}{\partial y_i^\alpha}$. Now, by differentiating (2.8), we see that $\frac{\partial}{\partial t} + \sum_{\alpha=1}^{n} \frac{\partial f_{ij}^\alpha}{\partial t}(y_j, t) \frac{\partial}{\partial y_i^\alpha}$ is a section over $W_i \cap W_j$ of $\Xi_G | W_i \cap W_j$, and $\tilde{\omega}\left(\frac{\partial}{\partial t} + \sum_{\alpha=1}^{n} \frac{\partial f_{ij}^\alpha}{\partial t}(y_j, t) \frac{\partial}{\partial y_i^\alpha}\right) = \frac{\partial}{\partial t}$ is a section of $T_D | W_i \cap W_j$. This all means that $\frac{\partial}{\partial t}$ is an element of $H^0(\mathscr{V}, T_D)$, as well as a global vector field on D. In (3.1) we let $\theta \in H^0(\mathscr{V}, \Xi_G)$ be such that $\tilde{\omega}(\theta) = \frac{\partial}{\partial t}$; θ exists since $\delta\left(\frac{\partial}{\partial t}\right) = 0$.

Now, since X is compact, we may choose $\epsilon > 0$ such that θ generates a 1-parameter group $\exp(s\theta)$ of transformations on $\mathscr{V}$. We observe also that $\exp(s\theta)$ "covers" the 1-parameter group $\exp\left(s \frac{\partial}{\partial t}\right)$ acting on D, in the sense that, for $x \in \mathscr{V}$, $\tilde{\omega}(\exp(s\theta)x) = \exp\left(s \frac{\partial}{\partial t}\right) \tilde{\omega}(x)$. From this it follows that $\exp(s\theta)$ is a fibre-preserving transformation group on $\mathscr{V} \xrightarrow{\tilde{\omega}} D$, and then we finally conclude that $\exp(s\theta)$ establishes a G-isomorphism $\exp(s\theta): X_t \xrightarrow{\approx} X_{t+s}$ ($X_\tau = \tilde{\omega}^{-1}(\tau)$), where now we have perhaps shrunk D to a smaller domain D^1.

Q.E.D.

Remark: If we let $\mathbf{N}$ be the nerve of the covering $\{W_i\}$, then in (3.1) $\delta\left(\frac{\partial}{\partial t}\right)$ is represented in $H^1(\mathbf{N}, \Psi_G)$ by the 1-cocycle $\{\xi_{ij}(y, t)\}$ where

$$\xi_{ij}(y, t) = \sum_{a=1}^{n} \frac{\partial f_{ij}^\alpha}{\partial t}(y_j, t) \frac{\partial}{\partial y_i^\alpha}.$$

We now let $\Theta_{G,t}$ be the sheaf corresponding to $\sigma(t): X \to B/G$, and observe that the restriction mapping $r_t: \mathscr{V} \to X_t = \tilde{\omega}^{-1}(t)$ takes Ψ_G into $\Theta_{G,t}$; thus we have $r_t: H^1(\mathscr{V}, \Psi_G) \to H^1(X_t, \Theta_{G,t})$.

Definition 3.1: $r_0\left(\delta\left(\frac{\partial}{\partial t}\right)\right) \in H^1(X, \Theta_G)$ is called the *infinitesimal deformation* of the family $\mathscr{V} \xrightarrow{\tilde{\omega}} D$ (see [7]).

2. Deformations and Sheaves of non-Abelian Groups. We now give another way of looking at the realation between deformation theory and sheaf theory; the idea of this approach goes back to A. Haefliger.

Proposition 3.2: *Let $G \to B_G \to X$ be a G-structure where X is compact. Then the germs of 1-parameter families of deformations of this structure may be identified with $H^1(X, \Gamma_G[t])$, where $\Gamma_G[t]$ was given in Definition 1.4.*

Proof: Let $D \supset \{o\}$ be an open set in $\mathbf{R}$, and let $\mathscr{V} \xrightarrow{\tilde{\omega}} D$ be a deformation of the G-structure on $X = X_0 = \tilde{\omega}^{-1}(0)$. Furthermore, let $\{U_i\}$ be a covering of X with nerve N, and let $\{U_i \times D\}$, $\{f_{ij}(X_j)\}$, $\{\varphi_i(t)\}$ $(t \in D)$ be a coordinate system of the first kind on $\mathscr{V}$. We form the 1-cochain $\gamma = \{\gamma_{ij}\}$ in $C^1(\mathrm{N}, \Gamma_G[t])$ by setting $\gamma_{ij}(t) = \varphi_i(t) \cdot \varphi_j(t)^{-1} (t \in D)$. Then clearly $\gamma \in Z^1(\mathrm{N}, \Gamma_G[t])$, and the set mapping $\mathscr{V} \to \gamma$ behaves under refinements of the covering. If now $\gamma \in B^1(\mathrm{N}, \Gamma_G[t])$, then $\gamma_{ij}(t) = \tau_i(t)^{-1} \cdot \tau_j(t)$ where $\tau_i(t): U_i \to U_i$ is a 1-parameter family of bi-G-Mappings in U_i with $\tau_i(0) =$ Identity. But in this case, in $U_i \cap U_j$, $\tau_i(t) \cdot \varphi_i = \tau_j(t) \cdot \varphi_j(t)$ and thus the family of mappings $\{\tau_i(t) \cdot \varphi_i(t)\}$ patch together to give a global bi-G-mapping $\varphi(t): X_0 \stackrel{\approx}{\to} X_t$ and then the deformation is trivial. Thus we have an injective set mapping $\mathscr{V} \to \gamma$ from germs of deformations into $H^1(X, \Gamma_G[t])$.

Conversely, given $\gamma = \{\gamma_{ij}\} \in Z^1(\mathrm{N}, \Gamma_y[t])$, we write $\gamma_{ij}(t)$ in the x_j-coordinates and then we have $\gamma_{ij}(t) \cdot f_{jk} \cdot \gamma_{jk}(t) = f_{jk} \cdot \gamma_{ik}(t)$. Then we may define transformations $f_{ij}(t)$ by $f_{ij}(t) = f_{ij} \cdot \gamma_{ij}(t)$, and

$$f_{ij}(t) \cdot f_{jk}(t) = f_{ij} \cdot \gamma_{ij}(t) \cdot f_{jk}(t) \cdot \gamma_{jk}(t) = f_{ik} \cdot \gamma_{ik}(t) = f_{ik}(t) \,.$$

From this, we may construct a deformation $\mathscr{V} \xrightarrow{\tilde{\omega}} D$ given in coordinates of the second kind by $\{y_i\}$, $\{f_{ij}(y_j, t) = f_{ij}(t)\,(y_j)\}$, and this proves the proposition.

There is a sheaf mapping $r: \Gamma_G[t] \to \Theta_G$ defined by sending a germ $f(t)$ in $\Gamma_G[t]$ into $\left.\frac{df}{\partial t}\right]_{t=0}$; thus we have $r: H^1(X, \Gamma_G[t]) \to H^1(X, \Theta_G)$.

Proposition 3.3: *For a germ of deformation* $\gamma = \gamma(t) \in H^1(X, \Gamma_G[t])$, $r(\gamma) \in H^1(X, \Theta_G)$ *is its infinitesimal deformation.*

Proof: Referring to the last paragraph in the proof of Proposition 3.2, we see that $r(\gamma) \in H^1(\mathrm{N}, \Theta_G)$ is given by the 1-cocycle.

$$(f_{ij})^* \left.\frac{\partial \gamma_{ij}^a(t)}{\partial t}\right]_{t=0} \frac{\partial}{\partial x_j^a} = \left.\frac{\partial f_{ij}^a(x_{i,t})}{\partial t}\right]_{t=0} \frac{\partial}{\partial x_j^a} \,. \qquad \text{Q. E. D.}$$

IV. The Higher Order Theory of Deformations

Let $G \to B_G \to X$ be a G-structure. The first order invariant of a deformation $\mathscr{V} \xrightarrow{\tilde{\omega}} D$ (§ II) is the class in $H^1(X, \Theta_G)$ representing the infinitesimal "tangent" to the deformation. We shall now construct a complete set of formal invariants for variation of structure. Knowing these invariants and the properties of X, one may theoretically determine the properties of the deformed manifolds $X_t = \tilde{w}^{-1}(t)$; an example of this will be given in § IV.3 below.

1. *Formal Local Theory of G-Automorphisms.* Let $U \subset \mathbf{R}^n$ be a relatively compact contractible domain, and let $G \to B_G \to U$ be a G-structure given by n-independent Pfaffians $w^1, \ldots, w^n$. We have defined sheaves Θ_G, $\Theta_G[t]$, Γ_G, $\Gamma_G[t]$; we let $\hat{\Theta}_G[t]$, $\hat{\Gamma}_G[t]$ be the corresponding *formal sheaves.* Thus, e.g., a germ $f(t) \in \hat{\Gamma}_G[t]$ is given by a formal series $f(t) = \sum_{\mu=0}^{\infty} f_\mu t^\mu (f_0 = 1)$ such

that, if for any $N > 0$, $f^N(t) = \sum_{\mu=0}^{N} f_\mu t^\mu$, then $f^N(t) \in \Gamma_G[t]$ (modulo t^{N+1}). If $\theta(t) = \sum_{\mu=0}^{\infty} \theta_\mu t^\mu$ is a germ in $\hat{\Theta}_G[t]$, we have

Lemma 4.1: *Each germ of a vector field θ_μ lies in Θ_G.*

Proof: It will suffice to show that $\frac{d\theta(t)}{dt} \in \hat{\Theta}_G[t]$. If

$$\theta^N(t) = \sum_{\mu=0}^{N} \theta_\mu t^\mu, \quad \mathscr{L}_{(t)}\, w^i \equiv \sum_{j=1}^{N} w^j g_j^i(t) \;(\mathrm{mod}\, t^{N+1})$$

where $g_j^i(t) \in \mathfrak{g}$, and thus

$$\mathscr{L}_{\frac{d\theta^N(t)}{dt}} \equiv \sum_{j=1}^{N} w^i \frac{dg_j^i}{dt} \;(\mathrm{mod}\, t^N)\,.$$

Since $\frac{dg_j^i(t)}{dt} \in \mathfrak{g}$, the Lemma follows.

The basic tool in the study of local G-automorphisms is the mapping $\exp : \hat{\Theta}_G[t] \to \hat{\Gamma}_G[t]$ (§ I.2) defined as follows. If $x_1, \ldots, x_n$ are local coordinates, and if $\theta(t) = \Sigma\, \theta^i(t) \frac{\partial}{\partial x_i}$, then $f(t) = \exp\theta(t)$ is defined locally by $f(0)$ = identity and

$$\frac{df^i(t)}{dt} = \theta^i(t) \circ f(t)\,. \tag{4.1}$$

We shall show that, by introducing the *enveloping algebra sheaf* Ω_G of Θ_G, the computations with exp. may be linearized. First we give an example.

Let $F \subset GL(n, \mathbf{R})$ be a linear Lie group with linear Lie algebra $\mathfrak{f} \subset gl(n, \mathbf{R})$. For $A \in \mathfrak{f}$, one defines a 1-parameter subgroup $f^{\#}(t) \subset F$ by $f^{\#}(t) = \mathrm{Exp}(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n t^n$. If $\mathscr{I} \subset gl(n, \mathbf{R})$ is the linear associative algebra generated by $\mathfrak{f}$ then $\mathrm{Exp}(A) \in \mathscr{I}[t]$.

On the other hand, we may think of $\mathfrak{f}$ as the right-invariant vector fields on F. On F we have a G-structure ($G = 1$) given by $m = \dim F$ left-invariant Pfaffians $w^1, \ldots, w^m$, and then the sheaf of germs of right-invariant vector fields on F is just Θ_G. Thus, given $A \in \mathfrak{f}$, we may define $\exp(A) = f(t) \in \Gamma_G[t]$ by (4.1). The connection between $f^{\#}(t)$ and $f(t)$ is simply that, for $x \in F$, $f(t)(x) = f^{\#}(t) \cdot x$ where $\cdot$ is multiplication in F. Furthermore, if we have $B \in \mathfrak{f}$, and if $g(t) = \exp(B)$, $g^{\#}(t) = \mathrm{Exp}(B)$, then $g(t) \circ f(t)\,(x) = g^{\#}(t) \cdot f^{\#}(t) \cdot x$ $(x \in F)$. In summary:

(4.2) The element $\exp A = f(t) \in \Gamma_G[t]$ may be locally expanded as a series in t with coefficients in the enveloping algebra sheaf of Θ_G;

(4.3) The composition $g(t) \circ f(t)$ is given by multiplying, in the enveloping algebra, the series expansions of $g(t)$ and $f(t)$.

Theorem 4.1 given below shows that (4.2) and (4.3), suitably interpreted, hold for general G-structures.

We consider again the G-structure $G \to B_G \to U$. The sheaf Θ_G, as a sheaf of vector fields, may be thought of as a sheaf of differential operators. Thus, over an open set $V \subset U$, the sections of $\Theta_G | V$ generate an associative algebra of differential operators, and we let Ω_G be the sheaf of associative algebras of differential operators so generated by Θ_G; upon adjoining to Ω_G a unit, it becomes the *enveloping algebra sheaf* of Θ_G. The symbol denotes the associative product in Ω_G.

Example: If $\eta = \sum \eta^i \frac{\partial}{\partial x_j}, \zeta = \sum \zeta^j \frac{\partial}{\partial x_i}$, then

$$\eta \cdot \zeta = \sum_{ij} \left(\eta^i \frac{\partial \zeta^j}{\partial x_j} \frac{\partial}{\partial x_i} + \eta^i \zeta^j \frac{\partial^2}{\partial x_i \partial x_j} \right), \text{ and } \eta \cdot \zeta - \zeta \cdot \eta = [\eta, \zeta].$$

There is a canonical filtration $\{\Omega_G^\mu\}$ of Ω_G with $\Omega_G^\mu \subset \Omega_G^{\mu+1}$ and $\Omega_G^\mu \cdot \Omega_G^\nu \subseteqq \Omega_G^{\mu+\nu}$; Ω_G^μ is linearly generated by all products $\theta^{i_1} \dots \theta^{i_\lambda} (\theta^{i_j} \in \Theta_G, \lambda \leqq \mu)$. Briefly, Ω_G^μ consists of the differential operators in Ω_G of degree $\leqq \mu$.

We may clearly define $\hat{\Omega}_G[t]$ as the enveloping algebra sheaf of $\Theta_G[t]$. Given $k > 0$ and a germ $\theta \in \Theta_G$, we define $\operatorname{Exp}^k \theta \in \Omega_G[t]$ by

$$\operatorname{Exp}^k \theta = \sum_{n=0}^{\infty} \times \tag{4.4}$$

$$\times \left(\sum_{k=1}^{n} \sum_{\nu_1 + \cdots + \nu_k = n-k} \frac{(n!)^k t^{kn-(n-k)} \theta^k}{(n-\nu_1)! \dots (n-\nu_k)! (\nu_1 + \cdots + \nu_k + k)(\nu_2 + \cdots + \nu_k + k - 1) \dots \nu_k + 1)} \right)$$

where $\theta^k = \underbrace{\theta \dots \theta}_{k}$. More generally, given $\theta(t) = \sum_{\mu=0}^{\infty} \theta_\mu t^\mu \in \hat{\Theta}_G[t]$, we define $\operatorname{Exp} \theta(t) \in \Omega_G[t]$ by

$$\operatorname{Exp} \theta(t) = \sum_{\lambda=0} \left(\sum_{k=1}^{\lambda} \sum_{\nu_1 + \cdots + \nu = \lambda - k} \frac{\theta \nu_1 \dots \theta \nu_k}{P(\nu_1, \dots, \nu_k)} \right) t^\lambda \tag{4.5}$$

where

$$P(\nu_1, \dots, \nu_k) = (\nu_1 + \cdots + \nu_k + k)(\nu_2 + \cdots + \nu_k + k - 1) - \cdots - (\nu_k + 1).$$

Then $\operatorname{Exp}^k \theta = \operatorname{Exp}(t^k \theta)$. We may now state the generalizations of (4.2) and (4.3):

Theorem 4.1: (CAMPBELL-HAUSDORFF; TAYLOR)

(4.6) (i) *The elements* $\operatorname{Exp}^k \theta$ $(k = 1, 2, \dots, \theta \in \Theta_G)$ *generate a sheaf of multiplicative subgroups* $\Omega_G^* \subset \Omega_G[t]$, *If* $\theta(t) \in \hat{\Theta}_G[t]$, $\operatorname{Exp} \theta(t) \in \Omega_G^*$.

(4.7) (ii) *There is a set isomorphism* $^\# : \Omega_G^* \to \Gamma_G[t]$ *such that* $\operatorname{Exp} \theta(t)^\# = \exp \theta(t)$.

(4.8) (iii) $^\#$ *is a group anti-isomorphism. That is, if* $\Phi(t), \Psi(t) \in \Omega_G^*$, *then* $\Psi(t)^\# \circ \Phi(t)^\# = (\Phi(t) \cdot \Psi(t))^\#$ *where* $\cdot$ *is multiplication in* $\Omega_G[t]$ *and* $\circ$ *is composition of mappings.*

Remarks: (*i*) Every element $f(t) \in \Gamma_G[t]$ has a formal expansion $f(t) = \prod_{k=1}^{\infty} \operatorname{Exp}^k \theta_k$ $(\theta_k \in \Theta_G)$. (ii) (4.6) is a variant of the usual Campbell-Hausdorff ([5], Chap. V); (4.7) is essentially Taylor's theorem; and (4.8) ties (4.6) and (4.7) together. (iii) (4.7) and (4.8) are the generalizations of (4.2) and (4.3) respectively. (iv) A proof of theorem 4.1 will be discussed in the Appendix to § IV below.

For applications, we need one final Proposition. There is an injection $i: \Theta_G \to \Omega_G$ and $i(\Theta_G)$ is an additive direct summand of Ω_G. The sections of Ω_G which belong to $\mathrm{Im}(i)$ are called *Lie elements*. Let $\theta(t)$, $\varphi(t)$, $\psi(t)$ be sections of $\hat{\Theta}_G[t]$; set $\Theta(t) = \sum_{\mu=0}^{\infty} \Theta_\mu t^\mu = \mathrm{Exp}\,\theta(t) \in \Omega_G^*[t]$, and define $\Phi(t)$, $\Psi(t)$ similarly. For any $N > 0$, set

$$\Gamma_N = \Theta_N + \Phi_N - \Psi_N + \sum_{\substack{\sigma+\tau=N \\ \sigma,\tau>0}} \Theta_\sigma \cdot \Phi_\tau \,. \tag{4.9}$$

Proposition 4.1: (i) $\Phi(t)^{\#} \circ \Theta(t)^{\#} = \Psi(t)^{\#}$ *if, and only if,* $\Gamma_N = 0$ *for all* N. (ii) *If* $\Gamma_N = 0$ *for* $1 \leqq N \leqq n$, *then* Γ_{n+1} *is a Lie element.*

Proof: (i) By Theorem 4.1, we may prove that $\Theta(t) \cdot \Phi(t) = \Psi(t)$ if, and only if, $\Gamma_N = 0$ for all N. But $\Theta(t) \cdot \Phi(t) = \left(\sum_{\mu=0}^{\infty} \Theta_\mu t^\mu\right) \cdot \left(\sum_{\nu=0}^{\infty} \Phi_\nu t^\nu\right)$ $= \sum_N \left(\Theta_N + \Phi_N + \sum_{\substack{\sigma+\tau=N \\ \sigma,\tau>0}} \Theta_\sigma \cdot \Phi_\tau\right) t^N$, and (i) clearly follows.

(ii) By (4.6), if $\Gamma(t) = I + \Gamma_N t^N + \Gamma_{N+1} t^{N+1} + \cdots \in \Omega_G^*$, then $\Gamma_N \in \Theta_G$. Define $\Gamma(t)$ by $\Theta(t) \cdot \Phi(t) = \Gamma(t) \cdot \Psi(t)$; then $\Gamma(t) = 1 + \Gamma_{n+1} t^{n+1} + \Gamma_{n+2} t^{n+1} +$, and Γ_{n+1} is a Lie element.

2. *Formal higher order of deformations.* Let $G \to B_G \to X$ be a G-structure. On X we have the sheaves Θ_G, $\hat{\Theta}_G[t]$, $\hat{\Gamma}_G[t]$, Ω_G, $\hat{\Omega}_G[t]$, and Ω_G^* as defined in § IV.1 above. Here $\Theta_G, \hat{\Theta}_G[t]$ are sheaves of Lie algebras, $\Omega_G, \hat{\Omega}_G[t]$ are sheaves of associative algebras, and Ω_G^*, $\hat{\Gamma}_G[t]$ are sheaves of multiplicative groups. There is an inclusion $\Omega_G^* \subset \hat{\Omega}_G[t]$, compatible with multiplication, and an anti-isomorphism $\# : \Omega_G^* \to \hat{\Gamma}_G[t]$. The set $H^1(X, \hat{\Gamma}_G[t])$ parametrizes the germs of *formal* deformation of structure.

Now let $\mathrm{U} = \{U_i\}$ be a sufficiently fine open covering of X with nerve N; also, define a "shift" operator $* : \Omega_G^* \to \hat{\Omega}_G[t]$ by setting $\Phi(t)^* = \Phi(t) - 1$ ($\Phi(t) \in \Omega_G^*$). The following is just a restatement of (i) in Proposition 4.1.

Proposition 4.2: *The elements* $f(t) \in Z^1(\mathrm{N}, \hat{\Gamma}_G[t])$ *are in a* 1—1 *correspondence with the elements* $\Phi(t) \in C^1(\mathrm{N}, \Omega_G^*)$ *satisfying*

$$\delta\Phi(t)^* + \Phi(t)^* \cdot \Phi(t)^* = 0 \quad in \quad C^2(\mathrm{N}, \hat{\Omega}_G[t]) \,. \tag{4.10}$$

The correspondence is given by $\Phi(t)^{\#} = f(t)$.

Remarks: (*i*) The expression $\Phi(t)^* \cdot \Phi(t)^*$ is the cup-product

$$C^1(\mathrm{N}, \hat{\Omega}_G[t]) \otimes C^1(\mathrm{N}, \hat{\Omega}_G[t]) \to C^2(\mathrm{N}, \hat{\Omega}_G[t]) \,.$$

(ii) If we consider formal deformations as given by $\Phi(t) \in C^1(\mathrm{N}, \Omega_G^*)$ satisfyiyng (4.10), then the infinitesimal deformation is given by

$$\varphi = \left.\frac{d\Phi(t)}{dt}\right]_{t=0} \in H^1(\mathrm{N}, \Theta_G) \,. \tag{4.11}$$

Again from Proposition 4.1, we have:

Proposition 4.3: *An element $\Phi(t) \in C^1(\mathrm{N}, \Omega_G^*)$ satisfies $\Phi(t)^\# \in \delta C^0(\mathrm{N}, \Gamma_G[t])$ if, and only if, there exists $\Psi(t) \in C^0(\mathrm{N}, \Omega_G^*)$ satisfying*

$$(4.12) \qquad \delta\Psi(t)^* + \Phi(t)^* \cdot \Psi(t)^* = \Phi(t)^* \quad in \quad C^1(\mathrm{N}, \hat{\Omega}_G[t]) \,.$$

Theorem 4.2: (i) *If $H^1(X, \Theta_G) = 0$, then, for any $\Phi(t) \in C^1(\mathrm{N}, \Omega_G^*)$ satisfying* (4.10). *there exists a $\Psi(t) \in C^0(\mathrm{N}, \Omega_G^*)$ such that* (4.12) *holds.* (ii) *If $H^2(X, \Theta_G) = 0$, then, for any $\varphi \in H^1(\mathrm{N}, \Theta_G)$, there exists a $\Phi(t) \in C^1(\mathrm{N}, \Omega_G^*)$ such that* (4.10) *and* (4.11) *hold.*

Remarks: The meaning of the above statements for deformation theory is the following: (a) If $H^1(X, \Theta_G) = 0$, then any deformation of the G-structure on X is formally trivial; and (b) if $H^2(X, \Theta_G) = 0$, then any class $\theta \in H^1(X, \Theta_G)$ is formally tangent to a deformation.

(ii) In § VIII, we shall show that, for certain G-structures, what has been done here formally can be done "actually".

Proof: (i) We define an element $\xi \in C^q(\mathrm{N}, \Omega_G)$ which lies in $i(C^q(\mathrm{N}, \Theta_G)) \subset C^q(\mathrm{N}, \Omega_G)$ to be a *Lie-cochain.*

Writing $\Phi(t)^* = \sum_{\mu=1}^{\infty} \Phi_\mu t^\mu$, then, by (4.10), $\delta\Phi_1 = 0$ and $\delta\Phi_\mu + \sum_{\sigma+\tau=\mu} \Phi_\sigma \cdot \Phi_\tau = 0$ $(\mu > 1)$. Now Φ_1 is a Lie cochain, and thus there exists a $\Psi_1 \in C^0(\mathrm{N}, \Theta_G)$ satisfying $\delta\Psi_1 = \Phi_1$. Then $\Psi^1(t) = \operatorname{Exp} \Psi_1 \in C^0(\mathrm{N}, \Omega_G^*)$ and satisfies

$$\delta\Psi^1(t)^* + \Phi(t)^* \cdot \Psi^1(t)^* \equiv \Phi(t)^* \pmod{t^2} \,.$$

Suppose now that we have $\Psi^N(t) \in C^0(\mathrm{N}, \Omega_G^*)$ satisfying $\delta\Psi^N(t)^* + {}$ ${}+ \Phi(t)^* \cdot \Psi^N(t)^* \equiv \Phi(t)^* \pmod{t^{N+1}}$. Let $\Psi^N(t)^* = \sum_{\mu=1}^{\infty} \Psi_\mu^N t^\mu$, and define

$$\eta_{N+1} \in C^1(\mathrm{N}, \Omega_G) \quad \text{by} - \eta_{N+1} = \delta\Psi_{N+1}^N + \sum_{\sigma+\tau=N+1} \Phi_\sigma \cdot \Psi_\tau^N \,.$$

Then:

Lemma 4.2: (i) $\delta\eta_{N+1} = 0$ *and* (ii) η_{N+1} *is a Lie cochain.*

Proof: (ii) follows directly from (ii) of Proposition (4.1). As for (i), we have

$$\begin{aligned}\delta\eta_{N+1} &= \sum_{\sigma+\tau=N+1} \delta\Phi_\sigma \cdot \Psi_\tau^N - \sum_{\sigma+\tau=N+1} \Phi_\sigma \cdot \delta\Psi_\tau^N \\ &= \sum_{\varrho+\sigma+\tau=N+1} \Phi_\varrho \cdot \Phi_\sigma \cdot \Psi_\tau^N - \Big(\sum_{\sigma+\varrho+\tau=N+1} \Phi_\sigma \cdot \Phi_\varrho \cdot \Psi_\tau^N \Big) = 0\end{aligned}$$

by (4.10) and the induction assumption. Q.E.D.

Now we set $\eta_{N+1} = \delta\Psi_{N+1}$ $(\Psi_{N+1} \in C^0(\mathrm{N}, \Theta_G)$ and define $\Psi^{N+1}(t) \in C^0(\mathrm{N}, \Omega_G^*)$ by $\Psi^{N+1}(t) = \Psi^N(t)$. $\operatorname{Exp}^{N+1} \ \Psi_{N+1}$; it then follows immediately that $\delta\Psi^{N+1}(t)^* + \Phi(t)^* \cdot \Psi^{N+1}(t)^* \equiv \Phi(t)^* \pmod{t^{N+2}}$, which proves (i).

To prove (ii), observe that $\operatorname{Exp} \varphi = \Phi^1(t) \in C^1(\mathrm{N}, \Omega_G^*)$ and satisfies $\delta\Phi^1(t)^* + \Phi^1(t)^* \circ \Phi^1(t)^* \equiv 0 \pmod{t^2}$. Suppose we have $\Phi^N(t) \in C^1(\mathrm{N}, \Omega_G^*)$ satisfying $\delta\Phi^N(t)^* + \Phi^N(t)^* \circ \Phi^N(t)^* \equiv 0 \pmod{t^{N+1}}$. Define $\eta_{N+1} \in C^2(\mathrm{N}, \Omega_G)$ by $\eta_{N+1} t^{N+1} \equiv \delta\Phi^N(t)^* + \Phi^N(t)^* \circ \Phi^N(t)^*) \pmod{t^{N+2}}$; η_{N+1} is called the N^{th} *obstruction* to deformation.

Lemma 4.3: (i) $\delta\eta_{N+1} = 0$ *and* (ii) η_{N+1} *is a Lie cochain.*

Proof: Again (ii) follows directly from (ii) of Proposition (4.1). Also, we have

$$\delta\eta_{N+1} = \sum_{\sigma+\tau=N+1} \delta\Phi_\sigma^{N+1}\cdot\Phi_\tau^{N+1} - \sum_{\sigma+\tau=N+1} \Phi_\sigma^{N+1}\cdot\delta\Phi_\tau^{N+1} = -\sum_{\varrho+\sigma+\tau=N+1} \times$$
$$\times\, \Phi_\sigma^{N+1}\cdot\Phi_\varrho^{N+1}\cdot\Phi_\tau^{N+1} + \sum_{\varrho+\sigma+\tau=N+1} \Phi_\sigma^{N+1}\cdot\Phi_\varrho^{N+1}\cdot\Phi_\tau^{N+1} = 0. \quad \text{Q.E.D.}$$

Now $\eta_{N+1} = \delta\varphi_{N+1}$ for some $\varphi_{N+1} \in C^1(\mathrm{N}, \Theta_G)$, and we define $\Phi^{N+1}(t) \in C^1(\mathrm{N}, \Omega_G^*)$ by $\Phi^{N+1}(t) = \Phi^N(t)\cdot\exp^{N+1}\varphi_{N+1}$; it follows immediately that

$$\delta\Phi^{N+1}(t)^* + \Phi^{N+1}(t)\cdot\Phi^{N+1}(t)^* \equiv 0 \pmod{t^{N+2}}. \qquad \text{Q.E.D.}$$

Remarks: By the Campbell-Hausdorff (see the Appendix below), the obstructions may be calculated. For example, if $\varphi = \{\varphi_{\alpha\beta}\} \in H^1(\mathrm{N}, \Theta_G)$, then the *primary obstruction* is $\eta_2 = [\varphi, \varphi] = \{[\varphi_{\alpha\beta}, \varphi_{\beta\gamma}]\} \in C^2(\mathrm{N}, \Theta_G)$. If $\eta_2 = \delta\theta$, then the *secondary obstruction* is given by $\eta_3 = \{(\eta)_{\alpha\beta\gamma}\}$ where $\eta_{\alpha\beta\gamma} = \frac{1}{2}[[\varphi_{\alpha\beta}, \theta_{\alpha\beta}], \theta_{\beta\gamma}] + \frac{1}{6}([\varphi_{\alpha\beta}, \theta_{\beta\gamma}] - 2[\varphi_{\alpha\beta}, \theta_{\beta\gamma}])$, etc.

3. Application: Stability of Infinitesimal Automorphisms. Let $G \to B_G \to X$ be a G-structure on a compact manifold X, and let $\theta \in H^0(X, \Theta_G)$ be an infinitesimal automorphism (i.a.) of the structure. Let $V \xrightarrow{\tilde{\omega}} D$ be a deformation of the G-structure on X; set $X_t = \tilde{\omega}^{-1}(t)$ $(t \in D)$ and $X_0 = X$.

Definition 4.1: The i.a. θ is *stable* under the deformation $\{X_t\}$ if there exists a family $\theta(t)$ of vector-fields on X such that $\theta(0) = \theta$ and, for fixed t, $\theta(t) \in H^0(X_t, \Theta_G)$.

The stability of θ is equivalent to the following: There exists on X a 2-parameter family of transformations $f(s, t)$ of X with the following properties: (i) $f(0, t) =$ Identity; and (ii) For each fixed t, $f(s, t)$ is a 1-parameter group of automorphisms of $G \to B_G(t) \to X_t$. (The family $f(s, t)$ (t fixed) is induced by $\theta(t)$.

We shall now investigate the *formal stability* of the i.a. θ. Following § IV.2, let the deformation $\{X_t\}$ be given formally by an element $\Phi(t) \in C^1(\mathrm{N}, \Omega_G^*)$ satisfying (4.10). Let $\Omega_G^*[s]$ be the sheaf of germs of elements $\Theta(s, t)$ where, for fixed t, $\Theta(s, t) \in \Omega_G^*$ (s is the variable) and $\Theta(s, t)\cdot\Theta(s', t) = \Theta(s + s', t)$. Thus $\Omega_G^*[s]$ is generated by elements of the form $\sum_{\mu=0}^{\infty} \frac{1}{\mu!}(\theta(t))^\mu s^\mu$ $(\theta(t) \in \hat{\Theta}_G[t])$.

Theorem 4.3: *An i.a.* $\theta \in H^0(X, \Theta_G)$ *is stable if, and only if, there exists a* $\Theta(s, t) \in C^0(\mathrm{N}, \Omega_G^*[s])$ *satisfying*

(4.13) (i) $\delta\Theta(s, t)^* + \Phi(t)^*\cdot\Theta(s, t)^* = 0$ (identically in s)

(ii) $\theta = \left.\frac{d\Theta(s, 0)}{ds}\right]_{s=0}$.

Proof: Let $\{(x_1^i, \ldots, x_n^i), f_{ij}(x^j)\}$ be a system of coordinates for X, and let $\{(x^k = (x_1^i, \ldots, x_n^i), f_{ij}(x^j, t)\}$ be a system of coordinates of the second kind on $\mathscr{V}$ (§ II). Then, if $\Phi(t) = \{\Phi_{ij}(t)\}$, we may write $f_{ij}(x^j, t) = (\Phi_{ij}(t)^\# \circ f_{ij})(x^j)$ by the proof of Proposition 3.2. If θ is stable, then we have a family $f(s, t)$ as described above, and, in coordinates, $f(s, t)$ is given by a family $\{f_i(s, t)\}$ of transformations satisfying $f_i(s,t) = f_{ij}(t) \circ f_j(s,t) = \Phi_{ij}(t)^{\#} \circ f_{ij}(t) \circ f_j(s,t)$. If we let $\Theta(s,t) = \{\Theta_i(s,t)\}$ be given by $\Theta_i(s,t)^{\#} = f_i(s,t)$, then $\Theta(s,t) \in C^0(\mathrm{N}, \Omega_G^*[s])$ and satisfies $\Theta_i(s,t)^{\#} = \Phi_{ij}(t)^{\#} \circ \Theta_j(s,t)^{\#}$. But, by theorem 4.1, this is equivalent to $\Theta_i(s,t) = \Phi_{ij}(t) \cdot \Theta_j(s,t)$, or $\delta\Theta(s,t)^* + \Phi(t)^* \cdot \Theta(s,t)^* = 0$. Since this argument is reversible, the theorem is proven. Q.E.D.

Remarks: (i) Just as in § IV.2, we may define *obstructions* to the stability of the i.a. θ. Indeed, the term of degree 0 in t in (4.13) says that $\delta\theta = 0$, which is satisfied. The term of degree 1 in t in (4.13) says that $[\varphi, \theta] = -\delta\theta_1$ for same $\theta_1 \in C^1(\mathrm{N}, \Theta_G)$. Thus $[\theta, \varphi]$ is the *primary obstruction* to the stability of θ.

(ii) This last remark has a geometric interpretation as follows: Let $f(t) = \exp\theta$; then $f(t)$ is a 1-parameter group of G-automorphisms of X; thus $f(t)$ acts on the sheaf Θ_G, and on the cohomology groups $H^q(X, \Theta_G)$. The infinitesimal form of this representation is given by $\theta \cdot (\zeta) = [\theta, \zeta] (\zeta \in H^q(X, \Theta_G))$. Thus, in order that θ be stable, it is *necessary* that the 1-parameter group $f(t)$ act trivially on the cocycle $\varphi \in H^1(X, \Theta_G)$ which is tangent to the deformation.

We close this section with a few related definitions which will be used below. Let $G \to B_G \to X$ and $\mathscr{V} \xrightarrow{\tilde{w}} D$ be as above, and let $f: X \to X$ be a G-automorphism. We say that f is *weakly stable* is there exists a mapping $F: \mathscr{V} \to \mathscr{V}$ which commutes with $\tilde{w}$, which preserves the G-structure along the fibers of $\mathscr{V}$, and which induces f on the fibre $X = \tilde{w}^{-1}(o)$. (The infinitesimal version of this was called stability above.) If the deformation $\{X_t\}$ is given by a family $\sigma(t): X \to B/G$, then weak stability means that we have a 1-parameter family $f(t)$ of diffeomorphisms of X satisfying $f(t)_* \sigma(t) = \sigma(t)$ and $f(o) = f$. If, for all t, $f(t) = f$, we say that f is *strongly stable.* (This corresponds to having $\theta(t) = \theta$ in Definition 4.1.)

We may also speak of a group M of G-automorphisms of X as being weakly or strongly stable. If M acts transitively, we may say that X is *weakly homogeneous* if M is weakly stable, or X is *strongly homogeneous* if M is strongly stable. Examples of these concepts will be taken up in § VI below.

Appendix to § IV. We shall outline here a proof of theorem 4.1.

1. Proof of (4.6)[1]. Let $x_1, \ldots, x_n$ be n-symbols considered as a basis for a real vector space V; the *free associative algebra* $\mathscr{A}$ on $x_1, \ldots, x_n$ is just the full tensor algebra of $V: \mathscr{A} = \bigoplus_{\mu} V^\mu$ where $V^\mu = \underbrace{V \otimes - \otimes V}_{\mu}$ and $V^0 = \mathbf{R}$. Set $\mathscr{A}' = \bigoplus_{\mu > 0} V^\mu$; $\mathscr{A}'$ is an ideal in $\mathscr{A}$. Now any associative algebra $\mathscr{B}$ may be canonically made into a Lie algebra $\mathscr{B}^L$; we let $\mathscr{L} \subset \mathscr{A}^L$ be the Lie subalgebra of $\mathscr{A}^L$ generated by $x_1, \ldots, x_n$. Then $\mathscr{L}$ is the *free Lie algebra* on $x_1, \ldots, x_n$ and $\mathscr{A}$ is the *universal enveloping algebra* of $\mathscr{L}$. If $y_1, y_2, -$ is a basis of $\mathscr{L}$, then the monoicals $y_1^{i_1} \ldots y_m^{i_m}$ form a basis for $\mathscr{A}$. An element $a \in \mathscr{A}$ is a *Lie element* if $a \in \mathscr{L}$; there are two criteria that $a \in \mathscr{A}$ be a Lie element, and we review these. Define a linear mapping

$\beta: \mathscr{A}' \to \mathscr{L}$ by $\beta(x_i) = x_i$ and $\beta(x_i \otimes \cdots \otimes x_{i_m}) = [[[x_i, x_{i_2}]x_{i_3}] \ldots x_{i_m}]$.

[1]) The reference for this Section is [5], Chapter V, §§ 4 and 5; the material of this Section may be viewed as a generalization of the discussion there.

Then (i) $a \in V^m$ is a Lie element if, and only, if $\beta(a) = ma$. The *diagonal mapping* is the unique algebra homomorphism $\delta : \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ satisfying $\delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$. Then; (ii) $a \in \mathscr{A}$ is a Lie element if, and only if, $\delta(a) = a \otimes 1 + 1 \otimes a$. These criteria are due to SPECHT-WEVER and FRIEDRICHS respectively.

Let t be a parameter; for a vector space W, let $W[t]$ be the vector space of formal series $w(t) = \sum_{\mu=0}^{\infty} w_\mu t^\mu (w_\mu \in W)$. We may obviously make $\mathscr{L}[t]$ into a Lie algebra, and $\mathscr{A}[t]$ into an associative algebra. We define $\operatorname{Exp} : \mathscr{A}[t] \to \mathscr{A}[t]$ as follows: if $a(t) \in \mathscr{A}[t]$, then $\operatorname{Exp} a(t) = b(t) = \sum_{\mu=0}^{\infty} b_\mu t^\mu$ is given by

$$b_0 = 1 \quad \text{and} \quad \frac{db(t)}{dt} = a(t) \cdot b(t). \tag{A.1}$$

Remark: $\operatorname{Exp} a(t) \neq e^{a(t)}$.

(A.2) **Proposition A.1:** $b_\mu = \sum_{k=1}^{\mu} \sum_{v_1+\cdots+v=\mu-k} \frac{a_{v_1} \cdots a_{v_k}}{\mathrm{P}(v_1, \ldots, v_k)}$

where $\mathrm{P}(v_1, \ldots, v_k) = (v_1 + \cdots + v_k + k)(v_2 + \cdots + v_k + k - 1) - (v_k + 1)$.

Proof: By induction. For $\mu = 1$, we get $b_1 = a_0$ which is immediate from (A.1) (at $t = 0$). Assume (A.2) for $\mu - 1$. Then, from (A.1), $\mu b_\mu = \sum_{\substack{\sigma+\tau=\mu-1 \\ \tau>0}} a_\sigma b_\tau + a_{\mu-1}$. By the induction assumption;

$$\begin{aligned} b_\mu &= \frac{1}{\mu} a_{\mu-1} + \frac{1}{\mu} \sum_{\substack{\sigma+\tau=\mu-1 \\ \tau>0}} \sum_{k=1}^{\tau} \sum_{\varrho_2+-+\varrho_k=\tau-k+1} \frac{a_\sigma \cdot a_{\varrho_2} \cdots a_{\varrho_k}}{(\varrho_2 + + \varrho_k + k - 1) -- (\varrho_k + 1)} \\ &= \frac{1}{\mu} a_{\mu-1} + \sum_{k=1}^{\mu} \sum_{\substack{\varrho_1+\cdots+\varrho_k=\mu-k \\ \varrho_1<\mu-1}} \frac{a_{\varrho_1} \cdots a_{\varrho_k}}{\mu(\varrho_2 + - + \varrho_k + k - 1) -- (\varrho_k + 1)} \\ &= \sum_{k=1}^{\mu} \sum_{\varrho_1+\cdots+\varrho_k=\mu-k} \frac{a_{\varrho_i} \cdots a_{\varrho_k}}{P(\varrho_1, \ldots, \varrho_k)}. \end{aligned}$$

Q. E. D.

Remark: Proposition A.1 makes contact with (4.5).

Now let $\lambda(t) = \sum_{\mu=0}^{\infty} \lambda_\mu t^\mu$, $\zeta(t) = \sum_{\nu=0}^{\infty} \zeta_\gamma t^\nu \in \mathscr{A}[t]$.

Proposition A.2: *If every term in* $\lambda(t)$ *commutes with every term in* $\xi(t)$, *then*

$$\operatorname{Exp}\lambda(t)\operatorname{Exp}\zeta(t) = \operatorname{Exp}(\zeta(t) + \lambda(t)) = \operatorname{Exp}\zeta(t) \cdot \operatorname{Exp}\lambda(t).^{2)} \tag{A.3}$$

Proof: Since

$$\frac{d(\operatorname{Exp}\lambda(t)\operatorname{Exp}\zeta(t))}{dt} = \lambda(t)\operatorname{Exp}\lambda(t) \cdot \operatorname{Exp}\zeta(t) + \operatorname{Exp}\lambda(t)\,\zeta(t)\operatorname{Exp}\zeta(t),$$

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[2]) *Remark:* It is *not* true that $\lambda(t)\,\xi(t) = \xi(t)\,\lambda(t)$ implies that $\operatorname{Exp}\lambda(t)\operatorname{Exp}\xi(t) = \operatorname{Exp}\xi(t)\operatorname{Exp}\lambda(t)$.

if we prove that $\operatorname{Exp}\lambda(t)\cdot\zeta(t)=\zeta(t)\cdot\operatorname{Exp}\lambda(t)$, then we will have

$$\frac{d(\operatorname{Exp}\lambda(t)\cdot\operatorname{Exp}\zeta(t))}{dt}=\lambda(t)+\zeta(t)\cdot\operatorname{Exp}\lambda(t)\cdot\operatorname{Exp}\zeta(t)$$

which will prove (A.3). By assumption, $\lambda_\sigma\zeta_\tau=\zeta_\tau a_\sigma$; from this and (A.2), we shall prove that $\zeta(t)\cdot\operatorname{Exp}\lambda(t)=\operatorname{Exp}\lambda(t)\cdot\zeta(t)$. The term of degree n in $\zeta(t)\operatorname{Exp}\lambda(t)$ is

$$\sum_{\sigma+\tau=n}\sum_{k=1}^{\tau}\sum_{v_1+\cdots+v_k=\tau-k}\frac{\zeta_\sigma\cdot\lambda_{v_1}\ldots\lambda_{v_k}}{P(v_1,\ldots,v_k)}=\sum_{\sigma+\tau=n}\sum_{k=1}^{\tau}\sum_{v_1+\cdots+v_k=\tau-k}\frac{\lambda_{v_1}\cdot\zeta_\sigma\lambda_{v_2}\ldots\lambda_{v_k}}{P(v_1,\ldots,v_k)}$$

$$=\sum_{\sigma+\tau=n}\sum_{k=1}^{\tau}\sum_{v_1+\cdots+v=\tau-k}\frac{\lambda_{v_1}\ldots\lambda_{v_k}\cdot\zeta_\sigma}{P(v_1,\ldots,v_k)}$$

$$=\text{ term of degree } n \text{ in } \operatorname{Exp}\lambda(t)\cdot\zeta(t). \qquad \text{Q. E. D.}$$

Let $\mathscr{A}^{\#}[t]\subset\mathscr{A}[t]$ be the subset of elements $a(t)=1+a_1t+\cdots$ with leading term 1. Then $\mathscr{A}^{\#}[t]\cdot\mathscr{A}^{\#}[t]\subseteq\mathscr{A}^{\#}[t]$. If $a(t)\in\mathscr{A}^{\#}[t]$, and, if we define $b(t)=\sum_{\mu=0}^{\infty}b_\mu t^\mu$ inductively by $b_0=1$, $b_\mu=-\sum_{\substack{\sigma+\tau-\mu\\ \sigma>0}}a_\sigma b_\tau$, then $a(t)\,b(t)=1=b(t)\,a(t)$. Thus $\mathscr{A}^{\#}[t]$ is a multiplicative subgroup of $\mathscr{A}[t]$. If $a(t)\in\mathscr{A}^{\#}[t]$, we define $\operatorname{Log}a(t)\in\mathscr{A}[t]$ by

$$\frac{da(t)}{dt}=\operatorname{Log}a(t)\cdot a(t)\,. \tag{A.4}$$

Setting $\operatorname{Log}a(t)=\lambda(t)=\sum_{\mu=0}^{\infty}\lambda_\mu t^\mu$, we have recursively

$$\lambda_{\mu-1}=\mu a_\mu-\sum_{\substack{\sigma+\tau=\mu-1\\ \tau>0}}\lambda_\sigma a_\tau\,. \tag{A.5}$$

Observe that $\operatorname{Log}\operatorname{Exp}\lambda(t)=\lambda(t)$.

Proposition A.3: *We have recursively*

$$\lambda_\mu=\sum_{v_1+\cdots+v_k=\mu+1}(-1)^{k+1}(v_1)\,a_{v_1},\ldots,a_{v_k}\,. \tag{A.6}$$

Proof: By induction, we assume (A.6) for $\mu-1$. Then, from (A.5),

$$\lambda_\mu=(\mu+1)a_{\mu+1}-\sum_{\substack{\sigma+\tau=\mu\\ \tau>0}}\sum_{v_1+\cdots+v_k=\sigma+1}(-1)^{k+1}(v_1)\,a_{v_1},\ldots,a_{v_k}\cdot a_\tau$$

$$=(\mu+1)a_{\mu+1}+\sum_{\substack{v_1+\cdots+v_{k+1}=\mu+1\\ v_k+1>0}}(-1)^{k+2}(v_1)\,a_{v_1},\ldots,a_{v_k}\cdot a_{v_{k+1}}$$

$$=\sum_{v_1+\cdots+v_k=\mu+1}(-1)^{k+1}(v_1)\,a_{v_1},\ldots,a_{v_k}\,. \qquad \text{Q.E.D.}$$

Now for $\lambda\in\mathscr{L}$, we define $\operatorname{Exp}^k\lambda$ by

$$\operatorname{Exp}^k\lambda=\sum_{n=0}^{\infty}\left(\sum_{k=1}^{n}\sum_{v_1+\cdots+v_k=n-k}\frac{(n!)^k t^{kn-(n-k)}\lambda^k}{(n-v_1)!-\ldots-(n-v_k)!\,(v_1,\ldots,v_k)}\right). \tag{A.7}$$

Observe that

$$\mathrm{Exp}^k \lambda = 1 + \frac{1}{(k+1)!} \lambda t^{k+1} + \cdots; \quad \mathrm{Exp}^0 \lambda = \mathrm{Exp}\,\lambda = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n t^n ,$$

and $\mathrm{Exp}^k \lambda = \mathrm{Exp}\, t^k \lambda$. Part (i) of theorem 4.1 is implied by

Theorem A.1: (i) *The elements* $\mathrm{Exp}\,\lambda(t)$ *for* $\lambda(t) \in \mathscr{L}[t]$ *form a multiplicative subgroup* $\mathscr{A}^*[t] \subset \mathscr{A}^{\#}[t] \subset \mathscr{A}[t]$.

(ii) $\mathscr{A}^*[t]$ *is generated by the elements* $\mathrm{Exp}^k \lambda$ *for* $\lambda \in \mathscr{L}$.

The proof of this theorem will occupy the remainder of this section. If $c(t) = \sum_{\mu=0}^{\infty} c_\mu t^\mu \in \mathscr{A}[t]$, we say that $c(t)$ is a *Lie element* if $c(t) \in \mathscr{L}[t]$. We define $\delta : \mathscr{A}[t] \to (\mathscr{A} \otimes \mathscr{A})[t]$ by sending $\sum a_\mu t^\mu$ into $\sum (\delta a_\mu) t^\mu$. It follows easily that $\lambda(t) \in \mathscr{A}[t]$ is a Lie element if, and only if, $\delta\lambda(t) = \lambda(t) \otimes 1 + 1 \otimes \lambda(t)$. If $a(t) \in \mathscr{A}^{\#}[t]$, we say that $a(t)$ is a *Lie exponential* if $\mathrm{Log}\, a(t)$ is a *Lie element.*

Proposition A.4: *If* $\lambda(t), \zeta(t) \in \mathscr{L}[t]$, *then* $\mathrm{Exp}\,\lambda(t)\,\mathrm{Exp}\,\zeta(t)$ *is a Lie exponential.*

Proof: We record two remarks to be used in the proof.

(i) $\mathrm{Exp}(\lambda(t) \otimes 1) = \mathrm{Exp}\,\lambda(t) \otimes 1 \in (\mathscr{A} \otimes \mathscr{A})[t]$.

Proof: $d\,\mathrm{Exp}\,\frac{\lambda}{dt}(t) \otimes 1 = (\lambda(t) \otimes 1)(\mathrm{Exp}\,\lambda(t)) \otimes 1$. Similarly, for $a(t) \in \mathscr{A}^{\#}[t]$, $\mathrm{Log}(a(t) \otimes 1) = (\mathrm{Log}\, a(t)) \otimes 1$.

(ii) $\delta\,\mathrm{Exp}\,\lambda(t) = \mathrm{Exp}\,\delta\lambda(t)$.

Proof: $\delta\lambda(t) \cdot \delta\,\mathrm{Exp}\,\lambda(t) = \delta \frac{d\,\mathrm{Exp}\,\lambda(t)}{dt} = \frac{d\,\delta\,\mathrm{Exp}\,\lambda(t)}{dt}$ and thus $\delta\,\mathrm{Exp}\,\lambda(t) = \mathrm{Exp}\,\delta\lambda(t)$.

We now prove the Proposition.

$$\begin{aligned}\delta(\mathrm{Exp}\,\lambda(t) \cdot \mathrm{Exp}\,\zeta(t)) &= \mathrm{Exp}\,\delta\lambda(t) \cdot \mathrm{Exp}\,\delta\zeta(t) \text{ (by (ii))} \\ &= \mathrm{Exp}(\lambda(t) \otimes 1 + 1 \otimes \lambda(t)) \cdot \mathrm{Exp}(\zeta(t) \otimes 1 + 1 \otimes \xi(t)) \\ &= \mathrm{Exp}(\lambda(t) \otimes 1)\,\mathrm{Exp}(\zeta(t) \otimes 1)\,\mathrm{Exp}(1 \otimes \lambda(t))\,\mathrm{Exp}(1 \otimes \zeta(t))\end{aligned}$$

(by Proposition A.2). Setting $w(t) = \mathrm{Log}\,(\mathrm{Exp}\,\lambda(t) \cdot \mathrm{Exp}\,\zeta(t))$, we get

$$\begin{aligned}\delta w(t) &= \mathrm{Log}\,\delta(\mathrm{Exp}\,\lambda(t) \cdot \mathrm{Exp}\,\zeta(t)) \\ &= \big(\mathrm{Log}\,(\mathrm{Exp}\,\lambda(t) \cdot \mathrm{Exp}\,\zeta(t)) \otimes 1\big) + \big(1 \otimes \mathrm{Log}(\mathrm{Exp}\,\lambda(t) \cdot \mathrm{Exp}\,\zeta(t))\big) \\ &= w(t) \otimes 1 + 1 \otimes w(t)\end{aligned}$$

since

$$\mathrm{Log}\,\big((a(t) \otimes 1) \cdot (1 \otimes b(t))\big) = \mathrm{Log}\,a(t) \otimes 1 + 1 \otimes \mathrm{Log}\,b(t) . \qquad \text{Q.E.D.}$$

Clearly (i) of Theorem A.1 follows from Proposition A.4. To prove (ii), we may show: Given $\lambda(t) \in \mathscr{L}[t]$, there exists a sequence $\{\zeta_\mu\} \in \mathscr{L}$ such that $\mathrm{Exp}\,\lambda(t) = \prod_{k=0}^{\infty} \mathrm{Exp}^k \zeta_k$.

If $\lambda(t) = \sum^{\infty} \lambda_\mu t^\mu$, we consider $(\mathrm{Exp}^0 \lambda_0)^{-1}\,\mathrm{Exp}\,\lambda(t) = \mathrm{Exp}\,\lambda^1(t)$ for some $\lambda^1(t) \in \mathscr{L}[t]$. But clearly $\lambda^1(t) = \lambda^1_1 t + \lambda^1_2 t^2 + -$, and we may consider

$(\mathrm{Exp}^1 \lambda_1^1)^{-1} (\mathrm{Exp}^0 \lambda_0)^{-1} \mathrm{Exp}\,\lambda(t) = \mathrm{Exp}\,\lambda^2(t)$ for some $\lambda^1(t) = \lambda_2^2 t + \lambda_3^2 t^3 + - \in \mathscr{L}[t]$. If we proceed inductively to determine λ_N^N for all N, and if we set $\zeta_N = \lambda_N^N$, then $\mathrm{Exp}\,\lambda(t) = \prod_{k=0}^{\infty} \mathrm{Exp}^k \zeta_k$. Q.E.D.

Remark: If $\lambda(t) \cdot \zeta(t) \in \mathscr{L}[t]$, then $\mathrm{Exp}\,\lambda(t) \cdot \mathrm{Exp}\zeta(t) = \mathrm{Exp}\,\Psi(t)$ for some $\Psi(t) = \sum_{\mu=0}^{\infty} \Psi_\mu t^\mu = \mathrm{Log}\,\mathrm{Exp}\,\lambda(t) \cdot \mathrm{Exp}\zeta(t) \in \mathscr{L}[t]$. The coefficients Ψ_μ may be determined by: (i) Writing out $\mathrm{Exp}\,\lambda(t) \cdot \mathrm{Exp}\zeta(t)$ using (A.2); (ii) Writing out Ψ_μ as an element of Ω_G using (A.6); and (iii) writing Ψ_μ as an element of $\mathscr{L}$ using the Specht-Wever criterion for a Lie element. The resulting formula may be called the generalized *Campbell-Hausdorff formula*; the first few terms were written out in the Remark following Theorem 4.2 above.

2. Proofs of (4.7) *and* (4.8). In order to prove the remainder of Theorem 4.1, we must define $\# : \Omega_G^* \to \Gamma_G[t]$ which satisfies $\mathrm{Exp}\,\theta(t)^\# = \exp\theta(t)$ (for $\theta(t) \in \hat{\Theta}_G[t]$) and $(\mathrm{Exp}\,\theta(t) \cdot \mathrm{Exp}\,\varphi(t))^\# = \exp\varphi(t) \circ \exp\theta(t)$ ($\varphi(t) \in \hat{\Theta}_G[t]$). What we shall actually do is define $\#^{-1} : \Gamma_G[t] \to \Omega_G^*$ by sending $\exp\theta(t)$ into its formal *Taylor's expansion*, which will be an element of Ω_G^*.

For a vector field Ψ and a function g, let $\Psi * g$ denote the action of the differential operator Ψ an g. Let $\theta(t) \in \hat{\Theta}_G[t]$:

Proposition A.5: *We have formally*

$$g \circ \exp\theta(t) = \mathrm{Exp}\,\theta(t) * g \qquad (\textit{Taylor's expansion}) \tag{A.8}$$

where $\exp\theta(t)$ *and* $\mathrm{Exp}\,\theta(t)$ *are given by* (4.1) *and* (4.5) *respectively.*

Proof: (3 steps) (i) Suppose first that $\theta(t) = \theta$ is independent of t, and let $x \in U$. If $\theta(x) = 0$, then (A.8) is clear at x since $\exp\theta(t) = x$. If $\theta(x) \neq 0$, then, for suitable local coordinates $x_1, \ldots, x_n$, we may assume $\theta = \frac{\partial}{\partial x_i}$; in this case, $\exp\theta(x_1, \ldots, x_n) = x_1, \ldots, x_i + t, \ldots, x_n$ and (A.8) is just the usual Taylor's formula. (ii) Suppose now that $\theta(t) = t^k\theta$ for some $k \geqq 0$, and let $x \in U$. If $\theta(x) = 0$, (A.8) is again trivial; if $\theta(x) \neq 0$, we may locally assume $\theta = \frac{\partial}{\partial x_i}$, and then $f(t)\,(x_1, \ldots, x_n) = \exp\theta(t)\,(x_1, \ldots, x_n) = (x_1, \ldots, x_1 + \frac{1}{k+1} t^{k+1}, \ldots, x_n)$. Let g be a function. Then

$$\frac{\partial g(f(t))}{\partial t} = \sum \frac{\partial g}{\partial x_i}(f(t)) \frac{\partial f^i(t)}{\partial x_i} = \sum \left(\theta^i(t) \frac{\partial g}{\partial x_i}\right) \circ (f(t)),$$

and, since $\theta(t)$ is *constant* the x-variables, we get $\frac{\partial g(f(t)\,(x))}{\partial t} = \theta(t) * g(f(t)\,(x))$. But now the very same argument as in the proof of Proposition A.1 above may be used to prove (A.8) when $\theta(t)$ is of the form $t^k\theta$. (iii) Now we have proved above that the elements $\mathrm{Exp}^k\theta$ ($k = \theta, 1, 2, \ldots, \theta \in \Theta_G$) generate Ω_G^*; clearly the elements $\exp(t^k\theta)$ ($k = 0, 1, 2, \ldots, \theta \in \Theta_G$) generate $\Gamma_G[t]$. Furthermore, if g is a function, and if $\theta, \varphi \in \Theta_G$, then by (ii) above $g \circ (\exp t^k\theta \circ \exp t^l\varphi) = \mathrm{Exp}\,^l\varphi * (g \circ \exp t^k\theta) = (\mathrm{Exp}\,^l\varphi \cdot \mathrm{Exp}^k\theta) * g$. This clearly implies that, for *any* $\theta(t) \in \hat{\Theta}_G[t]$, (A.8) holds. Q.E.D.

Proof of Theorem (4.1): We define $b = \#^{-1}$: $\hat{\Gamma}_G[t] \to \Omega_G^*$ by $\exp\theta(t)^b = \operatorname{Exp}\theta(t)$ where $\exp\theta(t)$ is given by (4.1) and $\operatorname{Exp}\theta(t)$ by (4.5). Clearly b is a set isomorphism, and, for $\theta(t)$, $\varphi(t) \in \hat{\Theta}_G[t]$, we have $(\operatorname{Exp}\theta(t) \cdot \operatorname{Exp}\varphi(t))^\# = \operatorname{Exp}\varphi(t)^\# \times$ $\times \operatorname{Exp}\theta(t)^\#$, just as in the proof of Proposition A.4. This completes the proof of Theorem 4.1.

Bibliography

[1] CALABAI, E.: On compact, Riemannian manifolds with constant curvature I. Proc. Symposia in Pure Math. Vol. III, 155—180 (1961).
[2] CARTAN, H.: Séminaire E. N. S. Paris 1961—62.
[3] CHERN, S. S.: Pseudo-groups continus infinis. Collogue de geometrie differentielle de Strasbourg (1953), 119—136.
[4] GRIFFITHS, P. A.: On certain homogeneous complex manifolds. To appear in Acta Math.
[5] JACOBSON, N.: Lie algebras. Interscience (1962).
[6] KOBAYASHI, S.: Canonical forms on frame bundles of higher order contact. Proc. Symposia in Pure Math., Vol. III, 186—193 (1961).
[7] KODAIRA, K., and D. C. SPENCER: Deformations of complex analytic structures I—II. Ann. Math. **67**, 328—466 (1958).
[8] —, L. NIRENBERG and D. C. SPENCER: On the existence of deformations of complex analytic structures. Ann. Math. **68**, 450—459 (1958).
[9] SINGER, I., and S. STERNBERG: Lecture notes on G-structures. Harvard (1961—62).
[10] SPENCER, D. C.: Some remarks on homological analysis and structures. Proc. Symposia in Pure Math., Vol. III, 56—86 (1961).
[11] — Deformation of structures on manifolds defined by transitive, continuous pseudo-groups, I—II. Ann. Math. **76** (1962).
[12] WANG, H. C.: Closed manifolds with homogeneous complex structure. Am. J. Math. **76**, 1—32 (1954).
[13] WEIL, A.: On discrete subgroups of Lie groups I. Ann. Math. **72**, 369—384 (1960).
[14] WOLF, J., and P. A. GRIFFITHS: Completeness of differentiable mappings. To appear in Mich. Math. J.

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