GRIFFITHS, P. A.
Math. Annalen 158, 326—351 (1965)

Deformations of G-Structures

Part B: Deformations of Geometric G-Structures

By

PHILLIP A. GRIFFITHS in Berkeley

Contents

Bibliography see Part A [Math. Ann. **155**, 292—315 (1964)]

V. Some Local Properties of Geometric G-Structures

It seems necessary, at the present time, to place some restrictions on our G-structures in order to be able to discuss the deeper facts in the deformation theory of these structures. One restriction which has certainly had considerable success is to assume that $G \subsetneqq GL(n, \mathbf{C}) \subset GL(2n, \mathbf{R})$, and that the structure is integrable. Because of this, and because our main purpose has been to define deformations of *non*-integrable *non*-transitive structures, we shall emphasize non-integrable structures. Furthermore, since our theory has been geometrically slanted, since many of the "classical" geometric structures occur when G is of finite type, and since we desire to have some worthwhile results, we shall henceforth work primarily with these *geometric G-structures*; i.e., G-structures of finite type. As an example of the difficulty when G is of infinite type, the author does not know whether or not, for X compact, $\dim H^1(X, \Theta_G)$ is finite for infinite G. For the remainder of this paragraph we shall derive some special properties, to be used later, of G structures of finite type.

1. Infinitesimal Automorphisms of Geometric Structures. Let X have a G-structure given by a cross section $\sigma : X \to B/G$. Then we have a principal fibering $G \to B_G \to X$; and, as was shown in [2], we have a G^1-structure on B_G and thus a bunde $G^1 \to B_{G^1} \to B_G$; and, in general, we have a G^k-structure on $B_{G^{k-1}}$ and thus a bundle $G^k \to B_{G^k} \to B_{G^{k-1}}$.

Proposition 5.1 ([3]): *G is of finite type of order q if, and only if, there is defined by the original G-structure a connexion in $G^{q-1} \to B_{G^{q-1}} \to B_{G^{q-2}}$.*

Corollary: If G is of finite type, then $\mathscr{R}_G$ is the constant sheaf $\mathbf{R}$.

Proof: From the fibering $G^k \to B_{G^k} \xrightarrow{\pi_k} B_{G^{k-1}}$, we have an injective mapping $\pi_k^* : \mathscr{R}_{G^k} \to \mathscr{R}_{G^{k+1}}$ where $\mathscr{R}_{G^k}$ is the sheaf on $B_{G^{k-1}}$. But, if q is the order of G, then, by the example given in § I.1, $\mathscr{R}_{G^q} = \mathbf{R}$ on $B_{G^{q-1}}$. Q.E.D.

Definition 5.1: If G is of finite type or order q, we set $d(G) = n + \dim(G) + + \cdots + \dim(G^{q-1})$.

Example: $d(O(n)) = n(n+1)/2$.

Proposition 5.2: Given $G \to B_G \to X$ where G is of finite type, then, for any $x \in X$, $\dim(\Theta_G)_x \leqq d(G) < \infty$.

Proof: If θ is a germ in Θ_G, then there is naturally induced a germ θ^1 in Θ_{G^1} on B_G as follows: $\exp t(\theta)$ acts on X and, since $\theta \in \Theta_{G^0}$, $(\exp t(\theta))_*$ acts on B_{G^1} and $\frac{d}{dt}(\exp t(\theta)_*)_{t=0} = \theta^1$ is a vector field on B_G; clearly θ^1 is a germ in Θ_{G^1} on B_G. Inductively, we get a germ θ^k in Θ_{G^k} on $B_{G^{k-1}}$. If the order of $G = q$, then there is defined on B_{G^q} a global parallelism given by vector fields $X_1, \ldots, X_d$ $(d = d(G))$, and the germ θ^q satisfies.

$$[\theta^q, X_j] = 0 \qquad (j = 1, \ldots, d)\,. \tag{5.1}$$

In fact, it is clear that the germs $\theta^q \in \Theta_{G^q}$ on B_{G^q} satisfying (5.1) are in a one-to-one correspondence with the germs $\theta \in \Theta_G$ on X. Our proposition will follow from;

Lemma 5.1: *Let $U \subset \mathbf{R}^m$ be a finite connected domain with coordinates $x^1, \ldots, x^m$, and let vector fields $X_1, \ldots, X_m$ define a global parallelism in U. Given $p \in U$ and the vector $\zeta = (\zeta^1, \ldots, \zeta^m) \in \mathbf{R}^m$, any vector field $Y = \sum_{j=1}^{m} \zeta^j(x) \frac{\partial}{\partial x^j}$. Such that $\zeta^j(p) = \zeta^j$ and $[Y, X_j] = 0$ $(j = 1, \ldots, m)$ is unique in U.*

Proof: Write $X_j = \sum_{j=1}^{m} \eta_j^l(x) \frac{\partial}{\partial x_l}$; then we have

$$\sum_{i,k} \left[\zeta^i(x) \frac{\partial}{\partial x^i}, \eta_j^k(x) \frac{\partial}{\partial x_k}\right] = 0 \qquad (j = 1, \ldots, m). \tag{5.2}$$

Now (5.2) gives that

$$\sum_i \zeta^i \frac{\partial \eta^k}{\partial x^j} = \sum_i \eta_j^i \frac{\partial \zeta^k}{\partial x^i} \qquad (j, k = 1, \ldots, m)\,. \tag{5.3}$$

Since the X_j give a parallelism, there exists a function $\hat{\eta}(x) : U \to GL(m, \mathbf{R})$ such that $\sum_j \hat{\eta}_j^i(x)\, \eta_k^j(x) = \delta_k^i$ in U. Letting ${}_rA = {}_rA_j^i(x)$ be the matrix ${}_rA_j^i = \sum_l \frac{\partial \eta_l^i}{\partial x^j} \hat{\eta}_r^l$ $(r = 1, \ldots, m)$, and letting $\zeta(x) = {}^t(\zeta^1(x), \ldots, \zeta^m(x))$, then

(5.3) is equivalent to the system

$$\frac{\partial \zeta}{\partial x^r} = {}_rA \cdot \zeta \qquad (r = 1, \ldots, m). \tag{5.4}$$

This (overdetermined) system was a unique solution (if any) with a prescribed value at a point. Q.E.D.

Remark: Lemma 5.1 may be proven geometrically as follows: It will suffice to show that $Y(p) = 0$ and $[X_j, Y] = 0$ $(j = 1, \ldots, m)$ imply that $Y \equiv 0$. For this, we shall show that $[x \in U \mid Y(x) = 0]$ is open. Now if p' is in a sufficiently small neighborhood of p, then $p' = \exp(t_1 X_1 + \cdots + t_m X_m)(p)$ for constants $t_1, \ldots, t_m$. Then $\exp t Y(p') = \exp(t_1 X_1 + \cdots + t_m X_m) \exp t Y(p)$ (since $[X_j, Y] = 0) = \exp(t_1 X_1 + \cdots + t_m X_m)(p)$ (since $Y(p) = 0) = p'$. Thus $Y(p') = 0$. Q.E.D.

Lemma 5.2: *Let B_t be the ball of radius t around* o *in* $\mathbf{R}^m$, *and let $X_1, \ldots, X_m$ be real analytic vector fields giving a parallelism in B_1. Consider the system of partial differential equations* (5.4) *where the matrices ${}_rA$ are constructed as in the proof of Lemma* 5.1. *Then, if a formal power series solution to* (5.4) *exists, this solution will converge in B_ε for some $\varepsilon > 0$. Furthermore, ε depends only on the initial values and the radius of convergence of the series for the X_i.*

Proof: Were this system not overdetermined, we could apply Cauchy-Kowaleski and be done; as it is, we shall construct a majorant problem, which also has a formal solution, and then show that this solution converges. We are considering the system

$$\frac{\partial \xi^l(x)}{\partial x_k} = \sum A^l_{kj}(x)\, \xi^j(x) \tag{5.5$_l$}$$

with initial value $\xi^i(0) = \xi^i_0$. We assume that a formal solution to (5.5) exists. Write

$$A^l_{kj}(x) = \sum_{\mu_1, \ldots, \mu_m} (A^l_{kj})_{\mu_1, \ldots, \mu_m} x_1^{\mu_1}, \ldots, x_m^{\mu_m}, \tag{5.6$^l_{kj}$}$$

and choose $r > 0$ such that the series $(5.6)^l_{kj}$ all converge absolutely for $|x_1| + \cdots + |x_m| < r$. Then choose $M > 0$ such that $|(A^l_{kj})_{\mu_1, \ldots, \mu_m}| \leqq \frac{M}{r^{\mu_1 + \cdots + \mu_m}}$ for all l, k, j and $\mu_1, \ldots, \mu_m$. Then, a fortiori, we have

$$|(A^l_{kj})_{\mu_1, \ldots, \mu_m}| \leqq \frac{M(\mu_1 + \cdots + \mu_m)!}{r^{\mu_1 + \cdots + \mu_m} (\mu_1)! \cdots (\mu_m)!}. \tag{5.7}$$

Set $M = nM'$ and

$$B(x) = M \sum_{\mu_1, \ldots, \mu_m} \left(\frac{x_1}{r}\right)^{\mu_1} \cdots \left(\frac{x_m}{r}\right)^{\mu_m} \frac{(\mu_1 + \cdots \mu_m)!}{(\mu_1)! \cdots (\mu_m)!} \tag{5.8}$$

We set $\xi_0 = \max_{j = 1, \ldots, m} |\xi^j_0|$, and consider the (again overdetermined) system

$$\frac{\partial \xi(x)}{\partial x_k} = B(x)\, \xi(x) \tag{5.9}$$

for the single function $\xi(x)$ with initial value $M \geqq \xi_0 > 0$. By our construction (5.9) is a majorant of each of the n systems $(5.5)_l$ $(l = 1, \ldots, n)$. Since any solution to $(5.5)_l$ is determined uniquely by its initial value, it will suffice to produce a convergent solution to (5.9). We tenatively write $\xi(x) = e^{H(x)}$ and try to determine a real analytic function $H(x)$ such that (5.9) is satisfied when $H(0) = \log \xi_0^M$. We have $\frac{\partial \xi(x)}{\partial x_k} = \xi(x)\frac{\partial H(x)}{\partial x_k}$, and thus we must have $\frac{\partial H(x)}{\partial x_k} = B(x)$ for $k = 1, \ldots, n$. Since $B(x) = M\left(\frac{1}{1-\left(\sum_{i=1}^{m}\frac{x_i}{r}\right)}\right)$, $\frac{\partial B(x)}{\partial x_j} = \frac{\partial B(x)}{\partial x_k}$ for all j, k and thus we may find $H(x)$. In fact, if we set

$$H(x) = -rM\log\left(1-\sum_{i=1}^{m}\frac{x_i}{r}\right) + \log M\,,$$

then $\frac{\partial H(x)}{\partial x_k} = B(x)$ for all k and $H(0) = \log M$. Q.E.D.

Corollary: *Let the data be as in Lemma 5.1, and real analytic, and assume that U is simply connected. Then any C^∞ local solution to the system $[Y, X_j] = 0$ $(j = 1, \ldots, m)$ is real analytic and may be analytically continued to a solution in all of U.*

Proposition 5.3: *Let G be of finite type, and let $G \to B_G \to X$ be a real analytic G-structure. Then:*

(i) *Any germ $\theta \in \Theta_G$ is real analytic,*

(ii) *The G-structure is normal.*

The proof is immediate from the Corollary to Lemma 5.2.

Let $G \to B_G \to X$ be a G-structure where G is of finite type. Then, for any $x \in X$, and any sequence of neighborhoods $\{U_n\}$ with $\overline{U_n} \subset U_{n-1}$ and $\cap U_n = \{x\}$, the restriction mapping $p_{jk} : \Theta_G \mid U_j \to \Theta_G \mid U_k$ $(k > j)$ is into, and is hence onto for j, k large enough. Thus every $x \in X$ has a neighborhood $N(x)$ such that, for any open U with $\{x\} \subset U \subset N(x)$, $\Theta_G \mid U \cong \Theta_G \mid N(x) \cong (\Theta_G)_x$.

Definition 5.2: Any such neighborhood $N(x)$ is called a *normal neighborhood*. A *normal covering* of X is a covering by normal neighborhoods.

Proposition 5.4: *Let $G \to B_G \to X$ be a normal G-structure where G is of finite type and X is simply connected. Then* $\dim H^0(X, \Theta_G) < \infty$, *and, if* $\mathfrak{g} = H^0(X, \Theta_G)$, *$\Theta_G$ is the trivial sheaf $X \times \mathfrak{g}$.*

Proof: Let $U = \{U_i\}$ be a normal covering of X. Then $\Theta_G \mid U_i \cong (\Theta_G)_x$ for any $x \in U_i$, and $\dim \Theta_G \mid U_i = d(x) = d$ for all i. Thus, we may define a d-dimensional vector bundle $\mathbf{E} \to X$ whose fibre $\mathbf{E}_x$ is just $(\Theta_G)_x$. The transition functions of $\mathbf{E}$ are given by a system $\{g_{ij}\}$ where $g_{ij} : U_i \cap U_j \to GL(d, \mathbf{R})$. But, by Lemma 5.1, the matrix functions $g_{ij}(x)$ on $U_i \cap U_j$ are constant. Since any vector bundle with constant transition functions, over a simply connected manifold is trivial, the result follows.

2. The Algebras $C^*(X, \Theta_G)$ and $H^*(X, \Theta_G)$. Let G be of finite type, let $G \to B_G \to X$ be a normal G-structure on a compact manifold X, and let $\mathbf{U} = \{U_i\}$ be a finite normal covering of X with nerve $\mathbf{N}$.

Proposition 5.5: (i) *The cochain groups $C^q(\mathbf{N}, \Theta_G)$ are finite dimensional vector spaces for all q.* (ii) *The natural mapping $H^q(\mathbf{N}, \Theta_G) \to H^q(X, \Theta_G)$ is an isomorphism for all q.*

Proof: (i) A *q-cochain* ζ is given by a collection $\{\zeta i_0 \ldots i_q\}$ where $\zeta i_0 \ldots i_q \in \Gamma(\Theta_G | U_{i_0 \ldots i_q})$ and $U_{i_0 \ldots i_q} = U_{i_0} \cap \ldots \cap U_{i_q}$. Now (i) follows since N is a finite complex. The assertion (ii) follows from Leray's theorem on acyclic coverings.

The sheaf Θ_G on X is not only a sheaf of vector spaces, but is also a sheaf of Lie algebras. Thus the usual formula defines a pairing

$[\ ,\ ] : C^p(\mathrm{N}, \Theta_G) \otimes C^q(\mathrm{N}, \Theta_G) \to C^{p+q}(\mathrm{N}, \Theta_G)$ satisfying

$$\delta[\zeta, \eta] = [\delta\zeta, \eta] + (-1)^p [\zeta, \delta\eta] \ (\zeta \in C^p(\mathrm{N}, \Theta_G)) .$$

There is thus an induced pairing $[\ ,\ ] : H^p(X, \Theta_G) \otimes H^q(X, \Theta_G) \to H^{p+q}(X, \Theta_G)$ which establishes the structure of a *graded Lie algebra* on $H^*(X, \Theta_G)$. This structure will be important in § VIII below.

For a finite normal covering $\mathrm{U} = \{U_i\}$ with nerve N, we choose a basis $\theta_i^1, \ldots, \theta_i^d$ of $\Theta_G | U_i$ for each i. Any germ θ of $\Theta_G | U_i$ may be written $\theta_i = \sum_{k=1}^{d} \zeta^k \theta_i^k$ ($\zeta^k \in \mathrm{R}$), and we set $\|\theta_i\|^2 = \sum_{k=1}^{d} |\zeta^k|^2$.

For any $\zeta = \{\zeta i_0 \ldots i_q\} \in C^q(\mathrm{N}, \Theta_G)$, we set $\|\zeta\|^2 = \sup_{i_0 \ldots i_q \in \mathrm{N}} \|\zeta i_0 \ldots i_q\|^2$. Then $\|\ \|^2$ is a norm on $C^q(\mathrm{N}, \Theta_G)$. Since $[\ ,\ ]$ is bi-linear, we have:

Proposition 5.6: *There exists a constant* C_1, *depending only on* U, *such that* $\|[\zeta, \eta]\| \leqq C_1 \|\zeta\| \cdot \|\eta\|$ $\zeta, \eta \in C^*(\mathrm{N}, \Theta_G)$.

Let $B^q(\mathrm{N}, \Theta_G) = \delta(C^{q-1}(\mathrm{N}, \Theta_G))$. Then we easily have

Proposition 5.7: *There exists a constant* C_2, *depending only on* U, *such that, for any* $\zeta \in B^q(\mathrm{N}, \Theta_G)$, *there exists an* $\eta_\zeta = \eta \in C^{q-1}(\mathrm{N}, \Theta_G)$ *with* $\delta(\eta) = \zeta$ *and* $\|\eta\| \leqq C_2 \|\zeta\|$.

3. A Resolution of Θ_G. Not unexpectedly, one of the salient features of geometric structures is that, in order to derive our main results, we shall not have to rely as heavily on the methods of analysis as has been the case for infinite pseudogroups. A notable exception here is the existence theorem given below in § VIII. We shall now give an injective resolution of Θ_G by differentiable cross sections of vector bundles. In § VIII below we shall, for a much more restricted class of G-structures, give somewhat modified but much more useful resolution of Θ_G.

Let $G \to B_G \to X$ be a normal G-structure where G is of finite type. If $\tilde{X}$ is the universal covering of X, and if $\pi_1(X) = \Gamma$, then we have a fibering $\Gamma \to \tilde{X} \xrightarrow{\pi} X$. There is naturally induced a normal G-structure $G \to \tilde{B}_G \to \tilde{X}$ and the sheaf $\tilde{\Theta}_G$ on $\tilde{X}$ admits a global basis, say $\tilde{\theta}_1, \ldots, \tilde{\theta}_d$, over R. We let $\tilde{\mathscr{A}}^q$ = sheaf of q forms an X, and we set $\mathscr{G}^q = \tilde{\Theta}_G \otimes \tilde{\mathscr{A}}^q$. Define $i : \tilde{\Theta}_G \to \mathscr{G}^0$ by $i(\tilde{\theta}) = \tilde{\theta} \otimes 1$, and define $D : \mathscr{G}^q \to \mathscr{G}^{q+1}$ by $D(\tilde{\theta} \otimes w) = \tilde{\theta} \otimes dw$. Then, by the Poincaré lemma,

Proposition 5.8: *The sequence*

$$0 \longrightarrow \tilde{\Theta}_G \xrightarrow{i} \mathscr{G}^0 \xrightarrow{D} \mathscr{G}^1 \longrightarrow \cdots \longrightarrow \mathscr{G}^q \xrightarrow{D} \mathscr{G}^{q+1} \longrightarrow \cdots$$

is exact and gives an injective fine resolution of $\tilde{\Theta}_G$.

The group Γ acts as a G-motions on $\tilde{X}$; thus Γ acts on $\tilde{\Theta}_G$ and $\tilde{\mathscr{A}}^q$, hence on $\tilde{\mathscr{G}}^q$. Obviously, as operators on $\tilde{\mathscr{G}}^q$, $D\gamma = \gamma D$ for any $\gamma \in \Gamma$. We define sheaves $\mathscr{G}^q = (\tilde{\mathscr{G}}^q)^\Gamma|$ on X as follows: over an open set $U \subset X$, the sections of $\mathscr{G}^q$ are the Γ-invariant sections of $\tilde{\mathscr{G}}^q$ over $\pi^{-1}(U)$.

Proposition 5.9: $(\tilde{\Theta}_G)^\Gamma \cong \Theta_G$ *and the sequence*

$$(5.10)\quad 0 \longrightarrow \Theta_G \xrightarrow{\ i\ } \mathscr{G}^0 \xrightarrow{\ D\ } \mathscr{G}^1 \longrightarrow \cdots \longrightarrow \mathscr{G}^q \xrightarrow{\ D\ } \mathscr{G}^{q+1} \longrightarrow \cdots$$

gives an injective fine resolution of Θ_G.

Set now $\mathrm{G}^q = H^0(X, \mathscr{G}^q)$; then $D: \mathrm{G}^q \to \mathrm{G}^{q+1}$, $D^2 = 0$, and we let $H^q(\mathrm{G})$ be the cohomology groups.

Corollary: *There is defined by* (5.10) *an anti-isomorphism of graded Lie algebras* $H^q(X, \Theta_G) \cong H^q(\mathrm{G})$.

4. Application: Rigidity of Structure. **Theorem 5.1:** *Let* $G \to B_G \to X$ *be a normal G-structure where G is of finite type and X is compact. Then, if* $H^1(X, \Theta_G) = 0$, *any* (C^∞) *germ of deformation of the G-structure an X is trivial.*

Proof: Let $\mathscr{V} \xrightarrow{\ \tilde{\omega}\ } D$ be a deformation of the G-structure $X = \tilde{\omega}^{-1}(\mathrm{o})$; we shall produce a neighborhood U of $\mathrm{o} \in D$ such that the deformation $\tilde{\omega}^{-1}(U) \xrightarrow{\ \tilde{\omega}\ } U$ is trivial. Now, in the notation of (5.10), the C^∞ sections of the sheaves $\mathscr{G}^q$ ($q = 0, 1, \ldots, n$) form the C^∞ sections of differentiable vector bundles $\mathbf{0}^q$ ($q = 0, 1, \ldots, n$) on X, and we have differential operators $D: \mathbf{0}^q \to \mathbf{0}^{q+1}$. By taking metrics on X and along the fibres of the bundles $\mathbf{0}^q$, we may form the adjoint D^* of D; $D^*: \mathbf{0}^q \to \mathbf{0}^{q-1}$, and $DD^* + D^*D$ is elliptic. From this, the theory of elliptic equations as applied in [7] tells us: (i) $\dim H^q(X_t, \Theta_{G,t}) \leqq$ $\leqq \dim H^q(X, \Theta_G)$ for t in a neighborhood U of $\mathrm{o} \in D$; and (ii) if $\dim H^q(X_t, \Theta_{G,t})$ is independent of t for $t \in \mathscr{V}$, then $\bigcup_{t \in D} H^q(X_t, \Theta_{G,t})$ forms a vector bundle $\mathscr{H}^q \to \mathscr{V}$ over $\mathscr{V}$, and $H^q(\mathscr{V}, \Psi_G)$ is isomorphic to the sheaf of C^∞ cross sections of $\mathscr{H}^q$. From these facts, we conclude that, if $H^1(X, \Theta_G) = 0$, then $H^1(\mathscr{V}, \Psi_G) = 0$, and applying Proposition 3.1, the theorem follows Q. E. D.

Remark: By using the estimates in Propositions 5.5 we can prove this Theorem without the use of elliptic theory for real-analytic deformations.

VI. Some Global Properties of Geometric G-Structures

1. Global Uniqueness of Structure. Let $G \to B_G \to X$ be a normal G-structure where G is of finite type (geometric structure) and X is simply connected but not necessarily compact. We shall see that, with a completeness assumption, deformations of X are trivial. To do this, we need to define a complete G-structure. Let $G \to B_G \to X$, $G \to B^1_G \to X^1$ be G-structures on n-manifolds X, X^1 respectively. Furthermore, let $f: X \to X^1$ be a mapping of maximal rank, which is also a *G-mapping*, i.e. $f_*(B_G) \subseteq B^1_G$.

Definition 6.1: The G-structure on X is *complete* if, for any X, f, X^1 as above, f is *surjective*; and if the topological group $\mathscr{G}$ of G-automorphisms is a Lie group with Lie algebra $H^0(X, \Theta_G)$.

Remarks: (i) In order to justify this definition, we recall that, from the theorem in [14], a Riemannian structure ($G = O(n)$) is complete in the usual sense if, and only if, Definition 6.1 is satisfied. Also, in the Riemannian case, if we assume that X is the universal covering of a compact Riemannian manifold in which the deck-transformations are isometries, then X is complete. (ii) An example of a normal G-structure which satisfies neither condition in Definition 6.1 is the following: Let $Y = \mathbf{R}^2$ with with coordinates (x, y), and let $X \subset Y = \{x, y) \mid y = 0$ implies $x < 0\}$; i.e. $X = \mathbf{R}^2$ minus the positive x-axis. Let $G = I$ and the parallelism is given by dx, dy. Then the injection $X \to Y$ is not onto. Furthermore, $H^0(X, \Theta_G) \cong \mathbf{R}^2$ (the translations), but $\mathcal{G}$ consists only of the identity. However, there is clearly no discrete group Γ of G-automorphisms of X such that X/Γ is compact. It may be that this latter condition in fact guarantees completeness.

Theorem 6.1: *Let $G \to B_G \to X$ be a complete, normal G-structure where G is of finite type and X is simply connected. Then:* (i) *If the structure is locally homogeneous, it is globally homogeneous,* (ii) *Any deformation of X is G-isomorphic to X.*

The discussion of the proof of this Theorem will occupy the remainder of this section. Let X, X^1 be n-manifolds. For $x \in X$, we consider the *r-jets* [6] at x of local bi-maps f of a neighborhood of x into X^1; for such an f, $j^r_x(f) = r$-jet of f at x. We set $J^r_x(X, X^1) = \bigcup_f j^r_x(f)$ and $J^r(X, X^1) = \bigcup_{x \in X} J^r_x(X, X^1)$; and we call $J^r(X, X^1)$ the bundle (over X) of invertible r-jets from X to X^1. Now let $G \subsetneq GL(n, \mathbf{R})$ and assume that we have G-structures $G \to B_G \to X$, $G \to \to B^1_G \to X^1$. Then we may clearly define the space of G *r-jets* $J^r_G(X, X^1)$. $J^r_G(X, X^1)$ is a topological subspace of $J^r(X, X^1)$, but it is not in general a bundle over X. However, we have (i) $J^r_G(X, X^1)$ is a fibre-space over X; (ii) If G is of finite type, then, for $r, r^1 \gg 0$[3]), $J^r_G(X, X^1) \cong J^{r^1}_G(X, X^1)$; (iii) If X and X^1 have the same local structure (in particular, if X^1 is a deformation of X), then $J^r_G(X, X^1)$ has local cross sections; (iv) If the G-structures on X and X^1 are both transitive (i.e. local homogeneity), then $J^r_G(X, X^1)$ is a fibre bundle.

We say that the G-structure on X is a *deformation* of the G-structure on X^1 if $X \underset{C^\infty}{\cong} X^1$ and there is a 1-parameter family $\sigma(t) : X \to B/G$ satisfying (2.1) such that $\sigma(0)$ gives B_G and $\sigma(1)$ gives B^1_G.

Proposition 6.1: *Let X be simply connected, and let $G \to B_G \to X$ be a complete transitive G-structure. Then* (i) *if X^1 is a deformation of X, then $X \underset{G}{\cong} X^1$, and* (ii) *the G-structure on X is globally homogeneous.*

Proof: Let X^1 be a deformation of X. Then $J^r_G(X, X^1) \cong J^\infty_G(X, X^1)$ $(r \gg 0)$ is a fibre bundle over X with constant transition functions, since a local bi-G-map is determined by its first $d = d(G)$ partial derivatives at a point. Since X is simply connected, this bundle is trivial, and thus there is a global section f. This f is a G-mapping $f : X \to X^1$ of maximal rank, and, by completeness, f is onto. Then (i) follows by the monodromy principal.

Now for $r \gg 0$, we may consider $J^r_G(X, X)$ as a fibre bundle over $X \times X$ (target and source projections), where it is a trivial bundle. Thus, given

[3]) The notation $N \gg 0$ means "for N sufficiently large".

$x, x^1 \in X$, there exists a global section f of $J^r_G(X, X^1)$ over X such that $f(x) = x^1$. The proof then follows from completeness Q. E. D.

We now give a direct proof (without using jets) of (ii) in Theorem 6.1. Thus, let $G \to B_G \to X$ be a complete normal G-structure, and let $G \to B^1_G \to X^1$ be a deformation of this structure. Then we may find an open covering $\mathbf{U} = \{U_i\}$ of X with the following property: For each i, there exists a family $f_i(t) : U_i \to X^1$ of local bi-G-mappings between X and $X_t (= G \to B_G(t) \to X)$ coming from $\sigma(t)$ such that $f_i(0) =$ identity. Also, we may assume that each $f_i(t)$ is defined on an open set $\hat{U} \supset U_i$ and $f_i(t)\hat{U} \supseteq U_i$. Set $f_i = f_i(1)$, and let $\mathbf{N} =$ nerve of $\mathbf{U}$. We define now an element $\tau = \{\tau_{ij}\} \in H^1(\mathbf{N}, \Gamma_G)$. For $(i, j) \in \mathbf{N}$, we choose $W_{ij} \supseteq U_i \cap U_j$ and let $\tau_{ij} = f_j^{-1} \circ f_i$, defined on W_{ij}. Then $\tau_{ij}(W_{ij}) \supseteq U_i \cap U_j$. Furthermore, for $(i, j, k) \in \mathbf{N}$, $\tau_{ij}\tau_{jk} = \tau_{ik}$. Thus $\tau \in H^1(\mathbf{N}, \Gamma_G)$. Now, since X is simply connected and the G-structure on X is complete and normal, Γ_G is a *constant sheaf* of non-abelian groups (infinitessimally generated by $\Theta_G | U_i$), say $\mathscr{G}$. Thus, $\tau \in H^1(\mathbf{N}, \mathscr{G})$ and gives rise to a bundle with locally constant transition functions; this bundle is therefore trivial. We may then find $\sigma = \{\sigma_i\} \in C^0(\mathbf{N}, \mathscr{G})$ such that, for $(i, j) \in \mathbf{N}$, $\tau_{ij} = \sigma_j \sigma_i^{-1}$. But then $f_j \cdot \sigma_j = f_i \cdot \sigma_i$ in $U_i \cap U_j$, and we have thus globally defined a G-mapping $g : X \to X^1$, and g is of maximal rank. Thus g is onto by completeness. Q. E. D.

2. Homogeneous G-Structures. Let X be a manifold with a G-structure $G \to B_G \to X$.

Definition 6.2: X is a *homogeneous G-manifold* if there exists a Lie group M which acts as a transitive group of G-motions on X.

If X is a homogeneous G-manifold, then we may write $X = M/V$ where $V \subset M$ is the isotropy group of point $x_0 \in X$. In this case, the G-structure on X is real-analytic, hence normal (§ V.1). Let $\mathrm{Ad}\#$ be the linear representation of V on $\mathbf{m}/\mathbf{v}$ induced by the adjoint representation of V on $\mathbf{m}$ (since $\mathrm{Ad}\, V(\mathbf{v}) \subseteq \mathbf{v}$).

Proposition 6.2: *X is G-isomorphic to a coset space M/V of connected Lie groups M, V if, and only if, X is differentiably isomorphic to M/V and* $\mathrm{Ad}\#(V) \subseteq G$, *where G is considered as acting on* $\mathbf{m}/\mathbf{v} \cong T_{x_0}(X)$.

Remark: If M is connected but V is not, we always have a *locally homogeneous* G-structure on M/V for any $G \supseteq \mathrm{Ad}\#(V^0)$ where V^0 is the identity component of V. If we take, in particular, $G = \mathrm{Ad}\#(V^0)$, then we call the corresponding G-structure on M/V the *canonical structure;*

Proposition 6.3[4]**:** *The canonical G-structures are all analytic of finite type.*

Proof: We may assume that V is connected. Let $V^1 \subset V$ be the connected kernel of $\mathrm{Ad}\#(V)$ on $\mathbf{m}/\mathbf{v}$. Then the fibration $G \to B_G \to M/V$ is just $V/V^1 \to M/V^1 \to M/V$. Repeating, the G^1 structure $G^1 \to B_{G^1} \to B/G$ is just $V^1/V^2 \to M/V^2 \to M/V^1$ where V^2 is the connected kernel of $\mathrm{Ad}\#(V^1)$ on $\mathbf{m}/\mathbf{v}^1$. Continuing, we get a sequence of the form $V^k/V^{k+1} \to M/V^{k+1} \to M/V^k$, and for some k, $V^{k+1} = \{1\}$ and we have a connexion in $V^k \to M \to M/V^k$, since M/V^k

[4]) This is not a G-structure in the strict sense but is generally a higher order pseudogroup structure.

will in this case be a reductive homogeneous space and the usual canonical affine connexion is just the connexion of the G_k-structure. Q. E. D.

Proposition 6.4: *Let M/V be connected. For the canonical G-structure on M/V, $H^0(M/V, \Theta_G) \cong \mathfrak{m}$.*

Proof: By Proposition 6.3, $H^0(M/V, \Theta_G) = \mathfrak{n}$ for some finite dimensional Lie algebra $\mathfrak{n}$; then we may assume that $\mathfrak{m} \subseteq \mathfrak{n}$ and that there is a Lie group N with Lie algebra $\mathfrak{n}$ such that $M/V \underset{G}{\cong} N/W$. Under the identification $\mathfrak{n}/\mathfrak{w} \cong T_0(M/V) \cong \mathfrak{m}/\mathfrak{v}$ ($\pi(1) = 0$ in $M \xrightarrow{\pi} M/V$), $\mathrm{Ad}\#(W) \subseteq \mathrm{Ad}\#(V)$. The proof now takes two steps. (i) We may construct the sequences $W \supset W^1 \supset \cdots \supset W^k$, $V \supset V^1 \supset \cdots \supset V^k$ as in the proof of Proposition 6.3, where we may assume that $W^j \supseteq V^j$ and $\mathrm{Ad}\#(W^j) \subseteq \mathrm{Ad}\#(V^j)$ for each j. We assert that $V^{k+1} = \{1\}$ implies that $\mathfrak{w}^{k+1} = 0$. Indeed, we have

$$\begin{tikzcd}
W^k/W^{k+1} \arrow[r] & N/W^{k+1} \arrow[r] \arrow[d, equal] & N/W^k \arrow[d, equal] \\
V^k \arrow[u] \arrow[r] & M \arrow[r] & M/V^k
\end{tikzcd}$$

and thus $\mathfrak{w}^{k+1}$ is an ideal in $\mathfrak{n}$, hence $\mathfrak{w}^{k+1} = 0$ (assuming, as we may, effective action of N on N/W). (ii) Since $\mathfrak{w}^{k+1} = 0$, $\mathrm{Ad}\#(\mathfrak{w}^k)$ is faithful, and $\mathrm{Ad}\#(\mathfrak{w}^k) \subset \mathrm{Ad}\#(\mathfrak{v}^k)$ implies that $\dim \mathfrak{w}^k \leqq \dim \mathfrak{v}^k$; i.e. $\dim \mathfrak{w}^k = \dim \mathfrak{v}^k$. Since $\mathfrak{w}^j/\mathfrak{w}^{j+1} \cong \mathfrak{v}^j/\mathfrak{v}^{j+1}$ for each j, we conclude that $\dim \mathfrak{v} = \dim \mathfrak{w}$. Thus $\dim \mathfrak{n} = \dim \mathfrak{m}$, and then $\mathfrak{n} = \mathfrak{m}$. Q. E. D.

3. Deformations of Homogeneous G-Structures. We shall not go into the construction of deformations on homogeneous G-manifolds, but shall give two simple propositions which put in a general setting the phenomenon observed in [4]. Thus let X be a homogeneous G-manifold M/V, and assume that the M-module $H^1(X, \Theta_G)$ is finite dimensional (G may not be of finite type!). Let $\mathscr{V} \xrightarrow{\tilde{\omega}} D$ be a deformation of the G-structure on X, and assume that $\mathscr{V}$ is weakly homogeneous with respect to M, i.e. M acts as automorphisms on $\mathscr{V}$ in a fibre-preserving manner. Thus, for each $t \in D$, M acts transitively as G-motions on $X_t = \tilde{\omega}^{-1}(t)$ and thus $X_t = M/V_t$.

Proposition 6.5: *$\mathscr{V}$ is strongly homogeneous with respect to M if, and only if, the subgroups V_t are conjugate in M by a 1-parameter family α_t of automorphisms of M.*

Proof: If we have such an $\{\alpha_t\}$ ($t \in D$), then the mapping $f_{t'}: M/V \to M/V_t$ defined by $f_t(mV) = \alpha_t(m)\,\alpha_t(V)$ is a bi-real-analytic mapping satisfying $m \circ f = f \circ m$ ($m \in M$). Thus, what we essentially have is a family $X_t (t \in D)$ of G-structures on the fixed homogeneous space M/V. This statement implies and is implied by strong homogeneity. Q. E. D.

Proposition 6.6: *If M is compact, then any weakly homogeneous family is strongly homogeneous, where we have perhaps shrunken D.*

The proof follows from Proposition 6.5.

Example: Let X be a C-space of H. C. WANG [12]. Then we may write $X = M/V = A/B$ where M is compact semi-simple, A is the complexification of M, and all groups are connected. If X is non-Kähler, then WANG exhibited a 1-parameter family homogeneous complex structures on the final C^∞ manifold M/V; these structures were obtained from a 1-parameter family of complex subgroups $B_t \subset A$ such that $B_0 = B$ and $M/V \cong_{C^\infty} A/B_t$. Furthermore, the B_t's are not conjugate to B_0 (for t small). Thus, this family of complex structures is weakly homogeneous with respect to A, but is not strongly homogeneous. We remark that there exists on $X = A/B$ a 1-parameter family of complex structures $\{X_t\}$ ($t \in D$) which is not even weakly homogeneous with respect to A. These are the non-homogeneous deformations constructed in [4].

VII. Lie Groups, Group Cohomology, and Deformations of Geometric Structures

1. The Groups $H^q(X, \Theta_G)$ for $0 \leqq q \leqq 2$. Let X have a normal G-structure $G \to B_G \to X$ where G is of finite type. Then, if $\Gamma = \pi_1(X)$ and $\tilde{X}$ is the universal covering of X, there is a principal fibration $\Gamma \to \tilde{X} \xrightarrow{\pi} X$. There is induced on $\tilde{X}$ a unique normal G-structure $G \to \tilde{B}_G \to \tilde{X}$ relative to which Γ acts as G-motions. Furthermore, Γ acts on $\mathfrak{g} = H^0(\tilde{X}, \tilde{\Theta}_G)$, and thus the group cohomology modules $H^q(\Gamma, \mathfrak{g})$ are defined.

Theorem 7.1: *For $q = 0, 1$, $H^q(X, \Theta_G) \cong H^q(\Gamma, \mathfrak{g})$. For $q = 2$, there is an injection $i : H^2(\Gamma, \mathfrak{g}) \to H^2(X, \Theta_G)$.*

Remarks: In general, i is not surjective.

Proof of Theorem 7.1: Let $\{\tilde{U}_i\}_{i \in I}$ be a finite system of open sets in $\tilde{X}$ such that: (i) the sets $\{\tilde{U}_i \gamma\}$ ($i \in I$, $\gamma \in \Gamma$) give a normal covering of $\tilde{X}$; and (ii) for each pair $ij \in I$, there is at most one $\gamma = \gamma_{ij} \in \Gamma$ such that $\tilde{U}_i \gamma \cap \tilde{U}_j \neq \emptyset$. We set $U_i = \pi(\tilde{U}_i)$, and then $\mathrm{U} = \pi\,\{U_i\}$ gives a finite normal covering of X with nerve N. To each $(i, j) \in \mathrm{N}$, there is associated a unique $\gamma_{ij} \in \Gamma$ such that (i) $\gamma_{ii} = e$, (ii) $\gamma_{ij} = \gamma_{ji}^{-1}$; and (iii) $\gamma_{ij}\gamma_{jk} = \gamma_{ik}$. Thus the system $\{\gamma_{ij}\} \in H^1(\mathrm{N}, \Gamma)$ and this element defines the principal bundle $\Gamma \to \tilde{X} \xrightarrow{\pi} X$.

Let $\tilde{\theta}_1, \ldots, \tilde{\theta}_r$ be a basis for $\mathfrak{g} = H^0(\tilde{X}, \tilde{\Theta}_G)$; then we may restrict $\tilde{\theta}_\alpha$ to $\tilde{U}_i$ and project $\tilde{\theta}_\alpha | \tilde{U}_i$ onto U_i by π_* to get a basis $\theta_1^i, \ldots, \theta_r^i$ of $\Theta_G | U_i$. The right action of Γ on $\tilde{X}$ induces a representation $\varrho : \Gamma \to GL(\mathfrak{g})$, and we have:

Proposition 7.1: $\theta_\alpha^i = \sum\limits_\beta \varrho(\gamma_{ij})_\alpha^\beta\, \theta_\beta^j$ *for* $(i, j) \in \mathrm{N}$.

Proof: Set $\tilde{\theta}_\alpha | \tilde{U}_i = \tilde{\theta}_\alpha^i$. Then $\theta_\alpha^i = \pi_*(\tilde{\theta}_\alpha^i) = \pi_*(\gamma_{ij})_*\,(\tilde{\theta}_\alpha^i) = \pi_* \sum \varrho(\gamma_{ij})_\alpha^\beta\, \tilde{\theta}_\beta^i = \sum\limits_\beta \varrho(\gamma_{ij})_\alpha^\beta\, \theta_\beta^j$. Q.E.D.

A q-cochain $\xi = \{\xi_{i_1 \ldots i_{q+1}}\} \in C^q(\mathrm{N}, \Theta_G)$ assigns to each $U_{i_1 \ldots i_{q+1}} = U_{i_1} \cap \cap \cdots \cap U_{i_{q+1}}$ a section $\xi_{i_1 \ldots i_{q+1}}$ of $\Theta_G | U_{i_1 \ldots i_{q+1}}$, written in the $\{\theta_\alpha^{i_1}\}$ basis. The cocycle conditions for $q = 0, 1$ are: (0) $\xi_i = \varrho(\gamma_{ij})\, \xi_j$; and (1) $\xi_{ij} + \varrho(\gamma_{ij})\, \xi_{jk} = \xi_{ik}$. The q-cochains $f \in C^q(\Gamma, \mathfrak{g})$ are alternating functions $f : \underbrace{\Gamma \times - \times \Gamma}_{q} \to \mathfrak{g}$ such that $f(\gamma_1, \ldots, \gamma_q) = 0$ if some $\gamma_j = e$. Then $\delta f \in C^{q+1}(\Gamma, \mathfrak{g})$ is given, for $q = 0, 1$ by: (0) $\delta f(\gamma) = \varrho(\gamma) f - f$; and (1) $\delta f(\gamma_1, \gamma_2) = \varrho(\gamma_1)\, f(\gamma_2) - f(\gamma_1 \gamma_2) + + f(\gamma_1)$.

We need to know the relation between the γ_{ij} and the group Γ; this is given in [13]. Fix $i_0 \in I$ and for each $i \in I$, fix a *chain* $C_i = (i_0, \ldots, i_{m-1}, i_m = i)$

where $(i_{s-1}, i_s) \in \mathrm{N}$, set $\delta_i = \gamma_{i_0 i_1} \gamma_{i_1 i_2} \cdots \gamma_{i_{m-1} i_m}$ and set $\sigma_{ij} = \delta_i \gamma_{ij} \delta_j^{-1}$. Then $\sigma_{ii} = e$; $\sigma_{ij} = \sigma_{ji}^{-1}$; $\sigma_{ij}\sigma_{jk} = \sigma_{ik}$; and $\sigma_{i_0 i_1}\sigma_{i_1 i_2} \cdots \sigma_{i_{m-1} i_m} = e$ whenever $i_m = i_0$. The elements σ_{ij}, $(i, j) \in \mathrm{N}$, generate Γ with the above as fundamental set of relations.

Let $\xi = \{\xi_{ij}\} \in Z^1(\mathrm{N}, \Theta_G)$; then $\xi_{ij} \in \mathfrak{g}$ and $\xi_{ij} + \varrho(\gamma_{ij})\, \xi_{jk} = \xi_{ik}$. If we have a chain $J = (j_0, \ldots, j_s)$ such that $(j_{k-1}, j_k) \in \mathrm{N}$ $(1 \leq k \leq s)$, we define $\xi_J \in \mathfrak{g}$ by

$$\xi_J = \sum_{k=0}^{s-1} \varrho(\gamma_{j_0 j_1} \cdots \gamma_{j_{k-1} j_k})\, \xi_{j_{k-1} j_k}. \tag{7.1}$$

If we have a k_0, $0 < k_0 \leq s$, and a $j \in I$ such that $\gamma_{j_{k_0-1} j_{k_0}} = \gamma_{j_{k_0-1} j}\, \gamma_{j j_{k_0}}$, then we have a chain $J^1 = (j_0, \ldots, j_{k_0-1}, j_{k_0}, \ldots, j_s)$ and $\xi_J = \xi_{J^1}$, since ξ_J is a cocycle. Thus ξ_J is invariant under "deformations" of the path J. We define $f_\xi \in Z^1(\Gamma, \mathfrak{g})$ as follows: (1) $f_\xi(e) = 0$, and (ii) let $\gamma = \delta_i \gamma_{ij} \delta_j^{-1}$, $J_{ij} = (i_0, \ldots, i_{m-1}, i, j, j_{r-1}, \ldots, j_0)$ $(j_0 = i_0)$, and set $f_\xi(\gamma) = \xi_{J_{ij}}$. Then $f_\xi(\gamma)$ is well-defined by the remark on continuous deformations of paths. In general, if $\gamma = \sigma_{i_1 i_2} \cdots \sigma_{i_{s-1} i_s}$, we may set $f_\xi(\gamma) = \sum \varrho(\sigma_{i_1 i_2} \cdots \sigma_{i_{k-1} i_k})\, f_\xi(\sigma_{i_{k-1} i_k})$. Clearly, $f_\xi \in Z^1(\Gamma, \mathfrak{g})$.

Now if $\xi = \delta\eta$ for some $\eta = \{\eta_i\} \in C^0(\mathrm{N}, \Theta_G)$, then $\xi_{ij} = \varrho(\gamma_{ij})\, \eta_j - \eta_i$ for $(i, j) \in \mathrm{N}$. But then $f_\xi(\sigma_{ij}) = \varrho(\sigma_{ij})\, \eta_{i_0} - \eta_{i_0}$ by (7.1) and it follows that $f_\xi = \delta\eta_{i0}$ for $\eta_{i0} \in C^0(\Gamma, \mathfrak{g}) \cong \mathfrak{g}$. It also follows easily that if $f_\xi \in \delta C^0(\Gamma, \mathfrak{g})$, then $\xi \in \delta C^0(\mathrm{N}, \Theta_G)$. Thus we have an injection $i : H^1(\mathrm{N}, \Theta_G) \to H^1(\Gamma, \mathfrak{g})$.

We now define a mapping $f \to \hat{f}$ of $Z^1(\Gamma, \mathfrak{g}) \to Z^1(\mathrm{N}, \Theta_G)$ such that the composite $\xi \to f_\xi \to \hat{f}_\xi$ is the identity. The mapping $f \to \hat{f}$ is given by $(\hat{f})_{ij} = \varrho(\delta_j^{-1}) \times f(\sigma_{ij}) - \varrho(\delta_i^{-1})\, f(\delta_i) - \varrho(\gamma_{ij})\, f(\delta_j^{-1})$ where $\sigma_{ij} = \delta_i \gamma_{ij} \delta_j^{-1}$. Then $\hat{f}_\xi = \xi$.

From the definition, $H^0(\Gamma, \mathfrak{g}) \cong H^0(\mathrm{N}, \Theta_G)$, and one checks that there is an injection $i : H^2(\Gamma, \mathfrak{g}) \to H^2(\mathrm{N}, \Theta_G)$ in a straightforward manner. This completes the proof of theorem.

An Application:

Proposition 7.2: *If $\pi_1(X)$ is finite, then any normal G-structure $G \to B_G \to X$ where G is of finite type is locally rigid.*

Proof. We may prove that $H^1(X, \Theta_G) = 0$. But $H^1(X, \Theta_G) \cong H^1(\Gamma, \mathfrak{g})$ and it is well known that, if Γ is a finite group of order q, and V is a torsion-free Γ-module, then $H^1(\Gamma, V) = 0$. In fact, if $f \in Z^1(\Gamma, V)$, then $f(\gamma \cdot \gamma') = f(y) + \gamma \cdot f(\gamma^1)$, and, if we set $\pi = \frac{1}{q} \sum_{\gamma \in \Gamma} f(\gamma)$, then $\pi \in C^0(\Gamma, \mathfrak{g})$ and $f(\gamma) = \gamma \cdot \pi - \pi = \delta\pi(\gamma)$. Q. E. D.

Remarks: (i) This Proposition generalizes known results about discrete subgroups of compact Lie groups. (ii) Theorem 7.1 may be proven using a spectral sequence argument; however, we shall use the constructions in the proof to prove a similar statement for sheaves of non-abelian groups, for which the spectral sequence is not available.

2. The Set $H^1(X, \Gamma_G[t])$. Assume now that the G-structure $G \to \tilde{B}_G \to \tilde{X}$ is complete, and let $\mathfrak{G}$ with Lie algebra $\mathfrak{g}$, be the Lie group of automorphisms of this structure. Then Γ is a subgroup of $\mathfrak{G}$, and we define a set $\mathrm{Hom}(\Gamma, \mathfrak{G})\,[t]$

as follows: $\mathrm{Hom}(\Gamma, \mathfrak{G})\,[t]$ consists of germs of 1-parameter families of mappings $f(t): \Gamma \to \mathfrak{G}$ such that: (i) $f(0) =$ identity; (ii) $f(t) \in \mathrm{Hom}(\Gamma, \mathfrak{G})$ for fixed t; and (iii) $f(t)$ is C^∞ (or real analytic) in t. We say that $f(t)$, $g(t) \in \mathrm{Hom}(\Gamma, \mathfrak{G})\,[t]$ are equivalent if there exists a smooth curve $\Psi(t) \in \mathfrak{G}$ such that $\Psi(0) = e \in \mathfrak{G}$ and, for $\gamma \in \Gamma$, $\Psi(t)\,(f(t)\,(\gamma)) = (g(t)\,(\gamma))\,\Psi(t)$.

Definition 7.1: The set $\mathrm{Hom}(\Gamma, \mathfrak{G})[t]$ is defined to be $\mathrm{Hom}(\Gamma, \mathfrak{G})\,[t]$ modulo the above equivalence relation.

Theorem 7.2: *If the G-structure $G \to \tilde{B}_G \to \tilde{X}$ is complete, then $H^1(X, \Gamma_G[t]) \cong$ $\cong \mathrm{Hom}(\Gamma, \mathfrak{G})\,[t]$ in a natural fashion.*

Proof of Theorem 7.2: We keep the same notations as in § VII (2) above. Thus $G \to B_G \to X$ is a normal G-structure such that the induced structure $G \to \tilde{B}_G \to \tilde{X}$ is complete. We have defined (see § I.2) sheaves $\Gamma_G[t]$ on X, and correspondingly, $\widetilde{\Gamma_G[t]}$ on $\tilde{X}$. Now, because of completeness, Γ_G and $\tilde{\Gamma}_G$ are both locally constant sheaves of finite dimensional Lie groups, say $\mathfrak{G}$. Thus, for $U \subset X$, $\Gamma_G \mid U = \mathfrak{G} \mid U$ is generated by the Lie algebra $\Theta_G \mid U$. A germ $f(t) \in \Gamma_G[t] \mid U$ is a 1-parameter family of local G-automorphisms of neighborhoods of U such that $f(0) =$ identity; or, $f(t)$ may be considered as a 1-parameter family of mappings into $\mathfrak{G}$ such that $f(0) = e$. However, writing a germ $f(t)$ in this second form necessitates a change of the value of $f(t)(x) \in \mathfrak{G}\,(x \in U)$ upon changing coordinates, and this is the whole point. To be more precise let $\mathfrak{U} = \{U_i\}$ be a normal open covering of X, and identify $\Gamma_G \mid U_i$ with $\mathfrak{G}$ (constant sheaf) in the same manner as was done when identifying $\Theta_G \mid U_i$ with $\mathfrak{g}$ in § VII (2) above. Then, as in Proposition 7.2, we have:

Proposition 7.3: *Let $f_j : U_j \to \mathfrak{G}$ be a section of Γ_G written in the coordinates in U_j. Then, for $(i, j) \in \mathrm{N}$, in $U_i \cap U_j$, f_j written in the coordinates in U_i is given by $\gamma_{ij} f_j \gamma_{ij}^{-1}$.*

An element $\xi(t) = \{\xi_{ij}(t)\} \in H^1(\mathrm{N}, \Gamma_G[t])$ is then given by a collection $\xi_{ij}(t) \in \mathfrak{G}$, $\xi_{ij}(0) = e$, satisfying $\xi_{ij}(t)\,\gamma_{ij}\,\xi_{jk}(t)\,\gamma_{ij}^{-1} = \xi_{jk}(t)$. Thus $\xi_{ij}(t)\,\gamma_{ij} \times$ $\times\ \xi_{jk}(t)\,\gamma_{jk} = \xi_{ik}(t)\,\gamma_{ik}$ for $(i, jk) \in \mathrm{N}$. Now given $\xi(t) \in H^1(\mathrm{N}, \Gamma_G[t])$, we define a collection $\{{}^{\#}\xi_{ij}(t)\}_{(i,j)} \in \mathrm{N}$ of elements in $\mathfrak{G}$ by ${}^{\#}\xi_{ij}(t) = \xi_{ij}(t)\,\gamma_{ij}$. Then, for $(i, j, k) \in \mathrm{N}$, ${}^{\#}\xi_{ij}(t)\,{}^{\#}\xi_{jk}(t) = {}^{\#}\xi_{ik}(t)$, and ${}^{\#}\xi_{ij}(0) = \gamma_{ij}$. For a chain $J = (i_1, i_2, \ldots, i_m)$, we set ${}^{\#}\xi_J(t) = \prod_{1 \leqq r < m} {}^{\#}\xi_{i_r j_{r+1}}(t)$, and then ${}^{\#}\xi_J(0)$ $= \prod_{1 \leqq r < m} \gamma_{i_r i_{r+1}}$ and ${}^{\#}\xi_J(t)$ is invariant under deformation of the path J. Now, for $(i, j) \in \mathrm{N}$, we have a chain $J_{ij} = (i_0, i_1, \ldots, i_m = i, j = i_s, i_{s-1}, \ldots, i_0)$, and we set ${}^{\#}\xi(t)\,(\sigma_{ij}) = {}^{\#}\xi_{J_{ij}}(t)$. Then ${}^{\#}\xi(t)\,(\sigma_{ij})\;{}^{\#}\xi(t)\,(\sigma_{jk}) = {}^{\#}\xi(t)\,(\sigma_{ik})$, and ${}^{\#}\xi(0)\,(\sigma_{ij}) = \sigma_{ij}$. In fact, we have in essence defined a mapping ${}^{\#}$: $Z^1(\mathrm{N}, \Gamma_G[t]) \to \mathrm{Hom}(\Gamma, \mathfrak{G})\,[t]$, by sending $\xi(t)$ into ${}^{\#}\xi(t)$.

Now if $\xi(t) \in \delta C^0(\mathrm{N}, \Gamma_G[t])$, then $\xi_{ij}(t) = \varrho_i(t)^{-1}\,\gamma_{ij}\varrho_j(t)\,\gamma_{ij}^{-1}$ for $\varrho(t) = \{\varrho_i(t)\} \in C^0(\mathrm{N}, \Gamma_G[t])$. But then ${}^{\#}\xi(t)\,(\sigma_{ij}) = \varrho_{i_0}(t)^{-1}\sigma_{ij}\varrho_{i_0}(t)$ or $\varrho_{i_0}(t)\,{}^{\#}\xi(t)\,(\sigma_j) = \sigma_{ij}\varrho_{i_0}(t)$, and thus, for all $\gamma \in \Gamma$, $\varrho_{i_0}(t)\,{}^{\#}\xi(t)\,(\gamma) = \gamma \cdot \varrho_{i_0}(t)$. Since $\varrho_{i_0}(0) = e$, we conclude that, if $\xi(t) = \eta(t)$ in $H^1(\mathrm{N}, \Gamma_G[t])$, then ${}^{\#}\xi(t)$ is related to ${}^{\#}\eta(t)$ in the manner described above theorem, and thus ${}^{\#}\xi(t)$ $= {}^{\#}\eta(t)$ in $\mathrm{Hom}(\Gamma, \mathfrak{G})\,[t]$.

The rest of the theorem proceeds along these very same lines, the end result being, as asserted, an identification between $H^1(X, \Gamma_G[t])$ and $\mathrm{Hom}(\Gamma, \mathfrak{G})\,[t]$. We omit the details and thus conclude the proof.

3. Formal Construction of $\mathrm{Hom}(\Gamma, \mathfrak{G})\,[t]$. We shall now give a global construction of $\mathrm{Hom}(\Gamma, \mathfrak{G})\,[t]$ analogous to the sheaf construction of $H^1(X, \Gamma_G[t])$ given in § IV. Although this construction will be carried out formally here, the convergence will be obtained by analytical methods below.

Theorem 7.3: *If* $H^2(\Gamma, \mathfrak{g}) = 0$, *then we may formally embed a neighborhood of* o *in* $H^1(\Gamma, \mathfrak{g})$ *into* $\mathrm{Hom}(\Gamma, \mathfrak{G})\,[t]$ *such that* o *corresponds to the distinguished element in* $\mathrm{Hom}(\Gamma, \mathfrak{G})\,[t]$.

Remarks: (i) The hypothesis in this theorem are only that $\mathfrak{G}$ is a Lie group with Lie algebra $\mathfrak{g}$, and Γ is a discrete subgroup of $\mathfrak{G}$. However, in order to simplify computations, we shall assume that $\mathfrak{G}$ is a linear matrix group; the proof in general is only notationally more complicated. Finally, in order to make the statement of the theorem more plausible, we shall associate, to each $f(t) \in \mathrm{Hom}(\Gamma, \mathfrak{G})\,[t]$, an element $f^\# \in H^1(\Gamma, \mathfrak{g})$. The proof of the theorem will then constitute a reversal of this process.

For $f(t) \in \mathrm{Hom}(\Gamma, \mathfrak{G})\,[t]$, write $f(t)\,(\gamma) = f_\gamma(t)$ $(\gamma \in \Gamma)$. Then $f_\gamma(t)\, f_\sigma(t) = f_{\gamma\sigma}(t)$ and $f_\gamma(0) = \gamma$. Define $f^\# \in C^1(\Gamma, \mathfrak{g})$ by $f^\#(\gamma) = \left.\frac{d f_\gamma(t)}{dt}\right]_{t=0} \gamma^{-1}$. This mapping $f(t) \to f^\#$ is called the *infinitessimal mapping.*

Proposition 7.4: $f^\#(\gamma\sigma) = f^\#(\gamma) + \gamma \cdot f^\#(\sigma) \cdot \gamma^{-1}$, *and thus* $f^\# \in H^1(\Gamma, \mathfrak{g})$.

Proof: Differentiate $f_\gamma(t)\, f_\sigma(t)\sigma^{-1} \cdot \gamma^{-1} = f_{\gamma\sigma}(t)\,(\gamma\sigma)^{-1}$ and set $t = 0$.

Proof of theorem 7.3: Let $\theta \in H^1(\Gamma, \mathfrak{g})$ be arbitrary. We shall construct, for each $N > 0$ and $\gamma \in \Gamma$, a series $f_\gamma^n(t) = \sum_{\mu=1}^{n} (f_\gamma^n)_\mu t^\mu \in \mathfrak{G}$ such that: (i) $(f_\gamma^n)_0 = \gamma$; (ii) $(f_\gamma^n)_1 \cdot \gamma^{-1} = \theta(\gamma)$; and (iii) $f_\gamma^n(t)\, f_\sigma^n(t) \equiv f_{\gamma\sigma}^n(t)$ (modulo t^{n+1}). Then $f_\gamma(t) = \underset{u}{\overleftarrow{\lim}}\, f_\gamma^n(t)$ will satisfy the requirements of the theorem. We shall write $\underset{t^n}{\equiv}$ to mean "congruent modulo t^n".

Set $f_\gamma^1(t) = (\exp\theta(\gamma) \cdot t) \cdot \gamma$. Then $f_\gamma^1(t) \in \mathfrak{G}$, $(f_\gamma^1)_0 = \gamma$, $(f_\gamma^1)_1 \cdot \gamma^{-1} = \theta(\gamma)$, and

$$\begin{aligned} f_\gamma^1(t)\, f_\sigma^1(t) &= (\exp\theta(\gamma) \cdot t) \cdot \gamma \cdot (\exp\theta(\sigma)t) \cdot \sigma \\ &= (\exp\theta(\gamma) \cdot t) \cdot \gamma \cdot (\exp\theta(\sigma)t) \cdot \gamma^{-1} \cdot (\gamma\sigma) \\ &\underset{t^2}{\equiv} \gamma\sigma + (\theta(\gamma) + \gamma\theta(\sigma) \cdot \gamma^{-1})\,(\gamma\sigma) \cdot t \\ &\underset{t^2}{\equiv} \exp(\theta(\gamma\sigma)t) \cdot \gamma\sigma = f_{\gamma\sigma}^1(t)\,. \end{aligned}$$

Suppose now that we have $f_\gamma^n(t)$; we try to construct $f_\gamma^{n+1}(t)$ $(\gamma \in \Gamma)$. We may inductively assume that $f_\gamma^n(t) = \exp(\varphi_\gamma^n(t)) \cdot \gamma$ where $\varphi_\gamma^n(t) = \sum_{\mu=0}^{n} \varphi_\mu(\gamma)t^\mu$, $\varphi_1(\gamma) = \theta(\gamma)$. Define $w_{\gamma,\sigma}^{n+1}$ by: $f_\gamma^n(t)\, f_\sigma^n(t) \underset{t^{n+2}}{\equiv} (1 + w_{\gamma,\sigma}^{n+1} t^{n+1})\, f_{\gamma\sigma}^n(t)$.

Proposition 7.5: $w_{\gamma,\sigma}^{n+1} \in \mathfrak{g}$ $(\gamma, \sigma \in \Gamma)$ *and, in fact,* $w_{\gamma,\sigma}^{n+1} \in H^2(\Gamma, \mathfrak{g})$.

Proof: We must show that $w_{\gamma\sigma,\tau}^{n+1} + w_{\gamma,\sigma}^{n+1} = w_{\gamma,\sigma\tau}^{n+1} + \gamma \cdot w_{\sigma,\tau}^{n+1} \cdot \gamma^{-1}$ $(\gamma, \sigma, \tau \in \Gamma)$.

Now

$$f^n_\gamma(t)\, f^n_\sigma(t)\, f^n_\tau(t) \underset{t^{n+2}}{\equiv} (1 + w^{n+1}_{\gamma,\sigma} t^{n+1})\, f^n_{\gamma\sigma}(t)\, f^n_\tau(t)$$

$$\underset{t^{n+2}}{\equiv} (1 + w^{n+1}_{\gamma,\sigma} t^{n+1})\, (1 + w^{n+1}_{\gamma\sigma,\tau} t^{n+1})\, f^n_{\gamma\sigma\tau}(t)\,.$$

On the other hand,

$$f^n_\gamma(t)\, f^n_\sigma(t)\, f^n_\tau(t) \underset{t^{n+2}}{\equiv} f^n_\gamma(t)\, (1 + w^{n+1}_{\sigma\tau} t^{n+1})\, f^n_{\sigma\tau}(t)$$

$$\underset{t^{n+2}}{\equiv} (1 + \gamma \cdot w^{n+1}_{\sigma,\tau} \cdot \gamma^{-1} t^{n+1}) f^n_\gamma(t)\, f^n_{\sigma\tau}(t)$$

$$\underset{t^{n+2}}{\equiv} (1 + \gamma \cdot w^{n+1}_{\sigma,\tau} \cdot \gamma^{-1} t^{n+1})\, (1 + w^{n+1}_{\gamma,\sigma\tau} t^{n+1})\, f^n_{\gamma\sigma\tau}(t)\,.$$

Thus $w^{n+1}_{\gamma,\sigma} + w^{n+1}_{\gamma\sigma,\tau} = w^{n+1}_{\gamma,\sigma\tau} + \gamma \cdot w^{n+1}_{\sigma,\tau} \cdot \gamma^{-1}$. Q. E. D.

By hypothesis, there exists $\varphi_{n+1} \in C^1(\Gamma, \mathfrak{g})$ such that $(\delta\varphi_{n+1})(\gamma, \sigma) = \varphi_{n+1}(\gamma\sigma) - \varphi_{n+1}(\gamma) - \gamma\varphi_{n+1}(\sigma) \cdot \gamma^{-1} = w^{n+1}_{\gamma,\sigma}$. Set $\varphi^{n+1}_\gamma(t) = \varphi^n_\gamma(t) + \varphi_{n+1}(\gamma) t^{n+1}$, and $f^{n+1}_\gamma(t) = \exp(\varphi^{n+1}_\gamma(t)) \cdot \gamma$. Then a direct calculation similar to that in Lemma 7.5 shows that $f^{n+1}_\gamma(t)\, f^{n+1}_\sigma(t) \underset{t^{n+2}}{\equiv} f^{n+1}_{\gamma\sigma}(t)$. Q. E. D.

4. Formal Completeness of Deformations in a Special Case. We have defined $\mathrm{Hom}(\Gamma, \mathfrak{G})\,[t]$ as the space of effective germs of 1-parameter families of homomorphisms of Γ into $\mathfrak{G}$ which reduce to the identity at $t = 0$. Also defined was the *infinitessimal mapping* $\mathrm{Hom}(\Gamma, \mathfrak{G})\,[t] \to H^1(\Gamma, \mathfrak{g})$ obtained by sending $f(t)$ into $f^\# = \frac{df(t)}{dt} \cdot f(t)^{-1}]_{t=0}$. If now $t = (t^1, \ldots, t^m)$ is a point varying in a neighborhood of $\mathbf{o} \in \mathbf{R}^m$, then we may clearly define $\mathrm{Hom}(\Gamma, \mathfrak{G})\,[t]$ as homomorphisms depending on m-parameters. There is also an infinitessimal mapping $i : \mathrm{Hom}(\Gamma, \mathfrak{G})\,[t] \times \mathbf{R}^m \to H^1(\Gamma, \mathfrak{g})$ defined by sending $(f(t); \xi^1, \ldots, \xi^m)$ into $i(f(t); \xi^1, \ldots, \xi^m) = \sum_{\alpha=1}^m \xi^\alpha \frac{\partial f(t)}{\partial t^\alpha} \cdot f(t)^{-1}]_{t=0}$. Let $f(t)$ be a germ in $\mathrm{Hom}(\Gamma, \mathfrak{G})[t]$, let s vary in a neighborhood of $\mathbf{o} \in \mathbf{R}$, and let $g(s)$ be an arbitrary germ in $\mathrm{Hom}(\Gamma, \mathfrak{G})\,[s]$.

Definition 7.2: $f(t)$ is a *complete germ of deformation* if, for any $g(s)$, there exists a mapping $\tau : s \to \tau(s) \in \mathbf{R}^m$, $\tau(\mathbf{o}) = \mathbf{o}$, such that $g(s) = f(\tau(s))$ in $\mathrm{Hom}(\Gamma, \mathfrak{G})\,[s]$.

Theorem 7.4: *If $i : f(t) \times \mathbf{R}^m \to H^1(\Gamma, \mathfrak{g})$ is onto, then $f(t)$ is formally a complete germ.*

Proof: We must produce formal mappings $\psi : s \to \psi(s) \in \mathbf{R}^m$, $\psi(\mathbf{o}) = 0$; and $\Psi : s \to \Psi(s) \in \mathfrak{G}$, $\Psi(0) = 1$, such that

$$\Psi(s)\, f(\psi(s)) = g(s)\, \Psi(s)\,. \tag{7.2}$$

Let $\xi \in \mathbf{R}^m$, and define $\theta_\xi \in Z^1(\Gamma, \mathfrak{g})$ by $\theta_\xi(\gamma) = \sum_{\alpha=1}^m \xi^\alpha \frac{\partial f_\gamma(t)}{\partial t^\alpha}\Big]_{t=0} \cdot \gamma^{-1}$. Given $w \in Z^1(\Gamma, \mathfrak{g})$, there exists $\xi \in \mathbf{R}^m$, and $\varphi \in C^0(\Gamma, \mathfrak{g}) = \mathfrak{g}$, such that $w = \theta_\xi + \delta\varphi$; i.e., for $\gamma \in \Gamma$, $w(\gamma) = \theta_\xi(\gamma) + \varphi - \gamma \cdot \varphi \cdot \gamma^{-1}$. We shall construct infinite sequences $\{\psi_\mu\}$ $(\psi_\mu \in \mathbf{R}^m)$, $\{\varphi_\gamma\}$ $(\varphi_\gamma \in \mathfrak{g})$ such that, setting $\Psi^n(s)$

$= \exp(\varphi^n(s))$ $(\varphi^n(s) = \varphi_1 s + \cdots + \varphi_n s^n)$, $\psi^n(s) = \sum_{\mu=1}^{n} \psi_\mu s^\mu$, we have

$$(7.3)^n \qquad \Psi^n(s)\, f_\gamma(\psi^n(s)) \underset{s^{n+1}}{\equiv} g_\gamma(s)\, \Psi^n(s) \qquad (\text{for all } \gamma \in \Gamma)\,.$$

The inductive construction may be assumed to begin with $n = 0$, in which case $(7.3)^n$ is satisfied. Suppose, therefore, that we have made the construction for n, and we shall do it for $n + 1$. Define w_γ^{n+1} by

$$\Psi^n(s)\, f_\gamma(\psi^n(s)) \underset{s^{n+2}}{\equiv} (1 + w_\gamma^{n+1}\, s^{n+1})\, g_\gamma(s)\, \Psi^n(s)\,.$$

Proposition 7.6: $w_\gamma^{n+1} \in \mathfrak{g}$ and $w^{n+1} \in C^1(\Gamma, \mathfrak{g})$ *defined by* $w^{n+1}(\gamma) = w_\gamma^{n+1}$ *satisfies* $\delta(w^{n+1}) = 0$.

Proof: $\Psi^n(s)\, f_{\gamma\sigma}(\psi^n(s)) \underset{s^{n+2}}{\equiv} (1 + w_{\gamma\sigma}^{n+1} s^{n+1})\, g_{\gamma\sigma}^n(s)\, \Psi^n(s)$. But

$$\begin{aligned}\Psi^n(s)\, f_{\gamma\sigma}(\psi^n(s)) &= \Psi^n(s)\, f_\gamma(\psi^n(s))\, f_\sigma(\psi^n(s))\\ &\underset{s^{n+2}}{\equiv} (1 + w_\gamma^{n+1}\, s^{n+1})\, g_\gamma(s)\, (1 + w_\sigma^{n+1}\, s^{n+1})\, g_\sigma(s)\, \Psi^n(s)\\ &\underset{s^{n+2}}{\equiv} (1 + w_\gamma^{n+1}\, s^{n+1})\, (1 + \gamma \cdot w_\sigma^{n+1} \cdot \gamma^{-1}\, s^{n+1})\, g_{\gamma\sigma}(s)\, \Psi^n(s)\,,\end{aligned}$$

and thus $w_{\gamma\sigma}^{n+1} = w_\gamma^{n+1} + \gamma \cdot w_\sigma^{n+1} \cdot \gamma^{-1}$. Q. E. D.

By hypothesis, there exists $\psi_{n+1} \in \mathbf{R}^m$, $\varphi_{n+1} \in \mathfrak{g}$, such that $-w_\gamma^{n+1} = \Theta_{\psi_{n+1}}(\gamma) + \varphi_{n+1} - \gamma \cdot \varphi_{n+1} \cdot \gamma^{-1}$. Set

$$\psi^{n+1}(s) = \psi^n(s) + \psi_{n+1}\, s^{n+1},\ \varphi^{n+1}(s) = \varphi^n(s) + \varphi_{n+1}\, s^{n+1}\,.$$

Proposition 7.7: $\Psi^{n+1}(s)\, f_\gamma(\psi^{n+1}(s)) \underset{s^{n+2}}{\equiv} g_\gamma(s)\, \Psi^{n+1}(s)$.

Proof: $\Psi^{n+1}(s)\, f_\gamma(\psi^{n+1}(s)) \underset{s^{n+2}}{\equiv} \exp(\varphi^n(s) + \varphi_{n+1} s^{n+1})\, f_\gamma(\psi^n(s) + \psi_{n+1} s^{n+1})$

$$\begin{aligned}&\underset{s^{n+2}}{\equiv} \big((\exp\varphi^n(s)) + \varphi_{n+1} s^{n+1}\big) \cdot \big(f_\gamma(\psi^n(s)) + \theta_{\psi_{n+1}}(\gamma) \cdot \gamma \cdot s^{n+1}\big)\\ &\underset{s^{n+2}}{\equiv} \Psi^n(s)\, f_\gamma(\psi^n(s)) + \varphi_{n+1} \cdot \gamma \cdot s^{n+1} + \theta_{\psi_{n+1}}(\gamma) \cdot \gamma \cdot s^{n+1}\\ &\underset{s^{n+2}}{\equiv} (1 + w_\gamma^{n+1} s^{n+1})\, g_\gamma(s)\, \Psi^n(s) + \varphi_{n+1} \cdot \gamma \cdot s^{n+1} + \theta_{\psi_{n+1}}(\gamma) \cdot \gamma \cdot s^{n+1}\\ &\underset{s^{n+2}}{\equiv} g_\gamma(s)\, \Psi^n(s) + w_\gamma^{n+1} \cdot \gamma \cdot s^{n+1} + \varphi_{n+1} \cdot \gamma \cdot s^{n+1} + \theta_{\psi_{n+1}}(\gamma) \cdot \gamma\, s^{n+1}\\ &\underset{s^{n+2}}{\equiv} g(s)\, \Psi^n(s) + \gamma \cdot \varphi_{n+1} \cdot s^{n+1} \underset{s^{n+2}}{\equiv} g_\gamma(s)\, \Psi^{n+1}(s) \qquad \text{Q. E. D.}\end{aligned}$$

Remarks: Although, we shall not do so here, it can be shown that convergence holds in theorem 7.4, although we are unable to prove convergence for theorem 7.3. For this it was necessary to assume that G is connected and that Γ is finitely generated and finitely related. If $\gamma_1, \ldots, \gamma_n$ are generators of Γ, we say that an expression $\gamma_{i_1} \ldots \gamma_{i_r}$ has *length* r, and choose R such that all relations in Γ have length $\leqq R$. Assume that we have a norm $|\ |$ on $\mathfrak{g}$; if $\varphi \in C^1(\Gamma, \mathfrak{g})$, define $\|\varphi\| = \sup_{r \leqq R} |\varphi(\gamma_{i_1} \ldots \gamma_{i_r})|$; and if $\psi \in C^2(\Gamma, \mathfrak{g})$, set $\|\psi\|$

$= \sup_{r \leqq R} |\psi(\gamma_{i_1} \ldots \gamma_{i_j}, \gamma_{i_{j+1}} \ldots \gamma_{i_r})|$. The crucial point in the convergence proof of theorem 7.3 is the following

Proposition 7.8: *There exists a $c > 0$ such that the following holds: for $q = 1, 2$ and for any $\psi \in \delta C^{q-1}(\Gamma, \mathfrak{g})$, there exists a $\varphi \in C^{q-1}(\Gamma, \mathfrak{g})$ such that $\delta\varphi = \psi$ and $\|\varphi\| \leqq C\|\psi\|$.*

Proof: Set $i(\psi) = \inf_{\delta\varphi = \psi} \|\varphi\|$. If the Proposition were false, then there exists an infinite sequence $\{\psi^\mu\} \in \delta C^{q-1}(\Gamma, \mathfrak{g})$ with $i(\psi^\mu) = 1$ and $\|\psi^\mu\| < \frac{1}{\mu}$. Choose $\{\varphi^\mu\} \in C^{q-1}(\Gamma, \mathfrak{g})$ such that $\|\varphi^\mu\| < 2$ and $\delta\varphi^\mu = \psi^\mu$. Assume $q = 2$ ($q = 1$ is similar). We have $\varphi^\mu(\gamma\sigma) - \varphi^\mu(\gamma) - \gamma\varphi^\mu(\sigma)\gamma^{-1} = \psi^\mu(\gamma, \sigma)$ $(\gamma, \sigma \in \Gamma)$. By the definition of $\| \ \|$, we may assume that, for $r \leqq R$, $\varphi^\mu(\gamma_{i_1} \cdots \gamma_{i_r}) \xrightarrow[\mu]{} \varphi(\gamma_{i_1} \cdots \gamma_{i_r})$; and, for $r \leqq R$ again, since $\|\psi^\mu\| \to 0$, $\varphi(\gamma_{i_1} \ldots \gamma_{i_r}) = \varphi(\gamma_{i_1} \ldots \gamma_{i_j}) + (\gamma_{i_1} \ldots \gamma_{i_j})\,\varphi(\gamma_{i_{j+1}} \ldots \gamma_{i_j})\,(\gamma_{i_1} \ldots \gamma_{i_j})^{-1}$. Since $\gamma_1, \ldots, \gamma_n$ generate Γ, and since all relations are of length $\leqq R$, we may uniquely extend φ to an affine homomorphism $\varphi : \Gamma \to \mathfrak{g}$; i.e. $\varphi \in H^1(\Gamma, \mathfrak{g})$. But then $\delta(\varphi^\mu - \varphi) = \psi^\mu$ and $\|\varphi^\mu - \varphi\| \to 0$. Contradiction.

VIII. Partial Differential Equations and Deformations of Geometric Structure

1. An Outline of the General Method. Let $G \to B_G \to X$ be a complete, normal G-structure where G is of finite type and X is compact, and assume that $H^2(X, \Theta_G) = 0$. In this section we shall use the method of partial differential equations, first initiated in [8], to prove Theorems 7.3 and 7.4; in other words, we shall prove that "formal implies actual."

Recall that a *graded Lie algebra* is a graded vector space $V = V^0 \oplus V^1 \oplus \cdots$ together with a bracket law $[\ ,\ ] : V^p \otimes V^q \to V^{p+q}$ satisfying: (i) $[\zeta, \eta] = (-1)^{pq+1}[\eta, \zeta]$ $(\zeta \in V^p, \eta \in V^q)$; and (ii) $(-1)^{pr}[[\zeta, \eta], \varrho] + (-1)^{qp}[[\eta, \varrho], \zeta] + (-1)^{qr}[[\varphi, \zeta], \eta] = 0$ for $\zeta \in V^p$, $\eta \in V^q$, $\varphi \in V^r$. For a vector bundle $\mathbf{E}$ over X, we let $\mathscr{E}$ be the sheaf of C^∞ sections of $\mathbf{E}$. A graded bundle $\mathbf{V} = \mathbf{V}^0 \oplus \mathbf{V}^1 \oplus \cdots$ over X is a *Lie bundle* if the sections of $\mathscr{V} = \mathscr{V}^0 \oplus \mathscr{V}^1 \oplus \cdots$ form a graded Lie algebra. We shall, in certain cases, associate to $G \to B_G \to X$ a Lie bundle $\mathbf{V}$ together with differential operators $D : \mathscr{V}^q \to \mathscr{V}^{q+1}$ satisfying the following conditions: (i) There is an injection $i : \Theta_G \to \mathscr{V}^0$ such that the sheaf sequence

$$0 \to \Theta_G \xrightarrow{i} \mathscr{V}^0 \xrightarrow{D} \mathscr{V}^1 \xrightarrow{D} \cdots \longrightarrow \mathscr{V}^q \xrightarrow{D} \mathscr{V}^{q+1} \tag{8.1}$$

is exact, and

$$D[\zeta, \eta] = [D\zeta, \eta] + (-1)^p[\zeta, D\eta] \quad (\zeta \in \mathscr{V}^p); \tag{8.2}$$

(ii) to state the second condition, we let $V^q = H^0(X, \mathscr{V}^q)$, and we let $V^q[t]$ be 1-parameter families $\varphi(t)$ of elements of V^q, depending smoothly on t, and satisfying $\varphi(0) = 0$. The second condition states that the non-linear equation

$$D\varphi(t) - [\varphi(t), \varphi(t)] = 0 \quad (\varphi(t) \in V^1[t]) \tag{8.3}$$

may be used to construct deformations. When (8.1)—(8.3) are satisfied, we say that $\mathscr{V}$ gives a *Lie resolution* of Θ_G. In general, we do not know that the resolution

(5.10) gives a Lie resolution of Θ_G, and it will be our task to modify (5.10) in certain cases to give a Lie resolution of Θ_G.

2. A Lie Resolution of Θ_G for Canonical Homogeneous G-Structures. We first give a resolution of the sheaf of infinitessimal automorphisms of a local Lie group. Let $U \subset \mathbf{R}^n$ be a relatively compact contractible domain, subject to shrinking, and let $w^1, \ldots, w^n$ be n-independent Pfaffians in U giving a parallelism and such that $dw^\alpha = c^\alpha_{\beta\gamma} w^\beta \wedge w^\gamma$ ($c^\alpha_{\beta\gamma}$ constant). Let $X_1, \ldots, X_n$ be a dual parallelism to $w^1, \ldots, w^n$, $\mathbf{T}^q =$ bundle of vector-valued q-forms on U, $\mathscr{T}^q$ and $T^q = H^0(U, \mathscr{T}^q)$ as above. Since we are shrinking U, we shall work with T^q in lieu of $\mathscr{T}^q$. If $\zeta \in T^q$, write $\zeta = \sum_{\alpha=1}^{n} = X_\alpha \otimes \zeta^\alpha$, and, if $\eta = \sum X_\beta \otimes \eta^\beta$, set $[\zeta, \eta] = \sum [X_\alpha, X_\beta] \otimes \zeta^\alpha \wedge \eta^\beta$. Then $T \oplus T^1 \oplus T^2 \oplus \cdots$ is a graded Lie algebra. We shall construct an injective Lie resolution

$$(8.4) \quad 0 \longrightarrow \Theta_G \xrightarrow{\ i\ } T \xrightarrow{D_1} T^1 \xrightarrow{D_2} T^2 \longrightarrow \cdots \longrightarrow T^q \xrightarrow{D_{q+1}} T^{q+1} \longrightarrow \cdots .$$

The injection $i : \Theta_G \to T$ is the injection of vector fields. For $\zeta \in T$, set $D_1(\zeta) = \sum X_\alpha \otimes L_\zeta w^\alpha$ where $L_\zeta w^\alpha$ is the Lie derivative of w^α along ζ. Then (8.4) is exact at Θ_G and T. We now find the differential equation which $D_1(\zeta)$ satisfies. For $\zeta \in T$, set $\eta^\alpha(t) = \exp(t\zeta)^* w^\alpha$. Then

$$(8.5) \qquad d\eta^\alpha(t) = c^\alpha_{\beta\gamma} \eta^\beta(t) \wedge \eta^\gamma(t)$$

and also

$$(8.6) \qquad \eta^\alpha(t) = w^\alpha + t L_\zeta w^\alpha + 0(t^2) .$$

Combining (8.5) and (8.6), we immediately get

$$(8.7) \qquad d(L_\zeta w^\alpha) = -c^\alpha_{\beta\gamma} w^\gamma \wedge L_\zeta w^\beta .$$

Furthermore, if $\varphi = \sum X_\alpha \otimes \varphi^\alpha \in T^1$, $\varphi^\alpha = L_\zeta w^\alpha$ $(\alpha = 1, \ldots, n)$ for some $\zeta \in T$ if, and only if,

$$(8.8) \qquad d\varphi^\alpha = -c^\alpha_{\beta\gamma} w^\gamma \wedge \varphi^\beta .$$

Thus, if we define $D_2 : T^1 \to T^2$ by

$$(8.9) \quad D_2\left(\sum X_\alpha \otimes \varphi^\alpha\right) = \sum X_\alpha \otimes d\varphi^\alpha + 2 \sum X_\alpha \otimes c^\alpha_{\beta\gamma} w^\gamma \wedge \varphi^\beta,\ D_2 D_1 = 0$$

and

$$0 \to \Theta_G \to T \to T^1 \to T^2$$

is exact.

From (8.9), we now see how to define $D_{q+1} : T^q \to T^{q+1}$; we first introduce some notation. Given $\zeta = \sum X_\alpha \otimes \zeta^\alpha \in T^q$, we set $d\zeta = \sum X_\alpha \otimes d\zeta^\alpha$, and we set $\Omega \barwedge \zeta = \sum X_\alpha \otimes c^\alpha_{\beta\gamma} w^\gamma \wedge \zeta^\beta$. Then we define $D_{q+1} = D : T^q \to T^{q+1}$ $(q \geqq 1)$ by

$$(8.10) \qquad D(\zeta) = d\zeta + 2\Omega \barwedge \zeta .$$

Proposition 8.1: $D^2 = 0$.

Proof: The proof is in three steps.

(i) **Lemma 8.1:** $d(\Omega \barwedge \zeta) = -\Omega \barwedge d\zeta - (\Omega \barwedge \Omega) \barwedge \zeta$.

Proof: The proof is a straightforward computation.

(ii) **Lemma 8.2:** $(\Omega \barwedge \Omega) \barwedge \zeta = 2\Omega \barwedge (\Omega \barwedge \zeta)$.

Proof: $((\Omega \barwedge \Omega) \barwedge \zeta)^\alpha = C^\alpha_{\beta\gamma} C^\beta_{\sigma\tau} w^\sigma \wedge w^\tau \wedge \zeta^\gamma$; and $(\Omega \barwedge (\Omega \barwedge \zeta))^\alpha = C^\alpha_{\beta\gamma} C^\beta_{\sigma\tau} w^\gamma \wedge w^\tau \wedge \zeta^\sigma$. From the Jacobi identity: $C^\alpha_{\beta\gamma} C^\beta_{\sigma\tau} = C^\alpha_{\tau\beta} C^\beta_{\gamma\sigma} + C^\alpha_{\sigma\beta} C^\beta_{\tau\gamma}$, $((\Omega \barwedge \Omega) \barwedge \zeta)^\alpha = C^\alpha_{\beta\tau} C^\beta_{\gamma\sigma} w^\tau \wedge w^\sigma \wedge \zeta^\gamma + C^\alpha_{\beta\sigma} C^\beta_{\gamma\tau} w^\sigma \wedge w^\tau \wedge \zeta^\gamma = 2(\Omega \barwedge (\Omega \barwedge \zeta))^\alpha$. Q.E.D.

(iii) $D(D\zeta) = D(d\zeta + 2\Omega \barwedge \zeta) + 2\Omega \barwedge (d\zeta + 2\Omega \barwedge \zeta) = -2\Omega \barwedge d\zeta - 2(\Omega \barwedge \Omega) \barwedge \zeta + 2\Omega \barwedge d\zeta + 4\Omega \barwedge (\Omega \barwedge \zeta) = 0$. Q. E. D.

Theorem 8.1: *The sequence* $0 \to \Theta_G \xrightarrow{i} T \xrightarrow{D} T^1 \xrightarrow{D} T^2 \to \cdots \to T^q \xrightarrow{D} \xrightarrow{D} T^{q+1} \to \cdots$ *is a Lie resolution of* Θ_G.

Proof: The proof of the exactness (i.e. the D-Poincaré lemma) follows from the d-Poincaré lemma and the relation between this resolution and the resolution (5.10); this is taken up in Lemma 9.2. § IX below. Thus, to prove the Theorem, we must prove:

$$D[\zeta, \eta] = [D\zeta, \eta] + (-1)^{\deg \xi} [\zeta, D\eta] \tag{8.11}$$

and that the equation

$$D\varphi(t) = [\varphi(t), \varphi(t)] \quad (\varphi(t) \in T^1[t]) \tag{8.12}$$

may be used to construct deformations.

For (8.11), we let $\xi \in T^q$, $\eta \in T^p$; and then
$D[\zeta, \eta] = \sum X_\alpha \otimes \{C^\alpha_{\beta\gamma} d\zeta^\beta \wedge \eta^\gamma + (-1)^q C^\alpha_{\beta\gamma} \xi^\beta \wedge d\eta^\gamma + 2C^\alpha_{\beta\gamma} C^\beta_{\sigma\tau} w^\gamma \wedge \xi^\sigma \wedge \eta^\tau\}$.
Now

$$[D\xi, \eta] = \sum X_\alpha \otimes \{C^\alpha_{\beta\gamma} d\zeta^\beta \wedge \eta^\gamma + 2C^\alpha_{\beta\sigma} C^\beta_{\gamma\tau} w^\tau \wedge \xi^\gamma \wedge \eta^\sigma\};$$

and

$$(-1)^q [\zeta, D\eta] = \sum X_\alpha \otimes \{(-1)^q C^\alpha_{\beta\gamma} \zeta^\beta \wedge \eta^\gamma + 2C^\alpha_{\beta\gamma} C^\gamma_{\sigma\tau} w^\tau \wedge \xi^\beta \wedge \eta^\sigma\}.$$

To prove (8.11), we must show:

$$C^\alpha_{\beta\gamma} C^\beta_{\sigma\tau} w^\gamma \wedge \zeta^\sigma \wedge \eta^\tau = C^\alpha_{\beta\gamma} C^\gamma_{\sigma\tau} w^\tau \wedge \zeta^\beta \wedge \eta^\sigma + C^\alpha_{\beta\sigma} \quad C^\beta_{\gamma\tau} w^\tau \zeta^\gamma \wedge \eta^\sigma;$$

and, as above, this equation is just the Jacobi-identity.

Now a deformation of the G-structure on U given by $w^1, \ldots, w^n$ is given by Pfaffians $\eta^1(t), \ldots, \eta^n(t)$ satisfying $\eta^\alpha(0) = w^\alpha$ and

$$d\eta^\alpha(t) = C^\alpha_{\beta\gamma} \eta^\beta(t) \wedge \eta^\gamma(t). \tag{8.13}$$

If we write $\varphi^\alpha(t) = \eta^\alpha(t) - w^\alpha$, then $\sum X_\alpha \otimes \varphi^\alpha(t) \in T^1[t]$ and from (8.13) we get

$$d\varphi^\alpha + 2C^\alpha_{\beta\gamma} w^\gamma \wedge \varphi^\beta(t) = C^\alpha_{\beta\gamma} \varphi^\beta(t) \wedge \varphi^\gamma(t). \tag{8.14}$$

However, (8.14) is just (8.12). Q. E. D.

Now let A be a connected Lie group, $B \subset A$ a closed connected Lie subgroup such that $X = A/B$ is simply connected. In § VI.2 we saw how to put a canonical homogeneous G-structure on X where $G = \mathrm{Ad}^\#$, the adjoint group of B acting on $\mathfrak{a}/\mathfrak{b}$. This structure is of finite type, and $H^0(X, \Theta_G) = \mathfrak{a}$. We shall construct a resolution of Θ_G from the local considerations above.

Over X we have the bundle $G \to B_G \to X$ $(G = B/B^1)$; over G we have the bundle $G^1 \to B_{G^1} \to B_G$; $\cdots$; and finally we have $G^k \to B_{G^k} \to B_{G^{k-1}}$ where

$G^k = I$ (and $B_{G^k} = A$). The groups G^j act on the right on the bundles B_{G^k} for $j \leqq k$. On A there is an absolute parallelism given by $w^1, \ldots, w^n$ where $N = d(G) + n$ ($n = \dim X$). Let $\mathfrak{G}$ be the linear Lie group of all homogeneous automorphisms of the flat G-structure on $\mathbf{R}^n$. Then $\mathfrak{G}$ acts on A on the right, and $A/G = X$. We let $\mathbf{T}^q$ be the bundle, on A, of vector valued q-forms. Then $\mathfrak{G}$ acts on $\mathbf{T}^q$, and we set $\mathbf{V}^q = \mathbf{T}^q/G$. Now, since the sheaf Θ_G on X lifts to precisely the sheaf of infinitessimal automorphisms of the structure on A given by $w^1, \ldots, w^n$, and since the operators $D : \mathscr{T}^q \to \mathscr{T}^{q+1}$ commute with action of $\mathfrak{G}$ on A, we see that $\mathbf{V} \oplus \mathbf{V}^1 \oplus \mathbf{V}^2 \oplus \cdots$ gives a Lie bundle on X, and

$$(8.15) \qquad 0 \longrightarrow \Theta_G \xrightarrow{i} \mathscr{V} \xrightarrow{D} \mathscr{V}^1 \longrightarrow \cdots \longrightarrow \mathscr{V} \xrightarrow{qD} \mathscr{V}^{q+1}$$

is a Lie resolution of Θ_G. Indeed, it is clear that the equation

$$(8.16) \qquad D\varphi(t) = [\varphi(t), \varphi(t)], \ \varphi(t) \in V^1[t]$$

may be used to construct deformations on X.

Theorem 8.2: *Let X have a complete, normal G-structure which is locally homogeneous and in fact is locally isomorphic to the the G-structure on A/B discussed above. Then* (8.15) *gives a Lie resolution of Θ_G on X. If $\varphi(t) \in V^1[t]$ satisfies* (8.16), *then the infinitessimal deformation to the family of G-structures $\{X_t\}$ given by $\varphi(t)$ is $\left.\frac{d\varphi(t)}{dt}\right]_{t=0} = \varphi$.*

Remark: $\varphi \in V^1$ and satisfies $D\varphi = 0$; thus φ gives a class in $H^1(X, \Theta_G)$ by the deRham theorem.

3. The Existence of Deformations. Let $G \to B_G \to X$ be a complete normal G-structure where X is compact and G is of finite type.

Theorem 8.3: *If $H^2(X, \Theta_G) = 0$, then a neighborhood of* o *in $H^1(X, \Theta_G)$ parametrizes a locally complete germ of deformation of the G-structure on X.*

Proof: We follow the notations established heretofore. In § VII.2, we proved that $H^1(X, \Gamma_G[t])$ was given by $\mathrm{Hom}(\Gamma, \mathfrak{G})[t]$. Also, from the formal considerations in § VII.2, we may assume that $\mathfrak{G}$ is connected and simply connected. Now $\mathfrak{G}$ acts as G-motions on $\tilde{X}$, and, since X is compact, we may find a point $\tilde{x} \in \tilde{X}$ such that, if $\mathbf{H}$ = stability group of $\tilde{x}$ in $\mathfrak{G}$, $\mathbf{H}$ is connected and the double coset space $Y = \Gamma^{\backslash \mathfrak{G}/\mathbf{H}}$ is compact. Now the compact manifold Y has a complete, normal $\hat{G}$-structure where now $\hat{G} = \mathrm{Ad}^{\#}$, the adjoint group of $\mathbf{H}$ acting on $\mathfrak{g}/\mathfrak{h}$. Since $H^1(X, \Theta_G) \cong H^1(\Gamma, \mathfrak{g}) \cong H^1(Y, \Theta_{\hat{G}})$, and since $H^1(X, \Gamma_G[t]) \cong$ $\cong \mathrm{Hom}(\Gamma, \mathfrak{G})[t] \cong H^1(Y, \Gamma_{\hat{G}}[t])$. The Theorem will follow from:

Proposition 8.2: *Let the resolution* (9.15) *of Θ_G be constructed on Y, and let $\varphi \in V^1$ with $D\varphi = 0$. Then there exists a $\varphi(t) \in V^1[t]$ satisfying $D\,\varphi(t) = [\varphi(t), \varphi(t)]$ and $\left.\frac{d\varphi(t)}{dt}\right]_{t=0} = \varphi$.*

Proof: By hypothesis, there are not obstructions to the formal constructions to be done below. This will mean that, given $\Psi \in V^2$ with $D\Psi = 0$, there will exist a $\zeta \in V^1$ with $D\zeta = \Psi$. We shall construct a sequence $\{\varphi^r(t)\}$ of elements $\varphi^r(t) = \sum_{\mu=1}^{r} \varphi_\mu t^\mu \in V^1[t]$ satisfying $\varphi_1 = \varphi$: and

$$(8.17)^r \qquad D\varphi^r(t) \equiv [\varphi^r(t), \varphi^r(t)] \ (\mathrm{mod}\, r^{r+1}),$$

then the formal series

$$\varphi(t) = \lim_{r\to\infty} \varphi^r(t) = \sum_{\mu=1}^{\infty} \varphi_\mu t^r$$

satisfies

$$\varphi_1 = \varphi, D\varphi(t) = [\varphi(t), \varphi(t)]$$

(in a formal sense) and we shall prove convergence in t and that $\varphi(t)$ is C^∞ on Y, so that formal is actual.

Since $(8.17)^1$ is fulfilled by $\varphi^1(t) = t\varphi$, we assume given $\varphi^r(t)$ satisfying $(8.17)^r$, and we shall construct $\varphi^{r+1}(t)$ satisfying $(8.17)^{r+1}$. Define $\Psi^{r+1} \in V^2$ by

(8.18) $$\Psi^{r+1}t^{r+1} \equiv D\varphi^r(t) - [\varphi^r(t), \varphi^r(t)] \,(\mathrm{mod}\, t^{r+1});\; \Psi^{r+1}$$

is called the $r + 1^{st}$ obstruction.

Lemma 8.3: $D\Psi^{r+1} = 0$.

Proof: $$D\Psi^{r+1}t^{r+1} \equiv -D[\Psi^r(t), \varphi^r(t)] \,(\mathrm{mod}\, t^{r+2})$$
$$\equiv -2[D\varphi^r(t), \varphi^r(t)] \,(\mathrm{mod}\, t^{r+2}) \quad (\text{by } (8.11))$$
$$\equiv -2[[\varphi^r(t), \varphi^r(t)], \varphi^r(t)] \,(\mathrm{mod}\, t^{r+2})$$

by $(8.17)^r \equiv 0 \,(\mathrm{mod}\, t^{r+2})$. Q.E.D.

If we then choose φ_{r+1} such that $D\varphi_{r+1} = -\Psi^{r+1}$; then we may easily verify that $\varphi^{r+1}(t) = \varphi_r(t) + \varphi_{r+1}t^{r+1}$ satisfies $(8.17)^{r+1}$. This completes the formal proof.

The proof of convergence is simpler than in [8]. Namely, $ds^2 = \sum_{\alpha=1}^{N} (w^\alpha)^2$ gives an $\mathbf{H}$ invariant metric on $\Gamma^{\backslash G}$, thus a metric in the bundles $\mathbf{V}^q$. We define an inner product $\langle\,,\rangle$ in V^q by setting

$$(\zeta, \eta) = \int_Y \langle \xi(y), \eta(y)\rangle \, d\mu\,,$$

and we also define $\|\zeta\|^2 = (\zeta, \zeta)$. If D^* is the adjoint of D, then $DD^* + D^*D$ is clearly elliptic, and by standard elliptic theory, we have:

There exists a $c > o$ such that, for any $\Psi \in V^2$ with $D\Psi = o$, there exists an unique $\eta \in V^1$ with $D^*\eta = o$, $D\eta = \zeta$, and $\|\eta\| < c\,\|\zeta\|$. Also, since $[\ ,\ ]$ involves no derivatives, we may assume $c > o$ is such that $\|[\zeta, \eta]\| < c\|\zeta\| \cdot \|\eta\|$. Then, assuming we choose the φ_{r+1} as above, one easily checks that, for b large enough, the series $b(\sum \|\varphi\|^\mu c^\mu t^\mu)$ dominates, in $\|\ \|$, the series for $\varphi(t)$ constructed inductively. Thus $\varphi(t)$ is real analytic in t, square integrable on Y, and satisfies $D\varphi(t) = [\varphi(t), \varphi(t)]$, where D is taken in the weak sense. However, since $\varphi(t)$ satisfies the equation

$$(DD^* + D^*D)\,\varphi(t) - D^*[\varphi(t), \varphi(t)] = t(DD^*\varphi_1)\,,$$

which is elliptic quasi-linear for (t) small, $\varphi(t)$ is actually C^∞ on Y. Q. E. D.

Remarks: (i) The success of the method just used is that it linearizes the equations in § VII.3 which we were unable to deal with exponentially. (ii) Given $\varphi \in H^1(Y, \Theta_G)$, the *primary obstruction* to φ is given by

$$[\varphi, \varphi] \in H^2(Y, \Theta_G)\,.$$

If $[\varphi, \varphi] = D\Psi$, then the *secondary obstruction* is given by $\frac{1}{2}[\varphi, \Psi]$.

IX. Examples, Automorphisms and Deformations

1. Automorphisms and Deformations. Let X be a compact manifold, and suppose that $G \to B_G \to X$ is a normal geometric G-structure. Under a mild restriction on X, one can associate to $G \to B_G \to X$ a real analytic (possibly reducible and/or singular) variety $\sum(G)$ which parametrizes a subset of the deformations of X coming, in some sense, from the automorphisms on X. We shall construct this *automorphism variety* $\sum(G)$ in certain special cases.

Let $\Gamma \to \tilde{X} \to X$ be the universal covering fibration of X, and let $\mathfrak{g} = H^0(\tilde{X}, \tilde{\Theta}_G)$. Then the Lie algebra $\mathfrak{g}$ is a finite dimensional Γ-module; if $\varrho : \Gamma \to GL(\mathfrak{g})$ is the action of Γ on $\mathfrak{g}$, then $\varrho(\gamma)[g, g^1] = [\varrho(\gamma)g, \varrho(\gamma)g^1]$ for $\gamma \in \Gamma, g, g^1 \in \mathfrak{g}$. Thus the subspace $\mathfrak{g}^{\#} = H^0(X, \Theta_G) = \{g \in \mathfrak{g} \mid \varrho(\gamma)g = g$ for all $\gamma \in \Gamma\}$ is the Lie algebra of the Lie group of G-automorphisms of X.

There is an exact sequence of Γ-modules

$$0 \to \mathfrak{g}^{\#} \to \mathfrak{g} \to \mathfrak{g}' \to 0; \tag{9.1}$$

and from the cohomology sequence, we get

$$0 \to H^1(\Gamma, \mathfrak{g}^{\#}) \to H^1(\Gamma, \mathfrak{g}) . \tag{9.2}$$

We shall examine the effect of the subspace $H^1(\Gamma, \mathfrak{g}^{\#})$ of $H^1(\Gamma, \mathfrak{g}) \cong$ $\cong H^1(X, \Theta_G)$ on deformations.

Lemma 9.1: $H^1(\Gamma, \mathfrak{g}^{\#}) \cong \mathfrak{g}^{\#} \otimes H^1(X, \mathbf{R})$.

Proof:

$$H^1(\Gamma, \mathfrak{g}^{\#}) \cong \operatorname{Hom}(\Gamma, \mathfrak{g}^{\#}) \cong \operatorname{Hom}(\Gamma/[\Gamma, \Gamma], \mathfrak{g}^{\#}) \cong \operatorname{Hom}(H_1(X, \mathbf{Z}), \mathfrak{g}^{\#}) \cong$$
$$\cong \mathfrak{g}^{\#} \otimes H^1(X, \mathbf{R})$$

(since $\mathfrak{g}^{\#}$ is a real vector space). Q.E.D.

Let $\mathfrak{G}^{\#}$ be the connected Lie group generated by $\mathfrak{g}^{\#}$ (X is compact!). We shall consider $H^q(X, \Theta_G)$ as given by vector-valued forms via the resolution of § V.3. Then, if $w \in H^q(X, \mathbf{R})$ is given by a form under the deRham resolution, and if $\theta \in H^0(X, \Theta_G)$, then $\theta \otimes w \in H^q(X, \Theta_G)$ via the resolution of § V.3.

Proposition 9.1: *For* $g \in \mathfrak{G}^{\#}$, $\gamma \in \mathfrak{g}^{\#}$, $w \in H^1(X, \mathbf{R})$,

$$g \circ (\gamma \otimes w) = Adg(\gamma) \otimes g^* w . \tag{9.3}$$

Proof: The proof is immediate from the definition of induced action on sheaf cohomology, together with the following remark. We consider $\mathfrak{G}^{\#}$ as acting on X on the left; then the Lie algebra $\mathfrak{g}^{\#}$ of $\mathfrak{G}^{\#}$ must be taken to be right-invariant vector fields (since these are infinitessimal left translations). But then the left-action of $\mathfrak{G}^{\#}$ on $\mathfrak{g}^{\#}$ is just the adjoint representation. Q.E.D.

Remark: $g^* w \sim w$ since the action of g is homotopic to the identity.

For a 1-form φ and a vector field θ, we let $L_\theta \varphi$ denote the Lie derivative of φ along θ. Then

Proposition 9.2: *For* $\gamma, \gamma^1 \in \mathfrak{g}^{\#}$; $w, w^1 \in H^1(X, \mathbf{R})$,

$$[\gamma \otimes w, \gamma^1 \otimes w^1] = [\gamma, \gamma^1] \otimes w \wedge w^1 + \gamma \otimes L_{\gamma^1} w \wedge w^1 + \gamma^1 \otimes L_\gamma w^1 \wedge w . \tag{9.4}$$

The bracket $[\gamma \otimes w, \gamma^1 \otimes w^1]$ is the cup product in cohomology; $[\ ,\ ]$: $H^1(X, \Theta_G) \otimes H^1(X, \Theta_G) \to H^2(X, \Theta_G)$. The proof of (9.4) is by a straightforward local calculation.

We now assume the following *condition* $\sum$ on X: There exists a basis $w^1, \ldots, w^r$ of $H^1(X, \mathbf{R})$ such that $g^* w^j = w^j$ for $j = 1, \ldots, r$ and for all $g \in \mathfrak{G}^\#$. (This will happen, e.g., if $\mathfrak{G}^\#$ acts as isometries relative to some metric.) Thus $L_\gamma w^j = 0$ for all $\gamma \in \mathfrak{g}^\#$ and $j = 1, \ldots, r$. Also, if $w \in H^1(X, \mathbf{R})$, $[\gamma \otimes w, \gamma \otimes w] = 0$ by (9.4). We shall construct a global 1-parameter family $X_t = X_t(\gamma, w)$ of G-structures on X whose infinitessimal deformation is $\gamma \otimes w$.

Let H be the torsion-free part of $H_1(X, \mathbf{Z})$, and let $X^* \to X$ be the covering space of X with deck-group H. We shall define a 1-parameter family of actions ϱ_t of H on X^* and $X_t(\gamma, w)$ will be $X^*/\varrho_t(H)$. We define ϱ_t as follows: for $z \in H$, $x \in X^*$,

$$\varrho_t(z) \cdot x = \left(\exp\left(t \int_z w\right) \gamma\right) \cdot x \cdot z\,. \tag{9.5}$$

We must explain this notation a little. The discrete group H acts on X^* on the right, and this action is the symbol $x \cdot z$ in (9.5). Also, $\mathfrak{G}^\#$ acts as G-motions on X^*, and this is the action $\exp\left(\left(t \int_z w\right) \gamma\right) \cdot x$ in (9.5). Set $X_t = X^*/\varrho_t(H)$.

Theorem 9.1: *The manifolds X_t give a deformation of the G-structure on $X = X_0$ whose infinitessimal tangent is $\gamma \otimes w \in H^1(X, \Theta_G)$.*

Proof: We first observe that $\varrho_t(z + z^1) \cdot x = \varrho_t(z)\, \varrho_t(z^1) \cdot x$ since $\exp\left(t \int_z w\right) \gamma \exp\left(t \int_{z^1} w\right) \gamma = \exp\left(t \int_{z+z^1} w\right) \gamma$. Since the actions $\varrho_t(H)$ on X^* are G-motions, and since $\varrho_0(H)$ is just the given action of H on X^*, we have so defined a smooth family X_t of G-structures on X, and this family clearly forms a deformation (i.e. local structure is preserved). We must compute the infinitessimal deformation. For fixed $z \in H$, the family of actions $t \to \varrho_t(z)$ is a 1-parameter group of G-motions on X^*, and this action is generated by a vector field ξ_z. Indeed, for a function f on X^* and for $x \in X^*$,

$$\begin{aligned}(\xi_z \cdot f)(x) &= \lim_{t \to 0} \frac{f(\varrho_t(z) \cdot x) - f(\varrho_0(z) \cdot x)}{t} \\ &= \lim_{t \to 0} \frac{f\left(\exp t \int_z w \cdot \gamma\right) \cdot x \cdot z) - f(x \cdot z)}{t} \\ &= R_z L_{\left(\int_z w\right)\gamma}(f)(x)\end{aligned}$$

where R_z of a function g is given by $R_z(g)(x) = g(x \cdot z)$. Thus, the deformation may be infinitessimally described by the equation $z \to \left(\left(\int_z w\right) \gamma\right) \circ R_z$. By considering the form of $H^1(X, \Theta_G)$ given in § V.3, it easily follows that the tangent to $\{X_t\}$ is $\gamma \otimes w$. Q. E. D.

2. Examples of Obstructions. We shall use the above example to show how obstructions, of any order, to deformation may be easily constructed. We shall also construct examples of obstructions to the stability of automorphisms.

Proposition 9.3: *If* $\mathfrak{g}^{\#}$ *is non-abelian and if the cup product* $H^1(X, \mathbf{R}) \otimes \otimes H^1(X, \mathbf{R}) \to H^2(X, \mathbf{R})$ *is non-trivial, then there are primary obstructions to deformation.*

Proof: Since the representation of $\mathcal{G}^{\#}$ on $H^1(X, \mathbf{R})$ is trivial, we see that, for $w \in H^1(X, \mathbf{R})$, $\gamma \in \mathfrak{g}^{\#}$, $L_\gamma w = df$ for some function f. Let $\gamma, \gamma^1 \in \mathfrak{g}^{\#}$ be such that $[\gamma, \gamma^1] \neq 0$, and let $w, w^1 \in H^1(X, \mathbf{R})$ be such that $w \wedge w^1 \neq 0$ in $H^2(X, \mathbf{R})$. Then, by (9.4), $\frac{1}{2}[\gamma \otimes w + \gamma^1 \otimes w^1, \gamma \otimes w + \gamma^1 \otimes w^1] \sim [\gamma \otimes w, \gamma^1 \otimes w^1] \sim \sim [\gamma, \gamma^1] \otimes w \wedge w^1 \neq 0$ in $H^2(X, \Theta_G)$ (Here, we are interpreting elements in $H^q(X, \Theta_G)$ via the resolution of Θ_G in § V.3.) Q. E. D.

Example 1: Let Z be a real n-torus ($n > 1$), and taken with an invariant metric; and let Y be a compact semi-simple Lie group with a bi-invariant metric. Then $X = Y \times Z$ is a Riemannian manifold satisfying the conditions of Proposition 9.3.

To construct an example of a secondary obstruction, we shall use the resolution of Θ_G constructed in § VIII.2. Thus, let X have a parallelism $w^1, \ldots, w^n$ with $dw^\alpha = C^\alpha_{\beta\gamma} w^\beta \wedge w^\gamma$ ($C^\alpha_{\beta\gamma}$ constant). Then (8.15) gives an exact sequence of sheaves

$$(9.6) \qquad 0 \longrightarrow \Theta_G \xrightarrow{i} T \xrightarrow{D} T^1 \to \cdots \to T^q \xrightarrow{D} T^{q+1} \to \cdots .$$

Let A^q be the global scalar q-forms on X.

Lemma 9.2: *There is an injection* $j\colon H^0(X, \Theta_G) \otimes A^q \to T^q$ *such that* $j(\theta \otimes dw) = Dj(\theta \otimes w)$ $(\theta \in H^0(X, \Theta_G), w \in A^q)$.

Proof: Let $X_1, \ldots, X_n$ be a dual parallelism to $w^1, \ldots, w^n$. For a vector field $\xi \in T$, $\zeta \in \Theta_G$ if, and only if, $L_\zeta w^\alpha = 0$ for $\alpha = 1, \ldots, n$. Writing $\zeta = \sum \zeta^\alpha X_\alpha (\zeta^\alpha \in C^\infty(X))$, we have, by the Cartan formula, $L_\zeta w^\alpha = i(\zeta)\, dw^\alpha + + di(\zeta) w^\alpha = 2 \sum C^\alpha_{\beta\gamma} w^\gamma \zeta^\beta + d\zeta^\alpha$. Now if $\theta = \zeta = \sum \zeta^\alpha X_\alpha \in H^0(X, \Theta_G)$, and if $w \in A^q$, then $j(\zeta \otimes dw) = \sum X_\alpha \otimes \zeta^\alpha dw$. On the other hand, by (8.10), $D(j(\zeta \otimes w)) = D(\sum X_\alpha \otimes \zeta^\alpha w) = d(\sum X_\alpha \otimes \zeta^\alpha w) + 2\Omega \barwedge (\sum X_\alpha \otimes \zeta^\alpha w) = \sum X_\alpha \otimes d\zeta^\alpha \wedge w + \sum X_\alpha \otimes \zeta^\alpha dw + 2 \sum X_\alpha \otimes C^\alpha_{\beta\gamma} w^\gamma \zeta^\beta = \sum X_\alpha \otimes \zeta^\alpha dw = j(\zeta \otimes dw)$. Q. E. D.

Corollary: *The mapping* j *defines a mapping* $j : \mathfrak{g}^{\#} \otimes H^q(X, \mathbf{R}) \to H^q(X, \Theta_G)$, *where the latter group is computed using the resolution* (9.6).

Now let $\zeta, \zeta^1 \in \mathfrak{g}^{\#}$; $w, w^1 \in A^1$, and suppose that

$$(9.7) \qquad L_\zeta w = L_\zeta w^1 = L_{\xi^1} w = L_{\zeta^1} w^1 = 0 \, .$$

Lemma 9.3: $[j(\zeta \otimes w), j(\zeta^1 \otimes w^1)] = 6j([\zeta, \zeta^1] \otimes w \wedge w^1)$.

Proof: Since $L_\zeta w^\alpha = 0 = L_{\xi^1}/w^\alpha$, it follows that $\xi^\alpha_{,\gamma} = 2C^\alpha_{\gamma\beta}\zeta^\beta$; $\zeta^{1\alpha}_{,\gamma} = 2C^\alpha_{\gamma\beta}\zeta^{1\beta}$ where $\zeta = \sum \zeta^\alpha X_\alpha$, $\zeta^1 = \sum \zeta^{1\beta} X_\beta$, and $\zeta^\alpha_{,\beta} = X_\beta(\zeta^\alpha)$, etc. Now $[j(\zeta \otimes w), j(\zeta^1 \otimes w^1)] = [\sum_\alpha X_\alpha \otimes \zeta^\alpha w, \sum_\beta X_\beta \otimes \zeta^{1\beta} w^1] = \sum X_\alpha \otimes C^\gamma_{\alpha\beta} \zeta^\alpha \zeta^{1\beta} w \wedge w^1$. On the other hand, $[\zeta, \zeta^1] = [\sum \zeta^\alpha X_\alpha, \sum \zeta^{1\beta} X_\beta] = 2 \sum (C^\gamma_{\alpha\beta} \zeta^\alpha \zeta^{1\beta}) X_\gamma + \sum \zeta^\alpha \zeta^{1\beta}_{,\alpha} X\beta - - \sum \zeta^{1\beta} \zeta^\alpha_{,\beta} X_\alpha = 2 \sum C^\gamma_{\alpha\beta} \zeta^\alpha \zeta^{1\beta} X_\gamma + 2 \sum \zeta^\alpha C^\beta_{\alpha\sigma} \zeta^{1\sigma} X_\beta - 2 \sum \zeta_{1\beta} C^\alpha_{\beta\sigma} \zeta^\sigma X_\alpha = 6 \sum X_\gamma \otimes C^\gamma_{\alpha\beta} \zeta^\alpha \zeta^{1\beta}$. Q. E. D.

Suppose now that we have ζ, ζ^1, w, w^1 where w, w^1 are both non-zero elements in $H^1(X, \mathbf{R})$ and (9.7) is satisfied. Suppose furthermore that: (i) $[\xi, \xi^1] = \xi^1$;

(ii) $w \wedge w^1 = d\eta$ for some $\eta \in A^1$; (iii) $L_\zeta \eta = 0 = L_{\zeta^1} \eta^1$; and (iv) $w \wedge \eta \neq 0$ in $H^2(X, \mathbf{R})$ (observe that $d(w \wedge \eta) = -w \wedge d\eta = -w \wedge w \wedge w^1 = 0$).

Proposition 9.4: *Under the above hypotheses, there are secondary obstructions to deformation.*

Proof: Recall first the Remark at the end of § VIII. 3 where formulae were given for primary and secondary obstructions (relative to D-cohomology). Let now $\varphi = j((\zeta \otimes w + \zeta^1 \otimes w^1))$, then $[\varphi, \varphi] = 12 j(\zeta^1 \otimes w \wedge w^1) = 12 D j(\zeta^1 \otimes \eta)$ by Lemmas 9.3 and 9.2. Thus the primary obstruction drops out. The secondary obstruction is given by

$$\frac{6.12}{2} [\varphi, j(\zeta^1 \otimes \eta)] = 36 j(\zeta^1 \otimes w \wedge \eta) \neq 0 \text{ in } H^2(X, \Theta_G). \quad \text{Q. E. D.}$$

Example 2: We shall explicitly construct a manifold satisfying the conditions given above Proposition 9.4. Let F be the Lie group of all matrices $f = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$. Then the right invariant Maurer-Cartan form on F is given by $df \cdot f^{-1} = \begin{pmatrix} 0 & w_1 & w_3 \\ 0 & 0 & w_2 \\ 0 & 0 & 0 \end{pmatrix}$ where $w_1 = dx$, $w_2 = dy$, and $w_3 = -y\,dx + dz$. Then $dw_2 = 0 = dw_2$ and $dw_3 = dy \wedge dx = w_1 \wedge w_2$.

Now let Γ be the discrete group of matrices

$$\sigma = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} (a, b, c \in \mathbf{Z}),$$

and set $Z = F/\Gamma$. Then Z has a G-structure, where $G = I$, given by w_1, w_2, w_3, which clearly project to Z. Also, Z is compact. On Z, we have the closed 2-form $w_2 \wedge w_3$ (since $d(w_2 \wedge w_3) = -w_2 \wedge dw_3 = -w_2 \wedge w_1 \wedge w_2 = 0$), but $w_2 \wedge w_3$ is *not* exact. In fact, $w_1 \wedge w_2 \wedge w_3$ gives a volume element on Z, and if $w_2 \wedge w_3 = d\varphi$ for some 1-form φ, then $w_1 \wedge w_2 \wedge w_3 = -d(w_1 \wedge \varphi)$ and thus $\int_Z w_1 \wedge \wedge w_2 \wedge w_3 = 0$. Contradiction.

Let $Y = SU(3)$ with the G-structure, where $G = I$, given by 8 left invariant Pfaffians on $SU(3)$. Then $X = Y \times Z$ has a G-structure where $G = I$, and, since there exist elements $\gamma, \gamma^1 \in SU(3)$ such that $[\gamma, \gamma^1] = \gamma^1$, the conditions above Proposition 9.4 are all met in a specific example.

Example 3: The above example may be generalized to give obstructions of any order $N > 0$. We shall not go into this here.

Example 4: Return to Example 1 constructed above. Let w be a harmonic 1-form on Z, and let γ be in the Lie algebra of Y. Then $[\gamma \otimes w, \gamma \otimes w] = 0$, and thus we may construct on $X = Y \times Z$ a 1-parameter family of G-structures (Riemannian metrics) X_t whose infinitessimal tangent is $\gamma \otimes w$ (by Theorem 9.1). Let $\gamma^1 \in \mathfrak{g}^\#$ be in the Lie algebra of Y and such that $[\gamma, \gamma^1] \neq 0$. Then the primary obstruction to the stability of the Killing field γ^1 on X is given by $[\gamma^1, \gamma \otimes w] = [\gamma^1, \gamma] \otimes w \neq 0$ in $H^1(X, \Theta_G)$. Thus γ^1 is *not* stable under the deformation X_t. (In fact, the manifolds $X_t (t \neq 0)$ are *non*-homogeneous.)

3. Examples of Deformation Spaces. Example 1: Return to Example 1 constructed in § IX.2 above, where $Y = M$ is a compact, connected semi-simple Lie group of dimension d, $Z = T$ is an n-torus ($n > 1$), and $X = M \times T$ is a Riemannian manifold ($G = SO(n + d)$). Let $\mathfrak{m}$ = Lie algebra of M, $\mathfrak{t}$ = Lie algebra of T, and let $w^1, \ldots, w^n$ be n-independent harmonic 1-forms on T. Then, for $i \neq j$, $w^i \wedge w^j$ is a non-zero harmonic 2-form on T. The Künneth formula shows that

$$H^q(X, \Theta_G) = \{\mathfrak{m} \otimes H^q(X, \mathbf{R})\} \oplus \{\mathfrak{t} \otimes H^q(X, \mathbf{R})\} \quad (q = 0, 1, 2)\,.$$

Thus, $\dim H^1(X, \Theta_G) = n^2 + nd$, and $\dim H^2(X, \Theta_G) = n^2 \frac{(n-1)}{2} + d \cdot n \frac{(n-1)}{2}$. Let $e_1, \ldots, e_d$ be a basis for $\mathfrak{m}$, and write $[e_\alpha, e_\beta] = C^\gamma_{\alpha\beta} e_\gamma$ ($\alpha, \beta, \gamma = 1, \ldots, d$). We consider $H^q(T, \mathbf{R})$ as identified with $\mathbf{R}^{n(n-1) - - (n-q+1)}$ by using the harmonic forms $w^1, \ldots, w^n$. Also, we identify $H^1(X, \Theta_G)$ with $\mathbf{R}^{n^2 + nd}$, and $\mathfrak{m} \otimes H^2(X, \mathbf{R})$ with $\mathbf{R}^{n \frac{(n-1)}{2} \cdot d}$.

Now we observe that, since $\mathfrak{t}$ is abelian, $[H^1(X, \Theta_G), H^1(X, \Theta_G)] \subseteq \mathfrak{m} \otimes H^2(X, \mathbf{R})$. Define $d\, n \frac{(n-1)}{2}$ quadratic functions f^α_{ij} ($\alpha = 1, \ldots, d$; $1 \leqq i < j \leqq n$) on $H^1(X, \Theta_G)$ as follows: For $\varphi \in H^1(X, \Theta_G)$,

$$(9.8) \qquad [\varphi, \varphi] = \frac{1}{3} \sum_{\alpha, i<j} f^\alpha_{ij}(\varphi) e_\alpha \otimes w^i \wedge w^j\,.$$

Theorem 9.2: *The zero locus* $f^\alpha_{ij} = 0$ ($\alpha = 1, \ldots, d$; $1 \leqq i < j \leqq n$) *on* $\mathbf{R}^{n^2 + nd}$ *gives a complete deformation space* $\mathscr{D}$ *of the G-structure on X.*

Proof: This Theorem will follow easily from Theorem 9.1. Let $\varphi = \sum \varphi_j \otimes w^j (\varphi_j \cdot \in H^0(X, \Theta_G)) \in H^1(X, \Theta_G)$. We shall follow the notations of Theorem 9.1 and define a 1-parameter of actions ϱ^φ_t of H on X^* by

$$\varrho^\varphi_t(z) x = \exp\left(\sum \left(t \int_z w^j\right) \varphi_j\right) \cdot x \cdot z\,.$$

Then it follows easily that, for all $z, z^1 \in H$, $\varrho^\varphi_t(z)\, \varrho^\varphi_t(z^1) = \varrho^\varphi_t(z + z^1) = \varrho^\varphi_t(z^1)\, \varrho^\varphi_t(z)$ if, and only if, $\varphi \in \mathscr{D}$. Thus, in case $\varphi \in \mathscr{D}$, we may define a 1-parameter of G-structures $X_t(\varphi)$ on X by setting $X_t(\varphi) = X^*/\varrho^\varphi_t(H)$. We then define a deformation family $\mathscr{V} \xrightarrow{\tilde{\omega}} \mathscr{D}$ by setting $\tilde{\omega}^{-1}(\varphi) = X_\varphi = X_1(\varphi)$. The remainder of the Theorem is now a straightforward checking of details, which we shall omit. Q. E. D.

Remarks: The space $\mathscr{D}$ has the following properties: (i) $\mathscr{D}$ is a real algebraic variety in $\mathbf{R}^{n^2 + nd}$, $\dim \mathscr{D} = n^2 + nl$ where $l = \operatorname{rank} M$; (ii) Writing $\mathbf{R}^{n^2 + nd} = \mathbf{R}^{n^2} \oplus \mathbf{R}^{nd}$, the Zariski tangent space at a point $(x, o) \in \mathbf{R}^{n^2} + \mathbf{R}^{nd}$ is a real vector space of dimension $n^2 + nd$. Thus $\mathscr{D}$ is singular along the subvariety of points (x, o), and along these points $\mathscr{D}$ is locally minimally embedded. These singular points are quadratic singularities, and (iii) this example shows that any general construction of deformation spaces must include singular and/or locally reducible varieties as parameter varieties.

Example 2: Let $X = Y \times Z$ be the Example 2 constructed in § IV.2 above. The general element $\varphi \in H^1(X, \Theta_G)$ may be written $\varphi = (\xi^1 + \sigma^1) \otimes w_1 +$

$+ (\xi^2 + \sigma^2) \otimes w_2$ where ξ^1, ξ^2 are left invariant vector fields on F, and σ^1, σ^2 are right invariant vector fields on $SU(3)$. By Lemma 9.3, $[\varphi, \varphi] = 12([\zeta^1, \zeta^2] + [\sigma^1, \sigma^2]) \otimes w_1 \wedge w_2$, and we let $\eta_\varphi = -12([\zeta^1, \zeta^2] + [\sigma^1, \sigma^2]) \otimes w_3$, so that $\mathscr{D}\eta_\varphi = -[\varphi, \varphi]$. Observe that, since $[\mathfrak{f}, [\mathfrak{f}, \mathfrak{f}]] = 0$, $[\varphi, \eta_\varphi] \in SU(3) \otimes H^2(X, \mathbf{R})$. Choose a basis $e_1, \ldots, e_8$ of the right invariant vector fields on $SU(3)$, and define 16 cubic functions $f_j^\alpha (\alpha = 1, \ldots, 8; j = 1, 2)$ on $H^1(X, \Theta_G)$ by

$$(9.9) \qquad [\varphi, \eta_\varphi] = \frac{1}{144} \sum_{\alpha, j} f_j^\alpha(\varphi) e_\alpha \otimes w_j \wedge w_3.$$

Let $\mathscr{D} \subset H^1(X, \Theta_G)$ be the locus $f_j^\alpha = 0 (\alpha = 1, \ldots, 8; j = 1, 2)$.

Theorem 9.3: *The real analytic variety $\mathscr{D}$ parametrizes a locally complete deformation space of the G-structure on X.*

Proof: Given $\varphi \in H^1(X, \Theta_G)$, define $\varphi(t) \in T^1[t]$ (notation of § VIII) by $\varphi(t) = t\varphi + t^2\eta_\varphi$. Then, from § IX.2, $D\varphi(t) - [\varphi(t), \varphi(t)] \equiv 0 \pmod{t^3}$ if, and only if, $\varphi \in \mathscr{D}$, in which case $D\varphi(t) = [\varphi(t), \varphi(t)]$. Thus, for these $\varphi \in \mathscr{D}$, we may, by Theorem 8.3, associate to φ a 1-parameter family of G-structures $X_{\varphi(t)}$ such that the family $\mathscr{V} \xrightarrow{\tilde{\omega}} \mathscr{D}$ defined by $\tilde{\omega}^{-1}(\varphi) = X_{\varphi(1)}$ gives a deformation of the G-structure on X. The Theorem now follows. Q. E. D.

Remark: The variety $\mathscr{D}$ is a locally minimally embedded, real algebraic variety having a cubic singularity along the subvariety $\mathfrak{f} \otimes H^1(X, \dot{\mathbf{R}}) \subset \mathscr{D} \subset H^1(X, \Theta_G)$.

(Received March 5, 1963)