DEFORMATIONS OF COMPLEX STRUCTURE

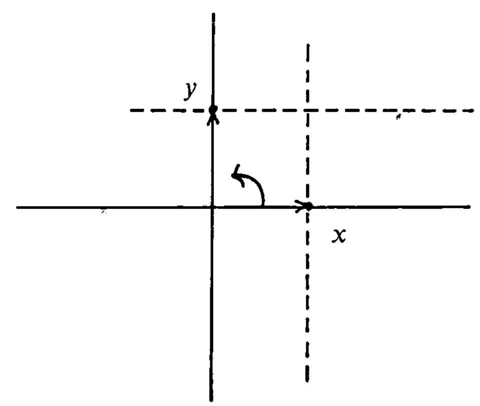
PHILLIP A. GRIFFITHS

1. Let V be a compact, connected C^∞ manifold. We want to talk about complex structures on V, so that we assume that V is of even dimension $2n$ and is oriented. The notion of a complex structure on V, which is compatible with the C^∞ structure and orientation, will mean that for each sufficiently small open set $U \subset V$ we are given the following data: complex-valued C^∞ one-forms $\omega^1, \ldots, \omega^n$ on U such that: (i) $(i/2)^n \omega^1 \wedge \bar{\omega}^1 \wedge \ldots \wedge \omega^n \wedge \bar{\omega}^n > 0$ (i.e. $\omega^1, \ldots, \omega^n, \bar{\omega}^1, \ldots, \bar{\omega}^n$ are linearly independent and define an orientation consistent with the given one on V); (ii) $d\omega^\alpha = \sum_{\beta=1}^n C_\beta^\alpha \wedge \omega^\beta$ where C_β^α are one-forms (this is the *integrability condition*, which may be written $d\omega^\alpha \equiv 0\,(\omega^1, \ldots, \omega^n)$). The holomorphic functions in U form a ring $\mathcal{O}_U$ where a C^∞ function f is in $\mathcal{O}_U$ if $f_{\bar{\beta}} = 0$ (*Cauchy-Riemann equations*) where the total differential $df = \sum_{\alpha=1}^n f_\alpha \omega^\alpha + \sum_{\beta=1}^n f_{\bar{\beta}} \bar{\omega}^\beta$. The fact that the equations $f_{\bar{\beta}} = 0$ have enough local solutions to give local coordinates on V is the *Newlander-Nirenberg theorem* (cf. Chern [**10**], Newlander-Nirenberg [**20**], and Kohn [**17**]).

Our general problem is this:

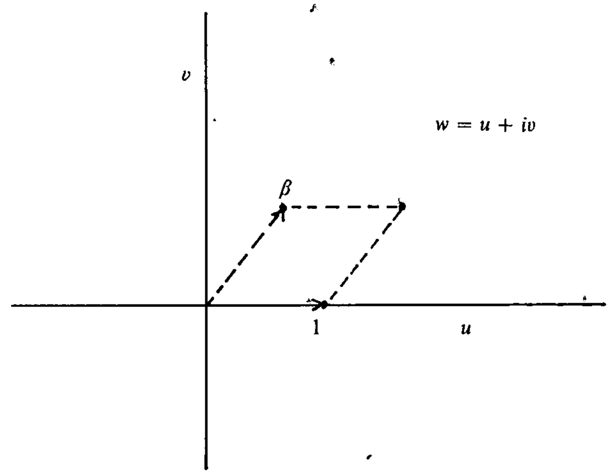
(I) Let $\Sigma(V)$ be the set of distinct complex structures on V which are consistent with the C^∞ structure and orientation. We want to describe $\Sigma(V)$.

EXAMPLE 1. In $\boldsymbol{R}^2$ with coordinates x, y we consider the standard lattice $\boldsymbol{L}$ generated by (1, 0) and (0, 1). Then $\boldsymbol{R}^2/\boldsymbol{L} = V$ is a C^∞ manifold, which is simply a real torus of dimension two. The orientation on V is given by $dx \wedge dy = (i/2) dz \wedge d\bar{z}$ where $z = x + iy$.



Let $\omega = dz + \alpha\, d\bar{z}$ where α is a complex number. Then ω is a complex valued one form on V and $\omega \wedge \bar{\omega} = (1 - |\alpha|^2) dz \wedge d\bar{z}$. Thus for $|\alpha| < 1$ we have a complex structure V_α on V; in fact, if Δ is the unit disc $|\alpha| < 1$, then $\{V_\alpha\}_{\alpha\in\Delta}$ is a "nice family" of complex structures on V. A standard theorem in the theory of Riemann surfaces is: Any complex structure on V is a V_α for $\alpha \in \Delta$. Thus, in our example, $\Sigma(V)$ is a quotient of Δ.

EXAMPLE 1 (CONTINUED). We want to know when $V_\alpha = V_{\alpha'}$. The change of variables $w = (1/(1+\alpha))(z + \alpha\bar{z})$ is a real linear transformation of $\boldsymbol{R}^2$ onto the complex w-plane $\boldsymbol{C}$. Since $w(1) = 1$ and $w(i) = ((1-\alpha)/(1+\alpha))\, i = \beta$, the lattice $\boldsymbol{L}$ in $\boldsymbol{R}^2$ goes into the lattice $\boldsymbol{L}_\beta$ generated by $w = 1$, $w = \beta$:



In fact, the transformation $\beta = ((1-\alpha)/(1+\alpha))i$, $|\alpha| < 1$ takes the disc Δ onto the upper-half-plane $\mathscr{H}$ and the torus $V_\alpha \cong \boldsymbol{C}/\boldsymbol{L}_\beta$ since $dw = (1/(1+\alpha))\omega$. Thus $\{V_\alpha\}_{\alpha\in\Delta} \cong \{V_\beta\}_{\beta\in\mathscr{H}}$ where $V_\beta = \boldsymbol{C}/\boldsymbol{L}_\beta$. Now it is a standard result that $V_\beta \cong V_{\beta'}$ if, and only if, $\beta' = (a\beta + b)/(c\beta + d)$ where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \boldsymbol{Z})$$

is an integral linear fractional transformation of $\mathscr{H}$ into itself. In other words, $\Sigma(V) = \mathscr{H}/\Gamma$ where $\Gamma = \mathrm{SL}(2, \boldsymbol{Z})$ is a properly discontinuous automorphism group acting on $\mathscr{H}$.

The proper way to interpret this example so far is this: Let $\Sigma(V)$ be the set of complex structures on $V = \boldsymbol{R}^2/\boldsymbol{L}$ as in (I), and let V be a complex structure on V. Then $V = V_\beta$ for some $\beta \in \mathscr{H}$ and β is determined up to $\mathrm{SL}(2, \boldsymbol{Z})$ acting on $\mathscr{H}$. This gives a set mapping $\Sigma(V) \xrightarrow{\mathfrak{P}} \mathscr{H}/\Gamma$ which, in this example, is one-to-one onto.

In general, rather than studying $\Sigma(V)$ directly, we shall map $\Sigma(V)$ into spaces (similar to $\mathscr{H}/\Gamma$) which are easy to describe and which give interesting invariants of a complex structure $V \in \Sigma(V)$.

We first do this in case $\dim V = 2$.

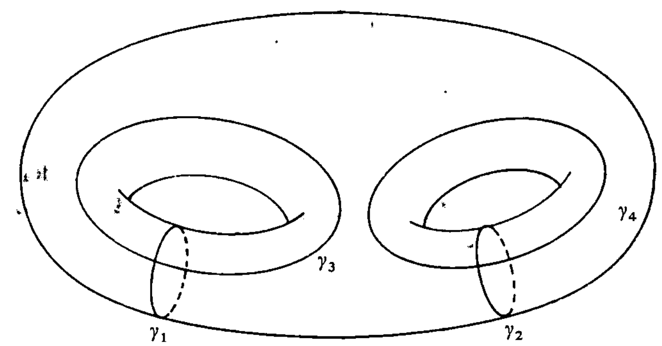
Let V be a compact, oriented two manifold of genus g and $\Sigma(V) = \Sigma_g$, the set of complex structures on V with the proper orientation. If $g = 0$, then every

$V \in \Sigma(V)$ is the standard structure on the Riemann sphere (Riemann mapping theorem) and Σ_0 is a single point.

Assume that $g \geqq 1$ and let $\gamma_1, \ldots, \gamma_{2g}$ be a canonical basis for the first homology group $H_1(V, \boldsymbol{Z})$ ($\cong \boldsymbol{Z} \oplus \ldots \oplus \boldsymbol{Z}$ ($2g$ terms)); by definition the intersection matrix $(\gamma_\rho \cdot \gamma_\sigma) = Q$ is given by

$$Q = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

The picture we have in mind is



The vector space of holomorphic differentials on any complex structure $V \in \Sigma_g$ is g-dimensional, and we choose a basis $\omega^1, \ldots, \omega^g$ for the holomorphic one forms. We then form the *period matrix*

$$\Omega = \overbrace{\begin{pmatrix} \int_{\gamma_1}\omega^1 & \cdots & \int_{\gamma_{2g}}\omega^1 \\ \vdots & & \vdots \\ \int_{\gamma_1}\omega^g & \cdots & \int_{\gamma_{2g}}\omega^g \end{pmatrix}}^{2g} \Bigg\} g;$$

this is a $g \times 2g$ matrix of rank g. A change of basis $\tilde{\omega}^\alpha = \sum_{\beta=1}^{g} A^\alpha_\beta \omega^\beta$ transforms Ω to $\tilde{\Omega} = A\Omega$; a change of homology basis $\tilde{\gamma}_\rho = \sum_{\sigma=1}^{2g} \Lambda^\sigma_\rho \gamma_\rho$ leads to $\tilde{\Omega} = \Omega\Lambda$. Here $A = (A^\alpha_\beta)$ is a nonsingular $g \times g$ matrix and $\Lambda = (\Lambda^\rho_\sigma)$ is a $2g \times 2g$ integral matrix which preserves the quadratic form Q; i.e. $\Lambda Q^t\Lambda = Q$. If we agree to call two $g \times 2g$ matrices $\tilde{\Omega}$, Ω *equivalent* if $\tilde{\Omega} = A\Omega\Lambda$; then the above procedure gives a set mapping: $\Sigma_g \xrightarrow{\mathfrak{P}}$ {equivalence classes of period matrices}.

The matrix Ω is not arbitrary but satisfies the *Riemann bilinear relations*

$$\text{(II)} \qquad \begin{aligned} \Omega Q {}^t\Omega &= 0, \\ i\Omega Q {}^t\bar{\Omega} &> 0. \end{aligned}$$

These relations are a restatement of

$$\int_V \omega \wedge \omega = 0, \qquad i \int_V \omega \wedge \bar{\omega} > 0,$$

for a holomorphic differential ω on V. We then define

$\mathscr{H}_g = \{$set of $g \times 2g$ matrices Ω satisfying the Riemann bilinear relations (II) and with the equivalence $\Omega \sim A\Omega\}$.

The group Γ_g of all $2g \times 2g$ integral matrices satisfying $\Lambda Q {}^t\Lambda = Q$ acts on Ω by $\Lambda(\Omega) = \Omega\Lambda$ and we have constructed the *period mapping*

$$\text{(III)} \qquad \Sigma_g \xrightarrow{\Phi} \mathscr{H}_g/\Gamma_g.$$

We now examine $\mathscr{H}_g$. Let Ω be a point in $\mathscr{H}_g$ and write $\Omega = (A, B)$ where A, B are $g \times g$ matrices. Then $i\Omega Q {}^t\bar{\Omega} = i(-B {}^t\bar{A} + A {}^t\bar{B})$ is positive definite, and it follows that A, B are each nonsingular. Thus $\Omega \sim A^{-1}\Omega = (I, Z)$ and each equivalence class $\Omega \in \mathscr{H}_g$ is given by a unique matrix (I, Z). The relations (II) then become:

$$\text{(II}'\text{)} \qquad \begin{aligned} Z &= {}^tZ, \\ Z &= X + iY, \quad Y > 0. \end{aligned}$$

Thus $\mathscr{H}_g \cong$ *Siegel generalized upper-half-plane* of genus g (cf. Siegel [22]). Clearly $\mathscr{H}_g$ is a convex open domain in $C^{g((g+1)/2)}$, and is equivalent to the bounded domain of all $g \times g$ matrices W satisfying $W = {}^tW$, $I - W\bar{W} > 0$ by a suitable linear fractional transformation.

We let G be the real, simple Lie group of all real $2g \times 2g$ matrices T which preserve Q; i.e. $TQ {}^tT = Q$ where

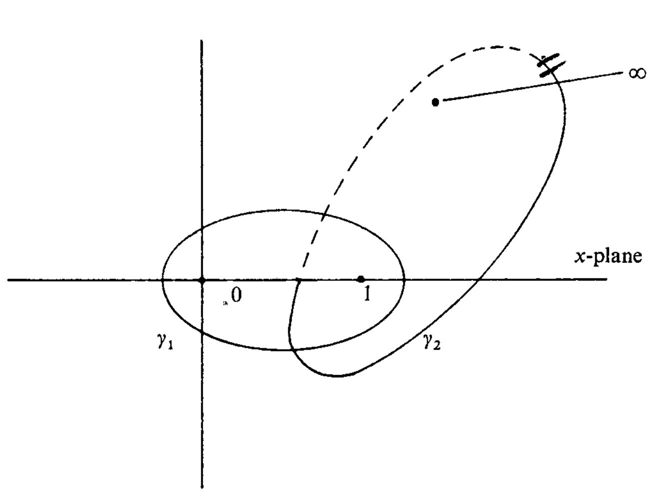
$$Q = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

Then G acts on $\mathscr{H}_g$ by $T(\Omega) = \Omega T$ (the relations (II) are obviously preserved). Writing $\Omega = (I, Z)$, $T(\Omega) \sim (I, (AZ + B)(CZ + D)^{-1})$ so that G acts as a transitive group of holomorphic linear fractional automorphisms of $\mathscr{H}_g$; thus $\mathscr{H}_g$ is the homogeneous complex manifold $H\backslash G$ where $H = \{T \in G: (I, iI)T \sim (I, iI)\}$ is the compact stability group of the point $iI_g \in \mathscr{H}_g$. (*Note*: $G \cong$ the symplectic group $\mathrm{Sp}(g, \boldsymbol{R})$ and $H \cong$ the unitary group $U(g)$.) The group Γ of integral matrices in G is a discrete subgroup which acts properly discontinuously on $\mathscr{H}_g = H\backslash G$; thus the quotient space $\mathscr{H}_g/\Gamma_g = H\backslash G/\Gamma$ is an *analytic space* (cf. Gunning-Rossi [**14**], H. Cartan [**8**]).

In summary, the period mapping (III) $\Sigma_g \xrightarrow{\Phi} \mathscr{H}_g/\Gamma_g$ maps the set of complex structures to the analytic space $\mathscr{H}_g/\Gamma_g$, which is a quotient of a homogeneous complex manifold by a properly discontinuous group of automorphisms.

EXAMPLE 1 (CONTINUED). Let V be a compact Riemann surface of genus 1. There is a unique holomorphic differential ω on V with the normalization $\int_{\gamma_1} \omega = 1$. Then the period matrix $\Omega(V) = (\int_{\gamma_1} \omega, \int_{\gamma_2} \omega) = (1, \beta)$ where $\operatorname{Im} \beta > 0$. Fixing a base point $*$ on V, the holomorphic mapping of $V \to V_\beta = C/L_\beta$ given by $x \to (\int_*^x \omega)/L_\beta$ is an analytic isomorphism of V with V_β and the mapping $\Sigma_1 \xrightarrow{\Phi} \mathscr{H}/\Gamma$ given above is just the period mapping.

EXAMPLE 2. An irreducible polynomial equation $f(x, y) = 0$ in C^2 gives a compact Riemann surface V_f in a well-known manner (cf. Weyl [**24**]). For instance, the equation $y^2 = x(x - 1)(x - \lambda)$ gives a V_f which is a two-sheeted covering of the x-plane branched at 0, 1, λ, and ∞:



This V_f is, for $\lambda \neq 0, 1, \infty$, a Riemann surface of genus one; we have drawn in above the one-cycles γ_1 and γ_2 with $\gamma_1 \cdot \gamma_2 = +1$. The holomorphic differential

$$\omega = \frac{dx}{y} = \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$$

is the usual integrand for the elliptic integral.

Now for $f(x, y; \lambda) = y^2 - x(x - 1)(x - \lambda)$ we have an *algebraic family of Riemann surfaces* of genus one; for each $\lambda \neq 0, 1, \infty$ we have $V_\lambda \in \Sigma_1$. In general, if we have an algebraic family of Riemann surfaces of genus g given by $f(x, y; \lambda_1, \ldots, \lambda_m) = 0$, there is an obvious mapping from the parameter space W to Σ_g.

Composing with the period mapping (III); we have $W \xrightarrow{\Phi} \mathscr{H}_g/\Gamma_g$ (we keep the same letter Φ for this mapping). If $f(x, y, \lambda) = y^2 - x(x-1)(x-\lambda)$, then $\Phi(\lambda) = (\int_{\gamma_2} dx/y)/(\int_{\gamma_1} dx/y)$ is the ratio of the elliptic integrals on V_λ. The resulting mapping $P_1 - (0, 1, \infty) \to \mathscr{H}/\Gamma$ given by $\Phi(\lambda) = (\int_{\gamma_2}\omega)/(\int_{\gamma_1}\omega)$ is evidently given by transcendental functions of λ (hypergeometric functions in this case). Thus (although the algebraic geometers disagree) we may think of the periods as being transcendental invariants defined on the space Σ_g.

The main general facts concerning the period mapping $\Sigma_g \xrightarrow{\Phi} \mathscr{H}_g/\Gamma_g$ are these:

(A) Φ is one-to-one into (i.e. injective on the set level); this is the Torelli theorem (cf. Andreotti [2]).

(B) The image $\Phi(\Sigma_g) \subset \mathscr{H}_g/\Gamma_g$ is a Zariski open on a $3g-3$ dimensional irreducible analytic subset $\overline{\Phi(\Sigma_g)}$ (i.e. $\Phi(\Sigma_g) = X - Y$ where $X \subset \mathscr{H}_g/\Gamma_g$ is an irreducible analytic set and $Y \subset X$ is an analytic subset). This result is due to Baily [4]; the irreducibility follows from the work of Bers and Ahlfors on Teichmuller spaces (cf. [1]). We mention here also the theorem of Andreotti-Mayer [3], which essentially gives very nice necessary and sufficient conditions on a matrix $\Omega \in \mathscr{H}_g/\Gamma_g$ in order that Ω be a period matrix of a Riemann surface of genus g.

The last general fact, which we shall call the *inversion of the periods*, needs some preliminary explanation. An *automorphic form* of weight m is given by an analytic function $f(Z)$ on $\mathscr{H}_g$ which satisfies the functional equation $f(T(Z)) = \det(CZ+D)^{-2m} f(Z)$ $(T \in \Gamma_g)$ (plus a condition at infinity if $g = 1$). If m is large these automorphic forms exist "in abundance" (cf. Séminaire Cartan [9]) and give a remarkable class of transcendental functions on $\mathscr{H}_g$. The quotient $\phi = f/g$ of two automorphic forms gives a meromorphic function on $\mathscr{H}_g/\Gamma_g$ (i.e. $\phi(T(Z)) = \phi(Z)$ for $T \in \Gamma_g$) and we have

(C) Let $\{V_\lambda\}_{\lambda\in W}$ be an algebraic family of Riemann surfaces of genus g and $\phi = f/g$ an automorphic function as above. Then $\phi(\Phi(\lambda))$ is a *rational* function of $\lambda \in W$.

In other words, the automorphic functions invert the period mapping up to rational functions. In particular, the functions of the form $\phi(\Phi(\lambda))$ give a subfield $\mathscr{F}$ of the field of rational functions $\mathscr{F}[W]$ on W such that: $V_\lambda \cong V_{\lambda'}$ if, and only if, $\psi(\lambda) = \psi(\lambda')$ for all $\psi \in \mathscr{F}$.

2. Let V be a compact, oriented C^∞ manifold and $\Sigma(V)$ the set of complex structures on V (cf. (I) above). We want to define a set mapping $\Sigma(V) \xrightarrow{\Phi}$ {suitable space} which gives good invariants of a point $V \in \Sigma(V)$ and which generalizes the periods given in Lecture 1 when $\dim V = 2$.

EXAMPLE 3. Let $\Omega \in \mathscr{H}_g$ be a $g \times 2g$ matrix satisfying the Riemann relations (II). Then $\Omega \sim (I, Z)$ where $Z = {}^tZ$, $\operatorname{Im} Z > 0$ (cf. (II')). Now Ω need not be a period matrix of a Riemann surface (this is generally the case if $g \geqq 4$). However, let $\pi_1, \dots, \pi_{2g}$ be the column vectors of Ω; then $\pi_\rho \in C^g$ and the vectors $\sum_{\rho=1}^{2g} n_\rho \pi_\rho$ $(n_\rho \in Z)$ give a lattice $L_\Omega \subset C^g$. The complex torus $T_\Omega = C^g/L_\Omega$ can

then be formed and the one cycles γ_ρ = projection of $\{t\pi_\rho\}$ $(0 \leqq t \leqq 1)$ give a basis of $H_1(T_\Omega, Z)$. Letting

$$w = \begin{pmatrix} w^1 \\ \vdots \\ w^g \end{pmatrix}$$

be the coordinates in C^g, the differentials $dw^1, \dots, dw^g$ give a basis for the holomorphic one-forms on T_Ω and clearly

$$\Omega = \begin{pmatrix} \int_{\gamma_1} dw^1 & \dots & \int_{\gamma_{2g}} dw^1 \\ \vdots & & \vdots \\ \int_{\gamma_1} dw^g & \dots & \int_{\gamma_{2g}} dw^g \end{pmatrix}$$

is the period matrix for the holomorphic one-forms on T_Ω.

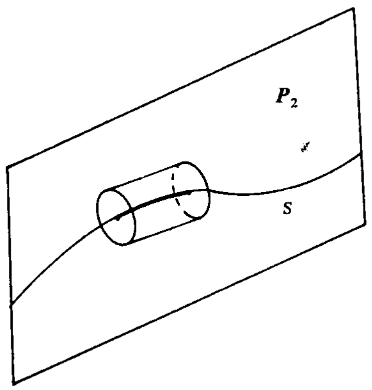
Now T_Ω is not an arbitrary complex torus; rather, there is a distinguished holomorphic embedding $f_\Omega: T_\Omega \to P_N$ $(N = 3^g - 1)$ which depends holomorphically on $\Omega \in \mathscr{H}_g$ (cf. Conforto [11]). Thus $\{T_\Omega\}_{\Omega\in\mathscr{H}_g}$ is an analytic family of projective algebraic manifolds. This suggests that we should reformulate (I) to read:

(I') Let $V \subset P_N$ be a nonsingular *algebraic manifold* and let $\Sigma(V)$ be the set of complex structures V on V such that there exists an analytic family $\{V_\lambda\}_{\lambda\in B}$ of projective algebraic manifolds $V_\lambda \subset P_N$, containing both V and V (cf. Kodaira-Spencer [16]). Then we want to find good invariants of a point $V \in \Sigma(V)$.

In particular, we want to assign to $V \in \Sigma(V)$ the "period matrix" of V and see how much of (A), (B), (C) in §1 still works. In a nutshell, we may say that (i) nothing essential from (A), (B), (C) is known to fail in higher dimensions; (ii) (A) and (B) have been proved in some special cases; (iii) (C) has been generalized a little, but the complete answer seems to involve knowledge of the discrete series representations of semisimple Lie groups (cf. Schmid [21]); and (iv) there are several totally new phenomena and many interesting problems which turn up.

Thus let $V \in \Sigma(V)$, so that we are given a projective embedding $V \subset P_N$ (technically, I am speaking about deformations of *polarized algebraic manifolds*). If $\gamma_1, \dots, \gamma_b$ is a basis for $H_q(V, Z)$/(torsion), then we want to look at period matrices $(\int_{\gamma_\rho} \omega^\alpha)$ where $\omega^1, \dots, \omega^m$ is a suitable set of differentials on V. By duality we may restrict q to $1 \leqq q \leqq n$. We could look at the holomorphic q-forms ω (such an ω is closed and is never exact; cf. Hodge [15]), but this will not be general enough for $n \geqq 3$. To see what we should use for differentials, we first look at

EXAMPLE 4. Let $S \subset P_2$ be a nonsingular plane curve of degree d given by $f(x, y) = 0$ in affine coordinates. For $x \in S$ there is a normal disc D_x with boundary ∂D_x. Thus as x traces out a one-cycle $\gamma \in H_1(S, \mathbf{Z})$, ∂D_x $(x \in \gamma)$ traces out a two-cycle $\tau(\gamma) \in H_2(\mathbf{P}_2 - S, \mathbf{Z})$:



this *tube mapping* $\tau: H_1(S, \mathbf{Z}) - H_2(\mathbf{P}_2 - S, \mathbf{Z})$ is an isomorphism, and we call the dual mapping $H^2(\mathbf{P}_2 - S) \xrightarrow{R} H^1(S)$ (the coefficients are $\mathbf{C}$ here) the *residue mapping*. The holomorphic differentials give a subspace $H^{1,0}(S) \subset H^1(S)$, and we want to know what the corresponding classes in $H^2(\mathbf{P}_2 - S)$ are.

Let ϕ be a rational two-form with a pole of order $k+1$ along S; in the above coordinate system,

$$\phi = P(x, y)\, dx\, dy / f(x, y)^{k+1}$$

where $\deg P \leqq (k+1)d - 3$. Since ϕ is holomorphic on $\mathbf{P}_2 - S$, ϕ gives a class in $H^2(\mathbf{P} - S)$. It is essentially a classical fact that the holomorphic differentials are the residues of the differentials $\phi = P(x, y)\, dx\, dy / f(x, y)$ with a first order pole along S.

EXAMPLE 5. To generalize Example 4, we let $V \subset \mathbf{P}_{n+1}$ be a nonsingular hypersurface of degree d given in affine coordinates by $f(x_1, \ldots, x_{n+1}) = 0$. The tube mapping $H_n(V) \xrightarrow{\tau} H_{n+1}(\mathbf{P}_{n+1} - V)$ exists as before and is essentially an isomorphism (technically, τ is an isomorphism on the *primitive part* of $H_n(V)$; cf. Hodge [15]). Again, as before, the cohomology group $H^{n+1}(P_{n+1} - V)$ is given by differentials

$$\phi = \frac{P(x) dx_1 \wedge \ldots \wedge dx_{n+1}}{f(x)^{k+1}} \qquad (\deg P \leqq d(k+1) - (n+2)),$$

taken modulo exact differentials. We let $F_k^n(V) \subset H^n(V)$ be the subspace of $H^n(V)$ given by the residues of classes ϕ with a pole of order $k+1$ along V. In [12] it is proved that: (i) $F_0^n(V) \subset F_1^n(V) \subset \ldots$ and $F_n^n(V) = F_{n+1}^n(V) = \ldots = F_{n+l}^n(V)$ for all $l > 0$; and (ii) the subspaces $F_q^n(V)$ for $0 \leqq q \leqq [(n-1)/2]$ determine all of the $F_k^n(V)$. Thus for $n = 1, 2$ we need only look at $F_0^n(V)$, but for $n = 3$ we must consider the two spaces $F_0^3(V) \subset F_1^3(V)$.

For this example $V \subset P_{n+1}$ we will use *all* of the subspaces $F_q^n(V)$, $0 \leqq q \leqq [(n-1)/2]$, as holomorphic differentials in forming the period matrix of V; i.e. we replace a vector space by a *filtration*. Thus let $\omega^1, \ldots, \omega^m$ be a basis for $F^n_{[(n-1)/2]}(V)$ such that $\omega^1, \ldots, \omega^{m_q}$ is a basis for $F_q^n(V)$ $(0 \leqq q \leqq [(n-1)/2])$; i.e. we choose a basis for the *flag* $F_0^n(V) \subset F_1^n(V) \subset \ldots \subset F^n_{[(n-1)/2]}(V)$. We may write $\omega^1 = R(\phi^1), \ldots, \omega^{m_q} = R(\phi^{m_q})$ where

$$\phi^\alpha = \frac{P_\alpha(x) dx_1 \ldots dx_{n+1}}{f(x)^{q+1}} \qquad (1 \leqq \alpha \leqq m_q).$$

Let

$$\Omega_q(V) = \begin{bmatrix} \int_{\gamma_1} \omega^1 & \ldots & \int_{\gamma_b} \omega^1 \\ \vdots & & \vdots \\ \int_{\gamma_1} \omega^{m_q} & \ldots & \int_{\gamma_b} \omega^{m_q} \end{bmatrix}$$

be the corresponding period matrix for $F_q^n(V)$, and $\Omega(V) = [\Omega_0(V), \ldots, \Omega_{[(n-1)/2]}(V)]$ be the total period matrix. As was the case for Riemann surfaces, we must allow the equivalences:

$$\begin{cases} \Omega \sim A\Omega \\ \Omega \sim \Omega \end{cases} \quad \text{where } A = \begin{pmatrix} A_0 & 0 \ldots 0 \\ * & \ddots & \vdots \\ * \cdots * & A_{[(n-1)/2]} \end{pmatrix}$$

is a linear transformation on the flag $F_0^n(V) \subset \ldots \subset F^n_{[(n-1)/2]}(V)$, and $\Lambda = (\Lambda_\alpha^\rho)$ is an integral matrix satisfying $\Lambda Q {}^t\Lambda = Q$ where $Q = (\gamma_\rho \cdot \gamma_\sigma)$ is the intersection matrix on the primitive cycles.

If we have polynomials $f(x_1, \ldots, x_{n+1}; \lambda)$ depending holomorphically on λ, we let V_λ be defined by $f(x; \lambda) = 0$; this is the sort of deformations considered in (I′). Then the periods

$$\int_{\gamma_\rho} \omega = \int_{\tau(\gamma_\rho)} \frac{P_\alpha(x) dx_1 \wedge \ldots \wedge dx_{n+1}}{f(x; \lambda)^{q+1}}$$

clearly depend holomorphically on λ. Furthermore, $\Omega = \Omega(V)$ satisfies the *generalized Riemann bilinear relations* (cf. Hodge [15]):

(II″) $$\Omega\, Q\, {}^t\Omega = 0, \qquad c_n \Omega\, Q\, {}^t\bar{\Omega} > 0.$$

Here c_n is a suitable power of i so chosen that $H = c_n \Omega\, Q\, {}^t\bar{\Omega}$ is Hermitian (for $n = 1$, $c_1 = i$; for $n = 2$, $c_2 = 1$, etc.), and the second relation in (II″) shall mean that the Hermitian matrix H has certain prescribed positive-definiteness properties (cf. Hodge, loc. cit.). For $n = 1, 2$, H is simply positive definite; however, for $n = 3$,

$$H = \begin{pmatrix} H_0 & * \\ \cdot & \\ * & * \end{pmatrix}$$

where H_0 is $m_0 \times m_0$ and is positive definite, whereas H has total signature $(m_0, m_1 - m_0)$.

We let $\mathscr{D}$ be the set of total period matrices $\Omega = [\Omega_0, \ldots, \Omega_{[(n-1)/2]}]$ satisfying the Riemann relations (II″) and with the equivalence $\Omega \sim A\Omega$ where

$$A = \begin{pmatrix} A_0 & 0 \ldots 0 \\ * & \ddots & \vdots \\ \vdots & & \vdots \\ * & \ldots & A_{[(n-1)/2]} \end{pmatrix}$$

If G is the identity component in the group of real $b \times b$ matrices T satisfying $TQ^tT = Q$ (G is a symplectic group or an orthogonal group depending on the parity of n), then G acts on $\mathscr{D}$ by $T(\Omega) = \Omega T$. Now G is a real, simple Lie group which is always noncompact of maximal rank (i.e. G contains a maximal torus). It may be seen that G acts transitively on $\mathscr{D}$ and the isotropy group of a fixed point $\Omega \in \mathscr{D}$ is compact; thus $\mathscr{D} \cong H\backslash G$ is a *homogeneous complex manifold*.

EXAMPLES. For $n = 1$, we are looking at Riemann surfaces and $\mathscr{D} \cong U(g)\backslash \mathrm{Sp}(g)$ as was seen in §1. For $n = 2$, we are considering the periods of holomorphic two-forms on an algebraic surface and $\mathscr{D} \cong U(m) \times \mathrm{SO}(k)\backslash \mathrm{SO}(2m, k)$ where $m = \dim F_0^2(V)$ and $b = 2m + k$ is the rank of the group of primitive two-cycles. In case $n = 3$, $\mathscr{D} \cong U(m_0) \times U(m - m_0)\backslash \mathrm{Sp}(m; \boldsymbol{R})$ where the homogeneous space $U(m_0) \times U(m - m_0)\backslash \mathrm{Sp}(m, \boldsymbol{R})$ has a somewhat subtle complex structure if $m_0 > 0$ ($m_0 = \dim F_0^3(V)$). In general, for $n > 1$, $\mathscr{D}$ is *not* an Hermitian symmetric space.

The group Γ of integral matrices in G is a discrete subgroup which acts properly discontinuously on $\mathscr{D}$; thus the quotient space $\mathscr{D}/\Gamma$ is naturally an analytic space (cf. H. Cartan, loc. cit.). The above may be summarized by saying that we have a set mapping:

(III′) $\qquad \{\text{nonsingular hypersurfaces of degree } d \text{ in } \boldsymbol{P}_{n+1}\} \xrightarrow{\Phi} \mathscr{D}/\Gamma.$

This period mapping coincides with the one constructed for Riemann surfaces in §1 in the case $n = 1$.

Referring now to (I′), given $V \subset \boldsymbol{P}_N$ we want to construct $\mathscr{D}/\Gamma$ so that the periods give a mapping:

(III″) $\qquad \Sigma(V) \xrightarrow{\Phi} \mathscr{D}/\Gamma;$

and which reduces to (III′) in case V is a nonsingular hypersurface. Given $V \in \Sigma(V)$ (i.e., V is a polarized deformation of V), we must define the filtration $F_0^q(V) \subset F_1^q(V) \subset \ldots \subset F_q^q(V) = H^q(V)$ without assuming that V is a hypersurface; here $1 \leq q \leq n$. To do this, we let A_k^q be the space of C^∞ q-forms ω on V which are of type $(q, 0) + \ldots + (q - k, k)$ (i.e. $\omega \wedge dz^{i_1} \wedge \ldots \wedge dz^{i_{-q+k+1+n}} = 0$ in any local holomorphic coordinate system). Then $A_{k-1}^q \subset A_k^q$, $d: A_k^q \to A_{k+1}^{q+1}$ and we let $\mathscr{F}_k^q(V) = Z_k^q / dA_{k-1}^{q-1}$ where Z_k^q are the closed forms in A_k^q. Using deRham's theorem,

there is a natural mapping $\mathscr{F}_k^q(V) \to H^q(V) \cong H^q(V)$, and these maps are injective (this uses that $V \subset \boldsymbol{P}_N$ is a Kahler manifold).

We let $F_k^q(V)$ be the image of $\mathscr{F}_k^q(V)$ in $H^q(V)$; then $F_0^q(V) \subset F_1^q(V) \subset \ldots \subset F_q^q(V) = H^q(V)$ and the $F_k^q(V)$ for $0 \leq k \leq [(q-1)/2]$ determine all of the $F_k^q(V)$ (cf. Hodge, loc. cit.). Having defined the flag $F_0^q(V) \subset \ldots \subset F_{[(q-1)/2]}^q(V)$, we may define the periods, the space $\mathscr{D}$ and automorphism group G, the discrete subgroup Γ, and everything just as before. As the notation suggests, in case $V \subset \boldsymbol{P}_{n+1}$ is a hypersurface and $q = n$, this intrinsic filtration on $H^n(V)$ is the same as the one given in Example 5 using the order of pole of rational differentials on $\boldsymbol{P}_{n+1}$ which are holomorphic on $\boldsymbol{P}_{n+1} - V$.

In summary, there is defined the *period mapping* $\Phi: \Sigma(V) \to \mathscr{D}/\Gamma$ where $\mathscr{D} = H\backslash G$ is a homogeneous complex manifold of a real, simple Lie group by a compact subgroup and $\Gamma \subset G$ is an arithmetically defined, properly discontinuous group of automorphisms of $\mathscr{D}$.

Two of the main properties of the period matrix $\Phi(V)$ are

(D) $\Phi(V)$ depends holomorphically on $V \in \Sigma(V)$ (i.e., if we have an analytic family $\{V_\lambda\}_{\lambda \in B}$ in the sense of Kodaira-Spencer **[16]**, then the mapping $\lambda \to \Phi(V_\lambda)$ is holomorphic from B to $\mathscr{D}/\Gamma$.) We have tried to motivate this by Example 5 above; the general argument uses the structure equations from the Kodaira-Spencer-Kuranishi local theory of deformations of complex structures **[18]**.

(E) In addition to the Riemann bilinear relations (II), the period mapping $V \to \Phi(V) = [\Omega_0(V), \ldots, \Omega_{[(q-1)/2]}(V)]$ satisfies the *infinitesimal bilinear relation*

$$\text{(IV)} \qquad \begin{aligned} d\Omega\, Q^t\Omega &= 0 \quad (\text{in case } q = 2p \text{ is even}), \\ d\Omega\, Q^t\Omega_{p-1} &= 0 \quad (\text{in case } q = 2p + 1 \text{ is odd}). \end{aligned}$$

Here $\Omega = \Omega_{[(q-1)/2]}$ and $d\Omega$ has the obvious meaning.

Now we don't have time to go into the matter, but the infinitesimal relations (IV) turn out to be the key to the study of periods in higher dimensions (cf. **[12]**).

We now return to (A), (B), (C) at the end of §1 to see how things look in higher dimensions.

(A) The *Torelli theorem* can at most be true *birationally* (look at nonsingular cubic surfaces), and even this fails (e.g. the Enriques surface—here one must use the periods of Prym differentials). Furthermore, the global Torelli theorem is closely related to the Hodge conjecture and so is presently inaccessible. However, we do have information on the *local* Torelli theorem (the differential of $\Sigma(V) \xrightarrow{\Phi} \mathscr{D}/\Gamma$ is injective), and it seems as though it may be true "in general." For the hypersurfaces in Example 5, the local Torelli theorem is true if the degree $d \geq 3$ ($n \neq 2$) or if $n = 2$ for $d \geq 4$. A good problem is to settle the local Torelli theorem (one way or the other) in case V is a simply-connected algebraic surface with *positive canonical bundle*.

(B) The assertion that $\Phi(\Sigma(V)) \subset \mathscr{D}/\Gamma$ is essentially an analytic subset is proved in many special cases and is most likely true in general. The method of proof used thus far is to make a very careful asymptotic analysis of the periods of an algebraic variety as it becomes singular, e.g.

this is done using the differential equations satisfied by the periods (*Picard-Fuchs equations*; cf. [**12**]).

(C) This is a very interesting problem which is wide open. In the first place there are *no* automorphic forms (or functions) in the classical sense, except for $q = 1$. It is conjectured that there are "generalized automorphic forms" of a certain type, but even these have not been constructed except in some special cases (cf. Schmid [**21**]). The existence of these generalized automorphic forms would follow from the so-called *Langlands conjecture* (cf. Langlands [**19**]) on the discrete series representations of semisimple Lie groups. However, even if we postulate the existence of these "generalized automorphic forms", it is not clear just how the inversion of the periods should look. What is clear is that the problem is very much related to the infinitesimal bilinear relations (IV) on the geometric side, as well as to $L^2(G)$ on the analytic side. There is also a rather convincing heuristic argument that there should be an inversion of periods, even though we have no idea how to do it.

3. Let $V \subset \boldsymbol{P}_N$ be a nonsingular algebraic manifold and $\boldsymbol{\Sigma}(V)$ the set of distinct complex structures V on the C^∞ manifold underlying V which are holomorphic deformations of V within $\boldsymbol{P}_N$ as explained in §2. Using the periods of the q-forms we defined a set mapping $\boldsymbol{\Sigma}(V) \to \mathcal{D}/\Gamma$, where $\mathcal{D}$ was a homogeneous complex manifold and Γ a properly discontinuous group of automorphisms of $\mathcal{D}$. In this lecture we want to give some geometric applications of this period mapping, plus some open problems and conjectures.

In particular, we want to talk about an *algebraic family of algebraic varieties*. In affine coordinates such a family will be given by a set of polynomial equations:

$$f_\alpha(x_1, \ldots, x_N; \lambda_1, \ldots, \lambda_M) = 0 \quad (\alpha = 1, \ldots),$$
$$g_j(\lambda_1, \ldots, \lambda_M) = 0 \quad (j = 1, \ldots).$$

The parameter space $\hat{B} \subset \boldsymbol{P}_M$ is given by the equations $g_j(\lambda) = 0$; for each $\lambda \in \hat{B}$, there is defined $V_\lambda \subset \boldsymbol{P}_N$ by $f_\alpha(x; \lambda) = 0$. To be precise, we let $\hat{\mathcal{F}} \subset \boldsymbol{P}_N \times \boldsymbol{P}_M$ be defined by $\{f(x; \lambda) = 0,\ g_j(\lambda) = 0\}$, and we assume that $\hat{\mathcal{F}}$, $\hat{B}$ are nonsingular. Then there is a rational holomorphic mapping $\hat{\mathcal{F}} \xrightarrow{\pi} \hat{B}$ such that $V_\lambda = \pi^{-1}(\lambda)$, and we assume that V_λ is generally nonsingular. Then there will be Zariski open sets $B \subset \hat{B}$, $\mathcal{F} = \pi^{-1}(B) \subset \hat{\mathcal{F}}$ such that we have

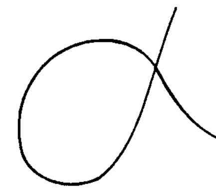
$$\begin{array}{ccc} \mathcal{F} & \subset & \hat{\mathcal{F}} \\ \downarrow\pi & & \downarrow\pi \\ B & \subset & \hat{B} \end{array}$$

where π has maximal rank on $\mathcal{F}$ so that the V_λ for $\lambda \in B$ are nonsingular. In particular, $\mathcal{F} \xrightarrow{\pi} B$ is a C^∞ fibre bundle (but *not* in general a holomorphic bundle), and we let $\lambda_0 \in B$ be a base point, $V = V_{\lambda_0}$ be the variety lying over λ_0.

EXAMPLE 6. Let $f(x, y; \lambda) = y^2 - x(x-1)(x-\lambda)$. Then $\hat{B} = \boldsymbol{P}_1$ and $\hat{\mathcal{F}}$ is a surface lying over $\boldsymbol{P}_1$ with elliptic curves as fibres. Since V_λ is nonsingular for $\lambda \neq \{0, 1, \infty\}$, we may take $B = \boldsymbol{P}_1 - \{0, 1, \infty\}$ so that $\mathcal{F} \subset \hat{\mathcal{F}}$ is an (open) algebraic surface fibered over $\boldsymbol{P}_1 - \{0, 1, \infty\}$ with nonsingular elliptic curves as fibres.

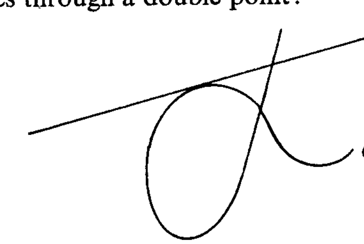
EXAMPLE 7. Let $[\xi_0, \ldots, \xi_n]$ be homogeneous coordinates in $\boldsymbol{P}_n$. A homogeneous form of degree d is given by $F_\lambda(\xi) = \sum_{i_0 \ldots i_d} \lambda_{i_0 \ldots i_d} \xi_{i_0} \cdots \xi_{i_d}$; here $\lambda = [\ldots, \lambda_{i_0 \ldots i_d} \ldots]$ is really a point in a big $\boldsymbol{P}_N$ since the variety $F_\lambda(\xi) = 0$ is the same as $F_{\alpha\lambda}(\xi) = 0$ $(\alpha \neq 0)$. In this case $\hat{B} = \boldsymbol{P}_N$ and $\hat{\mathcal{F}} \to \hat{B}$ is the family of all hypersurfaces of a fixed degree d in $\boldsymbol{P}_n$. The Zariski open set $B \subset \hat{B}$ is those $\lambda \in \hat{B}$ for which the hypersurface $V_\lambda \subset \boldsymbol{P}_n$ is nonsingular.

EXAMPLE 8. Let $C \subset \boldsymbol{P}_2$ be a plane curve of odd degree given by $h(x, y) = 0$; we assume that C has at most ordinary double points:

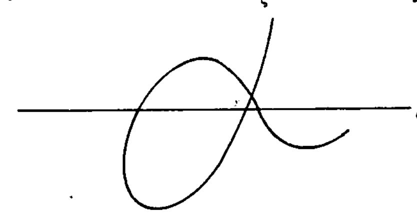


If $\xi = \xi(x, y)$ is a variable line (ξ is a point in $\boldsymbol{P}_2^*$, the dual projective space), then for each such ξ we can construct a surface V_ξ which is a two-sheeted covering of $\boldsymbol{P}_2$ with branch curve C_ξ given by $\xi(x, y)\, h(x, y) = 0$. Roughly speaking, V_ξ has an equation $z^2 = \xi(x, y)\, h(x, y)$; the assumption that $h(x, y)$ has odd degree means that the line at infinity in $\boldsymbol{P}_2$ is not part of the branch curve of $V_\xi \to \boldsymbol{P}_2$.

Now the *dual curve* $C^* \subset \boldsymbol{P}_2^*$ consists of all lines $\xi \in \boldsymbol{P}_2^*$ such that ξ is either tangent to C or passes through a double point:



If $\xi \in \boldsymbol{P}_2^* - C^*$, then the branch curve of V_ξ has ordinary double points:



and we may assume that V_ξ is nonsingular. Thus we get a family $\mathscr{F} \xrightarrow{\pi} B$ where $B = P_2^* - C^*$ and $\pi^{-1}(\xi) = V_\xi$ is a nonsingular surface.

Our problem is this: Given $\mathscr{F} \xrightarrow{\pi} B$ as above, using the periods of the q-forms there is defined a holomorphic mapping $B \xrightarrow{\Phi} \mathscr{D}/\Gamma$ (*period mapping*), and we want to study Φ.

For simplicity we shall consider the case $\dim V = 2 = q$, i.e. the periods of the holomorphic two-forms on a variable algebraic surface. We now describe Φ.

On each surface $V_\lambda \subset \boldsymbol{P}_N$ there is a distinguished homology class $H_\lambda = H\cdot V_\lambda$ where $H \subset \boldsymbol{P}_N$ is a general hyperplane ($H \cong \boldsymbol{P}_{N-1}$). Thus $H_\lambda \in H_2(V_\lambda, \boldsymbol{Q})$ is the two-cycle carried by the algebraic curve $H\cdot V_\lambda$ in V_λ. Since $\int_{H\cdot V_\lambda}\omega = 0$ for any holomorphic two-form ω on V_λ, the class H_λ should be ignored in describing the period mapping Φ. Thus let $H_2(V_\lambda, \boldsymbol{Q})_0 \subset H_2(V_\lambda, \boldsymbol{Q})$ be the classes γ such that the intersection number $\gamma\cdot H_\lambda = 0$; these are the so-called *primitive cycles* in the sense of Lefschetz (cf. Hodge [15]). Since $H_\lambda\cdot H_\lambda = d > 0$ (d = *degree* of V_λ), we have $\dim H_2(V_\lambda, \boldsymbol{Q})_0 = \dim H_2(V_\lambda, \boldsymbol{Q}) - 1$, and we may choose a basis $\gamma_1, \dots, \gamma_b$ for $H_2(V_\lambda, \boldsymbol{Q})_0$ such that $\gamma_1, \dots, \gamma_b, H_\lambda$ gives a basis for $H_2(V_\lambda, \boldsymbol{Q})$ and the intersection matrix is

$$\left(\begin{array}{c|c} (\gamma_\rho\cdot\gamma_\sigma) & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline 0\dots 0 & d^2 \end{array}\right)$$

in particular, the matrix $Q = (\gamma_\rho\cdot\gamma_\sigma)^{-1}$ is a rational, symmetric matrix.

On our reference variety $V = V_{\lambda_0}$ (λ_0 = base point on B) we choose a fixed basis $\gamma_1, \dots, \gamma_b$ for $H_2(V, \boldsymbol{Q})_0$. Since $\mathscr{F} \xrightarrow{\pi} B$ is a C^∞ bundle, the fundamental group $\pi_1(B) = \pi_1(B, \lambda_0)$ sets on $H_2(V, \boldsymbol{Q})$ by translating cycles on V around a loop in $\pi_1(B)$. The homology class of the hyperplane H_λ is obviously invariant, and so we see that $\pi_1(B)$ sets on the primitive homology $H_2(V, \boldsymbol{Q})_0$. Let $\Gamma \subset \mathrm{Aut}\,(H_2(V, \boldsymbol{Q})_0)$ be the group of automorphisms of $H_2(V, \boldsymbol{Q})_0$ obtained by the action of $\pi_1(B)$ on $H_2(V, \boldsymbol{Q})_0$. This group Γ is called the *monodromy group*, and the basis $\gamma_1, \dots, \gamma_b$ is determined up to the transformations in Γ.

Choose a basis $\omega^1, \dots, \omega^m$ for the holomorphic two-forms on V and form the period matrix

$$\Omega = \begin{pmatrix} \int_{\gamma_1}\omega^1 & \dots & \int_{\gamma_b}\omega^1 \\ \vdots & & \vdots \\ \int_{\gamma_1}\omega^m & \dots & \int_{\gamma_b}\omega^m \end{pmatrix}$$

Then Ω is determined up to transformations $\Omega \to A\Omega$ ($A \in \mathrm{GL}(m)$) and $\Omega \to \Omega\Lambda$ where $\Lambda \in \Gamma$, the monodromy group (this is a somewhat refined version of what we did in §2). Also Ω satisfies the Riemann relations:

$$\Omega Q^t\Omega = 0, \qquad \Omega Q^t\bar{\Omega} > 0.$$

Here $Q = (\gamma_\rho\cdot\gamma_\sigma)^{-1}$ is the matrix encountered above.

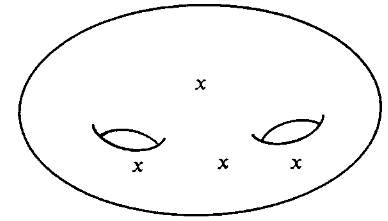
Let $G = \mathrm{SO}(Q)$ be the real special orthogonal group of Q; i.e. $G \subset \mathrm{GL}(b, \boldsymbol{R})$ is all $b \times b$ matrices T satisfying $T = \bar{T}$, $TQ^tT = Q$, $\det T = +1$. The *period matrix space* $\mathscr{D}$ is all $m \times b$ matrices Ω satisfying $\Omega Q^t\Omega = 0$, $\Omega Q^t\bar{\Omega} > 0$, and with the equivalence $\Omega \sim A\Omega$. Then G acts on $\mathscr{D}$ by $T(\Omega) = \Omega T$; this action is transitive, and so $\mathscr{D} \cong H\backslash G$ where $H \subset G$ is a connected, compact subgroup. The monodromy group Γ acts properly discontinuously on $\mathscr{D}$, and the period matrix $\Omega(V)$ is a well-defined point in $\mathscr{D}/\Gamma$.

We can be more explicit. By the *Hodge index theorem* (cf. Hodge [15]), the quadratic form Q has signature $(2m, k)$ $(2m + k = b)$; i.e. over the reals $\boldsymbol{R}$, Q is equivalent to $\sum_{\alpha=1}^{2m} x_\alpha^2 - \sum_{j=1}^{k} y_j^2$. Thus the real simple Lie group G is isomorphic to the indefinite orthogonal group $\mathrm{SO}(2m, k; \boldsymbol{R})$, and $H \cong U(m) \times \mathrm{SO}(k)$. The period matrix space $\mathscr{D} \cong H\ G$ is a homogeneous complex manifold (which is an Hermitian symmetric space if, and only if, $m = 1$), and the discrete subgroup Γ of G acts properly discontinuously on $\mathscr{D}$ so that the quotient space $\mathscr{D}/\Gamma$ is an analytic space.

With all these preliminaries, the periods give a holomorphic mapping $B \xrightarrow{\Phi} \mathscr{D}/\Gamma$ satisfying the infinitesimal period relation $d\Omega Q^t\Omega = 0$. The basic idea we want to illustrate is that the period mapping Φ and the monodromy group Γ are very closely related. A first theorem which illustrates this is:

(1) Theorem Φ *is constant if, and only if,* Γ *is a finite group.*

Outline of Proof. Suppose that $\Gamma = \{I\}$ is trivial and let us set out to prove that Φ is constant. Assume also that $\dim B = 1$, so that B is a compact Riemann surface with a finite number of points deleted:



Since $\Gamma = \{I\}$, the period mapping lifts to $\Phi: B \to \mathscr{D}$ and the proof consists of two steps:

(a) the period mapping Φ extends to all of $\hat{B}$ (i.e. the deleted points are removable singularities of Φ, this is a local question);

(b) any holomorphic mapping $\Phi: \hat{B} \to \mathscr{D}$ of a closed variety into $\mathscr{D}$ which satisfies $d\Omega Q^t\Omega = 0$ is constant.

To prove (a), we will use a result of W. Schmid that there is a G-invariant Hermitian metric on $\mathscr{D}$ such that the holomorphic sectional curvatures are all *negative* on the subspace $d\Omega Q^t\Omega = 0$ of the tangent space $T_\Omega(\mathscr{D})$; the point here is that, whereas not all sectional curvatures on $\mathscr{D}$ are negative, they are negative on the images $\Phi_* T_\lambda(B) \subset T_{\Phi(\lambda)}(\mathscr{D})$ of the period mapping. Thus the period mapping $\Phi: B \to \mathscr{D}$ is *negatively curved* (this is a differential-geometric analogue of mapping into the unit disc, which is well known to have constant negative curvature). Given that $\Phi: B \hookrightarrow \mathscr{D}$ is *negatively curved*, results of Wu, Kobayashi, and Kwack imply (a). The proof of (b) is similar.

Another application of periods is:

(2) APPLICATION. Suppose that $\dim B = l$ and that the differential Φ_* of the period mapping is nonsingular at one point. Then any holomorphic mapping $C^l \xrightarrow{f} B$ is degenerate (i.e. f_* is everywhere singular).

PROOF. We have a diagram

$$\begin{array}{ccc} C^l & \xrightarrow{F} & \mathscr{D} \\ \downarrow f & & \downarrow \\ B & \xrightarrow{\Phi} & \mathscr{D}/\Gamma \end{array}$$

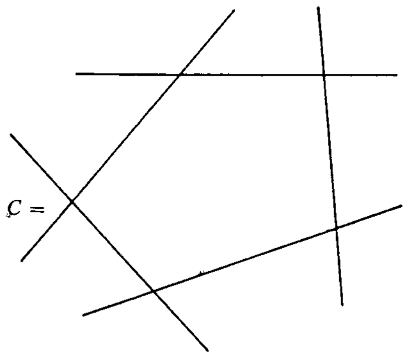
(monodromy principle); thus F is negatively curved since Φ is. By a theorem of Wu (cf. [24]), F is degenerate, and hence the original mapping f is since Φ_* is nonsingular almost everywhere.

EXAMPLE 9. Consider the family $\mathscr{F} \xrightarrow{\pi} B$ of Example 7. Then $B = P_1 - \{0, 1, \infty\}$ and (2) gives the classical Picard theorem: any holomorphic mapping $C \xrightarrow{f} P_1 - \{0, 1, \infty\}$ is constant.

(3) APPLICATION. If the differential Φ_* is everywhere injective, then any covering space $\tilde{B} \to B$ is negatively curved (i.e. all holomorphic sectional curvatures are $\leqq -1$).

EXAMPLE 10. In Example 9 we find that the universal covering of $P_1 - \{0, 1, \infty\}$ is the unit disc in C (the fact that $\Phi_* \neq 0$ in this case is not difficult).

EXAMPLE 11. In Example 8, let $h(x, y) = L_1 \ldots L_5$ be a product of five linear forms; the curve

$C =$ 

and the dual curve C^* is 15 lines in general position in P_2^* (the set of lines through a point in P_2 is a line in P_2^*). The surface V_ξ is, in this case, a *Kummer surface* having the unique (up to multiples) holomorphic two-form $\omega = dx\, dy/z$. Thus $m = 1$ and the period matrix space $\mathscr{D} \cong SO(2) \times SO(19)\backslash SO(2, 19; R)$ is a bounded symmetric domain of type IV. From (3) we find that: The universal covering space of $P_2^* - (15 \text{ lines})$ can be immersed in a bounded domain.

There is an old conjecture that the universal covering of $P_2 - (4 \text{ lines})$ is a bounded domain. The above illustrates a rather simple application of periods to mappings in several complex variables, and it seems to me that the interplay between periods and Picard type theorems has only been superficially explored.

Another application of periods is to prove geometric analogues of the so-called Tate conjectures (cf. Tate [23]; the geometric analogues were suggested by Grothendieck). Tate's situation deals with an algebraic variety X defined over a finite field k; letting $\bar{k}$ be the algebraic closure of k, the equations defining X also define a variety $\bar{X}$ over $\bar{k}$. The Galois group $\mathscr{G}$ of $\bar{k}$ over k acts on $\bar{X}$, and $\mathscr{G}$ also operates on the l-adic cohomology groups $H^*(\bar{X}, Q_l)$ which the algebraic geometers use. In particular, a subvariety $S \subset X$ lifts to $\bar{S} \subset \bar{X}$ and defines a class in $H^{\text{even}}(\bar{X}, Q_l)$ which is invariant under $\mathscr{G}$. Tate's conjecture is the converse: if $\gamma \in H^{\text{even}}(\bar{X}, Q_l)$ and is invariant under $\mathscr{G}$, then γ comes from a subvariety $S \subset X$. In general, Tate's conjectures say that the Galois group should have very strong action on the cohomology $H^*(\bar{X}, Q_l)$.

The geometric analogue is concerned with a family $\mathscr{F} \to B$ of the sort we have been discussing above; the Galois group acting on $H^*(\bar{X}, Q_l)$ is replaced by the fundamental group $\pi_1(B)$ acting on the homology $H_*(V, Q)$, and the following holds (we are still looking at the case where V is a surface):

(4) THEOREM. *Let $\gamma \in H_2(V, Q)$ be a homology class invariant under the action of the fundamental group $\pi_1(B)$. Then γ determines a class $\gamma_\lambda \in H_2(V_\lambda, Q)$ for all $\lambda \in B$, and we assume that γ_λ is a curve at one point $\lambda_0 \in B$ (i.e. there is a curve C_{λ_0} such that $C_{\lambda_0} = \gamma_{\lambda_0}$ in $H_2(V_{\lambda_0}, Q)$). From this it follows that γ_λ is everywhere a curve, and in fact there exists an algebraic surface $S \subset \mathscr{F}$ such that $S \cdot V_\lambda$ is a curve in the homology class γ_λ.*

REMARK. The assumption that γ_λ is algebraic at one point is necessary, because it might be that $\mathscr{F} \cong V \times B$ is a trivial family, in which case all of $H_2(V, Q)$ is invariant.

OUTLINE OF PROOF. Let $\omega^1, \ldots, \omega^m$ be a basis for the holomorphic two-forms on V. A famous theorem of Lefschetz says that γ is a curve if, and only if, $\int_\gamma \omega^\alpha = 0$ for $\alpha = 1, \ldots, m$. Thus γ is a curve if, and only if, the column vector

$$\xi = \begin{pmatrix} \vdots \\ \int_\gamma \omega^\alpha \\ \vdots \end{pmatrix}$$

is zero. Changing our basis by $\tilde{\omega}^\alpha = \sum_{\beta=1}^m A_\beta^\alpha \omega^\beta$ transforms ξ into $A\xi$.

Consider the period matrix Ω and let $J = \Omega Q {}^t\bar{\Omega}$. Then J is Hermitian positive definite, as is $H = J^{-1}$. A change of basis transforms H into ${}^t\bar{A}^{-1} H A^{-1}$; thus the length $|\xi|^2 = {}^t\bar{\xi} H \xi$ is well defined. In other words, the invariant cycle $\gamma \in H_2(V, Q)$ defines a nonnegative function ψ $(=|\xi|^2)$ on B such that $\psi(\lambda) = 0$ if, and only if, $\gamma_\lambda \in H_2(V_\lambda, Q)$ is an algebraic curve.

To prove (4) we will show:

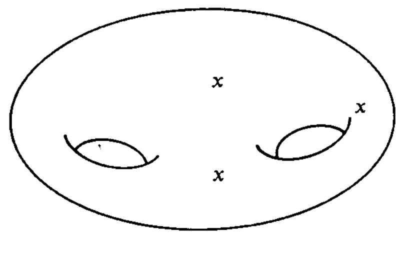
(a) ψ is plurisubharmonic;

(b) ψ is bounded on B, and thus extends to a plurisubharmonic function on the closed manifold $\hat{B}$.

If (a) and (b) are proved, then we use the maximum principle to conclude that ψ is constant, which gives our theorem.

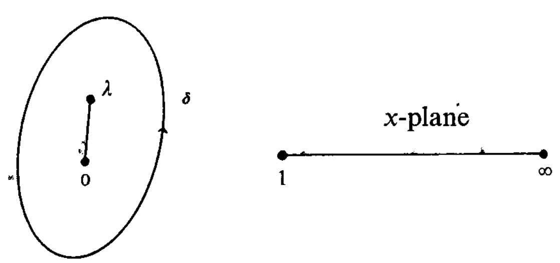
The proof of (a) is a differential-geometric computation; the crucial step is to use the infinitesimal period relation $d\Omega Q {}^t\Omega = 0$.

To see what is involved in (b), we suppose that $\dim B = 1$. Thus B is a compact Riemann surface $\hat{B}$ with finitely many points deleted:



These deleted points correspond to singular fibres in the total family $\hat{\mathscr{F}} \to \hat{B}$, and the proof of (b) is made by looking into the so-called *vanishing cycles.* These are cycles (homology classes) on the nonsingular algebraic surfaces V_λ which disappear as V_λ becomes singular.

EXAMPLE 12. In the family $y^2 = x(x-1)(x-\lambda)$, the following cycle δ vanishes as $\lambda \to 0$,



(5) APPLICATION. As an application of Theorem 4, we consider two families $\mathscr{F}_1 \xrightarrow{\pi_1} B, \mathscr{F}_2 \xrightarrow{\pi_2} B$. We assume that: (i) For some point $\lambda_0 \in B$, the fibres $V_1 = \pi_1^{-1}(\lambda_0)$ and $V_2 = \pi_2^{-1}(\lambda_0)$ are isomorphic; (ii) the fundamental group $\pi_1(B)$ acts the same on $H_*(V_1, Q)$ and $H_*(V_2, Q)$. Then the period mappings $\Phi_1: B \to \mathscr{D}/\Gamma$ and $\Phi_2: B \to \mathscr{D}/\Gamma$ are the same.

REMARK. Briefly, we may say that the period mapping $\Phi: B \to \mathscr{D}/\Gamma$ is completely determined by its value at one point and by the monodromy group. This rigidity theorem suggests a very strong interplay between periods and monodromy.

This result was proved in case V is an abelian variety by Grothendieck [13] and Borel-Narasimhan [7].

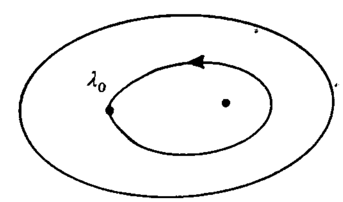
PROOF OF APPLICATION. Consider the product family $\mathscr{F} \xrightarrow{\pi} B$ where $\pi^{-1}(\lambda) = V_{1,\lambda} \times V_{2,\lambda}$. The graph of the isomorphism $V_{1,\lambda_0} \cong V_{2,\lambda_0}$ gives an algebraic homology class $\gamma_{\lambda_0} \in H_*(V_{\lambda_0}, Q) = H_*(V_{1,\lambda_0} \times V_{2,\lambda_0}, Q)$. Since the fundamental group $\pi_1(B)$ acts the same in both families, γ_{λ_0} is invariant, algebraic at one point, and the result follows from (4).

To close, we would like to mention some problems and conjectures (sic!). The main one is

(6) Let $\mathscr{F} \xrightarrow{\pi} B$ be an algebraic family of algebraic manifolds and $\Phi: B \to \mathscr{D}/\Gamma$ the period mapping. Then the closure $\overline{\Phi(B)} \subset \mathscr{D}/\Gamma$ is an analytic subvariety in which $\Phi(B)$ is a Zariski open set (i.e. $\overline{\Phi(B)} = \Phi(B) \cup Y$ where $Y \subset \mathscr{D}/\Gamma$ is an analytic subvariety of dimension strictly less than $\dim \overline{\Phi(B)}$).

This conjecture has been proved in many special cases, and we given an outline of how one might proceed. Essentially we may think of B as a compact Riemann surface $\hat{B}$ with finitely many *critical points* $\lambda_1, \ldots, \lambda_N$ (corresponding to the *singular fibres* V_{λ_α}) deleted. The idea is to investigate the period mapping $\Phi(\lambda)$ as λ tends to a critical point λ_α. In other words, we want to give an asymptotic analysis of the periods on V_λ as V_λ degenerates into a singular variety.

This is a local problem on $\hat{B}$; we let $\Delta \subset \hat{B}$ be a disc, with coordinate λ, centered at λ_α and we denote by $\mathscr{F}_\Delta = \pi^{-1}(\Delta)$ the restriction of $\hat{\mathscr{F}}$ to Δ. Thus we have an analytic family $\{V_\lambda\}$ of algebraic manifolds over the disc Δ with V_0 a singular variety. Let $V = V_{\lambda_0}$ be a reference variety lying over a base point $\lambda_0 \neq 0$ and $T: H_q(V, Q) \to H_q(V, Q)$ the transformation on homology obtained by looping around $\lambda = 0$:



This transformation T is called a *Picard-Lefschetz transformation,* and T gives a topological reflection of the singularities of V_0. A remarkable theorem of Landman is

(7) All eigenvalues of T are roots of unity and the elementary divisors of T are less than or equal to q.

In matrix terms, we see that $(T^N - I)^{q+1} = 0$ for some $N > 0$. Replacing λ by ξ^N changes T into T^N and does not really change our problem; thus we may assume that the Picard-Lefschetz transformation T is a *rational, unipotent matrix.*

Locally, the period mapping $\Phi: B \to \mathscr{D}/\Gamma$ lifts to $\Phi: \dot{\Delta} \to \{\mathscr{D}/T^n\}$ where $\dot{\Delta}$ is the punctured disc and $\{T^n\}$ is the cyclic group generated by T.

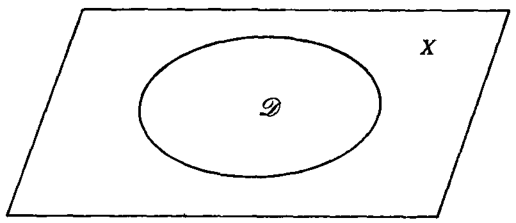
A local version of (1) is the following.

(8) The period mapping $\Phi: B \to \mathscr{D}/\Gamma$ extends across the critical point λ_α if, and only if, the Picard-Lefschetz transformation around λ_α is of finite order.

PROOF. This follows from (a) in the proof of (1).

Thus, in analyzing the period mapping $\Phi(\lambda)$ as $\lambda \to 0$, we may assume that T is of infinite order and, by (8), the periods $\Phi(\lambda)$ go to infinity in $\mathscr{D}$ as $\lambda \to 0$.

Now the period matrix space $\mathscr{D}$ is an open submanifold of a closed algebraic variety X (think of the usual upper-half plane sitting in the Riemann sphere). The manifold X, in the case of surfaces, is all $m \times b$ matrices Ω satisfying $\Omega Q {}^t\Omega = 0$ and with the equivalence $\Omega \sim A\Omega$. The closure $\bar{\mathscr{D}}$ of $\mathscr{D}$ in X is all Ω satisfying $\Omega Q {}^t\Omega = 0$, $\Omega Q {}^t\bar{Q} \geqq 0$, and with the equivalence $\Omega \sim A\Omega$. Pictorially, the situation is

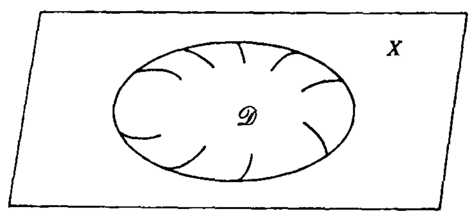


Note that the group G of automorphisms of $\mathscr{D}$ acts on X; in fact, X is acted transitively upon by the larger group of all complex matrices S satisfying $SQ {}^tS = Q$, and so X is a rational homogeneous algebraic manifold.

Return now to our period mapping $\Phi: \dot{\Delta} \to \mathscr{D}/\{T^n\}$. Using the differential equations of the periods (*Picard-Fuchs equations*), it can be shown that

(9) As $\lambda \to 0$, the period matrix $\Phi(\lambda)$ tends to a unique point in $\bar{\mathscr{D}}$ modulo $\{T^n\}$.

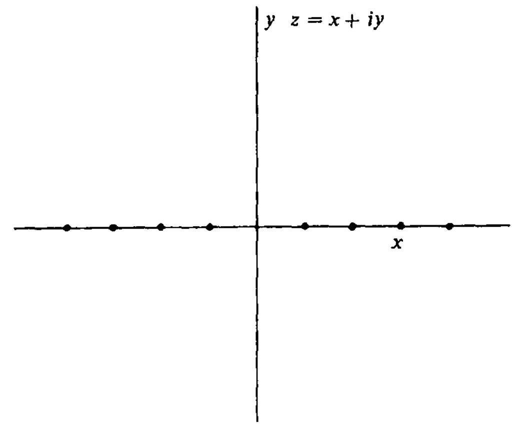
Pictorially, the inverse image of $\Phi(\lambda)$ in $\bar{\mathscr{D}}$ as $\lambda \to 0$ will look like



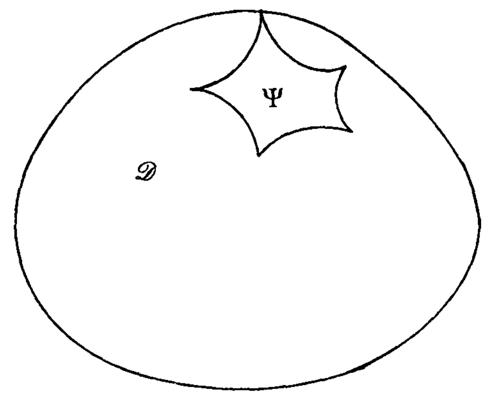
(there will be infinitely many determinations of $\Phi(\lambda)$ in $\mathscr{D}$).

This picture, although quite nice, is not enough to prove (6); what we need to know is that the limit point $\Omega_0 = \lim_{\lambda \to 0} \Phi(\lambda)$ is in a *rational boundary component* (cf. Borel-Baily [5]) of $\mathscr{D}$; here $\Omega_0 \in \partial\mathscr{D} = \bar{\mathscr{D}} - \mathscr{D}$.

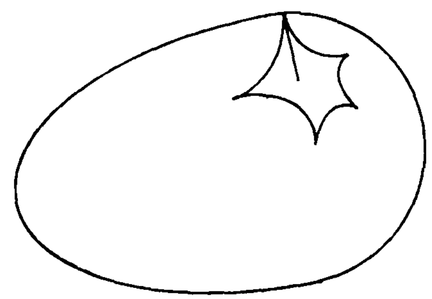
EXAMPLE 13. In the case of the ordinary upper half-plane, the rational boundary components are the rational points on the real axis, plus the point at infinity:



Another more geometric way of saying the above is the following: The period mapping goes from the parameter space B to $\mathscr{D}/\Gamma$ (using the full monodromy group now), and we look at a *fundamental domain* $\Psi \subset \mathscr{D}$:



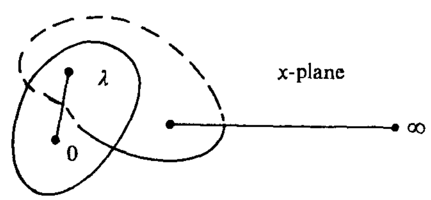
Then each point $\Phi(\lambda)$ determines a unique point $\Omega(\lambda) \in \Psi$, and the condition that Ω_0 belong to a rational boundary component essentially means that the analytic curve $\Omega(\lambda) \subset \Psi$ goes nicely to the boundary:



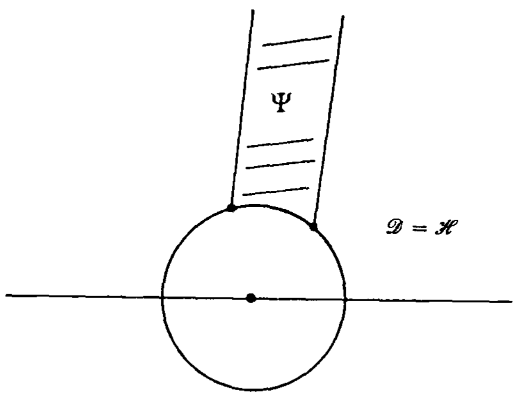
Return to the Picard-Lefschetz transformation T. It is easy to see that $T(\Omega_0) = \Omega_0$. On the other hand, Ω_0 belongs to a unique boundary component $F(\Omega_0)$ (cf. Borel [6]), and we could probably prove everything if it could be shown that

(10) The boundary component $F(\Omega_0)$ is the fixed point set of the rational unipotent matrix T.

EXAMPLE 14. Look at the family of elliptic curves V_λ given by $y^2 = x(x-1)(x-\lambda)$:



(cf. Example 6). In this case the period matrix space $\mathscr{D}$ is the usual upper half-plane $\mathscr{H}$, $\Gamma = \mathrm{SL}(2, \mathbf{Z})$ is the modular group, and so we get the familiar picture



of the fundamental domain for $\mathrm{SL}(2, \mathbf{Z})$. As $\lambda \to 0$ it can be seen that

$$\Omega(\lambda) \sim -\frac{i}{2\pi}\log\lambda = \frac{\arg\lambda}{2\pi} - i\log|\lambda|.$$

Thus $\Omega(\lambda)$ goes uniformly to infinity in Ψ, and this is the sort of picture we would like to see work in general.

Note that, in this example,

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and so ∞ is the fixed set of T; this verifies a special case of (10).

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