DEFORMATIONS OF COMPLEX STRUCTURE

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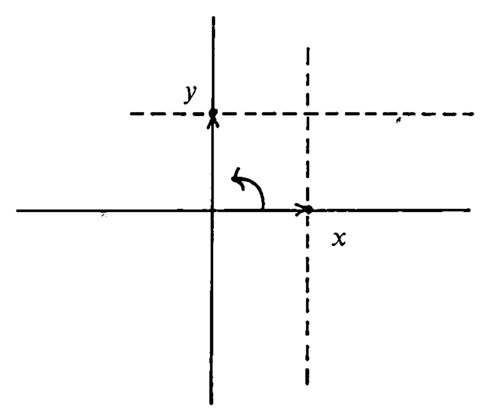
1. Let V be a compact, connected C^∞ manifold. We want to talk about complex structures on V, so that we assume that V is of even dimension $2n$ and is oriented. The notion of a complex structure on V, which is compatible with the C^∞ structure and orientation, will mean that for each sufficiently small open set $U \subset V$ we are given the following data: complex-valued C^∞ one-forms $\omega^1, \ldots, \omega^n$ on U such that: (i) $(i/2)^n \omega^1 \wedge \bar{\omega}^1 \wedge \ldots \wedge \omega^n \wedge \bar{\omega}^n > 0$ (i.e. $\omega^1, \ldots, \omega^n, \bar{\omega}^1, \ldots, \bar{\omega}^n$ are linearly independent and define an orientation consistent with the given one on V); (ii) $d\omega^\alpha = \sum_{\beta=1}^n C_\beta^\alpha \wedge \omega^\beta$ where C_β^α are one-forms (this is the *integrability condition*, which may be written $d\omega^\alpha \equiv 0\,(\omega^1, \ldots, \omega^n)$). The holomorphic functions in U form a ring $\mathcal{O}_U$ where a C^∞ function f is in $\mathcal{O}_U$ if $f_{\bar{\beta}} = 0$ (*Cauchy-Riemann equations*) where the total differential $df = \sum_{\alpha=1}^n f_\alpha \omega^\alpha + \sum_{\beta=1}^n f_{\bar{\beta}} \bar{\omega}^\beta$. The fact that the equations $f_{\bar{\beta}} = 0$ have enough local solutions to give local coordinates on V is the *Newlander-Nirenberg theorem* (cf. Chern [**10**], Newlander-Nirenberg [**20**], and Kohn [**17**]).

Our general problem is this:

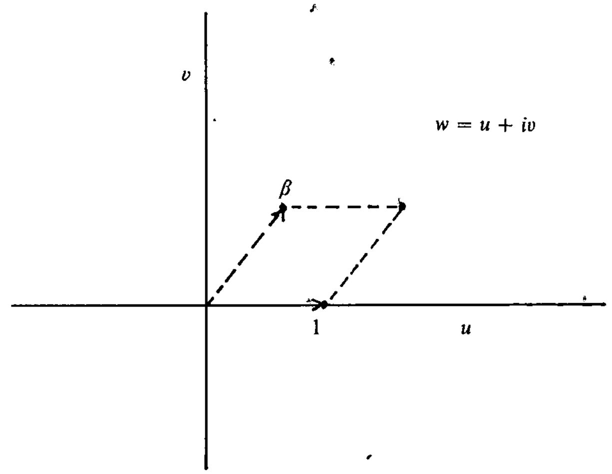
(I) Let $\Sigma(V)$ be the set of distinct complex structures on V which are consistent with the C^∞ structure and orientation. We want to describe $\Sigma(V)$.

EXAMPLE 1. In $\boldsymbol{R}^2$ with coordinates x, y we consider the standard lattice $\boldsymbol{L}$ generated by $(1, 0)$ and $(0, 1)$. Then $\boldsymbol{R}^2/\boldsymbol{L} = V$ is a C^∞ manifold, which is simply a real torus of dimension two. The orientation on V is given by $dx \wedge dy = (i/2) dz \wedge d\bar{z}$ where $z = x + iy$.



Let $\omega = dz + \alpha\, d\bar{z}$ where α is a complex number. Then ω is a complex valued one form on V and $\omega \wedge \bar{\omega} = (1 - |\alpha|^2) dz \wedge d\bar{z}$. Thus for $|\alpha| < 1$ we have a complex structure V_α on V; in fact, if Δ is the unit disc $|\alpha| < 1$, then $\{V_\alpha\}_{\alpha\in\Delta}$ is a "nice family" of complex structures on V. A standard theorem in the theory of Riemann surfaces is: Any complex structure on V is a V_α for $\alpha \in \Delta$. Thus, in our example, $\Sigma(V)$ is a quotient of Δ.

EXAMPLE 1 (CONTINUED). We want to know when $V_\alpha = V_{\alpha'}$. The change of variables $w = (1/(1+\alpha))(z + \alpha\bar{z})$ is a real linear transformation of $\boldsymbol{R}^2$ onto the complex w-plane $\boldsymbol{C}$. Since $w(1) = 1$ and $w(i) = ((1-\alpha)/(1+\alpha))\, i = \beta$, the lattice $\boldsymbol{L}$ in $\boldsymbol{R}^2$ goes into the lattice $\boldsymbol{L}_\beta$ generated by $w = 1$, $w = \beta$:



In fact, the transformation $\beta = ((1-\alpha)/(1+\alpha))i$, $|\alpha| < 1$ takes the disc Δ onto the upper-half-plane $\mathscr{H}$ and the torus $V_\alpha \cong \boldsymbol{C}/\boldsymbol{L}_\beta$ since $dw = (1/(1+\alpha))\omega$. Thus $\{V_\alpha\}_{\alpha\in\Delta} \cong \{V_\beta\}_{\beta\in\mathscr{H}}$ where $V_\beta = \boldsymbol{C}/\boldsymbol{L}_\beta$. Now it is a standard result that $V_\beta \cong V_{\beta'}$ if, and only if, $\beta' = (a\beta + b)/(c\beta + d)$ where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \boldsymbol{Z})$$

is an integral linear fractional transformation of $\mathscr{H}$ into itself. In other words, $\Sigma(V) = \mathscr{H}/\Gamma$ where $\Gamma = \mathrm{SL}(2, \boldsymbol{Z})$ is a properly discontinuous automorphism group acting on $\mathscr{H}$.

The proper way to interpret this example so far is this: Let $\Sigma(V)$ be the set of complex structures on $V = \boldsymbol{R}^2/\boldsymbol{L}$ as in (I), and let V be a complex structure on V. Then $V = V_\beta$ for some $\beta \in \mathscr{H}$ and β is determined up to $\mathrm{SL}(2, \boldsymbol{Z})$ acting on $\mathscr{H}$. This gives a set mapping $\Sigma(V) \xrightarrow{\mathfrak{P}} \mathscr{H}/\Gamma$ which, in this example, is one-to-one onto.

In general, rather than studying $\Sigma(V)$ directly, we shall map $\Sigma(V)$ into spaces (similar to $\mathscr{H}/\Gamma$) which are easy to describe and which give interesting invariants of a complex structure $V \in \Sigma(V)$.

We first do this in case $\dim V = 2$.

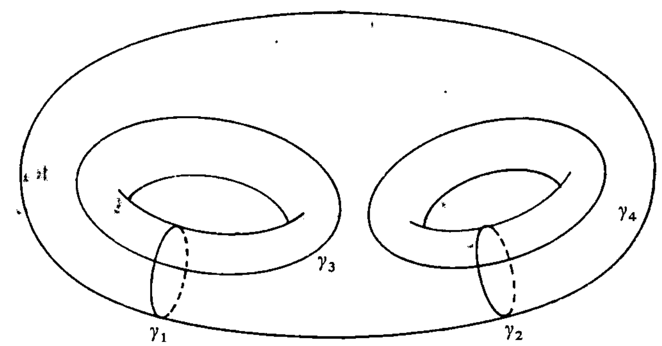
Let V be a compact, oriented two manifold of genus g and $\Sigma(V) = \Sigma_g$, the set of complex structures on V with the proper orientation. If $g = 0$, then every

$V \in \Sigma(V)$ is the standard structure on the Riemann sphere (Riemann mapping theorem) and Σ_0 is a single point.

Assume that $g \geqq 1$ and let $\gamma_1, \ldots, \gamma_{2g}$ be a canonical basis for the first homology group $H_1(V, \boldsymbol{Z})$ $(\cong \boldsymbol{Z} \oplus \ldots \oplus \boldsymbol{Z}$ ($2g$ terms)); by definition the intersection matrix $(\gamma_\rho \cdot \gamma_\sigma) = Q$ is given by

$$Q = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

The picture we have in mind is



The vector space of holomorphic differentials on any complex structure $V \in \Sigma_g$ is g-dimensional, and we choose a basis $\omega^1, \ldots, \omega^g$ for the holomorphic one forms. We then form the *period matrix*

$$\Omega = \overbrace{\begin{pmatrix} \int_{\gamma_1} \omega^1 & \ldots & \int_{\gamma_{2g}} \omega^1 \\ \vdots & & \vdots \\ \int_{\gamma_1} \omega^g & \ldots & \int_{\gamma_{2g}} \omega^g \end{pmatrix}}^{2g} \Bigg\} g;$$

this is a $g \times 2g$ matrix of rank g. A change of basis $\tilde{\omega}^\alpha = \sum_{\beta=1}^{g} A_\beta^\alpha \omega^\beta$ transforms Ω to $\tilde{\Omega} = A\Omega$; a change of homology basis $\tilde{\gamma}_\rho = \sum_{\sigma=1}^{2g} \Lambda_\rho^\sigma \gamma_\rho$ leads to $\tilde{\Omega} = \Omega\Lambda$. Here $A = (A_\beta^\alpha)$ is a nonsingular $g \times g$ matrix and $\Lambda = (\Lambda_\sigma^\rho)$ is a $2g \times 2g$ integral matrix which preserves the quadratic form Q; i.e. $\Lambda Q {}^t\Lambda = Q$. If we agree to call two $g \times 2g$ matrices $\tilde{\Omega}$, Ω *equivalent* if $\tilde{\Omega} = A\Omega\Lambda$; then the above procedure gives a set mapping: $\Sigma_g \xrightarrow{\mathfrak{P}} \{\text{equivalence classes of period matrices}\}$.

The matrix Ω is not arbitrary but satisfies the *Riemann bilinear relations*

$$\text{(II)} \qquad \Omega Q^t\Omega = 0, \quad i\Omega Q^t\bar{\Omega} > 0.$$

These relations are a restatement of

$$\int_V \omega \wedge \omega = 0, \qquad i\int_V \omega \wedge \bar{\omega} > 0,$$

for a holomorphic differential ω on V. We then define

$\mathscr{H}_g =$ {set of $g \times 2g$ matrices Ω satisfying the Riemann bilinear relations (II) and with the equivalence $\Omega \sim A\Omega$}.

The group Γ_g of all $2g \times 2g$ integral matrices satisfying $\Lambda Q^t\Lambda = Q$ acts on Ω by $\Lambda(\Omega) = \Omega\Lambda$ and we have constructed the *period mapping*

$$\text{(III)} \qquad \Sigma_g \xrightarrow{\Phi} \mathscr{H}_g/\Gamma_g.$$

We now examine $\mathscr{H}_g$. Let Ω be a point in $\mathscr{H}_g$ and write $\Omega = (A, B)$ where A, B are $g \times g$ matrices. Then $i\Omega Q^t\bar{\Omega} = i(-B^t\bar{A} + A^t\bar{B})$ is positive definite, and it follows that A, B are each nonsingular. Thus $\Omega \sim A^{-1}\Omega = (I, Z)$ and each equivalence class $\Omega \in \mathscr{H}_g$ is given by a unique matrix (I, Z). The relations (II) then become:

$$\text{(II')} \qquad Z = {}^tZ, \quad Z = X + iY, \quad Y > 0.$$

Thus $\mathscr{H}_g \cong$ *Siegel generalized upper-half-plane* of genus g (cf. Siegel [22]). Clearly $\mathscr{H}_g$ is a convex open domain in $C^{g((g+1)/2)}$, and is equivalent to the bounded domain of all $g \times g$ matrices W satisfying $W = {}^tW$, $I - W\bar{W} > 0$ by a suitable linear fractional transformation.

We let G be the real, simple Lie group of all real $2g \times 2g$ matrices T which preserve Q; i.e. $TQ^tT = Q$ where

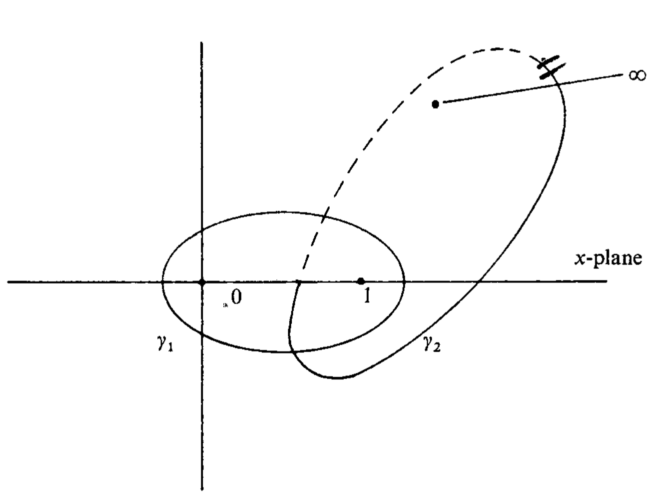
$$Q = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

Then G acts on $\mathscr{H}_g$ by $T(\Omega) = \Omega T$ (the relations (II) are obviously preserved). Writing $\Omega = (I, Z)$, $T(\Omega) \sim (I, (AZ + B)(CZ + D)^{-1})$ so that G acts as a transitive group of holomorphic linear fractional automorphisms of $\mathscr{H}_g$; thus $\mathscr{H}_g$ is the homogeneous complex manifold $H\backslash G$ where $H = \{T \in G : (I, iI)T \sim (I, iI)\}$ is the compact stability group of the point $iI_g \in \mathscr{H}_g$. (*Note*: $G \cong$ the symplectic group $\mathrm{Sp}(g, \boldsymbol{R})$ and $H \cong$ the unitary group $U(g)$.) The group Γ of integral matrices in G is a discrete subgroup which acts properly discontinuously on $\mathscr{H}_g = H\backslash G$; thus the quotient space $\mathscr{H}_g/\Gamma_g = H\backslash G/\Gamma$ is an *analytic space* (cf. Gunning-Rossi [**14**], H. Cartan [**8**]).

In summary, the period mapping (III) $\Sigma_g \xrightarrow{\Phi} \mathscr{H}_g/\Gamma_g$ maps the set of complex structures to the analytic space $\mathscr{H}_g/\Gamma_g$, which is a quotient of a homogeneous complex manifold by a properly discontinuous group of automorphisms.

EXAMPLE 1 (CONTINUED). Let V be a compact Riemann surface of genus 1. There is a unique holomorphic differential ω on V with the normalization $\int_{\gamma_1} \omega = 1$. Then the period matrix $\Omega(V) = (\int_{\gamma_1}\omega, \int_{\gamma_2}\omega) = (1, \beta)$ where $\mathrm{Im}\,\beta > 0$. Fixing a base point $*$ on V, the holomorphic mapping of $V \to V_\beta = C/L_\beta$ given by $x \to (\int_*^x \omega)/L_\beta$ is an analytic isomorphism of V with V_β and the mapping $\Sigma_1 \xrightarrow{\Phi} \mathscr{H}/\Gamma$ given above is just the period mapping.

EXAMPLE 2. An irreducible polynomial equation $f(x, y) = 0$ in C^2 gives a compact Riemann surface V_f in a well-known manner (cf. Weyl [**24**]). For instance, the equation $y^2 = x(x-1)(x-\lambda)$ gives a V_f which is a two-sheeted covering of the x-plane branched at 0, 1, λ, and ∞:



This V_f is, for $\lambda \neq 0, 1, \infty$, a Riemann surface of genus one; we have drawn in above the one-cycles γ_1 and γ_2 with $\gamma_1 \cdot \gamma_2 = +1$. The holomorphic differential

$$\omega = \frac{dx}{y} = \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$$

is the usual integrand for the elliptic integral.

Now for $f(x, y; \lambda) = y^2 - x(x-1)(x-\lambda)$ we have an *algebraic family of Riemann surfaces* of genus one; for each $\lambda \neq 0, 1, \infty$ we have $V_\lambda \in \Sigma_1$. In general, if we have an algebraic family of Riemann surfaces of genus g given by $f(x, y; \lambda_1, \dots, \lambda_m) = 0$, there is an obvious mapping from the parameter space W to Σ_g.

Composing with the period mapping (III); we have $W \xrightarrow{\Phi} \mathscr{H}_g/\Gamma_g$ (we keep the same letter Φ for this mapping). If $f(x, y, \lambda) = y^2 - x(x-1)(x-\lambda)$, then $\Phi(\lambda) = (\int_{\gamma_2} dx/y)/(\int_{\gamma_1} dx/y)$ is the ratio of the elliptic integrals on V_λ. The resulting mapping $P_1 - (0, 1, \infty) \to \mathscr{H}/\Gamma$ given by $\Phi(\lambda) = (\int_{\gamma_2}\omega)/(\int_{\gamma_1}\omega)$ is evidently given by transcendental functions of λ (hypergeometric functions in this case). Thus (although the algebraic geometers disagree) we may think of the periods as being transcendental invariants defined on the space Σ_g.

The main general facts concerning the period mapping $\Sigma_g \xrightarrow{\Phi} \mathscr{H}_g/\Gamma_g$ are these:

(A) Φ is one-to-one into (i.e. injective on the set level); this is the Torelli theorem (cf. Andreotti [2]).

(B) The image $\Phi(\Sigma_g) \subset \mathscr{H}_g/\Gamma_g$ is a Zariski open on a $3g-3$ dimensional irreducible analytic subset $\overline{\Phi(\Sigma_g)}$ (i.e. $\Phi(\Sigma_g) = X - Y$ where $X \subset \mathscr{H}_g/\Gamma_g$ is an irreducible analytic set and $Y \subset X$ is an analytic subset). This result is due to Baily [4]; the irreducibility follows from the work of Bers and Ahlfors on Teichmuller spaces (cf. [1]). We mention here also the theorem of Andreotti-Mayer [3], which essentially gives very nice necessary and sufficient conditions on a matrix $\Omega \in \mathscr{H}_g/\Gamma_g$ in order that Ω be a period matrix of a Riemann surface of genus g.

The last general fact, which we shall call the *inversion of the periods*, needs some preliminary explanation. An *automorphic form* of weight m is given by an analytic function $f(Z)$ on $\mathscr{H}_g$ which satisfies the functional equation $f(T(Z)) = \det(CZ+D)^{-2m} f(Z)$ $(T \in \Gamma_g)$ (plus a condition at infinity if $g = 1$). If m is large these automorphic forms exist "in abundance" (cf. Séminaire Cartan [9]) and give a remarkable class of transcendental functions on $\mathscr{H}_g$. The quotient $\phi = f/g$ of two automorphic forms gives a meromorphic function on $\mathscr{H}_g/\Gamma_g$ (i.e. $\phi(T(Z)) = \phi(Z)$ for $T \in \Gamma_g$) and we have

(C) Let $\{V_\lambda\}_{\lambda \in W}$ be an algebraic family of Riemann surfaces of genus g and $\phi = f/g$ an automorphic function as above. Then $\phi(\Phi(\lambda))$ is a *rational* function of $\lambda \in W$.

In other words, the automorphic functions invert the period mapping up to rational functions. In particular, the functions of the form $\phi(\Phi(\lambda))$ give a subfield $\mathscr{F}$ of the field of rational functions $\mathscr{F}[W]$ on W such that: $V_\lambda \cong V_{\lambda'}$ if, and only if, $\psi(\lambda) = \psi(\lambda')$ for all $\psi \in \mathscr{F}$.

2. Let V be a compact, oriented C^∞ manifold and $\Sigma(V)$ the set of complex structures on V (cf. (I) above). We want to define a set mapping $\Sigma(V) \xrightarrow{\Phi}$ {suitable space} which gives good invariants of a point $V \in \Sigma(V)$ and which generalizes the periods given in Lecture 1 when $\dim V = 2$.

EXAMPLE 3. Let $\Omega \in \mathscr{H}_g$ be a $g \times 2g$ matrix satisfying the Riemann relations (II). Then $\Omega \sim (I, Z)$ where $Z = {}^tZ$, $\operatorname{Im} Z > 0$ (cf. (II')). Now Ω need not be a period matrix of a Riemann surface (this is generally the case if $g \geqq 4$). However, let $\pi_1, \ldots, \pi_{2g}$ be the column vectors of Ω; then $\pi_\rho \in C^g$ and the vectors $\sum_{\rho=1}^{2g} n_\rho \pi_\rho$ $(n_\rho \in Z)$ give a lattice $L_\Omega \subset C^g$. The complex torus $T_\Omega = C^g/L_\Omega$ can

then be formed and the one cycles γ_ρ = projection of $\{t\pi_\rho\}$ $(0 \leqq t \leqq 1)$ give a basis of $H_1(T_\Omega, Z)$. Letting

$$w = \begin{pmatrix} w^1 \\ \vdots \\ w^g \end{pmatrix}$$

be the coordinates in C^g, the differentials $dw^1, \ldots, dw^g$ give a basis for the holomorphic one-forms on T_Ω and clearly

$$\Omega = \begin{pmatrix} \int_{\gamma_1} dw^1 & \cdots & \int_{\gamma_{2g}} dw^1 \\ \vdots & & \vdots \\ \int_{\gamma_1} dw^g & \cdots & \int_{\gamma_{2g}} dw^g \end{pmatrix}$$

is the period matrix for the holomorphic one-forms on T_Ω.

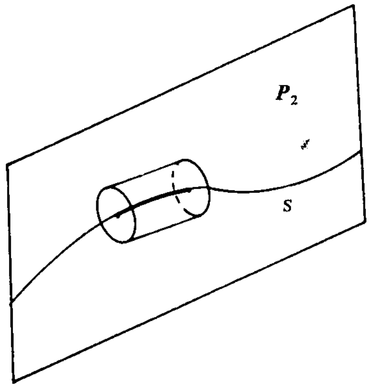
Now T_Ω is not an arbitrary complex torus; rather, there is a distinguished holomorphic embedding $f_\Omega: T_\Omega \to P_N$ $(N = 3^g - 1)$ which depends holomorphically on $\Omega \in \mathscr{H}_g$ (cf. Conforto [11]). Thus $\{T_\Omega\}_{\Omega \in \mathscr{H}_g}$ is an analytic family of projective algebraic manifolds. This suggests that we should reformulate (I) to read:

(I') Let $V \subset P_N$ be a nonsingular *algebraic manifold* and let $\Sigma(V)$ be the set of complex structures V on V such that there exists an analytic family $\{V_\lambda\}_{\lambda \in B}$ of projective algebraic manifolds $V_\lambda \subset P_N$, containing both V and V (cf. Kodaira-Spencer [16]). Then we want to find good invariants of a point $V \in \Sigma(V)$.

In particular, we want to assign to $V \in \Sigma(V)$ the "period matrix" of V and see how much of (A), (B), (C) in §1 still works. In a nutshell, we may say that (i) nothing essential from (A), (B), (C) is known to fail in higher dimensions; (ii) (A) and (B) have been proved in some special cases; (iii) (C) has been generalized a little, but the complete answer seems to involve knowledge of the discrete series representations of semisimple Lie groups (cf. Schmid [21]); and (iv) there are several totally new phenomena and many interesting problems which turn up.

Thus let $V \in \Sigma(V)$, so that we are given a projective embedding $V \subset P_N$ (technically, I am speaking about deformations of *polarized algebraic manifolds*). If $\gamma_1, \ldots, \gamma_b$ is a basis for $H_q(V, Z)$/(torsion), then we want to look at period matrices $(\int_{\gamma_\rho} \omega^\alpha)$ where $\omega^1, \ldots, \omega^m$ is a suitable set of differentials on V. By duality we may restrict q to $1 \leqq q \leqq n$. We could look at the holomorphic q-forms ω (such an ω is closed and is never exact; cf. Hodge [15]), but this will not be general enough for $n \geqq 3$. To see what we should use for differentials, we first look at

EXAMPLE 4. Let $S \subset P_2$ be a nonsingular plane curve of degree d given by $f(x, y) = 0$ in affine coordinates. For $x \in S$ there is a normal disc D_x with boundary ∂D_x. Thus as x traces out a one-cycle $\gamma \in H_1(S, \mathbf{Z})$, ∂D_x $(x \in \gamma)$ traces out a two-cycle $\tau(\gamma) \in H_2(\mathbf{P}_2 - S, \mathbf{Z})$:



this *tube mapping* $\tau: H_1(S, \mathbf{Z}) - H_2(\mathbf{P}_2 - S, \mathbf{Z})$ is an isomorphism, and we call the dual mapping $H^2(\mathbf{P}_2 - S) \xrightarrow{R} H^1(S)$ (the coefficients are $\mathbf{C}$ here) the *residue mapping*. The holomorphic differentials give a subspace $H^{1,0}(S) \subset H^1(S)$, and we want to know what the corresponding classes in $H^2(\mathbf{P}_2 - S)$ are.

Let ϕ be a rational two-form with a pole of order $k + 1$ along S; in the above coordinate system,

$$\phi = P(x, y)\, dx\, dy / f(x, y)^{k+1}$$

where $\deg P \leqq (k + 1)d - 3$. Since ϕ is holomorphic on $\mathbf{P}_2 - S$, ϕ gives a class in $H^2(\mathbf{P} - S)$. It is essentially a classical fact that the holomorphic differentials are the residues of the differentials $\phi = P(x, y)\, dx\, dy / f(x, y)$ with a first order pole along S.

EXAMPLE 5. To generalize Example 4, we let $V \subset \mathbf{P}_{n+1}$ be a nonsingular hypersurface of degree d given in affine coordinates by $f(x_1, \ldots, x_{n+1}) = 0$. The tube mapping $H_n(V) \xrightarrow{\tau} H_{n+1}(\mathbf{P}_{n+1} - V)$ exists as before and is essentially an isomorphism (technically, τ is an isomorphism on the *primitive part* of $H_n(V)$; cf. Hodge [15]). Again, as before, the cohomology group $H^{n+1}(P_{n+1} - V)$ is given by differentials

$$\phi = \frac{P(x) dx_1 \wedge \ldots \wedge dx_{n+1}}{f(x)^{k+1}} \qquad (\deg P \leqq d(k + 1) - (n + 2)),$$

taken modulo exact differentials. We let $F_k^n(V) \subset H^n(V)$ be the subspace of $H^n(V)$ given by the residues of classes ϕ with a pole of order $k + 1$ along V. In [12] it is proved that: (i) $F_0^n(V) \subset F_1^n(V) \subset \ldots$ and $F_n^n(V) = F_{n+1}^n(V) = \ldots = F_{n+l}^n(V)$ for all $l > 0$; and (ii) the subspaces $F_q^n(V)$ for $0 \leqq q \leqq [(n - 1)/2]$ determine all of the $F_k^n(V)$. Thus for $n = 1, 2$ we need only look at $F_0^n(V)$, but for $n = 3$ we must consider the two spaces $F_0^3(V) \subset F_1^3(V)$.

For this example $V \subset P_{n+1}$ we will use *all* of the subspaces $F_q^n(V)$, $0 \leqq q \leqq [(n - 1)/2]$, as holomorphic differentials in forming the period matrix of V; i.e. we replace a vector space by a *filtration*. Thus let $\omega^1, \ldots, \omega^m$ be a basis for $F^n_{[(n-1)/2]}(V)$ such that $\omega^1, \ldots, \omega^{m_q}$ is a basis for $F_q^n(V)$ $(0 \leqq q \leqq [(n - 1)/2])$; i.e. we choose a basis for the *flag* $F_0^n(V) \subset F_1^n(V) \subset \ldots \subset F^n_{[(n-1)/2]}(V)$. We may write $\omega^1 = R(\phi^1), \ldots, \omega^{m_q} = R(\phi^{m_q})$ where

$$\phi^\alpha = \frac{P_\alpha(x) dx_1 \ldots dx_{n+1}}{f(x)^{q+1}} \qquad (1 \leqq \alpha \leqq m_q).$$

Let

$$\Omega_q(V) = \begin{bmatrix} \int_{\gamma_1} \omega^1 & \ldots & \int_{\gamma_b} \omega^1 \\ \vdots & & \vdots \\ \int_{\gamma_1} \omega^{m_q} & \ldots & \int_{\gamma_b} \omega^{m_q} \end{bmatrix}$$

be the corresponding period matrix for $F_q^n(V)$, and $\Omega(V) = [\Omega_0(V), \ldots, \Omega_{[(n-1)/2]}(V)]$ be the total period matrix. As was the case for Riemann surfaces, we must allow the equivalences:

$$\begin{cases} \Omega \sim A\Omega \\ \Omega \sim \Omega \end{cases} \quad \text{where } A = \begin{pmatrix} A_0 & 0 \ldots 0 \\ * \ddots & \vdots \\ * \cdots * & A_{[(n-1)/2]} \end{pmatrix}$$

is a linear transformation on the flag $F_0^n(V) \subset \ldots \subset F^n_{[(n-1)/2]}(V)$, and $\Lambda = (\Lambda_\sigma^\rho)$ is an integral matrix satisfying $\Lambda Q {}^t\Lambda = Q$ where $Q = (\gamma_\rho \cdot \gamma_\sigma)$ is the intersection matrix on the primitive cycles.

If we have polynomials $f(x_1, \ldots, x_{n+1}; \lambda)$ depending holomorphically on λ, we let V_λ be defined by $f(x; \lambda) = 0$; this is the sort of deformations considered in (I'). Then the periods

$$\int_{\gamma_\rho} \omega = \int_{\tau(\gamma_\rho)} \frac{P_\alpha(x) dx_1 \wedge \ldots \wedge dx_{n+1}}{f(x; \lambda)^{q+1}}$$

clearly depend holomorphically on λ. Furthermore, $\Omega = \Omega(V)$ satisfies the *generalized Riemann bilinear relations* (cf. Hodge [15]):

(II'') $$\Omega Q {}^t\Omega = 0, \qquad c_n \Omega Q {}^t\bar{\Omega} > 0.$$

Here c_n is a suitable power of i so chosen that $H = c_n \Omega Q {}^t\bar{\Omega}$ is Hermitian (for $n = 1$, $c_1 = i$; for $n = 2$, $c_2 = 1$, etc.), and the second relation in (II'') shall mean that the Hermitian matrix H has certain prescribed positive-definiteness properties (cf. Hodge, loc. cit.). For $n = 1, 2$, H is simply positive definite; however, for $n = 3$,

$$H = \begin{pmatrix} H_0 & * \\ \cdot & \\ * & * \end{pmatrix}$$