DEFORMATIONS OF HOLOMORPHIC MAPPINGS

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The questions this paper seeks to answer are roughly the following:

(i) Given an arbitrary holomorphic mapping $f : M \to M'$ between fixed complex manifolds M, M' where M is compact, upon how many, if any, parameters can f depend?

(ii) If such parameters exist, what specifically are the deformations of the holomorphic mapping f?

We shall, under a cohomological restriction, answer (i) (Theorem 2); and, under the same cohomological restriction plus a restriction on f, give a solution to (ii) (Theorem 4). As examples show, the cohomological restriction is necessary; however, the restriction on f under (ii) is unsatisfactory and is probably unnecessary.

1.1. Let M and M' be two nonsingular complex manifolds of complex dimensions m and m' respectively, and assume that M is compact. Suppose that we are given a holomorphic mapping

$$f : M \to M'$$

and a complex space V with a distinguished point $v_0 \in V$.

DEFINITION 1. A deformation, with parameter space V, of the holomorphic mapping $f : M \to M'$ is given by a holomorphic mapping

$$g : M \times V \to M'$$

such that $g(m, v_0) = f(m)$ for all $m \in M$.

For each $v \in V$ we consider the holomorphic mapping

$$f_v : M \to M'$$

defined by $f_v(m) = g(m, v)$. Then $f_{v_0} = f$. Let $\{U_\alpha\}$, $\{U_j\}$ be open coordinate coverings of M, M' respectively, and let

$$Z_\alpha = (Z_\alpha^1, \cdots, Z_\alpha^m), \qquad W_j = (W_j^1, \cdots, W_j^{m'})$$

be holomorphic coordinates in U_α, U_j. The open sets $U_{\alpha,j} = U_\alpha \times U_j$ give a coordinate covering of $M \times M'$. If $g(U_\alpha \times V) \cap U_j \neq \emptyset$, then g is given as a mapping of $U_\alpha \times V$ to U_j by

$$W_j = g_{\alpha,j}(Z_\alpha, v) = (f_v)_{\alpha,j}(Z_\alpha), \tag{1.1}$$

and in particular

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$$(1.1)' \qquad W_j = g_{\alpha,j}(Z_\alpha, v_0) = f_{\alpha,j}(Z_\alpha).$$

Here, and from now on where applicable, the vector notation will be used.

1.2. Let Y be a complex manifold of dimension $n + n'$, and X a compact nonsingular complex submanifold of Y of dimension n. As above, let V be a nonsingular analytic space with a distinguished point $v_0 \in V$.

Definition 2. By a deformation, with parameter space V, of X in Y, we mean an analytic submanifold W of $Y \times Y$ of codimension n' satisfying the following:

(i) For each $v \in V$, the intersection $W \cap (Y \times \{v\})$ is a connected compact nonsingular submanifold of $Y \times \{v\}$ of dimension n,

(ii) $W \cap (Y \times \{v_0\}) = X \times \{v_0\}$ as a submanifold,

(iii) for each point $p \in W$, there exists a neighborhood

$$N_p = U_p \times V_p \subset Y \times V$$

with analytic coordinates $w = (w^1, \cdots, w^{n+n'})$ in U_p, v in V_p, and there exist n' holomorphic functions

$$f_1 = f_1(w, v), \quad \cdots, \quad f_{n'} = f_{n'}(w, v)$$

such that

$$\operatorname{rank} \partial(f_1, \cdots, f_{n'})/\partial(w^1, \cdots, w^{n+n'}) = n',$$

and finally

$$N_p \cap W = \{(w, v) \mid f_1(w, v) = 0 = \cdots = f_{n'}(w, v)\}.$$

Let $X \subset Y$ be as above. We may choose a covering $\{U_\alpha\}$ of X in Y such that

(i) $\bigcup_\alpha U_\alpha = U$ is a neighborhood of X in Y,

(ii) $U_\alpha \cap U_\mu = \emptyset \Leftrightarrow U_\alpha \cap U_\mu \cap X = \emptyset$,

(iii) there exist holomorphic coordinates

$$(Z_\alpha, W_\alpha) = (Z_\alpha^1, \cdots, Z_\alpha^n; W_\alpha^1, \cdots, W_\alpha^{n'})$$

in U_α such that $U_\alpha \cap X = \{(Z_\alpha, 0)\}$.

The transition functions from U_μ to U_α may be written

$$(1.2) \qquad Z_\alpha = g_{\alpha\mu}(Z_\mu, W_\mu),$$

$$(1.2)' \qquad W_\alpha = h_{\alpha\mu}(Z_\mu, W_\mu) \qquad \text{where } h_{\alpha\mu}(Z_\mu, 0) = 0.$$

Then a family W as described in Definition 2 is given as follows: In each U_α we have a $\mathbf{C}^{n'}$-valued holomorphic function $\varphi_\alpha = \varphi_\alpha(Z_\alpha, v)$ such that

(i) $\varphi_\alpha(Z_\alpha, 0) = 0$,

(ii) if X_v is defined by $W \cap Y_v \times \{v\} = X_v \times \{v\}$, then $X_v \cap U_\alpha$ is given by the set $\{(Z_\alpha, W_\alpha) \in U_\alpha \mid W_\alpha = \varphi_\alpha(Z_\alpha, v)\}$.

The equations of consistency in passing from U_μ to U_α may be written as

$$\varphi_\alpha(Z_\alpha, v) = h_{\alpha\mu}(Z_\mu, \varphi_\mu(Z_\mu, v)), \tag{1.3}$$

$$Z_\alpha = g_{\alpha\mu}(Z_\mu, \varphi_\mu(Z_\mu, v)), \tag{1.4}$$

or combined into

$$\varphi_\alpha(g_{\alpha\mu}(Z_\mu, \varphi_\mu(Z_\mu, v)), v) = h_{\alpha\mu}(Z_\mu, \varphi_\mu(Z_\mu, v)). \tag{1.5}$$

Let, for fixed $v \in V$,

$$\zeta_{\alpha,v} = \zeta_{\alpha,v}(Z_\alpha, W_\alpha, v) = W_\alpha - \varphi_\alpha(Z_\alpha, v);$$

then the functions $(Z_\alpha, \zeta_{\alpha,v})$ give a new coordinate system in U_α such that $X_v \cap U_\alpha = \{(Z_\alpha, 0)\}$. This is true since

$$\frac{\partial(Z_\alpha, \zeta_{\alpha,v})}{\partial(Z_\alpha, W_\alpha)} = \begin{pmatrix} I & 0 \\ -\partial\varphi_\alpha/\partial Z_\alpha & I \end{pmatrix}.$$

1.3. Let X, Y be as in §1.2, and follow the notation established there. The matrices

$$N_{\alpha\mu}(Z) = \partial W_\alpha(Z_\mu, W_\mu)/\partial W_\mu]_{W=0} = \partial h_{\alpha\mu}/\partial W_\mu]_{W=0}$$

satisfy $N_{\mu\xi} = N_{\alpha\mu} \circ N_{\mu\xi}$ and define the normal bundle $\mathbf{N}_Y$ of X in Y. Differentiating (1.3), we have, on setting

$$\psi_\alpha = \partial\varphi_\alpha/\partial\bar{v}]_{v_0} \tag{1.6}$$

where $\bar{v}$ is any direction[1] in V at v_0, that $\psi_\alpha = N_{\alpha\mu} \circ \psi_\mu$, and thus the collection $\psi = \{\psi_\alpha\}$ defines an element of $H^0(X, \mathfrak{N}_Y)$ where $\mathfrak{N}_Y$ is the analytic sheaf associated with $\mathbf{N}_Y$. (In general, the analytic sheaf associated with a holomorphic vector bundle $\mathbf{B}$ will be denoted by $\mathfrak{B}$.) The process (1.6) gives a linear mapping

$$\Delta : \mathbf{T}_{v_0}(V) \to H^0(X), \mathfrak{N}_Y) \tag{1.7}$$

of the holomorphic tangent space to V at v_0 into $H^0(X, \mathfrak{N}_Y)$. The converse question of "integrating" elements of $H^0(X, \mathfrak{N}_Y)$ so as to give a variation of X in Y has been taken up in [2], and we may state the result as follows:

THEOREM (Kodaira). *Let the assumptions and notations be as in* §§1.2 *and* 1.3. *If $H^1(X, \mathfrak{N}_Y) = 0$, then there exists a locally complete analytic family X_v $(v \in V)$ of compact submanifolds of Y such that $X_{v_0} = X$ and the mapping* (1.7) *is an isomorphism.*

Remark. The method of proof of this theorem is to construct, using the fact that $H^1(X, \mathfrak{N}_Y) = 0$ implies no obstructions, a formal variation of X in Y through any vector in $H^0(X, \mathfrak{N}_Y)$, and then to show that this formal solution converges.

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[1] By this we mean the following: If $v^1, \cdots, v^r$ are coordinates in a neighborhood of v_0 in V_1, then a direction at v_0 is given by $\bar{v} = (\bar{v}^1, \cdots, \bar{v}^r)$, and

$$\partial\varphi_\alpha/\partial\bar{v}]_{v_0} = \sum \bar{v}^i\, \partial\varphi_\alpha/\partial v^i]_{v_0}.$$

1.4. Return now to the situation of §1.1 where we have M, M' and $f : M \to M'$. The set $G_f = \{(m, f(m)) \in M \times M' \mid m \in M\}$ is the *graph* of f, and it is a nonsingular subvariety of dimension m of $M \times M'$. The open set

$$U = \bigcup U_\alpha \times U_j \qquad \text{(union over } \alpha, j \text{ such that } f(U_\alpha) \cap U_j \neq \emptyset,)$$

is an open neighborhood of G_f in $M \times M'$. On $U_{\alpha,j} = U_\alpha \times U_j$ we have coordinates (Z_α, W_j), and $G_f \cap U_{\alpha,j} = \{(Z_\alpha, W_j) \mid W_j = f_{\alpha,j}(Z_\alpha)\}$. Setting $\zeta_{\alpha,j} = \zeta_{\alpha,j}(Z_\alpha, W_j) = W_j - f_{\alpha,j}(Z_\alpha)$, we have new coordinates $(Z_\alpha, \zeta_{\alpha,j})$ in $U_{\alpha,j}$ such that $G_f \cap U_{\alpha,j} = \{(Z_\alpha, 0)\}$. The object of this section is to prove the following:

THEOREM 1. *There is a local* (1-1) *correspondene between variations of the analytic function* $f : M \to M'$ *(Definition* 1*) and deformations of the graph* G_f *as a submanifold of* $M \times M'$.

Proof. Assume that $g : M \times V \to M'$ gives a deformation of f. Then we may take in Definition 2, $Y = M \times M'$, $X = G_f$, and $W = G_g$ upon identifying $M \times M' \times V$ with $M \times V \times M'$.

Before proving the converse, we make some remarks. If the transition functions from U_β to U_α on M are $\tau_{\alpha\beta}$, and if the transition functions from U_j to U_i on M' are σ_{ij}, then the transition functions from $U_{\beta,j}$ to $U_{\alpha,i}$ on $M \times M'$ are $(\tau_{\alpha\beta}, \sigma_{ij})$. This is all in terms of the (Z, W) coordinate systems. If we let $(1, q_{\beta,j}) : U_{\beta,j} \to U_{\beta,j}$ be defined by

$$(1, q_{\beta,j})(Z_\beta, W_j) = (Z_\beta, W_j - f_{\beta,j}(Z_\beta)),$$

then $(1, q_{\beta,j})$ introduces new coordinates in $U_{\beta,j}$, and the transformation from $(Z_\beta, \zeta_{\beta,j})$ to $(Z_\alpha, \zeta_{\alpha,i})$ may be written as

$$Z_\alpha = \tau_{\alpha\beta}(Z_\beta), \qquad \zeta_{\alpha,i} = \hat{\sigma}_{\alpha\beta,ij}(Z_\beta, \zeta_{\beta,j}).$$

Then the following diagram is commutative:

$$(1.8) \qquad \begin{array}{ccc} (Z_\beta, \zeta_{\beta,j}) & \xrightarrow{(\tau_{\alpha\beta}, \hat{\sigma}_{\alpha\beta,ij})} & (Z_\alpha, \zeta_{\alpha,i}) \\ {\scriptstyle (1, q_{\beta,j})}\uparrow & & \uparrow{\scriptstyle (1, q_{\alpha,i})} \\ (Z_\beta, W_j) & \xrightarrow{(\tau_{\alpha\beta}, \sigma_{i,j})} & (Z_\alpha, W_i) \end{array}$$

Now $G_f \cap U_{\alpha,i}$ is defined by $\zeta_{\alpha,i} = 0$, and, as in §1.2, a deformation of G_f is given by functions

$$\varphi_{\alpha,i} = \varphi_{\alpha,i}(Z_\alpha, v) \qquad (v \in V),$$

and then $(G_{f_v}) \cap (U_\alpha \times U_i)$ is given by $\zeta_{\alpha,i} = \varphi_{\alpha,i}(Z_\alpha, v)$. We have, by (1.3)–(1.5), that

$$\varphi_{\alpha,i}(Z_\alpha, v) = \hat{\sigma}_{\alpha\beta,ij}(Z_\beta, \varphi_{\beta,j}(Z_\beta, v)), \tag{1.9}$$

$$\varphi_{\alpha,i}(\tau_{\alpha\beta}(Z_\beta), v) = \hat{\sigma}_{\alpha\beta,ij}(Z_\beta, \varphi_{\beta,j}(Z_\beta, v)). \tag{1.10}$$

We may define *local* functions

$$(f_v)_{\beta,j} : U_\beta \to U_j$$

by $(f_v)_{\beta,j}(Z_\beta) = \varphi_{\beta,j}(Z_\beta, v) + f_{\beta,j}(Z_\beta)$. If we can sew these local functions together on both M and M', then we will have defined global functions $f_v : M \to M'$ ($v \in V$), and from these a holomorphic function $g : M \times V \to M'$ such that $g \mid M \times v_0 = f$, which will prove the theorem. In order to sew these functions together, we must show

$$(f_v)_{\alpha,i} = \sigma_{ij}((f_v)_{\alpha,j}), \tag{1.11}$$

$$(f_v)_{\beta,i} = (f_v)_{\alpha,i} \circ \tau_{\alpha\beta}. \tag{1.12}$$

For (1.11), we have

$$\begin{aligned} \sigma_{ij}((f_v)_{\alpha,j}(Z_\alpha)) &= q_{\alpha,i}^{-1} \hat{\sigma}_{\alpha\alpha,ij}(Z_\alpha, q_{\alpha,j}((f_v)_{\alpha,j}(Z_\alpha))) && \text{(by (1.8))} \\ &= q_{\alpha,i}^{-1} \hat{\sigma}_{\alpha\alpha,ij}(Z_\alpha, \varphi_{\alpha,j}(Z_\alpha, v)) \\ &= q_{\alpha,i}^{-1} \varphi_{\alpha,i}(Z_\alpha, v) && \text{(by (1.9))} \\ &= (f_v)_{\alpha,i}(Z_\alpha). \end{aligned}$$

For (1.12), we go in reverse order: Since $(f_v)_{\alpha,i} = \varphi_{\alpha,i}(\ , v) + f_{\alpha,i}$, and since $f_{\beta,i} = f_{\alpha,i} \circ \tau_{\alpha\beta}$, it will suffice to show that $\varphi_{\beta,i} = \varphi_{\alpha,i} \circ \tau_{\alpha\beta}$. By (1.10), we have

$$\begin{aligned} \varphi_{\alpha,i}(\tau_{\alpha\beta}(Z_\beta), v) &= \hat{\sigma}_{\alpha\beta,ii}(Z_\beta, \varphi_{\beta,i}(Z_\beta, v)) \\ &= q_{\alpha,i} \sigma_{ii} q_{\beta,i}^{-1}(\varphi_{\beta,i}(Z_\beta, v)) \\ &= q_{\beta,i} \sigma_{ii}(\varphi_{\beta,i}(Z_\beta, v) + f_{\beta,i}(Z_\beta)) \\ &= q_{\beta,i}(\varphi_{\beta,i}(Z_\beta, v) + f_{\beta,i}(Z_\beta)) \\ &= \varphi_{\beta,i}(Z_\beta, v) + f_{\beta,i}(\tau_{\alpha\beta}(Z_\beta)) \\ &= \varphi_{\beta,i}(Z_\beta, v), && \text{Q.E.D.} \end{aligned}$$

1.5. We shall see in §2 below that, under the natural biregular correspondence between M and G_f, the normal bundle to G_f in $M \times M'$ corresponds to the bundle $f^{-1}(\mathbf{T}')$ over M where $\mathbf{T}'$ is the holomorphic tangent bundle to M'. Combining §§1.3 and 1.4, we have

THEOREM 2. *Assume that $H^1(M, f^{-1}(\mathfrak{T}')) = 0$. Then there exists an analytic family of maps*

$$f_v : M \to M' \qquad (v \in V)$$

such that $f_{v_0} = f$, and such that the mapping Δ of §1.3 carries $\mathbf{T}_{v_0}(V)$ iso-

morphically onto $H^0(M, f^{-1}(\mathfrak{Z}'))$. *The mapping* Δ *may be described as follows*: *If* $\bar{v} \in \mathbf{T}_{v_0}(V)$, *then*

$$\Delta(\bar{v}) \in H^0(M, f^{-1}(\mathfrak{Z}')) \cong H^0(f(M), \mathfrak{Z}' \mid f(M))$$

is given by the holomorphic vector field $\partial f_v / \partial \bar{v}]_{v_0}$.

2.1. We shall now examine what relation the normal bundle to the graph has to invariants of the mapping, and then in §3 we shall use this information to strengthen and interpret Theorem 2.

Let M, M' be topological spaces and $f : M \to M'$ a continuous proper mapping. Let $\mathbf{V}$, $\mathbf{V}'$ be vector bundles over M, M' such that we have

$$\begin{array}{ccc} \mathbf{V} & \xrightarrow{\quad f \quad} & \mathbf{V}' \\ \downarrow & & \downarrow \\ M & \xrightarrow{\qquad} & M', \end{array}$$

and let $d = \dim \mathbf{V}$, $d' = \dim \mathbf{V}'$ (fibre dimensions in each case).

For any vector bundle $\mathbf{B}$ on $M \times M'$, we let $\mathbf{B}^\wedge = \mathbf{B} \mid G_f$ where $G_f \subset M \times M'$ is the graph of f. Both $\mathbf{V}$ and $\mathbf{V}'$ may be considered as vector bundles on $M \times M'$, and define a subset $\mathbf{D}_f \subset (\mathbf{V} \oplus \mathbf{V}')^\wedge$ as follows:

DEFINITION 3. Over $(m, f(m)) \in G_f$, we set

$$(\mathbf{D}_f)_{(m, f(m))} = \{(v, f(v)) \mid v \in \mathbf{V}_m\}.$$

It is immediate that $(\mathbf{D}_f)_{(m, f(m))} \subset (\mathbf{V} \oplus \mathbf{V}_{(m, f(m))})^\wedge$ is a subspace of dimension d, and in fact $\mathbf{D}_f$ is a d-dimensional subbundle of $(\mathbf{V} \oplus \mathbf{V}')^\wedge$; we have

$$\begin{array}{ccc} 0 \to \mathbf{D}_f & \to & (\mathbf{V} \oplus \mathbf{V}')^\wedge \\ \downarrow & & \downarrow \\ G_f & \xrightarrow{\text{id}} & G_f . \end{array}$$

There is a projection $\pi_M : M \times M' \to M$, and $\pi_M \mid G_f$ is a bi-map with inverse j_M defined by

$$j_M(m) = (m, f(m)) \in G_f \qquad \text{for } m \in M.$$

Denoting by $\pi_M^{-1}(\mathbf{V})$ the induced bundle over G_f, we have

$$\begin{array}{ccc} \pi_M^{-1}(\mathbf{V}) & \xrightarrow{\pi_M} & \mathbf{V} \\ \downarrow & & \downarrow \\ G_f & \xrightarrow{\pi_M} & M. \end{array}$$

PROPOSITION 1. *There is a natural bundle isomorphism*

$$\pi_M^{-1}(\mathbf{V}) \cong \mathbf{D}_f .$$

Proof. By definition, $\pi_M^{-1}(\mathbf{V}) \subset G_f \times \mathbf{V}$ consists of the objects $\langle (m, f(m)), v\rangle$ such that $v \in \mathbf{V}_m$. Define

$$\sigma : \pi_M^{-1}(\mathbf{V}) \to \mathbf{D}_f \quad \text{by} \quad \sigma\langle (m, f(m)), v\rangle = (v, f(v)) \in (\mathbf{D}_f)_{(m, f(m))} ;$$

then σ is an isomorphism.

2.2. We now consider $f^{-1}(\mathbf{V}') \subset M \times \mathbf{V}'$ consisting of the set

$$\{ (m, v') \mid v' \in \mathbf{V}'_{f(m)}\}.$$

There is a bundle mapping $\tau : \mathbf{V} \to f^{-1}(\mathbf{V}')$ defined by

$$\tau(v) = (m, f(v)) \in f^{-1}(\mathbf{V}')_m \quad \text{if} \quad v \in \mathbf{V}_m .$$

DEFINITION 4. We define a subset $\mathbf{D}'_f \subset (f^{-1}(\mathbf{V}') \oplus \mathbf{V}')^{\wedge}$ as follows: $(\mathbf{D}'_f)_{(m, f(m))}$ are objects of the form

$$\langle (m, v'), -v'\rangle \in (f^{-1}(\mathbf{V}') \oplus \mathbf{V}'_{(m, f(m))})^{\wedge}.$$

Clearly $\mathbf{D}'_f$ is a d'-dimensional vector bundle over G_f.

PROPOSITION 2. *There is a natural isomorphism*

$$\pi_M^{-1}(f_M^{-1}(\mathbf{V}')) \cong \mathbf{D}'_f .$$

Proof. We have a commutative diagram

$$\begin{array}{ccc} G_f & \xrightarrow{\pi_{M'}} & M' \\ \downarrow{\scriptstyle \pi_M} & \nearrow{\scriptstyle f} & \\ M & & \end{array}$$

and it will suffice to show that $(\pi_{M'}^{-1})(\mathbf{V}') = \mathbf{D}_f$. Now

$$G_f \times \mathbf{V}' \supset (\pi_{M'}^{-1})(\mathbf{V}') = \{\langle (m, f(m)), v'\rangle \mid v' \in \mathbf{V}'_{(m)}\}.$$

If we define

$$\eta : (\pi_{M'}^{-1})(\mathbf{V}') \to \mathbf{D}'_f \quad \text{by} \quad \eta\, \langle (m, f(m)), v'\rangle = \langle (m, v'), -v\rangle,$$

then η is an isomorphism.

2.3. We keep the hypotheses of §2.1 together with the additional assumption that $f : \mathbf{V} \to \mathbf{V}'$ is fibrewise surjective.

PROPOSITION 3. *There is a bundle mapping* $\varphi : (\mathbf{V} \oplus \mathbf{V}')^{\wedge} \to \mathbf{D}'_f$ *such that we have*

$$0 \to \mathbf{D}_f \to (\mathbf{V} \oplus \mathbf{V}')^{\wedge} \xrightarrow{\varphi} \mathbf{D}'_f \to 0.$$

Proof. By assumption, any pair $(v, v') \in (\mathbf{V} \oplus \mathbf{V}'_{(m, f(m))})^{\wedge}$ may be *non-uniquely* written as $(v, v') = (v, f(\tilde{v}))$ $(\tilde{v} \in \mathbf{V}_m)$. Then we have

$$(v, f(\tilde{v})) = \tfrac{1}{2}(v + \tilde{v}, f(v + \tilde{v})) + \tfrac{1}{2}(v - \tilde{v}, f(-v + \tilde{v})),$$

and we define

$$\varphi(v, v') = \langle (m, f(-v + \tilde{v}), f(v - \tilde{v})\rangle \in (\mathbf{D}'_f)_{(m, f(m))}\,.$$

It is easily seen that φ is surjective and well defined, and, if $\varphi(v, v') = 0$, then $f(v) = f(\tilde{v}) = v'$, and thus $(v, v') = (v, f(v)) \in (\mathbf{D}_f)_{(m, f(m))}$, which proves exactness.

2.4. We now assume that M, M' are ringed spaces with sheaves of rings $\mathcal{O}$, $\mathcal{O}'$ and that f is a morphism. We also suppose that the vector bundles $\mathbf{V}$, $\mathbf{V}'$ are over $\mathcal{O}$, $\mathcal{O}'$. For any $\mathcal{O}$-bundle $\mathbf{W}$ on M, we denote by $\mathcal{W}$ the corresponding $\mathcal{O}$-sheaf, the same for M'. Then $\mathcal{V}$, $\mathcal{V}'$ are $\mathcal{O}$-, $\mathcal{O}'$-coherent respectively, we have

$$\begin{array}{ccc} \mathcal{V} & \longrightarrow & \mathcal{V}' \\ \downarrow & & \downarrow \\ M & \longrightarrow & M', \end{array}$$

and $f(\mathcal{V})$ is an $\mathcal{O}'$-coherent subsheaf of $\mathcal{V}'$. We denote the quotient sheaf $\mathcal{V}'/f(\mathcal{V})$ by $\mathfrak{N}$; $\mathfrak{N}$ is $\mathcal{O}'$-coherent and we have

$$0 \to f(\mathcal{V}) \to \mathcal{V}' \to \mathfrak{N} \to 0.$$

We observe that we could have carried through §§2.1–2.3 on the coherent sheaf level, and Proposition 3 still holds with $\mathbf{V}'$ replaced by $f(\mathcal{V})$. Thus we have

$$0 \to \mathfrak{D}_f \to (\mathcal{V} \oplus f(\mathcal{V}))^\wedge \to \mathfrak{D}'_f \to 0.$$

PROPOSITION 4. *The following is a commutative diagram*:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & \mathfrak{D}_f & \longrightarrow & \mathfrak{D}_f & \longrightarrow & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & (\mathcal{V} \oplus f(\mathcal{V}))^\wedge & \to & (\mathcal{V} \oplus \mathcal{V}')^\wedge & \to & \pi_{M'}^{-1}(\mathfrak{N}) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{D}'_f & \longrightarrow & \mathfrak{Q} & \longrightarrow & \pi_{M'}^{-1}(\mathfrak{N}) & \to & 0. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

2.5. Let now M, M' be as in §1.1; $\mathcal{O}$, $\mathcal{O}'$ are sheaves of germs of holomorphic functions on M, M' respectively. We take $\mathbf{V} = \mathbf{T} = \mathbf{T}(M)$ and $\mathbf{V}' = \mathbf{T}' = \mathbf{T}(M')$ where $\mathbf{T}(\ )$ = holomorphic tangent bundle of $(\ )$; we have

$$\begin{array}{ccc} \mathfrak{T} & \xrightarrow{\ f\ } & \mathfrak{T}' \\ \downarrow & & \downarrow \\ M & \xrightarrow{\ f\ } & M'. \end{array}$$

Since f is a proper holomorphic mapping, $f(M)$ is a (perhaps singular) subvariety of M'; $f(\mathfrak{I}') =$ (notation) $\mathfrak{I}_f =$ tangents to $f(M)$, and $\mathfrak{N}$ is the normal sheaf of $f(M)$ in M'. Both $f(\mathfrak{I}')$ and $\mathfrak{N}$ are $\mathcal{O}'$-coherent, but are not necessarily $\mathcal{O}'$-free. The concept of tangents and normals to singular analytic spaces is taken up in [1], and it is in this sense that we speak of these objects.

We observe furthermore that $\mathbf{D}_f = \mathbf{T}(G_f)$, and thus $\mathfrak{D}_f = \mathfrak{I}(G_f)$; $\mathfrak{Q}$ is a free sheaf and is in fact the normal sheaf of G_f in $M \times M'$. This is because, if $X \subset Y$ is any nonsingular subvariety, the normal bundle $\mathbf{N}_Y$ is defined by $0 \to \mathbf{T}(X) \to \mathbf{T}(Y) \mid X \to \mathbf{N}_Y \to 0$, and

$$(\mathbf{T}(X) \oplus \mathbf{T}(Y))^{\wedge} = \mathbf{T}(X \times Y) \mid G_f .$$

THEOREM 3. *Let $f : M \to M'$ be as in* §1.1, *and let $\mathfrak{I}_f$ be the sheaf of holomorphic tangents to $f(M)$, $\mathfrak{Q}$ the normal sheaf of G_f in $M \times M'$, and $\mathbf{N}$ the normal sheaf of $f(M)$ in M'. Then $j_M^{-1}(\mathfrak{Q}) = f^{-1}(\mathfrak{I}')$, and we have*

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathfrak{D}_f' & \longrightarrow & \mathfrak{Q} & \longrightarrow & \pi_{M'}^{-1}(\mathfrak{N}) & \to 0 \\ & \downarrow j_M^{-1} & & \downarrow j_M^{-1} & & \downarrow j_M^{-1} & \\ 0 \to & f^{-1}(\mathfrak{I}_f) & \to & f^{-1}(\mathfrak{I}') & \to & f^{-1}(\mathfrak{N}) & \to 0. \end{array}$$

Finally,

$$H^q(G_f, \mathfrak{Q}) \cong H^q(M, f^{-1}(\mathfrak{I}')).$$

Proof. Since $f = \pi_{M'} \circ j_M$, the theorem will follow from Propositions 2 and 4 when we prove that $j_M^{-1}(\mathfrak{Q}) = f^{-1}(\mathfrak{I}')$. Letting $\mathbf{N}$ be the normal bundle of G_f in $M \times M'$, we shall show that $j_M^{-1}(\mathbf{N}) \cong f^{-1}(\mathbf{T}')$. Let $\{U_\alpha\}$, $\{U_j\}$, $\{Z_\alpha\}$, $\{W_j\}$, and $\zeta_{\alpha,j} = W_j - f_{\alpha,j}(Z_\alpha)$ be as in §1. Then $\mathbf{N}$ has transition functions

$$\psi_{ij}^{\alpha\beta} = \partial(\zeta_{\alpha,i})/\partial(\zeta_{\beta,j})]_{\zeta=0} \qquad \text{(see §1.3).}$$

We have the local maps:

$$\begin{aligned} (1, q_{\beta,j}^{-1}) &: (Z_{\beta,j}\, \zeta_{\beta,j}) \to (Z_\beta, W_j), \\ (\tau_{\alpha\beta}, \sigma_{i,j}) &: (Z_\beta, W_j) \to (Z_\alpha, W_i), \\ (1, q_{\alpha,i}) &: (Z_\alpha, W_i) \to (Z_\alpha, \zeta_{\alpha,i}) \qquad \text{(see (1.18)).} \end{aligned}$$

Denoting the holomorphic Jacobian of any analytic mapping $\xi : \mathbf{C}^r \to \mathbf{C}^r$ by $J(\xi)$, we have that

$$\begin{aligned} J(\tau_{\alpha\beta}, \hat{\sigma}_{\alpha\beta,ij}) &= J(1, q_{\alpha,i}) J(\tau_{\alpha\beta}, \sigma_{ij}) J(1, q_{\beta,j}^{-1}) \\ &= \begin{pmatrix} I & -\partial f_{\alpha,i}/\partial Z_\alpha \\ 0 & I \end{pmatrix} \begin{pmatrix} J(\tau_{\alpha\beta}) & 0 \\ 0 & J(\sigma_{ij}) \end{pmatrix} \begin{pmatrix} I & \partial f_{\beta,j}/\partial Z_\beta \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} J(\tau_{\alpha\beta}) & J(\tau_{\alpha\beta})(\partial f_{\beta,j}/\partial Z_\beta) - (\partial f_{\alpha,i}/\partial Z_\alpha) J(\sigma_{i,j}) \\ 0 & J(\sigma_{ij}) \end{pmatrix}. \end{aligned}$$

Thus

$$\partial(\zeta_{\alpha,i})/\partial(\zeta_{\beta,1})]_{\zeta=0} = J(\sigma_{ij})]_{f(U_\alpha)\cap f(U_\beta)\cap U_i\cap U_j}\,.$$

By checking through the definition of the induced bundle, we see that we are done.

3.1. From Theorems 2 and 3, the cohomology group in which we are interested is $H^0(M, f^{-1}(\mathfrak{Z}'))$. We have

$$\begin{array}{ccccccccc} 0 \rightarrow & f^{-1}(\mathfrak{Z}_f) & \rightarrow & f^{-1}(\mathfrak{Z}') & \rightarrow & f^{-1}(\mathfrak{N}) & \rightarrow 0 \\ & \uparrow & & \uparrow & & \updownarrow & \\ 0 \rightarrow & \mathfrak{Z}_f & \rightarrow & \mathfrak{Z}' \,|\, f(M) & \rightarrow & \mathfrak{N} & \rightarrow 0 \end{array} \tag{3.1}$$

and

$$\begin{array}{ccccccc} 0 \rightarrow H^0(M,f^{-1}(\mathfrak{Z}_f)) & \rightarrow & H^0(M,f^{-1}(\mathfrak{Z}')) & \rightarrow & H^0(M,f^{-1}(\mathfrak{N})) & \xrightarrow{\bar\delta} & H^1(M,f^{-1}(\mathfrak{Z}_f)) \\ \uparrow & & \uparrow & & \uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\rho} \\ 0 \rightarrow H^0(f(M),\mathfrak{Z}_f) & \rightarrow & H^0(f(M),\mathfrak{Z}'\,|\,f(M)) & \rightarrow & H^0(f(M),\mathfrak{N}) & \xrightarrow{\delta} & H^1(f(M),\mathfrak{Z}_f). \end{array} \tag{3.2}$$

We then have that

$$H^0(M, f^{-1}(\mathfrak{Z}')) = A \oplus B, \tag{3.3}$$

where

$$A = H^0(M, f^{-1}(\mathfrak{Z}_f)) \quad \text{and} \quad B = \ker\bar\delta \subseteq H^0(M, f^{-1}(\mathfrak{N})). \tag{3.4}$$

3.2. In Kodaira's theory, where seeking the ways of embedding X in Y, he assumed that $H^1(X, \mathfrak{N}_Y) = 0$ where $\mathfrak{N}_Y$ is the normal sheaf of X of Y. In our context, we have split up $\mathfrak{N}_Y = f^{-1}(\mathfrak{Z}')$ by the exact sequence (3.1); thus we shall replace Kodaira's assumption by two assumptions, one each to insure no obstructions in A and B. These two hypotheses taken together will be weaker than the assumption of Theorem 2.

We have a diagram

$$\begin{array}{ccc} \mathfrak{Z} & \longrightarrow & \mathfrak{Z}_f \\ \downarrow & & \downarrow \\ M & \xrightarrow{\quad f \quad} & f(M), \end{array}$$

and there is an induced mapping

$$\begin{array}{l} \bar f : \mathfrak{Z} \rightarrow f^{-1}(\mathfrak{Z}_f) \rightarrow 0 \\ \qquad\ \downarrow \qquad \downarrow \\ \qquad M \xrightarrow{\text{id}} M. \end{array}$$

We let $\mathfrak{K}_f = \ker \bar f$; then $\mathfrak{K}_f$ is $\mathfrak{O}$-coherent, and we have

$$0 \to \mathcal{K}_f \to \mathfrak{I} \to f^{-1}(\mathfrak{I}_f) \to 0.$$

Geometrically, $\mathcal{K}_f$ is the sheaf of holomorphic tangents to the fibres of the mapping $f : M \to M$.

(3.5) *Assumption* 1. $H^1(M, \mathcal{K}_f) = 0$.

Assumption 1 is satisfied in the following cases:
(i) f is an injection or a covering map;
(ii) f is the projection in an analytic fibration

$$F \to M \xrightarrow{f} f(M),$$

and either (α) or (β) below is satisfied:
(α) $H^1(F, \mathfrak{I}(F)) = 0 = H^1(F, \mathfrak{I}(F))$,
(β)[2] $H^1(f(M), \Theta(f(M))) = 0 = H^1(F, \mathfrak{I}(F))$.
$(\Theta(f(M)) = \Theta'/\mathcal{I}(f(M))$ where $\mathcal{I}(f(M))$ is the sheaf of ideals of $f(M)$.)
Both (α) and (β) are satisfied in a large number of examples.
(iii) f is a *blowing down* map collapsing finitely many subvarieties $V_1, \cdots, V_s$ of M, each having the property that $H^1(V_j, \mathfrak{I}(V_j)) = 0$. Such is the case with the usual blowing down maps.
There are, of course, important cases where (3.5) fails.

DEFINITION 5. Let $R \subset H^0(M, \mathfrak{I})$ be the set

$$\{\theta \in H^0(M, \mathfrak{I}) \mid f(m) = f(\bar{m}) \Rightarrow f(\theta_m) = f(\theta_{\bar{m}});\, m, \bar{m} \in M\}.$$

Geometrically, R is the set of all holomorphic vector fields which are constant along the fibres of the mapping

$$f : M \to M'.$$

PROPOSITION 5. *R is a complex subalgebra of $H^0(M, \mathfrak{I})$. Under the mapping $\bar{f} : H^0(M, \mathfrak{I}) \to H^0(M, f^{-1}(\mathfrak{I}_f)$,*

$$\bar{f}(R) = \bar{f}(H^0(M, \mathfrak{I})),$$

and, if (3.5) *is satisfied,*

$$\bar{f}(R) = H^0(M, f^{-1}(\mathfrak{I}_f)).$$

Assuming (3.5), *if $\theta \in R$, we let $g_t = \exp t\theta$, $h_t = \exp t(f\theta)$. Then*

$$f \circ g_t = h_t \circ f.$$

This proposition follows easily from the exact cohomology sequence of

$$0 \to \mathcal{K}_f \to \mathfrak{I} \to f^{-1}(\mathfrak{I}_f) \to 0.$$

We may interpret the results of this paragraph in the following statement:

I. *Let A be as in* §3.1, (3.4), *and suppose that* (3.5) *is satisfied. Then the deformations of the holomorphic mapping $f : M \to M'$ which are infinitesimally*

[2] We assume that $\pi_1(f(M))$ acts trivially on the fibres.

parametrized by A constitute the complex subgroup A_f of[3] *$A^0(M)$ consisting of $a \in A^0(M)$ which leave fixed the fibres of the mapping f. If a_t is a curve in A_f, then the corresponding deformation is given by the maps $g_t = f \circ a_t$.*

3.3. We return to the situation of §3.1. Corresponding to B, we have

(3.6) *Assumption* 2. $H^1(f(M), \mathfrak{N}) = 0.$

(Assumption 2 is satisfied, e.g., if $\mathfrak{N}$ is a positive sheaf in the sense of [1].) Here we encounter a problem. The result of Kodaira (§1.3) held under the assumption that X was nonsingularly embedded in Y. However, as his proof did not involve any derivatives, harmonic theory, or the like, presumably the result might hold when X is singular in Y.[4]

We wish to apply Kodaira's result not only to $G_f \subset M \times M'$, but also to $f(M) \subset M'$, where the restriction of nonsingularity of $f(M)$ in M' severely limits f. Thus we shall make

(3.7) *Assumption* N.S. *$f(M)$ is nonsingularly embedded in M'.*[4]

We now determine $B = \ker \delta$ in (3.2).

PROPOSITION 6. *There is a surjective mapping*

$$\rho : H^1(f(M), \mathfrak{J}_f) \oplus H^0(f(M), \Sigma) \to H^1(M, f^{-1}(\mathfrak{J}_f)) \to 0$$

where $\Sigma \to f(M)$ is given by the presheaf

$$U \to H^1(f^{-1}(U), f^{-1}(\mathfrak{J}_f) \mid f^{-1}(U)) \qquad \text{for } U \subset f(M).$$

Furthermore, $\rho \mid H^1(f(M), \mathfrak{J}_f)$ is injective, and this ρ is the same as that given in (3.2).

Proof. The proof follows immediately from the Leray Spectral Sequence.

COROLLARY. *In the notation of* (3.2),

$$\bar{\delta} = \rho \circ \delta \quad \text{and} \quad \rho \ker \bar{\delta} = \alpha(\ker \delta).$$

Under the assumptions (3.6) and (3.7), the subspace B has an interpretation which we now discuss. Under the mapping

$$\delta : H^0(f(M), \mathfrak{N}) \to H^1(f(M), \mathfrak{J}_f),$$

we may, by choosing a complement, write

$$H^0(f(M), \mathfrak{N}) = \ker \delta \oplus H^1_m(f(M), \mathfrak{J}_f).$$

The subspace $H^1_m(f(M), \mathfrak{J}_f)$ has been discussed in [3, §12], where, under certain conditions, it was termed the space of *relative moduli* of $f(M)$ in M'. If

[3] As usual, $A^0(M)$ denotes the identity component of the complex automorphism group of M.

[4] *Added in proof.* This can be done under the weaker assumption

$$H^0(f(M), \mathrm{Tor}^1(\mathfrak{N})) = 0.$$

(3.6) and (3.7) are satisfied, the following interpretation holds: If, by §1.3, we think of $H^0(f(M), \mathfrak{N})$ as infinitesimally parametrizing the variations of $f(M)$ in M', then ker δ parametrizes those deformations keeping the complex structure on $f(M)$ fixed, and $H^1_m(f(M), \mathfrak{J}_f)$ infinitesimally parametrizes those variations of $f(M)$ in M' obtained by varying the complex structure of $f(M)$. Thus we have

II. *Under assumptions* 2 *and* N.S., *the deformations of the holomorphic mapping* $f : M \to M'$ *which are infinitesimally parametrized by* B ((3.4)) *are given locally by a* dim B *family* B_f *of injections of* $f(M)$ *in* M *retaining the same complex structure on* $f(M)$.

Thus, in this case, the deformations of the holomorphic mapping f do not *cover* the variations of structure of $f(M)$ in M'.

3.4. Combining I and II, we have finally

THEOREM 4. *Under the assumptions* (3.5), (3.6), *and* (3.7), *there exists a locally complete family* $f_v = g_v$ $(v \in V)$ *of holomorphic mappings of* M *into* M' *varying the holomorphic mapping* $f : M \to M'$. *The space* V *may be chosen to have the property that* $V = A_f \times B_f$ *with the following interpretation: Any* 1*-parameter family* g *such that* $g_0 = f$ *may be factored as* $g_t = b_t \circ f \circ a_t$ *where* a_t, b_t *are* 1*-parameter families in* A_f, B_f *respectively, and* A_f, B_f *were given in* I, II *of* §§3.2, 3.3.

REFERENCES

1. H. GRAUERT, *Über Modifikationen und exzeptionelle analytische Mengen*, Math. Ann., vol. 146 (1962), pp. 331–368.
2. K. KODAIRA, *A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds*, mimeographed notes, Princeton University, 1962.
3. K. KODAIRA AND D. C. SPENCER, *On deformations of complex analytic structures, I*, Ann. of Math. (2), vol. 67 (1958), pp. 328–401.

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