Dear Пятницкий-Шапиро,

Except at 2, I have now a good understanding at the bad primes of the $\ell$-adic representations attached to modular forms (for $GL(2, \mathbb{Q})$).

A. Isogenies

**Defn.** The category of *elliptic curves up to isogeny* is obtained from that of elliptic curves by inverting isogenies i.e.,

a) an elliptic curve $E$ defines an elliptic curve up to isogeny $E \otimes \mathbb{Q}$

b) $\text{Hom}(E \otimes \mathbb{Q}, F \otimes \mathbb{Q}) = \text{Hom}(E, F) \otimes \mathbb{Q}$

hence if $F$ is a functor (elliptic curves) $\rightarrow (....)$, and $F$ (any isogeny) is an isomorphism, $F$ makes sense for elliptic curves up to isogeny.

Notations:

- $T_\ell(E) = \lim_{\leftarrow n} E_{\ell^n}$ (for $\ell$ prime to $p$, $E/k$ algebraically closed of char $p$, it is a free module of rank 2 on $\mathbb{Z}_\ell$)

- $V_\ell(E) = T_\ell(E) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, makes sense for elliptic curves up to isogeny

- $\hat{T}_{\ell'} = \lim_{(p,n)=1} E_n$

- $\hat{V}_{\ell'}(E) = \hat{T}_{\ell'} \otimes_{\mathbb{Z}} \mathbb{Q}$, a $\mathbb{A}_{\ell'}$-module of rank 2, makes sense for all curves up to isogeny

- $\mathbb{Z}_\ell(1) = \lim_{\leftarrow n} \mu_{\ell^n}$

- $\mathbb{Q}_\ell(1) = \mathbb{Z}_\ell(1) \otimes \mathbb{Q}_\ell$

- $(1) = \otimes \mathbb{Q}_\ell(1) \text{ or } \otimes \mathbb{Z}_\ell(1)$

- $d:\begin{cases} d(E) = \mathbb{Z} \\
If f : E \rightarrow F \text{ then } d(f) : d(E) \rightarrow d(F) \text{ is } \deg(f) \text{ (0 if } f = 0)\end{cases}$

Then

a) $d(E) \otimes \mathbb{Q}$ makes sense for elliptic curves up to isogeny: notation $d(E \otimes \mathbb{Q}) = d(E) \otimes \mathbb{Q}$

b) for $E_0$ an elliptic curve up to isogeny, the $e_n$-pairings, and their behavior under isogeny, enables one to define

$$\bigwedge^2 V_\ell(E_0) \simeq d(E_0) \otimes \mathbb{Q}_\ell(1).$$

At $p$: Let us look at the case of supersingular curves of char $p$, then, a substitute for $T_p(E)$ is the formal group of $E$ (a height 2 dim 1 formal group). The formalism of $d$ has the following analogue
a) a height 2 dim 1 formal group defines \( d_p(F) \), a rank one module over \( \mathbb{Z}_p \): \( d_p(F) = \text{set of } F \overset{u}{\to} F^* \) (\( F^* = \text{Pontryagin dual} \) with \( u = -u \).

b) if \( \tilde{F}/S \) is a deformation of \( F \) over \( S \) (local complete), and if \( T_p(\tilde{F}) \) is the corresponding local system on \( S \otimes \mathbb{Z}_p \mathbb{Q}_p \), then

\[
\bigwedge^2 T_p(\tilde{F}) \simeq d_p(F)(1)
\]

c) for \( E \) a supersingular elliptic curve, with corresponding formal group \( \hat{E} \),

\[
d(E) \otimes \mathbb{Z}_p \simeq d_p(\hat{E}).
\]

**Recovering \( E \).** Let \( E_0 \) be a supersingular elliptic curve up to isogeny. An elliptic curve \( E \) with an isomorphism \( E \otimes \mathbb{Q} \overset{\beta}{\sim} E_0 \) defines

a) a “lattice” \( \hat{T}_p'(E) \) in \( \hat{V}_p'(E_0) \)

b) a “lattice” \( d(E) \) in \( d(E_0) \) (the \( p' \)-part of it is determined by a): \( d(E) \otimes \mathbb{Z}_p(1) \simeq \bigwedge^2 T(E) \subset \bigwedge^2 V(E) = d(E_0) \otimes \mathbb{Q}(1) \).

**Lemma 1.** It amounts to the same to give either \( [E_0 \text{ supersingular}] \)

a) \( (E, \beta) \)

b) the lattices \( \hat{T}_p'(E) \subset \hat{V}_p'(E_0) \) and \( d(E) \otimes \mathbb{Z}_p \subset d(E_0) \otimes \mathbb{Q}_p \).

The reason why no more information is required at \( p \) is that, for supersingular \( E \), the only degree \( p' \)-isogeny of source \( E \) is \( E \to E^{(p')} \).

**Variant:** Let \( F_0 \) be a height 2 dim 1 formal group law, up to isogeny. Then, to give \( F \) defining \( F_0 \) amounts to give \( d_p(F)(\sim \mathbb{Z}_p) \subset d_p(F_0)(\sim \mathbb{Q}_p) \).

**B. The fundamental local construction**

Let:

\( \overline{\mathbb{Q}}_p \) be an algebraic closure of \( \mathbb{Q}_p \)

\( \overline{\mathbb{F}}_p \) be the algebraic closure of \( \mathbb{F}_p \), residue field of \( \overline{\mathbb{Q}}_p \)

\( k \) be an algebraic closure of \( \mathbb{F}_p \), provided with a class modulo integral powers of Frobenius of allowed isomorphisms \( k \overset{\sim}{\to} \mathbb{F}_p \). The class is denoted \( \text{Isom}(k, \mathbb{F}_p) \).

\( V_0 \) be a 2-dimensional vector space over \( \mathbb{Q}_p \)

\( E_0 \) be a supersingular elliptic curve up to isogeny over \( k \)

\( F_0 \) be the formal group up to isogeny /\( k \) defined by \( E_0 \).

To be able to use the “transport de structure”, I prefer not to take \( k = \mathbb{F}_p \) nor \( V_0 = \mathbb{Q}_p^2 \).
1. First construction: Let $\sigma \in \text{Isom}(k, \mathbb{F}_p)$ and $\beta \in \text{Isom}(d(E_0) \otimes \mathbb{Q}_p(1), \wedge^2 V_0)$. Here $\mathbb{Q}_p(1)$ is relative to $\mathbb{Q}_p$: $\mathbb{Q}_p(1) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \lim_{\leftarrow n} (\text{group of } p^n\text{-roots of unity of } \overline{\mathbb{Q}_p})$.

$\sigma$ and $\beta$ do define $E_0(\sigma, \beta) = (\sigma(E_0), \sigma(\beta))$ where

\[
\begin{cases}
\sigma(E_0) & \text{is an elliptic curve up to isogeny over } \mathbb{F}_p \\
\sigma(\beta) & \text{is an isomorphism of one-dimensional vector spaces over } \mathbb{Q}_p,
\end{cases}
\]

\[d(\sigma(E_0)) \otimes \mathbb{Q}_p(1) \xrightarrow{\sigma} d(E_0) \otimes \mathbb{Q}_p(1) \xrightarrow{2} \wedge^2 V_0.\]

Let $\varphi$ be the Frobenius substitution $\varphi : x \mapsto x^p; k \to k$. Then, for any elliptic curve $E/k$, $\varphi(E)$ is $E^{(p)}$ and one disposes of [has at one’s disposal] the Frobenius isogeny $F : E \to E^{(p)}$.

The diagram

\[
\begin{array}{ccc}
\text{d}(E) & \overset{\varphi}{\longrightarrow} & \text{d}(E^{(p)}) \\
\downarrow & & \downarrow \\
\mathbb{Z} & \overset{p}{\longrightarrow} & \mathbb{Z}
\end{array}
\]

is commutative. In particular, $F$ is an isomorphism $F : E_0 \xrightarrow{\sim} \varphi(E_0)$, and the diagram

\[
\begin{array}{ccc}
d(E_0) & \overset{\varphi}{\longrightarrow} & d(\varphi(E_0)) \\
\downarrow & & \downarrow \\
d(E_0) & \overset{F}{\longrightarrow} & d(\varphi(E_0))
\end{array}
\]

is commutative.

$F$ hence induces an isomorphism

\[F \text{ or } \sigma F : (\sigma(E_0), \sigma(\beta)) \xrightarrow{\sim} (\sigma \varphi(E_0), \sigma(p^{-1}\beta))\]

Definition. $D_p$ is the one dimensional vector space over $\mathbb{Q}_p$, quotient of $\text{Isom}(k, \overline{\mathbb{F}_p}) \times \text{Isom}(d(E_0) \otimes \mathbb{Q}_p(1), \wedge^2 V_0)$ by the equivalence relation $(\sigma, \beta) \sim (\sigma \varphi^k, p^{-k}\beta)$ for $k \in \mathbb{Z}$.

(1) defines an isomorphism between $E_0(\sigma, \beta)$ and $E_0(\sigma', \beta')$ for $(\sigma, \beta) \sim (\sigma', \beta')$, and those isomorphisms form a transitive system of isomorphisms. They hence allow us to define $(E_0(\delta), \beta(\delta))$ for $\delta \in D_p$, where

\[
\begin{cases}
E_0(\delta) & \text{is a (supersingular) elliptic curve up to isogeny on } \overline{\mathbb{F}_p}, \text{ and} \\
\beta(\delta) & \text{is an isomorphism } d(E_0(\delta)) \otimes \mathbb{Q}_p(1) \xrightarrow{\sim} \wedge^2 V_0.
\end{cases}
\]

Remark. The following groups are acting on $D_p$ (by “transport de structure”)

a) [crossed out]
\( \beta \) \( W(\mathbb{Q}_p/\mathbb{Q}_p) \) via its actions both on \( Isom(k, \mathbb{F}_p) \) and \( \mathbb{Q}_p(1) \). If the isomorphism \( cl \, W(\mathbb{Q}_p/\mathbb{Q}_p)^{ab} = \mathbb{Q}_p^* \) is normalized \((\pm)\) so that Frobeniuses \( \varphi \) correspond to \textit{inverse} of uniformizing parameter, then
\[
\sigma \cdot \delta = cl(\sigma)^{-1} \cdot \delta.
\]
\( \gamma \) \( GL(V_0) \) via its action on \( \Lambda^2 V_0 \). One has
\[
g \cdot \delta = \det(g) \cdot \delta.
\]
\( \delta \) \( Aut(E_0) \), via its action on \( d(E_0) \). We define
\[
H = Aut(E_0).
\]
It is the multiplicative group of the quaternion algebra ramified at \( p \) and at \( \infty \). One has
\[
g \cdot \delta = \text{Nrd}(g)^{-1} \cdot \delta \quad \text{(reduced norm)}.
\]

\textbf{Remark 2} Assume that a lattice \( \Lambda \simeq \mathbb{Z}_p \) has been chosen in \( \bigwedge^2 V_0 \). Then, \( (E_0, \beta)/\mathbb{F}_p \) as above define an elliptic curve up to \( p' \)-isogeny on \( \mathbb{F}_p \), corresponding to the lattice \( \beta^{-1}(\Lambda)(-1) \subset d(E_0) \otimes \mathbb{Q}_p \). \( p' \)-isogeny means isogeny of degree prime to \( p \) (cf. Lemma A1).

\( I' \). Variant. Let us start with \( F_0 \) instead of \( E_0 \). \( D_p \) is defined as before, using \( d_p(F_0) \) instead of \( d(E_0) \otimes \mathbb{Q}_p \), and is the same as before, via the isomorphism \( d(E_0) \otimes \mathbb{Q}_p \simeq d_p(F_0) \) \( (F_0 = E_0) \). This time, \( D_p \) is acted on by \( W(\mathbb{Q}_p/\mathbb{Q}_p) \), \( GL(V_0) \) and \( Aut(F_0) = H(\mathbb{Q}_p) \). The formulae are the same as before.

\( \delta \in D_p \) defines
\[
\begin{cases}
F_0(\delta), & \text{a (supersingular) formal group law up to isogeny on } \mathbb{F}_p \\
\beta(\delta), & d(F_0(\delta)) \otimes \mathbb{Q}_p(1) \sim \bigwedge^2 V_0.
\end{cases}
\]
If \( \Lambda \) has been chosen in \( \bigwedge^2 V_0 \), one gets
\[
\begin{cases}
F(\delta), & \text{a (supersingular) formal group law on } \mathbb{F}_p \\
\beta(\delta), & d(F(\delta)) \otimes \mathbb{Z}_p(1) \sim \Lambda.
\end{cases}
\]

\[ 2. \text{Second construction.} \]

I have now to define “vanishing cycles varieties” and vanishing cycles groups. Eventually, for \( K^0 \) an open compact subgroup of \( SL(V_0) \), and for \( \delta \in D_p \), a scheme \( V(K^0, \delta) \) over \( \mathbb{Q}_p \) will be defined. For \( K \) an open compact subgroup of \( GL(V_0) \) such that \( K^0 = K \cap SL(V_0) \), isomorphisms
\[
V(K^0, \delta) \xrightarrow{(K)} V(K^0, \alpha \delta) \quad \text{(for } \alpha \in \text{det } K^0 \subset \mathbb{Z}_p^* \text{)}
\]
are defined. For variable \( \delta \), the \( V(K^0, \delta) \)'s will thus be a “local system” of schemes on \( D_p \). The \( V(K^0, \delta) \) are not of finite type, but of the type usual in the vanishing cycle theory. However we won’t need it, here is a description of the \( \mathbb{Q}_p \)-valued points of \( V(K^0, \delta) \).

\[ a) \] \( \delta \) provides us with \( (E_0(\delta), \beta(\delta)) \) on \( \mathbb{F}_p \)
\[ b) \] a point of \( V(K^0, \delta) \) is an isomorphism class of systems consisting in
\[ a) \] an elliptic curve up to isogeny on \( \mathbb{Q}_p \): \( E \)
\[ b) \] an isomorphism \( \alpha : V_p(E) \sim V_0 \), given mod \( K^0 \)
\[ c) \] an isomorphism of the reduction of \( E \) with \( E_0(\delta) \);
\[ \psi : E|\mathbb{F}_p \simeq E_0(\delta) \]
δ) required: the diagram

\[(*) \quad \Lambda^2 V_p(E) \xrightarrow{\wedge^2 \alpha} \Lambda^2 V_0 \xrightarrow{\beta(\delta)} d(E)(1) \xrightarrow{\sim} d(E_0(\delta))(1) \]

is commutative.

If K is as above, it amounts to the same to give α only \( \mod K \), (**) being required only \( \mod \det(K) \).

To actually construct \( V(K, \delta) \), let us start with

\( K \) an open compact subgroup of \( GL(V_0) \)

\( E_0 \) a supersingular curve up to isogeny on \( \mathbb{F}_p \) (in practice, \( E_0(\delta) \))

\( \beta : d(E_0) \otimes \mathbb{Q}_p(1) \xrightarrow{\sim} \wedge^2 V_0 \) (in practice, \( \beta(\delta) \))

\( \overline{\beta} = \beta \mod \det(K) \subset \mathbb{Z}_p^\times \) (only \( \overline{\beta} \), not \( \beta \), will be used).

The construction will involve the auxiliary data of

\( L \) a \( K \)-stable lattice in \( V_0 \).

a) \( \beta^{-1}(\wedge^2 L)(-1) \) is a lattice in \( d(E_0) \otimes \mathbb{Q}_p \) and defines an elliptic curve up to \( p' \)-isogeny \( E' \), with \( E' \otimes \mathbb{Q} \simeq E_0 \). Let me choose an elliptic curve \( E \) with level \( n \) structure \( \alpha_n \) \( (n \geq 3, (n, p) = 1) \) with \( E \otimes \mathbb{Z}(p) = E' \). [The construction will be shown to be independent of \( L, E', E, \alpha_n \).] Let \( M_n \) be the modular scheme for elliptic curves with level \( n \) structure and \( e \in M_n(\mathbb{F}_p) \) be the point of \( M_n \) defined by \( (E, \alpha_n) \).

b) \( M(E, \alpha_n) \) is the spectrum of the henselization of the local ring at \( e \) of \( M_n \otimes \mathbb{Z} W(\mathbb{F}_p) \).

c) The completion of \( M(E, \alpha_n) \) is isomorphic to \( W(\mathbb{F}_p)[[t]] \).

d) Assume that \( (E_1, \alpha^1_n) \) and \( (E_2, \alpha^2_n) \) are two systems as above, and that \( \varphi : E_1 \to E_2 \) is a \( p' \)-isogeny. There is then one and only one isomorphism \( \overline{\varphi} : M(E_1, \alpha^1_n) \to M(E_2, \alpha^2_n) \) fitting in a commutative diagram.
In this diagram, \( \ast \) means \( Spec(F_p) \) (= point), and \( \tilde{E}_1, \tilde{E}_2 \) are the pull-back over \( M(E_1, \alpha_1^1), M(E_2, \alpha_2^2) \) of the universal curve over \( M_n \).

This can be better expressed by saying that \( M(E_1, \alpha_1^1) \) is the parameter space of the universal deformation of the elliptic curve up to \( p' \)-isogeny \( E'/F_p \) (universal is with respect to deformation over henselian local \( W(F_p) \)-algebras). This allows us to write simply \( M(E') \) for \( M(E_1, \alpha_1^1) \) and \( \tilde{E}' \) for the elliptic curve up to \( p' \)-isogeny over \( M(E') \) defined by (any) \( E_1 \).

For \( n \) large enough, \( K \) is the subgroup of \( GL(L) \) inverse image of a suitable subgroup \( \overline{K} \) of \( GL(L/p^n L) \).

Let \( K(F_p) \) be the field of fractions of \( W(F_p) \), and let \( \mathbb{Z}_p \) be the ring of integers in \( \overline{F}_p \). The group scheme \( E_{p^n}(\mathbb{Z}_p) \) over \( M(E') \) is finite etale over \( M(E') \otimes K(F_p) \), hence

\[
\text{Isom}(E_{p^n}, L/p^n L)
\]

is a finite etale Galois covering of \( M(E') \otimes K(F_p) \), with Galois group \( GL(L/p^n L) \). Let us rather consider

\[
\text{Isom}_{M(E')} \otimes \mathbb{Z}_p(E_{p^n}(\mathbb{Z}_p), L/p^n L).
\]

This time, we get a \textit{disconnected} covering of \( M(E') \otimes \mathbb{Z}_p \). A piece of it can be picked as follows: via \( \beta \), one has \( \mathbb{K}^2 E_{p^n} = d(E') \otimes \mathbb{Z}_p(1) = \mathbb{K}^2 L/p^n L \), and one considers only isomorphisms of “determinant 1”. Similarly, \( \overline{\beta} \) enables one to pick a component of

\[
\overline{K} \setminus \text{Isom}_{M(E')} \otimes \mathbb{Z}_p(E_{p^n}, L/p^n L).
\]

We will call the component \( V_L(K, E_{0}', \beta) \). It is also the quotient by \( \overline{K} \cap SL(L/p^n L) \) of the picked component of \( \text{Isom}_{M(E')} \otimes \mathbb{Z}_p(E_{p^n}, L/p^n L) \) of \( W(F_p) \).

\textit{Summary.} \( V_L(K, E_{0}', \beta) \) depends on \( K \subset GL(V_0) \), compact open, \( E_{0}' \) up to isogeny on \( F_p \), \( \beta : d(E') \otimes \mathbb{Q}_p(1) \to \mathbb{K}^2 V_0 \), and of a lattice \( L \) in \( V_0 \), stable by \( K \). It does not depend on the whole of \( \beta \), but only on \( \overline{\beta} = \beta \mod \det(K) \). For \( K' \subset K \), one has a map

\[
V_L(K', E', \beta) \to V_L(K, E', \beta).
\]

\textit{Construction} \( V_L(K, E_{0}', \beta) \) is independent of \( L \).

More precisely, the system consisting of

\[
\begin{align*}
V_L(K, E', \beta) \\
\text{the elliptic curve up to isogeny } \tilde{E}_{0}' \text{ over } V_L(K, E', \beta) \\
\text{the universal isomorphism } \alpha \text{ given } \mod K : V_p(\tilde{E}_{0}') \to V_0 \\
\text{(deduced from } \pi : \tilde{E}_{p^n} \cong L/p^n L, \mod \overline{K}, \text{ or } \overline{\pi} : T_p(\tilde{E}') \cong L, \mod K) 
\end{align*}
\]

is independent of \( L \).

Let \( L_1 \) and \( L_2 \) be two \( K \)-invariant lattices, and let \( E_1 \) and \( E_2 \) be the corresponding elliptic curves up to \( p' \)-isogeny over \( F_p \). We are to define an isomorphism

\[
K \setminus \text{Isom}_{M(E_1)} \otimes K(F_p) \left( V_p(\tilde{E}_1), V_0 \cup T_p(\tilde{E}_1) \right) \cong K \setminus \text{Isom}_{M(E_2)} \otimes K(F_p) \left( V_p(\tilde{E}_2), V_0 \cup T_p(\tilde{E}_2) \right)
\]

\[
M(E_1) \quad M(E_2)
\]
For simplicity, we assume \( L_1 \subset L_2 \). Then, over \( K \setminus \text{Isom}_{\mathcal{M}(E_1)} \cdots \), \( \tilde{E}_1 \) is provided with a subgroup isomorphic to \( L_2/L_1 \), call it \( H \). Let \( \tilde{W}_1(K, E_0, \beta) \) be the normalization of \( M(E_1) \) in \( K \setminus \text{Isom} \cdots \). Due to the normality of \( \tilde{W} \) and the fact that an elliptic curve has only finitely many subgroup-schemes of a given order, the subgroup-scheme \( H \) of \( \tilde{E}_1 \) on \( \tilde{W}_1(K, E_0, \beta) \otimes K(\overline{\mathbb{F}}_p) \) extends as a subgroup-scheme \( H \) on \( \tilde{E}_1 \) on \( \tilde{W}_1(K, E_0, \beta) \). The quotient \( \tilde{E}_1/H \) is a deformation of \( E_2 \), hence a map
\[
\tilde{W}_1(K, E_0, \beta) \twoheadrightarrow M(E_2).
\]

Further, \( V_p(\tilde{E}_1/H) = V_p(\tilde{E}_1) \), and, over \( K \setminus \text{Isom} \quad  \begin{pmatrix} V_p(\tilde{E}_1) & V_0 \\ \cup & \cup \\ T_p(\tilde{E}_1) & L_1 \end{pmatrix} \), isomorphism \( \alpha : V_p(\tilde{E}_1) \to V_0 \) carrying \( T_p(\tilde{E}_1) \) to \( L_1 \) do carry \( T_p(\tilde{E}_1/H) \) to \( L_2 \); if \( W_2 \) is defined as \( W_1 \), one has
\[
K \setminus \text{Isom}_{\mathcal{M}(E_1) \otimes \mathbb{Q}_p} \quad \begin{pmatrix} V_p(\tilde{E}_1) & V_0 \\ \cup & \cup \\ T_p(\tilde{E}_1) & L_1 \end{pmatrix} \quad \sim \quad K \setminus \text{Isom}_{\mathcal{M}(E_2) \otimes \mathbb{Q}_p} \quad \begin{pmatrix} V_p(\tilde{E}_2) & V_0 \\ \cup & \cup \\ T_p(\tilde{E}_2) & L_2 \end{pmatrix}
\]

This defines the dotted maps; they are the seeked for isomorphisms.

By extending the scalars to \( \overline{\mathbb{Q}}_p \) and taking one component, one gets the isomorphisms expressing that \( V_L(K, E_0, \beta) \) is independent of \( L \). For \( \delta \in D_p \), we note
\[
V(K, \delta) = V_L(K, E_0(\delta), \beta(\delta)).
\]

**Summary.** \( V(K, \delta) \) is a scheme over \( \overline{\mathbb{Q}}_p \); it depends on \( K \) compact open in \( \text{GL}(V_0) \) and on \( \delta \in D_p \), given modulo multiplication by elements of \( \text{det}(K) \subset \mathbb{Z}_p^* \). For \( K \) smaller and smaller, the \( V(K, \delta) \) form a projective system. Over \( V(K, \delta) \) is given an elliptic curve up to isogeny \( \tilde{E} \), provided with an isomorphism, given mod \( K \), \( \alpha : V_p(\tilde{E}) \cong V_0 \). In a sense, \( \tilde{E} \) is a deformation of \( E_0(\delta) \), in particular \( d(E_0(\delta)) = d(\tilde{E}) \). The morphism \( \alpha \) is compatible with \( \beta(\delta) \)
\[
\wedge^2 V_p(\tilde{E}) = d(E_0(\delta)) \otimes \mathbb{Q}_p(1) \xrightarrow{\text{det}(\alpha)} \wedge^2 V_0 \quad (\text{mod det}(K))
\]

In fact, the statement that \( \tilde{E} \) is a deformation of \( E_0 \) can be made more precise by introducing a suitable \( \tilde{E}/\tilde{V}(K, \delta)/\mathbb{Z}_p \).

2'. **Variant.** Starting with \( F_0 \) instead of \( E_0 \), one can construct analogues of the \( V(K, \delta) \), called \( \tilde{V}(K, \delta) \), with complete local rings replacing henselian local rings.

3. **Third construction.**

We are interested in the \( \ell \)-adic cohomology groups
\[
H^1(V(K, \delta), \mathbb{Q}_\ell).
\]
These groups are finite dimensional, and locally constant as functions of \(\delta\) (as \(V(K, \delta)\) itself is). If \(K'\) is a distinguished subgroup of \(K\), one clearly has

\[
H^1(V(K, \delta), \mathbb{Q}_\ell) = H^1(V(K', \delta), \mathbb{Q}_\ell)^{K \cap \text{SL}(V_0)/K' \cap \text{SL}(V_0)} \quad \text{(invariants)}.
\]

\((V(K, \delta)\) depends also only on \(K \cap \text{SL}(V_0)\) and \(\delta\).)

The local fundamental object is the “bundle” \(\mathcal{H}/D_p\), with

\[
\mathcal{H}_d = \lim_{\overset{\longrightarrow}{K}} H^1(V(K, \delta), \mathbb{Q}_\ell).
\]

There is a notion of “locally constant section of \(\mathcal{H}_d/D_p\)”. It is a function \(\varphi(\delta)(\delta \in D_p, \varphi(\delta) \in \mathcal{H}_d)\) with locally \(\varphi(\delta)\) in \(H^1(V(K, \delta), \mathbb{Q}_\ell)\) and locally constant. The space of locally section is noted \(\Gamma(D_p, \mathcal{H})\)

On the local fundamental object are acting, by “transport de structure”:

a) \(W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\)
b) \(GL(V_0)\)
c) \(\text{Aut}(E_0) = H\).

The actions over \(D_p\) being those already described.

The actions of \(W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\) and \(GL(V_0)\) are “continuous” (the latter, will respect the notion of locally constant section).

**Proposition.** The action of \(\text{Aut}(E_0)\) extends as a continuous action of \(H(\mathbb{Q}_p)\) on \(\mathcal{H}/D_p\).

I don’t have a satisfactory proof. My idea of proof would be to go back to the \(W\) introduced earlier and to express \(\mathcal{H}\) in terms of special fibre of stable models of \(W\) over suitable ramified extensions of \(W(\overline{\mathbb{F}}_p)\).

Intuitively, one may argue that \(V(K, \delta)\) and \(\tilde{V}(K, \delta)\) could have the same cohomology, that \(\text{Aut}(F_0) = H(\mathbb{Q}_p)\) acts on \(\tilde{V}(K, \delta)\), hence on \(H^1(\tilde{V}(K, \delta), \mathbb{Q}_\ell)\), and that it would be hell if the action were not continuous.

**C. Statement of the local results.** (The proofs will be of a global nature.)

It will be easier to work not with \(\mathbb{Q}_p\)-cohomology, but with \(\overline{\mathbb{Q}}_p\)-cohomology, obtained by extending \(\mathbb{Q}_p\) to an algebraic closure \(\overline{\mathbb{Q}}_\ell\). By abuse of language, we will again denote by \(\mathcal{H}/D_p\) the “admissible” bundle over \(D_p\) with fibre

\[
\mathcal{H}_d = \lim_{\overset{\longrightarrow}{K}} H^1(V(K, \delta), \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell.
\]

The groups \(W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\), \(GL(V_0)\), \(H(\mathbb{Q}_p)\) act admissibly on \(\mathcal{H}/D_p\). The actions on \(D_p\) have been computed. Of course, the actions commute the one with the other. If \(a \in \mathbb{Q}_p^*\), the action of the elements \(a \in GL(V_0)\) and \(a \in H(\mathbb{Q}_p)\) are inverse the one of the other. In terms of a fixed \(\delta_0 \in D_p\), this could as well be expressed by saying

A/ the subgroup of \(W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \times GL(V_0) \times H(\mathbb{Q}_p)\) formed by the elements such that

\[
cl(\sigma)^{-1}, \det(g), \text{Nrd}(h)^{-1} = 1
\]

acts on \(\mathcal{H}_{\delta_0}\)

B/ the subgroup \((1, 1, a, a)\) acts trivially.
Mnemonic way to check the action on the $V(K, \delta)/D_p$ it is good to view a point of $\lim \frac{V(K, \delta)}{D_p}$ as consisting of

- $\delta \in D_p$
- $\tilde{E}$: elliptic curve up to isogeny on $\overline{\mathbb{Q}}_p$
- $\alpha : V'_p(\tilde{E}) \sim V_0$
- $\psi$ “specialization map” $\tilde{E} - \psi \to \delta(E_0)$

with a compatibility between $\alpha$, $\psi$, and $\beta(\delta)$.

Let $\chi : \mathbb{Q}_p^* \to \overline{\mathbb{Q}}_p^*$ be a quasi-character (with open kernel).

We denote by

$$\Gamma(D_p, \mathcal{H})$$

the $\overline{\mathbb{Q}}_\ell$-vector space of locally constant sections of $\mathcal{H}/D_p$

$$\Gamma_\chi(D_p, \mathcal{H})$$

the subspace of those sections for which, for any $a \in \mathbb{Q}_p^*$, with image $z(a)$ in the center(?) of $GL(V_0)$, $z(a)f = \chi(a)f$, i.e. $z(a)f(\delta) = \chi(a)f(a^2\delta)$.

**Theorem.** (i) $\Gamma_\chi(D_p, \mathcal{H})$ is a direct sum of triple tensor products

$$\Gamma_\chi(D_p, \mathcal{H}) = \bigoplus_{\varphi \in \Phi} V_\varphi \otimes V'_\varphi \otimes W_\varphi$$

where

- $\alpha$) $V_\varphi$ is an admissible irreducible of $GL(V_0)$, the center of $GL(V_0)$ acting by the character $\chi$, and $V_\varphi$ being of the discrete series (=special or supercuspidal)
- $\beta$) $V'_\varphi$ is an admissible irreducible (finite dimensional) representation of $H(\mathbb{Q}_p)$ with the center acting by $\chi^{-1}$
- $\gamma$) $W_\varphi$ is a 2-dimensional (or 1-dimensional) continuous $\overline{\mathbb{Q}}_\ell$-adic irreducible representation of $W(\overline{\mathbb{Q}}_p/Q_p)$.
  If it is 2-dimensional, then $W(\overline{\mathbb{Q}}_p/Q_p)$ acts on $\Lambda^2 W_\varphi$ by the character . If 1-dimensional, defined by a character $\nu$, then $\nu = (\Lambda^2(W_\varphi \otimes W_\varphi(1))$ corresponds again to $\chi$.

(ii) $V'_\varphi$ runs (once and only once) through the representations said in (i)
(iii) The same holds for $V_\varphi$, and $V_\varphi$ and $V'_\varphi$ correspond by the Weil representation construction (suitably normalized)
(iv) $W_\varphi$ one dimensional $\iff$ $V'_\varphi$ is $\iff$ $V_\varphi$ is special, and if $V'_\varphi$ is $\mu(Nrd)$, then $W_\varphi$ is the character of $W(\overline{\mathbb{Q}}_p/Q_p) \cong Q_p^*$.
(v) If $V_\varphi$ is defined by a quasi-character of a [...] line cut off [...].

Except for $p = 2$, this gives a complete determination of $\Gamma_\chi(D_p, \mathcal{H})$.

For $\mu$ a quasi-character of $\mathbb{Q}_p^*$ and $\delta_0 \in D_p$, multiplication by the function $\mu(\delta^{-1})$ on $D_p$ provides an isomorphism

$$\Gamma_\chi(D_p, \mathcal{H}) \to \Gamma_{\chi\mu^{-1}}(D_p, \mathcal{H});$$

if $V_\varphi \otimes V'_\varphi \otimes W_\varphi$ occurs in $\Gamma_\chi(D_p, \mathcal{H})$, then

$$(V_\varphi \otimes \mu^{-1} \det(g)) \otimes (V'_\varphi \otimes \mu Nrd) \otimes (W_\varphi \otimes \mu cl)$$
occurs in $\Gamma_{\chi_{\mu,2}}(D_p,\mathcal{H})$.

**D. Global Theory.**

We consider

1. $K$ open compact subgroup of $GL(2,A_f)$
2. $X^\pm$ Poincaré upper and lower half-plane.
   
   $X^\pm = \text{Isom}_R(\mathbb{Z}^2 \otimes \mathbb{R}, \mathbb{C})/\mathbb{C}^* \subset \text{Hom}(\mathbb{Z}^2, \mathbb{C})/\mathbb{C}^*$
   
   ($GL(2,\mathbb{R})$ acts on the right on $\mathbb{Z}^2 \otimes \mathbb{R}$ via its action on $\mathbb{Z}^2 \otimes \mathbb{R} = \mathbb{R}^2$).
3. $M_k^0(\mathbb{C}) = K \backslash X^\pm \times GL(2,A_f)/GL(2,\mathbb{Q})$
4. $k$ an integer ($k \geq 0$)
5. $\mu$ the representation $\text{Sim}^k$ (dual of obvious representation) of $GL(2,\mathbb{Q})$
6. $F^Q_\mu$ the corresponding local system on $M_k^0(\mathbb{C})$
7. $M_K(\mathbb{C}) =$ the Satake compactification of $M_k^0(\mathbb{C})$; $j : M^0_K(\mathbb{C}) \hookrightarrow M_K(\mathbb{C})$
8. $F^Q_\mu$ the sheaf $j^* F^Q_\mu$ on $M_K(\mathbb{C})$
9. $\mathcal{H}^Q(\mu) = \lim_{\to} H^1(M_K(\mathbb{C}), F^Q_\mu)$ (an admissible representation of $GL(2,A_f)$ defined $\mathbb{Q}$)
10. $\mathcal{H}^C(\mu) = \mathcal{H}(\mu) \otimes \mathbb{C}$; $\mathcal{H}^{Q_\ell} = \mathcal{H}(\mu) \otimes \mathbb{Q_\ell}$; $F^Q_\mu = F_\mu \otimes \mathbb{Q_\ell}$.

One has a decomposition

$$\mathcal{H}^C(\mu) = \mathcal{H}^{k+1,0} \oplus \mathcal{H}^{0,k+1}$$

of $\mathcal{H}^C$ into 2 complex conjugate subspaces; $\mathcal{H}^{k+1,0}$ is a complex admissible representation of $GL(2,A_f)$, which can be defined over $\mathbb{Q}$; it is hence isomorphic to the complex conjugate representation $\mathcal{H}^{0,k+1}$. It corresponds to holomorphic modular cusp forms of weight $k + 2$ (note that $k + 2 \geq 2$). For some explicit admissible representation $D_{k+2}$ of $GL(2,\mathbb{R})$, of the discrete series,

$$\mathcal{H}^{k+1,0} = \text{Hom}_{GL(2,\mathbb{R})}(D_{k+2}, L_0(GL(2,A_f))/GL(2,\mathbb{Q})).$$

Further, $\{M_K(\mathbb{C})\}$ is naturally defined $\mathbb{Q}$ (with its $GL(2,A_f)$-action: $M_K \xrightarrow{\sim} M_{gKg^{-1}}$, and $F^{Q_\ell}_\mu$ is an $\ell$-adic sheaf, defined $\mathbb{Q}$). Hence,

$$\mathcal{H}^{Q_\ell}(\mu)$$

carries a $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action, commuting with $GL(2,A_f)$, acting by “transport de structure”. After extension of $\mathbb{Q}_\ell$ to $\overline{\mathbb{Q}}_\ell$, one has a decomposition

$$\mathcal{H}^{Q_\ell}(\mu) = \bigoplus_{f \in F} (\otimes_p V_{f,p}) \otimes W_f$$

$V_{f,p}$: irreducible admissible representation of $GL(2,\mathbb{Q}_p)$
$W_f$: 2-dimensional representation of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ ($F=$spectrum of $GL(2)$)
By Eichler–Shimura–Kuga–Deligne–Ihara–Vötter–Langlands, if $V_{f,p}$ is of the principal series, then $W_f|\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ (restriction to the decomposition group) is a sum of 2 corresponding characters. If $V_{f,p}$ is special, then $W_f|\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is the corresponding special $\ell$-adic representation of $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

I can now prove

(A) If $V_{f,p}$ is supercuspidal, then $W_f|\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is irreducible, and, with the notation of C. Theorem, if $V_{f,p} \sim V_{\phi}$, then $W_f \sim W_{\phi}$.

(Hence $V_{f,p}$ determines, by a local rule, $W_{f,p}$ by the Weil construction to a character of a quadratic extension.)

The method is to use the theory of vanishing cycles to prove that a space to be described below is a quotient of $H(\mu) \otimes \mathbb{Q}_\ell$. (This is accurate only for $k \neq 0$; I will not bother much about $k = 0$ and the phenomena related to special representations.)

We keep the notation of B., except that now $V_0 = \mathbb{Q}_p^2$. Let us consider

$$\text{Isom}(V_{p}(E_0), (\mathbb{A}_f^I)^2) \times D_p$$

and the right action of $H$ on it (by composition for the first factor, and the inverse of the already defined action on $D_p$). On this space, we have the following $H$-equivariant local system

$$\text{Sim}^k(V_{\ell}(E)^*) \otimes \lim_{\longrightarrow} V(K, \delta)$$

(on the second factor, a right action is required, one takes the inverse of the one already constructed). The space and local system is acted by $GL(2, \mathbb{A}_f)$:

$$\begin{cases}
\text{space} : & \text{composition on 1st factor, already described on } D_p \\
\ell. \text{ system} : & \text{trivial on the first factor, described on 2nd}
\end{cases}$$

We now take as announced representation of $GL(2, \mathbb{A}_f)$:

$$H^0 \left( \begin{array}{c}
\text{local system } \text{Sim}^k(V_{\ell}(E_0)^*) \otimes \lim_{\longrightarrow} V(K, \delta) \\
\text{on } \text{Isom}(V_{p}(E_0), (\mathbb{A}_f^I)^2) \times D_p
\end{array} \right) / H$$

The action of $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is via its action on $V(K, \delta)/D_p$.

I now wish to relate that $H^0$ with the spectrum of $H(\mathbb{A})/H(\mathbb{Q})$ and $\Gamma(D_p, H)$.

For simplicity, let me in any way extend the scalars from $\mathbb{Q}_\ell$ to $\mathbb{C}$ (not to lose the $\text{Gal}$ action, I have to use here that it is continuous for the discrete topology of $\mathbb{Q}_\ell$ – which I can do directly). Let me also choose an isomorphism

$$\begin{cases}
V_{p}(E_0) \simeq (\mathbb{A}_f^I)^2, & \text{hence} \\
H(\mathbb{Q}_\ell) \simeq GL(2, \mathbb{Q}_\ell) & (\ell \neq p)
\end{cases}$$
Then
\[
H^0 = \{ \text{functions } H(A^f, p) \to Sim^k(\ast) \otimes \Gamma(D_p, \mathcal{H}) \text{ with } f(x\gamma) = f(x)\gamma \}
\]
\[
= \{ \text{functions } H(A) \to Sim^k(\ast) \otimes \Gamma(D_p, \mathcal{H}) \text{ with } f(x\gamma) = f(x) \text{ and } f(g_\infty g_p x) = g_\infty g_p f(x) \}
\]
\[
= \bigoplus_{(\mathbb{A}/\mathbb{Q})^\times} \{ \}_{X}
\]
\[
= \bigoplus_{(x, H) \in H(\mathbb{R}) \times H(\mathbb{Q})} \text{Hom}_{H(\mathbb{R}) \times H(\mathbb{Q})}(Sim^k(V) \otimes \Gamma_{X_p^{-1}}(D_p, \mathcal{H})^\dagger, L_0^H(H(A)/H(\mathbb{Q}))).
\]

This can be expressed as follows: In $L_0(H(\mathbb{A})/H(\mathbb{Q}))$ as a given vector of a given representation of $H(\mathbb{R})$. The representation of $H(A^f)$ so obtained may be written
\[
\bigoplus_{f_0 \in F_0} (\otimes_p V_{f_0, p}),
\]
the following representation occurs in $H(\mu)$:
\[
\bigoplus_{f_0 \in F_0, \ell \neq p} V_{f_0, \ell} \otimes \bigoplus_{V_\varphi \sim V_{f_0, p}} (V_\varphi \otimes W_\varphi)
\]

Now, comparison with Jacquet-Langlands §16, plus the fact that supercuspidal representations cannot occur outside $H(\mu)$ (this is given by Pierrepont-Heegner or Langlands + vanishing cycle theory) one gets (i) $\alpha, (\beta)$ (ii) (iii), 2-dimensionality in (i) $\gamma$ of Theorem C. One also gets statement (v). By checking these general rule against modular forms attached to $L$ functions with grossencharacter of imaginary quadratic fields (+ the end remark of C), one gets statement (v). The proof of (i) $\gamma$ in characteristic 2, in cases not covered by (v), uses entirely different ideas, which I cannot explain here.

Yours sincerely

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