Princeton, March 25, 1973

Dear Пятецкий-Шапиро,

Except at 2, I have now a good understanding at the bad primes of the ℓ-adic representations attached to modular forms (for $GL(2,\mathbb{Q})$).

A. Isogenies

Defn. The category of *elliptic curves up to isogeny* is obtained from that of elliptic curves by inverting isogenies
i.e.,

α) an elliptic curve E defines an elliptic curve up to isogeny $E \otimes \mathbb{Q}$
β) $Hom(E\otimes\mathbb{Q}, F\otimes\mathbb{Q}) = Hom(E,F)\otimes\mathbb{Q}$

hence if F is a functor (elliptic curves) $\to$ (....), and F(any isogeny) is an isomorphism, F makes sense for elliptic curves up to isogeny.

Notations:

$T_\ell(E) = \varprojlim E_{\ell^n}$ (for ℓ prime to p, E/k algebraically closed of char p, it is a free module of rank 2 on $\mathbb{Z}_\ell$)

$V_\ell(E) = T_\ell(E)\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$, makes sense for elliptic curves up to isogeny

$$\widehat{T}_{p'} = \varprojlim_{(p,n)=1} E_n$$

$\widehat{V}_{p'}(E) = \widehat{T}_{p'}\otimes_{\mathbb{Z}}\mathbb{Q}$, a $\mathbb{A}^{f,p'} = \prod_{\ell\neq p,\infty}\mathbb{Q}_\ell$–module of rank 2, makes sense for all curves up to isogeny

$\mathbb{Z}_\ell(1) = \varprojlim \mu_{\ell^n}$

$\mathbb{Q}_\ell(1) = \mathbb{Z}_\ell(1)\otimes\mathbb{Q}_\ell$

$(1) = \otimes\mathbb{Q}_\ell(1)$ or $\otimes\mathbb{Z}_\ell(1)$

$$d:\begin{cases} d(E)=\mathbb{Z} \\ \text{If } f:E\to F \text{ then } d(f):d(E)\to d(F) \text{ is } \deg(f)\ (0 \text{ if } f=0)\end{cases}$$

Then

a) $d(E)\otimes\mathbb{Q}$ makes sense for elliptic curves up to isogeny: notation $d(E\otimes\mathbb{Q}) = d(E)\otimes\mathbb{Q}$
b) for E_0 an elliptic curve up to isogeny, the e_n-pairings, and their behavior under isogeny, enables one to define

$$\bigwedge^2 V_\ell(E_0)\simeq d(E_0)\otimes\mathbb{Q}_\ell(1).$$

At p: Let us look at the case of supersingular curves of char p, then, a substitute for $T_p(E)$ is the formal group of E (a height 2 dim 1 formal group). The formalism of d has the following analogue

a) a height 2 dim 1 formal group defines $d_p(F)$, a rank one module over $\mathbb{Z}_p$: $d_p(F) =$ set of $F \xrightarrow{u} F^*$ ($F^* =$ Pontryagin dual), with ${}^t u = -u$.
b) if $\widetilde{F}/S$ is a deformation of F over S (local complete), and if $T_p(\widetilde{F})$ is the corresponding local system on $S \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, then

$$\bigwedge^2 T_p(\widetilde{F}) \simeq d_p(F)(1)$$

c) for E a supersingular elliptic curve, with corresponding formal group $\widehat{E}$,

$$d(E) \otimes \mathbb{Z}_p \simeq d_p(\widehat{E}).$$

Recovering E. Let E_0 be a supersingular elliptic curve up to isogeny. An elliptic curve E with an isomorphism $E \otimes \mathbb{Q} \xrightarrow[\beta]{\sim} E_0$ defines

a) a "lattice" $\widehat{T}_{p'}(E)$ in $\widehat{V}_{p'}(E_0)$
b) a "lattice" $d(E)$ in $d(E_0)$ (the p'–part of it is determined by a): $d(E) \otimes \mathbb{Z}_\ell(1) \simeq \bigwedge^2 T_\ell(E) \subset \bigwedge^2 V_\ell(E) = d(E_0) \otimes \mathbb{Q}_\ell(1)$).

Lemma 1. It amounts to the same to give either [E_0 supersingular]

α) (E, β)
β) the lattices $\widehat{T}_{p'}(E) \subset \widehat{V}_{p'}(E_0)$ and $d(E) \otimes \mathbb{Z}_p \subset d(E_0) \otimes \mathbb{Q}_p$.

The reason why no more information is required at p is that, for supersingular E, the only degree p^k–isogeny of source E is $E \to E^{(p^k)}$.

Variant: Let F_0 be a height 2 dim 1 formal group law, up to isogeny. Then, to give F defining F_0 amounts to give $d_p(F)(\sim \mathbb{Z}_p) \subset d_p(F_0)(\sim \mathbb{Q}_p)$.

B. The fundamental local construction

Let:

$\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$

$\overline{\mathbb{F}}_p$ be the algebraic closure of $\mathbb{F}_p$, residue field of $\overline{\mathbb{Q}}_p$

k be an algebraic closure of $\mathbb{F}_p$, provided with a class modulo integral powers of Frobenius of allowed isomorphisms $k \xrightarrow{\sim} \overline{\mathbb{F}}_p$. The class is denoted $Isom(k, \overline{\mathbb{F}}_p)$.

V_0 be a 2-dimensional vector space over $\mathbb{Q}_p$

E_0 be a supersingular elliptic curve up to isogeny over k

F_0 be the formal group up to isogeny $/k$ defined by E_0.

To be able to use the "transport de structure", I prefer *not* to take $k = \overline{\mathbb{F}}_p$ nor $V_0 = \mathbb{Q}_p^2$.

1. First construction: Let $\sigma \in Isom(k, \overline{\mathbb{F}_p})$ and $\beta \in Isom(d(E_0) \otimes \mathbb{Q}_p(1), \bigwedge^2 V_0)$. Here $\mathbb{Q}_p(1)$ is *relative to* $\overline{\mathbb{Q}_p}$:

$$\mathbb{Q}_p(1) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} (\varprojlim_n (\text{group of } p^n\text{-roots of unity of } \overline{\mathbb{Q}}_p)).$$

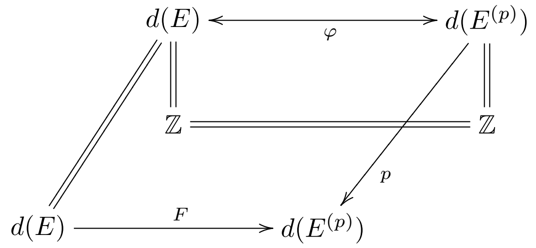
σ and β do define $E_0(\sigma, \beta) = (\sigma(E_0), \sigma(\beta))$ where

$$\begin{cases} \sigma(E_0) & \text{is an elliptic curve up to isogeny over } \overline{\mathbb{F}}_p \\ \sigma(\beta) & \text{is an isomorphism of one-dimensional vector spaces over } \mathbb{Q}_p \end{cases}$$

$$d(\sigma(E_0)) \otimes \mathbb{Q}_p(1) \underset{\sigma}{=} d(E_0) \otimes \mathbb{Q}_p(1) \xrightarrow{\sim} \bigwedge^2 V_0.$$

Let φ be the Frobenius substitution $\varphi : x \mapsto x^p; k \to k$. Then, for any elliptic curve E/k, $\varphi(E)$ is $E^{(p)}$ and one disposes of [has at one's disposal] the Frobenius isogeny $F : E \to E^{(p)}$.

The diagram



is commutative. In particular, F is an isomorphism $F : E_0 \xrightarrow{\sim} \varphi(E_0)$, and the diagram

$$\begin{array}{ccc} d(E_0) & \overset{\varphi}{\longleftrightarrow} & d(\varphi(E_0)) \\ \| & & \downarrow p \\ d(E_0) & \xrightarrow{F} & d(\varphi(E_0)) \end{array}$$

is commutative.

F hence induces an isomorphism

(1) $$F \text{ or } \sigma F : (\sigma(E_0), \sigma(\beta)) \xrightarrow{\sim} (\sigma\varphi(E_0), \sigma(p^{-1}\beta))$$

Definition. D_p is the one dimensional vector space over $\mathbb{Q}_p$, quotient of $Isom(k, \overline{\mathbb{F}}_p) \times Isom(d(E_0) \otimes \mathbb{Q}_p(1), \bigwedge^2 V_0)$ by the equivalence relation $(\sigma, \beta) \sim (\sigma\varphi^k, p^{-k}\beta)$ for $k \in \mathbb{Z}$.

(1) defines an isomorphism between $E_0(\sigma, \beta)$ and $E_0(\sigma', \beta')$ for $(\sigma, \beta) \sim (\sigma', \beta')$, and those isomorphisms form a transitive system of isomorphisms. They hence allow us to define $(E_0(\delta), \beta(\delta))$ for $\delta \in D_p$, where

$$\begin{cases} E_0(\delta) & \text{is a (supersingular) elliptic curve up to isogeny on } \overline{\mathbb{F}}_p, \text{ and} \\ \beta(\delta) & \text{is an isomorphism } d(E_0(\delta)) \otimes \mathbb{Q}_p(1) \xrightarrow{\sim} \bigwedge^2 V_0. \end{cases}$$

Remark. The following groups are acting on D_p (by "transport de structure")

α) [crossed out]

β) $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ via its actions both on $Isom(k, \overline{\mathbb{F}}_p)$ and $\mathbb{Q}_p(1)$. If the isomorphism $cl\ W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)^{ab} = \mathbb{Q}_p^\times$ is normalized (±) so that Frobeniuses φ correspond to *inverse* of uniformizing parameter, then

$$\sigma.\delta = cl(\sigma)^{-1}.\delta.$$

γ) $GL(V_0)$ via its action on $\bigwedge^2 V_0$. One has

$$g.\delta = \det(g).\delta.$$

δ) $Aut(E_0)$, via its action on $d(E_0)$. We define

$$H = Aut(E_0).$$

It is the multiplicative group of the quaternion algebra ramified at p and at ∞. One has

$$g.\delta = Nrd(g)^{-1}.\delta \quad \text{(reduced norm)}.$$

Remark 2. Assume that a lattice $\Lambda \simeq \mathbb{Z}_p$ has been chosen in $\bigwedge^2 V_0$. Then, $(E_0, \beta)/\overline{\mathbb{F}}_p$ as above define an elliptic curve up to p'-isogeny on $\overline{\mathbb{F}}_p$, corresponding to the lattice $\beta^{-1}(\Lambda)(-1) \subset d(E_0) \otimes \mathbb{Q}_p$. p'-isogeny means isogeny of degree prime to p (cf. Lemma A1).

1′. Variant. Let us start with F_0 instead of E_0. D_p is defined as before, using $d_p(F_0)$ instead of $d(E_0) \otimes \mathbb{Q}_p$, and is the same as before, via the isomorphism $d(E_0) \otimes \mathbb{Q}_p \simeq d_p(F_0)$ $(F_0 = \widehat{E}_0)$. This time, D_p is acted on by $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, $GL(V_0)$ and $\mathrm{Aut}(F_0) = H(\mathbb{Q}_p)$. The formuli are the same as before.

$\delta \in D_p$ defines

$$\begin{cases} F_0(\delta), & \text{a (supersingular) formal group law up to isogeny on } \overline{\mathbb{F}}_p \\ \beta(\delta), & d(F_0(\delta)) \otimes \mathbb{Q}_p(1) \xrightarrow{\sim} \bigwedge^2 V_0. \end{cases}$$

If Λ has been chosen in $\bigwedge^2 V_0$, one gets

$$\begin{cases} F(\delta), & \text{a (supersingular) formal group law on } \overline{\mathbb{F}}_p \\ \beta(\delta), & d(F(\delta)) \otimes \mathbb{Z}_p(1) \xrightarrow{\sim} \Lambda. \end{cases}$$

2. Second construction.

I have now to define *"vanishing cycles varieties"* and vanishing cycles groups. Eventually, for K^0 an open compact subgroup of $SL(V_0)$, and for $\delta \in D_p$, a scheme $V(K^0, \delta)$ over $\overline{\mathbb{Q}}_p$ will be defined. For K an open compact subgroup of $GL(V_0)$ such that $K^0 = K \cap SL(V_0)$, isomorphisms

$$V(K^0, \delta) \overset{(K)}{\longleftrightarrow} V(K^0, \alpha\delta) \qquad (\text{for } \alpha \in \det K^0 \subset \mathbb{Z}_p^\times)$$

are defined. For variable δ, the $V(K^0, \delta)$'s will thus be a "local system" of schemes on D_p. The $V(K^0, \delta)$ are not of finite type, but of the type usual in the vanishing cycle theory. However we won't need it, here is a description of the $\overline{\mathbb{Q}}_p$-valued points of $V(K^0, \delta)$.

a) δ provides us with $(E_0(\delta), \beta(\delta))$ on $\overline{\mathbb{F}}_p$
b) a point of $V(K^0, \delta)$ is an isomorphism class of systems consisting in
  α) an elliptic curve up to isogeny on $\overline{\mathbb{Q}}_p$: E
  β) an isomorphism $\alpha : V_p(E) \xrightarrow{\sim} V_0$, given mod K^0
  γ) an isomorphism of the reduction of E with $E_0(\delta)$: $\psi : E|\overline{\mathbb{F}}_p \simeq E_0(\delta)$

δ) required: the diagram

$$(*)\qquad \begin{array}{ccc} \bigwedge^2 V_p(E) & \xrightarrow{\bigwedge^2\alpha} & \bigwedge^2 V_0 \\ \Vert & & \wr\uparrow \beta(\delta) \\ d(E)(1) & \xrightarrow[\psi]{\sim} & d(E_0(\delta))(1) \end{array}$$

is commutative.

If K is as above, it amounts to the same to give α only $\mod K$, $(*)$ being required only $\mod \det(K)$.

To actually construct $V(K,\delta)$, let us start with

K an open compact subgroup of $GL(V_0)$

E_0 a supersingular curve up to isogeny on $\overline{\mathbb{F}}_p$ (in practice, $E_0(\delta)$)

$\beta : d(E_0)\otimes\mathbb{Q}_p(1) \xrightarrow{\sim} \bigwedge^2 V_0$ (in practice, $\beta(\delta)$)

$\overline{\beta} = \beta \mod \det(K) \subset \mathbb{Z}_p^\times$ (only $\overline{\beta}$, not β, will be used).

The construction will involve the auxiliary data of

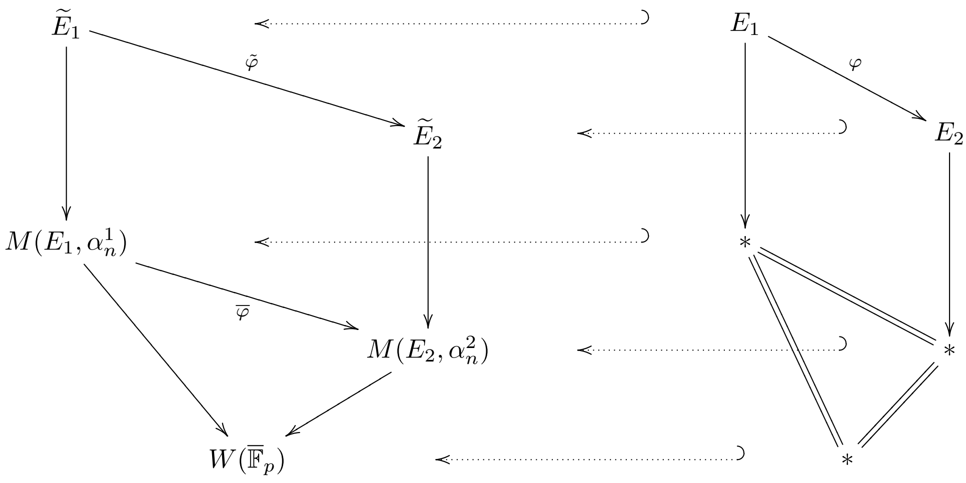
L a K-stable lattice in V_0.

a) $\beta^{-1}(\bigwedge^2 L)(-1)$ is a lattice in $d(E_0)\otimes\mathbb{Q}_p$ and defines an elliptic curve up to p'-isogeny E', with $E'\otimes\mathbb{Q}\simeq E_0$. Let me choose an elliptic curve E with level n structure α_n ($n\geq 3$, $(n,p)=1$) with $E\otimes\mathbb{Z}_{(p)}=E'$. [The construction will be shown to be independent of L, E', E, α_n.] Let M_n be the modular scheme for elliptic curves with level n structure and $e\in M_n(\overline{\mathbb{F}}_p)$ be the point of M_n defined by (E,α_n).

b) $M(E,\alpha_n)$ *is* the spectrum of the henselization of the local ring at e of $M_n\otimes_{\mathbb{Z}} W(\overline{\mathbb{F}}_p)$.

c) The completion of $M(E,\alpha_n)$ is isomorphic to $W(\overline{\mathbb{F}}_p)[[t]]$.

d) Assume that (E_1,α_n^1) and (E_2,α_n^2) are two systems as above, and that $\varphi : E_1\to E_2$ is a p'-isogeny. There is then one and only one isomorphism $\overline{\varphi} : M(E_1,\alpha_n^1)\to M(E_2,\alpha_n^2)$ fitting in a commutative diagram



In this diagram, $*$ means $Spec(\mathbb{F}_p)$ (= point), and $\widetilde{E}_1$, $\widetilde{E}_2$ are the pull-back over $M(E_1,\alpha_n^1)$, $M(E_2,\alpha_n^2)$ of the universal curve over M_n.

This can be better expressed by saying that $M(E_1,\alpha_n^1)$ is the parameter space of the universal deformation of the elliptic curve up to p'-isogeny $E'/\overline{\mathbb{F}}_p$ (universal is with respect to deformation over henselian local $W(\overline{\mathbb{F}}_p)$-algebras). This allows us to write simply $M(E')$ for $M(E_1,\alpha_n^1)$ and $\widetilde{E}'$ for the elliptic curve up to p'-isogeny over $M(E')$ defined by (any) E_1.

For n large enough, K is the subgroup of $GL(L)$ inverse image of a suitable subgroup $\overline{K}$ of $GL(L/p^nL)$.

Let $K(\overline{\mathbb{F}}_p)$ be the field of fractions of $W(\overline{\mathbb{F}}_p)$, and let $\overline{\mathbb{Z}}_p$ be the ring of integers in $\overline{\mathbb{Q}}_p$. The group scheme E'_{p^n} over $M(E')$ is finite etale over $M(E')\otimes K(\overline{\mathbb{F}}_p)$, hence

$$\underline{Isom}(E'_{p^n}, L/p^nL)$$

is a finite etale Galois covering of $M(E')\otimes K(\overline{\mathbb{F}}_p)$, with Galois group $GL(L/p^nL)$. Let us rather consider

$$\underline{Isom}_{M(E')\otimes\overline{\mathbb{Q}}_p}(E'_{p^n}, L/p^nL).$$

This time, we get a *disconnected* covering of $M(E')\otimes\overline{\mathbb{Q}}_p$. A piece of it can be picked as follows: via β, one has $\bigwedge^2 E'_{p^n} = d(E')_p\otimes\mathbb{Z}/p^n(1) = \bigwedge^2 L/p^nL$, and one considers only isomorphisms of "determinant 1". Similarly, $\overline{\beta}$ enables one to pick a component of

$$\overline{K}\backslash\underline{Isom}_{M(E')\otimes\overline{\mathbb{Q}}_p}(E'_{p^n}, L/p^nL).$$

We will call the component $V_L(K,E'_0,\beta)$. It is also the quotient by $\overline{K}\cap SL(L/p^nL)$ of the picked component of $\underline{Isom}_{M(E')}(E'_{p^n}, L/p^nL)\otimes_{W(\overline{\mathbb{F}}_p)}\overline{\mathbb{Q}}_p$.

Summary. $V_L(K,E'_0,\beta)$ depends on $K\subset GL(V_0)$, compact open, E'_0 up to isogeny on $\overline{\mathbb{F}}_p$, $\beta : d(E')\otimes\mathbb{Q}_p(1)\xrightarrow{\sim}\bigwedge^2 V_0$, and of a lattice L in V_0, stable by K. It does not depend on the whole of β, but only on $\overline{\beta}=\beta \mod \det(K)$. For $K'\subset K$, one has a map

$$V_L(K',E',\beta)\longrightarrow V_L(K,E',\beta).$$

Construction $V_L(K,E'_0,\beta)$ is independent of L.

More precisely, the system consisting of

$$\begin{cases} V_L(K,E',\beta) \\ \text{the elliptic curve up to isogeny } \widetilde{E}'_0 \text{ over} V_L(K,E',\beta) \\ \text{the universal isomorphism } \alpha \text{ given} \quad \mod K : V_p(\widetilde{E}'_0)\to V_0 \\ \qquad \text{(deduced from } \overline{\alpha} : \widetilde{E}'_{p^n}\xrightarrow{\sim} L/p^nL, \quad \mod \overline{K}, \quad \text{or } \overline{\alpha} : T_p(\widetilde{E}')\xrightarrow{\sim} L, \quad \mod K). \end{cases}$$

is independent of L.

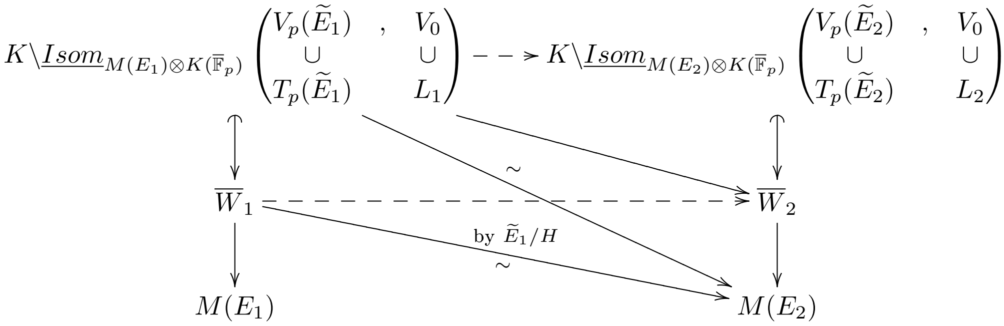
Let L_1 and L_2 be two K-invariant lattices, and let E_1 and E_2 be the corresponding elliptic curves up to p'-isogeny over $\overline{\mathbb{F}}_p$. We are to define an isomorphism

$$\begin{array}{ccc} K\backslash\underline{Isom}_{M(E_1)\otimes K(\overline{\mathbb{F}}_p)}\begin{pmatrix} V_p(\widetilde{E}_1) & , & V_0 \\ \cup & & \cup \\ T_p(\widetilde{E}_1) & & L_1 \end{pmatrix} & \longleftrightarrow & K\backslash\underline{Isom}_{M(E_2)\otimes K(\overline{\mathbb{F}}_p)}\begin{pmatrix} V_p(\widetilde{E}_2) & , & V_0 \\ \cup & & \cup \\ T_p(\widetilde{E}_2) & & L_2 \end{pmatrix} \\ \downarrow & & \downarrow \\ M(E_1) & & M(E_2) \end{array}$$

For simplicity, we assume $L_1 \subset L_2$. Then, over $K\backslash\underline{Isom}_{M(E_1)}\cdots$, $\widetilde{E}_1$ is provided with a subgroup isomorphic to L_2/L_1, call it H. Let $\overline{W}_1(K,E_0,\beta)$ be the normalization of $M(E_1)$ in $K\backslash\underline{Isom}\cdots$. Due to the normality of $\overline{W}$ and the fact that an elliptic curve has only *finitely many subgroup-schemes of a given order, the subgroup-scheme H of $\widetilde{E}_1$ on $\overline{W}_1(K,E_0,\beta)\otimes K(\overline{\mathbb{F}}_p)$ extends* as a subgroup-scheme H on $\widetilde{E}_1$ on $\overline{W}_1(K,E_0,\beta)$. The quotient $\widetilde{E}_1/H$ is a deformation of E_2, hence a map

$$\overline{W}_1(K,E_0,\beta) \longrightarrow M(E_2).$$

Further, $V_p(\widetilde{E}_1/H)=V_p(\widetilde{E}_1)$, and, over $K\backslash\underline{Isom}\begin{pmatrix} V_p(\widetilde{E}_1) & , & V_0 \\ \cup & & \cup \\ T_p(\widetilde{E}_1) & & L_1\end{pmatrix}$, isomorphism $\alpha : V_p(E_1)\to V_0$ carrying $T_p(\widetilde{E}_1)$ to L_1 do carry $T_p(\widetilde{E}_1/H)$ to L_2: if W_2 is defined as W_1, one has



This defines the dotted maps; they are the seeked for isomorphisms.

By extending the scalars to $\overline{\mathbb{Q}}_p$ and taking one component, one gets the isomorphisms expressing that $V_L(K,E_0,\beta)$ is independent of L. For $\delta\in D_p$, we note

$$V(K,\delta)=V_L(K,E_0(\delta),\beta(\delta)).$$

Summary. $V(K,\delta)$ is a scheme over $\overline{\mathbb{Q}}_p$, it depends on K compact open in $GL(V_0)$ and on $\delta\in D_p$, given modulo multiplication by elements of $\det(K)\subset\mathbb{Z}_p^\times$. For K smaller and smaller, the $V(K,\delta)$ form a projective system. Over $V(K,\delta)$ is given an elliptic curve up to isogeny $\widetilde{E}$, provided with an isomorphism, given mod K, $\alpha : V_p(\widetilde{E}) \xrightarrow{\sim} V_0$. In a sense, $\widetilde{E}$ is a deformation of $E_0(\delta)$, in particular $d(E_0(\delta))=d(\widetilde{E})$. The morphism α is compatible with $\beta(\delta)$

$$\textstyle\bigwedge^2 V_p(\widetilde{E}) = d(E_0(\delta))\otimes\mathbb{Q}_p(1) \;\overset{\det(\alpha)}{\underset{\beta(\delta)}{\overset{\longrightarrow}{\underset{\longrightarrow}{||}}}}\; \bigwedge^2 V_0 \quad (\mathrm{mod}\,\det(K))$$

In fact, the statement that $\widetilde{E}$ is a deformation of E_0 can be made more precise by introducing a suitable $\widetilde{E}/\overline{V}(K,\delta)/\overline{\mathbb{Z}}_p$.

$2'$. *Variant.* Starting with F_0 instead of E_0, one can construct analogues of the $V(K,\delta)$, called $\widehat{V}(K,\delta)$, with complete local rings replacing henselian local rings.

3. Third construction.

We are interested in the ℓ-adic cohomology groups

$$H^1(V(K,\delta),\mathbb{Q}_\ell).$$

These groups are finite dimensional, and locally constant as a functions of δ (as $V(K,\delta)$ itself is). If K' is a distinguished subgroup of K, one clearly has

$$H^1(V(K,\delta),\mathbb{Q}_\ell) = H^1(V(K',\delta),\mathbb{Q}_\ell)^{K\cap SL(V_0)/K'\cap SL(V_0)} \qquad \text{(invariants)}.$$

($V(K,\delta)$ depends also only on $K\cap SL(V_0)$ and δ.)

The *local fundamental object* is the "bundle" $\mathcal{H}/D_p$, with

$$\mathcal{H}_\delta = \varinjlim_K H^1(V(K,\delta),\mathbb{Q}_\ell).$$

There is a notion of "locally constant section of $\mathcal{H}_\delta/D_p$". It is a function $\varphi(\delta)(\delta\in D_p,\ \varphi(\delta)\in\mathcal{H}_\delta)$ with locally $\varphi(\delta)$ in a $H^1(V(K,\delta),\mathbb{Q}_\ell)$ and locally constant. The space of locally section is noted $\Gamma(D_p,\mathcal{H})$

On the local fundamental object are acting, by "transport de structure":

a) $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$
b) $GL(V_0)$
c) $Aut(E_0)=H$.

The actions over D_p being those already described.

The actions of $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and $GL(V_0)$ are "continuous" (the latter, will respect the notion of locally constant section).

Proposition. The action of $Aut(E_0)$ extends as a continuous action of $H(\mathbb{Q}_p)$ on $\mathcal{H}/D_p$.

I don't have a satisfactory proof. My idea of proof would be to go back to the $\overline{W}$ introduced earlier and to express $\mathcal{H}$ in terms of special fibre of stable models of $\overline{W}$ over suitable ramified extensions of $W(\overline{\mathbb{F}}_p)$.

Intuitively, one may argue that $V(K,\delta)$ and $\widehat{V}(K,\delta)$ could have the same cohomology, that $Aut(F_0)=H(\mathbb{Q}_p)$ acts on $\widehat{V}(K,\delta)$, hence on $H^1(\widehat{V}(K,\delta),\mathbb{Q}_\ell)$, and that it would be hell if the action were not continuous.

C. Statement of the local results. (The proofs will be of a global nature.)

It will be easier to work not with $\mathbb{Q}_\ell$-cohomology, but with $\overline{\mathbb{Q}}_\ell$-cohomology, obtained by extending $\mathbb{Q}_\ell$ to an algebraic closure $\overline{\mathbb{Q}}_\ell$. By abuse of language, we will again denote by $\mathcal{H}/D_p$ the "admissible" bundle over D_p with fibre

$$\mathcal{H}_\delta = \varinjlim_K H^1(V(K,\delta),\mathbb{Q}_\ell)\otimes_{\mathbb{Q}_\ell}\overline{\mathbb{Q}}_\ell.$$

The groups $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, $GL(V_0)$, $H(\mathbb{Q}_p)$ act admissibly on $\mathcal{H}/D_p$. The actions on D_p have been computed. Of course, the actions commute the one with the other. If $a\in\mathbb{Q}_p^*$, the action of the elements $a\in GL(V_0)$ and $a\in H(\mathbb{Q}_p)$ are *inverse* the one of the other. In terms of a fixed $\delta_0\in D_p$, this could as well be expressed by saying

A/ the subgroup of $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\times GL(V_0)\times H(\mathbb{Q}_p)$ formed by the elements such that

$$cl(\sigma)^{-1}.\det(g).Nrd(h)^{-1}=1$$

acts on $\mathcal{H}_{\delta_0}$

B/ the subgroup $(1,1,a,a)$ acts trivially.

Mnemotechnik way to check the action on the $V(K,\delta)/D_p$, it is good to view a point of $\varprojlim V(K,\delta)/D_p$ as consisting of

- $\delta \in D_p$
- $\widetilde{E}$: elliptic curve up to isogeny on $\overline{\mathbb{Q}}_p$
- $\alpha : V_p(\widetilde{E}) \xrightarrow{\sim} V_0$
- ψ "specialization map" $\widetilde{E} \overset{\psi}{\dashrightarrow} \delta(E_0)$

with a compatibility between α, ψ, and $\beta(\delta)$.

Let $\chi : \mathbb{Q}_p^* \longrightarrow \overline{\mathbb{Q}}_p^*$ be a quasi-character (with open kernel).

We denote by

$\Gamma(D_p,\mathcal{H})$ the $\overline{\mathbb{Q}}_\ell$-vector space of locally constant sections of $\mathcal{H}/D_p$

$\Gamma_\chi(D_p,\mathcal{H})$ the subspace of those sections for which, for any $a \in \mathbb{Q}_p^*$, with image $z(a)$ in the center(?) of $GL(V_0)$,

$$z(a).f = \chi(a).f, \quad i.e. \quad z(a)f(\delta) = \chi(a)f(a^2\delta).$$

Theorem. (i) $\Gamma_\chi(D_p,\mathcal{H})$ is a direct sum of triple tensor products

$$\Gamma_\chi(D_p,\mathcal{H}) = \bigoplus_{\varphi\in\Phi} V_\varphi \otimes V'_\varphi \otimes W_\varphi$$

where

α) V_φ is an admissible irreducible of $GL(V_0)$, the center of $GL(V_0)$ acting by the character χ, and V_φ being of the discrete series (=special or supercuspidal)

β) V'_φ is an admissible irreducible (finite dimensional) representation of $H(\mathbb{Q}_p)$ with the center acting by χ^{-1}

γ) W_φ is a 2-dimensional (or 1-dimensional) continuous $\overline{\mathbb{Q}}_\ell$-adic irreducible representation of $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. If it is 2-dimensional, then $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts on $\bigwedge^2 W_\varphi$ by the character . If 1-dimensional, defined by a character ν, then $\nu =$ $(\bigwedge^2(W_\varphi \oplus W_\varphi(1))$ corresponds again to).

(ii) V'_φ runs (once and only once) through the representations said in (i)
(iii) The same holds for V_φ, and V_φ and V'_φ correspond by the Weil representation construction (suitably normalized)
(iv) W_φ one dimensional $\Leftrightarrow$ V'_φ is $\Leftrightarrow$ V_φ is special, and if V'_φ is $\mu(Nrd)$, then W_φ is the character of $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \xrightarrow[cl]{\sim} \mathbb{Q}_p^*$.
(v) If V_φ is defined by a quasi-character of a [... line cut off ...].

Except for $p=2$, this gives a complete determination of $\Gamma_\chi(D_p,\mathcal{H})$.

For μ a quasi-character of $\mathbb{Q}_\ell^*$ and $\delta_0 \in D_p$, multiplication by the function $\mu(\delta\delta_0^{-1})$ on D_p provides an isomorphism

$$\Gamma_\chi(D_p,\mathcal{H}) \longrightarrow \Gamma_{\chi\mu^{-2}}(D_p,\mathcal{H});$$

if $V_\varphi \otimes V'_\varphi \otimes W_\varphi$ occurs in $\Gamma_\chi(D_p,\mathcal{H})$, then

$$(V_\varphi \otimes \mu^{-1}\det(g)) \otimes (V'_\varphi \otimes \mu\ Nrd) \otimes (W_\varphi \otimes \mu\ cl)$$

occurs in $\Gamma_{\chi\mu^{-2}}(D_p, \mathcal{H})$.

D. Global Theory.

We consider

K open compact subgroup of $GL(2, \mathbb{A}^f)$

$X^\pm$ Poincaré upper and lower half-plane.

$$X^\pm = Isom_{\mathbb{R}}(\mathbb{Z}^2 \otimes \mathbb{R}, \mathbb{C})/\mathbb{C}^* \subset Hom(\mathbb{Z}^2, \mathbb{C})/\mathbb{C}^*$$

($GL(2, \mathbb{R})$ acts on the *right* on $\mathbb{Z}^2 \otimes \mathbb{R}$ via its action on $\mathbb{Z}^2 \otimes \mathbb{R} = \mathbb{R}^2$).

$M_k^0(\mathbb{C}) \;=\; K\backslash X^\pm \times GL(2, \mathbb{A}^f)/GL(2, \mathbb{Q})$

k an integer ($k \geq 0$)

μ the representation Sim^k (dual of obvious representation) of $GL(2, \mathbb{Q})$

$F_\mu^{\mathbb{Q}}$ the corresponding local system on $M_K^0(\mathbb{C})$

$M_K(\mathbb{C}) =$ the Satake compactification of $M_K^0(\mathbb{C})$; $j : M_K^0(\mathbb{C}) \hookrightarrow M_K(\mathbb{C})$

$F_\mu^{\mathbb{Q}}$ the sheaf $j_* F_\mu^{\mathbb{Q}}$ on $M_K(\mathbb{C})$

$\mathcal{H}^{\mathbb{Q}}(\mu) = \varinjlim_K H^1(M_K(\mathbb{C}), F_\mu^{\mathbb{Q}})$ (an admissible representation of $GL(2, \mathbb{A}^f)$ defined $/\mathbb{Q}$))

$\mathcal{H}^{\mathbb{C}}(\mu) = \mathcal{H}(\mu) \otimes \mathbb{C}$; $\mathcal{H}^{\mathbb{Q}_\ell} = \mathcal{H}(\mu) \otimes \mathbb{Q}_\ell$; $F_\mu^{\mathbb{Q}_\ell} = F_\mu \otimes \mathbb{Q}_\ell$.

One has a decomposition

$$\mathcal{H}^{\mathbb{C}}(\mu) = \mathcal{H}^{k+1,0} \oplus \mathcal{H}^{0,k+1}$$

of $\mathcal{H}^{\mathbb{C}}$ into 2 complex conjugate subspaces; $\mathcal{H}^{k+1,0}$ is a complex admissible representation of $GL(2, \mathbb{A}^f)$, which can be defined over $\mathbb{Q}$; it is hence isomorphic to the complex conjugate representation $\mathcal{H}^{0,k+1}$. It corresponds to holomorphic modular cusp forms of weight $k+2$ (note that $k+2 \geq 2$!). For some explicit admissible representation D_{k+2} of $GL(2, \mathbb{R})$, of the discrete series,

$$\mathcal{H}^{k+1,0} = Hom_{GL(2,\mathbb{R})}(D_{k+2}, L_0(GL(2, \mathbb{A})/GL(2, \mathbb{Q})).$$

Further, $\{M_K(\mathbb{C})\}$ is naturally defined $/\mathbb{Q}$ (with its $GL(2, \mathbb{A}^f)$-action: $M_K \xrightarrow[g]{\sim} M_{gKg^{-1}}$), and $F_\mu^{\mathbb{Q}_\ell}$ is an ℓ-adic sheaf, defined $/\mathbb{Q}$. Hence,

$\mathcal{H}^{\mathbb{Q}_\ell}(\mu)$ carries a $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-action, commuting with $GL(2, \mathbb{A}^f)$, acting by "transport de structure". After extension of $\mathbb{Q}_\ell$ to $\overline{\mathbb{Q}}_\ell$, one has a decomposition

$$\mathcal{H}^{\mathbb{Q}_\ell}(\mu) = \bigoplus_{f \in F} (\otimes_p V_{f,p}) \otimes W_f$$

$V_{f,p}$: irreducible admissible representation of $GL(2, \mathbb{Q}_p)$
W_f: 2-dimensional representation of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$
(F=spectrum of $GL(2)$)

By Eichler–Shimura–Kuga–Deligne–Ihara–Пятецкий-Шапиро–Langlands, if $V_{f,p}$ is of the principal series, then $W_f|Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ (restriction to the decomposition group) is a sum of 2 corresponding characters. If $V_{f,p}$ is special, then $W_f|Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is the corresponding special ℓ-adic representation of $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

I can now prove

(A) If $V_{f,p}$ is supercuspidal, then $W_f|Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is irreducible, and, with the notation of C.Theorem, if $V_{f,p} \sim V_\varphi$, then $W_f \sim W_\varphi$.

(Hence $V_{f,p}$ determines, by a local rule, $W_{f,p} = W_f|Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, and the rule is the obvious one (up to a normalization) if $V_{f,p}$ corresponds by the Weil construction to a character of a quadratic extension.)

The method is to use the theory of vanishing cycles to prove that a space to be described below is a quotient of $\mathcal{H}(\mu) \otimes \mathbb{Q}_\ell$. (This is accurate only for $k \neq 0$; I will not bother much about $k = 0$ and the phenomena related to special representations.)

We keep the notation of B., except that now $V_0 = \mathbb{Q}_p^2$. Let us consider

$$Isom(V_{p'}(E_0), (\mathbb{A}^{f,p'})^2) \times D_p$$

and the right action of H on it (by composition for the first factor, and the inverse of the already defined action on D_p). On this space, we have the following H-equivariant local system

$$Sim^k(V_\ell(E)^*) \otimes \varinjlim_K V(K, \delta)$$

(on the second factor, a right action is required, one takes the inverse of the one already constructed). The space and local system is acted by $GL(2, \mathbb{A}^f)$

$$\begin{cases} \text{space :} & \text{composition on } 1^{st} \text{ factor, already described on } D_p \\ \ell. \text{ system :} & \text{trivial on the first factor, described on } 2^{nd} \end{cases}.$$

We now take as announced representation of $GL(2, \mathbb{A}^f)$:

$$H^0 \left(\begin{array}{c} \text{local system } Sim^k(V_\ell(E_0)^*) \otimes \varinjlim_K V(K, \delta) \\ \text{on} \\ Isom(V_{p'}(E_0), (\mathbb{A}^{f,p'})^2) \times D_p \end{array} \quad /H \right)$$

The action of $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is via its action on $V(K, \delta)/D_p$

I now wish to relate that H^0 with the spectrum of $H(\mathbb{A})/H(\mathbb{Q})$ and $\Gamma(D_p, \mathcal{H})$.

For simplicity, let me in any way extend the scalars from $\mathbb{Q}_\ell$ to $\mathbb{C}$ (not to lose the Gal action, I have to use here that it is continuous for the discrete topology of $\mathbb{Q}_\ell$ – which I can do directly). Let me also choose an isomorphism

$$\begin{cases} V_{p'}(E_0) \simeq (\mathbb{A}^{f,p'})^2, & \text{hence} \\ H(\mathbb{Q}_\ell) \simeq GL(2, \mathbb{Q}_\ell) & (\ell \neq p) \end{cases}$$

Then

$$H^0 = \{ \text{ functions } H(\mathbb{A}^{f,p}) \to Sim^k(\ ^*) \otimes \Gamma(D_p, \mathcal{H}) \text{ with } f(x\gamma) = f(x)\gamma\}$$

$$= \{ \text{ functions } H(\mathbb{A}) \to Sim^k(\ ^*) \otimes \Gamma(D_p, \mathcal{H}) \text{ with } f(x\gamma) = f(x) \text{ and } f(g_\infty g_p x) = g_\infty g_p f(x)\}$$

$$= \bigoplus_{(\mathbb{A}^*/\mathbb{Q}^*)^\wedge} \{\ \}_\chi$$

$$= \bigoplus_{\substack{\chi \\ (\chi_\infty = \cdots)}} Hom_{H(\mathbb{R})\times H(\mathbb{Q}_p)}(Sim^k(V) \otimes \Gamma_{\chi_p^{-1}}(D_p, \mathcal{H})^\vee, L_0^\chi(H(\mathbb{A})/H(\mathbb{Q})).$$

This can be expressed as follows: In $L_0(H(\mathbb{A})/H(\mathbb{Q}))$ [...cut off line? ...] as a given vector of a given representation of $H(\mathbb{R})$. The representation of $H(\mathbb{A}^f)$ so obtained may be written

$$\bigoplus_{f_0 \in F_0} (\otimes_p V_{f_0,p}),$$

the following representation occurs in $\mathcal{H}(\mu)$:

$$\boxed{\bigoplus_{f_0 \in F} (\bigotimes_{\ell \neq p} V_{f_0,\ell}) \otimes \bigoplus_{V_\varphi \sim V_{f_0,p}} (V_\varphi \otimes W_\varphi)}$$

Now, comparison with Jacquet-Langlands §16, plus the fact that supercuspidal representations cannot occur outside $\mathcal{H}(\mu)$ (this is given by Пятецкий-Шапиро or Langlands + vanishing cycle theory) one gets (i)(α)(β)(ii)(iii), 2 dimensionality in (i)γ) of Theorem C. One also gets statement (A). By checking these general rule against modular forms attached to L functions with grossencharacter of imaginary quadratic fields (+ the end remark of C), one gets statement (v). The proof of (i)γ) in characteristic 2, in cases not covered by (v), uses entirely different ideas, which I cannot explain here.

Yours sincerely

P. Deligne