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DIFFERENTIAL GEOMETRY AND COMPLEX ANALYSIS*

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0. Introduction. The theme of this paper will be negative curvature and complex analysis. The basic reasons underlying this theme may be schematically expressed as follows:

negative curvature $\Rightarrow$ suitable functions are plurisubharmonic,
plurisubharmonic functions $\Rightarrow$ consequences in function theory.

At the risk of oversimplification and omission, the theory as it presently exists may be outlined in the following way:

Negative curvature and complex analysis

- Holomorphic mappings
  - Hodge theory and Kähler manifolds
  - Ahlfors lemma and the Kobayashi metric (little Picard theorems)
  - Big Picard theorems and Hartogs-type results
  - Nevanlinna theory
    - Equidimensional case
    - Holomorphic curves
- Vanishing theorems and study of $\bar{\partial}$ by L^2 methods
  - Hodge theory and Kähler manifolds
  - Greene-Wu theory of simply-connected negatively curved complex manifolds

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*General lecture given at the Institute.

The reader may note the special position of Hodge theory and its subsequent applications to algebraic geometry in the outline. It is here that occurs what to me are the most interesting applications of the philosophy of negative curvature in all of its facets—both the linear and geometric theories appear, but time will not permit exploration of this subject (cf. [G-S] for heuristic discussion and further references). Similarly we will be unable to discuss the recent work of Greene and Wu, and for this we refer to their recent announcements in the *Bulletin of the American Mathematical Society*.

The plan for this paper is

1. Ahlfors lemma and applications.
2. Kobayashi metric and volume forms.
3. Nevanlinna theory—the equidimensional case.
4. Nevanlinna theory—holomorphic curves.

Generally speaking, I shall attempt to give the relevant definitions, some representative proofs, but shall omit all straightforward computations. The background references are [C], [K], [Co-G], and [G-K] (letters refer to the bibliography at the end).

1. Ahlfors lemma and applications.

A. *Hermitian metrics.* Let M be a complex manifold with holomorphic coordinates $z_1, \cdots, z_m$. A Hermitian metric on M is given locally by

$$ds^2 = \sum h_{ij}\, dz_i\, d\bar{z}_j = \sum \varphi_i \bar{\varphi}_i$$

where (h_{ij}) is a C^∞ positive definite Hermitian matrix, and the φ_i are C^∞ (1, 0) forms which diagonalize the metric. Given such a metric, there are two basic consequences:

(i) There exists a unique connection compatible with the metric and complex structure. The structure equation for this connection is

$$d\varphi_i = \sum_j \varphi_{ij} \wedge \varphi_j + t_i \qquad (\varphi_{ij} + \bar{\varphi}_{ji} = 0)$$

where (φ_{ij}) is the connection matrix and the t_i are forms of type (2, 0) (torsion forms). The curvature matrix is defined by

$$\Phi_{ij} = d\varphi_{ij} - \sum_k \varphi_{ik} \wedge \varphi_{kj}.$$

The most important scalar quantities arising from the curvature matrix are the holomorphic sectional curvatures

$$\Phi(\xi) = \frac{1}{|\xi|^2}\left\{\sum \Phi_{ijkl}\, \xi^i \bar{\xi}^j \xi^k \bar{\xi}^l\right\}$$

determined by the (1, 0) vector ξ. The complex manifold M is said to be *negatively curved* in case there exists a Hermitian metric all of whose holomorphic sectional curvatures satisfy

$$\Phi(\xi) \leqq -A < 0$$

for some positive constant A. Multiplying the metric by $1/A$ allows us to assume that $A = 1$, and this will always be done.

(ii) If $S \subset M$ is a complex submanifold with induced metric, then we may choose the φ_i such that $\varphi_{s+1} = \cdots = \varphi_m = 0$ along S. Using the index range $1 \leqq \alpha, \beta \leqq s$ and the obvious notations, the curvature matrix for S is given by a formula

$$\Phi(S)_{\alpha\beta} = \Phi(M)_{\alpha\beta} - \sum_\nu A_{\alpha\nu} \wedge \bar{A}_{\nu\beta}$$

where $(A_{\alpha\nu})$ is a matrix of (1, 0) forms. In particular, if ξ is tangent to S, then $\Phi(S, \xi) \leqq \Phi(M, \xi)$, so that S is negatively curved in case M is. This principle that *curvatures decrease on complex submanifolds* is of fundamental importance, and perhaps may be explained as reflecting the ellipticity of the $\bar{\partial}$-operator. Indeed, in $\boldsymbol{C} \times \boldsymbol{C}$ the graph $(z, f(z))$ of a holomorphic function has negative curvature exactly because of $\partial f/\partial \bar{z} = 0$.

In case M is a Riemann surface, a Hermitian metric is $ds^2 = h\, dz\, d\bar{z}$ with associated (1, 1) form $\Omega = \frac{1}{2}(-1)^{1/2}\, h\, dz \wedge d\bar{z}$. The curvature matrix is a global (1, 1) form Φ and $\Phi = K \cdot \Omega$ where

$$K = -\frac{c}{h}\frac{\partial^2 \log h}{\partial z \partial \bar{z}} \qquad (c > 0)$$

is the Gaussian curvature. Thus

$$K \leqq 0 \Leftrightarrow \log h \text{ is subharmonic}$$

which

(a) using (ii) above shows that negative curvature has to do with plurisubharmonic functions, and

(b) marks the first appearance of the logarithm function, which is ubiquitous in the theory.

EXAMPLES.

(i) The Poincaré metric,

$$\begin{aligned}\pi(\rho) &= \frac{c\rho^2\, dz\, d\bar{z}}{(\rho^2 - |z|^2)^2} && \text{on } \Delta_\rho = \{|z| < \rho\}, \\ &= c_1 \frac{dx\, dy}{y^2} && \text{on } H = \{z = x + iy : y > 0\},\end{aligned}$$

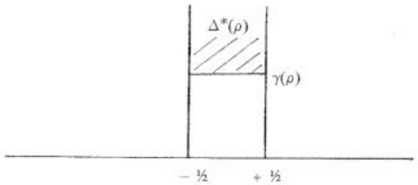
has constant Gaussian curvature $K = -1$ for suitable constants c, c_1.

(ii) The punctured disc $\Delta^* = \{0 < |t| < 1\}$ has universal covering H, and the induced Poincaré metric is

$$\pi(\Delta^*) = \frac{c_2\, dt\, d\bar{t}}{|t|^2(\log 1/|t|^2)^2}.$$

Two trivial but important properties of this metric deal with the circles $\gamma(\rho) = \{|t| = \rho\}$ and concentric punctured discs $\Delta^*(\rho) = \{0 < |t| < \rho\}$, and these are

(a) the length $l(\gamma(\rho)) \to 0$ as $\rho \to 0$,
(b) the area $\int_{\Delta^*(\rho)} \Omega < \infty$ for $\rho < 1$.
Both properties are immediate from the usual picture



of the fundamental domain for $H \to \exp \Delta^*$ and the formula given above for the Poincaré metric on H.

(iii) On the Riemann sphere $\boldsymbol{P}^1 = \boldsymbol{C} \cup \{\infty\}$, the metric

$$\pi = \frac{c\, dz\, d\bar{z}}{(1 + |z|^2)^2}$$

has Gaussian curvature $K = +1$. Given a point $a \neq \infty$, we set

$$\sigma(a) = \frac{|a|^2}{1 + |a|^2}, \qquad \rho(a) = \sigma(a)(\log \mu/\sigma(a))^2$$

where $\mu > 1$ is constant. If a, b, c are distinct finite points, we set

$$\pi(a, b, c) = \left[\frac{1}{\rho(a)\rho(b)\rho(c)}\right]\pi.$$

At each point a, b, c this metric has asymptotically the same singularity as has the Poincaré metric $\pi(\Delta^*)$ at $t = 0$, and a computation shows that the Gaussian curvature of $\pi(a, b, c)$ is $\leqq -1$ for a suitable choice of μ. The fact that $\boldsymbol{P}^1 - \{a, b, c\}$ is negatively curved was traditionally deduced from the uniformization theorem, but the above elementary procedure of writing down negatively curved metrics globalizing the singularity of the Poincaré metric $\pi(\Delta^*)$ at $t = 0$ will work in situations where there is no uniformization.

B. *Ahlfors generalization of the Schwarz lemma.* A *pseudo-metric* on a Riemann surface is the same as a metric except that the coefficient function is allowed to vanish at isolated points. If $f: \Delta_\rho \to M$ is a nonconstant holomorphic mapping into a Hermitian manifold M, then $f^*(ds^2)$ is a pseudo-metric.

Let $h\, dz\, d\bar{z}$ be a pseudo-metric on Δ_ρ with Gaussian curvature K and $\pi(\rho) = h(\rho)\, dz\, d\bar{z}$ the Poincaré metric with constant Gaussian curvature $K_\rho \equiv -1$.

AHLFORS LEMMA. *If $K \leqq -1$, then $h \leqq h(\rho)$.*

PROOF. It will suffice to prove the case $\rho = 1$. For this we write $h = u(\sigma)h(\sigma)$ $(\sigma \leqq 1)$ and observe that
(i) $\lim_{\sigma \to 1} u(\sigma)(z) = u(1)(z)$,
(ii) $\lim_{|z| \to \sigma} u(\sigma)(z) = 0$ $(\sigma < 1)$.
Because of (i) it will suffice to show that $u(\sigma) \leqq 1$ for $\sigma < 1$, while (ii) implies that, for $\sigma < 1$, $u(\sigma)$ has an interior maximum at some point $z_0 = z(\sigma)$. By the maximum principle

$$0 \geqq \frac{\partial^2 \log u(\sigma)(z_0)}{\partial z \partial \bar{z}} = -K(z_0)h(z_0) + K_\sigma u(\sigma)(z_0)$$

which, using $K_\sigma \equiv -1$ and $K \leqq -1$, implies that

$$h(\sigma)(z_0) \geqq h(z_0),$$

or equivalently

$$u(\sigma)(z_0) \leqq 1. \qquad \text{Q.E.D.}$$

COROLLARY. *If M is a negatively curved complex manifold, then a holomorphic mapping $f: \Delta \to M$ is distance decreasing relative to the Poincaré metric on Δ and given metric on M.*

Applying this corollary to a holomorphic mapping $f: \Delta \to \Delta$, we find that

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leqq \frac{1}{1 - |z|^2},$$

which is the intrinsic form of the Schwarz lemma due to Pick.

C. *Some applications of the Ahlfors lemma.*

SCHOTTKY-LANDAU THEOREM. *Let a, b, c be distinct points on the Riemann sphere $\boldsymbol{P}^1$ and $f: \Delta_r \to \boldsymbol{P}^1 - \{a, b, c\}$ a holomorphic mapping with $f'(0) \neq 0$. Then $r \leqq R(a, b, c, f(0), f'(0))$.*

PROOF. If $\pi(a, b, c)$ is the metric on $\boldsymbol{P}^1 - \{a, b, c\}$ constructed in example (iii) above, then the Ahlfors lemma applied to $f^*\pi(a, b, c) = h\, dz\, d\bar{z}$ gives

$$h(z) \leqq \pi(r)(z) \Rightarrow h(0) \leqq \frac{c}{r^2}$$
$$\Rightarrow r \leqq \left(\frac{c}{h(0)}\right)^{1/2} = R(a, b, c, f(0), f'(0)).$$

COROLLARY (LITTLE PICARD THEOREM). *A holomorphic mapping $f: \boldsymbol{C} \to \boldsymbol{P}^1 - \{a, b, c\}$ is constant.*

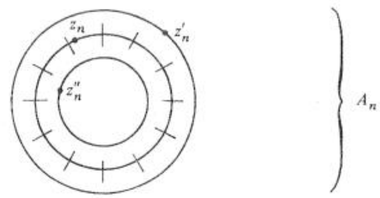
In general, we shall say that if a complex manifold M has the property that, for any holomorphic mapping $f: \Delta_r \to M$ with $f'(0) \neq 0$, the radius satisfies $r \leqq R(M, f(0), f'(0))$, then M has the *Schottky-Landau property*. The Ahlfors lemma implies that any negatively curved complex manifold has the Schottky-Landau property.

A second application of the Ahlfors lemma is the following extension result:

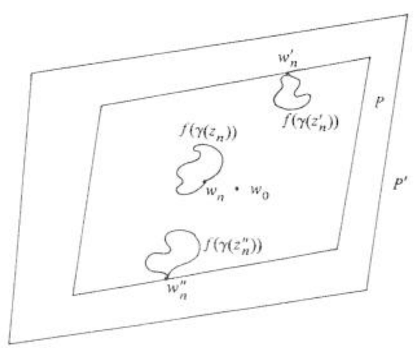
Kwack's Theorem. *If M is a compact negatively curved complex manifold, then any holomorphic mapping $f : \Delta^* \to M$ has a removable singularity at the origin.*

Proof. Using the compactness of M and passing to subsequences when necessary, we may assume that any sequence of points $\{z_n\}$ tending to zero in Δ^* has images $w_n = f(z_n)$ tending to some point w_0. We must prove that w_0 is the same for any such sequence $\{z_n\}$.

To begin with, we observe from the distance decreasing property of f and the first property mentioned above of the Poincaré metric on Δ^* that the circles $\gamma(z_n)$ passing through z_n have images also tending to w_0. Let $P \subset\subset P'$ be concentric polycylindrical coordinate systems around w_0 in M. By what was just said, an annular ring A_n around the circle $\gamma(z_n)$ may be assumed to be mapped into P.



If our result were false, then in trying to let A_n become as large as possible and still be mapped into P we will encounter points z'_n and z''_n on the outer and inner boundaries of A_n whose images $w'_n = f(z'_n)$ and $w''_n = f(z''_n)$ lie on the boundary of P. Passing again to subsequences, we may assume that $w'_n \to w'_0$ and $w''_n \to w''_0$.



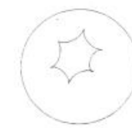
Let g be a holomorphic function on P' which vanishes at w_n ($n \gg 0$) but not at w'_n or w''_n, and set $h = g \circ f$. By the argument principle, the number of zeroes of h in A_n is given by

$$\frac{1}{2\pi}\int_{\gamma(z_n')} d \arg h - \frac{1}{2\pi}\int_{\gamma(z_n'')} d \arg h = 0.$$

On the other hand, $h(z_n) = 0$ by construction. Q.E.D.

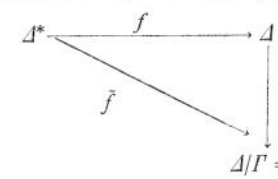
Corollary (Riemann Extension Theorem). *A holomorphic mapping $f : \Delta^* \to \Delta$ has a removable singularity at the origin.*

Proof. Let $\Gamma \subset \mathrm{Aut}(\Delta)$ be a properly discontinuous group operating without fixed points and having compact quotient.



FUNDAMENTAL DOMAIN FOR Γ

Applying Kwack's theorem to the composed mapping



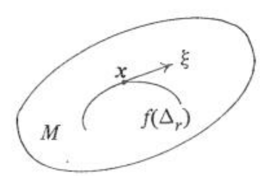
gives an extension of $\tilde{f}$, and hence one for f. Q.E.D.

It is amusing to compare the geometric argument just given with the usual analytic proof utilizing Laurent series.

2. The Kobayashi metric and volume forms.

A. *The Kobayashi metric.* Let M be a complex manifold and $T_x(M)$ the holomorphic tangent space at a point $x \in M$. For each vector $\xi \in T_x(M)$ we will, following Royden, define the *Kobayashi length* $F(x, \xi)$ as follows:

Let $\mathscr{F} = \mathscr{F}(x, \xi)$ be the class of all holomorphic mappings $f : \Delta_r \to M$ which satisfy $f(0) = x, f_*(\partial/\partial z) = \xi$.



Then we set

$$F(x, \xi) = \inf_{f \in \mathscr{F}} \frac{1}{r}.$$

Royden has proved that $F(x, \xi)$ is semicontinuous on the tangent bundle $T(M)$.

Consequently, if $\gamma : [0, 1] \to M$ is a piecewise smooth path, then the integral

$$l(\gamma) = \int_0^1 F(\gamma(t), \gamma'(t)^{(1,0)})\, dt$$

exists and allows us to define the *Kobayashi pseudo-distance*

$$\rho(x, y) \qquad (x, y \in M)$$

by the usual procedure of minimizing the lengths of paths joining x and y.

The manifold M is *hyperbolic* in case $\rho(x, y)$ is a distance. Manifolds satisfying the Schottky-Landau property are hyperbolic—in particular, this is the case if M is negatively curved. Especially noteworthy are manifolds which are complete hyperbolic.

The Kobayashi metric is intrinsically defined on any complex manifold, and has many pleasant properties of which we wish to mention two:

(i) holomorphic mappings are distance decreasing relative to the Kobayashi metric; and

(ii) the metric ρ "behaves well" with respect to products, covering spaces, submanifolds, etc.

Time will not permit us to give many of the results concerning the Kobayashi metric, for which we refer to [K], but we do want to call attention to two recent pretty applications. The first is Royden's theorem [R] that the Kobayashi metric equals the Teichmüller metric on the Teichmüller space. The proof is noteworthy for the insight which it gives into both the Teichmüller space and Kobayashi metric.

The second application, which will be discussed in some detail, is a recent Big Picard type result due to Borel in a form proved by Kobayashi and Ochiai (cf. [K-O] and [K-K]). We begin with the observation that, because of the distance decreasing property of ρ, Kwack's theorem discussed above holds with the same proof in case M is a compact hyperbolic manifold. A significant generalization occurs when M is an open set in a compact, complex space N. Then we say that M is *hyperbolically embedded* in case: (i) M is hyperbolic, and (ii) if $\{x_n\}$, $\{y_n\}$ are sequences of points in M with $x_n \to x$, $y_n \to y$, and $\rho(x_n, y_n) \to 0$, then $x = y$. Intuitively, ρ should distinguish points on the closure $\bar{M}$.

If M is hyperbolically embedded in N, then the proof of Kwack's theorem still applies to prove that a holomorphic mapping $f : \Delta^* \to M$ extends to a holomorphic mapping $f : \Delta \to N$.

To apply this result, we consider a bounded symmetric domain $D \subset \mathbf{C}^n$ and arithmetic group Γ of automorphisms. The quotient space $M = D/\Gamma$ is a negatively curved, quasi-projective algebraic variety admitting the Baily-Borel compactification $N = \overline{D/\Gamma}$. Kobayashi and Ochiai proved that M is hyperbolically embedded in N, thus proving

BOREL'S THEOREM. *A holomorphic mapping $f : \Delta^* \to D/\Gamma$ extends to $f : \Delta \to \overline{D/\Gamma}$.*

A special case of Borel's theorem is when $M = \mathbf{P}^1 - \{a, b, c\}$ and $N = \mathbf{P}^1$. In this situation the fact that M is hyperbolically embedded in N is obvious from the metric written down in the first lecture, and the above result is the usual Big Picard Theorem.

B. *Volume forms.* Thus far our discussion has centered around holomorphic mappings where the domain is one-dimensional. In the general several variables case, the first situation to study is the *equidimensional case* of a holomorphic mapping $f : D \to M$ between complex manifolds of the same dimension, and where f is assumed to be *nondegenerate* in the sense that the Jacobian determinant $J(f)$ is not identically zero. In this situation volume forms will play the analogous role of metrics in the one-dimensional case, with the Ricci form being the analogue of the curvature.

A *volume form* on a complex manifold M is given by a positive $C^\infty(n, n)$ form Ω. Locally $\Omega = h(z)\,\Phi(z)$ where h is a positive C^∞ function and

$$\Phi(z) = \prod_{\nu=1}^{m} \left(((-1)^{1/2}/2)\, dz_\nu \wedge d\bar{z}_\nu\right)$$

is the Euclidean volume form. A *pseudo-volume form* is the same, except that locally $h = |g|^2 h_0$ where h_0 is positive and g is a holomorphic function. If $f : D \to M$ is an equidimensional, nondegenerate holomorphic mapping and Ω is a volume form on M, then $f^*\Omega$ is a pseudo-volume form on D.

Associated to a pseudo-volume form Ω is its *Ricci form* Ric Ω, a global $C^\infty(1, 1)$ form defined locally by

$$\operatorname{Ric} \Omega = dd^c \log h$$

where $d^c = ((-1)^{1/2}/4\pi)(\bar{\partial} - \partial)$. (The operator $dd^c = ((-1)^{1/2}/2\pi)\partial\bar{\partial}$ is intrinsically defined by the complex structure, and plays in several variables the analogous role to the Laplacian in one variable.) Ricci forms are functorial in the sense that $f^*(\operatorname{Ric} \Omega) = \operatorname{Ric}(f^*\Omega)$.

The conditions

$$(*) \qquad \operatorname{Ric} \Omega > 0, \qquad (\operatorname{Ric} \Omega)^n = \underbrace{\operatorname{Ric} \Omega \wedge \cdots \wedge \operatorname{Ric} \Omega}_{n} \geqq \Omega$$

will play the analogous role to the Gaussian curvature condition $K \leqq -1$ in the one variable case.

EXAMPLES. (i) When M is a Riemann surface with Hermitian metric $h\, dz\, d\bar{z}$, the associated $(1, 1)$ form $\Omega = ((-1)^{1/2}/2)\, h\, dz \wedge d\bar{z}$ is a volume form and

$$\operatorname{Ric} \Omega = c(-K)\Omega$$

where $c > 0$ is a constant and K is the Gaussian curvature. Our signs have been chosen so as to keep as many as possible of them positive during the discussion.

(ii) On the unit ball $B \subset \mathbf{C}^n$ or unit polycylinder $P \subset \mathbf{C}^n$ there is a unique volume form Π, the *Poincaré-Bergman volume form*, which is invariant under the biholomorphic automorphism group and which satisfies

$$\mathrm{Ric}\, \Pi > 0, \qquad (\mathrm{Ric}\, \Pi)^n = \Pi.$$

(iii) On the complex projective space P^n with homogeneous coordinates $Z = [Z_0, \cdots, Z_n]$, the differential form $\phi = dd^c \log \|Z\|^2$ is the (1, 1) form associated to the Fubini-Study Kähler metric on P^n. The volume form $\Psi = \phi^n$ satisfies $\mathrm{Ric}\, \Psi = -(n+1)\phi$. (To check the signs, recall that the usual metric on P^1 has positive curvature, hence negative Ricci form.)

Hyperplanes A in P^n are given by linear equations

$$\langle A, Z \rangle = A_0 Z_0 + \cdots + A_n Z_n = 0,$$

and we set (cf. § 1)

$$\sigma(A) = \frac{|\langle A, Z\rangle|^2}{\|A\|^2 \|Z\|^2}, \qquad \rho(A) = \sigma(A)\left(\log \frac{\mu}{\sigma(A)}\right)^2.$$

Given $n+2$ hyperplanes $\{A_\nu\}$ in general position, i.e. no $n+1$ are linearly dependent, define

$$\Omega(A_\nu) = \left[\prod_{\nu=1}^{n+2} \frac{1}{\rho(A_\nu)}\right] \Psi.$$

This is a volume form on $P^n - \bigcup_{\nu=1}^{n+2} A_\nu$ having singularities along the A_ν of the same character as those of the Poincaré volume form on $(\Delta^*)^k \times \Delta^{n-k}$. For suitable choice of constant μ, one checks directly that

$$\mathrm{Ric}\, \Omega(A_\nu) > 0, \qquad \mathrm{Ric}\, \Omega(A_\nu)^n \geqq \Omega(A_\nu).$$

In the case $n = 1$, our construction reduces to the singular metric $\pi(a, b, c)$ on P^1 given in § 1. The condition of $n+2$ hyperplanes comes from the $n+1$ factor in $\mathrm{Ric}\, \Psi = -(n+1)\phi$. Analogues of $\Omega(A_\nu)$ exist on general smooth projective varieties with the anticanonical divisor playing the role of the "$n+1$" in the present case—cf. **[Ca-G]** for further discussion.

C. *The Ahlfors lemma for volume forms and applications.* Let D be either the unit ball $B \subset C^n$ or unit polycylinder $P \subset C^n$ with Poincaré-Bergman volume form Π, and let Ω be a pseudo-volume form on D.

LEMMA (CHERN-KOBAYASHI). *If Ω satisfies the conditions (*) above, then $\Omega \leqq \Pi$.*

PROOF. Writing $\Omega = u \cdot \Pi$, the proof is almost exactly the same as that of the Ahlfors lemma, where the only new step uses the Hadamard inequality

$$[\mathrm{Trace}\ (h_{i\bar{j}})/n] \geqq [\det\ (h_{i\bar{j}})]^{1/n}$$

for a positive Hermitian matrix $(h_{i\bar{j}})$.

As an application of the Ahlfors lemma for volume forms, we let B_r be the ball of radius r in C^n, $\{A_\nu\}$ a set of $n+2$ hyperplanes in general position in P^n, and $f: B_r \to P^n - \bigcup_{\nu=1}^{n+2} A_\nu$ a holomorphic mapping with Jacobian determinant $Jf(0) \neq 0$. Then the same proof as that of the Schottky-Landau theorem above leads to the

COROLLARY. *Under the above conditions, $r \leqq R(\{A_\nu\}, f(0), Jf(0))$.*

In particular, an entire holomorphic mapping $f: C^n \to P^n - \bigcup_\nu A_\nu$ is necessarily degenerate, a result due to A. Bloch (1926), and which has been recently rediscovered by Fujimoto and Green.

3. Equidimensional Nevanlinna theory.

A. *General philosophy.* In § 2 we saw that a nondegenerate holomorphic mapping $f: C^n \to P^n$ must meet at least one of $n+2$ hyperplanes $\{A_\nu\}$ in general position. More precisely, given $f(0)$ and $Jf(0)$, there is a largest r such that the ball $B_r = \{z \in C^n : \|z\| \leqq r\}$ of radius r can miss $f^{-1}(A_1 + \cdots + A_{n+2})$. Applying the same reasoning to balls centered around other points in C^n, we arrive in principle at a lower bound on the size of $f^{-1}(A_1 + \cdots + A_{n+2})$.

Nevanlinna theory is a precise and far reaching quantitative study of the size of $f^{-1}(A_1 + \cdots + A_{n+2})$. The *First Main Theorem* (= F.M.T.) gives an upper bound on the magnitude of $f^{-1}(A)$ for any hyperplane A. The *Second Main Theorem* (= S.M.T.) gives a lower bound on $f^{-1}(A_1 + \cdots + A_{n+2})$, and when played off against one another these two estimates yield the famous *defect relation* of Rolf Nevanlinna.

In one complex variable, what is being studied are the solutions to the equation

$$f(z) = a \qquad (z \in C,\ a \in P^1) \tag{3.1}$$

where f is an entire meromorphic function (hence the synonym *value distribution theory* for the subject). The size of $f^{-1}(a)$ in this case means the number $n(a, r)$ of solutions to (3.1) in the disc $|z| \leqq r$. The F.M.T. bounds $n(a, r)$ from above by the *order function* $T(f, r)$, an increasing convex function of $\log r$ which plays for general meromorphic functions a role analogous to the degree of a rational function or maximum modulus of an entire holomorphic function. The S.M.T. gives a lower bound of approximately the sort

$$n(a, r) + n(b, r) + n(c, r) \geqq T(f, r). \tag{3.2}$$

The F.M.T. may be viewed as a noncompact version of the *Wirtinger theorem*, which says that the area of an algebraic curve $C \subset P^2$ is equal to the intersection number of C with any line. The lower bound (3.2) is proved as follows: Given $f: C \to P^1$ and the metric $\pi(a, b, c)$ on $P^1 - \{a, b, c\}$, set $f^*\pi(a, b, c) = h\, dz\, d\bar{z}$. Then Picard's theorem says that h must have some singularities, and the S.M.T. gives a formula for the size of the singular set of h in terms of $T(f, r)$.

In these two remaining lectures we shall discuss in more detail how the theory works for nondegenerate, equidimensional holomorphic mappings $f: C^n \to P^n$ and nondegenerate holomorphic curves. When coupled with a standard discussion regarding line bundles and divisors on complex manifolds, the equidimensional theory goes through whenever P^n is replaced by an arbitrary algebraic variety with the anticanonical divisor replacing the $n+1$ hyperplanes in general position, but the situation regarding holomorphic curves in general algebraic varieties is still pretty much open.

B. *The order function and F.M.T.* Let $f: C \to P^n$ be a holomorphic mapping and $\psi = dd^c \log \|Z\|^2$ the standard Kähler form on P^n. Wirtinger's theorem suggests that the quantity $t(f, r) = \int_{\Delta_r} f^*\psi$ should be related to the number $n(A, r)$ of points of intersection of the analytic curve $f(\Delta_r)$ with a hyperplane A in P^n. This is made precise by *Crofton's formula*

$$t(f, r) = \int_A n(A, r)\Psi(A)$$

expressing the area of the piece of analytic curve $f(\Delta_r)$ as the average number of points of intersection with hyperplanes $A \in P^{n*}$, the dual projective space.

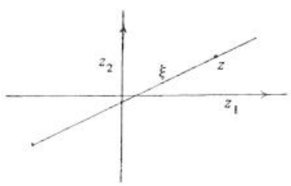
For reasons arising from Jensen's theorem, and ultimately related to twice integrating the operator dd^c, the growth of f is more conveniently measured by the *order function*

$$T(f, r) = \int_0^r t(f, \rho)\, d\rho/\rho.$$

For an entire holomorphic function $f: C \to C \subset P^1$, it is an easy consequence of Crofton's formula and the Poisson-Jensen formula that the order function $T(f, r)$ is essentially the maximum of $\log |f|$ in Δ_r.

To define the order function for a holomorphic mapping $f: C^m \to P^n$, we shall restrict f to the lines through the origin and use the 1-dimensional order function just introduced. Although not the most natural definition of the order function, this is the quickest and will suffice for our purposes.

In C^m, we let P^{m-1} be the projective space of lines ξ passing through the origin and $\Psi(\xi) = (dd^c \log \|z\|^2)^{m-1}$ the canonical density on P^{m-1}.



Given $f: C^m \to P^n$, for each line ξ we may restrict f to ξ and then define the order function $T(f, r, \xi)$ as above. Using this, the order function for f is given by

$$T(f, r) = \int_\xi T(f, r, \xi)\Psi(\xi).$$

We will now describe how one measures the size of an analytic hypersurface $V \subset C^m$; for simplicity, we shall always assume that $0 \notin V$. Denoting by $V[r] = V \cap B_r$ that part of V in the ball of radius r, we set

$$n(V, r) = \frac{\text{vol } V[r]}{r^{2m-2}}, \qquad N(V, r) = \int_0^r n(V, \rho)\frac{d\rho}{\rho}.$$

For each line $\xi \in P^{m-1}$, the intersection $V_\xi = V \cap \xi$ is a discrete set of points in ξ, and a variant of Crofton's formula is

(i) $$n(V, r) = \int_\xi n(V_\xi, r)\Psi(\xi).$$

Moreover, results of Lelong and Stoll (cf. [G-K]) show that

(ii) V is algebraic of degree $d \Leftrightarrow n(V, r) \leqq d$.

Because of (i) and (ii) it seems reasonable to use the *counting function* $N(V, r)$ to measure the growth of the analytic hypersurface V.

Now let $f: C^m \to P^n$ be a holomorphic mapping which is nondegenerate in the sense that the image does not lie in a hyperplane. For each such hyperplane A, the inverse image $A_f = f^{-1}(A)$ is an analytic hypersurface in C^m, and we set

$$n(A, r) = n(A_f, r), \qquad N(A, r) = N(A_f, r).$$

Crofton's formula and (i) above imply that

(3.3) $$T(f, r) = \int_A N(A, r)\Psi(A).$$

The F.M.T. expresses the relation between $T(f, r)$ and $N(A, r)$ for a *particular* hyperplane A. To state this formula we recall the function

$$\sigma(A) = |\langle A, Z\rangle|^2/\|A\|^2\|Z\|^2$$

on P^n, and we denote by Λ the unique closed $2m - 1$ on $C^m - \{0\}$ which is invariant under unitary transformations and satisfies $\int_{\|z\|=r} \Lambda = 1$ for all radii r. The F.M.T. is the formula

(F.M.T) $$N(A, r) + \int_{\|z\|=r} \log \frac{1}{\sigma(A)}\Lambda = T(f, r) + O(1, A).$$

This equation is proved quite easily by integrating twice the Poincaré equation of currents

(3.4) $$dd^c \log \frac{1}{\sigma(A)} = f^*\psi - A_f,$$

and, as mentioned previously, should be viewed as a noncompact form of Wirtinger's theorem. Since $\sigma(A) \leqq 1$, a corollary of the F.M.T. is the famous *Nevanlinna inequality*

(3.5) $$N(A, r) \leqq T(f, r) + O(1, A).$$

The reader may wish to compare (3.3) and (3.5) .For an entire meromorphic function $f(z)$, (3.5) bounds the number of zeroes of f in the disc $|z| \leqq r$ by the maximum modulus of f.

C. *The S.M.T. and defect relation.* Let $f: C^n \to P^n$ be an equidimensional holomorphic mapping whose Jacobian determinant Jf is not identically zero. This implies that $f(C^n)$ cannot lie in a hyperplane. We want to measure how much this image meets a set $\{A_\nu\}$ of $n + 2$ hyperplanes in general position.

Let $\Omega(A_\nu)$ be the volume form on P^n with singularities on $A_1 + \cdots + A_{n+2}$

which was constructed in the second section. The pull-back $f^*\Omega(A_\nu) = \Omega_f(A_\nu)$ is a singular pseudo-volume form on C^n, having "zeroes" along the ramification divisor $R = \{Jf = 0\}$ and "poles" on $f^{-1}(A_1 + \cdots + A_{n+2})$.

It was proved in § 2 that $f^{-1}(A_1 + \cdots + A_{n+2})$ must be nonempty. To measure the size of this analytic hypersurface, we write $\Omega_f(A_\nu) = h\Phi$ where $\Phi = \prod_{j=1}^n((-1)^{1/2}/2)\, dz_j \wedge d\bar{z}_j$ is the Euclidean volume form. The function h is non-negative with zeroes on R and poles on $f^{-1}(A_1 + \cdots + A_{n+2})$. Considering the locally L^1-function $\log h$ as a distribution, we arrive at the equation of currents (cf. (3.4))

$$dd^c \log h + \sum_\nu f^{-1}(A_\nu) = R + f^* \operatorname{Ric} \Omega(A_\nu). \tag{3.6}$$

Integrating (3.6) twice yields the

$$\begin{aligned}\int_{\|z\|=r} \log h \cdot \Lambda + \sum_\nu N(A_\nu, r) \\ = N(R, r) + \int_0^r \Big\{\int_{B_\rho} \operatorname{Ric} \Omega_f(A_\nu) \wedge \psi^{n-1}\Big\}\frac{d\rho}{\rho}\end{aligned} \tag{S.M.T}$$

where for simplicity we have assumed that $h(0) = 1$.

To see better how the S.M.T. leads to a lower bound on $\sum_\nu N(A_\nu, r)$, we shall restrict to the case $n = 1$, although the final estimates (3.12) below are the same in the general case. In fact, the general situation is done in basically the same way, the only new ingredient being the use of the Hadamard inequality as in the extension of the Ahlfors lemma to volume forms.

In the case $n = 1$, $\operatorname{Ric} \Omega_f(A_\nu) \geqq \Omega_f(A_\nu)$ by the condition $(*)$ on negative curvature. Since $N(R, r) \geqq 0$, the S.M.T. implies the estimate

$$\int_0^r \Big(\int_{\Delta_\rho} h\, dz\, d\bar{z}\Big)\frac{d\rho}{\rho} \leqq \sum_\nu N(A_\nu, r) + \frac{1}{2\pi}\int_{|z|=r} \log h\, d\theta. \tag{3.7}$$

At this point we use the notation

$$T^\#(r) = \int_0^r \Big(\int_{\Delta_\rho} h\, dz\, d\bar{z}\Big)\frac{d\rho}{\rho}.$$

If we use the ubiquitous *concavity of the logarithm*

$$\frac{1}{2\pi}\int \log h\, d\theta \leqq \log\Big(\frac{1}{2\pi}\int h\, d\theta\Big)$$

and obvious computation

$$\frac{1}{2\pi}\int_{|z|=r} h\, d\theta = \frac{1}{r^2}\frac{d^2T^\#(r)}{(d\log r)^2}$$

in (3.7), we arrive at the inequality ($r \geqq 1$)

$$T^\#(r) \leqq \sum_\nu N(A_\nu, r) + \log\Big[\frac{d^2T^\#(r)}{(d\log r)^2}\Big]. \tag{3.8}$$

If there were no derivatives in the last term on the R.H.S. of (3.8), then we would obviously have a lower bound on $\sum_\nu N(A_\nu, r)$. Even so, a clever but simple calculus argument gives

$$\frac{d^2T^\#(r)}{(d \log r)^2} \leqq [T^\#(r)]^{1+\delta} \quad \|,$$

where the notation "$\|$" means that the stated inequality holds outside an exceptional open set E satisfying $\int_E dr/r < +\infty$. Combining this inequality with (3.8) gives, for any $\varepsilon > 0$, the lower bound estimate

$$(1 - \varepsilon)T^\#(r) \leqq \sum_\nu N(A_\nu, r) \quad \| \tag{3.10}$$

where the exceptional intervals depend of course on the particular ε chosen.

Now in principle we are done. The inequality (3.10) gives a lower bound on the size of $f^{-1}(A_1 + \cdots + A_{n+2})$. To obtain the defect relation, one first proves that

$$T^\#(r) = T(f, r) + (\text{negligible terms}) \tag{3.11}$$

by explicitly taking into account the form of $\Omega(A_\nu)$. Combining this with (3.5) and (3.10) gives the simultaneous ineqaulities

$$\begin{aligned} N(A_\nu, r) &\leqq T(f, r) + O(1), \\ \sum N(A_\nu, r) &\geqq (1 - \varepsilon)\, T(f, r) \quad \|. \end{aligned} \tag{3.12}$$

Using the first inequality in (3.12), we may define the *Nevanlinna defect*

$$\delta(A_\nu) = 1 - \limsup_{r\to\infty} \frac{N(A_\nu, r)}{T(f, r)}$$

with the properties that

$$0 \leqq \delta(A_\nu) \leqq 1,$$
$$\delta(A_\nu) = 1 \text{ if } f \text{ omits the hyperplane } A_\nu.$$

Using the second inequality in (3.12),

$$\begin{aligned}\sum_\nu \delta(A_\nu) &\leqq (n + 2) - \limsup_{n\to\infty} \{\sum N(A_\nu, r)/T(f, r)\} \\ &\leqq n + 1 + \varepsilon,\end{aligned}$$

which yields the *defect relation*

$$\sum_\nu \delta(A_\nu) \leqq n + 1$$

in its usual form.

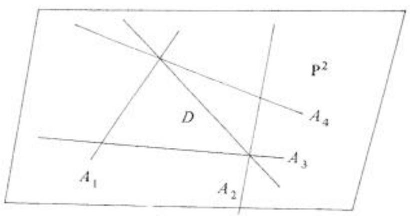
4. Holomorphic curves and some open problems.

A. *Statement of the theorems of Ahlfors and Bloch.* A holomorphic mapping $f : \Delta_r \to P^n$ will be called a holomorphic curve. We say that the curve is *entire* in case $r = +\infty$, and *nondegenerate* in case the image does not lie in a linear hyperplane. A classical theorem of E. Borel (1896) states that a nondegenerate entire holomorphic curve must meet at least one of $n + 2$ hyperplanes $\{A_\nu\}$ in general position. For $n = 1$ this reduces to the usual Picard theorem. Following a preliminary attempt by H. and J. Weyl, the corresponding defect relation

$$\sum_{\nu} \delta(A_\nu) \leqq n + 1 \tag{4.0}$$

was proved by Ahlfors (1941). An analogue of the Schottky-Landau theorem for $f: \Delta_r \to P^n - (A_1 + \cdots + A_{n+2})$ was proved by A. Bloch (1926). We shall discuss how the method of negative curvature leads to a proof of Ahlfors' theorem in a similar fashion to the equidimensional case treated in the last lecture. Such an approach has thus far failed to yield the Bloch theorem, and there are some nice open problems in this area.

For simplicity we shall usually restrict ourselves to the case $n = 2$ of holomorphic curves in the projective plane. Our $n + 2$ hyperplanes in general position then become four lines A_1, A_2, A_3, A_4 spanning a quadrilateral Q.



The diagonals D of this quadrilateral give lines P^1 meeting Q in 2 points. Thus $P^1 - P^1 \cap Q \cong C^*$, and in this way we find nonconstant but necessarily degenerate maps $f: C \to P^2 - Q$. The proof of Borel's theorem shows that any such f must map into one of the diagonals.

Recalling the Kobayashi metric $F(x, \xi)$ from § 2, Bloch's theorem is:

$F(x, \xi) > 0$ *unless x is on a diagonal D and ξ is tangent to D.*

This is a beautiful result, and the only proof of which I am aware is the highly nontransparent one given originally by Bloch. Finding a more conceptual argument for this theorem is an open problem for the theory.

Returning to the general case of a nondegenerate entire holomorphic curve $f: C \to P^n$, the order function $T(f, r)$ and counting function $N(A, r)$ have been defined and satisfy the Nevanlinna inequality $N(A, r) \leqq T(f, r) + C$. As before we may define the *defect*

$$\delta(A) = 1 - \limsup_{r \to \infty} \frac{N(A, r)}{T(f, r)}$$

with the properties that

$$0 \leqq \delta(A) \leqq 1, \qquad \delta(A) = 1 \quad \text{if } f(C) \text{ misses } A.$$

The Ahlfors defect relation (4.1) is a quantitative refinement of the Borel theorem. This result concerns a 1-dimensional curve in a high-dimensional space, and the interpolation between dimensions 1 and n is accomplished via the *osculating curves* associated to the holomorphic curve. We shall see how these osculating curves arise naturally when one attempts to use the method of negative curvature which proved successful in the equidimensional case.

B. *Negative curvature and the Ahlfors theorem.* Let $f: C \to P^2$ be a nondegenerate entire holomorphic mapping given by a homogeneous coordinate vector $Z(t) = [z_0(t), z_1(t), z_2(t)]$ ($t \in C$). Suppose that $\{A_\nu\}$ is a set of N lines in general position, A is a generic line, and recall the notations

$$\psi_0 = dd^c \log \|Z(t)\|^2 \quad (= f^* \text{ (standard Kähler form on } P^2)),$$

$$\rho_0(A) = \frac{|\langle Z(t), A\rangle|^2}{\|Z(t)\|^2 \|A\|^2} \quad (= 0 \Leftrightarrow Z(t) \text{ lies on the line } A).$$

Motivated by the equidimensional case and explicit expression

$$\frac{c\, dz\, d\bar{z}}{|z|^2(\log(1/|z|^2))^2}$$

for the Poincaré metric on the punctured disc, we are prompted to consider the singular metric

$$\omega_0 = \Big\{\prod_{\nu} \frac{1}{\rho_0(A_\nu)[\log(\mu/\rho_0(A_\nu))]^2}\Big\} \psi_0.$$

In computing the Ricci forms of this and all future metrics, we shall ignore the $[\log(\mu/\rho_0(A_\nu))]^{-2}$ terms, the reason being that these terms essentially always help to make curvatures more negative—especially around the singular points. To assist in computing Ric ω_0, we shall also use the following comments:

(i) $\mathrm{Ric}(u\psi) = dd^c \log u + \mathrm{Ric}\, \psi$ where u is a positive function and ψ is a positive (1, 1) form;

(ii) $\mathrm{Ric}\, \psi_0 \geqq -2\psi_0$, since the holomorphic sectional curvatures of P^n are all equal to $+2$, and curvatures decrease on complex submanifolds; and

(iii) $dd^c \log(1/\rho_0(A)) = \psi_0$, by definition.

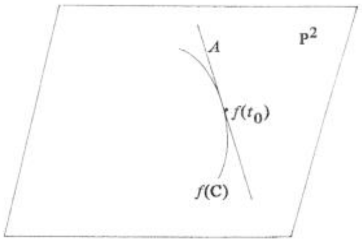
Using (i)—(iii) and ignoring the $[\log(\mu/\rho_0(A_\nu))]^{-2}$ terms, we find that

$$\mathrm{Ric}\, \omega_0 \geqq (N - 3)\psi_0. \tag{4.1}$$

Consequently, for $N \geqq 3$ the metric ω_0 has negative curvature.

However, this curvature cannot be bounded away from zero for the following reason:

Given on the punctured disc $\Delta^* = \{0 < |z| < 1\}$ a metric $h\, dz\, d\bar{z}$ with Gaussian curvature $K \leqq -1$, then $\int_{|z| \leqq 1-\varepsilon} h\, dz\, d\bar{z} < \infty$ by the Ahlfors lemma and second property of the Poincaré metric on Δ^*. Now, on the other hand, the holomorphic curve may, at some point t_0, be unramified so that $\psi_0(t_0) \neq 0$, but have arbitrarily high order of contact with one of the lines, say A_1. Then the denominator $\rho(A_1)\, [\log(\mu/\rho(A_1))]^2$ in ω_0 becomes zero to arbitrarily high order and so $\int_{|t-t_0|<\delta} \omega^0 = +\infty$.



This suggests that we additionally consider the functions

$$\rho_1(A) = \frac{\|Z(t) \wedge Z'(t), A\|^2}{\|Z(t) \wedge Z'(t)\|^2\|A\|^2},$$

which vanish at points t_0 where $f(t_0)$ meets A tangentially. Using the notations

$$\psi_1(A) = dd^c \log\|Z(t) \wedge Z'(t), A\|^2$$
$$\psi_1 = dd^c \log\|Z(t) \wedge Z'(t)\|^2,$$

the singular metric

$$\omega_1 = \left\{\prod_\nu \frac{\rho_1(A_\nu)}{\rho_0(A_\nu)[\log(\mu/\rho_0(A_\nu))^2]}\right\}\psi_0$$

is always integrable. The Ricci form

(4.2) $$\mathrm{Ric}\,\omega_1 = N\psi_0 + \mathrm{Ric}\,\psi_0 - N\psi_1 + \sum_\nu \psi_1(A_\nu),$$

where we continue to ignore the $[\log(\mu/\rho_0(A_\nu))]^{-2}$ terms. To compensate for the term $-N\psi_1$, we are thus prompted to consider a second metric

$$\omega_2 = \left\{\prod_\nu \frac{1}{\rho_1(A_\nu)\,[\log(\mu/\rho_1(A_\nu))]^2}\right\}\psi_1.$$

Using (4.2),

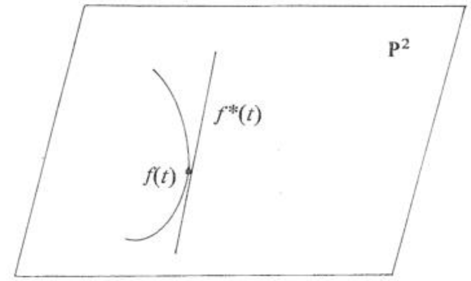
(4.3) $$\mathrm{Ric}\,\omega_1 + \mathrm{Ric}\,\omega_2 \geqq N\psi_0 + \mathrm{Ric}\,\psi_0 + \mathrm{Ric}\,\psi_1.$$

In order to conclude that ω_1 and ω_2 form, so to speak, a *negatively curved pair of metrics*—thereby forcing a defect relation as before—it is necessary to relate $\mathrm{Ric}\,\psi_0$ and $\mathrm{Ric}\,\psi_1$.

Now $\psi_0 = dd^c \log \|Z(t)\|^2$ is the pull-back of the standard Kähler metric on $\boldsymbol{P}^2$ under the given mapping f. Similarly, $\psi_1 = dd^c \log \|Z(t) \wedge Z'(t)\|^2$ is the pull-back of the standard Kähler metric on the dual projective space $\boldsymbol{P}^{2*}$ of lines in $\boldsymbol{P}^2$ under the *dual curve mapping* $f^* : \boldsymbol{C} \to \boldsymbol{P}^{2*}$, which is given by

$$f^*(t) = \text{tangent line to } f(\boldsymbol{C}) \text{ at } f(t).$$

In classical algebraic geometry, the relation between degree (C) and degree (C^*), for an algebraic curve C and its dual C^* is provided by the *Plücker formulae*, whose



extension to holomorphic curves was given by H. and J. Weyl. For plane curves the relevant relations are

(4.4) $$\mathrm{Ric}\,\psi_0 = -2\psi_0 + \psi_1, \qquad \mathrm{Ric}\,\psi_1 = -2\psi_1 + \psi_0.$$

Plugging these into (4.3) gives

(4.5) $$\mathrm{Ric}\,\omega_1 + \mathrm{Ric}\,\omega_2 \geqq (N-1)\psi_0 - \psi_1.$$

On the other hand, following the same procedure as in the equidimensional case, the first equation in (4.4) gives

$$\int_0^r\left(\int_{\Delta_\rho}\psi_1\right)\frac{d\rho}{\rho} < 2\int_0^r\left(\int_{\Delta_\rho}\psi_0\right)\frac{d\rho}{\rho} + C\log\left[\int_0^r\left(\int_{\Delta_\rho}\psi_0\right)\frac{d\rho}{\rho}\right]\Big\|,$$

which we shall write as

(4.6) $$\psi_1 < (2+\delta)\psi_0 \;\|.$$

Combining (4.5) and (4.6) yields

(4.7) $$\mathrm{Ric}\,\omega_1 + \mathrm{Ric}\,\omega_2 \geqq (N-3-\delta)\,\psi_0 \;\|.$$

Thus, for $N \geqq 4$, the pair of metrics ω_1, ω_2 taken together has negative curvature and both metrics are integrable. In fact, one easily calculates the curvature is bounded away from zero except at intersection points $A_i \cap A_j$.

To circumvent this final difficulty, Mike Cowen introduced Hölder exponents with the following conclusion: Setting

$$\varphi_1 = c_1\left\{\prod_\nu \frac{\rho_1(A_\nu)}{\rho_0(A_\nu)[\log(\mu/\rho_0(A_\nu))]^2}\right\}^{1/2}\psi_0,$$

$$\varphi_2 = c_2\left\{\prod_\nu \frac{1}{\rho_1(A_\nu)[\log(\mu/\rho_1(A_\nu))]}\right\}\psi_1,$$

for suitable choice of constants c_1, c_2, μ,

(4.8) $$2\,\mathrm{Ric}\,\varphi_1 + \mathrm{Ric}\,\varphi_2 \geqq (N-3-\varepsilon)\psi_0 + (\varphi_1+\varphi_2)\;\|.$$

Since the singular divisor of 2 Ric φ_1 + Ric φ_2 is just $\sum_\nu f^{-1}(A_\nu)$, (4.8) may be rewritten in distributional language as

$$\sum_\nu f^{-1}(A_\nu) \geqq (N - 3 - \varepsilon)\, \phi_0 \; \|,$$

which then leads to the Ahlfors defect relation as in the equidimensional case.

C. *Some problems.* The following collection of problems, which to the best of my knowledge are still open, deal for the most part with the relationship between holomorphic mappings and the Kobayashi metric and algebraic geometry. The basic underlying question is *to understand in terms of the algebro-geometric properties of a variety V the possible holomorphic mappings $f : D \to V$ from open sets D in C^k into V*, generalizing as far as possible the understanding of the case when V is an algebraic curve obtained through the classical uniformization theorem.

(i) Let M be a complex Hermitian manifold. The holomorphic tangent spaces are denoted by $T_x(M)$, and $\zeta^{1,0}$, $\zeta^{0,1}$ denote the projections of a tangent vector ζ into $T(M)$, $\overline{T(M)}$ respectively. A complex manifold M' is said to be *ε-quasi-conformally equivalent to M* if there is a diffeomorphism $f : M' \to M$ such that $\|f_*(\xi)^{0,1}\| / \|f_*(\xi)\| < \varepsilon$ for every $\xi \in T_x(M')$.

PROBLEM. Suppose that M is a compact, complex manifold which is hyperbolic in the sense of Kobayashi. Then for sufficiently small ε, is any complex manifold M' which is ε-quasi-conformally equivalent to M necessarily hyperbolic?

It seems to me quite likely that the answer to this question is yes. If so, then given an analytic family $\{M_t\}_{t \in \Delta}$ of compact, complex manifolds when M_0 is hyperbolic, the M_t would be hyperbolic for sufficiently small t.

(ii) The next two problems deal with complex manifolds (usually algebraic varieties) M which are *hyperbolic on a Zariski open set* in the sense that the Kobayashi length $F(x, \xi) > 0$ unless x lies on a subvariety S and ξ is tangent to S there. This definition is prompted by the Bloch theorem concerning $P^2 - \{4$ lines in general position$\}$. Because of his result and similar examples of Mark Green [G], it seems that being hyperbolic on a Zariski open set may be more fruitful in studying algebraic varieties than the requirement of strict hyperbolicity. In any case, being hyperbolic on a Zariski open set is invariant under *birational transformations*, thus affording some additional flexibility.

PROBLEM. Let $\{M_t\}_{t \in \Delta}$ be an analytic family of compact, complex manifolds where M_t is hyperbolic on a Zariski open set. In particular, the M_t may be hyperbolic. Then is M_0 hyperbolic on a Zariski open set?

(iii) Recall that an algebraic surface M is of *general type* if the graded canonical ring $\bigoplus_n H^0(M, \mathcal{O}(nK))$ has transcendence degree two. Examples are nonsingular surfaces of degree $\geqq 5$ in P^3.

PROBLEM. Is an algebraic surface of general type hyperbolic on a Zariski open set? Is the complement $P^2 - C$ of a smooth plane curve of degree $\geqq 5$ hyperbolic on a Zariski open set? Is a general smooth surface of degree $\geqq 5$ in P^3 hyperbolic?

An affirmative answer to problems (i) and (ii) should allow one to obtain information on at least the last part of this problem by checking a few special surfaces and the applying techniques from deformation theory.

(iv) There is a notion of *measure hyperbolic* due to Pelles (cf. [E]) which bears the same relation to volume forms Ω satisfying (*) in §2 as does hyperbolic to negatively curved metrics. In particular, any surface of general type is measure hyperbolic.

PROBLEM. If M is an algebraic surface which is measure hyperbolic, then is M of general type?

By looking at the classification of algebraic surfaces, one sees that this problem is equivalent to showing that *a $K3$ surface M is not measure hyperbolic* (elliptic surfaces are never measure hyperbolic). Letting $P(r_1, r_2) = \{(z_1, z_2) : |z_1| < r_1, |z_2| < r_2\}$, roughly speaking one must construct nondegenerate holomorphic maps $f : P(r_1, r_2) \to M$ where the product $r_1 r_2 \to \infty$, which is a uniformization type of question. In any case, a dense set of $K3$'s is *not* measure hyperbolic.

(v) Let M be a compact algebraic variety, and $D \subset M$ a divisor with normal crossings such that $M - D$ is complete (in a suitable sense) hyperbolic on a Zariski open set.

(EXAMPLE. $M = P^n$ and $D = (n + 2)$ hyperplanes in general position.)

PROBLEM. Can one find a lower bound on the size of $f^{-1}(D)$ for a nondegenerate entire holomorphic curve $f : C \to M$?

If so, then in order to prove defect relations one might not need to construct negatively curved metrics so explicitly as has been the case thus far. Solving this problem would probably necessitate obtaining information on the following two questions:

(vi) PROBLEM. Let M be a compact algebraic variety and $D \subset M$ a divisor with normal crossings such that $M - D$ is complete hyperbolic. Then can one estimate the Kobayashi length $F(x, \xi)$ for $M - D$ as x tends to D?

In this connection, we should like to point out that A. Sommese (Princeton thesis, 1973) has shown that given a *complete*, negatively curved ds^2 on $M - D$, then this metric is, in a suitable sense, asymptotic to the Poincaré metric on Δ^* when one approaches D from a normal direction.

(vii) PROBLEM. In what, if any, sense is the Kobayashi metric $F(x, \xi)$ of a hyperbolic manifold M negatively curved? Specifically, given a holomorphic mapping $f : \Delta \to M$ and setting $h(z) = F(f(z), f_*(\partial/\partial z))$, then can one say anything about $\Delta \log h$ (taken in the distributional sense)?

(viii) PROBLEM. Let M be a compact, complex manifold and $D \subset M$ a divisor such that $M - D$ is complete hyperbolic. Then does any holomorphic mapping $f : \Delta^* \to M - D$ extend to $f : \Delta \to M$? If $M - D$ is only assumed to be complete hyperbolic on a Zariski open set but f is taken to be nondegenerate, then does the same extension theorem hold?

EXAMPLES. $P^2 - \{5$ lines in general position$\}$ and $P^2 - \{4$ lines in general position$\}$.

BIBLIOGRAPHY

This bibliography is not intended to be complete, but rather to give references for background material together with a few references for the topics discussed in the lectures through which additional sources can be found.

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The second reference contains a general discussion of the formalism of Nevanlinna theory and gives some applications to Big Picard theorems including a proof of the Borel theorem discussed in lecture 2. The theory of holomorphic curves together with an extensive historical bibliography is given in

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ON THE CURVATURE OF RATIONAL SURFACES

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1. Introduction. Among the differential-geometric vanishing theorems for Kähler manifolds we have the following:

(a) If a Kähler manifold X has positive *scalar curvature*, the plurigenera P_m vanish for $m > 0$.

(b) If X has positive *holomorphic sectional curvature*, the fundamental group π_1 is trivial and $P_m = 0$ for $m > 0$.

(c) If X has positive *Ricci curvature*, the dimension of the space of holomorphic p-forms $h^{p,0}$ is zero for $p > 0$, $\pi_1 = 1$ and $P_m = 0$ for $m > 0$.

We notice now that the objects which vanish are not only invariants of the complex structure but also *birational* invariants. In particular, they are all zero for rational algebraic varieties. This leads one to conjecture that rational varieties are characterized by admitting a Kähler metric with some positivity of curvature which will force the vanishing of one or more of these invariants.

For curves this is clearly true—a one-dimensional Kähler manifold with positive curvature is biholomorphically equivalent to P^1 by any of the above arguments, and conversely P^1 admits a Kähler metric of positive curvature. For regular surfaces (i.e., the first Betti number $b_1 = 0$) with positive scalar curvature, vanishing theorem (a) above together with Kōdaira's classification implies rationality. The question we are concerned with is the converse. Do all rational surfaces admit a Kähler metric of positive scalar curvature? We prove the following:

THEOREM. *Almost all rational surfaces admit a Hodge metric of positive scalar curvature.*

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