

# DIMENSION OF SPACES OF AUTOMORPHIC FORMS

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I will first formulate a problem in the theory of group representations and show how to solve it; then I will discuss the relation of this problem to the theory of automorphic forms. Since there is no point in striving for maximum generality, I start with a connected semisimple group  $G$  with finite center. An irreducible unitary representation  $\pi$  of  $G$  on the Hilbert space  $H$  is said to be square-integrable if for one and hence, as one can show, every pair  $u$  and  $v$  of nonzero vectors in  $H$  the function  $(\pi(g)u, v)$  is square-integrable on  $G$ . It is said to be integrable if for one such pair  $(\pi(g)u, v)$  is integrable.

Suppose  $\Gamma$  is a discrete subgroup of  $G$  and  $\Gamma \backslash G$  is compact. As was shown by Godement in an earlier lecture the representation  $\pi$  of the previous paragraph occurs a finite number of times, say  $N(\pi)$ , in the regular representation on  $L^2(\Gamma \backslash G)$ . The problem is first to find a closed formula for  $N(\pi)$ . The method which I will now describe of obtaining such a formula is valid only when  $\pi$  is actually integrable.

Square integrable representations are similar in some respects to representations of compact groups; in particular they satisfy a form of the Schur orthogonality relations. There is a constant  $d_\pi$  called the formal degree of  $\pi$  such that if  $u', v', u$ , and  $v$  belong to  $H$  then

$$\int_G (\pi(g)u', v') \overline{(\pi(g)u, v)} dg = d_\pi^{-1}(u', u)(v, v').$$

If  $u$  and  $v$  are such that  $(\pi(g)u, v)$  is integrable and  $\pi'$  is unitary representation of  $G$  on  $H'$  which does not contain  $\pi$ , then

$$\int_G (\pi'(g)u', v') \overline{(\pi(g)u, v)} dg = 0$$

for all  $u', v'$  in  $H'$ .

Let  $L_i$ ,  $1 \leq i \leq N(\pi)$ , be a family of mutually orthogonal invariant subspaces of  $L^2(\Gamma \backslash G)$  which are such that the action of  $G$  on each of them is equivalent to  $\pi$ . Suppose that  $\pi$  does not occur in the orthogonal complement of

$$\bigoplus_{i=1}^{N(\pi)} L_i.$$

If  $\pi$  is integrable there is a unit vector  $v$  in  $H$  such that  $(\pi(g)v, v)$  is integrable. Let  $v_i$  be a unit vector in  $L_i$  corresponding to  $v$  under some equivalence between  $H$  and  $L_i$ . The

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orthogonality relations imply that the operator  $\Phi \rightarrow \Phi'$  with

$$\begin{aligned}\Phi'(g) &= d_\pi \int_G \Phi(gh) \overline{(\pi(g)v, v)} dh, \\ &= \int_{\Gamma \backslash G} \Phi(h) \left\{ \sum_\Gamma \xi(g^{-1}\gamma h) \right\} dh,\end{aligned}$$

if  $\xi(g) = d_\pi \overline{(\pi(g)v, v)}$ , is an orthogonal projection on the space spanned by  $v_1, \dots, v_{N(\pi)}$ . For our purposes it may be assumed that  $v$  transforms according to a finite-dimensional representation of some maximal compact subgroup of  $G$ . Then the argument used by Borel in a previous lecture shows that

$$\sum_\Gamma \xi(g^{-1}\gamma h)$$

converges absolutely uniformly on compact subsets of  $G \times G$ . Hence  $v_1, \dots, v_{N(\pi)}$  may be supposed continuous. As a consequence

$$\sum_{i=1}^{N(\pi)} v_i(g) \bar{v}_i(g) = \sum_\Gamma \xi(g^{-1}\gamma h).$$

Set  $h = g$  and integrate over  $\Gamma \backslash G$  to obtain

$$N(\pi) = \int_{\Gamma \backslash G} \sum_\Gamma \xi(g^{-1}\gamma g) dg.$$

The sum in the integrand may be rearranged at will. If  $\Sigma$  is a set of representatives for the conjugacy classes in  $\Gamma$  the integral on the right equals

$$\begin{aligned}\int_{\Gamma \backslash G} \sum_{\gamma \in \Sigma} \sum_{\delta \in \Gamma_\gamma \backslash \Gamma} \xi(g^{-1}\delta^{-1}\gamma\delta g) dg &= \sum_{\gamma \in \Sigma} \int_{\Gamma_\gamma \backslash G} \xi(g^{-1}\gamma g) dg \\ &= \sum_{\gamma \in \Sigma} \mu(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} \xi(g^{-1}\gamma g) dg,\end{aligned}$$

if  $\Gamma_\gamma$  and  $G_\gamma$  are the centralizers of  $\gamma$  in  $\Gamma$  and  $G$  respectively. The equality of  $N(\pi)$  and the final expression is of course a special case of a formula of Selberg and has been known for some time.

The problem of evaluating  $\mu(\Gamma_\gamma \backslash G_\gamma)$ , the volume of  $\Gamma_\gamma \backslash G_\gamma$ , has been discussed in the lectures on Tamagawa numbers. So we shall not worry about it now. Since  $\Gamma \backslash G$  is compact every element of  $\Gamma$  is semisimple; thus our problem is to express the integral

$$\int_{G_\gamma \backslash G} \xi(g^{-1}\gamma g) dg$$

in elementary terms when  $\gamma$  is a semisimple element of  $G$ .

If  $\pi$  is a square-integrable representation of  $G$  on  $H$ ,  $v$  is a vector in  $H$  which transforms according to a finite-dimensional representation of some maximal compact subgroup of  $G$ , and

$$\xi(g) = d_\pi \overline{(\pi(g)v, v)},$$

then a recent theorem of Harish-Chandra states that

$$(a) \quad \int_{G_\gamma \backslash G} \xi(g^{-1}\gamma g) dg$$

exists for  $\gamma$  semisimple and vanishes unless  $\gamma$  is elliptic, that is, belongs to some compact subgroup of  $G$ . Since  $\Sigma$  contains only a finite number of elliptic elements the sum in the expression for  $N(\pi)$  is finite. We still require a closed expression for the integrals appearing in it.

Let  $K$  be a maximal compact subgroup of  $G$ . Since  $G$  has a square-integrable representation there is a Cartan subgroup  $T$  of  $G$  contained in  $K$ . It is enough to compute the integral (a) for  $\gamma$  in  $T$ . There is a limit formula of Harish-Chandra which allows one to compute its value at the singular elements once its values at the regular elements are known. Thus we need only evaluate it when  $\gamma$  is regular. It should be remarked that in this limit formula there is a constant which depends on the choice of Haar measure on  $G_\gamma$ . The exact relation of this constant to the choice of Haar measure has never been determined; until it is, our problem cannot be regarded as completely solved.

If  $\gamma$  is regular and the measure on  $G_\gamma$  is so normalized that the volume of  $G_\gamma$  is one, then

$$\int_{G_\gamma \backslash G} \xi(g^{-1}\gamma g) dg = \chi_\pi(\gamma^{-1})$$

if  $\chi_\pi$  is the character of  $\pi$ . An explicit expression for the right-hand side has recently been obtained.

Let  $\mathfrak{h}$  be the Lie algebra of  $T$ ; choose an order on the roots of  $\mathfrak{h}_\mathbb{C}$ ; and let  $\Lambda$  be a linear function on  $\mathfrak{h}_\mathbb{C}$  such that  $\Lambda + \rho$ ,  $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$ , extends to a character of  $T$  and so that  $(\Lambda + \rho, \alpha) \neq 0$  for all roots  $\alpha$ . Assume, for simplicity, that  $\rho$  also extends to a character of  $T$ . To each such  $\Lambda$  there is associated a square-integrable representation  $\pi_\Lambda$  and if  $H \in \mathfrak{h}$

$$\chi_{\pi_\Lambda}(\exp H) = (-1)^m \epsilon(\Lambda) \sum_{\sigma \in W} \frac{\operatorname{sgn} \sigma \exp(\sigma(\Lambda + \rho))(H)}{\prod_{\alpha > 0} \left\{ \left( \exp(\alpha(G)/2) - \exp(-\alpha(G)/2) \right) \right\}}.$$

Here  $m = \frac{1}{2} \dim G/K$ ,  $\epsilon(\Lambda) = \operatorname{sgn}(\prod_{\alpha > 0} (\Lambda + \rho, \alpha))$ , and  $W$  is the Weyl group of  $K$ . Every square-integrable representation is equivalent to  $\pi_\Lambda$  for some  $\Lambda$ . However the values of  $\Lambda$  for which  $\pi_\Lambda$  is integrable are not yet known. For some special cases see [1] and [2].

The geometrical meaning of the numbers  $N(\pi_\Lambda)$  is not yet completely clear. I would like to close this lecture with some suggestions as to what it might be. Since the evidence at present is rather meagre, they are only tentative. If  $\mathfrak{g}_\mathbb{C}$  is the complexification of the Lie algebra of  $\mathfrak{g}$ , the elements of  $\mathfrak{g}_\mathbb{C}$  may be regarded as left-invariant complex vector fields on  $G$  and  $G/T$  may be turned into a complex manifold in such a way that the space of antiholomorphic tangent vectors at  $\bar{g} = gT$  is the image of  $\mathfrak{n}_\mathbb{C}^-$  if  $\mathfrak{n}_\mathbb{C}^-$  is the subalgebra of  $\mathfrak{g}_\mathbb{C}$  generated by root vectors belonging to negative roots. Let  $V^*$  be the bundle of antiholomorphic cotangent vectors and introduce a  $G$ -invariant metric in  $V^*$  and hence in  $\bigwedge^q V^*$ . Let  $B$  be the line bundle over  $G/T$  associated to the character  $\xi(\exp H) = \exp(\Lambda(H))$  of  $T$ . If  $\Gamma$  is a discrete subgroup of  $G$  let  $C^q(\Lambda, \Gamma)$  be the space of  $\Gamma$ -invariant cross-sections of  $B \otimes \bigwedge^q V^*$  which are square integrable over  $\Gamma \backslash G/T$ . There is a unique closed operator  $\bar{\partial}$  from  $C^q(\Lambda, \Gamma)$  to  $C^{q+1}(\Lambda, \Gamma)$  whose domain contains the infinitely differentiable cross-sections of compact support on which  $\bar{\partial}$  is to have its usual meaning and whose adjoint is defined on the infinitely differentiable cross-sections of  $C^{q+1}(\Lambda, \Gamma)$  with compact support.

Set  $C^q(\Lambda, \{1\}) = C^q(\Lambda)$ . I expect, although I do not know how to prove it, that when  $\Lambda + \rho$  is nonsingular the range of  $\bar{\partial}$  is closed for every  $q$ . If this is so then the cohomology groups  $H^1(\Lambda)$  will be Hilbert spaces on which  $G$  acts. Is it true that they vanish for all but one value of  $q$ , say  $q = q_\Lambda$ , and that the representation  $\pi'_\Lambda$  of  $G$  on  $H^{q_\Lambda}(\Lambda)$  is equivalent to  $\pi_\Lambda$ ? The following theorem is a clue to the value of  $q_\Lambda$ .

**Theorem (P. Griffiths).** *Let  $a_1$  be the number of noncompact positive roots for which  $(\Lambda + \rho, \alpha) > 0$  and let  $a_2$  be the number of compact positive roots for which  $(\Lambda + \rho, \alpha) < 0$ . There is a constant  $c$  such that if  $|(\Lambda + \rho, \alpha)| > c$  for every simple root,  $\Gamma \backslash G$  is compact, and  $\Gamma$  acts freely on  $G/T$ , then  $H^q(\Lambda, \Gamma) = 0$  unless  $q = a_1 + a_2$ .*

It is, I think, worthy of remark that if one assumes that  $H^q(\Lambda) = \{0\}$  for  $q \neq q_\Lambda = a_1 + a_2$ , then a formal application of the Woods Hole fixed point formula shows that if  $\gamma$  is a regular element of  $T$ , then the value at  $\gamma$  of the character of  $\pi'_\Lambda$  is  $\chi_{\pi_\Lambda}(\gamma)$ . By the way, it is known that  $H^0(\Lambda) = 0$  unless  $q_\Lambda = 0$  and that if  $q_\Lambda = 0$  the representation of  $G$  on  $H^0(\Lambda)$  is in fact  $\pi_\Lambda$ .

Finally one will want to show that when  $\pi_\Lambda$  is integrable and  $\Gamma \backslash G$  is compact the number  $N(\pi_\Lambda)$  is equal to the dimension of  $H^{q_\Lambda}(\Lambda, \Gamma)$ . This can be done when  $q_\Lambda = 0$ ; in this case  $H^0(\Gamma, \Lambda)$  is a space of automorphic forms.

It should be possible, although I have not done so, to test these suggestions for groups whose unitary representations are well understood, in particular, for  $\mathrm{SL}(2, \mathbf{R})$  and the De Sitter group. To do this one might make use of an idea basic to Kostant's proof of the (generalized) Borel-Weil theorem for compact groups. Suppose  $\sigma$  is a unitary representation of  $G$  on a Hilbert space  $V$ . Let  $C^q(V)$  be the space of all linear maps from  $\bigwedge^q \mathfrak{n}_{\mathbf{C}}^-$  to  $V$ .  $C^q(V)$  is a Hilbert space. The usual coboundary operator from  $C^q(V)$  to  $C^{q+1}(V)$  can be defined on those elements of  $C^q(V)$  which take values in the Gårding subspace of  $V$ . The closure  $d$  of this operator is the adjoint of the restriction of its formal adjoint to those elements of  $C^{q+1}(V)$  which take values in the Gårding subspace. Of course  $T$  acts on  $\bigwedge^q \mathfrak{n}_{\mathbf{C}}^-$ . If  $f \in C^q(V)$  define  $tf = f'$  by  $f'(X) = tf(t^{-1}X)$ ,  $X \in \bigwedge^q \mathfrak{n}_{\mathbf{C}}^-$ . There is a natural identification of  $C^q(\Lambda)$  with the set of  $f$  in  $C^q(L^2(G))$  such that  $tf = \exp(-\Lambda(H))f$  if  $t = \exp H$  belongs to  $T$  and of  $C^q(\Lambda, \Gamma)$  with the set of  $f$  in  $C^q(L^2(\Gamma \backslash G))$  such that  $tf = \exp(-\Lambda(H))f$ . Moreover the following diagrams are commutative.

$$\begin{array}{ccc} C^q(\Lambda) & \xrightarrow{\bar{\partial}} & C^{q+1}(\Lambda) & & C^q(\Lambda, \Gamma) & \xrightarrow{\bar{\partial}} & C^{q+1}(\Lambda, \Gamma) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C^q(L^2(G)) & \xrightarrow{d} & C^{q+1}(L^2(G)) & & C^q(L^2(\Gamma \backslash G)) & \xrightarrow{d} & C^{q+1}(L^2(\Gamma \backslash G)) \end{array}$$

The point is that  $d$  is easier to study than  $\bar{\partial}$  because to study  $d$  we can decompose  $V$  into irreducible representations and study the action of  $d$  on each part.

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