DIMENSION OF SPACES OF AUTOMORPHIC FORMS

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I will first formulate a problem in the theory of group representations and show how to solve it; then I will discuss the relation of this problem to the theory of automorphic forms. Since there is no point in striving for maximum generality, I start with a connected semisimple group G with finite center. An irreducible unitary representation π of G on the Hilbert space H is said to be square-integrable if for one and hence, as one can show, every pair u and v of nonzero vectors in H the function $\bigl(\pi(g)u, v\bigr)$ is square-integrable on G. It is said to be integrable if for one such pair $\bigl(\pi(g)u, v\bigr)$ is integrable.

Suppose Γ is a discrete subgroup of G and $\Gamma\backslash G$ is compact. As was shown by Godement in an earlier lecture the representation π of the previous paragraph occurs a finite number of times, say $N(\pi)$, in the regular representation on $L^2(\Gamma\backslash G)$. The problem is first to find a closed formula for $N(\pi)$. The method which I will now describe of obtaining such a formula is valid only when π is actually integrable.

Square integrable representations are similar in some respects to representations of compact groups; in particular they satisfy a form of the Schur orthogonality relations. There is a constant d_π called the formal degree of π such that if u', v', u, and v belong to H then

$$\int_G \bigl(\pi(g)u', v'\bigr)\overline{\bigl(\pi(g)u, v\bigr)}\, dg = d_\pi^{-1}(u', u)(v, v').$$

If u and v are such that $\bigl(\pi(g)u, v\bigr)$ is integrable and π' is unitary representation of G on H' which does not contain π, then

$$\int_G \bigl(\pi'(g)u', v'\bigr)\overline{\bigl(\pi(g)u, v\bigr)}\, dg = 0$$

for all u', v' in H.

Let L_i, $1 \leqslant i \leqslant N(\pi)$, be a family of mutually orthogonal invariant subspaces of $L^2(\Gamma\backslash G)$ which are such that the action of G on each of them is equivalent to π. Suppose that π does not occur in the orthogonal complement of

$$\bigoplus_{i=1}^{N(\pi)} L_i.$$

If π is integrable there is a unit vector v in H such that $\bigl(\pi(g)v, v\bigr)$ is integrable. Let v_i be a unit vector in L_i corresponding to v under some equivalence between H and L_i. The

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orthogonality relations imply that the operator $\Phi \to \Phi'$ with

$$\Phi'(g) = d_\pi \int_G \Phi(gh)\overline{\big(\pi(g)v, v\big)}\, dh,$$
$$= \int_{\Gamma\backslash G} \Phi(h)\left\{\sum_\Gamma \xi(g^{-1}\gamma h)\right\} dh,$$

if $\xi(g) = d_\pi\overline{\big(\pi(g)v, v\big)}$, is an orthogonal projection on the space spanned by $v_1, \ldots, v_{N(\pi)}$. For our purposes it may be assumed that v transforms according to a finite-dimensional representation of some maximal compact subgroup of G. Then the argument used by Borel in a previous lecture shows that

$$\sum_\Gamma \xi(g^{-1}\gamma h)$$

converges absolutely uniformly on compact subsets of $G \times G$. Hence $v_1, \ldots, v_{N(\pi)}$ may be supposed continuous. As a consequence

$$\sum_{i=1}^{N(\pi)} v_i(g)\overline{v}_i(g) = \sum_\Gamma \xi(g^{-1}\gamma h).$$

Set $h = g$ and integrate over $\Gamma\backslash G$ to obtain

$$N(\pi) = \int_{\Gamma\backslash G} \sum_\Gamma \xi(g^{-1}\gamma g)\, dg.$$

The sum in the integrand may be rearranged at will. If Σ is a set of representatives for the conjugacy classes in Γ the integral on the right equals

$$\int_{\Gamma\backslash G} \sum_{\gamma\in\Sigma} \sum_{\delta\in\Gamma_\gamma\backslash\Gamma} \xi(g^{-1}\delta^{-1}\gamma\delta g)\, dg = \sum_{\gamma\in\Sigma} \int_{\Gamma_\gamma\backslash G} \xi(g^{-1}\gamma g)\, dg$$
$$= \sum_{\gamma\in\Sigma} \mu(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} \xi(g^{-1}\gamma g)\, dg,$$

if Γ_γ and G_γ are the centralizers of γ in Γ and G respectively. The equality of $N(\pi)$ and the final expression is of course a special case of a formula of Selberg and has been known for some time.

The problem of evaluating $\mu(\Gamma_\gamma\backslash G_\gamma)$, the volume of $\Gamma_\gamma\backslash G_\gamma$, has been discussed in the lectures on Tamagawa numbers. So we shall not worry about it now. Since $\Gamma\backslash G$ is compact every element of Γ is semisimple; thus our problem is to express the integral

$$\int_{G_\gamma\backslash G} \xi(g^{-1}\gamma g)\, dg$$

in elementary terms when γ is a semisimple element of G.

If π is a square-integrable representation of G on H, v is a vector in H which transforms according to a finite-dimensional representation of some maximal compact subgroup of G, and

$$\xi(g) = d_\pi\overline{\big(\pi(g)v, v\big)},$$

then a recent theorem of Harish-Chandra states that

$$\int_{G_\gamma\backslash G} \xi(g^{-1}\gamma g)\,dg \tag{a}$$

exists for γ semisimple and vanishes unless γ is elliptic, that is, belongs to some compact subgroup of G. Since Σ contains only a finite number of elliptic elements the sum in the expression for $N(\pi)$ is finite. We still require a closed expression for the integrals appearing in it.

Let K be a maximal compact subgroup of G. Since G has a square-integrable representation there is a Cartan subgroup T of G contained in K. It is enough to compute the integral (a) for γ in T. There is a limit formula of Harish-Chandra which allows one to compute its value at the singular elements once its values at the regular elements are known. Thus we need only evaluate it when γ is regular. It should be remarked that in this limit formula there is a constant which depends on the choice of Haar measure on G_γ. The exact relation of this constant to the choice of Haar measure has never been determined; until it is, our problem cannot be regarded as completely solved.

If γ is regular and the measure on G_γ is so normalized that the volume of G_γ is one, then

$$\int_{G_\gamma\backslash G} \xi(g^{-1}\gamma g)\,dg = \chi_\pi(\gamma^{-1})$$

if χ_π is the character of π. An explicit expression for the right-hand side has recently been obtained.

Let $\mathfrak{h}$ be the Lie algebra of T; choose an order on the roots of $\mathfrak{h}_{\mathbf{C}}$; and let Λ be a linear function on $\mathfrak{h}_{\mathbf{C}}$ such that $\Lambda+\rho$, $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$, extends to a character of T and so that $(\Gamma+\rho,\alpha)\neq 0$ for all roots α. Assume, for simplicity, that ρ also extends to a character of T. To each such Λ there is associated a square-integrable representation π_Λ and if $H\in\mathfrak{h}$

$$\chi_{\pi_\Lambda}(\exp H) = (-1)^m\epsilon(\Lambda)\sum_{\sigma\in W}\frac{\operatorname{sgn}\sigma\exp\bigl(\sigma(\Lambda+\rho)\bigr)(H)}{\prod_{\alpha>0}\Bigl\{\Bigl(\exp\bigl(\alpha(G)/2\bigr)-\exp\bigl(-\alpha(G)/2\bigr)\Bigr)\Bigr\}}.$$

Here $m=\frac{1}{2}\dim G/K$, $\epsilon(\Lambda)=\operatorname{sgn}\bigl(\prod_{\alpha>0}(\Lambda+\rho,\alpha)\bigr)$, and W is the Weyl group of K. Every square-integrable representation is equivalent to π_Λ for some Λ. However the values of Λ for which π_Λ is integrable are not yet known. For some special cases see [1] and [2].

The geometrical meaning of the numbers $N(\pi_\Lambda)$ is not yet completely clear. I would like to close this lecture with some suggestions as to what it might be. Since the evidence at present is rather meagre, they are only tentative. If $\mathfrak{g}_{\mathbf{C}}$ is the complexification of the Lie algebra of $\mathfrak{g}$, the elements of $\mathfrak{g}_{\mathbf{C}}$ may be regarded as left-invariant complex vector fields on G and G/T may be turned into a complex manifold in such a way that the space of antiholomorphic tangent vectors at $\overline{g}=gT$ is the image of $\mathfrak{n}_{\mathbf{C}}^-$ if $\mathfrak{n}_{\mathbf{C}}^-$ is the subalgebra of $\mathfrak{g}_{\mathbf{C}}$ generated by root vectors belonging to negative roots. Let V^* be the bundle of antiholomorphic cotangent vectors and introduce a G-invariant metric in V^* and hence in $\bigwedge^q V^*$. Let B be the line bundle over G/T associated to the character $\xi(\exp H)=\exp\bigl(\Lambda(H)\bigr)$ of T. If Γ is a discrete subgroup of G let $C^q(\Lambda,\Gamma)$ be the space of Γ-invariant cross-sections of $B\otimes\bigwedge^q V^*$ which are square integrable over $\Gamma\backslash G/T$. There is a unique closed operator $\overline{\partial}$ from $C^q(\Lambda,\Gamma)$ to $C^{q+1}(\Lambda,\Gamma)$ whose domain contains the infinitely differentiable cross-sections of compact support on which $\overline{\partial}$ is to have its usual meaning and whose adjoint is defined on the infinitely differentiable cross-sections of $C^{q+1}(\Lambda,\Gamma)$ with compact support.

Set $C^q\big(\Lambda,\{1\}\big) = C^q(\Lambda)$. I expect, although I do not know how to prove it, that when $\Lambda+\rho$ is nonsingular the range of $\overline{\partial}$ is closed for every q. If this is so then the cohomology groups $H^1(\Lambda)$ will be Hilbert spaces on which G acts. Is it true that they vanish for all but one value of q, say $q=q_\Lambda$, and that the representation π'_Λ of G on $H^{q_\Lambda}(\Lambda)$ is equivalent to π_Λ? The following theorem is a clue to the value of q_Λ.

Theorem (P. Griffiths). *Let a_1 be the number of noncompact positive roots for which $(\Lambda+\rho,\alpha)>0$ and let a_2 be the number of compact positive roots for which $(\Lambda+\rho,\alpha)<0$. There is a constant c such that if $\big|(\Lambda+\rho,\alpha)\big|>c$ for every simple root, $\Gamma\backslash G$ is compact, and Γ acts freely on G/T, then $H^q(\Lambda,\Gamma)=0$ unless $q=a_1+a_2$.*

It is, I think, worthy of remark that if one assumes that $H^q(\Lambda)=\{0\}$ for $q\neq q_\Lambda=a_1+a_2$, then a formal application of the Woods Hole fixed point formula shows that if γ is a regular element of T, then the value at γ of the character of π'_Λ is $\chi_{\pi_\Lambda}(\gamma)$. By the way, it is known that $H^0(\Lambda)=0$ unless $q_\Lambda=0$ and that if $q_\Lambda=0$ the representation of G on $H^0(\Lambda)$ is in fact π_Λ.

Finally one will want to show that when π_Λ is integrable and $\Gamma\backslash G$ is compact the number $N(\pi_\Lambda)$ is equal to the dimension of $H^{q_\Lambda}(\Lambda,\Gamma)$. This can be done when $q_\Lambda=0$; in this case $H^0(\Gamma,\Lambda)$ is a space of automorphic forms.

It should be possible, although I have not done so, to test these suggestions for groups whose unitary representations are well understood, in particular, for $\mathrm{SL}(2,\mathbf{R})$ and the De Sitter group. To do this one might make use of an idea basic to Kostant's proof of the (generalized) Borel-Weil theorem for compact groups. Suppose σ is a unitary representation of G on a Hilbert space V. Let $C^q(V)$ be the space of all linear maps from $\bigwedge^q \mathfrak{n}^-_{\mathbf{C}}$ to V. $C^q(V)$ is a Hilbert space. The usual coboundary operator from $C^q(V)$ to $C^{q+1}(V)$ can be defined on those elements of $C^q(V)$ which take values in the Gårding subspace of V. The closure d of this operator is the adjoint of the restriction of its formal adjoint to those elements of $C^{q+1}(V)$ which take values in the Gårding subspace. Of course T acts on $\bigwedge^q \mathfrak{n}^-_{\mathbf{C}}$. If $f\in C^q(V)$ define $tf=f'$ by $f'(X)=tf(t^{-1}X)$, $X\in\bigwedge^q\mathfrak{n}^-_{\mathbf{C}}$. There is a natural identification of $C^q(\Lambda)$ with the set of f in $C^q\big(L^2(G)\big)$ such that $tf=\exp\big(-\Lambda(H)\big)f$ if $t=\exp H$ belongs to T and of $C^q(\Lambda,\Gamma)$ with the set of f in $C^q\big(L^2(\Gamma\backslash G)\big)$ such that $tf=\exp\big(-\Lambda(H)\big)f$. Moreover the following diagrams are commutative.

$$\begin{array}{ccc} C^q(\Lambda) & \xrightarrow{\ \overline{\partial}\ } & C^{q+1}(\Lambda) \\ \big\downarrow & & \big\downarrow \\ C^q\big(L^2(G)\big) & \xrightarrow{\ d\ } & C^{q+1}\big(L^2(G)\big) \end{array} \qquad \begin{array}{ccc} C^q(\Lambda,\Gamma) & \xrightarrow{\ \overline{\partial}\ } & C^{q+1}(\Lambda,\Gamma) \\ \big\downarrow & & \big\downarrow \\ C^q\big(L^2(\Gamma\backslash G)\big) & \xrightarrow{\ d\ } & C^{q+1}\big(L^2(\Gamma\backslash G)\big) \end{array}$$

The point is that d is easier to study than $\overline{\partial}$ because to study d we can decompose V into irreducible representations and study the action of d on each part.

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