

DIMENSION OF SPACES OF AUTOMORPHIC FORMS

ROBERT P. LANGLANDS

I will first formulate a problem in the theory of group representations and show how to solve it; then I will discuss the relation of this problem to the theory of automorphic forms. Since there is no point in striving for maximum generality, I start with a connected semisimple group G with finite centre. An irreducible unitary representation π of G on the Hilbert space H is said to be square-integrable if for one and hence, as one can show, every pair u and v of nonzero vectors in H the function $(\pi(g)u, v)$ is square-integrable on G . It is said to be integrable if for one such pair $(\pi(g)u, v)$ is integrable.

Suppose Γ is a discrete subgroup of G and $\Gamma \backslash G$ is compact. As was shown by Godement in an earlier lecture the representation π of the previous paragraph occurs a finite number of times, say $N(\pi)$, in the regular representation on $L^2(\Gamma \backslash G)$. The problem is first to find a closed formula for $N(\pi)$. The method which I will now describe of obtaining such a formula is valid only when π is actually integrable.

Square integrable representations are similar in some respects to representations of compact groups; in particular they satisfy a form of the Schur orthogonality relations. There is a constant d_π called the formal degree of π such that if u', v', u , and v belong to H then

$$\int_G (\pi(g)u', v') \overline{(\pi(g)u, v)} dg = d_\pi^{-1}(u', u)(v, v').$$

If u and v are such that $(\pi(g)u, v)$ is integrable and π' is unitary representation of G on H' which does not contain π , then

$$\int_G (\pi'(g)u', v') \overline{(\pi(g)u, v)} dg = 0$$

for all u', v' in H .

Let L_i , $1 \leq i \leq N(\pi)$, be a family of mutually orthogonal invariant subspaces of $L^2(\Gamma \backslash G)$ which are such that the action of G on each of them is equivalent to π . Suppose that π does not occur in the orthogonal complement of

$$\bigoplus_{i=1}^{N(\pi)} L_i.$$

If π is integrable there is a unit vector v in H such that $(\pi(g)v, v)$ is integrable. Let v_i be a unit vector in L_i corresponding to v under some equivalence between H and L_i . The

Appeared first in Proc. of Symp. in Pure Mathematics, vol. IX, Amer. Math. Soc. (1966).

Based on notes of the original lecture taken by J. Shalika.

Miller Fellow.

orthogonality relations imply that the operator $\Phi \rightarrow \Phi'$ with

$$\begin{aligned}\Phi'(g) &= d_\pi \int_G \Phi(gh) \overline{(\pi(g)v, v)} dh, \\ &= \int_{\Gamma \backslash G} \Phi(h) \left\{ \sum_{\Gamma} \xi(g^{-1}\gamma h) \right\} dh,\end{aligned}$$

if $\xi(g) = d_\pi \overline{(\pi(g)v, v)}$, is an orthogonal projection on the space spanned by $v_1, \dots, v_{N(\pi)}$. For our purposes it may be assumed that v transforms according to a finite-dimensional representation of some maximal compact subgroup of G . Then the argument used by Borel in a previous lecture shows that

$$\sum_{\Gamma} \xi(g^{-1}\gamma h)$$

converges absolutely uniformly on compact subsets of $G \times G$. Hence $v_1, \dots, v_{N(\pi)}$ may be supposed continuous. As a consequence

$$\sum_{i=1}^{N(\pi)} v_i(g) \bar{v}_i(g) = \sum_{\Gamma} \xi(g^{-1}\gamma h).$$

Set $h = g$ and integrate over $\Gamma \backslash G$ to obtain

$$N(\pi) = \int_{\Gamma \backslash G} \sum_{\Gamma} \xi(g^{-1}\gamma g) dg.$$

The sum in the integrand may be rearranged at will. If Σ is a set of representatives for the conjugacy classes in Γ the integral on the right equals

$$\begin{aligned}\int_{\Gamma \backslash G} \sum_{\gamma \in \Sigma} \sum_{\delta \in \Gamma_\gamma \backslash \Gamma} \xi(g^{-1}\delta^{-1}\gamma\delta g) dg &= \sum_{\gamma \in \Sigma} \int_{\Gamma_\gamma \backslash G} \xi(g^{-1}\gamma g) dg \\ &= \sum_{\gamma \in \Sigma} \mu(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} \xi(g^{-1}\gamma g) dg,\end{aligned}$$

if Γ_γ and G_γ are the centralizers of γ in Γ and G respectively. The equality of $N(\pi)$ and the final expression is of course a special case of a formula of Selberg and has been known for some time.

The problem of evaluating $\mu(\Gamma_\gamma \backslash G_\gamma)$, the volume of $\Gamma_\gamma \backslash G_\gamma$, has been discussed in the lectures on Tamagawa numbers. So we shall not worry about it now. Since $\Gamma \backslash G$ is compact every element of Γ is semisimple; thus our problem is to express the integral

$$\int_{G_\gamma \backslash G} \xi(g^{-1}\gamma g) dg$$

in elementary terms when γ is a semisimple element of G .

If π is a square-integrable representation of G on H , v is a vector in H which transforms according to a finite-dimensional representation of some maximal compact subgroup of G , and

$$\xi(g) = d_\pi \overline{(\pi(g)v, v)},$$

then a recent theorem of Harish-Chandra states that

$$(a) \quad \int_{G_\gamma \backslash G} \xi(g^{-1}\gamma g) dg$$

exists for γ semisimple and vanishes unless γ is elliptic, that is, belongs to some compact subgroup of G . Since Σ contains only a finite number of elliptic elements the sum in the expression for $N(\pi)$ is finite. We still require a closed expression for the integrals appearing in it.

Let K be a maximal compact subgroup of G . Since G has a square-integrable representation there is a Cartan subgroup T of G contained in K . It is enough to compute the integral (a) for γ in T . There is a limit formula of Harish-Chandra which allows one to compute its value at the singular elements once its values at the regular elements are known. Thus we need only evaluate it when γ is regular. It should be remarked that in this limit formula there is a constant which depends on the choice of Haar measure on G_γ . The exact relation of this constant to the choice of Haar measure has never been determined; until it is, our problem cannot be regarded as completely solved.

If γ is regular and the measure on G_γ is so normalized that the volume of G_γ is one, then

$$\int_{G_\gamma \backslash G} \xi(g^{-1}\gamma g) dg = \chi_\pi(\gamma^{-1})$$

if χ_π is the character of π . An explicit expression for the right-hand side has recently been obtained.

Let \mathfrak{h} be the Lie algebra of T ; choose an order on the roots of $\mathfrak{h}_\mathbb{C}$; and let Λ be a linear function on $\mathfrak{h}_\mathbb{C}$ such that $\Lambda + \rho$, $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$, extends to a character of T and so that $(\Gamma + \rho, \alpha) \neq 0$ for all roots α . Assume, for simplicity, that ρ also extends to a character of T . To each such Λ there is associated a square-integrable representation π_Λ and if $H \in \mathfrak{h}$

$$\chi_{\pi_\Lambda}(\exp H) = (-1)^m \epsilon(\Lambda) \sum_{\sigma \in W} \frac{\operatorname{sgn} \sigma \exp(\sigma(\Lambda + \rho))(H)}{\prod_{\alpha > 0} \left\{ \left(\exp(\alpha(G)/2) - \exp(-\alpha(G)/2) \right) \right\}}.$$

Here $m = \frac{1}{2} \dim G/K$, $\epsilon(\Lambda) = \operatorname{sgn}(\prod_{\alpha > 0} (\Lambda + \rho, \alpha))$, and W is the Weyl group of K . Every square-integrable representation is equivalent to π_Λ for some Λ . However the values of Λ for which π_Λ is integrable are not yet known. For some special cases see [1] and [2].

The geometrical meaning of the numbers $N(\pi_\Lambda)$ is not yet completely clear. I would like to close this lecture with some suggestions as to what it might be. Since the evidence at present is rather meagre, they are only tentative. If $\mathfrak{g}_\mathbb{C}$ is the complexification of the Lie algebra of \mathfrak{g} , the elements of $\mathfrak{g}_\mathbb{C}$ may be regarded as left-invariant complex vector fields on G and G/T may be turned into a complex manifold in such a way that the space of antiholomorphic tangent vectors at $\bar{g} = gT$ is the image of $\mathfrak{n}_\mathbb{C}^-$ if $\mathfrak{n}_\mathbb{C}^-$ is the subalgebra of $\mathfrak{g}_\mathbb{C}$ generated by root vectors belonging to negative roots. Let V^* be the bundle of antiholomorphic cotangent vectors and introduce a G -invariant metric in V^* and hence in $\bigwedge^q V^*$. Let B be the line bundle over G/T associated to the character $\xi(\exp H) = \exp(\Lambda(H))$ of T . If Γ is a discrete subgroup of G let $C^q(\Lambda, \Gamma)$ be the space of Γ -invariant cross-sections of $B \otimes \bigwedge^q V^*$ which are square integrable over $\Gamma \backslash G/T$. There is a unique closed operator $\bar{\partial}$ from $C^q(\Lambda, \Gamma)$ to $C^{q+1}(\Lambda, \Gamma)$ whose domain contains the infinitely differentiable cross-sections of compact support on which $\bar{\partial}$ is to have its usual meaning and whose adjoint is defined on the infinitely differentiable cross-sections of $C^{q+1}(\Lambda, \Gamma)$ with compact support.

Set $C^q(\Lambda, \{1\}) = C^q(\Lambda)$. I expect, although I do not know how to prove it, that when $\Lambda + \rho$ is nonsingular the range of $\bar{\partial}$ is closed for every q . If this is so then the cohomology groups $H^1(\Lambda)$ will be Hilbert spaces on which G acts. Is it true that they vanish for all but one value of q , say $q = q_\Lambda$, and that the representation π'_Λ of G on $H^{q_\Lambda}(\Lambda)$ is equivalent to π_Λ ? The following theorem is a clue to the value of q_Λ .

Theorem (P. Griffiths). *Let a_1 be the number of noncompact positive roots for which $(\Lambda + \rho, \alpha) > 0$ and let a_2 be the number of compact positive roots for which $(\Lambda + \rho, \alpha) < 0$. There is a constant c such that if $|(\Lambda + \rho, \alpha)| > c$ for every simple root, $\Gamma \backslash G$ is compact, and Γ acts freely on G/T , then $H^q(\Lambda, \Gamma) = 0$ unless $q = a_1 + a_2$.*

It is, I think, worthy of remark that if one assumes that $H^q(\Lambda) = \{0\}$ for $q \neq q_\Lambda = a_1 + a_2$, then a formal application of the Woods Hole fixed point formula shows that if γ is a regular element of T , then the value at γ of the character of π'_Λ is $\chi_{\pi_\Lambda}(\gamma)$. By the way, it is known that $H^0(\Lambda) = 0$ unless $q_\Lambda = 0$ and that if $q_\Lambda = 0$ the representation of G on $H^0(\Lambda)$ is in fact π_Λ .

Finally one will want to show that when π_Λ is integrable and $\Gamma \backslash G$ is compact the number $N(\pi_\Lambda)$ is equal to the dimension of $H^{q_\Lambda}(\Lambda, \Gamma)$. This can be done when $q_\Lambda = 0$; in this case $H^0(\Gamma, \Lambda)$ is a space of automorphic forms.

It should be possible, although I have not done so, to test these suggestions for groups whose unitary representations are well understood, in particular, for $\mathrm{SL}(2, \mathbf{R})$ and the De Sitter group. To do this one might make use of an idea basic to Kostant's proof of the (generalized) Borel-Weil theorem for compact groups. Suppose σ is a unitary representation of G on a Hilbert space V . Let $C^q(V)$ be the space of all linear maps from $\bigwedge^q \mathfrak{n}_\mathbb{C}^-$ to V . $C^q(V)$ is a Hilbert space. The usual coboundary operator from $C^q(V)$ to $C^{q+1}(V)$ can be defined on those elements of $C^q(V)$ which take values in the Gårding subspace of V . The closure d of this operator is the adjoint of the restriction of its formal adjoint to those elements of $C^{q+1}(V)$ which take values in the Gårding subspace. Of course T acts on $\bigwedge^q \mathfrak{n}_\mathbb{C}^-$. If $f \in C^q(V)$ define $tf = f'$ by $f'(X) = tf(t^{-1}X)$, $X \in \bigwedge^q \mathfrak{n}_\mathbb{C}^-$. There is a natural identification of $C^q(\Lambda)$ with the set of f in $C^q(L^2(G))$ such that $tf = \exp(-\Lambda(H))f$ if $t = \exp H$ belongs to T and of $C^q(\Lambda, \Gamma)$ with the set of f in $C^q(L^2(\Gamma \backslash G))$ such that $tf = \exp(-\Lambda(H))f$. Moreover the following diagrams are commutative.

$$\begin{array}{ccc} C^q(\Lambda) & \xrightarrow{\bar{\partial}} & C^{q+1}(\Lambda) \\ \downarrow & & \downarrow \\ C^q(L^2(G)) & \xrightarrow{d} & C^{q+1}(L^2(G)) \end{array} \quad \begin{array}{ccc} C^q(\Lambda, \Gamma) & \xrightarrow{\bar{\partial}} & C^{q+1}(\Lambda, \Gamma) \\ \downarrow & & \downarrow \\ C^q(L^2(\Gamma \backslash G)) & \xrightarrow{d} & C^{q+1}(L^2(\Gamma \backslash G)) \end{array}$$

The point is that d is easier to study than $\bar{\partial}$ because to study d we can decompose V into irreducible representations and study the action of d on each part.

REFERENCES

- [1] J. Dixmier, *Représentations intégrables du groupe de De Sitter*, Bull. Soc. Math., France **89** (1961), 9–41.
- [2] Harish-Chandra, *Representations of semisimple Lie groups*, VI, Amer. J. Math. **78** (1956), 564–628.
- [3] ———, *Discrete series for semisimple Lie groups*, II. Acta Math. (to appear) [Acta Mathematica, vol. 116, December 1966, pp. 1–111].
- [4] Bertram Kostant, *Lie algebra cohomology and the generalized Borel-Weil theorem*, Ann. of Math. (2) **74** (1961), 329–387.

- [5] R. P. Langlands, *The dimension of spaces of automorphic forms*, Amer. J. Math. **85** (1963), 99–125.

Compiled on November 17, 2025.