Last or very well last thoughts on the theory of automorphic forms.

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These thoughts are related to two documents neither of which is yet in its final form. They can be found at

http://publications.ias.edu/rpl/section/2659

The first, in Russian, will be accessible to relatively few mathematicians; the second, in Turkish, to even fewer. The first is still somewhat rough, but I hope it will ultimately appear. The second is even rougher. It was intended as the notes for a lecture that was ultimately delivered extemporaneously. It was, however, only during the preparation of the lecture that I began to appreciate the nature of the difference between the arithmetic theory, with which I have spent almost of my mathematical career, and the geometric theory, with which I have only recently become familiar but about which I have distinct and strong views.

The goal of the first document, a paper submitted for publication, is clear. It is a step towards proving the following assertion in the geometric theory. For each non-singular complete algebraic curve over $\mathbf{C}$ there is a group Γ_{aut}, a *galoisian group* whose unitary representations in the group ${}^LG$ parametrize the automorphic representations of G. I considered only unramified representations, only two groups, $GL(1)$, which is of course very easy, and $GL(2)$, for which considerable time and effort were necessary, and only elliptic curves. Although this is the simplest possible case, to understand it was not easy, at least for me.

As I observed, the second document was originally intended as notes for a lecture in Istanbul. It is evident that it contains far more material than can be explained in an hour. Besides much was irrelevant. The lecture itself was ultimately extemporaneous. The preparation was, however, of considerable importance for me, because I began to understand the difference between the geometric theory and the arithmetic theory. I began with the belief that such a galoisian group would appear in the arithmetic theory as well, but was ultimately convinced that the nature of the arithmetic theory lies to some extent in the circumstance that there was no such group in the arithmetic theory, that the group was replaced by a Tannakian category, for which a pertinent reference is available on the web.

Deligne-Milne, *Tannakian Categories.*

These are categories of which the representations of a given group is the standard example. The notion of *functoriality* in the theory of automorphic forms, a theory of which we have as yet only intimations, is the example pertinent in the present context. The text

http://publications.ias.edu/rpl/paper/91

is an introduction. Otherwise it is best to consult the writings of Arthur, especially

The endoscopic classification of representations,

and others. The passage from functoriality to Tannakian categories does not appear explicitly in these early papers but on reflection it is evident. In the context of the geometric theory of automorphic forms or, at the moment more to the point, the arithmetic theory of automorphic forms it is the functoriality associated to ${}^LG \to {}^LG'$ and automorphic representations of the associated quasi-split groups G and G' over a given number field of finite degree over $\mathbf{Q}$. For a precise discussion it may be best to fix ${}^LG$ and to consider the various representations of ${}^LG$ in $GL(n)$, n a positive integer for these determine, once functoriality is established, the various L-functions associated to representations of ${}^LG$ in ${}^LG'$, $G' = GL(n)$. These were studied by Godement-Jacquet. I repeat that the paper **Об аналитическом виде геометрической теории автоморфных форм** was meant as an initial step towards a proof of the assertion that the for a function field this Tannakian category was the category of representations of a specific group, a *galoisian group*.

Over a number field this is not so. For them there is a second Tannakian category, that defined by the cohomology groups of smooth, complete varieties over the given field and that is not associated to a galoisian group. What has to be shown is that this second category maps to the first, that derived from the theory of automorphic forms, in such a way that the associated L-functions, Hasse-Weil L-functions on one hand and automorphic L-functions on the other are equal. This would establish the basic properties of the Hasse-Weil L-functions and the number theorists would have to take it from there! This will be an enormous task! I repeat that this was my conclusion after attempting unsuccessfully to prepare the lecture along the lines of the second text mentioned in the introductory paragraph.