

# **Euler Products**

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First published in 1971 as a Yale Mathematical Monograph by Yale University Press.  
James K. Whittemore Lectures in Mathematics given at Yale University.

## Contents

Chapter 1.	Introduction	1
Chapter 2.	Some Euler products	5
Chapter 3.	Spherical functions	9
Chapter 4.	The formula of Gindikin and Karpelevich	21
Chapter 5.	A review of Eisenstein series	25
Chapter 6.	Examples	31



## CHAPTER 1

### Introduction

The first undisguised automorphic forms met by most of us are the modular forms. A modular form of weight  $k$  is an analytic function on the upper half-plane which satisfies

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all integral matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of determinant 1. Actually there is a supplementary condition, of no importance to us, on its rate of growth as  $\text{Im}(z) \rightarrow \infty$ . Nowadays the Siegel modular forms are met soon afterwards. A Siegel modular form of weight  $k$  is a complex analytic function on the space of complex  $n \times n$  symmetric matrices with positive definite real part which satisfies

$$f((AZ+B)(CZ+D)^{-1}) = \det(CZ+D)^k f(Z)$$

for all integral  $2n \times 2n$  symplectic matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

For some purposes it is best to consider not  $f$  but an associated function  $\phi_f$  on the group  $G$  of real  $2n \times 2n$  symplectic matrices defined by

$$\phi_f(g) = \det(Ci+D)^{-k} f((Ai+B)(Ci+D)^{-1})$$

if

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

If  $\Gamma$  is the group of integral matrices in  $G$  then  $\phi_f$  is a function on  $\Gamma \backslash G$ . The functions  $\phi_f$  associated to those  $f$  which have a finite norm in the Petersson metric can be characterized in terms of the representations of  $G$ . Associating to each  $h$  in  $G$  the operator  $\lambda(h)$  on  $L^2(\Gamma \backslash G)$  defined by

$$(\lambda(h)\phi)(g) = \phi(gh),$$

we obtain a representation of  $G$  on  $L^2(\Gamma \backslash G)$ . There is a representation  $\pi_k$  of  $G$  on a Hilbert space  $H_k$  and a distinguished subspace  $H_k^0$  of  $H_k$  such that  $\phi$  is a  $\phi_f$  if and only if there is a  $G$ -invariant map of  $H_k$  to  $L^2(\Gamma \backslash G)$  which takes  $H_k^0$  to the space generated by  $\phi$ .

This is the first hint that it might not be entirely unprofitable to study automorphic forms in the context of the theory of group representations. The Hecke operators, which play a major role in the study of modular forms, provide a second. To see this we have to introduce the adèle group of  $2n \times 2n$  symplectic matrices. It will be convenient to change the notation a little. If  $R$  is a commutative ring let  $G_R$  be the group of  $2n \times 2n$  symplectic matrices with entries from  $R$ . Thus the groups  $\Gamma$  and  $G$  become  $G_{\mathbf{Z}}$  and  $G_{\mathbf{R}}$ . If  $p$  is a prime,

finite or infinite, let  $\mathbf{Q}_p$  be the corresponding completion of  $\mathbf{Q}$ . The adèle ring  $\mathbf{A}$  is the set of elements  $\{a_p\}$  in  $\prod_p \mathbf{Q}_p$  such that  $a_p$  is integral for all but a finite number of  $p$ . The diagonal map defines an imbedding of  $\mathbf{Q}$  in  $\mathbf{A}$ . There is a standard topology on  $G_{\mathbf{A}}$  which turns it into a locally compact group with  $G_{\mathbf{Q}}$  as a discrete subgroup. Let

$$\mathbf{A}^0 = \mathbf{R} \times \prod_{p \text{ finite}} \mathbf{Z}_p.$$

It is known that  $G_{\mathbf{A}} = G_{\mathbf{Q}}G_{\mathbf{A}^0}$ . Any function  $\phi$  on  $G_{\mathbf{Z}} \backslash G_{\mathbf{R}}$  can be regarded as a function on  $G_{\mathbf{Q}} \backslash G_{\mathbf{A}}$  if one sets  $\phi(g) = \phi(g'_2)$  when  $g = g_1g_2$ ,  $g_1 \in G_{\mathbf{Q}}$ ,  $g_2 \in G_{\mathbf{A}^0}$ , and  $g'_2$  is the projection of  $g_2$  on  $G_{\mathbf{R}}$ . The functions so obtained are characterized by their right invariance under

$$U = 1 \times \prod_{p \text{ finite}} G_{\mathbf{Z}_p}.$$

If  $f$  is a function, with compact support, on

$$G^0 = \left\{ 1 \times \prod_{p \text{ finite}} G_{\mathbf{Q}_p} \right\} \cap G_{\mathbf{A}},$$

which is invariant on the left and right under  $U$ , and if  $\lambda(f)\phi$  is defined by

$$\lambda(f)\phi(g) = \int_{G^0} \phi(gh)f(h) dh,$$

then  $\lambda(f)\phi$  is invariant on the right under  $U$  if  $\phi$  is. Thus the operators  $\lambda(f)$ , the Hecke operators, act on the functions on  $G_{\mathbf{Z}} \backslash G_{\mathbf{R}}$ . If  $\phi$  belongs to a subspace  $H$  of  $L^2(G_{\mathbf{Q}} \backslash G_{\mathbf{A}})$  which is invariant and irreducible under the action of  $G_{\mathbf{A}}$  then  $\phi$  is an eigenfunction of all the Hecke operators and the corresponding eigenvalues are determined by the equivalence class of the representation of  $G_{\mathbf{A}}$  on  $H$ .

The theory of modular forms and the operators  $\lambda(f)$  is far from complete. Indeed very little attempt has been made, so far as I can see, to understand what the goals of the theory should be. Once it is put in the above form it is clear that the concepts at least admit of extension to any reductive algebraic group defined over a number field. It may be possible to give some coherency to the subject by introducing the simple principle, implicit in the work of Harish-Chandra, that what can be done for one reductive group should be done for all.

The simplest reductive group over  $\mathbf{Q}$  is  $\text{GL}(1)$ . Since  $G_{\mathbf{A}}$  is abelian, the space  $G_{\mathbf{Q}} \backslash G_{\mathbf{A}}$  is a locally compact abelian group  $C$ , the group of idèle classes of  $\mathbf{Q}$ . According to the Plancherel theorem for abelian groups the characters of  $C$  can be used to decompose  $L^2(G_{\mathbf{Q}} \backslash G_{\mathbf{A}})$  under the action of  $G_{\mathbf{A}}$  and the characters of  $C$  must be regarded as the basic automorphic forms.

Since  $\mathbf{Q}_p^\times$ , the multiplicative group of  $\mathbf{Q}_p$ , is contained in  $G_{\mathbf{A}}$ , each character  $\chi$  of  $C$  defines a character  $\chi_p$  of  $\mathbf{Q}_p^\times$ . If  $p$  does not divide  $f_\chi$ , the conductor of  $\chi$ ,  $\chi_p$  is trivial on the units of  $\mathbf{Q}_p^\times$ . It is customary to associate to  $\chi$  the  $L$ -series

$$L(s, \chi) = \prod_{p \nmid f_\chi} \frac{1}{1 - \frac{\chi_p(p)}{p^s}}$$

and the function

$$\xi(s, \chi) = \left( \frac{\pi}{f_\chi} \right)^{-\frac{s+\alpha+\beta}{2}} \Gamma\left( \frac{s+\alpha+\beta}{2} \right) L(s, \chi)$$

if

$$\chi_\infty(x) = (\operatorname{sgn} x)^\alpha |x|^\beta.$$

The number  $\alpha$  is 0 or 1. The function  $\xi(s, \chi)$  is known to be meromorphic in the entire complex plane and to satisfy

$$\xi(s, \chi) = \epsilon(\chi) \xi(1 - s, \chi^{-1})$$

where  $\epsilon(\chi)$  is a constant of absolute value 1.

For a general group  $L^2(G_{\mathbf{Q}} \backslash G_{\mathbf{A}})$  decomposes not into a direct integral of one-dimensional spaces but into a direct integral of Hilbert spaces on each of which  $G_{\mathbf{A}}$  acts irreducibly. Can we associate to each of these Hilbert spaces an Euler product with the same analytic properties as the functions  $L(s, \chi)$ ? In these lectures I would like to present a little evidence, far from conclusive, that the answer is “yes”. Let me observe that each power of a character is also a character, thus to  $\chi$  is associated the whole collection  $\{L(s, \chi^n) \mid n \in \mathbf{Z}\}$ , and that  $\mathbf{Z}$  parametrizes not only the powers of  $\chi$  but also the rational representations of  $\mathrm{GL}(1)$ .

Before beginning the substantial part of these lectures let me make, without committing myself, a further observation. The Euler products mentioned above are defined by means of the Hecke operators. Thus they are defined in an entirely different manner than those of Artin or Hasse-Weil. An assertion that an Euler product of the latter type is equal to one of those associated to an automorphic form is tantamount to a reciprocity law (for one equation in one variable in the case of the Artin  $L$ -series and for several equations in several variables in the case of the Hasse-Weil  $L$ -series).





## CHAPTER 2

### Some Euler products

Suppose  $\mathfrak{g}$  is a split semisimple Lie algebra over  $\mathbf{Q}$  and  $G$  its adjoint group. Fix a split Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  and choose a Chevalley basis for  $\mathfrak{g}$  with respect to  $\mathfrak{h}$ . Let  $M$  be the lattice generated by the Chevalley basis over  $\mathbf{Z}$ . If  $p$  is a finite prime let  $G_{\mathbf{Z}_p}$  be the stabilizer of  $M \times \mathbf{Z}_p$  in  $G_{\mathbf{Q}_p}$ . If  $p$  is the infinite prime let

$$G_{\mathbf{Z}_p} \subseteq G_{\mathbf{Q}_p} \equiv G_{\mathbf{R}}$$

be the maximal compact subgroup of  $G_{\mathbf{R}}$  corresponding to the involution associated to the Chevalley basis. Let

$$U = \prod_p G_{\mathbf{Z}_p} \subseteq G_{\mathbf{A}},$$

be the adèle group of  $G$ . Fix a Borel subgroup  $B$  containing  $T$ , the Cartan subgroup with Lie algebra  $\mathfrak{h}$ . It is known that

$$G_{\mathbf{Q}_p} = B_{\mathbf{Q}_p} G_{\mathbf{Z}_p}$$

for each  $p$ . For this and various other facts about reduction theory over local fields see the article by F. Bruhat in the collection *Algebraic groups and discontinuous subgroups*. As a consequence  $G_{\mathbf{A}} = B_{\mathbf{A}} U$ . Moreover  $B_{\mathbf{A}} = B_{\mathbf{Q}} B_{\mathbf{R}} (B_{\mathbf{A}} \cap U)$ ; hence  $G_{\mathbf{A}} = B_{\mathbf{Q}} G_{\mathbf{R}} U$  and  $G_{\mathbf{Q}} = B_{\mathbf{Q}} G_{\mathbf{Z}}$  if  $G_{\mathbf{Z}}$  is the stabilizer of  $M$ . In particular any function on  $G_{\mathbf{Q}} \backslash G_{\mathbf{A}} / U$  is determined by its restriction to  $G_{\mathbf{R}}$ .

Let  $L$  be the space of all square integrable functions on  $G_{\mathbf{Q}} \backslash G_{\mathbf{A}}$  which are invariant under right translations by elements of  $U$ . Let  $P$  be a parabolic group containing  $B$  and let  $N$  be its unipotent radical. If  $\phi$  lies in  $L$  then

$$\int_{N_{\mathbf{Q}} \backslash N_{\mathbf{A}}} \phi(ng) \, dn$$

vanishes for almost all  $g$  in  $G_{\mathbf{A}}$  if and only if

$$\int_{N_{\mathbf{Z}} \backslash N_{\mathbf{R}}} \phi(ng) \, dn$$

vanishes for almost all  $g$  in  $G_{\mathbf{R}}$ . If these integrals vanish for almost all  $g$  for all choices of  $P$  except  $G$  itself we say that  $\phi$  is a cusp form. The set of cusp forms is a closed subspace  $L_0$  of  $L$ .

If  $p$  is a prime, finite or infinite, let  $H_p$  be the algebra of all compactly supported regular Borel measures on  $G_{\mathbf{Q}_p}$  invariant under left and right translations by elements of  $G_{\mathbf{Z}_p}$ . Multiplication is given by convolution. If  $\mu$  lies in  $H_p$ , define the operator  $\lambda(\mu)$  on  $L_0$  by

$$\lambda(\mu)\phi(g) = \int_{G_{\mathbf{Q}_p}} \phi(gh) \, d\mu(h).$$

If  $\mu$  is the measure associated with an  $L^1$  function  $f$  we shall sometimes write  $\lambda(f)$  instead of  $\lambda(\mu)$ . Moreover  $f$  will be regarded as an element of  $H_p$ . If  $p$  is finite all the measures in

$H_p$  are absolutely continuous with respect to Haar measure. There is an orthonormal basis  $\phi_1, \phi_2, \dots$  of  $L_0$  such that each  $\phi_i$  is, for all  $p$ , an eigenfunction of  $\lambda(\mu)$  for all  $\mu$  in  $H_p$ .

Fix one element  $\phi$  of this basis. If  $\mu$  belongs to  $H_p$  let  $\lambda(\mu)\phi = \chi_p(\mu)\phi$ . The map  $\mu \rightarrow \chi_p(\mu)$  is a homomorphism of  $H_p$  into the complex numbers. Let me remind you of the standard method of obtaining all such homomorphisms. Let  $V$  be the unipotent radical of  $B$ . Now  $V_{\mathbf{Q}_p} \backslash B_{\mathbf{Q}_p}$  is isomorphic to  $T_{\mathbf{Q}_p}$ . Thus any homomorphism  $w$  of  $T_{\mathbf{Q}_p}/T_{\mathbf{Z}_p}$  into the complex numbers determines a homomorphism of  $B_{\mathbf{Q}_p}$  into the complex numbers which we again denote by  $w$ . If  $b$  belongs to  $B$  let  $\xi(b)$  be the determinant of the restriction of  $\text{Ad}(b)$  to  $\mathfrak{v}$ , the Lie algebra of  $V$ . If  $g$  lies in  $G_{\mathbf{Q}_p}$  then  $g$  can be written as a product  $bk$  with  $b$  in  $B_{\mathbf{Q}_p}$  and  $k$  in  $G_{\mathbf{Z}_p}$ . Set

$$\psi_w(g) = w(b)|\xi(b)|^{1/2}.$$

The function  $\psi_w$  is well-defined and any other function  $\psi$  on  $G_{\mathbf{Q}_p}$  satisfying

$$(1) \quad \psi(bgk) = w(b)|\xi(b)|^{1/2}\psi(g)$$

for all  $b, g$ , and  $k$  is a scalar multiple of  $\psi_w$ . If  $\mu$  lies in  $H_p$  define  $\lambda(\mu)\psi_w$  by

$$(\lambda(\mu)\psi_w)(g) = \int_{G_{\mathbf{Q}_p}} \psi_w(gh) d\mu(h).$$

The function  $\lambda(\mu)\psi_w$  satisfies (1) so there is a scalar  $\chi_w(\mu)$  such that

$$\lambda(\mu)\psi_w = \chi_w(\mu)\psi_w.$$

The map  $\mu \rightarrow \chi_w(\mu)$  is a homomorphism of  $H_p$  into  $\mathbf{C}$ . All homomorphisms of  $H_p$  into the complex numbers which are continuous in the weak topology are obtained in this way. The homomorphism  $\chi_w$  equals  $\chi_{w'}$  if and only if there is a  $\sigma$  in the Weyl group such that  $w(t) = w'(t^\sigma)$  for all  $t$  in  $T_{\mathbf{Q}_p}$ .

Suppose  $p$  is finite. If  $L$  is the lattice generated by the roots of  $\mathfrak{h}$  there is a homomorphism  $\lambda$  from  $\bar{T} = T_{\mathbf{Q}_p}/T_{\mathbf{Z}_p}$ , or from  $T_{\mathbf{Q}_p}$ , to  ${}^c\mathcal{L} = \text{Hom}(L, \mathbf{Z})$  such that  $|\xi_\alpha(t)| = p^{\lambda(t)(\alpha)}$  if  $\alpha$  is a root. Here  $\xi_\alpha$  is the character of  $T$  associated to  $\alpha$ . If  $\alpha$  is a root let  $H_\alpha$  be, in the language of Chevalley, the *copoid* attached to  $\alpha$ . Let  $\alpha_1, \dots, \alpha_n$  be the simple roots. The matrix

$$(A_{ij}) = \left( \begin{array}{c} (\alpha_i, \alpha_j) \\ (\alpha_i, \alpha_i) \end{array} \right)$$

is the Cartan matrix of  $\mathfrak{g}$ . The matrix

$$\left( \begin{array}{c} (H_{\alpha_i}, H_{\alpha_j}) \\ (H_{\alpha_i}, H_{\alpha_i}) \end{array} \right)$$

is the transpose of  $(A_{ij})$  and the Cartan matrix of another Lie algebra  ${}^c\mathfrak{g}$ . The lattice  ${}^c\mathcal{L}'$  generated by the roots of a split Cartan subalgebra  ${}^c\mathfrak{h}$  of  ${}^c\mathfrak{g}$  can be identified with the lattice in  $\mathfrak{h}_{\mathbf{R}}$  generated by  $H_{\alpha_1}, \dots, H_{\alpha_n}$  in such a way that the roots of  ${}^c\mathfrak{h}$  correspond to the elements  $H_\alpha$ . Moreover  ${}^c\mathcal{L}$  can be regarded as a lattice in  $\mathfrak{h}_{\mathbf{R}}$ . It contains  ${}^c\mathcal{L}'$  and can in fact be regarded as the lattice of weights of  ${}^c\mathfrak{h}$  so  $\mathfrak{h}_{\mathbf{R}} \supseteq {}^c\mathcal{L} \supseteq {}^c\mathcal{L}'$ . In the same way  ${}^c\mathfrak{h}_{\mathbf{R}}$  may be identified with  $\text{Hom}({}^c\mathcal{L}, \mathbf{R})$  so  ${}^c\mathfrak{h}_{\mathbf{R}} \supseteq L' \supseteq L$  if  $L'$  is the lattice of weights of  $\mathfrak{h}$ . Let  ${}^cG$  be the simply connected group with Lie algebra  ${}^c\mathfrak{g}$  and let  ${}^cT$  be the Cartan subgroup corresponding to  ${}^c\mathfrak{h}$ . There is an isomorphism  $\sigma \rightarrow {}^c\sigma$  of the Weyl group of  $T$  in  $G$  with that of  ${}^cT$  in  ${}^cG$  such that

$${}^c\sigma(\lambda(t)) = \lambda(\sigma t), \quad t \in T_{\mathbf{Q}_p}.$$

If  $w$  is a homomorphism of  $\bar{T}$  into the complex numbers there is a unique point  $g$  in  ${}^cT_{\mathbf{C}}$  such that  $w(t) = \xi_{\lambda}(g)$  for all  $t$ . Here  $\lambda = \lambda(t)$  and  $\xi_{\lambda}$  is the rational character of  ${}^cT$  associated to  $\lambda$ . Thus associated to each homomorphism of  $H_p$  into the complex numbers is an orbit of the Weyl group in  ${}^cT_{\mathbf{C}}$  or, as I prefer, a semisimple conjugacy class in  ${}^cG_{\mathbf{C}}$ .

The automorphic form  $\phi$  determined for each  $p$  a homomorphism  $\chi_p$  of  $H_p$  into  $\mathbf{C}$ . If  $p$  is a finite prime let  $\{g_p\}$  be the conjugacy class in  ${}^cG_{\mathbf{C}}$  corresponding to  $\chi_p$ . If  $\pi$  is a finite-dimensional complex representation of  ${}^cG_{\mathbf{C}}$ , we can consider the Euler product

$$(2) \quad \prod_p \frac{1}{\det\left(1 - \frac{\pi(g_p)}{p^s}\right)} = L(s, \pi, \phi).$$

As we shall see, this Euler product is absolutely convergent for  $\text{Re}(s)$  sufficiently large. I do not know in general what the analytic properties of the function  $L(s, \pi, \phi)$  are. I will show, however, that for all but three of the simple groups there is at least one nontrivial representation for which  $L(s, \pi, \phi)$  is meromorphic in the whole complex plane. For some groups there are several such representations.

Let me first introduce a  $\Gamma$ -factor to go with  $L(S, \pi, \phi)$ . If  $p$  is the infinite prime there is a homomorphism  $\lambda$  of

$$\bar{T} = T_{\mathbf{Q}_p}/T_{\mathbf{Q}_p} \cap G_{\mathbf{Z}_p}$$

into  $\mathfrak{h}_{\mathbf{R}} = \text{Hom}(L, \mathbf{R})$  such that  $|\xi_{\alpha}(t)| = e^{\lambda(t)(\alpha)}$  if  $\alpha$  is a root. Since  $L$  is a lattice in  ${}^c\mathfrak{h}_{\mathbf{R}}$ , every homomorphism of  $\bar{T}$  into  $\mathbf{C}$  is of the form

$$w(t) = e^{\lambda(t)(X)}$$

for some  $X$  in  ${}^c\mathfrak{h}_{\mathbf{C}}$ . Thus to every homomorphism of  $H_p$  into  $\mathbf{C}$  there is associated an orbit of the Weyl group in  ${}^c\mathfrak{h}_{\mathbf{C}}$  or a semisimple conjugacy class in  ${}^c\mathfrak{g}_{\mathbf{C}}$ . If  $\chi_p$  is the homomorphism associated to the automorphic form  $\phi$ , let  $\{X\}$  be the associated conjugacy class and set

$$\frac{1}{\Gamma(s, \pi, \phi)} = \pi^{\text{trace}\left(\frac{s-\pi(X)}{2}\right)} \det\left(\frac{s-\pi(X)}{2}\right) e^{\gamma \text{trace}\left(\frac{s-\pi(X)}{2}\right)} \cdot \prod_{n=1}^{\infty} \left\{ \det\left(I + \frac{s-\pi(X)}{2n}\right) e^{-\text{trace}\left(\frac{s-\pi(X)}{2n}\right)} \right\}$$

where  $\gamma$  is Euler's constant. The function  $\Gamma(s, \pi, \phi)$  can be expressed as a product of  $\Gamma$ -functions. Set

$$\xi(s, \pi, \phi) = \Gamma(s, \pi, \phi)L(s, \pi, \phi).$$

The functional equation to expect is

$$\xi(s, \pi, \phi) = \xi(1-s, \tilde{\pi}, \phi)$$

if  $\tilde{\pi}$  is the representation contragredient to  $\pi$ .



## CHAPTER 3

### Spherical functions

Each  $g_p$  that occurs in the expression on the left of (2) can be chosen to lie in  ${}^cT_{\mathbf{C}}$ . To see that the product converges in a half-plane it would be enough to show that for all  $\lambda$  in  ${}^cL$  and all  $p$

$$|\xi_{\lambda}(g_p)| \leq p^{\bar{\lambda}(\rho)}$$

where  $\bar{\lambda}$  is that element in the orbit of  $\lambda$  under the Weyl group which lies in the positive Weyl chamber and  $\rho$  is one-half the sum of the positive roots. We can associate to each  $g$  in  ${}^cT_{\mathbf{C}}$  a point  $\mu = \mu(g)$  in  ${}^c\mathfrak{h}_{\mathbf{C}}$  so that<sup>1</sup>  $\xi_{\lambda}(g) = p^{\lambda(\mu)}$  for all  $\lambda$  in  ${}^cL$ . The point  $\mu$  is not uniquely determined by  $g$  but its real part is. If  $\mu_p = \mu(g_p)$  we have to show that

$$\operatorname{Re} \lambda(\mu_p) \leq \bar{\lambda}(\rho).$$

The class  $\{g_p\}$  is associated to the homomorphism  $\chi_p$  of  $H_p$  into  $\mathbf{C}$  determined by  $\phi$ . This homomorphism has the property, not shared by all homomorphisms, that

$$|\chi_p(f)| \leq c \int_{G_{\mathbf{Q}_p}} |f(g)| dg$$

for all  $f$  in  $H_p$ . The factor  $c$  is a fixed constant. To prove this we must recall that  $\phi$  is a cusp form and hence bounded. If  $M$  is a bound for  $\phi$  and if  $\phi(g_0) \neq 0$  then

$$|\chi_p(f)\phi(g_0)| = \left| \int_{G_{\mathbf{Q}_p}} \phi(g_0h)f(h) dh \right| \leq M \int_{G_{\mathbf{Q}_p}} |f(h)| dh$$

and the assertion follows.

**Lemma.** *Suppose  $\mu$  in  ${}^c\mathfrak{h}_{\mathbf{C}}$  is associated to the homomorphism  $\chi_{\mu}$  of  $H_p$  into  $\mathbf{C}$ . If there is a constant  $c$  such that*

$$|\chi_{\mu}(f)| \leq c \int_{G_{\mathbf{Q}_p}} |f(g)| dg$$

for all  $f$ , then

$$\operatorname{Re} \lambda(\mu) \leq \bar{\lambda}(\rho)$$

for all  $\lambda$ .

If  $w$  is the homomorphism defined by  $w(t) = p^{\lambda(t)(\mu)}$ , set

$$\psi_{\mu}(g) = \psi_w(g).$$

By definition

$$\chi_{\mu}(f)\psi_{\mu}(g) = \int_{G_{\mathbf{Q}_p}} \psi_{\mu}(gh)f(h) dh.$$

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<sup>1</sup>At the infinite prime we would take  $\mu = X$  with  $X$  as above.

If

$$\phi_\mu(g) = \int_{G_{\mathbf{Z}_p}} \psi_\mu(kg) dk,$$

then  $\phi_\mu(1) = 1$ , and  $\phi_\mu(k_1 g k_2) = \phi_\mu(g)$  if  $k_1$  and  $k_2$  lie in  $G_{\mathbf{Z}_p}$ . Moreover it is easily verified that

$$\chi_\mu(f)\phi_\mu(g) = \int_{G_{\mathbf{Q}_p}} \phi_\mu(gh)f(h) dh.$$

If  $\mu$  satisfies the assumption of the lemma, take  $f_g$  to be the characteristic function of the double coset  $G_{\mathbf{Z}_p} g G_{\mathbf{Z}_p}$ . Then  $\chi_\mu(f_g)$  is equal to  $\chi_\mu(f_g)\phi_\mu(1) = \phi_\mu(g)$  times the measure of  $(G_{\mathbf{Z}_p} g G_{\mathbf{Z}_p})$ . It follows immediately that  $|\phi_\mu(g)| \leq c$  for all  $g$ .

To prove the lemma, it will be necessary to study the asymptotic behavior of  $\phi_\mu$  for general values of  $\mu$ . Let  $T_{\mathbf{Q}_p}^-$  be the set of  $t$  in  $T_{\mathbf{Q}_p}$  for which  $-\lambda(t)$  lies in the positive Weyl chamber. Since

$$G_{\mathbf{Q}_p} = G_{\mathbf{Z}_p} T_{\mathbf{Q}_p}^- G_{\mathbf{Z}_p}$$

it is sufficient to study the function  $\phi_\mu$  on  $T_{\mathbf{Q}_p}^-$ .

So far we are free to choose Haar measures in any manner we like. We so choose them on  $V_{\mathbf{Q}_p}$ ,  $T_{\mathbf{Q}_p}$  and  $G_{\mathbf{Z}_p}$  so that the total measures of  $V_{\mathbf{Z}_p}$ ,  $T_{\mathbf{Z}_p}$  and  $G_{\mathbf{Z}_p}$  are 1. Then we choose the Haar measure on  $G_{\mathbf{Q}_p}$  so that

$$\int_{G_{\mathbf{Q}_p}} f(g) dg = \int_{V_{\mathbf{Q}_p}} \int_{T_{\mathbf{Q}_p}} \int_{G_{\mathbf{Z}_p}} f(vtk) p^{-2\lambda(t)(\rho)} dv dt dk.$$

Choose an  $f$  in  $H_p$  and let  $C$  be a compact set in  $V_{\mathbf{Q}_p}$  such that the support of  $f$  is contained in  $CT_{\mathbf{Q}_p}G_{\mathbf{Z}_p}$ . There is a constant  $c_f > 0$  such that if  $|\xi_\alpha(t)| \leq c_f$  for  $\alpha > 0$ , then

$$tCt^{-1} \subseteq G_{\mathbf{Z}_p}.$$

Choose a  $t$  satisfying this condition. Then

$$\begin{aligned} \chi_\mu(f)\phi_\mu(t) &= \int_{V_{\mathbf{Q}_p}} \int_{T_{\mathbf{Q}_p}} \phi_\mu(tvs) f(vs) p^{-2\lambda(s)(\rho)} dv ds \\ &= \int_C \int_{T_{\mathbf{Q}_p}} \phi_\mu(tvt^{-1}ts) f(vs) p^{-2\lambda(s)(\rho)} dv ds \\ &= \int_C \int_{T_{\mathbf{Q}_p}} \phi_\mu(ts) f(vs) p^{-2\lambda(s)(\rho)} dv ds. \end{aligned}$$

Set  $\tilde{\phi}_\mu(t) = \phi_\mu(t) p^{-\lambda(t)(\rho)}$ . Replacing the integral over  $C$  by an integral over  $V_{\mathbf{Q}_p}$  we see that

$$\chi_\mu(f)\tilde{\phi}_\mu(t) = \int_{T_{\mathbf{Q}_p}} \tilde{\phi}_\mu(ts) \tilde{f}(s) ds = \int_{\bar{T}} \tilde{\phi}_\mu(ts) \tilde{f}(s) ds$$

if  $f \rightarrow \tilde{f}$  is the Satake homomorphism of  $H_p$  into the group algebra  $H'_p$  of  $\bar{T}$  defined by

$$\tilde{f}(s) = p^{-\lambda(s)(\rho)} \int_{V_{\mathbf{Q}_p}} f(vs) dv.$$

If  $\psi_1$  and  $\psi_2$  are two functions on  $\bar{T}$  and  $c > 0$ , we shall say that  $\psi_1 \simeq_c \psi_2$  if  $\psi_1(t) = \psi_2(t)$  whenever  $|\xi_\alpha(t)| \leq c$  for all positive roots  $\alpha$ . The set of equivalence classes forms a vector

space  $W_c$ . Let  $W_\infty$  be the injective limit of the spaces  $W_c$ . If  $f$  lies in  $H'_p$  and  $\psi$  is a function on  $\overline{T}$ , then we define  $\lambda(f)\psi$  by

$$\lambda(f)\psi(t) = \int_T \psi(ts)f(s) ds.$$

We can also regard  $\lambda(f)$  as an operator on  $W_\infty$ . If  $\Phi_\mu$  is the image of  $\tilde{\phi}_\mu$  in  $W_\infty$ , then  $\lambda(\tilde{f})\Phi_\mu = \chi_\mu(f)\Phi_\mu$  for all  $f$  in  $H_p$ . Since  $H'_p$  is a finite module over the image of  $H_p$ , the space

$$W = \left\{ \lambda(f)\Phi_\mu \mid f \in H'_p \right\}$$

is a finite-dimensional subspace of  $W_\infty$ .

Choose  $t_1, \dots, t_n$  such that

$$|\xi_{\alpha_i}(t_j)| = p^{-\delta_{ij}}.$$

The points  $\lambda(t_1), \dots, \lambda(t_n)$  are a basis of  ${}^cL$  over  $\mathbf{Z}$ . Let  $\delta_i$  be the characteristic function of  $\{\overline{t}_i\}$  and let  $S_i$  and  $N_i$  be respectively the semisimple and nilpotent parts of the restriction of  $\lambda(\delta_i)$  to  $W$ . The matrices  $S_i, N_i$ , where  $1 \leq i \leq n$ , all commute. Choose a basis  $\Psi_1, \dots, \Psi_\ell$  of  $W$  with respect to which  $S_1, \dots, S_n$  are in diagonal form. Let

$$S_i \Psi_j = p^{\gamma_{ij}} \Psi_j.$$

If  $a_i$  is the smallest integer satisfying  $N_i^{a_i+1} = 0$ , then

$$(S_i + N_i)^{\ell_i} = \sum_{r_i=0}^{a_i} \binom{\ell_i}{r_i} S_i^{\ell_i-r_i} N_i^{r_i}.$$

Let  $\psi_1, \dots, \psi_\ell$  be representatives of  $\Psi_1, \dots, \Psi_\ell$ , and choose  $c_0 > 0$  such that

$$\psi_i(tt_k) \simeq_{c_0} \sum_j (s_k^{j_i} + n_k^{j_i}) \psi_j(t)$$

if  $|\xi_\alpha(t)| \leq c_0$  for  $\alpha$  positive. Choose  $t_0$  such that  $|\xi_\alpha(t_0)| \leq c_0$  if  $\alpha$  is positive. If

$$\Omega_0 = (\psi_1(t_0), \dots, \psi_\ell(t_0)), \quad \Omega(t) = (\psi_1(t), \dots, \psi_\ell(t)),$$

and  $\ell_1 \geq 0, \dots, \ell_n \geq 0$  then

$$\begin{aligned} \Omega \left( t_0 \prod_{k=1}^n t_k^{\ell_k} \right) &= \Omega_0 \prod_{k=1}^n (S_k + N_k)^{\ell_k} \\ &= \Omega_0 \left\{ \sum_{r_1=0}^{a_1} \dots \sum_{r_n=0}^{a_n} \binom{\ell_1}{r_1} \dots \binom{\ell_n}{r_n} S_1^{-r_1} \dots S_n^{-r_n} N_1^{r_1} \dots N_n^{r_n} \right\} \prod_{k=1}^n S_k^{\ell_k} \\ &= \Theta(\ell_1, \dots, \ell_n) \prod_{k=1}^n S_k^{\ell_k} \end{aligned}$$

where  $\Theta(\ell_1, \dots, \ell_n)$  is a row vector with entries which are polynomials in  $\ell_1, \dots, \ell_n$ . Choose  $\mu_j$  such that  $\lambda(t_i)(\mu_j) = \gamma_{ij}$ , for  $1 \leq i \leq n$ . If  $|\xi_\alpha(t)| \leq |\xi_\alpha(t_0)|$  for  $\alpha$  positive, then

$$\psi_j(t) = p^{\lambda(t)(\mu_j)} \theta_j(\lambda(t))$$

where  $\theta_j(\lambda(t))$  is a polynomial in  $\lambda(t)$ . Thus there is a constant  $c_1$  and polynomials  $\xi_1(\lambda(t)), \dots, \xi_\ell(\lambda(t))$  such that

$$\tilde{\phi}_\mu(t) \simeq_{c_1} \sum_{j=1}^{\ell} p^{\lambda(t)(\mu_j)} \xi_j(\lambda(t)).$$

For our purposes it will be necessary to know the relation between  $\mu_1, \dots, \mu_\ell$  and  $\mu$  and to have a more or less explicit formula for the polynomials  $\xi_j(\lambda(t))$ . Let  ${}^c\Omega$  be the Weyl group of  ${}^cT$ . For notational convenience let us use the map  $\sigma \rightarrow {}^c\sigma$  to identify  ${}^c\Omega$  and  $\Omega$ , the Weyl group of  $T$ . Let

$$A = \{\mu \in {}^c\mathfrak{h}_{\mathbf{R}} \mid \sigma\mu = t\mu, \sigma, t \in \Omega \text{ implies } \sigma = t\}.$$

Let  $\bar{t}_i$  be the image of  $t_i$  in  $\bar{T}$  and let

$$\bar{t}_i^{(1)}, \dots, \bar{t}_i^{(b_i)}$$

be the distinct elements in the orbit of  $\bar{t}_i$  under  $\Omega$ . Set

$$B = \left\{ \mu \in {}^c\mathfrak{h}_{\mathbf{R}} \mid \lambda(\bar{t}_i^{(k)}) (\mu) = \lambda(\bar{t}_j^{(1)}) (\mu) \text{ implies } i = j \text{ and } k = \ell \right\}.$$

Define  $\mu_{k_1, \dots, k_n}$  by the condition that

$$\lambda(\bar{t}_i^{(k_i)}) (\mu) = \lambda(\bar{t}_i) (\mu_{k_1, \dots, k_n}) \quad 1 \leq i \leq n$$

and set

$$C = \left\{ \mu \in {}^c\mathfrak{h}_{\mathbf{R}} \mid \sigma\mu = \mu_{k_1, \dots, k_n} \text{ implies } \sigma\bar{t}_i^{(k_i)} = \bar{t}_i, 1 \leq i \leq n \right\}.$$

The complements of  $A, B$ , and  $C$  are the union of a finite number of proper affine subspaces of  ${}^c\mathfrak{h}_{\mathbf{R}}$ . Thus there is a point  $\mu^0$  in  $A \cap B \cap C$ . Choose  $t_0$  such that

- (i)  $\lambda(t_0)(\sigma\mu^0) = \lambda(t_0)(\tau\mu^0)$  implies  $\sigma = \tau$ ;
- (ii)  $\lambda(t_0)({}^c\sigma\mu^0) = \lambda(t_0)(\mu_{k_1, \dots, k_n}^0)$  implies  $\sigma\bar{t}_i^{(k_i)} = \bar{t}_i, 1 \leq i \leq n$ .

Let  $S$  be the collection of points  $\mu$  in  ${}^c\mathfrak{h}_{\mathbf{C}}$  satisfying

- (i)  $p^{\lambda(t_0)(\sigma\mu)} = p^{\lambda(t_0)(\tau\mu)}$  implies  $\sigma = \tau$ ;
- (ii)  $p^{\lambda(t_0)(\sigma\mu)} = p^{\lambda(t_0)(\mu_{k_1, \dots, k_n})}$  implies  $\sigma\bar{t}_i^{(k_i)} = \bar{t}_i, 1 \leq i \leq n$ ;
- (iii)  $p^{\lambda(\bar{t}_i^{(k)})} (\mu) = p^{\lambda(\bar{t}_j^{(\ell)})} (\mu)$  implies  $i = j$  and  $k = \ell$ .

Then  $S$  is an open, dense, and connected subset of  ${}^c\mathfrak{h}_{\mathbf{C}}$ .

Suppose  $\mu$  lies in  $S$ . Since the coefficients of the polynomial

$$p_i(X) = \prod_{j=1}^{b_i} \left( X - \delta(\bar{t}_i^{(j)}) \right)$$

lie in the image of  $H_p$ , the equation

$$\lambda(p_i(X)) \Phi_\mu = \prod_{j=1}^{b_i} \left( X - p^{\lambda(\bar{t}_i^{(j)})} (\mu) \right) \Phi_\mu$$



is satisfied. It is satisfied not only by  $\Phi_\mu$  but by every element of  $W$ . Since  $p_i(\delta(\bar{t}_i)) = 0$ , the minimal polynomial of the restriction of  $\lambda(\delta(\bar{t}_i))$  to  $W$  divides

$$\prod_{j=1}^{b_i} \left( X - p^{\lambda(\bar{t}_i^{(j)})}(\mu) \right).$$

Since this polynomial has no multiple root,  $N_i = 0$  and

$$\lambda(\delta_t)\Psi_j = p^{\lambda(t)(\mu_j)}\Psi_j$$

for all  $t$ . Here  $\delta_t$  is the characteristic function of  $\{\bar{t}\}$ . The point  $\mu_j$  must be equivalent modulo  $2\pi iL/\log p$  to an element in the orbit, under  $\Omega$ , of  $\mu$ . If not, there is a  $t$  such that

$$p^{\lambda(t)(\sigma\mu)} \neq p^{\lambda(t)(\mu_j)}$$

for any  $\sigma$  in  $\Omega$ . This is impossible because the minimal polynomial of the restriction of  $\lambda(\delta_t)$  is divisible by

$$X - p^{\lambda(t)(\mu_j)}$$

and divides

$$\prod_{\sigma} \left( X - p^{\lambda(t)(\sigma\mu)} \right).$$

Thus there is a constant  $c_\mu$  and constants  $a_\sigma(\mu)$  such that

$$\tilde{\phi}_\mu(t) \simeq_{c_\mu} \sum_{\sigma} a_\sigma(\mu) p^{\lambda(t)(\sigma\mu)}.$$

We need to prove also that the constant  $c_\mu$  can be chosen to be independent of  $\mu$ . Since the coefficients of  $p_i(X)$  are independent of  $\mu$ , there is a constant  $c_1$  such that

$$\lambda(p_i(X))\tilde{\phi}_\mu \simeq_{c_1} \prod_{j=1}^{b_i} \left( X - p^{\lambda(\bar{t}_i^{(j)})}(\mu) \right) \tilde{\phi}_\mu$$

for all  $\mu$ . If

$$\tilde{\phi}_\mu(\cdot, k_1, \dots, k_n) = \prod_{i=1}^n \left\{ \prod_{\substack{j_i \neq k_i \\ 1 \leq j_i \leq b_i}} \left( \frac{\lambda(\delta_i) - p^{\lambda(\bar{t}_i^{(j_i)})}(\mu)}{p^{\lambda(t_i^{(k_i)})}(\mu) - p^{\lambda(t_i^{(j_i)})}(\mu)} \right) \right\} \tilde{\phi}_\mu$$

then

$$\tilde{\phi}_\mu = \sum_{k_1=1}^{b_1} \cdots \sum_{k_n=1}^{b_n} \tilde{\phi}_\mu(\cdot, k_1, \dots, k_n)$$

and, for some constant  $c_2$  which does not depend on  $\mu$ ,

$$\lambda(\delta_i)\tilde{\phi}_\mu(\cdot, k_1, \dots, k_n) \simeq_{c_2} p^{\lambda(t_i)(\mu_{k_1, \dots, k_n})} \tilde{\phi}_\mu, \quad 1 \leq i \leq n.$$

If

$$p_0(X) = \prod_{\sigma} \left( X - p^{\lambda(t_0)(\sigma\mu)} \right)$$

there is a constant  $c_3$  such that

$$p_0(\lambda(\delta_{t_0}))\tilde{\phi}_\mu(\cdot, k, \dots, k_n) \simeq_{c_3} 0$$

and

$$p_0(\lambda(\delta_{t_0}))\tilde{\phi}_\mu(\cdot, k_1, \dots, k_n) \simeq_{c_3} p_0\left(p^{\lambda(t_0)(\mu_{k_1, \dots, k_n})}\right)\tilde{\phi}_\mu(\cdot, k_1, \dots, k_n).$$

Since  $\mu$  lies in  $S$ ,

$$\tilde{\phi}_\mu(\cdot, k_1, \dots, k_n) \simeq_{c_3} 0$$

unless  $\sigma \bar{t}_i^{k_i} = \bar{t}_i$ ,  $1 \leq i \leq n$ , for some  $\sigma$  in  $\Omega$ . Then  $\mu_{k_1, \dots, k_n} = \sigma\mu$ . For a given  $\sigma$ , let  $k_1(\sigma), \dots, k_n(\sigma)$  be the indices satisfying  $\sigma \bar{t}_i^{(k_i(\sigma))} = \bar{t}_i$  and set

$$\tilde{\phi}_\mu(\cdot, \sigma) = \tilde{\phi}_\mu(\cdot, k_1(\sigma), \dots, k_n(\sigma)).$$

There is a constant  $c_4$  such that

$$\tilde{\phi}_\mu \simeq_{c_4} \sum_{\sigma} \tilde{\phi}_\mu(\cdot, \sigma)$$

and

$$\tilde{\phi}_\mu(\cdot, \sigma) \simeq_{c_4} a_\sigma(\mu) p^{\lambda(t)(\sigma\mu)}$$

for all  $\mu$  in  $S$ .

The next step is to evaluate the coefficients  $a_\sigma(\mu)$ . Since  $\phi_{\sigma\mu} = \phi_\mu$ , the same is true of  $\tilde{\phi}_\mu$ . If  $\mu$  lies in  $\bigcap_{\sigma} \sigma S$  which is an open, dense, and connected set, then

$$\sum_{\tau} a_\tau(\mu) p^{\lambda(t)(\mu)} \simeq_{c_4} \sum_{\tau} a_\tau(\sigma\mu) p^{\lambda(t)(\tau\sigma\mu)}.$$

As a consequence  $a_{\tau\sigma}(\mu) = a_\tau(\sigma\mu)$ . Thus it is enough to evaluate  $a(\mu) = a_{\sigma_0}(\mu)$  if  $\sigma_0$  is the element of the Weyl group which takes every positive root to a negative root. Since  $a(\mu)$  is an analytic function, it is enough to evaluate it when  $\text{Re } \mu$  lies in the interior of the positive Weyl chamber.

Suppose  $\lambda(t)$  lies in the interior of the positive Weyl chamber. Since

$$\phi_\mu(t^n) = \phi_\mu((\sigma_0 t)^n) = \tilde{\phi}_\mu((\sigma_0 t)^n) p^{-n\lambda(t)(\rho)}$$

the relation

$$\phi_\mu(t^n) = \sum_{\sigma} a_\sigma(\mu) p^{n\lambda(t)(\sigma_0^{-1}\sigma\mu - \rho)}$$

is valid for sufficiently large  $n$  and

$$\lim_{n \rightarrow \infty} p^{-n\lambda(t)(\mu - \rho)} \phi_\mu(t^n) = a_{\sigma_0}(\mu) = a(\mu)$$

because  $\text{Re } \lambda(t)(\mu - \sigma\mu) > 0$ . We shall evaluate this limit in another way and obtain  $a(\mu)$ . Recall that

$$\phi_\mu(t) = \int_{G_{\mathbf{Z}_p}} \psi_\mu(kt) dk.$$

Following Harish-Chandra we study this integral by means of the following easily proved lemma.

**Lemma.** *Suppose  $\bar{V}$  is the unipotent radical of the parabolic group opposed to  $B$ . If  $\bar{v}$  lies in  $\bar{V}_{\mathbf{Q}_p}$ , let  $\bar{v} = v(\bar{v})t(\bar{v})k(\bar{v})$  with  $v(\bar{v})$  in  $V_{\mathbf{Q}_p}$ ,  $t(\bar{v})$  in  $T_{\mathbf{Q}_p}$ , and  $k(\bar{v})$  in  $G_{\mathbf{Z}_p}$ . Set  $\lambda(\bar{v}) = \lambda(t(\bar{v}))$ . There is a constant  $a$  such that if  $\psi$  is any integrable function on  $B_{\mathbf{Z}_p} \backslash G_{\mathbf{Z}_p}$  then*

$$a \int_{G_{\mathbf{Z}_p}} \psi(k) dk = \int_{\bar{V}_{\mathbf{Q}_p}} \psi(k(\bar{v})) p^{\lambda(\bar{v})(2\rho)} d\bar{v}.$$

I ask you to bear in mind for a while that  $a$  must necessarily equal

$$\int_{\overline{V}_{\mathbf{Q}_p}} p^{\lambda(\overline{v})(2\rho)} d\overline{v}.$$

Using the lemma we see that

$$\begin{aligned} \phi_\mu(t) &= \frac{1}{a} \int_{\overline{V}_{\mathbf{Q}_p}} \psi_\mu(t^{-1}(\overline{v})\overline{v}t) p^{\lambda(\overline{v})(2\rho)} d\overline{v} \\ &= \frac{1}{a} \int_{\overline{V}_{\mathbf{Q}_p}} \psi_\mu(\overline{v}t) p^{-\lambda(\overline{v})(\mu-\rho)} d\overline{v} \\ &= \frac{p^{\lambda(t)(\mu+\rho)}}{a} \int_{\overline{V}_{\mathbf{Q}_p}} \psi_\mu(t^{-1}\overline{v}t) p^{-\lambda(\overline{v})(\mu-\rho)} d\overline{v} \\ &= \frac{p^{\lambda(t)(\mu-\rho)}}{a} \int_{\overline{V}_{\mathbf{Q}_p}} \psi_\mu(\overline{v}) p^{-\lambda(t\overline{v}t^{-1})(\mu-\rho)} d\overline{v} \\ &= \frac{p^{\lambda(t)(\mu-\rho)}}{a} \int_{\overline{V}_{\mathbf{Q}_p}} p^{\lambda(\overline{v})(\mu+\rho)} p^{-\lambda(t\overline{v}t^{-1})(\mu-\rho)} d\overline{v}. \end{aligned}$$

Thus

$$(3) \quad a(\mu) = \frac{1}{a} \lim_{n \rightarrow \infty} \int_{\overline{V}_{\mathbf{Q}_p}} p^{\lambda(\overline{v})(\mu+\rho)} p^{-\lambda(t^n \overline{v} t^{-n})(\mu-\rho)} d\overline{v}$$

if  $t$  lies in the interior of the positive Weyl chamber.

**Lemma.** *If  $\nu$  lies in the positive Weyl chamber, if  $\lambda(t)$  does also, and if  $\overline{v}$  lies in  $\overline{V}_{\mathbf{Q}_p}$  then  $\lambda(\overline{v})(\nu) \leq 0$  and  $\lambda(\overline{v})(\nu) \leq \lambda(t\overline{v}t^{-1})(\nu)$ .*

If  $g$  lies in  $G_{\mathbf{Q}_p}$  and  $g = vsk$ , with  $v$  in  $T_{\mathbf{Q}_p}$ ,  $s$  in  $T_{\mathbf{Q}_p}$ , and  $k$  in  $G_{\mathbf{Z}_p}$  set  $\lambda(g) = \lambda(s)$ . It is known that if  $t$  satisfies the condition of the lemma, then

$$\lambda(kt)(\nu) \leq \lambda(t)(\nu).$$

Since

$$\lambda(gt) = \lambda(g) + \lambda(kt),$$

we have

$$\lambda(g)(\nu) + \lambda(t)(\nu) \geq \lambda(gt)(\nu).$$

Moreover

$$\lambda(t^{-1}gt)(\nu) = -\lambda(t)(\nu) + \lambda(gt)(\nu) \leq \lambda(g)(\nu).$$

The second assertion of the lemma follows. If  $\overline{v}$  lies in  $\overline{V}_{\mathbf{Q}_p}$ , there is a  $t$  with  $\lambda(t)$  in the positive Weyl chamber such that  $t\overline{v}t^{-1}$  lies in  $G_{\mathbf{Z}_p}$ . Since  $\lambda(t\overline{v}t^{-1})$  is then zero, the first assertion follows from the second.

If  $R$  is any open half-space in  ${}^c\mathfrak{h}_{\mathbf{R}}$  which is bounded by a hyperplane passing through zero and if  $\overline{R}$  is its closure, let  $\Sigma_R$  be the set of roots lying in  $R$ , let  $\Sigma_R^+$  be the set of positive roots lying in  $\overline{R}$ , and let  $\Sigma_R^-$  be the set of negative roots lying in  $R$ . Let  $\overline{\mathfrak{n}}(R)$  be the Lie algebra spanned by the root vectors corresponding to roots in  $\Sigma_R^-$  and let  $\overline{N}(R)$  be the group with Lie algebra  $\overline{\mathfrak{n}}(R)$ . If  $p$  is any prime, finite or infinite, we consider

$$\int_{\overline{N}_{\mathbf{Q}_p}(R)} \psi_\mu(\overline{n}) d\overline{n} = \delta_R(\mu).$$

It has been shown by Gindikin and Karpelevich that, when  $p = \infty$ , this integral converges if  $\operatorname{Re} \mu(H_\alpha) > 0$  for every positive root  $\alpha$  and is equal to

$$\prod_{-\alpha \in \Sigma_R^-} \frac{\pi^{1/2} \Gamma\left(\frac{\mu(H_\alpha)}{2}\right)}{\Gamma\left(\frac{1+\mu(H_\alpha)}{2}\right)}.$$

If  $X_\alpha$  are the root vectors belonging to the Chevalley basis, the Haar measure is that associated to the form which takes the value 1 on

$$\bigwedge_{\alpha \in \Sigma_R^-} X_\alpha.$$

We shall imitate their proof and show that the integral converges in the same region when  $p$  is finite and is equal to

$$(4) \quad \prod_{-\alpha \in \Sigma_R^-} \frac{1 - \frac{1}{p^{\mu(H_\alpha)+1}}}{1 - \frac{1}{p^{\mu(H_\alpha)}}}.$$

For the moment we shall assume this and complete our evaluation of the limit (3). Choose  $\epsilon > 0$  such that  $\operatorname{Re}(\mu) - \epsilon\rho$  lies in the positive Weyl chamber. Then

$$\operatorname{Re}\{\lambda(\bar{\nu})(\mu + \rho) - \lambda(t^n \bar{\nu} t^{-n})(\mu - \rho)\}$$

is the sum of

$$\operatorname{Re}\left\{(\lambda(\bar{\nu}) - \lambda(t^n \bar{\nu} t^{-n}))(\mu - \epsilon\rho) + \lambda(t^n \bar{\nu} t^{-n})(\rho - \epsilon\rho)\right\}$$

and

$$\operatorname{Re}\{\lambda(\bar{\nu})(\rho + \epsilon\rho)\}.$$

It follows from the lemma that the first expression is less than or equal to zero. Since

$$\int_{\bar{V}_{\mathbf{Q}_p}} p^{\operatorname{Re}\{\lambda(\bar{\nu})(\rho + \epsilon\rho)\}} d\bar{\nu}$$

is finite, we can take the limit under the integral sign in (3) to obtain

$$\begin{aligned} a(\mu) &= \frac{1}{a} \int_{\bar{V}_{\mathbf{Q}_p}} p^{\lambda(\bar{\nu})(\mu + \rho)} d\bar{\nu} \\ &= \frac{1}{a} \prod_{\alpha > 0} \frac{1 - \frac{1}{p^{\mu(H_\alpha)+1}}}{1 - \frac{1}{p^{\mu(H_\alpha)}}} \end{aligned}$$

Thus there is a constant  $c$  such that if  $|\xi_\alpha(t)| \geq c$  for  $\alpha$  positive then

$$(5) \quad \phi_\mu(t) = \frac{1}{a} \sum_\sigma \left\{ \prod_{\alpha > 0} \frac{1 - \frac{1}{p^{\sigma\mu(H_\alpha)+1}}}{1 - \frac{1}{p^{\sigma\mu(H_\alpha)}}} \right\} p^{\lambda(t)(\sigma\mu - \rho)}.$$

Let

$$\prod_{\alpha > 0} \left(1 - \frac{1}{p^{\nu(H_\alpha)+1}}\right) = \sum_{\bar{s} \in \bar{T}} b_{\bar{s}} p^{\lambda(\bar{s})(\nu)}.$$

Only a finite number of the coefficients are not zero. If  ${}^c\alpha$  is the root of  ${}^c\mathfrak{h}$  corresponding to  $\alpha$  the formula (5) may be written

$$(6) \quad \phi_\mu(t) = \frac{1}{a} \sum_{\bar{s}} b_{\bar{s}} \left\{ \frac{\sum_{\sigma} \operatorname{sgn} \sigma p^{(\lambda(\bar{s}) + {}^c\rho)(\sigma\mu)}}{\prod_{c_{\alpha} > 0} \left( p^{\frac{c_{\alpha}(\mu)}{2}} - p^{-\frac{c_{\alpha}(\mu)}{2}} \right)} \right\} p^{-\lambda(t)\rho}.$$

This formula is valid for all  $\mu$ . The relation of this formula to the Weyl character formula need not be pointed out to the knowledgeable reader.

I do not know if it is valid for all  $t$ . However it is valid for  $t = 1$ . To prove this we show that the right side is 1 when  $t = 1$ . First of all, it follows from the formulae for  $\delta_R(\mu)$  that

$$a = \prod_{\alpha > 0} \frac{1 - \frac{1}{p^{\rho(H_{\alpha})+1}}}{1 - \frac{1}{p^{\rho(H_{\alpha})}}} = \prod_{c_{\alpha} > 0} \frac{1 - \frac{1}{p^{c_{\alpha}(\rho)+1}}}{1 - \frac{1}{p^{c_{\alpha}(\rho)}}}.$$

Now  $b_{\bar{s}}$  is zero unless

$$\lambda(\bar{s}) = \sum_{c_{\alpha} \in w} c_{\alpha}$$

where  $w$  is a subset of the positive roots of  ${}^c\mathfrak{h}$ . Then  ${}^c\rho + \lambda(\bar{s})$  is either singular or in the orbit of  ${}^c\rho$  under  ${}^c\Omega$ . To prove this<sup>2</sup> we recall that Kostant has shown in lemma 5.9 of his paper on the Borel-Weil theorem that every element in the orbit of  ${}^c\rho + \lambda(\bar{s})$  is of the form  ${}^c\rho + \lambda(\bar{s}')$  with

$$\lambda(\bar{s}') = - \sum_{c_{\alpha} \in w'} c_{\alpha}$$

and suppose that  ${}^c\rho + \lambda(\bar{s})$  lies in the positive Weyl chamber. If it is nonsingular, it equals  ${}^c\rho + \lambda$  with  $\lambda$  in the positive Weyl chamber. Then  $\lambda = \lambda(\bar{s})$ . It follows immediately that  $\lambda = \lambda(\bar{s}) = 0$ . If  $b_{\bar{s}}$  is not zero, the corresponding term in brackets on the right side of (5) is zero when  $t = 1$  and  $\lambda(\bar{s}) + {}^c\rho$  is singular and is  $\pm 1$  when  $t = 1$  and  $\lambda(\bar{s}) + {}^c\rho$  is in the orbit of  ${}^c\rho$ . In any case if, for brevity, we denote the right side of (6) by  $\Theta_\mu(t)$ , then  $\Theta_\mu(1)$  is independent of  $\mu$ . Thus

$$\Theta_\mu(1) = \Theta_\rho(1) = \frac{1}{a} \sum_{\sigma} \left\{ \prod_{\alpha > 0} \frac{1 - \frac{1}{p^{\sigma\rho(H_{\alpha})+1}}}{1 - \frac{1}{p^{\sigma\rho(H_{\alpha})}}} \right\}.$$

Suppose  $\sigma \neq 1$ . Then, for some simple root  $\alpha_0$ ,  $\sigma\alpha_0 = -\beta_0$  is negative and

$$\sigma\rho(H_{\beta_0}) = -\rho(H_{\alpha_0}) = -1$$

and the corresponding term in the above sum is zero. Thus

$$\Theta_\mu(1) = \frac{1}{a} \left\{ \prod_{\alpha > 0} \frac{1 - \frac{1}{p^{\rho(H_{\alpha})+1}}}{1 - \frac{1}{p^{\rho(H_{\alpha})}}} \right\} = 1.$$

Since  $\Theta_\mu(t)$  is a linear combination of products of exponentials and polynomials in  $\lambda(t)$  it cannot vanish in an open cone without vanishing identically. This the last formula shows it cannot do.

We are now in a position to show that if  $\phi_\mu$  is bounded then

$$\operatorname{Re} \lambda(\mu) \leq \bar{\lambda}(\rho)$$

<sup>2</sup>[Added 1970] I now notice that I made the matter unnecessarily complicated.

for all  $\lambda$ . We may suppose that  $\operatorname{Re} \mu$  lies in the positive Weyl chamber. Then

$$\operatorname{Re} \lambda(\mu) \leq \operatorname{Re} \bar{\lambda}(\mu),$$

and we need only consider  $\lambda$  lying in the positive Weyl chamber. It will be simpler to consider only  $\lambda$  lying in the interior of the positive Weyl chamber. The assertion for points on the boundary can be obtained by a simple limiting argument. Then if  $\operatorname{Re}(\sigma\mu) \neq \operatorname{Re} \mu$ , for some  $\sigma$  in  $\Omega$ ,  $\operatorname{Re} \lambda(\sigma\mu) < \operatorname{Re} \lambda(\mu)$ .

Let  $w$  be the set of simple roots  $\alpha$  for which  $\operatorname{Re} \alpha(\mu) = 0$ . Let  $\Sigma_0^+(w)$  be the set of all positive roots which are linear combinations of the elements of  $w$ , and let  $\Sigma^+(w)$  be the other positive roots. Let  $G_1$  be the subgroup of  $G$  corresponding to the Lie algebra generated by the root vectors associated to the elements of  $\Sigma^+(w)$  and their negatives and let  $\Omega_1 \subseteq \Omega$  be the Weyl group of  $G_1$ . If  $\sigma$  belongs to  $\Omega$  and  $\operatorname{Re} \sigma(\mu) = \operatorname{Re} \mu$ , then  $\sigma$  belongs to  $\Omega_1$ . Set  $\Theta'_\nu(t)$  equal to

$$\frac{1}{a} \sum_{\sigma \in \Omega_1} \left\{ \prod_{\alpha \in \Sigma^+(w)} \frac{1 - \frac{1}{p^{\sigma\nu(H\alpha)+1}}}{1 - \frac{1}{p^{\sigma\nu(H\alpha)}}} \right\} \left\{ \prod_{\alpha \in \Sigma_0^+(w)} \frac{1 - \frac{1}{p^{\sigma\nu(H\alpha)+1}}}{1 - \frac{1}{p^{\sigma\nu(H\alpha)}}} \right\} p^{\lambda(t)(\sigma\nu-\rho)}.$$

Since  $\lambda(t) = \lambda_1 + \lambda_2$  with  $\lambda_1 = \lambda(t_1)$  for some  $t_1$  in the adjoint group of  $G_1$  and  $\lambda_2(\alpha) = 0$  for  $\alpha$  in  $w$ , we can write  $\Theta'_\nu(t)$  as the product of

$$\frac{1}{a} \left\{ \prod_{\alpha \in \Sigma^+(w)} \frac{1 - \frac{1}{p^{\nu(H\alpha)+1}}}{1 - \frac{1}{p^{\nu(H\alpha)}}} \right\}$$

and

$$\left\{ \sum_{\sigma \in \Omega_1} \left( \prod_{\alpha \in \Sigma_0^+(w)} \frac{1 - \frac{1}{p^{\sigma\nu(H\alpha)+1}}}{1 - \frac{1}{p^{\sigma\nu(H\alpha)}}} \right) p^{\lambda_1(\sigma\nu-\rho)} \right\} p^{\lambda_2(\nu-\rho)}.$$

Applying the previous discussion to  $G_1$  instead of  $G$ , we see that  $\Theta'_\nu(t)$  is analytic at  $\mu$ . Moreover  $\Theta'_\nu(t)$  does not vanish, as a function of  $t$ , for  $\lambda(t)$  in an open cone. Set  $\Theta''_\nu(t) = \Theta_\nu(t) - \Theta'_\nu(t)$ ;  $\Theta''_\nu(t)$  is also analytic at  $\mu$ . As a function of  $t$ ,  $\Theta''_\nu(t)$  is a linear combination of terms of the form  $p(\lambda(t))p^{\lambda(t)(\mu'-\rho)}$  where  $\mu'$  is an element in the orbit of  $\mu$  with  $\operatorname{Re} \mu' \neq \operatorname{Re} \mu$  and  $p(\lambda(t))$  is a polynomial in  $\lambda(t)$ . Thus, if  $\lambda(t)$  lies in the interior of the positive Weyl chamber,

$$\lim_{n \rightarrow \infty} p^{-n\lambda(t)(\mu-\rho)} \Theta''_\nu(t^n) = 0$$

and

$$\lim_{n \rightarrow \infty} p^{-n\lambda(t)(\mu-\rho)} \phi_\mu(t^n) = \lim_{n \rightarrow \infty} p^{-n\lambda(t)(\mu-\rho)} \Theta'_\nu(t^n).$$

Suppose  $\phi_\mu$  were bounded and for some  $\lambda$  in the positive Weyl chamber  $\operatorname{Re} \lambda(\rho - \mu)$  were less than zero. Then there would exist a  $t$  such that  $\operatorname{Re} \lambda(t)(\rho - \mu) < 0$  and  $\Theta'_\nu(t^n)$  did not vanish identically. Then

$$\lim_{n \rightarrow \infty} p^{-n\lambda(t)(\mu-\rho)} \Theta'_\nu(t^n) = 0.$$

On the other hand if  $t$  is fixed

$$p^{-n\lambda(t)(\mu-\rho)} \Theta'_\nu(t^n) = \sum_{k=0}^q \varphi_k(n) n^k$$

where  $\varphi_k(m)$  is a linear combination of exponentials  $e^{i\alpha m}$  with  $\alpha$  real. We can suppose  $\varphi_q(n) \not\equiv 0$ . Certainly

$$\lim_{n \rightarrow \infty} \varphi_q(n) = 0.$$

Let

$$\varphi_q(n) = \sum_{j=1}^r a_j e^{i\alpha_j n}$$

with  $\alpha_1, \dots, \alpha_r$  real and incongruent modulo  $2\pi$ . Then

$$0 = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \varphi_q(n) e^{-i\alpha_j n} = a_j.$$

This is a contradiction.





## CHAPTER 4

### The formula of Gindikin and Karpelevich

To complete the proof of the lemma and to prepare ourselves for the next stage of the argument, we must evaluate the functions  $\delta_R(\mu)$  in closed form. The argument is an induction on the number of elements in  $\Sigma_R^-$ . Since  $\delta_R(\mu)$  is certainly 1 when  $\Sigma_R^-$  is empty, we can start immediately with the induction step. Let  $C_R^-$  and  $C_R^+$  be the convex cones with vertex at the origin generated by  $\Sigma_R^-$  and  $\Sigma_R^+$  respectively. Let

$$\begin{aligned} D_R^+ &= \{ \lambda \in \mathfrak{h}_{\mathbf{R}} \mid \lambda(\mu) \geq 0 \text{ for all } \mu \in C_R^+ \}, \\ D_R^- &= \{ \lambda \in \mathfrak{h}_{\mathbf{R}} \mid \lambda(\mu) \geq 0 \text{ for all } \mu \in C_R^- \}. \end{aligned}$$

If, as before,  ${}^c\rho = \frac{1}{2} \sum_{\alpha > 0} {}^c\alpha$ , then  ${}^c\rho$  lies in the interior of  $D_R^+$  and, if  $\Sigma_R^-$  is not empty, in the exterior of  $D_R^-$ . If

$$R = \{ \mu \mid \lambda_0(\mu) \geq 0 \},$$

then  $\lambda_0$  lies in the intersection of  $D_R^+$  with the interior of  $D_R^-$ . Joining  ${}^c\rho$  to  $\lambda_0$ , we pass through a point of the boundary of  $D_R^-$  which lies in the interior of  $D_R^+$ . Since  $D_R^-$  is polygonal, there is a point  $\lambda_1$  near this boundary point which lies inside an  $n - 1$ -dimensional side of  $D_R^-$  and in the interior of  $D_R^+$ . Then  $\Sigma_R^-$  is the set of all negative roots satisfying  $\lambda_1(\alpha) \geq 0$ . There is exactly one negative root  $-\alpha_0$  such that  $\lambda_1(-\alpha_0) = 0$ . Let

$$S = \{ \mu \mid \lambda_1(\mu) > 0 \}.$$

Then  $\Sigma_R^-$  is the union of  $-\alpha_0$  and  $\Sigma_S^-$ .

To establish the formula (4) we show that

$$\delta_R(\mu) = \frac{1 - \frac{1}{p^{\mu(H_{\alpha_0})+1}}}{1 - \frac{1}{p^{\mu(H_{\alpha_0})}}} \delta_S(\mu).$$

We shall also see that the integral defining  $\delta_R(\mu)$  converges if that defining  $\delta_S(\mu)$  does and

$$\operatorname{Re} \mu(H_{\alpha_0}) > 0.$$

Let  $\bar{N}^0$  be the one parameter group generated by the root vector  $X_{-\alpha_0}$  belonging to  $-\alpha_0$ . Let  $G^0$  be the group corresponding to the Lie algebra spanned by  $X_{\alpha_0}$ ,  $X_{-\alpha_0}$ , and  $H_{\alpha_0}$ . As usual there is a mapping of  $\operatorname{SL}(2)$  into  $G^0$  such that the image of

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$$

is  $\exp xX_{-\alpha_0}$  and the image of

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

is  $\exp xX_{\alpha_0}$ . The image of  $\mathrm{SL}(2, \mathbf{Z}_p)$  is contained in  $G_{\mathbf{Z}_p}$ . If  $\bar{n}_1 = \exp xX_{-\alpha_0}$ , let  $a_1$  be the identity if  $x$  lies in  $\mathbf{Z}_p$  and let  $a_1$  be the image of

$$\begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}$$

if  $x$  is not in  $\mathbf{Z}_p$ . Let  $n_1$  be the identity if  $x$  is in  $\mathbf{Z}_p$  and let  $n_1$  be the image of

$$\begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix}$$

if  $x$  is not in  $\mathbf{Z}_p$ . In all cases  $\bar{n}_1$  lies in  $n_1 a_1 G_{\mathbf{Z}_p}$ . Thus, if  $\bar{n}_2$  lies in  $\bar{N}_{\mathbf{Q}_p}(S)$ ,

$$\psi_\mu(\bar{n}_2 \bar{n}_1) = \psi_\mu(\bar{n}_2 n_1 a_1) = \psi_\mu(a_1) \psi_\mu(a_1^{-1} n_1^{-1} \bar{n}_2 n_1 a_1).$$

Consequently

$$\begin{aligned} \int_{\bar{N}_{\mathbf{Q}_p}(R)} \psi_\mu(\bar{n}) d\bar{n} &= \int_{\bar{N}_{\mathbf{Q}_p}^0} \left\{ \int_{\bar{N}_{\mathbf{Q}_p}(S)} \psi_\mu(\bar{n}_2 \bar{n}_1) d\bar{n}_2 \right\} d\bar{n}_1 \\ &= \int_{\bar{N}_{\mathbf{Q}_p}^0} \psi_\mu(a_1) \left\{ \int_{\bar{N}_{\mathbf{Q}_p}(S)} \psi_\mu(a_1^{-1} n_1^{-1} \bar{n}_2 n_1 a_1) d\bar{n}_2 \right\} d\bar{n}_1. \end{aligned}$$

Let

$$\mathfrak{n} = \sum_{\lambda_1(\alpha) > 0} \mathbf{Q}_p X_\alpha, \quad \mathfrak{a} = \sum_{\substack{\lambda_1(\alpha) > 0 \\ \alpha > 0}} \mathbf{Q}_p X_\alpha, \quad \mathfrak{b} = \sum_{\substack{\lambda_1(\alpha) > 0 \\ \alpha < 0}} \mathbf{Q}_p X_\alpha.$$

Here  $\mathfrak{n}$  is the direct sum of  $\mathfrak{a}$  and  $\mathfrak{b}$ . If  $Q$  is a closed half-space contained in  $S$ , let  $\Theta_Q$  be the set of roots contained in  $Q$ . Let the distinct collections of roots obtained in this way be, in decreasing order,  $\Theta_0, \Theta_1, \dots, \Theta_\ell, \Theta_{\ell+1} = \emptyset$ , and set

$$\mathfrak{n}_k = \sum_{\alpha \in \Theta_k} \mathbf{Q}_p X_\alpha.$$

The relations  $[\mathfrak{n}, \mathfrak{n}_k] \subseteq \mathfrak{n}_{k+1}$  and  $\mathfrak{n}_k = \mathfrak{a} \cap \mathfrak{n}_k + \mathfrak{b} \cap \mathfrak{n}_k$  are clear; in particular,  $\mathfrak{n}$  is nilpotent. The following rather complicated lemma is an easy consequence of the Campbell-Hausdorff formula.

**Lemma.** *Suppose  $\mathfrak{n}$  is a Lie algebra of nilpotent transformations of a vector space  $V$  over a field  $k$  of characteristic zero and  $N$  is the associated group of linear transformations. Suppose*

$$\mathfrak{n} = \mathfrak{n}_0 \supsetneq \mathfrak{n}_1 \supsetneq \dots \supsetneq \mathfrak{n}_{\ell+1} = \{0\}$$

*is a decreasing sequence of ideals in  $\mathfrak{n}$  and  $[\mathfrak{n}, \mathfrak{n}_k] \subseteq \mathfrak{n}_{k+1}$ . Suppose that  $\mathfrak{a}$  and  $\mathfrak{b}$  are two subspaces of  $\mathfrak{n}$  and*

$$\mathfrak{n}_k = \mathfrak{n}_k \cap \mathfrak{a} \oplus \mathfrak{n}_k \cap \mathfrak{b}$$

*for each  $k$ . Set  $\mathfrak{a}_k = \mathfrak{n}_k \cap \mathfrak{a}$ ,  $\mathfrak{b}_k = \mathfrak{n}_k \cap \mathfrak{b}$ , and choose  $\tilde{\mathfrak{a}}_i, \tilde{\mathfrak{b}}_i$  such that*

$$\mathfrak{a}_k = \bigoplus_{i=k}^{\ell} \tilde{\mathfrak{a}}_i \quad \text{and} \quad \mathfrak{b}_k = \bigoplus_{i=k}^{\ell} \tilde{\mathfrak{b}}_i.$$

*Then every element of  $N$  can be written uniquely as*

$$n = \exp X_0 \exp X_1 \cdots \exp X_\ell \exp Y_\ell \cdots \exp Y_0$$

with  $X_i$  in  $\tilde{\mathfrak{a}}_i$ ,  $Y_i$  in  $\tilde{\mathfrak{b}}_i$ . If  $X \rightarrow X^a$  is an automorphism of  $\mathfrak{n}$  leaving each  $\mathfrak{n}_k$  invariant let  $X_k + Y_k \rightarrow X'_k + Y'_k$  be the induced transformation on

$$\mathfrak{n}_k/\mathfrak{n}_{k+1} \simeq \tilde{\mathfrak{a}}_k \oplus \tilde{\mathfrak{b}}_k.$$

If

$$n^a = \exp X''_0 \cdots \exp X''_\ell \exp Y''_\ell \cdots \exp Y''_0,$$

then

$$X''_k = X'_k + f(X_0, \dots, X_{k-1}, Y_0, \dots, Y_{k-1}),$$

$$Y''_k = Y'_k + g(X_0, \dots, X_{k-1}, Y_0, \dots, Y_{k-1})$$

with some polynomial functions  $f$  and  $g$ .

In the case of concern to us, both  $\mathfrak{a}$  and  $\mathfrak{b}$  are subalgebras of  $\mathfrak{n}$ . The group corresponding to  $\mathfrak{b}$  is  $\overline{N}_{\mathbf{Q}_p}(S)$ . There is a subgroup  $N(S)$  of  $G$  such that the group corresponding to  $\mathfrak{a}$  is  $N_{\mathbf{Q}_p}(S)$ . Moreover  $N(S)$  is contained in  $V$  the unipotent radical of  $B$ . As a particular consequence of the lemma

$$N = N_{\mathbf{Q}_p}(S)\overline{N}_{\mathbf{Q}_p}(S).$$

If  $\bar{n}_2$  lies in  $\overline{N}_{\mathbf{Q}_p}(S)$  then  $a_1^{-1}n_1^{-1}\bar{n}_2n_1a_1$  lies in  $N$ . Project it onto  $\overline{N}_{\mathbf{Q}_p}(S)$  to obtain  $\bar{n}'_2$ . It is an easy consequence of the lemma that the map  $\bar{n}_2 \rightarrow \bar{n}'_2$  is a one-to-one mapping of  $\overline{N}_{\mathbf{Q}_p}(S)$  onto itself and that

$$d\bar{n}_2 = \prod_{\substack{\lambda_1(\alpha) < 0 \\ \alpha > 0}} |\xi_\alpha(a_1)|^{-1} d\bar{n}'_2.$$

Now  $\psi_\mu(a_1) = p^{\lambda(a_1)(\mu+p)}$  and

$$\rho - \sum_{\substack{\lambda_1(\alpha) < 0 \\ \alpha > 0}} \alpha = \frac{1}{2} \left( \alpha_0 + \sum_{\substack{\lambda_1(\alpha) > 0 \\ \alpha > 0}} \alpha - \sum_{\substack{\lambda_1(\alpha) < 0 \\ \alpha > 0}} \alpha \right) = \frac{1}{2} \left( \alpha_0 + \sum_{\lambda_1(\alpha) > 0} \alpha \right).$$

Since

$$\sum_{\lambda_1(\alpha) > 0} \lambda(a_1)(\alpha) = 0,$$

we have

$$\int_{\overline{N}_{\mathbf{Q}_p}(R)} \psi_\mu(\bar{n}) d\bar{n} = \left\{ \int_{\overline{N}_{\mathbf{Q}_p}^0} p^{\lambda(a_1)(\mu + \frac{\alpha_0}{2})} d\bar{n}_1 \right\} \left\{ \int_{\overline{N}_{\mathbf{Q}_p}(S)} \psi_\mu(\bar{n}_2) d\bar{n}_2 \right\}.$$

The first integral is equal to

$$\int_{\mathbf{Z}_p} 1 + \int_{\substack{x \in \mathbf{Q}_p \\ |x| > 1}} |x|^{-\mu(H_{\alpha_0})-1} dx,$$

which is

$$1 + \left(1 - \frac{1}{p}\right) \sum_{n=1}^{\infty} \frac{p^n}{p^{n(\mu(H_{\alpha_0})+1)}} = 1 + \left(\frac{1 - \frac{1}{p}}{p^{\mu(H_{\alpha_0})}}\right) \frac{1}{1 - \frac{1}{p^{\mu(H_{\alpha_0})}}}$$

which equals

$$\frac{1 - \frac{1}{p^{\mu(H_{\alpha_0})+1}}}{1 - \frac{1}{p^{\mu(H_{\alpha_0})}}}$$

if  $\operatorname{Re} \mu(H_{\alpha_0}) > 0$ .

## CHAPTER 5

### A review of Eisenstein series

Let  $\Delta$  be the set of simple roots of  $\mathfrak{h}$ . If  $\alpha_0$  belongs to  $\Delta$ , there is a parabolic group  $P = P(\alpha_0)$  of rank one which contains  $B$  associated to  $\alpha_0$ . Let  $N$  be the unipotent radical of  $P$ . Then  $P$  is the semidirect product of  $N$  and a reductive group  $M$ . It is convenient to suppose that  $M$  contains  $T$ . Let  $A$  be the centre of  $M$  and let  ${}^0G = A \backslash M$ . There is a map from  $P$  to  ${}^0G$ . Furthermore,  ${}^0G$  is the adjoint group of a split Lie algebra of rank one less than  $G$ . Its Dynkin diagram is obtained by deleting  $\alpha_0$  from the Dynkin diagram of  $G$ .

**Lemma.** *Each of the maps*

$$P_{\mathbf{Q}_p} \rightarrow {}^0G_{\mathbf{Q}_p}, \quad P_{\mathbf{Z}_p} \rightarrow {}^0G_{\mathbf{Z}_p}, \quad P_{\mathbf{A}} \rightarrow {}^0G_{\mathbf{A}}$$

*is surjective.*

It is enough to verify this for the first two maps. Let  ${}^0T$  be the image of  $T$  in  ${}^0G$ . Using the Bruhat decomposition one readily shows that the map  $P_{\mathbf{Q}_p} \rightarrow {}^0G_{\mathbf{Q}_p}$  is surjective if the map  $T_{\mathbf{Q}_p} \rightarrow {}^0T_{\mathbf{Q}_p}$  is surjective. If  ${}^0t$  lies in  ${}^0T_{\mathbf{Q}_p}$  then  $\xi_\beta({}^0t)$  is given for  $\beta$  in  $\Delta - \{\alpha\}$ . There is certainly a  $t$  in  $T_{\mathbf{Q}_p}$  such that  $\xi_\beta(t) = \xi_\beta({}^0t)$  for these  $\beta$ ;  $t$  is mapped to  ${}^0t$ .

Suppose  $u$  in  ${}^0G_{\mathbf{Z}_p}$  is the image of  $p = vtk$  with  $v$  in  $V_{\mathbf{Q}_p}$ ,  $t$  in  $T_{\mathbf{Q}_p}$ ,  $k$  in  $G_{\mathbf{Z}_p}$ . For the purposes of the lemma we may suppose that  $k = 1$ . If  $\alpha$  is a root of  ${}^0G$ , then  $|\xi_\alpha(t)| = 1$ . Let  $t_0$  be such that  $\xi_\alpha(t_0) = |\xi_\alpha(t)|$  for each root  $\alpha$ ;  $t_0$  must lie in the centre of  $M$ . Replacing  $t$  by  $t_0t$ , we may suppose that  $t$  lies in  $G_{\mathbf{Z}_p}$  or, even better, that  $t$  is 1 and  $p = v$ . If  $p$  is infinite,  $v$  must lie in  $N_{\mathbf{Q}_p}$  and if  $p$  is finite  $v$  must be congruent modulo  $N_{\mathbf{Q}_p}$  to an element of  $G_{\mathbf{Z}_p}$ . (See C. Chevalley, Séminaire Bourbaki, Exposé 219.)

If  $p$  belongs to  $P$  let  $\chi(p)$  be the determinant of the restriction of  $\text{Ad } p$  to the Lie algebra of  $N$ . Every element of  $G_{\mathbf{A}}$  is a product  $g = bu$  with  $b$  in  $B_{\mathbf{A}}$  and  $u$  in  $U$ ; set  $\xi_s(g) = \xi_s(b) = \prod_p |\chi(b_p)|^{s + \frac{1}{2}}$ . The product is taken over all primes including the infinite one. The function  $\xi_s$  is well-defined and is a function on  $P_{\mathbf{Q}} \backslash G_{\mathbf{A}}$ . Let  $\phi$  be one of the basis elements for the cusp forms on  ${}^0G_{\mathbf{Q}} \backslash {}^0G_{\mathbf{A}}$ . Of course  $\phi$  is supposed to be invariant on the right under  ${}^0U$ . Also  $\phi$  may be lifted to a function on  $N_{\mathbf{A}} P_{\mathbf{Q}} \backslash P_{\mathbf{A}}$ . If  $g = bu$ , set

$$F(g, s, \phi) = \xi_s(g) \phi(b).$$

This function is well-defined. The sum

$$E(g, s, \phi) = \sum_{\gamma \in P_{\mathbf{Q}} \backslash G_{\mathbf{Q}}} F(\gamma g, s, \phi)$$

is called an Eisenstein series. It converges absolutely for  $\text{Re } s > \frac{1}{2}$  but the function on the left is actually a meromorphic function of  $s$  for all  $g$ . By the way, if  $g$  belongs to  $G_{\mathbf{R}}$ ,

$$E(g, s, \phi) = \sum_{\gamma \in P_{\mathbf{Z}} \backslash G_{\mathbf{Z}}} F(\gamma g, s, \phi).$$

Suppose  $P' = P(\alpha'_0)$  is another parabolic group of rank one, and  $N'$  is its unipotent radical. Then

$$\int_{N'_Q \backslash N'_A} E(ng, s, \phi) dn$$

is for each  $g$  a meromorphic function of  $s$ . If  $\operatorname{Re} s > \frac{1}{2}$ , it is equal to

$$\int_{N'_Q \backslash N'_A} \sum_{P_Q \backslash G_Q} F(\gamma ng, s, \phi) dn,$$

which equals

$$\int_{N'_Q \backslash N'_A} \sum_{P_Q \backslash G_Q / N'_Q} \sum_{\gamma^{-1} P_Q \gamma \cap N'_Q \backslash N'_Q} F(\gamma \delta ng, s, \phi) dn$$

or

$$\sum_{P_Q \backslash G_Q / N'_Q} \int_{\gamma^{-1} P_Q \gamma \cap N'_Q \backslash N'_A} F(\gamma ng, s, \phi) dn.$$

Because of the Bruhat decomposition we can suppose that each  $\gamma$  is of the form  $\gamma = w\gamma'$  with  $w$  in the intersection of  $G_Z$  and the normalizer of  $T$  and  $\gamma'$  in  $P'_{Q_p}$ . Then a typical term equals<sup>1</sup>

$$\int_{w^{-1} P_Q w \cap N'_Q \backslash N'_A} F(wng, s, \phi) dn.$$

There is an order on the roots of  ${}^0G$  such that the positive roots are those of the form  $w\alpha$  where  $\alpha$  is a positive root of  $G$ . Multiplying  $w$  on the left by an element in the normalizer of  $T$  in  $G_Z \cap M$ , we may suppose this is the order induced from the original order on the roots of  $\mathfrak{h}$ . Let  ${}^0\Sigma_+$  be the roots of  ${}^0G$  of the form  $w\alpha$  where  $\alpha$  is a positive root of  $G$  which is not a root of  ${}^0G'$  and let  ${}^0\Sigma_0$  be the roots of  ${}^0G$  of the form  $w\alpha$  where  $\alpha$  is a positive root of  $G$  which is a root of  ${}^0G'$ . If  $\alpha$  belongs to  ${}^0\Sigma_+$  and  $\beta$  belongs to  ${}^0\Sigma_+$  or to  ${}^0\Sigma_0$  and  $\alpha + \beta$  is a root, then  $\alpha + \beta$  belongs to  ${}^0\Sigma_+$ ; on the other hand, if  $\alpha$  and  $\beta$  both belong to  ${}^0\Sigma_0$  and  $\alpha + \beta$  is a root, then  $\alpha + \beta$  belongs to  ${}^0\Sigma_0$ . As a consequence, the group  $N''$  whose Lie algebra is the span of  $\{X_\alpha \mid \alpha \in {}^0\Sigma_+\}$  is the unipotent radical of a parabolic subgroup of  ${}^0G$ . Since  $w^{-1}N''w$  is contained in  $N'$  and

$$w^{-1}P_Q w \cap w^{-1}N''_A w = w^{-1}N''_A w,$$

our integral equals

$$\int_{(w^{-1}P_Q w \cap N'_Q)w^{-1}N''_A w \backslash N'_A} \left\{ \int_{N''_Q \backslash N''_A} F(n_1 wng, s, \phi) dn_1 \right\} dn.$$

If  $wng = bk$  with  $b$  in  $B_A$  and  $k$  in  $U$ , the inner integral equals

$$\xi_s(b) \int_{N''_Q \backslash N''_A} \phi(n_1 b) dn_1$$

which is zero if  $N'' \neq \{1\}$  because  $\phi$  is a cusp form. Thus the integral vanishes identically unless every positive root of  ${}^0G$  is of the form  $w\alpha$  where  $\alpha$  is a root of  ${}^0G'$ . Then, if  $\alpha$  is a positive root of  ${}^0G'$ ,  $w\alpha$  is a linear combination of roots of  ${}^0G$  and thus a root of  ${}^0G$ . As a

<sup>1</sup>[Added 1970] This statement is, I now notice, an oversimplification. There should be a factor in front which depends on  $\gamma$  and  $g$  should be replaced by  $\gamma'g$ .

consequence,  $wM'w^{-1} = M$ . For these terms we can take  $\gamma' = 1$ . If  $P = P'$ , then  $w = 1$  is one possibility and the resulting integral is  $F(g, s, \phi)$ , which is for each  $g$  an entire function of  $s$ . The only other possibility is that  $wP'w^{-1}$  is the parabolic group opposed to  $P$ . This is the case we are interested in. Then  $w^{-1}P_{\mathbf{Q}}w \cap N'_{\mathbf{A}} = \{1\}$  and

$$\int_{N'_{\mathbf{A}}} F(wng, s, \phi) dn$$

is for each  $g$  a meromorphic function of  $s$  in the whole complex plane.

We have demanded that  $w$  lie in  $G_{\mathbf{Z}}$ . We can also demand that it lie in  $G_{\mathbf{Z}_{\infty}}$ ; this will make it easier to evaluate the integral, for then  $w$  lies in  $U$  and

$$F(wng, s, \phi) = F(wngw^{-1}, s, \phi).$$

It is enough to evaluate the integral for  $g = m'$  in  $M'_{\mathbf{A}}$ . Set  $m = wm'w^{-1}$ ;  $m$  lies in  $M_{\mathbf{A}}$ . A simple change of variable shows that the integral equals

$$\prod_p |\chi(m_p)|^{-1} \int_{N'_{\mathbf{A}}} F(mwnw^{-1}, s, \phi) dn.$$

The product is taken over all primes including the one at infinity. If  $\bar{N}$  is the unipotent radical of the group opposed to  $P$ , this may also be written as

$$(7) \quad \prod_p |\chi(m_p)|^{-1} \int_{\bar{N}_{\mathbf{A}}} F(m\bar{n}, s, \phi) d\bar{n}.$$

The map  $T \rightarrow {}^0T$  determines a map  $\mathfrak{h}_{\mathbf{R}} \rightarrow {}^0\mathfrak{h}_{\mathbf{R}}$ . Since we have more or less consistently viewed  ${}^c\mathfrak{h}_{\mathbf{R}}$  and  ${}^c({}^0\mathfrak{h})_{\mathbf{R}}$  as the duals of  $\mathfrak{h}_{\mathbf{R}}$  and  ${}^0\mathfrak{h}_{\mathbf{R}}$ , we can agree that this determines a map  ${}^c({}^0\mathfrak{h})_{\mathbf{C}}$  into  ${}^c\mathfrak{h}_{\mathbf{C}}$ . If  $\chi_p$  is for each  $p$  the homomorphism of the Hecke algebra into the complex numbers determined by  $\phi$ , let  ${}^0\mu_p$  be one of the elements in  ${}^c({}^0\mathfrak{h})_{\mathbf{C}}$  associated to  $\chi_p$ . Its image in  ${}^c\mathfrak{h}_{\mathbf{C}}$  will again be denoted by  ${}^0\mu_p$ . If  $\nu$  is the sum of those roots whose root vectors belong to the Lie algebra of  $N$ , we set  $\mu_p(s) = {}^0\mu_p + s\nu$ . Denote this set of roots by  $\Sigma$ . If

$$M(s) = \left\{ \prod_{\alpha \in \Sigma} \frac{\pi^{1/2} \Gamma\left(\frac{\mu_{\infty}(s)(H_{\alpha})}{2}\right)}{\Gamma\left(\frac{\mu_{\infty}(s)(H_{\alpha})+1}{2}\right)} \right\} \prod_{p \text{ finite}} \left\{ \prod_{\alpha \in \Sigma} \frac{1 - \frac{1}{p^{\mu_p(s)(H_{\alpha})+1}}}{1 - \frac{1}{p^{\mu_p(s)(H_{\alpha})}}}\right\},$$

the integral in the expression (7) is equal to

$$M(s)F(m, s, \phi).$$

This is not too difficult to prove. Observe first that if  $S_k$  is the set consisting of the infinite prime and the first  $k$  finite primes the integral in (7) is equal to

$$\lim_{k \rightarrow \infty} \int_{\prod_{p \in S_k} \bar{N}_{\mathbf{Q}_p}} F(m\bar{n}, s, \phi) d\bar{n}.$$

So to prove our assertion all we need to do is show that, if  $h$  lies in  $\prod_{q \neq p} G_{\mathbf{Q}_q} \cap G_{\mathbf{A}}$  and  $m$  lies in  $M_{\mathbf{Q}_p}$  then

$$\int_{\bar{N}_{\mathbf{Q}_p}} F(hm\bar{n}, s, \phi) d\bar{n} = \left\{ \prod_{\alpha \in \Sigma} \frac{\pi^{1/2} \Gamma\left(\frac{\mu_p(s)(H_{\alpha})}{2}\right)}{\Gamma\left(\frac{\mu_p(s)(H_{\alpha})+1}{2}\right)} \right\} F(hm, s, \phi)$$

if  $p$  is the infinite prime and

$$\int_{\bar{N}_{\mathbf{Q}_p}} F(hm\bar{n}, s, \phi) d\bar{n} = \left\{ \prod_{\alpha \in \Sigma} \frac{1 - \frac{1}{p^{\mu_p(s)(H_\alpha)+1}}}{1 - \frac{1}{p^{\mu_p(s)(H_\alpha)}}} \right\} F(hm, s, \phi)$$

if  $p$  is a finite prime.

Fix a prime, finite or infinite; fix  $h$  in

$$\prod_{q \neq p} G_{\mathbf{Q}_q} \cap G_{\mathbf{A}}$$

and consider the function  $F(hg, s, \phi)$ ,  $g$  in  $G_{\mathbf{Q}_p}$ .

If  $h = bu$ ,  $b$  in  $B_{\mathbf{A}}$ ,  $u$  in  $U$ , if  $\bar{b}$  is the projection of  $b$  on  $M_{\mathbf{A}}$ , and if  $g = n(g)m(g)k(g)$  with  $n(g)$  in  $N_{\mathbf{Q}_p}$ ,  $m(g)$  in  $M_{\mathbf{Q}_p}$ , and  $k(g)$  in  $G_{\mathbf{Z}_p}$  then it equals

$$\xi_s(h)\xi_s(m(g))\phi(\bar{b}m(g)).$$

If  $m$  belongs to  $M_{\mathbf{Q}_p}$ , set  $\psi(m) = \phi(\bar{b}m)$ . If it is convenient, we can regard  $\psi(m)$  as a function on  ${}^0G_{\mathbf{Q}_p}$ . We are reduced to evaluating

$$(8) \quad \int_{\bar{N}_{\mathbf{Q}_p}} \xi_s(mm(\bar{n}))\psi(mm(\bar{n})) d\bar{n}$$

if  $\psi$  is a function on  ${}^0G_{\mathbf{Q}_p}$  invariant under right translations by elements of  ${}^0G_{\mathbf{Z}_p}$  which is an eigenfunction of the operators  $\lambda(f)$  for  $f$  in  ${}^0H_p$  associated to the homomorphism  $\chi_p$  of  ${}^0H_p$  into  $\mathbf{C}$  determined by  ${}^0\mu_p$ . Of course we assume that the integral converges absolutely. Recall that

$$\lambda(f)\psi(g) = \int_{{}^0G_{\mathbf{Q}_p}} \psi(gh)f(h) dh$$

if  $g$  belongs to  ${}^0G_{\mathbf{Q}_p}$ .

Let  $M = M_{\mathbf{Q}_p}$  and let

$$K = G_{\mathbf{Z}_p} \cap P_{\mathbf{Q}_p} / G_{\mathbf{Z}_p} \cap N_{\mathbf{Q}_p}.$$

Define a measure  $\mu$  on  $M/K$  by setting  $\mu(E)$  equal to the measure of

$$\left\{ \bar{n} \in \bar{N}_{\mathbf{Q}_p} \mid m(\bar{n})E \right\}.$$

Suppose  $k$  lies in  $K$  and  $\bar{n}$  in  $\bar{N}$  equals  $n(\bar{n})m(\bar{n})k(\bar{n})$ ; let  $k$  be the coset of  $\bar{k}$ . Since

$$\bar{k}\bar{n}\bar{k}^{-1} = (\bar{k}n(\bar{n})\bar{k}^{-1})(\bar{k}m(\bar{n})\bar{k}^{-1})(\bar{k}k(\bar{n})\bar{k}^{-1}),$$

the sets  $\{\bar{n} \mid m(\bar{n}) \in kE\}$  and  $\{\bar{k}\bar{n}\bar{k}^{-1} \mid m(\bar{n}) \in E\}$  are the same and  $\mu$  is left-invariant under  $K$ . Define a measure on  $M$ , again called  $\mu$ , which is invariant under left and right translations by elements of  $K$  by setting

$$\mu(E) = \int_{M/K} \left\{ \int_K \chi_E(mk) dk \right\} d\mu(m)$$

if  $\chi_E$  is the characteristic function of  $E$ .

The integral (8) is equal to

$$\int_M \xi_s(mm_1)\psi(mm_1) d\mu(m_1).$$



If  $F$  is a subset of  ${}^0G_{\mathbf{Q}_p} = A_{\mathbf{Q}_p} \backslash M_{\mathbf{Q}_p}$  and  $E$  is the inverse image of  $F$  in  $M_{\mathbf{Q}_p}$ , set

$$\nu_s(F) = \int_E \xi_s(m_1) d\mu(m_1).$$

Since, as we observed earlier,  $K$  maps onto  ${}^0G_{\mathbf{Z}_p}$ ,  $\nu_s$  is invariant on the left and the right under  ${}^0G_{\mathbf{Z}_p}$ . The integral (8) equals

$$\xi_s(m) \int_{{}^0G_{\mathbf{Q}_p}} \psi(\bar{m}h) d\nu_s(h)$$

if  $\bar{m}$  is the image of  $m$  in  ${}^0G_{\mathbf{Q}_p}$ .

Let  $F_1 \subseteq F_2 \subseteq \dots$  be an increasing sequence of compact sets (we assume that  $F_i = {}^0G_{\mathbf{Z}_p} F_i {}^0G_{\mathbf{Z}_p}$ ) whose union exhausts  ${}^0G_{\mathbf{Q}_p}$  and define the measure  $\nu_s^n$  by  $\nu_s^n(F) = \nu_s(F \cap F_n)$ . Since  $\nu_s^n$  belongs to  $H_p$

$$\int_{{}^0G_{\mathbf{Q}_p}} \psi(\bar{m}h) d\nu_s(h) = \lim_{n \rightarrow \infty} \int_{{}^0G_{\mathbf{Q}_p}} \psi(\bar{m}h) d\nu_s^n(h) = \psi(\bar{m}) \lim_{n \rightarrow \infty} \chi_p(\nu_s^n)$$

and the integral (8) equals

$$\xi_s(m) \psi(m) \lim_{n \rightarrow \infty} \chi_p(\nu_s^n).$$

To evaluate the limit, take  $\psi$  to be the function  $\psi_{\mu_p}$  of Section 3; then

$$\xi_s(m(g)) \psi_{\mu_p}(m(g)) = \psi_{\mu_p(s)}(g)$$

and

$$\lim_{n \rightarrow \infty} \chi_p(\nu_s^n) = \int_{\bar{N}_{\mathbf{Q}_p}} \psi_{\mu_p(s)}(\bar{n}) d\bar{n}.$$

The integral on the right can be evaluated by the formula of Gindikin and Karpelevich if  $\operatorname{Re} s$  is sufficiently large. Retracing our steps, we see that the integral in (7) is indeed equal to  $M(s)F(m, s, \phi)$  and conclude that  $M(s)$  is a meromorphic function in the whole complex plane.

J. Tits pointed out a way of expressing  $M(s)$  which is more convenient for our purposes. We observed that there was a map of  ${}^c({}^0\mathfrak{h})_{\mathbf{C}}$  into  ${}^c\mathfrak{h}_{\mathbf{C}}$ . It is easy to see that it is induced by an imbedding of  ${}^c({}^0\mathfrak{g})$  in  ${}^c\mathfrak{g}$ . Since  ${}^c({}^0G)$  is simply connected, there is an associated map of  ${}^c({}^0G)$  into  ${}^cG$ . Let  ${}^c\mathfrak{n}$  be the Lie algebra spanned by the root vectors belonging to positive roots of  ${}^cG$  which are not roots of  ${}^c({}^0G)$ . These are the roots  ${}^c\alpha$  corresponding to roots in  $\Sigma$ . Let  $H_{c\alpha}$  be the *copoid* attached to  ${}^c\alpha$  and set

$$H_0 = \sum_{\alpha \in \Sigma} H_{c\alpha}.$$

Let  $\mathfrak{n}_1, \dots, \mathfrak{n}_r$  be the eigenspaces of  $\operatorname{ad}(H_0)$  in  ${}^c\mathfrak{n}$ . Let  $a_i$  be the eigenvalue of  $\operatorname{ad}(H_0)$  corresponding to  $\mathfrak{n}_i$ . Each of the subspaces  $\mathfrak{n}_i$  is invariant under  ${}^c({}^0G)$ ; let  $\pi_i$  be the representation of  ${}^c({}^0G)$  on  $\mathfrak{n}_i$ . If  $\tilde{\pi}_i$  is the representation contragredient to  $\pi_i$ , then  $M(s)$  can be written as

$$\prod_{i=1}^r \frac{\xi(a_i s, \tilde{\pi}_i, \phi)}{\xi(a_i s + 1, \tilde{\pi}_i, \phi)}.$$



## CHAPTER 6

### Examples

If  $r$  is 1, then

$$M(s) = \frac{\xi(a_1 s, \tilde{\pi}_1, \phi)}{\xi(a_1 s + 1, \tilde{\pi}_1, \phi)}$$

is meromorphic in the whole plane and

$$\xi(s, \tilde{\pi}_1, \phi) = M\left(\frac{s}{a_1}\right)\xi(s + 1, \tilde{\pi}_1, \phi).$$

Since we already know that  $\xi(s, \tilde{\pi}_1, \phi)$  is analytic in a half-plane, we can conclude that it is meromorphic in the whole plane.

If  $r = 2$  and  $\xi(s, \tilde{\pi}_1, \phi)$  is known to be meromorphic in the whole plane, the same argument shows that  $\xi(s, \tilde{\pi}_2, \phi)$  is meromorphic in the whole plane. Thus every time we can adjoin a point to the Dynkin diagram of  ${}^0G$  to obtain the Dynkin diagram of a group of rank 1 greater, we can expect to find a representation  $\pi$  of  ${}^c({}^0G)$  for which  $\xi(s, \pi, \phi)$  is meromorphic in the whole plane. Before listing the possibilities, there is one further remark I should make.

If we define the function  $\xi'_s$  in the same manner as  $\xi_s$ , the expression (7) is easily seen to equal

$$M(s)\xi'_{-s}(m')\phi(wm'w^{-1}) = M(s)\xi'_{-s}(m')\phi'(m').$$

Recall that  $m = wm'w^{-1}$ . Of course  $\phi'$  is a function on  $A'_\mathbf{A} \backslash M'_\mathbf{A}$  and thus a function on  ${}^0G'_\mathbf{A}$ . It satisfies the same conditions as  $\phi$ . Associated to it is the function

$$M'(s) = \prod_{i=1}^{r'} \frac{\xi(a'_i s, \tilde{\pi}'_i, \phi')}{\xi(a'_i s + 1, \tilde{\pi}'_i, \phi')}.$$

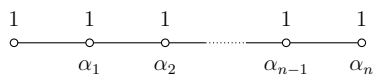
But  $m' \rightarrow wm'w^{-1}$  defines an isomorphism of  $M'$  with  $M$  and an isomorphism of  ${}^0G'$  with  ${}^0G$ . Thus  $\phi$  and  $\phi'$  are essentially the same. Moreover  ${}^c({}^0G')$  and  ${}^c({}^0G)$  are isomorphic, such that a representation of  ${}^c({}^0G')$  may be regarded as a representation of  ${}^c({}^0G)$ . Recalling that the elements of the adjoint group of  ${}^cG$  are orthogonal with respect to the Killing form and that the Killing form turns  $\text{Ad } w({}^c\mathfrak{n}')$  into the dual of  ${}^c\mathfrak{n}$ , one sees readily that  $r = r'$ , that, with a suitable order,  $a'_i = a_i$ , and that  $\pi'_i$  is the contragredient of  $\pi_i$ . Thus

$$M'(s) = \prod_{i=1}^r \frac{\xi(a_i s, \pi_i, \phi)}{\xi(a_i s + 1, \pi_i, \phi)}.$$

It is known that  $M(s)M'(-s) = 1$ . This is implied by, but does not imply, the relation  $\xi(s, \pi_i, \phi) = \xi(1 - s, \tilde{\pi}_i, \phi)$ .

In the examples we shall give the Dynkin diagram of  $G$  with the points belonging to the Dynkin diagram of  ${}^0G$  labeled. We give the number  $r$ , the numbers  $a_i$ , and the highest weight  $\lambda_i$  of the representations  $\pi_i$  as a linear combination of the fundamental weights  $\delta_j$ . In the examples considered  $\pi_i$  is always irreducible. Do not forget that  $\pi_i$  is not a representation of  ${}^0G$  but a representation of  ${}^c({}^0G)$ .

(i)

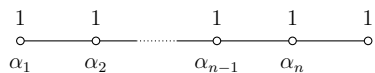


$r = 1$

$a_1 = n + 2$

$\lambda_1 = \delta_n$

(ii)

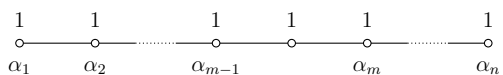


$r = 1$

$a_1 = n + 2$

$\lambda_1 = \delta_1$

(iii)

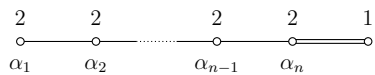


$r = 1$

$a_1 = n + 2$

$\lambda_1 = \delta_1 + \delta_n$

(iv)

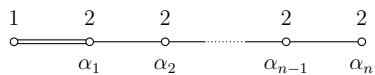


$r = 1$

$a_1 = 2(n + 1)$

$\lambda_1 = 2\delta_1$

(v)

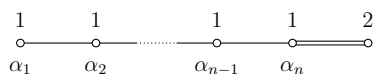


$r = 1$

$a_1 = 2(n + 1)$

$\lambda_1 = 2\delta_n$

(vi)



$r = 2$

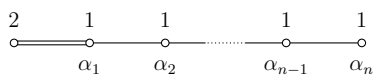
$a_1 = 2(n + 2)$

$\lambda_1 = \delta_2$

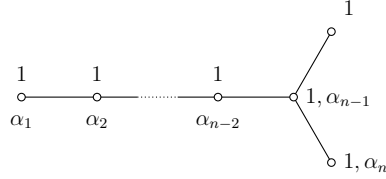
$a_2 = n + 2$

$\lambda_2 = \delta_1$

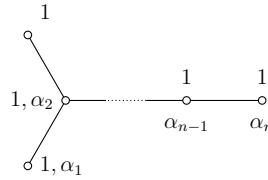
(vii)



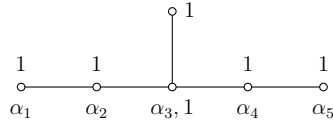
(viii)  $r = 2$   $a_1 = 2(n + 2)$   $\lambda_1 = \delta_{n-1}$   
 $a_2 = n + 2$   $\lambda_2 = \delta_n$



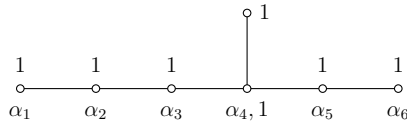
(ix)  $r = 1$   $a_1 = 2n$   $\lambda_1 = \delta_2$



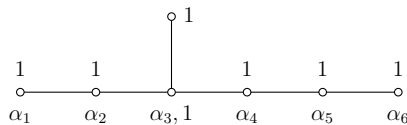
(x)  $r = 1$   $a_1 = 2n$   $\lambda_1 = \delta_{n-1}$



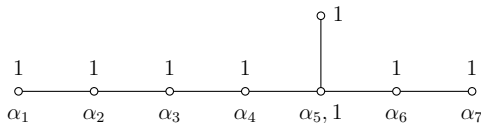
(xi)  $r = 2$   $a_1 = 11$   $\lambda_1 = \delta_3$   
 $a_2 = 22$   $\lambda_2 = 0$



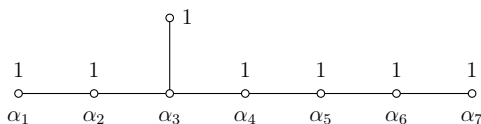
(xii)  $r = 2$   $a_1 = 14$   $\lambda_1 = \delta_3$   
 $a_2 = 28$   $\lambda_2 = \delta_6$



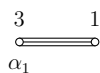
(xiii)  $r = 2$   $a_1 = 14$   $\lambda_1 = \delta_4$   
 $a_2 = 28$   $\lambda_2 = \delta_1$



(xiv)  $r = 3$   $a_1 = 51$   $\lambda_1 = \delta_1$   
 $a_2 = 34$   $\lambda_2 = \delta_6$   
 $a_3 = 17$   $\lambda_3 = \delta_3$

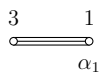


(xv)  $r = 3$   $a_1 = 51$   $\lambda_1 = \delta_1$   
 $a_2 = 34$   $\lambda_2 = \delta_2$   
 $a_3 = 17$   $\lambda_3 = \delta_5$

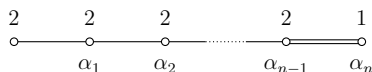


(xvi)  $r = 2$   $a_1 = 10$   $\lambda_1 = 0$   
 $a_2 = 5$   $\lambda_2 = 3\delta_1$

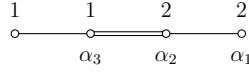
This example is particularly striking.



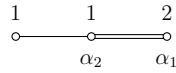
(xvii)  $r = 3$   $a_1 = 3$   $\lambda_1 = \delta_1$   
 $a_2 = 6$   $\lambda_2 = 0$   
 $a_3 = 9$   $\lambda_3 = \delta_1$



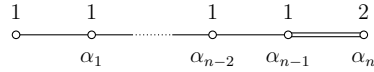
(xviii)  $r = 2$   $a_1 = 2n + 1$   $\lambda_1 = 0$   
 $a_2 = 4n + 2$   $\lambda_2 = \delta_1$



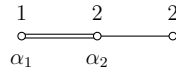
(xix)  $r = 2$   $a_1 = 22$   $\lambda_1 = 0$   
 $a_2 = 11$   $\lambda_2 = \delta_3$



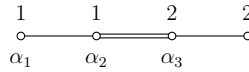
(xx)  $r = 1$   $a_1 = 4$   $\lambda_1 = \delta_2$



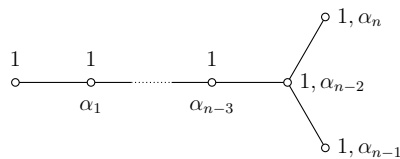
(xxi)  $r = 1$   $a_1 = 2(n + 1)$   $\lambda_1 = \delta_1$



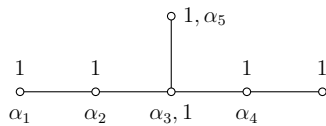
(xxii)  $r = 2$   $a_1 = 5$   $\lambda_1 = 0$   
 $a_2 = 10$   $\lambda_2 = \delta_2$



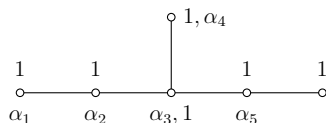
(xxiii)  $r = 2$   $a_1 = 16$   $\lambda_1 = \delta_1$   
 $a_2 = 8$   $\lambda_2 = \delta_3$



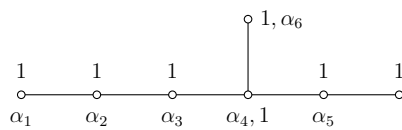
(xxiv)  $r = 1$   $a_1 = 2n$   $\lambda_1 = \delta_1$



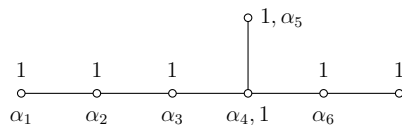
(xxv)  $r = 1$   $a_1 = 12$   $\lambda_1 = \delta_5$



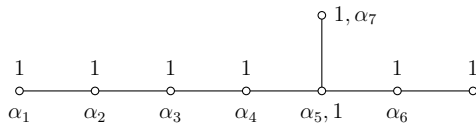
(xxvi)  $r = 1$   $a_1 = 12$   $\lambda_1 = \delta_4$



(xxvii)  $r = 2$   $a_1 = 34$   $\lambda_1 = 0$   
 $a_2 = 17$   $\lambda_2 = \delta_5$



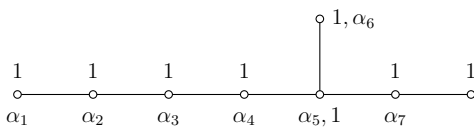
(xxviii)  $r = 2$   $a_1 = 34$   $\lambda_1 = 0$   
 $a_2 = 17$   $\lambda_2 = \delta_6$



$r = 2$   $a_1 = 46$   $\lambda_1 = \delta_1$   
 $a_2 = 23$   $\lambda_2 = \delta_7$



(xxix)



$r = 2$

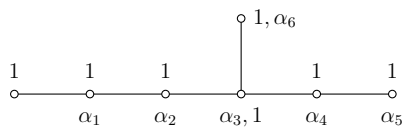
$a_1 = 46$

$\lambda_1 = \delta_1$

$a_2 = 23$

$\lambda_2 = \delta_6$

(xxx)

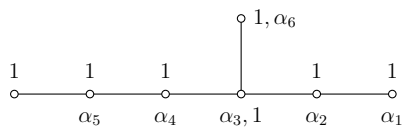


$r = 1$

$a_1 = 18$

$\lambda_1 = \delta_5$

(xxxii)

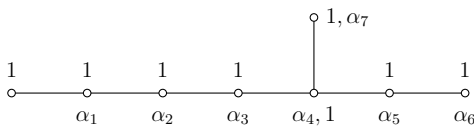


$r = 1$

$a_1 = 18$

$\lambda_1 = \delta_1$

(xxxiii)



$r = 2$

$a_1 = 29$

$\lambda_1 = \delta_1$

$a_2 = 58$

$\lambda_2 = 0$

Compiled on December 22, 2023.