The Extension Problem for Compact Submanifolds of Complex Manifolds I

(The Case of a Trivial Normal Bundle)*

By

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Let X be a compact, complex submanifold of a V. We wish to consider over X certain *analytic objects*, such as: (i) a holomorphic vector bundle $E \to X$ (the notations are explained in § 1 below); (ii) a subspace $S \subset H^q(X, \mathscr{E})$; or (iii) a holomorphic mapping $f: X \to Y$ for some complex manifold Y. The extension problem we consider is, given an analytic object α over X, to find a corresponding analytic object β over V such that β restricted to X gives α.

We shall be primarily interested in the extension problem when V is a *germ of a neighborhood* of X, a concept which we now make precise. Let $\mathcal{O}$ be the sheaf of local rings of holomorphic functions on V and $\mathscr{I} \subset \mathcal{O}$ the ideal sheaf of X. Denote by $\mathscr{I}^\mu$ the μ^{th} power of $\mathscr{I}$ and set $\mathcal{O}^\mu = \mathcal{O}/\mathscr{I}^{\mu+1}$ (sheaf of jets of order μ in the normal parameter along X). The pair $(X, \mathcal{O}^\mu)$ then forms a ringed space X^μ, $X^\circ = X$. Also we set $\mathcal{O}^* = \mathcal{O}\,|\,X$ and denote by X^* the generalized complex space $(X, \mathcal{O}^*)$. Then X^* is a germ of a neighborhood of X and X^μ is a neighborhood of order μ of X in X^*.

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Suppose now that we have an extension α^μ to X^μ of the analytic object α on X. Then the obstruction to extending α^μ to $X^{\mu+1}$ is given by a cohomology class $\omega(\alpha^\mu)$. In case (i), $\omega(\alpha^\mu) \in H^2(X, \text{Hom}(\mathscr{E}, \mathscr{E}) \otimes \otimes \mathscr{I}^{\mu+1}/\mathscr{I}^{\mu+2})$; in case (ii), $\omega(\alpha^\mu) \in H^{q+1}(X, \mathscr{E} \otimes \mathscr{I}^{\mu+1}/\mathscr{I}^{\mu+2})$; and in (iii) $\omega(\alpha^\mu) \in H^1(X, \text{Hom}(\mathscr{T}(V), \mathscr{T}(Y)) \otimes \mathscr{I}^{\mu+1}/\mathscr{I}^{\mu+2})$.

There are two general statements which may hold for analytic objects of types (i), (ii), or (iii) and a germ of embedding $X \subset V$; these are

(I) there are only finitely many obstructions to the extension problem;

(II) a local extension exists if, and only if, a formal extension exists.

Of course, (I) and (II) are not always true; there are easy counter-examples to (I) and HIRONAKA has a counter-example to (II). Our program is to investigate the extension problem and its applications after making assumptions on the normal bundle N of X in V.

In this paper we shall essentially assume that N is trivial; this means, at least when $H^1(X, \mathcal{O}) = 0$, that V may be considered as an analytic fibre space over an analytic set D with one fibre being X. Thus the techniques in the theory of deformations of complex structure ([7] and [9]) are available to treat the extension problem. We are then able to prove I and II for analytic objects of types (i) or (ii) and to also derive several other results peculiar to the case of a trivial normal bundle.

For example, suppose that D is non-singular and of dimension 1 with parameter t. Write $\mathscr{V} = \bigcup_{t \in D} X_t$ where $X_0 = X$ and the X_t are the fibres of the projection of V onto D. Let $\mathscr{S}$ be a locally free analytic sheaf on V and $\mathscr{S}_0 = \mathscr{S}/I \cdot \mathscr{S}$. Denote by E^q the subspace of $H^q(X, \mathscr{S}_0)$ composed of extendible classes, and denote by J^q the subspace of E^q composed of extendible classes whose restrictions to X_t vanish for $t \neq 0$. (These may be called the jump classes.) Then there is a natural isomorphism

$$H^q(X, \mathscr{S}_0)/E^q \cong J^{q+1}.$$

If a class belongs to E^q, it is represented by a q-cocycle Z_t and, if it belongs to J^q, then Z_t is the coboundary of a $(q-1)$-cochain which has a pole at the point $t = 0$. If $\dim D = m > 1$, then Z_t is the coboundary of a $(q-1)$-cochain which has as polar locus an analytic set of dimension $(m-1)$ through 0.

As another illustration of our results, we are able to construct, for an analytic object α of type (i) or (ii), a maximal analytic subset V_α of V with $X \subset V_\alpha$ and such that α may be extended to V_α.

1. Notations and terminology

The basic object on which we shall work will be a compact, complex manifold. Let J be the almost-complex structure tensor of X acting on the complex tangent bundle $\boldsymbol{T}^\#(X)$; then $\boldsymbol{T}^\#(X) = \boldsymbol{T} + \boldsymbol{T}^*$ where $\boldsymbol{T}$ is the complex tangent bundle, which is the $\sqrt{-1}$ eigenbundle of J, and $\boldsymbol{T}^*$ is the conjugate bundle. A general holomorphic vector bundle over X will be written $E \to \boldsymbol{E} \to X$ where E, a complex vector space, is a typical fibre and $\boldsymbol{E}$ is the total space. The dual bundle is denoted by $E' \to \boldsymbol{E}' \to X$ and $\mathscr{E}$ is the sheaf of germs of holomorphic cross-sections of $E \to \boldsymbol{E} \to X$. We use the standard conventions: $\mathscr{T} = \Theta$, $\Lambda^P \mathscr{T}' = \Omega^P$, and $\mathcal{O}_X =$ sheaf of germs of holomorphic functions. For a holomorphic vector bundle $E \to \boldsymbol{E} \to X$, $\mathscr{A}^q(\boldsymbol{E})$ is the sheaf of germs of C^∞ $\boldsymbol{E}$-valued (o, q) forms over X.

We denote holomorphic principal bundles by $G \to P \xrightarrow{\pi} X$ where the complex Lie group G acts holomorphically on the right on the total space P. Over X, we have the holomorphic vector bundle $\boldsymbol{Q} = \boldsymbol{T}(P)/G$, and there is an onto bundle homomorphism $\pi: \boldsymbol{Q} \to \boldsymbol{T}$ with kernel $\boldsymbol{L} = P \times_G \mathfrak{g}$ where $\mathfrak{g}$ is the complex Lie algebra of G. Thus, we have the *fundamental bundle sequence* [*1*]

$$0 \to \boldsymbol{L} \to \boldsymbol{Q} \to \boldsymbol{T} \to 0\,.$$

The sheaf $\mathscr{A}^\circ(\boldsymbol{Q})$ ($= C^\infty$ germs of sections of $\boldsymbol{Q}$) acts on $\mathscr{A}^\circ(\boldsymbol{E})$ as follows: a germ σ in $\mathscr{A}^\circ(\boldsymbol{E})$ is given by an E-valued C^∞ function $\hat{\sigma}$ on P satisfying $\hat{\sigma}(p \cdot g) = \varrho(g)\hat{\sigma}(p)\,(p \in P, g \in G)$. Let $\hat{\xi}$ be a germ of a right-invariant vector field on P ($=$ germ ξ in $\mathscr{A}^\circ(\boldsymbol{Q})$); then $\hat{\xi} \cdot \hat{\sigma}$ is again an E-valued C^∞ function on P satisfying the equivariance condition, and $\widehat{\xi \cdot \sigma} = \hat{\xi} \cdot \hat{\sigma}$. This action may be extended to a pairing $[\ ,\ ]: \mathscr{A}^p(\boldsymbol{Q}) \otimes \mathscr{A}^q(\boldsymbol{E}) \to \mathscr{A}^{p+q}(\boldsymbol{E})$ ([*11*]). In particular, we get a pairing $[\ ,\ ]: A^p(\boldsymbol{T}) \otimes A^q(\boldsymbol{T}) \to A^{p+q}(\boldsymbol{T})$.

A *deformation* $\{\mathscr{V} \xrightarrow{\tilde{\omega}} D\}$ of X is given by the following data: (i) An analytic subset D of an open neighborhood U of the origin in $\boldsymbol{C}^m$; (ii) An analytic space $\mathscr{V}$ and a proper holomorphic mapping $\tilde{\omega}: \mathscr{V} \to D$ such that $\tilde{\omega}$ has maximal rank and connected fibres $X_t = \tilde{\omega}^{-1}(t)\,(t \in D)$; and (iii) A holomorphic embedding $\iota: X \to \mathscr{V}$ such that $\tilde{\omega} \circ \iota = 0 \in D$. A mapping
$$\begin{array}{ccc} F: \mathscr{V} & \to & \mathscr{V}' \\ \downarrow \tilde{\omega} & & \downarrow \tilde{\omega}' \\ G: D & \to & D' \end{array}$$
between deformation spaces $\{\mathscr{V} \xrightarrow{\tilde{\omega}} D\}$ and $\{\mathscr{V}' \xrightarrow{\tilde{\omega}'} D'\}$ is given by a pair of holomorphic mappings $F: \mathscr{V} \to \mathscr{V}'$ and $G: D \to D'$ such that $\tilde{\omega}' \circ F = G \circ \tilde{\omega}$, $F \circ \iota = \iota'$, and such that F is biholomorphic on fibres.

For technical reasons, we introduce the notion of an *almost-complex deformation* $\{\mathscr{W} \xrightarrow{\tilde{\omega}} U\}$. This is given by:

(i) An open neighborhood U of 0 in $\boldsymbol{C}^m$;

(ii) An almost-complex manifold $\mathscr{W}$ and an almost-complex mapping $\tilde{\omega}: \mathscr{W} \to U$ such that $\tilde{\omega}$ has maximal rank, connected fibres, and such

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that each fibre $X_t = \tilde{\omega}^{-1}(t)\,(t \in U)$ is an almost-complex submanifold; and (iii) An almost-complex injection $\tau: X \to \mathscr{W}$ such that $\tilde{\omega} \circ \tau = 0 \in U$. If $\{\mathscr{V} \overset{\tilde{\omega}}{\to} D\}$ is a deformation, we say that
$$\begin{array}{ccc} F: \mathscr{V} & \to & \mathscr{W} \\ \downarrow \tilde{\omega} & & \downarrow \tilde{\omega} \\ G: D & \to & U \end{array}$$
embeds $\{\mathscr{V} \overset{\tilde{\omega}}{\to} D\}$ into $\{\mathscr{W} \overset{\tilde{\omega}}{\to} U\}$ if G is the injection of an analytic subset D of U into U, if F is differentiable embedding of $\mathscr{V}$ into $\mathscr{W}$ which induces an almost-complex injection on each fibre, and if the above diagram commutes. For any deformation $\{\mathscr{V} \overset{\tilde{\omega}}{\to} D\}$, we shall always *assume* that there exists an almost-complex deformation $\{\mathscr{W} \overset{\tilde{\omega}}{\to} U\}$ into which $\{\mathscr{V} \overset{\tilde{\omega}}{\to} D\}$ can be embedded.

Given a holomorphic principal bundle $G \to P \to X$, we define a *deformation* $\{G \to \mathscr{P} \overset{\pi}{\to} \mathscr{V} \overset{\tilde{\omega}}{\to} D\}$ (or just $\{G \to \mathscr{P} \to D\}$) to consist of a deformation $\{\mathscr{V} \overset{\tilde{\omega}}{\to} D\}$ of X together with a holomorphic principal bundle $G \to \mathscr{P} \overset{\pi}{\to} \mathscr{V}$ such that $\iota^{-1}(G \to \mathscr{P} \to \mathscr{V}) = G \to P \to X$. The auxiliary discussion above about deformations of X may now be carried over to deformations of $G \to P \to X$. In particular, the analogue of the assumption about the existence of ambient almost-complex deformations of X will be assumed to hold for deformations of $G \to P \to X$.

1.1. Graded complexes and Lie algebras

We recall that a *graded Lie algebra* is given by a graded vector space $A = \sum_{p \geq 0} A^p$ together with a bracket operation $[\,,\,]: A^p \otimes A^q \to A^{p+q}$ such that:

$$[\varphi, \psi] = (-1)^{pq+1}[\psi, \varphi] \qquad (\varphi \in A^p,\ \psi \in A^q) \tag{1.1}$$

$$\begin{gathered}(-1)^{pr}[\varphi,[\psi,\eta]] + (-1)^{qr}[\eta,[\varphi,\psi]] + (-1)^{pq}[\psi,[\eta,\varphi]] = 0 \\ (\varphi \in A^p,\ \psi \in A^q,\ \eta \in A^r)\,.\end{gathered} \tag{1.2}$$

The notion of a homomorphism between graded Lie algebras is clear. If $K = \sum_{q \geq 0} K^q$ is a graded vector space, we say that K is over the graded Lie algebra A if there is a pairing $[\,,\,]: A^p \otimes K^q \to K^{p+q}$ such that

$$\begin{gathered}[\varphi,[\psi,\gamma]] + (-1)^{pq+1}[\psi,[\varphi,\gamma]] = [[\varphi,\psi],\gamma] \\ (\varphi \in A^p,\ \psi \in A^q,\ \gamma \in K)\,.\end{gathered} \tag{1.3}$$

We define $K = \sum_{q \geq 0} K^q$ to be a *graded complex* (K, δ) if there are linear mappings $\delta: K^q \to K^{q+1}$ satisfying $\delta \circ \delta = 0$. We set $Z(K^p) =$ kernel $\delta: K^p \to K^{p+1}$, $H^p(K) = Z(K^p)/\delta K^{p-1}$, and $H(K) = \sum_{p \geq 0} H^p(K)$.

A graded Lie algebra A which is also a graded complex (A, d) is called a *graded Lie algebra complex* if the following rule holds:

$$d[\varphi, \psi] = [d\varphi, \psi] + (-1)^p [\varphi, d\psi] \qquad (\varphi \in A^p,\ \psi \in A)\,. \tag{1.4}$$

Finally, if (K, δ) is a graded complex and (A, d) is a graded Lie algebra complex, then (K, δ) is over (A, d) if K is over A and if

$$\delta[\varphi, \gamma] = [d\varphi, \gamma] + (-1)^p [\varphi, \delta\gamma] \qquad (\varphi \in A^p,\ \gamma \in K)\,. \tag{1.5}$$

Suppose now that (K, δ) is over (A, d) as above, and let $t = (t^1, \ldots, t^m)$ be a variable point in $\boldsymbol{C}^m$. We write $K[t]$ for the graded vector space of formal power series in t with coefficients in K; $K\{t\}$ is the subspace of $K[t]$ consisting of the series with constant term equal to zero. We define $A[t]$ and $A\{t\}$ similarly, so that we have, e.g., $[A\{t\}, K[t]] \subset K\{t\}$. Let $\Phi(t) \in A^1\{t\}$.

Definition: We let

$$\Delta\Phi(t) = d\Phi(t) - [\Phi(t), \Phi(t)] \tag{1.6}$$

and say that $\Phi(t)$ is *integrable* if $\Delta\Phi(t) = 0$.

Let now $\Phi(t) \in A^1\{t\}$ and define $\delta_\Phi : K^q[t] \to K^{q+1}[t]$ by

$$\delta_\Phi(\Gamma(t)) = \delta\Gamma(t) - 2[\Phi(t), \Gamma(t)] \quad (\Gamma(t) \in K^q[t])\,. \tag{1.6$'$}$$

Lemma 1.1. $\delta_\Phi \circ \delta_\Phi \Gamma(t) = -2[\Delta\Phi(t), \Gamma(t)]$.

Proof:
$$\begin{aligned}
\delta_\Phi(\delta_\Phi\Gamma(t)) &= \delta_\Phi(\delta\Gamma(t) - 2[\Phi(t), \Gamma(t)]) \\
&= \delta^2\Gamma(t) - 2\,\delta[\Phi(t), \Gamma(t)] - 2[\Phi(t), \delta\Gamma(t)] \\
&+ 4[\Phi(t), [\Phi(t), \Gamma(t)]] = -2[\delta\Phi(t), \Gamma(t)] \\
&+ 2[[\Phi(t), \Phi(t)], \Gamma(t)] \quad \text{by} \quad (1.4) \text{ and } (1.2). \qquad \text{Q.E.D.}
\end{aligned}$$

If we now assume that $\Delta\Phi(t) = 0$, then $(K[t], \delta_\Phi)$ becomes again a graded complex; we set $Z^q{}_\Phi(K) = Z(K^q[t])$, $H^q{}_\Phi(K) = H^q(K[t])$, where the differential operator δ_Φ is given by (1.6). Our immediate goal is to study the relationship between $H^q(K)$ and $H^q{}_\Phi(K)$.

1.2. The formal extension problem in cohomology

Let (K, δ) be a graded complex over a graded Lie algebra complex (A, d), and let $\Phi(t) \in A^1\{t\}$ be an integrable element. Given $\gamma \in Z(K^q)$, we say that γ is *extendible* if there exists $\Gamma(t) \in K^q[t]$ such that $\Gamma(0) = \gamma$ and $\delta_\Phi\Gamma(t) = 0$. If $\sigma \in K^{q-1}$, then $\delta_\Phi(\sigma)$ is an extension of $\delta(\sigma)$, so that the following is justified:

Definition. A class $[\gamma] \in H^q(K)$ is *extendible* if there exists $\gamma \in Z(K^q)$ which represents $[\gamma]$ and is itself extendible.

Suppose now that $s \in \boldsymbol{C}$ is a single complex parameter, and let $K(s)$ be the formal power series in s and with coefficients in K which have finitely many terms with negative s-exponents; a $\sum(s) \in K(s)$ is thus written as

$$\sum(s) = \sum_{\mu=-N}^{\infty} \gamma_\mu s^\mu \qquad (\gamma_\mu \in K).$$

Definition. Let $\Gamma(s) \in Z^q_\Phi(K)$ be an extension of $\gamma \in Z(K^q)$. Then Γ is said to be a *jump extension* if there exists $\sum \in K^{q-1}(s)$ with $\delta_\Phi \sum = \Gamma$. We say that $\gamma \in Z(K^q)$ is a *jump cocycle* if there exists a jump extension Γ of γ.

If $\sigma \in K^{q-1}$, then $\delta_\Phi \sigma$ is a jump extension of $\delta\sigma$, so that we may define what is meant by a *jump class* in $H^q(K)$.

We let $E^q_\Phi \subset H^q(K)$ be the subspace of extendible classes, and $J^q_\Phi \subset E^q_\Phi$ the space of jump classes. Our first main result is the following

Theorem 1.1. *Assume that $H(K)$ is finite dimensional.*
(i) *There are only finitely many obstructions to extending a class in $H^q(K)$;*
(ii) *If $G^q_\Phi = H^q(K)/E^q_\Phi$, then there are on G^q_Φ and J^{q+1}_Φ canonical filtrations whose associated graded modules are naturally isomorphic.*

(1.2) Proof of Theorem 1.1. Let $\gamma \in Z(K^q)$. We wish to find $\Gamma(s) = \sum_{\mu=0}^{\infty} \gamma_\mu s^\mu$ with $\gamma_0 = \gamma$ and $\delta_\Phi \Gamma = 0$. Thus we must recursively solve the equations

$$\delta\gamma_N = 2 \sum_{\substack{\sigma + \iota = N \\ \sigma > 0}}^{\infty} [\varphi \ , \gamma_\iota] \qquad (1.7)^N$$

with $\gamma_0 = \gamma$, and where we have written $\Phi(s) = \sum_{\nu=1}^{\infty} \varphi_\nu s^\nu$. Suppose that we have $\gamma_0, \dots, \gamma_{N-1}$ such that $(1.7)^\mu$ is satisfied for $1 \leqq \mu \leqq N-1$; we say then that γ is *extendible to order* $N-1$. Let

$$\omega_N = 2 \sum_{\sigma+\iota=N} [\varphi_\sigma, \gamma_\iota]; \qquad (1.8)$$

then ω_N is the N^{th} *obstruction to extending* γ, or the *obstruction to extending γ to order N.*

Lemma 1.2. $\omega_N \in K^{q+1}$ and $\delta\omega_N = 0$.

Proof. We use (1.3) and (1.5) to calculate:

$$\begin{aligned}\delta\omega_N &= 2 \sum_{\sigma+\iota=N} [d\varphi_\sigma, \gamma_\iota] - \sum_{\sigma+\iota=N} [\varphi_\sigma, \delta\gamma_\iota] \\ &= 2 \sum_{\varrho+\sigma+\iota=N} [[\varphi_\varrho, \varphi_\sigma], \gamma_\iota] - 4 \sum_{\varrho+\sigma+\iota=N} [\varphi_\varrho, [\varphi_\sigma, \gamma_\iota]] = 0.\end{aligned}$$

In the middle step we have used $(1.7)^\mu$ for $1 \leqq \mu \leqq N-1$. Q. E. D.

We come now to the main points in the formal theory.

Lemma 1.3. $\omega \in Z(K^{q+1})$ is a jump cocycle if, and only if, ω is an obstruction to extending some $\gamma \in Z(K^q)$.

Proof. Suppose first that ω is an obstruction. Then there exists $\gamma = \gamma_0$, $\gamma_1, \ldots, \gamma_{N-1} \in K^q$ with $\delta\gamma = 0$, $\delta\gamma_\mu = 2\sum_{\sigma+\iota=\mu} [\varphi_\sigma, \gamma_\iota]$ $(1 \leqq \mu \leqq N-1)$, and $\omega = 2\sum_{\sigma+\iota=N} [\varphi_\sigma, \gamma_\iota]$. Let $\gamma^{N-1}(s) = \sum_{\mu=0}^{N-1} \gamma_\mu s^\mu$, and define $\sum(s) \in K^q[D]$ by $\sum(s) = \underset{s}{\perp}{}_N\ \gamma^{N-1}(s)$. Then $\delta_\Phi \sum(s) = \delta\sum(s) - 2[\Phi(s), \sum(s)]$ $= \underset{s}{\perp}{}_N(\delta\gamma^{N-1}(s) - 2[\Phi(s), \gamma^{N-1}(s)]) = \underset{s}{\perp}{}_N(\omega s^N + 0(s^{N+1}))$ $(0(s^{N+1})$ $=$ terms divisible by $s^{N+1}) = \Omega(s) \in K^{q+1}(D)$. Since $\Omega(0) = \omega$, the pair $(\Omega(s), \sum(s))$ makes ω into a jump cocycle.

Now suppose conversely that ω is a jump cocycle. Then there exists $\Omega(s) \in K^{q+1}(D)$, $\sum(s) \in K^q[D]$ such that:

(i) $\Omega(0) = \omega$;

(ii) $\delta_\Phi \Omega(s) = 0$; and

(iii) $\delta_\Phi \sum(s) = \Omega(s)$ $(s \neq 0)$.

Let N be the order of the pole of $\sum(s)$ at $s = 0$, and define $\gamma \in Z(K^q)$ by $\gamma = \lim_{s\to 0} s^N \sum(s)$. We may then reverse the above argument to find that ω is an N^{th} obstruction to extending $\gamma \in Z(K^q)$. Q. E. D.

Now the primary obstruction to extending $\gamma \in Z(K^q)$ is a well-defined class in $H^{q+1}(K)$, depending only on $[\gamma] \in H^q(K)$; this obstruction is given by $\omega_1 = [\varphi_1, \gamma]$. However, the higher obstructions do not depend upon $[\gamma] \in H^q(K)$ alone; if e. g., $\omega_1 = \delta\gamma_1$, then $\omega_2 = [\varphi_1, \gamma_1] + [\varphi_2, \gamma]$ is *a* secondary obstruction. But so is $\omega_2' = [\varphi_1, \gamma_1 + \sigma] + [\varphi_2, \gamma]$ for any $\sigma \in Z(K^q)$. We now show how to deal with this situation. First we observe:

Lemma 1.4. If the primary obstruction to extending $\gamma \in Z(K^q)$ is a non-zero class in $H^{q+1}(K)$, then no solution of the extension problem exists for γ.

Lemma 1.5. If ω is an N^{th} obstruction to extending $\gamma \in Z(K^q)$, and if, by different choices, γ is extendible to order N, then ω is an $N-1^{\text{st}}$ obstruction to extending some $\varrho \in Z(K^q)$.

Proof. We are given: (i) $\gamma = \gamma_0, \gamma_1, \ldots, \gamma_{N-1} \in K^q$ such that $(1.7)^\mu$ is satisfied for $1 \leqq \mu \leqq N-1$ and such that $\omega = 2\sum_{\sigma+\tau=N} [\varphi_\sigma, \gamma_\tau]$; and (ii) $\gamma = \gamma_0', \gamma_1', \ldots, \gamma_N' \in K^q$ such that $(1.7)^\mu$ is satisfied for $1 \leqq \mu \leqq N$.

Let $\varrho = \varrho_0 = \gamma_1 - \gamma_1'$, $\varrho_\mu = \gamma_{\mu+1}'$ $(1 \leqq \mu \leqq N-2)$. Then $\delta\varrho = \delta\gamma_1 - \delta\gamma_1' = [\varphi_1, \gamma] - [\varphi_1, \gamma] = 0$. Also, for $1 \leqq \mu \leqq N-2$, $2\sum_{\sigma+\tau=\mu} [\varphi_\sigma, \varrho_\tau]$ $= 2(\sum_{\sigma+\tau=\mu+1} [\varphi_\sigma, \gamma_\tau] - [\varphi_\sigma, \gamma_\tau']) = \delta\gamma_{\mu+1}' - \delta\gamma_{\mu+1}' = \delta\varrho_\mu$. Thus ϱ is extended to order $N-2$. But $2\sum_{\sigma+\tau=N-1}[\varphi_\sigma, \varrho_\tau] = 2\sum_{\sigma+\tau=N}[\varphi_\sigma, \gamma_\tau] - [\varphi_\sigma, \gamma'_\tau] = \omega - \delta\gamma'_N$ and thus ω is an $N-1^{\text{st}}$ obstruction to extending $\varrho \in Z(K^q)$. Q. E. D.

Corollary. If $\omega \in Z(K^{q+1})$ is a jump cocycle then there exists $\gamma^{\#} \in Z(K^q)$ such that the extension problem cannot be solved for $\gamma^{\#}$ and such that ω is an N^{th} obstruction to extending $\gamma^{\#}$ where N is maximal, i. e., there exists no extension of order N of $\gamma^{\#}$.

For $\omega \in Z(K^{q+1})$, $\gamma \in Z(K^q)$, we set: $N(\omega) = \inf\{N \in Z^+ \mid \omega$ is an N^{th} obstruction to extending some $\gamma' \in Z(K^q)\}$, $N^{\#}(\gamma) = \sup\{N \in Z^+ \mid \gamma$ is extendible to order $N-1\}$.

Lemma 1.6. (i) If ω is an obstruction of order $N(\omega)$ to extending $\gamma \in Z(K^q)$, then $N^{\#}(\gamma) = N(\omega)$. (ii) If ω is an obstruction of order $N^{\#}(\gamma)$ to extending $\gamma \in Z(K^q)$, then $N(\omega) = N^{\#}(\gamma)$.

Proof. (i) is just a restatement of Lemma 1.5. To see (ii), we assume that $N(\omega) < N^{\#}(\gamma)$ (since, in any case, $N(\omega) \leqq N^{\#}(\gamma)$); and we let $r = N^{\#}(\gamma) - N(\omega) > 0$. We know the following: (a) there exists $\gamma = \gamma_0, \gamma_1, \dots, \gamma_{N-1} (N = N^{\#}(\gamma))$ such that $(1.7)^\mu$ is satisfied for $1 \leqq \mu \leqq N-1$ and $\omega = 2\sum_{\sigma+\tau=N}[\varphi_\sigma, \gamma_\tau]$; (b) there exists $\varrho = \varrho_0, \varrho_1, \dots, \varrho_{M-1} (M = N(\omega))$ such that $(1.7)^\mu$ is satisfied for $1 \leqq \mu \leqq M-1$ and $\omega = 2\sum_{\sigma+\tau=M}[\varphi_\sigma, \varrho_\tau]$. Define now $\gamma = \gamma'_0, \gamma'_1, \dots, \gamma'_{N-1} \in K^q$ as follows: $\gamma'_\mu = \gamma_\mu$ for $0 \leqq \mu \leqq r-1$; and $\gamma'_\nu = \nu_\nu - \varrho_{\nu-r} (r \leqq \nu \leqq N-1)$. Then, for $1 \leqq \mu \leqq r-1$, $(1.7)^\mu$ is satisfied. For $r \leqq \nu \leqq N-1$, $\sum_{\sigma+\tau=\nu}[\varphi_\sigma, \gamma'_\tau] = \sum_{\sigma+\tau=\nu}[\varrho_\sigma, \gamma'_\tau] - \sum_{\sigma+\tau=\nu-r}[\varphi_\sigma, \varrho_\tau] = \delta\gamma_\nu - \delta\varrho_{\nu-r} = \delta(\gamma'_\nu)$. But, also, we have: $\sum_{\sigma+\tau=N}[\varphi_\sigma, \gamma'_\tau] = \sum_{\sigma+\tau=N}[\varphi_\sigma, \gamma_\tau] - \sum_{\sigma+\tau=N-r-M}[\varphi_\sigma, \varrho_\tau] = \omega - \omega = 0$. Thus γ is extendible to order N, which is a contradiction. Q. E. D.

Now J^{q+1}_Φ is a finite dimensional vector space; we choose a basis $\omega_1, \dots, \omega_r$ and set $N_i = N(\omega_i)$ for $1 \leqq i \leqq R$. Then, to each ω_i, we may associate $\Omega_i(s) \in K^{q+1}[s]$, $\Sigma_i(s) \in K^q(s)$ such that $\Omega_i(0) = \omega_i$, $\delta_\Phi \Sigma_i = \Omega_i$, and $\Sigma_i(s)$ has a pole of order N_i in s. Since $N(\omega + \omega') \leqq \max(N(\omega), N(\omega'))$, if we set $N(q+1) = \max_{1 \leqq i \leqq R} N_i$, then, for each $\omega \in J^{q+1}_\Phi$, we may associate $\Omega(s) \in K^{q+1}[s]$, $\Sigma(s) \in K^q(s)$ such that $\Omega(0) = \omega$, $\delta_\Phi \Sigma = \Omega$, and such that $\Sigma(s)$ has a pole of order less than or equal to $N(q+1)$ in s.

Lemma 1.7. Let $\gamma \in Z(K^q)$. If, for some $N \geqq N(q+1)$, γ has an extension of order N, then γ may be extended.

Proof. If not, then $N(\gamma) < \infty$ and $N(\gamma) > N(q+1)$. Thus there exists $\omega \in J^{q+1}_\Phi$ which is an obstruction of order $N(\gamma)$ to extending γ. By Lemma 1.6, $N(\omega) = N(\gamma) > N(q+1)$, which is a contradiction. Q.E.D.

It is in the sense of Lemma 1.7 that there are only finitely many obstructions to the extension of cohomology.

Set now $M(q) = \sup_{\gamma \in G^q_\Phi} N(\gamma)$; by Lemma 1.7, $M(q) < \infty$. We define a filtration $\{G^q_M\}_{1 \leqq M \leqq M(q)}$ of G^q_Φ as follows: $G^q_M = \{\gamma \in G^q_\Phi : N^{\#}(\gamma) \geqq M\}$. Since $N^{\#}(\gamma + \gamma') \geqq \min(N^{\#}(\gamma), N^{\#}(\gamma'))$, the G^q_M are subspaces of G^q_Φ. Clearly $G_{M+1}^{q} \subset G^q_M$, and we set $G^q(M) = G^q_M / G_{M+1}^{q}$.

We also define a filtration $\{J^{q+1}_N\}_{1 \leqq N \leqq N(q+1)}$ of J^{q+1}_Φ: For $1 \leqq N \leqq$ $\leqq N(q+1)$, $J^{q+1}_N = \{\omega \in J^{q+1}_\Phi : N(\omega) \leqq N\}$. Clearly J^{q+1}_N is a vector subspace of J^{q+1}_Φ, and $J^{q+1}_{N+1} \supset J^{q+1}_N$. Set $J^{q+1}(N) = J^{q+1}_N / J^{q+1}_{N-1}$.

Lemma 1.8. Let Θ^N be the mapping which associates to each $\gamma \in G^q(N)$ its obstruction of order N in $J^{q+1}(N)$. Then Θ_N is well-defined, linear, and is an isomorphism for each N.

This result follows from Lemmas 1.2—1.7, and completes the proof of Theorem 1.1.

2.1. Norms and a harmonic theory

On our graded complexes (K, δ) we shall now assume the existence of a graded Banach space with norm $\| \ \|$ such that $\delta : K^q \to K^{q+1}$ is a bounded operator. In the case of a graded Lie algebra complex (A, d), we also assume that $[,] : A^p \times A^q \to A^{p+q}$ is bounded in each factor for $p + q \geqq 1$.

We shall also assume on (K, δ) a harmonic theory, given by linear transformations $\delta^* : K^{q+1} \to K^q$, $\boldsymbol{G} : K^q \to K^q$, and a projection $\pi_H : K^q \to \boldsymbol{H}^q$ onto a finite dimensional subspace $\boldsymbol{H}^q \subset K^q$ such that the following hold:

(i) $\delta^* \circ \delta^* = 0$,

(ii) $\delta \circ \boldsymbol{G} = \boldsymbol{G} \circ \delta$ and $\delta^* \circ \boldsymbol{G} = \boldsymbol{G} \circ \delta^*$,

(iii) $\pi_H \circ \boldsymbol{G} = \boldsymbol{G} \circ \pi_H = \pi_H \circ \delta = \delta \circ \pi_H = \pi_H \circ \delta^* = \delta^* \circ \pi_H = 0$,

and (iv) every $\gamma \in K^q$ has a unique representation (Hodge decomposition)

$$\gamma = \pi_H(\gamma) + \delta^* \delta \boldsymbol{G}(\gamma) + \delta \delta^* \boldsymbol{G}(\gamma) . \tag{2.1}$$

We also assume that $\delta^* \boldsymbol{G} : K^q \to K^q$ is bounded so that we have a Banach space direct sum decomposition $K^q = \boldsymbol{H}^q \oplus \delta K^{q-1} \oplus \delta^* K^{q+1}$. We denote by π_H, π_δ, and π_{δ^*} the respective projection operators.

The concept of homomorphism between graded complexes or graded Lie algebra complexes will now refer to bounded transformations which commute with the harmonic theories.

Let now (K, δ) be over (A, d), and let $\Phi(t) \in A^1\{t\}$ be convergent for small t. Then, in addition to the cohomology groups $H^q(K)$ and $H^q_\Phi(K)$, we may obviously form a whole family $H^q_t(K)$ of cohomology groups for each fixed t $(H^q_0(K) = H^q(K))$. Our object is now to establish the relationship between these three.

2.2. Existence theorems

Following the suggestion of Nijenhuis and Richardson, we shall now use the implicit function theorem in Banach spaces [2] (rather than doing successive approximations directly) to derive certain existence theorems. For notation, we denote by $N(*)$ a generic neighborhood of the origin in a Banach space $*$.

Let (A, d) be a normed graded Lie algebra complex. Define $\varphi \in A^1$ to be *semi-integrable* if $d\varphi - \pi_d[\varphi, \varphi] = 0$.

Lemma 2.1 ([12]). There exists $N(\pi_H(A^1))$, $N(\pi_{d^*}(A^1))$, and a differentiable mapping $p: N(\pi_H(A^1)) \to N(\pi_{d^*}(A^1))$ such that, if $\varphi \in N(\pi_H(A^1))$, $\psi \in N(\pi_{d^*}(A^1))$, then $\varphi + \psi$ is semi-integrable if, and only if, $\psi = p(\varphi)$.

Proof. Define a differentiable mapping $q: \pi_H(A^1) \times \pi_{d^*}(A^1) \to \pi_d(A^2)$ by $q(\varphi, \psi) = d(\varphi + \psi) - \pi_d[\varphi + \psi, \varphi + \psi]$. Then $D_2 q(0, 0) = d: \pi_{d^*}(A^1) \to \pi_d(A^2)$ is an isomorphism, and, by the implicit function theorem, there exist $N(\pi_H(A^1))$, $N(\pi_{d^*}(A^1))$, and $p: N(\pi_H(A^1)) \to N(\pi_{d^*}(A^1))$ such that, for $\varphi \in N(\pi_H(A^1))$, $\psi \in N(\pi_{d^*}(A^1))$, $q(\varphi, \psi) = 0$ if and only if $\psi = p(\varphi)$. But $q(\varphi, \psi) = 0$ if, and only if, $\varphi + \psi$ is semi-integrable.

Lemma 2.2. $p(\varphi)$ is defined by the equation

$$p(\varphi) = d^* G[\varphi + p(\varphi), \varphi + p(\varphi)]. \tag{2.2}$$

Proof. $d(\varphi + p(\varphi)) - \pi_d[\varphi + p(\varphi), \varphi + p(\varphi)] = 0$. Q.E.D.

For $\varphi \in N(\pi_H(A^1))$, we set $P(\varphi) = \varphi + p(\varphi)$, and we define a vector-valued holomorphic function h on $N(\pi_H(A^1))$ by

$$h(\varphi) = \pi_H[P(\varphi), P(\varphi)]. \tag{2.3}$$

Lemma 2.3. [9]. There exists $N(\pi_H(A^1))$ such that, for $\varphi \in N(\pi_H(A^1))$, $h(\varphi) = 0$ if, and only if, $P(\varphi)$ is integrable.

Proof. If φ is integrable, then, since $P(\varphi)$ is semi-integrable, we get that $\pi_{d^*}[P(\varphi), P(\varphi)] = -\pi_H[P(\varphi), P(\varphi)] = -h(\varphi)$. Thus $h(\varphi) = 0$.

Now we have that $d^*dG[P(\varphi), P(\varphi)] = 2d^*G[dP(\varphi), P(\varphi)] = 2d^*G[dd^*G[P(\varphi), P(\varphi)], P(\varphi)] = 2d^*G\{[h(\varphi), P(\varphi)] + [d^*dG[P(\varphi), P(\varphi)], P(\varphi)]\}$ by (1.1), (1.2), and (1.4). Setting $F(\varphi) = d^*dG[P(\varphi), P(\varphi)]$, if $h(\varphi) = 0$ we get $F(\varphi) = 2d^*G[F(\varphi), P(\varphi)]$ and thus $\|F(\varphi)\| \leq c\|P(\varphi)\| \cdot \|F(\varphi)\|$. However, this implies that $F(\varphi) = 0$ if $h(\varphi) = 0$ for $\varphi \in N(\pi_H(A^1))$. By (2.2), $\Delta P(\varphi) = h(\varphi) + F(\varphi)$. Q.E.D.

We let $V \subset N(\pi_H(A^1))$ be the analytic set through the origin defined by the zeroes of $h(\varphi)$; to each $\varphi \in V$ we have associated the integrable element $P(\varphi) \in A^1$.

Let now the graded complex (K, δ) be over the graded Lie algebra complex (A, d). Let $\varphi \in A^1$. We say that $\gamma \in K^q$ is *semi-closed relative to* φ if $\delta\gamma - \pi_\delta[\varphi, \gamma] = 0$.

For Banach spaces S_1, S_2, $L(S_1, S_2)$ is the Banach space of bounded linear transformations $T: S_1 \to S_2$.

Lemma 2.4. There exist $N(A^1)$, $N(L(Z^q, \pi_{\delta^*}(K^q))$ and a differentiable mapping $r: N(A^1) \to N(L(Z^q, \pi_{\delta^*}(K^q)))$ such that the following holds: If $\varphi \in N(A^1)$, $\gamma \in Z^q$, and $\sigma \in \pi_{\delta^*}(K^q)$, then $\gamma + \sigma$ is semi-closed relative to φ if, and only if, $\sigma = r(\varphi)\gamma$.

Proof. Define $s: A^1 \times Z^q \times \pi_{\delta^*}(K^q) \to \pi_\delta(K^{q+1})$ by $s(\varphi, \gamma, \sigma) = \delta(\gamma + \sigma) - 2\pi_\delta[\varphi, \gamma + \sigma]$. Then s is differentiable and $D_3 s(0, 0, 0) = \delta: \pi_{\delta^*}(K^q) \to \pi_\delta(K^{q+1})$ is an isomorphism. The existence of r now follows again from the implicit function theorem.

Lemma 2.5. $r(\varphi)\gamma$ is defined by

$$r(\varphi)\gamma = 2\delta^* G[\varphi, \gamma + r(\varphi)\gamma]. \tag{2.4}$$

Proof. $\delta(\gamma + r(\varphi)\gamma) - 2\pi_\delta[\varphi, \gamma + r(\varphi)\gamma] = 0$. Q.E.D.

For $\varphi \in N(A^1), \gamma \in Z^q$, we set $R_\Phi(\gamma) = \gamma + r(\varphi)\gamma$.

Now, for each $\gamma \in \boldsymbol{H}^q \subset Z^q$, we define a vector valued holomorphic function $h_\gamma(\varphi)$ on $N(\pi_{\boldsymbol{H}}(A^1))$ by

$$h_\gamma(\varphi) = \pi_{\boldsymbol{H}}[P(\varphi), R_{\boldsymbol{\varphi}}(\gamma)], \tag{2.5}$$

where, by way of notation, we set $R_{\boldsymbol{\varphi}}(\gamma) = R_{P(\varphi)}(\gamma)$.

Lemma 2.6. There exists a neighborhood $N(\pi_{\boldsymbol{H}}(A^1))$ such that, for $\varphi \in N(\pi_{\boldsymbol{H}}(A^1)) \cap V$, $\gamma \in \boldsymbol{H}^q$, $\delta R_{\boldsymbol{\varphi}}(\gamma) - 2[P(\varphi), R_{\boldsymbol{\varphi}}(\gamma)] = 0$ if, and only if, $h_\gamma(\varphi) = 0$.

Proof. If $\delta R_{\boldsymbol{\varphi}}(\gamma) - 2[P(\varphi), R_{\boldsymbol{\varphi}}(\gamma)] = 0$, then $\pi_{\boldsymbol{H}}[P(\varphi), R_{\boldsymbol{\varphi}}(\gamma)] = -\pi_\delta[P(\varphi), R_{\boldsymbol{\varphi}}(\gamma)] = 0$ since $R_{\boldsymbol{\varphi}}(\gamma)$ is semi-closed relative to $P(\varphi)$.

Now assume that $\delta P(\varphi) = [P(\varphi), P(\varphi)]$. Then $\delta^*\delta G[P(\varphi), R_{\boldsymbol{\varphi}}(\gamma)] = \delta^* G[[P(\varphi), P(\varphi)], R_{\boldsymbol{\varphi}}(\gamma)] - 2\delta^* G\{[P(\varphi), [P(\varphi), R_{\boldsymbol{\varphi}}(\gamma)]] + [P(\varphi), h_\gamma(\varphi)] + [P(\varphi), \delta^*\delta G[P(\varphi), R_{\boldsymbol{\varphi}}(\gamma)]]\}$. Setting $E_\gamma(\varphi) = \delta^*\delta G[P(\varphi), R_{\boldsymbol{\varphi}}(\gamma)]$, we conclude that, if $h_\gamma(\varphi) = 0$, $E_\gamma(\varphi) = 2\delta^* G[P(\varphi), E_\gamma(\varphi)]$, from which it follows that $E_\gamma(\varphi) = 0$ for $\varphi \in N(\pi_{\boldsymbol{H}}(A^1)) \cap V$. But $\delta R_{\boldsymbol{\varphi}}(\gamma) - 2[P(\varphi), R_{\boldsymbol{\varphi}}(\gamma)] = 2\{h_\gamma(\varphi) + E_\gamma(\varphi)\}$. Q.E.D.

For a subspace $S \subset \boldsymbol{H}^q$, we let $V(S) \subset V$ be the analytic set defined by $V(S) = \{\varphi \in V: h_\gamma(\varphi) = 0 \text{ for all } \gamma \in S\}$. Then, to each $\varphi \in V(S)$, $\gamma \in S$, we have associated an element $R(\gamma)$ which satisfies

$$\delta_{P(\varphi)} R_{\boldsymbol{\varphi}}(\gamma) = \delta R_{\boldsymbol{\varphi}}(\gamma) - 2[P(\varphi), R_{\boldsymbol{\varphi}}(\gamma)] = 0 \tag{2.6}$$

We close this section with the following remark. Suppose that we are given a neighborhood U of the origin in any $\boldsymbol{C}^m$ and a holomorphic mapping $t \to \Phi(t)$ of U into A^1 such that the locus $Z = \{t \in U: \Delta\Phi(t) = 0\}$

is an analytic set Z. Then, for any subspace $S \subset \boldsymbol{H}^q$, we may construct the analytic set $Z(S) \subset Z$ just as $V(S) \subset V$ was constructed above.

2.3. A continuity property of cohomology

We shall now combine the results in 1.2 and 2.2. Before doing this, we must first establish a certain continuity property for cohomology.

Assume that $\varphi \in N(A^1)$ is integrable, and recall that any $\xi \in K^q$ which is semi-closed relative to φ may be uniquely written as $\xi = R_\varphi(\gamma) = \gamma + r(\varphi)\gamma$ where $\gamma \in Z(K^q)$ and $r(\varphi)\gamma \in \pi_{\delta^*}(K^q)$. Since φ is integrable, if $\tau \in K^{q-1}$, then $\delta_\varphi(\tau) = \delta\tau - 2[\varphi, \tau]$ is closed relative to φ, and we may write $\delta_\varphi(\tau) = R_\varphi(\gamma)$ where $\gamma \in Z(K^q)$, and, in fact,

$$\gamma = \delta\tau - 2\,\delta\delta^* G[\varphi, \tau] - 2\,\pi_H[\varphi, \tau]\,. \tag{2.7}$$

Define a linear mapping $\Lambda_\varphi: \pi_{\delta^*}(K^{q-1}) \subset Z(K^q)$ by $\Lambda_\varphi(\tau) = \delta\tau - 2\,\delta\delta^* G[\varphi, \tau] - 2\pi_H[\varphi, \tau]$, so that $R_\varphi \Lambda_\varphi(\tau) = \delta_\varphi(\tau)$. Finally, define a bounded linear mapping

$$\begin{aligned} &\equiv_\varphi: \pi_{\delta^*}(K^{q-1}) \to \pi_\delta(K^q) \quad \text{by} \\ &\equiv_\varphi(\tau) = \delta\tau - 2\,\delta\delta^* G[\varphi, \tau]\,. \end{aligned} \tag{2.8}$$

For $\varphi \in N(A^1)$, $\equiv_\varphi$ is an isomorphism.

Now let $Z_\varphi(K^q) \to K^q$ be the kernel of δ_φ. Then every $\xi \in Z_\varphi(K^q)$ may be written as $\xi = R_\varphi \gamma$ for some $\gamma \in Z(K^q)$. Clearly Λ_φ is an injection of $\pi_{\delta^*}(K^{q-1})$ into $Z_\varphi(K^q)$, and, since $R_\varphi(\Lambda_\varphi(\pi_{\delta^*}(K^{q-1}))) \subset \delta_\varphi(K^{q-1})$, we have $\dim H^q_\varphi(K) \leqq \dim \{Z(K^q)/\Lambda_\varphi(\pi_{\varphi^*}(K^{q-1}))\}$.

Lemma 2.7. $\dim\{Z(K^q)/\Lambda_\varphi(\pi_{\delta^*}(K^{q-1}))\} = \dim H^q(K)$.

Proof. Write $Z^q(K) = \pi_H(K^q) \times \pi_\delta(K^q) = \pi_H(K^q) \times \delta(\pi_{\delta^*}(K^{q-1}))$. We define $\Theta_\varphi: \pi_{\delta^*}(K^{q-1}) \to \pi_H(K^q)$ by $\Theta_\varphi(\tau) = \pi_H[\varphi, \tau]$. We then write $\pi_{\delta^*}(K^{q-1}) = W_\varphi \times V_\varphi$ where W_φ is the kernel of Θ_φ and where V_φ is a complementary finite dimensional subspace. On W_φ, we have then that $\Lambda_\varphi = \equiv_\varphi$. If we let $X_\varphi = \Lambda_\varphi(W_\varphi)$; then X_φ is a closed subspace with codim (X_φ) in $\pi_{\delta^*}(K^{q-1}) = \dim V_\varphi$, and

$$\dim\{Z(K^q)/\Lambda_\varphi(\pi_{\delta^*}(K^{q-1}))\} \leqq \dim H^q(K) + \dim V_\varphi.$$

In fact, it is easily seen that
$\dim\{Z(K^q)/\Lambda_\varphi(\pi_{\delta^*}(K^{q-1}))\} = \dim H^q(K) + \dim V_\varphi - \dim \Lambda_\varphi(V_\varphi) = \dim H^q(K)$, provided that φ is sufficiently near to 0. This proves the Lemma.

Thus we have a diagram

$$\begin{array}{ccc} Z^q(K) & \xrightarrow{R_\varphi} & Z^q_\varphi(K) \\ \uparrow{\scriptstyle\delta} & & \uparrow \\ \pi_{\delta^*}(K^{q-1}) & \xrightarrow{R\varphi \circ \Lambda\varphi} & \delta_\varphi(K^{q-1}) \end{array}$$

and a mapping

$\sigma: Z^q(K)/\pi_{\delta^*}(K^{q-1}) \cong Z^q_\varphi(K)/R_\varphi \circ \Lambda_\varphi(\pi_{\delta^*}(K^{q-1})) \xrightarrow{\text{onto}} Z^q_\varphi(K)/\delta_\varphi(K^{q-1})$; i.e. $\sigma: H^q(K) \xrightarrow{\text{onto}} H^q_\varphi(K)$. This is the continuity property of cohomology which we were seeking.

2.4. The extension problem in cohomology for normed complexes

Let U be an open neighborhood of the origin in $\boldsymbol{C}^m$ and $t \to \Phi(t)$ holomorphic mapping of U into A^1. Suppose furthermore that the locus $D = \{t \in U : \Delta\Phi(t) = 0\}$ is an analytic set. The *extension problem in cohomology* is the following: Given $\gamma \in H^q(K)$, to find a holomorphic mapping $\Gamma: U \to K^q$ such that $\Gamma(0) = \gamma$ and $\delta\Gamma(t) - 2[\Phi(t), \Gamma(t)] = 0$ for $t \in D$.

For fixed $t \in D$, denote by $H^q_t(K)$ the cohomology computed with the differential operator $\delta - 2[\Phi(t),\ ]$. We summarize our results in the following theorems:

Theorem 2.1. *There are finitely many obstructions to the extension problem for $\gamma \in H^q(K)$. Furthermore, if a formal solution exists, then an actual solution exists.*

Theorem 2.2. *The extension problem for $\gamma \in H^q(K)$ can be solved if dim $H^{q+1}_t(K)$ is independent of t. If dim $H^{q+1}_t(K)$ is independent of t, then dim $H^q_t(K)$ is independent of t if, and only if, dim $H^{q-1}_t(K)$ is independent of t.*

Finally, suppose that $m = 1$ and $D = U$.

Theorem 2.3. *In the notations of Theorem* 1.1, $H^q_t(K) = E^q_\Phi/J^q_\Phi$ *for* $t \neq 0$.

The proofs of Theorems 2.1 – 3 are immediate from what we have done; the continuity property of cohomology, together with the finitely many obstructions, are sufficient to assure convergence in the formal statement of Theorem 1.1. It is perhaps worth noting that Lemma 2.7 includes, in particular, the usual statements about upper-semi-continuity of cohomology, while Theorem 2.3 implies the invariance of the Euler characteristic $\chi_t(K) = \sum_{q=0}(-1)^q \dim H^q_t(K)$. Indeed, for $t \neq 0$, $\chi_t(K)$ $= \sum_q (-1)^q \{\dim E^q_\Phi - \dim J^q_\Phi\} = \sum_q (-1)^q \{\dim H^q(K) - \dim G^q_\Phi(K) - - \dim J^q_\Phi\} = \sum_q (-1)^q \{\dim H^q(K) - \dim J^{q+1}_\Phi(K) - \dim J^q_\Phi(K)\}$ $= \sum_q (-1)^q \dim H^q(K)$.

Finally, we have the following

Theorem 2.4. *Let $S \subset H^q(K)$ be a subspace. Then the extension problem can be solved over $D(S) \subset D$, and $D(S)$ is a maximal such analytic set.*

2.5. The extension problem for exact sequences of graded Lie algebra complexes

Let (A, d), (B, d), and (C, d) be graded Lie algebra complexes which form an exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$. Thus π and i are bounded mappings which commute with the harmonic theories. Clearly, π maps integrable elements of B^1 into integrable elements in C^1; we wish to know when π maps the integrable elements of B^1 *onto* those in C^1.

Theorem 2.5. *Assume $H^2(A) = 0$. Then there exist $N(C^1)$, $N(B^1)$, and a differentiable mapping $T: N(C^1) \to N(B^1)$ such that $\pi \circ T$ = Indentity and such that, for $\varphi \in N(C^1)$ $T(\varphi)$ is integrable if, and only if, φ is integrable.*

Proof. We first make an assumption concerning the exact sequence $0 \to A \to B \to C \to 0$ which will be satisfied in our applications.

Namely, we suppose that there exist bounded linear maps $\omega \in \mathrm{Hom}(C^i, B^i)$ $(i = 1, 2)$ and $\Omega \in \mathrm{Hom}(C^1, A^2)$ which satisfy the following: $\pi \circ \omega = \text{Identity}$ and $d(\omega\varphi) = \Omega(\varphi) + \omega(d\varphi)$ for $\varphi \in C^1$. Thus the general element ψ in B^1 such that $\pi(\psi) = \varphi \in C^1$ is of the form $\psi = \omega(\varphi) + \gamma$ where $\gamma \in A^1$. We recall that $\psi \in B^1$ is semi-integrable if $d(\psi) - \pi_d[\psi, \psi] = 0$.

Lemma 2.8. There exist $N(C^1)$ and $N(\pi_{d^*}(A^1))$ and a differentiable mapping $t: N(C^1) \to N(\pi_{d^*}(A^1))$ such that, for $\varphi \in N(C^1)$, $\gamma \in N(\pi_{d^*}(A^1))$, $\omega(\varphi) + \gamma \in B^1$ is semi-integrable if, and only if, $\gamma = t(\varphi)$.

Proof. If we define a differentiable mapping $u: C^1 \times \pi_{d^*}(A^1) \to \pi_d(B^1)$ by $u(\varphi, \gamma) = d(\omega(\varphi) + \gamma) - \pi_d[\omega(\varphi) + \gamma, \omega(\varphi) + \gamma]$, then $D_2 u(0, 0) = d$. The result then follows from the implicit function theorem.

Lemma 2.9. If φ is integrable, then $\gamma = t(\varphi)$ is given by $\gamma = d^*G([\gamma + \omega(\varphi), \gamma + \omega(\varphi)] - \omega[\varphi, \varphi] - \Omega(\varphi))$.

Proof. $\pi([\gamma + \omega(\varphi), \gamma + \omega(\varphi)] - \omega[\varphi, \varphi] - \Omega(\varphi)) = [\varphi, \varphi] - [\varphi, \varphi] = 0$; thus $\gamma \in \pi_{d^*}(A^1)$. We must show that $\gamma + \omega(\varphi)$ is semi-integrable if $d\varphi = [\varphi, \varphi]$. We have $d(\omega(\varphi)) = \omega([\varphi, \varphi]) + \Omega(\varphi)$, and thus $d(\gamma + \omega(\varphi)) - \pi_d[\gamma + \omega(\varphi), \gamma + \omega(\varphi)] = d(\omega(\varphi)) + dd^*G([\gamma + \omega(\varphi), \gamma + \omega(\varphi)]) - dd^*G(d(\omega(\varphi)) - \pi_d[\gamma + \omega(\varphi), \gamma + \omega(\varphi)] = 0$. Q.E.D.

For $\varphi \in N(C^1)$, we set $T(\varphi) = \varphi + t(\varphi) \in B^1$.

Lemma 2.10. Assume $H^2(A) = 0$. Then there exists $N'(C^1) \subset N(C^1)$ such that, for $\varphi \in N'(C^1)$, $T(\varphi)$ is integrable if, and only if, φ is.

Proof. Assume that φ is integrable; then $dT(\varphi) - dd^*G[T(\varphi), T(\varphi)] = 0$ by Lemma 2.9. Thus $\Delta T(\varphi) = \pi_H[T(\varphi), T(\varphi)] + d^*dG[T(\varphi), T(\varphi)]$. But, since $H^2(A) = 0$, $\pi_H[T(\varphi), T(\varphi)] = \pi_H(\pi[T(\varphi), T(\varphi)]) = \pi_H[\pi T(\varphi), \pi T(\varphi)] = \pi_H[\varphi, \varphi] = 0$. Set $\Lambda = d^*dG[T(\varphi), T(\varphi)]$. Then $\Delta T(\varphi) = \Lambda$. But $\Lambda = 2d^*G[dT(\varphi), T(\varphi)] = 2d^*G[[T(\varphi), T(\varphi)],$

$T(\varphi)] + 2d^*G[\Lambda, T(\varphi)]$. Thus $\|\Lambda\| \leqq c\|T(\varphi)\|\|\Lambda\|$; since $T(0) = 0$, the Lemma follows. Q.E.D.

This completes the proof of Theorem 2.5.

3. Some results on complex manifolds, fibre bundles, and deformations of complex structures

3.1. Deformations of complex structures

Let X be a compact, complex manifold. Denote by $\mathscr{A}^q(\boldsymbol{T})$ the sheaf of germs of vector-valued $(0, q)$ forms of class $C^{k-q+\alpha}$ (in the sense of [8]), and set $C^q = H^\circ(X, \mathscr{A}^q(\boldsymbol{T}))$. Then the graded vector space $C = \sum_{p \geqq 0} C^p$ has a differential operator $\bar{\partial}: C^p \to C^{p+1}$ and a bracket $[\, ,]: C^p \otimes C^q \to \to C^{p+q}$ such that (1.1), (1.2), and (1.4) are satisfied.

We may define $\|\ \|$ on C^q by setting $\|\ \| = \|\ \|_{k-q+\alpha} (k \gg 0)$ where the latter norm was defined in [8], § 4. We set $d = \bar{\partial}$ on C and, by taking a C^∞ Hermitian metric on X, we may define d^* as the adjoint $\mathbf{d}$ of $\bar{\partial}$. The harmonic theory for (C, d) is then taken as the harmonic theory relative to the Hermitian metric on X; e. g., $\boldsymbol{G}$ is the usual Green's operator. In fact, using the potential-theoretic lemma in [8], § 4, it is easily seen that (C, d) becomes a normed graded Lie algebra complex as prescribed in § 1.

Let J_0 be the almost complex structure underlying the complex structure on X.

Lemma 3.1. There is a one-to-one correspondence between almost-complex structures J on X, which are sufficiently close to J_0, and elements $\Phi \in C^1$ which are near to 0. The integrability condition is

$$\bar{\partial}\Phi - [\Phi, \Phi] = 0\,. \tag{3.1}$$

Proof. An almost-complex structure J is given by a family of "admissible frames" $e^{\#} = (e_1, \ldots, e_n;\ e_1^*, \ldots, e_n^*)$ where the e_α are complex tangent vectors and e_α^* is the complex conjugate of e_α. We write $e^{\#} = (e, e^*)$; given $e^{\#}$, the admissible frames are of the form $(A\,e, \bar{A}\,e^*)$ where $A \in GL(n, \boldsymbol{C})$.

We let P_0 and Q_0 be the projections, associated to J_0, onto the vectors of type (1,0) and (0,1) respectively; P and Q fulfill similar functions for J. Let $z = (z^1, \ldots, z^n)$ be local holomorphic coordinates on X. If J is close to J_0, then Q_0 will be non-singular on Image (Q), and we may uniquely choose a J-admissible frame $e^{\#} = (e(z), e^*(z))$ such that $Q_0(e^*) = \left(\frac{\partial}{\partial \bar{z}^1}, \ldots, \frac{\partial}{\partial \bar{z}^n}\right)$. Then

$$e_\alpha^* = \frac{\partial}{\partial \bar{z}^\alpha} - \sum_{\beta=1}^{n} \Phi_\alpha^\beta \frac{\partial}{\partial z^\beta}\,, \quad \text{and} \quad e_\alpha = \frac{\partial}{\partial z^\alpha} - \sum_{\beta=1}^{n} \overline{\Phi_\alpha^\beta} \frac{\partial}{\partial \bar{z}^\beta}\,. \tag{3.2}$$

From (3.2), it follows that $\Phi = \sum_{\alpha, \beta} \Phi^\alpha_{\bar\beta} \frac{\partial}{\partial z^\alpha} \otimes \overline{dz}^\beta$ is a tensor and defines an element of C^1. (An intrinsic representation of Φ is given in Lemma 3.3 below.)

Now the co-frame $\omega^\# = (\omega, \omega^*)$ where $\omega = (\omega^1, \ldots, \omega^n)$ which is dual to $e^\#$ is defined by $\langle \omega^\alpha, e_\beta \rangle = \delta^\alpha_\beta$ $(\alpha, \beta = 1, \ldots, n)$. It then follows that $\omega^\alpha = dz^\alpha + \sum \Phi^\alpha_{\bar\beta} d\bar z^\beta$. The *Frobenius integrability condition* is written symbolically as $d\omega^\alpha \equiv 0 (\mathrm{mod}\ \omega)$, which means that $d\omega^\alpha$ should be in the exterior ideal generated by $\omega^1, \ldots, \omega^n$. We have that

$$d\omega^\alpha = \sum \frac{\partial \Phi^\alpha_{\bar\beta}}{\partial z^\gamma} dz^\gamma \wedge d\bar z^\beta + \sum \frac{\partial \Phi^\alpha_{\bar\beta}}{\partial \bar z^\tau} d\bar z^\tau \wedge d\bar z^\beta .$$

Since $-dz^\alpha \equiv \sum \Phi^\alpha_{\bar\beta}\, d\bar z^\beta$ (mod ω) and $d\bar z^\gamma \equiv \bar\omega^\gamma (\mathrm{mod}\ \omega)$, it follows that

$$d\omega^\alpha \equiv \sum \left(\frac{\partial \Phi^\alpha_{\bar\beta}}{\partial \bar z^\tau} d\bar z^\tau \right) \wedge d\bar z^\beta - \sum \Phi^{\ \gamma}_{\bar\tau} \frac{\partial \Phi^\alpha_{\bar\beta}}{\partial z^\gamma} d\bar z^\tau \wedge d\bar z^\beta \ (\mathrm{mod}\ \omega);$$

this equation says, *by definition*, that $d\omega \equiv \bar\partial \Phi - [\Phi, \Phi] (\mathrm{mod}\ \omega)$. Since this argument is reversible, the lemma follows. Q. E. D.

Remarks. (i) If J is integrable and, if $\xi^1, \ldots, \xi^n$ are local holomorphic coordinates for J, then $\Phi = \{\Phi^\alpha_{\bar\beta}\}$ is defined locally by

$$\frac{\partial \xi^\alpha}{\partial \bar z^\tau} = \sum_{\gamma=1}^{n} \frac{\partial \xi^\alpha}{\partial z^\gamma} \Phi^\gamma_{\bar\tau} . \tag{3.3}$$

Thus, in an intrinsic form, introducing holomorphic coordinates for J is *equivalent* to solving locally the linear equation $\bar\partial \xi = 2[\Phi, \xi]$, for a local vector-valued function ξ which gives a differentiable coordinate for the C^∞ structure on X.

(ii) If J is an almost-complex structure near to J_0, and if $f: X \to X$ is a diffeomorphism near the identity, then f transforms J into a new almost-complex structure $J \circ f_*$ near to J_0. If $\Phi \in C^1$ corresponds to J, we denote by $f_*(\Phi) \in C^1$ the element corresponding to $J \circ f_*$.

Let now $V \subset \pi_H(C^1)$ be the germ of an analytic set defined in § 2.2.

Proposition A. There exists a deformation $\{\mathscr{V} \xrightarrow{\tilde\omega} V\}$ of X such that $\tilde\omega^{-1}(\varphi)$ has the integrable almost-complex structure $P(\varphi)$ given in Lemma 2.2.

Following KURANISHI, we call an element $\varphi \in C^1$ which satisfies $d^*\varphi = 0$ *extremal*. If $\varphi \in \pi_H(C^1)$, then $P(\varphi)$ is clearly extremal.

Proposition B. (KURANISHI). Let $\varphi \in N(C^1)$. Then there exists a diffeomorphism $f: X \to X$ such that $f_*(\varphi)$ is extremal. If furthermore $\varphi = \varphi(s)$ depends differentiably on s, then we may assume that $f(s)$ depends differentiably on s.

From this, we have

Proposition 3.1. The family $\{\mathscr{V} \xrightarrow{\tilde{\omega}} V\}$ is universal for differentiable families of complex structures.

Proof. Let $\{\mathscr{V}' \xrightarrow{\tilde{\omega}} D\}$ be a differentiable family (cf. [7]) such that $X_d = \tilde{\omega}^{-1}(d)$ $(d \in D)$ has the almost complex structure represented by $\Phi(d) \in C^1$. Then $\Phi(d)$ is differentiable in d, and we may find diffeomorphisms $f(d)$ such that $f(d)_* \Phi(d)$ is extremal. But $f(d)_* \Phi(d)$ is integrable, hence semi-integrable, and thus, by Lemma 2.1, $f(d)_* \Phi(d) = P(\varphi(d))$ where $\varphi(d) \in \pi_H(C^1)$ and is differentiable in $d \in D$. If we define $G: D \to V$ by $G(d) = \varphi(d)$, then we may define

$$\begin{array}{ccc} F: \mathscr{V}' & \to & \mathscr{V} \\ \downarrow & & \downarrow \\ G: D & \to & V \end{array} \qquad \text{by } F(d, x) = (G(d), f(d)x) \qquad (d \in D,\ x \in X),$$

and this proves the universality of $\{\mathscr{V} \xrightarrow{\tilde{\omega}} V\}$.

Finally, we also need

Proposition C. If $\{\mathscr{V}' \xrightarrow{\tilde{\omega}} D\}$ is a complex analytic deformation, then we may choose

$$\begin{array}{ccc} F: \mathscr{V}' & \to & \mathscr{V} \\ \downarrow & & \downarrow \\ G: D & \to & V \end{array}$$

to be a complex analytic mapping.

Propositions A, B, and C are consequences of the results of KURANISHI given in his paper in this volume.

3.2. Deformations of holomorphic fibre bundles

Let X be a compact complex manifold, G a complex Lie group, and $G \to P \to X$ a holomorphic principal fibre bundle. We consider the fundamental bundle sequence (see [1] and [11])

$$0 \to \mathbf{L} \to \mathbf{Q} \to \mathbf{T} \to 0. \tag{3.4}$$

The sheaves $\mathscr{L}, \mathscr{Q}, \mathscr{T}$ are sheaves of Lie algebras and $0 \to \mathscr{L} \to \mathscr{Q} \to \mathscr{T} \to 0$ is an exact sequence of sheaves of Lie algebras. We set $A^q = H^0(X, \mathscr{A}^q(\mathbf{L}))$, $B^q = H^0(X, \mathscr{A}^q(\mathbf{Q}))$. Then, if $A = \sum_{q \geqq 0} A^q$ and $B = \sum_{q \geqq 0} B^q$, and if $d = \bar{\partial}: A^q \to A^{q+1}$ and $d = \bar{\partial}: B^q \to B^{q+1}$, (A, d) and (B, d) become graded Lie algebra complexes. In fact, there are natural homomorphisms of graded Lie algebras $\iota: A \to B$ and $\pi: B \to C$ such that

$$0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0 \tag{3.5}$$

is an exact sequence of complexes.

Now let J, J' be almost-complex structures on the differentiable manifolds P, X respectively.

Definition 3.1. The pair (J, J') makes $G \to P \to X$ an *almost-complex fibre bundle* if the following three conditions are satisfied: (i) J is G-invariant where G acts on P on the right; (ii) the almost-complex structure on P/G induced by J is J' (i. e. $\pi_* J = J' \pi_*$); and (iii) J restricted to a fibre gives the integrable almost-complex structure on G (this makes sense by (i) and since G is a complex Lie group).

Let (J_0, J_0') be the given integrable almost-complex fibre bundle structure on $G \to P \to X$.

Lemma 3.2. (i) There is a one-to-one correspondence between almost-complex fibre bundle structures (J, J') on P, which are sufficiently close to (J_0, J_0'), and elements $\psi \in B^1$ which are sufficiently near 0 and which satisfy $\pi(\psi) = \Phi$ where Φ corresponds to J' using Lemma 3.1.

(ii) J is integrable if, and only if,

$$\bar{\partial}\psi - [\psi, \psi] = 0 . \tag{3.6}$$

Proof. Using the above notation and Lemma 3.1, J is given by an element $\psi \in D^1 = \Gamma_\infty(P, \mathrm{Hom}(\boldsymbol{T}(P)^*, \boldsymbol{T}(P)))$. By (i) in Definition 3.1, $\psi \in \Gamma_\infty(P, \mathrm{Hom}(\boldsymbol{T}(P)^*/G, \boldsymbol{T}(P)^*/G)) \cong \Gamma_\infty(X, \mathrm{Hom}(\boldsymbol{Q}^*, \boldsymbol{Q}))$. By (iii) in the Definition, ψ will annihilate vertical vectors, and thus $\psi \in$ $\in \Gamma_\infty(X, \mathrm{Hom}(\boldsymbol{Q}^*/\boldsymbol{L}^*, \boldsymbol{Q})) \cong B^1$. Finally, from the proof of Lemma 3.1, it is clear that (ii) implies that $\pi(\psi) = \Phi$. By reversing this argument, we get (i) in Lemma 3.2.

Also, (ii) in the Lemma follows from (ii) in Lemma 3.1 by using the embedding $B^1 \subset D^1$ and the fact (mentioned above) that $\bar{\partial}$ and $[\ ,\ ]$ on B are induced from these operations on D. Q. E. D.

By speaking of $(0, q)$ forms with values in $\boldsymbol{Q}$ of class $C^{k-q+\alpha}$, we may, just as above, make (B, d) (where $B = \sum_{p \geqq 0} B^p$ and $d = \bar{\partial}$) into a graded Lie algebra complex in the sense of § 1. Furthermore, the same remarks which were made about (C, d) and deformations of complex structure now make sense and are true for (B, d) and deformations of bundle structure. For example, where we spoke above of diffeomorphisms of X, we must now speak of bundle diffeomorphisms; one such is given by a diffeomorphism $f: P \to P$ such that $f(pg) = f(p)g$ $(p \in P, g \in G)$. In particular, we have now analogues of Propositions A, B, C, and 3.1; we shall assume that the reader has translated these into the language of complex fibre bundles, and we shall refer to them as Propositions A', B', C' and 3.1′ respectively. The proofs of Propositions A', B', and C' are similar to those of Propositions A, B, and C; as above, Proposition 3.1′ follows from the other three.

3.3. Perturbation of differential operators

Let X_0 be a fixed C^∞ manifold, G a complex Lie group, and let $G \to P \to X$, $G' \to P' \to X'$ be two "close" (cf. § 3.2) complex fibre bundle structures on a fixed C^∞ principal bundle $G \to P_0 \to X_0$. If $\varrho : G \to GL(E)$ is a finite-dimensional holomorphic linear representation of G, then we may construct two holomorphic vector bundles: $E \to \boldsymbol{E} = P \times_G E \to X$, and $E \to \boldsymbol{E}' = P' \times_G E \to X'$. These are both the same C^∞ bundle $E \to \boldsymbol{E}_0 \to X_0$. If $\mathscr{A}^0(\boldsymbol{E})$ is the sheaf of C^∞ cross-sections of $E \to \boldsymbol{E}_0 \to X_0$, then there are differential operators $\bar\partial$ and $\bar\partial'$ on $\mathscr{A}^0(\boldsymbol{E})$ corresponding to the two complex structures involved. From Lemma 3.2, we know that $G \to P' \to X'$ is uniquely prescribed from $G \to P \to K$ by an element $\psi \in B^1$ which satisfies (3.6). We want to find an expression for $\bar\partial'$ in terms of $\bar\partial$ and ψ.

Let $U \subset\subset \boldsymbol{R}^{2n}$ be a contractible open set, and let J, J' be two integrable almost-complex structures on U. Let $z = (z_1, \ldots, z_n)$ and $\xi = (\xi_1, \ldots, \xi_n)$ be holomorphic coordinates for J, J' respectively. Define a section $\partial\xi$ of $\mathrm{Hom}(\boldsymbol{T}, \boldsymbol{T}')$ by $\partial\xi = \sum_{\alpha=1} \frac{\partial}{\partial\xi_\alpha} \oplus \partial\xi_\alpha$ where ∂ is taken with respect to J. Assume that $\partial\xi$ is non-singular, i. e. J' is close to J. Define now a section $\bar\partial\xi$ of $\mathrm{Hom}(\boldsymbol{T}^*, \boldsymbol{T}')$ by $\bar\partial\xi = \sum_{\alpha=1} \frac{\partial}{\partial\xi_\alpha} \oplus \bar\partial\xi_\alpha$. Then the element $\Phi \in C^1$ (relative to J) which defines J' from J is given by

$$\bar\partial\xi = \partial\xi \circ \Phi \qquad (\text{see } (3.3))\,. \tag{3.7}$$

We now write $J' = J_\Phi$, $\boldsymbol{T}' = \boldsymbol{T}_\Phi$, $\boldsymbol{T}'^* = \boldsymbol{T}_\Phi^*$, and $\bar\partial' = \bar\partial_\Phi$. Let P, Q be the projections of $\boldsymbol{T}^\#$ on $\boldsymbol{T}, \boldsymbol{T}^*$ respectively; and let P_Φ, Q_Φ be the projections of $\boldsymbol{T}^\#$ on $\boldsymbol{T}_\Phi, \boldsymbol{T}_\Phi^*$. Now, for a function f and a vector v, $\langle \bar\partial_\Phi f, v\rangle = \langle df, Q_\Phi(v)\rangle$; thus, we seek to find Q_Φ in terms of Φ, P, and Q.

Lemma 3.3. $\boldsymbol{T}_\Phi^* = (I - \Phi)\,\boldsymbol{T}^*$.

Proof. A vector v lies in $\boldsymbol{T}_\Phi^*$ if, and only if, $\langle d\xi_\alpha, v\rangle = 0$ $(\alpha = 1, \ldots, n)$. But $\langle d\xi_\alpha, v\rangle = \langle \bar\partial\xi_\alpha + \partial\xi_\alpha, v\rangle = \langle \partial\xi_\alpha, (I + \Phi)v\rangle$ (by (3.7)). Since $\Phi \circ \Phi = 0$, $\boldsymbol{T}_\Phi^* = (I - \Phi)\,\boldsymbol{T}^*$. Q. E. D.

We now determine P_Φ and Q_Φ in a purely algebraic fashion. On a $2n$-dimensional real vector space V, let J_1 and J_2 be complex structures ($J_1^2 = -I = J_2^2$) where J_2 is close to J_1. Write:

$$\mathfrak{V} = V \otimes_{\boldsymbol{R}} \boldsymbol{C} = \begin{cases} U_1 \oplus W_1; & J_1\text{-decomposition where } W_1^* = U_1 \\ U_2 \oplus W_2; & J_2\text{-decomposition where } W_2^* = U_2 \,. \end{cases}$$

Let P_1, Q_1 and P_2, Q_2 be the projection operators associated to J_1 and J_2 respectively, and suppose that we have $T \in \mathrm{Hom}(W_1, U_1)$ such that $W_2 = (I - T)\,W_1$. (Lemma 3.3). Letting now $*$ be conjugation with respect to J_1, we then have: $U_2 = (I - T^*) U_1$, $U_1 = (I + T^*) U_2$, and $W_1 = (I + T) W_2$. Since J_2 is close to J_1, $(I - T T^*)$ is invertible, and we set $S_1 = (I - T T^*)^{-1}$. Also, we define $S_2 = T^* \circ S_1 = S_1^* \circ T^*$ (using $(I - A)^{-1} = I + A^2 + A^2 + \ldots$ for a linear transformation A). Then, since $I = (I - T T^*) S_1 = (I - T^* + T^* - T T^*) S_1 = (I - T^*) S_1 + (I - T) S_2$, we get:

$$\begin{aligned} P_1 &= (I - T^*) S_1 P_1 + (I - T) S_2 P_1 \\ Q_1 &= (I - T) S_1^* W_1 + (I - T^*) S_2^* Q_1 . \end{aligned} \tag{3.8}$$

Lemma 3.4. $P_2 = (I - T^*) S_1 P_1 + (I - T^*) S_2^* Q_1$ and $Q_2 = (I - T) S_2 P_1 + (I - T) S_1^* Q_1$.

Proof. Let $V \in U_1$. Then, by (3.8),

$$v = (I - T^*) A_1 P_1(v) + (I - T) S_2 P_1(v)$$

and, since

$$(I - T^*) S_1 P_1(v) \in U_2 \quad \text{and} \quad (I - T) S_2 P_1(v) \in W_2 ,$$

we have verified the formula for P_2 on U_1. The other verifications are similar. Q. E. D.

Now, setting $S' = (I + T) Q_2$, $S' \in \operatorname{Hom}(W_2, W_1)$ and establishes a vector space isomorphism. We have that

$$S' = S_2 \circ P_1 + S_1^* \circ Q_1 \tag{3.9}$$

by Lemma 4.4 and since $T^2 = 0$. Similarly, $S'^* \in \operatorname{Hom}(U_2, U_1)$ and is an isomorphism. Combining, we get an isomorphism

$$S : \Lambda^p U_1' \oplus \Lambda^q W_1' \to \Lambda^p U_2' \oplus \Lambda^q W_2' .$$

Applying now this result to $U \subset\subset \mathbf{R}^{2n}$, we get an isomorphism $S_\Phi : \mathscr{A}^{p,q} \to \mathscr{A}_\Phi^{p,q}$ where $\mathscr{A}_\Phi^{p,q}$ is the sheaf of C^∞ (p, q) forms relative to J_Φ.

Lemma 3.5. For $\omega \in \mathscr{A}^{p,q}$, $\bar\partial_\Phi (S_\Phi \omega) = S_\Phi (\bar\partial \omega - 2[\Phi, \omega])$.

Remark. As in [3], $[\Phi, \omega] = d\omega \wedge \Phi + (-1)^{p+q} d(\omega \wedge \Phi)$ where $\wedge$ is the contraction operation given by equation (2.7) in [3].

Proof. Since $\bar\partial^2 = 0 = \bar\partial_\Phi^2$, $\bar\partial_\Phi d = - d\bar\partial_\Phi$, and, by Proposition 4.5 in [3], we may prove Lemma 3.5 when $\omega = f$ is a function. In this case, as operators on $T^\#$, $S_\Phi (\bar\partial f - 2[\Phi, f]) = (\bar\partial f - df \wedge \Phi) \circ S_\Phi$. On the other hand, $\bar\partial_\Phi (S_\Phi f) = \bar\partial_\Phi f = df \circ Q_\Phi =$ (by Lemma 3.4) $df \circ (I - \Phi) \circ \circ S_2 \circ P_1 + (I - \Phi) \circ S_1^* \circ Q_1) = df \circ (I - \Phi) \circ (S_2 \circ P_1 + S_1^* \circ Q_1) = (\partial f + \bar\partial f) \circ (I - \Phi) \circ S_\Phi (\text{by (4.9)})) = (\bar\partial f - \partial f \circ \Phi) \circ S_\Phi$ (since $\bar\partial f \circ \Phi = 0 = \partial f \circ S_\Phi) = (\bar\partial f - df \wedge \Phi) \circ S_\Phi = S_\Phi (\bar\partial f - 2[\Phi, f])$. Q. E. D.

Return now to the situation at the beginning of this section. We let $\mathscr{A}^q(E)$ be the sheaf of germs of $C^{k-q+\alpha}(0, q)$ forms with values in $E \to \to E \to X$ and $\mathscr{A}^q(E')$ the sheaf of germs of $C^{k-q+\alpha}(0, q)'$ forms (i. e. $(0, q)$ forms relative to the J' structure) with values in $E \to E' \to X'$.

Theorem 3.1. *There exists a linear isomorphism of sheaves $S_\varphi : \mathscr{A}^q(\boldsymbol{E}) \to \to \mathscr{A}^q(\boldsymbol{E}')$ which is bounded in $\| \;\|$ and which has the property: For a germ η in $\mathscr{A}^q(\boldsymbol{E})$,*

$$\bar{\partial}' S_\Psi(\eta) = S_\Psi(\bar{\partial}\eta - 2[\Psi, \eta]). \tag{3.10}$$

Proof. The proof is an easy consequence of Lemma 3.5 together with the standard remarks to the effect that: (i) bundle-valued forms on X' are given by ordinary vector-valued forms on P' which satisfy an equivariance condition, and (ii) the operator $\bar{\partial}'$ on bundle-valued forms on X' corresponds to $\bar{\partial}'$ on ordinary forms on P'. Q. E. D.

3.4. Complexes over analytic sets and the cohomology of fibered analytic spaces

Let $G \to P \to X$ be a holomorphic principal bundle, let $\Psi(t) \in B^1\{t\}$, and suppose that the equation $\Delta\Psi(t) = 0$ defines a germ $D_\Psi = D$ of an analytic set in $\boldsymbol{C}^m$; by § 3.2, we then get a deformation $\{\mathscr{P} \to \mathscr{V} \to D\}$ of $G \to P \to X$. Let $\varrho : G \to GL(E)$ be a holomorphic representation and consider the holomorphic vector bundle $E \to \boldsymbol{E}^\# = \mathscr{P} \times_G E \to \mathscr{V}$. We wish to describe the groups $H^q(\mathscr{V}, \mathscr{E}^\#)$ in terms of differential forms.

With no loss of generality we may assume that D is Stein. Furthermore, we may choose a differentiable isomorphism $h : \mathscr{V} \underset{C^\infty}{\simeq} X \times D$ such that: (i) $h | \tilde{\omega}^{-1}(0)$ is holomorphic; and (ii) if $E \to \boldsymbol{E} \to X \times D$ is the trivial extension of $E \to \boldsymbol{E} = P \times_G E \to X$, then $h^{-1}(E \to \boldsymbol{E} \to X) \underset{C^\infty}{\simeq} E \to \to \boldsymbol{E}^\# \to \mathscr{V}$.

Consider now the sheaf $\mathscr{A}^q(\boldsymbol{E})$ of germs of $C^\infty(0, q)$ forms with values in $\boldsymbol{E}$. Over an open set $U_1 \subset X$, the sections of this sheaf form a Fréchèt space $\boldsymbol{A}^q(\boldsymbol{E})(U_1)$ by taking the family of norms to be $\{\| \;\|_{k+\alpha}^{V_\gamma}\}$ where $k = 0, 1, \ldots$ and V_γ runs over a countable family of compact subsets of U_1 which generate the topology of U_1. We now define the sheaves $\mathscr{A}^q(\boldsymbol{E}) \hat{\otimes} \mathscr{O}_D$ over V; this definition will be done via h. If $U_1 \subset X$ and $U_2 \subset D$, then the sections of $\mathscr{A}^q(\boldsymbol{E}) \hat{\otimes} \mathscr{O}_D$ over $h^{-1}(U_1 \times U_2)$ are given by the holomorphic functions $\varphi : U_2 \to \boldsymbol{A}^q(\boldsymbol{E})(U_1)$. Thus we may write $\varphi(x, t) = \sum_{\mu=0}^{\infty} f_\mu(t)\, \varphi_\mu(x)$ where $f_\mu(t)$ is holomorphic in U_2, $\varphi_\mu(x) \in \in \boldsymbol{A}^q(\boldsymbol{E})(U_1)$, and, for compact sets $K_1 \subset U_1$, $K_2 \subset U_2$ and for an integer k, $\sum_\mu |f_\mu(t)| \, \|\varphi_\mu(x)\|_{k+\alpha}^{K_1}$ converges uniformly for $t \in K_2$.

We define $\boldsymbol{D} : \mathscr{A}^q(\boldsymbol{E}) \hat{\otimes} \mathscr{O}_D \to \mathscr{A}^{q+1}(\boldsymbol{E}) \hat{\otimes} \mathscr{O}_D$ by

$$\boldsymbol{D}\,\varphi(x, t) = \bar{\partial}\varphi(x, t) - 2[\Psi(t), \varphi(x, t)]. \tag{3.11}$$

From our definition, it is clear that $\boldsymbol{D}\varphi(x, t)$ is a germ of section of $\mathscr{A}^{q+1}(\boldsymbol{E}) \hat{\otimes} \mathscr{O}_D$; by the remark at the end of § 3.3, $\boldsymbol{D}^2 = 0$.

Lemma 3.6. D satisfies a Poincaré lemma.

Proof. Let $\varphi(x, t)$ be a section of $\mathscr{A}^q(E) \hat{\otimes} \mathscr{O}_D$ over $U_1 \times U_2$ which satisfies $D\varphi(x, t) = 0$.

Now we shall use the definition of deformation of complex structure given in [8]. Namely, there exists open neighborhoods $U_1' \subset U_1$, $U_2' \subset U_2$ and a bi-C^∞ mapping $h_{U'}: U' \to P' \times U_2'(U' = U_1' \times U_2')$ where $P' \subset C^n$ is a polycylinder, and such that $h_{U'}$ locally trivializes the deformation in the following sense: The transform of D by $h_{U'}$ is the operator $\bar{\partial}'$ in the holomorphic coordinates z' in P'. This follows from the Newlander-Nirenberg theorem [10] together with Theorem 3.1. Also $\varphi(x, t)$ is transformed into $\varphi(x', t)$ $(x' \in P', t \in U_2')$ where $\varphi(x', t)$ is still holomorphic in t. From this point of view, the D-Poincaré lemma is essentially the $\bar{\partial}$-Poincaré lemma with holomorphic dependence on t. As is well-known, this is permissible. Q.E.D.

It is perhaps worth remarking that, conversely, the D-Poincaré lemma implies the Newlander-Nirenberg theorem.

Theorem 3.2. (i) *There exists an injection* $j: \mathscr{E}^\# \to \mathscr{A}^0(E) \hat{\otimes} \mathscr{O}_D$ *such that:*

$$0 \to \mathscr{E}^\# \to \mathscr{A}^0(E) \hat{\otimes} \mathscr{O}_D \xrightarrow{D} \cdots \xrightarrow{D} A^q(E) \hat{\otimes} \mathscr{O}_D \xrightarrow{D} \cdots \tag{3.12}$$

is an exact sequence of sheaves over V.

(ii) $H^q(V, \mathscr{A}^q(E) \hat{\otimes} \mathscr{O}_D) = 0 \quad (q \geqq 1)$; *and*

(iii) *The* D*-cohomology of the complex* $\ldots \to \Gamma(\mathscr{V}, \mathscr{A}^q(E) \hat{\otimes} \mathscr{O}_D) \xrightarrow{D} \Gamma(\mathscr{V}, \mathscr{A}^{q+1}(E) \hat{\otimes} \mathscr{O}_D) \to \cdots$ *represents the sheaf cohomology* $H^*(\mathscr{V}, \mathscr{E}^\#)$.

Proof. (iii) follows from (i) and (ii) by the standard sheaf argument ([6]). By Lemma 3.6, (3.12) is exact except perhaps at $\mathscr{E}^\# \xrightarrow{j} A^0(E) \hat{\otimes} \mathscr{O}_D \xrightarrow{D} \mathscr{A}^1(E) \hat{\otimes} \mathscr{O}_D$; exactness here follows from (3.10). Finally, it has been pointed out to me by L. BUNGART that (ii) follows from a suitable generalization, given in BUNGART's thesis (Princeton University, 1962), of the Künneth formula of GROTHENDIECK ([5]). We use here that $H^p(X, \mathscr{A}^q(E)) = 0 = H^p(D, \mathscr{O}_D)$ $(p \geqq 1)$. Q.E.D.

4. The extension problem for fibered complex-analytic varieties

4.1. The extension problem in cohomology

Let $X \subset \mathscr{V}$ be a germ of an embedding such that $\mathscr{V}$ may be considered as a deformation $\{\mathscr{V} \xrightarrow{\bar{\omega}} D\}$ of the compact, complex manifold X. Let $E^\# \to \mathscr{V}$ be a holomorphic vector bundle and $E = E^\# | X$. By Theorems 3.3 and 3.2, the extension problem for $H(X, \mathscr{E})$ fits into the formal framework built in §§ 1 and 2. Our main results are then the following:

Theorem 4.1. *There are only finitely many obstructions to extending a class $\gamma \in H^q(X, \mathscr{E})$ to $H^q(\mathscr{V}, \mathscr{E}^{\#})$. Furthermore, if a formal extension exists, then an actual one does also.*

Theorem 4.2. *Let $S \subset H^q(X, \mathscr{E})$ be a subspace. Then the extension problem for S can be solved over $D(S) \subset D$ and $D(S)$ is a maximal such analytic set.*

Set $X_t = \tilde{\omega}^{-1}(t)$ $(t \in D)$ and $\boldsymbol{E}_t = \boldsymbol{E}^{\#} | X_t$.

Theorem 4.3. *The extension problem for $H^q(X, \mathscr{E})$ can be solved if dim $H^{q+1}(X_t, \mathscr{E}_t)$ is independent of t. If dim $H^{q+1}(X_t, \mathscr{E}_t)$ is independent of t, then dim $H^q(X_t, \mathscr{E}_t)$ is locally constant if, and only if, dim $H^{q-1}(X_t, \mathscr{E}_t)$ is locally constant.*

Suppose that dim $D = 1$.

Theorem 4.4. *For $t \neq 0$, $H^q(X_t, \mathscr{E}_t)$ is isomorphic to {extendable classes in $H^q(X, \mathscr{E})$}/{Jump classes in $H^q(X, \mathscr{E})$}.*

4.2. The extension problem for analytic fibre bundles

Let $X \subset \mathscr{V}$ be as above in 4.1, and suppose that $G \to P \to X$ is a holomorphic principal bundle. We wish to find a principal bundle $G \to \mathscr{P} \to \mathscr{V}$ such that $\mathscr{P} | X = P$.

From the fundamental bundle sequence $0 \to \boldsymbol{L} \to \boldsymbol{Q} \to \boldsymbol{T} \to 0$ (§ 1) we have seen (§ 3.2) that we get an exact sequence of graded Lie algebra complexes.

$$0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0 . \tag{4.1}$$

The deformation $\{\mathscr{V} \xrightarrow{\tilde{\omega}} D\}$ may, by § 3.1, be given by an element $\Phi(t) \in C^1\{t\}$. By Lemma 3.2, the extension problem for $G \to P \to X$ is the same as the extension problem for exact sequences of graded Lie algebra complexes considered in § 2.5. In order to apply the result there and in § 3.2, we must first settle two technical points: (i) We must assure that the mappings in (4.1) may be made compatible with harmonic theories on A, B, and C; and (ii) We must produce the maps $\omega \in \mathrm{Hom}(C^i, B^i)$ $(i = 1, 2)$ and $\Omega \in \mathrm{Hom}(C^1, A^2)$ which satisfy $\pi \circ \omega =$ Identity and $d(\omega\varphi) = \Omega(\varphi) + \omega(d\varphi)$ $(\varphi \in C^1)$.

Now (i) is easily arranged by choosing an Hermitian metric in B which induces a C^∞ splitting of (4.1) so that, for each $x \in X$, $\boldsymbol{Q}_x = \boldsymbol{L}_x \oplus \boldsymbol{T}_x$. In this case, ι and π clearly commute with d^* (= adjoint of $\bar{\partial}$), hence with $\square$, and finally with the Green's operators $\boldsymbol{G}$.

We may thus deal with (ii).

For a holomorphic vector bundle $E \to \boldsymbol{E} \to X$, we set $A^q(\boldsymbol{E}) = H^0(X, \mathscr{A}^q(\boldsymbol{E}))$; $H^q(\boldsymbol{A}(\boldsymbol{E}))$ is the q^{th} Dolbeault cohomology group. Let $0 \to \boldsymbol{E}' \xrightarrow{\iota} \boldsymbol{E} \xrightarrow{\pi} \boldsymbol{E}'' \to 0$ be an exact sequence of analytic vector bundles over X. From the exact sequences $0 \to \boldsymbol{A}^q(\boldsymbol{E}') \to \boldsymbol{A}^q(\boldsymbol{E}) \to \boldsymbol{A}^q(\boldsymbol{E}'') \to 0$, we get the exact cohomology sequence $\to H^q(\boldsymbol{A}(\boldsymbol{E})) \to H^q(\boldsymbol{A}(\boldsymbol{E}'')) \xrightarrow{\delta^q} H^{q+1}(\boldsymbol{A}(\boldsymbol{E}')) \to$. Consider the bundle $\mathrm{Hom}(\boldsymbol{E}'', \boldsymbol{E}')$. There is a natural pairing $\circ$: $\boldsymbol{A}^p(\mathrm{Hom}(\boldsymbol{E}'', \boldsymbol{E}')) \otimes \boldsymbol{A}^q(\boldsymbol{E}'') \to \boldsymbol{A}^{p+q}(\boldsymbol{E}')$ which satisfies

$$\bar\partial(\xi \circ \eta) = \bar\partial\xi \circ \eta + (-1)^p \xi \circ \bar\partial\eta \quad (\xi \in A^p(\mathrm{Hom}(\boldsymbol{E}'', \boldsymbol{E}')), \eta \in A^q(\boldsymbol{E}'')) . \tag{4.2}$$

Lemma 4.1. There exists an element $\Omega \in H^1(\boldsymbol{A}(\mathrm{Hom}(\boldsymbol{E}'', \boldsymbol{E}'))$ such that: (i) For any $\eta \in H^q(\boldsymbol{A}(\boldsymbol{E}''))$, $\Omega \circ \eta \in H^{q+1}(\boldsymbol{A}(\boldsymbol{E}'))$;
(ii) $\delta^q(\eta) = \Omega \circ \eta$.

Proof. Let I be the identity in $H^0(A(\boldsymbol{E}'', \boldsymbol{E}''))$ and choose $\omega \in A^0(\mathrm{Hom}(\boldsymbol{E}'', \boldsymbol{E}))$ such that $\pi \circ \omega = I$. Then $\Omega = \bar\partial\omega \in \boldsymbol{A}^1(\mathrm{Hom}(\boldsymbol{E}'', \boldsymbol{E}'))$ and it is easily checked that it satisfies the required conditions. Q.E.D.

In the case of the fundamental sequence $0 \to \boldsymbol{L} \to \boldsymbol{Q} \to \boldsymbol{T} \to 0$, the element $\omega \in \Gamma_\infty(X, \mathrm{Hom}(\boldsymbol{T}, \boldsymbol{T}(P)/G))$ geometrically gives a C^∞ connection of type $(1, 0)$ in $G \to P \to X$. The tensor Ω is a (1,1) form with values in $\boldsymbol{L}$, and gives the curvature of ω. Clearly ω and Ω satisfy the requirements of (ii) above.

Theorem 4.5. *The extension problem for $P \to X$ can be solved if $H^2(X, \mathscr{L}) = 0$. If $H^1(X, \mathscr{L}) = 0$, then any solution to the extension problem is unique.*

Proof. We need only prove uniqueness. Suppose that we have two analytic principal bundles $G \to \mathscr{P} \to \mathscr{V}$, $G \to \tilde{\mathscr{P}} \to \mathscr{V}$ which are both extensions of $G \to P \to X$. Let $\varrho: G \to GL(E)$ be any holomorphic linear representation, and form the associated bundles $E \to \boldsymbol{E}^\# \to V$, $E \to \tilde{\boldsymbol{E}}^\# \to V$. The vector bundle $\mathrm{Hom}(\boldsymbol{E}^\#, \tilde{\boldsymbol{E}}^\#)$ is an extension to $\mathscr{V}$ of $\mathrm{Hom}(\boldsymbol{E}, \boldsymbol{E}) \cong \boldsymbol{L}$ over $X \cong \tilde\omega^{-1}(0)$. Since $H^1(X, \mathscr{L}) = 0$, there exists, by Theorem 4.1, an extension $\Gamma \in H^0(\mathscr{V}, \mathrm{Hom}(\mathscr{E}^\#, \tilde{\mathscr{E}}^\#))$ of any class $\gamma \in H^0(X, \mathrm{Hom}(\mathscr{E}, \mathscr{E}))$. Taking $\gamma =$ identity, it follows that Γ establishes a bundle equivalence between $\boldsymbol{E}^\#$ and $\tilde{\boldsymbol{E}}^\#$. Q.E.D.

5. Some examples and applications

5.1. The extension problem for the groups $H^p(X, \Omega^q)$

Let $\{\mathscr{V} \xrightarrow{\tilde\omega} D\}$ be a deformation of X and let Ω^q be the sheaf of germs of holomorphic $(q, 0)$ forms on X. We set $h^{p,q} = \dim H^p(X, \Omega^q)$ and also $h_t^{p,q} = \dim H^p(X_t, \Omega_t^q)$ $(t \in D)$. Finally, we let $b_r = \dim H^r(X, \boldsymbol{C})$.

Proposition 5.1. If, for some r, $\sum_{p+q=r} h^{p,q} \leqq b_r$, then the extension problem can be solved for all the groups $H^p(X, \Omega^q)$ $(p + q = r)$. In particular, if X is Kähler, then the extension problem can be solved for the groups $H^p(X, \Omega^q)$.

Proof. We recall the inequality of FRÖLICHER: $\sum_{p+q=r} h^{p,q} \geqq b_r$. From this, using upper semi-continuity, it follows that $h_t^{p,q}$ is locally a constant function of t. We complete the proof by giving a proof of the inequality $\sum_{p+q=r} h^{p,q} \geqq b_r$.

If $A^{p,q}$ is the vector space of global $C^\infty(p,q)$ forms on X, then $A = \sum_{p,q} A^{p,q}$ forms a double complex with differential operators $d, \partial, \bar\partial$ satisfying $d = \partial + \bar\partial$ and $\partial\bar\partial + \bar\partial\partial = 0$. By [4], § 4.8, there is a spectral sequence $\{E_r^{p,q}\}$ such that E_∞ is associated to $H^*(A) \simeq H^*(X, \boldsymbol{C})$ (de Rham), and such that $E_1^{p,q} = H_{\bar\partial}(A^{p,q}) \simeq H^p(X, \Omega^q)$ (Theorem of DOLBEAULT). But then clearly $\sum_{p+q=r} \dim E_1^{p,q} \geqq \dim H^r(X, \boldsymbol{C})$. Q.E.D.

5.2. An example of a non-extendible abelian differential

Let F be the complex Lie group of complex matrices

$f = \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix}$; let $\Gamma \subset F$ be the discrete subgroup

of matrices $\gamma = \begin{pmatrix} 1 & \gamma_1 & \gamma_3 \\ 0 & 1 & \gamma_2 \\ 0 & 0 & 1 \end{pmatrix}$ where the γ_i are

Gaussian integers. The manifold $X = F/\Gamma$ is a compact, complex manifold which was first discussed by Iwasawa. A basis for the abelian differentials on X is given by the right-invariant holomorphic Maurer-Cartan forms on X. These are: $\omega_1 = dz_1$, $\omega_2 = dz_2$, $\omega_3 = -z_2 dz_1 + dz_3$. The dual holomorphic vector fields are:

$$\Theta_1 = \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_3}, \Theta_2 = \frac{\partial}{\partial z_2}, \Theta_3 = \frac{\partial}{\partial z_3}.$$

The element

$$\varphi = \Theta_2 \otimes \bar\omega_2 \left(= \frac{\partial}{\partial z_2} \otimes \overline{dz_2}\right)$$

gives a non-zero element of $H^1(X, \Theta)$, and $[\varphi, \varphi] \equiv 0$. Thus $\Phi(t) = t\varphi$ gives a 1-parameter family $\{\mathscr{V} \to D\}$ of deformations of X.

Consider the abelian differential $\omega = \omega_3 = -z_2 dz_1 + dz_3 \in H^0(X, \Omega^1)$. We have: $[\varphi, \omega] = d\omega \bar\wedge \varphi - d(\omega \bar\wedge \varphi) = \partial\omega \bar\wedge \varphi = (\omega_1 \wedge \omega_2) \bar\wedge \Theta_2 \otimes \bar\omega_2 = \omega_1 \otimes \bar\omega_2 \neq 0$ in $H^1(X, \Omega^1)$. Thus the extension problem for ω cannot be solved over $\{\mathscr{V} \to D\}$.

5.3. Stability of automorphisms under deformations

Let X be as above and let Γ be the identity component of the complex Lie group of analytic automorphisms of X. Then Γ acts analytically in the bundle $C^n \to T \to X$, and there are induced holomorphic representations $\varrho^q : \Gamma \to GL(H^q(X, \Theta))$.

Suppose that we have a deformation of $X, \{\mathscr{V} \to D\}$, given by $\Phi(t) = \sum_{\mu=1}^{\infty} \varphi_\mu(t) \in C^1\{t\}$, and assume that the classes φ_1^α span $H^1(X, \Theta)$ $(\varphi_1(t) = \sum_{\alpha=1}^{m} \varphi_1^\alpha t_\alpha)$. Let Γ_t be the identity component of the analytic automorphism group of $X_t (t \in D)$.

Proposition 5.2. For $t \neq 0$, $\Gamma_t \subseteqq \operatorname{Ker}(\varrho^1)$.

Remark. This Theorem shows how we may think of automorphisms as being "exceptional phenomena".

Proof. Let $\mathfrak{g}_t$ be the complex Lie algebra of $\Gamma_t : \mathfrak{g}_0 = \mathfrak{g}$. The infinitesimal representation of Γ on $H^1(X, \Theta)$ is given by the bracket [,]; i.e. for $\gamma \in \mathfrak{g}$, $\varphi \in H^1(X, \Theta)$, we have that $d\varrho^1(\gamma)(\varphi) = [\gamma, \varphi]$. Thus the subspace of $H^0(X, \Theta)$ for which the extension problem can be solved is a subspace of $\{\operatorname{Ker} d\varrho^1\} \subseteqq \mathfrak{g}$. The theorem now follows. Q.E.D.

Corollary. The "general" deformation of a simply-connected compact homogeneous complex manifold is non-homogeneous.

5.4. Extension of holomorphic mappings

Let X be a compact, complex manifold, let $\boldsymbol{G}$ be a Grassmann variety, and let $f : X \to \boldsymbol{G}$ be a holomorphic mapping. Let $G \to B \to \boldsymbol{G}$ be the universal principal bundle over $\boldsymbol{G}$ and let $E \to \boldsymbol{F} \to \boldsymbol{G}$ be the universal vector bundle. Set $P = f^{-1}(B)$, $\boldsymbol{E} = f^{-1}(\boldsymbol{F})$. Let $0 \to \boldsymbol{L} \to \boldsymbol{Q} \to \boldsymbol{T} \to 0$ be the fundamental bundle sequence of $G \to P \to X$.

Theorem 5.1. (i) *There exists a maximal germ of deformation* $\{\mathscr{V}_f \xrightarrow{\tilde{\omega}} D_f\}$ *of* X *for which there is a holomorphic mapping* $F : \mathscr{V}_f \to \boldsymbol{G}$ *such that* $F \mid \tilde{\omega}^{-1}(0) = f$.

(ii) *If* f *is an embedding, then* F *is an embedding on fibres.*

(ii) $\{\mathscr{V}_f \to D_f\}$ *coincides with Kuranishi's family if*

$$H^2(X, \mathscr{L}) = 0 = H^1(X, \mathscr{E}).$$

Proof. Giving $f : X \to \boldsymbol{G}$ is equivalent to giving an analytic vector bundle $E \to \boldsymbol{E} \to X$ such that the sections $H^0(X, \mathscr{E})$ generate each fibre $\boldsymbol{E}_x (x \in X)$. Thus, in order to extend f to a deformation $\{\mathscr{V} \to D\}$ of X, we must solve the extension problems for $E \to \boldsymbol{E} \to X$ and $H^0(X, \boldsymbol{E})$ over $\mathscr{V}$. We first solve the extension problem for $E \to \boldsymbol{E} \to X$ in a maximal way, and then we solve the extension problem for $H^0(X, \mathscr{E})$ in a maximal way. The rest is clear. Q.E.D.

5.5. The direct image theorem for fibered analytic spaces

Let $\mathscr{V}$ be a complex space and let $\tilde{\omega}:\mathscr{V} \to D$ be a proper holomorphic mapping of maximal rank whose fibres are connected and non-singular. Let $\boldsymbol{E}^{\#} \to V$ be a holomorphic vector bundle; set $X_t = \omega^{-1}(t)$ and $\boldsymbol{E}_t = \boldsymbol{E}^{\#} | X_t$ for $t \in D$.

Proposition 5.3. The sets $D^{q,k} = \{t \in D \,|\, \dim H^q(X_t, \mathscr{E}_t) \geqq k\}$ are analytic subsets of D.

In fact, from §§2 and 3 we have the following more general result, which is still a special case of Grauert's Theorem [*Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen.* Inst. Hautes Études Sci., Publ. Math **5**, 1–64 (1960)].

Proposition 5.4. The direct image sheaves $\mathscr{R}^q(\tilde{\omega}, \mathscr{E}^{\#})$ are coherent analytic sheaves over D.

Remark. In §1.3 we have exhibited explicitly a finite set of generators and relations for each stalk

$$\mathscr{R}^q(\tilde{\omega}, \mathscr{E}^{\#})_t = \lim_{U \supset \{t\}} H^q(\tilde{\omega}^{-1}(U), \mathscr{E}^{\#} \,|\, \tilde{\omega}^{-1}(U)).$$

5.6. On the local triviality of certain analytic fibre spaces

We give another application which generalizes a theorem of Kodaira-Spencer. Let $\mathscr{V}$, D be compact, complex manifolds and $\tilde{\omega}:\mathscr{V} \to D$ a proper holomorphic mapping of maximal rank. Set $X_t = \tilde{\omega}^{-1}(t)$ $(t \in D)$ and suppose that $H^1(X_t, \Theta_t) = 0$ for all $t \in D$. Suppose furthermore that, for some t_0, X_{t_0} is bi-holomorphically equivalent to a rational homogeneous manifold G/U where G and U are suitable complex algebraic Lie groups. (Then automatically $H^1(X_{t_0}, \Theta_{t_0}) = 0$.)

Theorem 5.2. *$\mathscr{V}$ is a locally trivial fibre bundle over D with typical fibre G/U. In fact, G acts on $\mathscr{V}$ as a complex Lie group of bi-holomorphic transformations and $D = \mathscr{V}/G$.*

Remark. The assumption $H^1(X_t, \Theta_t) = 0$ is necessary, as the family of Hirzebruch surfaces shows (cf. § 5.8 below).

Proof. Since $\dim H^1(X_t, \Theta_t) = 0$, we may locally solve the extension problem for $H^0(X_{t_0}, \Theta_{t_0}) \cong \mathfrak{g}$ where $\mathfrak{g}$ is the complex Lie algebra of G. But then it is easy to see that there exists on $\mathscr{V}$ a complex Lie algebra, isomorphic to $\mathfrak{g}$, of vertical holomorphic vector fields. From this, it follows that G acts on $\mathscr{V}$ effectively as a group of bi-regular transformations. Then we may form the holomorphic vector bundle $\boldsymbol{T}(\mathscr{V})/G$ over D, and there is an onto bundle mapping $\boldsymbol{T}(\mathscr{V})/G \xrightarrow{\pi} \boldsymbol{T}(D) \to 0$ $(\pi = \tilde{\omega}_*)$. Thus we get over D an exact sequence of holomorphic vector bundles $0 \to \boldsymbol{L} \to \boldsymbol{T}(\mathscr{V})/G \to \boldsymbol{T}(D) \to 0$, and we let $0 \to \mathscr{L} \to \Xi \to \Sigma \to 0$ be the corresponding exact sheaf sequence. It is easy to see that, since

$H^1(X_t, \Theta_t) = 0$, this exact sheaf sequence is locally split. Thus, given locally n independent holomorphic vector fields $\sigma_1, \ldots, \sigma_n$ on D, there exist n independent G-invariant holomorphic vector fields $\gamma_1, \ldots, \gamma_n$ on $\mathscr{V}$ such that $\tilde{\omega}_*(\gamma_j) = \sigma_j$. But then these holomorphic vector fields may be used to give a local holomorphic cross-section $\sigma: D \to \mathscr{V}$ passing through any point $v \in \mathscr{V}$. The Theorem now follows.

5.7. An interpretation of the integrability equation 1.6

Let (A, d) be a graded Lie algebra complex and let $\varphi(t) = \sum_{\mu=1}^{\infty} \varphi_\mu t^\mu \in$ $\in A^1\{t\}$ satisfy

$$d\varphi(t) - [\varphi(t), \varphi(t)] = 0 . \tag{5.1}$$

If we let $\psi(t) = \frac{\partial\varphi(t)}{\partial t} \in A^1[t]$, then by differentiating (5.1) we get

$$d\psi(t) - 2[\varphi(t), \psi(t)] = 0 . \tag{5.2}$$

Since $\psi(0) = \varphi_1 = \left.\frac{\partial\varphi(t)}{\partial t}\right]_{t=0} \in H^1(A)$, we have

Proposition 5.5. If $\varphi_1 \in H^1(A)$ is tangent to an integrable family given by $\varphi(t) \in A^1\{t\}$, then φ_1 is extendible along this family.

Remark. This Proposition is rather obvious geometrically. Moreover, it is clear that (in case $\varphi(t)$ converges), for each fixed t_0, $\psi(t_0) \in H^1{}_{\varphi(t_0)}(A)$ is tangent to an integrable family based at $\varphi(t_0)$. Observe also that the converse to Proposition 5.5 is true.

5.8. Automorphisms and jumping of structure

Our next application concerns the following remark of Mumford: Let $\{\mathscr{V} \xrightarrow{\tilde{\omega}} D\}$ be an algebraic family of algebraic varieties and X a variety such that $X_t \cong X$ for $t \neq 0$ but $X_0 \neq X$ (this is a so-called *jumping of structure*). Then $\dim H^0(X_0, \Theta_0) \geqq 1$.

By the theory of § 1 we can show explicitly where the holomorphic vector field on X_0 comes from:

Proposition 5.6. Let $\{\mathscr{V} \xrightarrow{\tilde{\omega}} D\}$ be an analytic family of compact, complex manifolds where there is a jumping of structure. Then $\dim H^i(X_0, \Theta_0) \geqq \dim H^1(X_t, \Theta_t) + 1$ $(i = 0, 1; t \neq 0)$. More precisely, there exists a jump class $\psi \in H^1(X_0, \Theta_0)$ which obstructs an element $\theta \in H^0(X_0, \Theta_0)$.

Proof. For simplicity, assume $\dim D = 1$. We record two obvious remarks: (i) if a germ of deformation $\mathscr{V} \xrightarrow{\tilde{\omega}} D$ is trivial, then the tangent $\tau_0 \in H^1(X_0, \Theta_0)$ is zero; (ii) if a germ of deformation $\mathscr{V} \xrightarrow{\tilde{\omega}} D$ is not trivial, then, for any neighborhood U of $0 \in D$, there exists a $t \in U$ such that $\tau_t \neq 0$ in $H^1(X_t, \Theta_t)$.

The family $\{\mathscr{V} \xrightarrow{\tilde{\omega}} D\}$ is given by a holomorphic function $\varphi(t) \in A^1\{t\}$ satisfying $d\varphi(t) - [\varphi(t), \varphi(t)] = 0$. By § 5.7 above, the elements $\psi_t = \frac{\partial \varphi(t)}{\partial t}$ give a family of classes in $H^1(X_t, \Theta_t)$ which are tangent at X_t to the deformation $\{\mathscr{V} \xrightarrow{\tilde{\omega}} D\}$. The germ of deformation which $\{\mathscr{V} \xrightarrow{\tilde{\omega}} D\}$ defines is not trivial at 0 but is trivial at any $t \neq 0$. Thus $\psi = \psi_0$ is a non-zero class in $H^1(X_0, \Theta_0)$ but ψ_t defines the zero class in $H^1(X_t, \Theta_t)$ for $t \neq 0$. Thus the element $\psi \in H^1(X_0, \Theta_0)$ is a jump class (i. e. it drops off into a boundary for $t \neq 0$), and by § 1 it obstructs an element $\theta \in H^0(X_0, \Theta_0)$. Q. E. D.

Remark. It is perhaps interesting to compare this with Mumford's argument in the algebraic case. Let D be the affine line and $\mathscr{W} = X \times D$. Since $\{\mathscr{V} \xrightarrow{\tilde{\omega}} D\}$ ($\tilde{\omega}$ is now a regular map) is a jumping of structure, there is a meromorphic mapping $F: \mathscr{W} \to \mathscr{V}$ defined except on $X \times \{0\}$. Now $\partial/\partial t$ is a regular vector field on $\mathscr{W}$ and so $F_*(\partial/\partial t)$ has at worst a finite pole on $X_0 = \tilde{\omega}^{-1}(0)$. Thus, for some smallest integer m, $t^m F_*(\partial/\partial t)$ is a regular vector field θ on $\mathscr{V}$; we shall show that θ is tangent to X_0.

Now, since
$$\begin{array}{ccc} \mathscr{W} & \xrightarrow{F} & \mathscr{V} \\ \downarrow & & \downarrow \tilde{\omega} \\ D & \to & D \end{array}$$
commutes, θ restricted to each fibre X_t has constant projection on D. If $\tilde{\omega}_*(\theta) \neq 0$ on X_0, then the local 1-parameter group generated by θ would move X_0 biregularly onto X_{t_0} for some $t_0 \neq 0$. However, this is impossible by assumption. Q. E. D.

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