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THE EXTENSION PROBLEM IN COMPLEX ANALYSIS II; EMBEDDINGS WITH POSITIVE NORMAL BUNDLE.

By PHILLIP A. GRIFFITHS.

The extension problem we have in mind is simply to extend certain "analytic objects" from a compact analytic submanifold X of a complex manifold W to a neighborhood of X in W. The analytic objects are to be such things as holomorphic vector bundles, cohomology classes, or holomorphic mappings. This problem arises naturally when one considers the following question: How much about W can we determine knowing X and the normal bundle of X in W? In [7] we have discussed the case of a trivial normal bundle; here we shall mainly be concerned with the situation when the normal bundle is positive. The main results of this paper are stated in § II.

Now the extension problem falls naturally into two pieces. On is the formal problem which is easy to treat using sheaf theory. The point is that the obstructions to extending analytic objects are cohomology classes with coefficients in a coherent analytic sheaf which involves the symmetric powers of the dual normal bundle (§ 1). The other part of the extension problem is the question of existence. Here our approach has been to use geometric arguments (based on projective embeddings and the theory of continuous systems) to reduce the existence question to linear problems (§ III). By use of monoidal transformations (§ IV) we are able to reduce the linear problems to proving a "vanishing theorem," an open manifolds, for cohomology with coefficients in the sheaf of germs of holomorphic cross sections of an analytic vector bundle. Our technique for this is to use the theory of harmonic integrals developed by Joe Kohn [15] together with a systematic treatment of Hermitian differential geometry (§§ VI and VII). In § VIII we complete the proof of our vanishing theorem.

The central part of the present paper is, we feel, the various differential-geometric calculations which give a precise analytic reflection of the geometric assumptions in the problem. As one example, in the appendix to § IV we use curvature arguments to characterize positive vector bundles as those for which a suitable set of integrals over subvarieties are positive. Again, in § V,

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we use curvature calculations to prove the following result: If $X \subset W$ is a non-singular hypersurface in a closed algebraic variety W, if $X \subset W$ looks as though it should be the monoidal transform of some $X' \subset W'$, and if the (hypothetical) normal bundle of $X' \subset W'$ is positive, then X' and W' exist and $X \subset W$ is the monoidal transformation along X' in W'. Finally, curvature arguments lead to two integral formulae (§§ VII. 3 and VII. 5) which, among other things, yield the required vanishing theorems.

To close the introduction, we should like to remark on the question which was the original motivation for the present study. This is the rigidity problem, of Nirenberg and Spencer [17], which goes as follows: Given a germ of embedding $X \subset W (\dim X \geqq 2)$ with positive normal bundle, is W determined by a finite neighborhood of X? Hironaka has answered this question affimatively for hypersurfaces, and we give also an affirmative answer in general, provided that the normal bundle of X in W is sufficiently positive (geometrically, the complete continuous system generated by X must be large).

Concerning notation, $X, W, Y, \cdots$ are non-singular complex manifolds and $\mathfrak{O}_X, \mathfrak{O}_W, \mathfrak{O}_Y, \cdots$ are the structure sheaves. (As usual, we write $\mathfrak{O}$ for $\mathfrak{O}_X$ if no confusion is possible.) A holomorphic vector bundle over X will be denoted by $\boldsymbol{E} \to X$; $\Omega(\boldsymbol{E})$ is the sheaf of germs of holomorphic sections of $\boldsymbol{E}$. We let $\boldsymbol{T}(X)$ be the holomorphic tangent bundle of X and $A^{p,q}(\boldsymbol{E})$ the sheaf of germs of C^∞ sections of $\boldsymbol{E} \otimes \wedge^p \boldsymbol{T}(X)^* \otimes \wedge^q \overline{\boldsymbol{T}(X)}^*$ (here $\boldsymbol{E}^*$ is the dual bundle to $\boldsymbol{E}$). Also we set $A^q(\boldsymbol{E}) = A^{0,q}(\boldsymbol{E})$. The other notations are the standard ones from [9].

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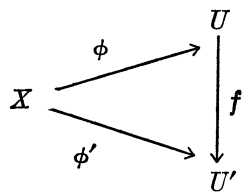
Bibliography.

I. The Extension Problem for Analytic Objects.

1. Germs of embeddings and the system of jet neighborhoods of a Submanifold. Let X be a compact, complex manifold and consider an embedding $\phi: X \to W$ of X in a complex manifold W. We shall consider another such embedding $\phi': X \to W'$ as equivalent to $\phi: X \to W$ if there exist open neighborhoods U of $\phi(X)$ in W and U' of $\phi'(X)$ in W' and a bi-holomorphic mapping $f: U \to U'$ such that



commutes. Thus we are considering *germs of embeddings,* and we shall usually suppress the mapping ϕ and write $X \subset W$ as a representative of the germ of embedding defined by $\phi: X \to W$. If we let $\mathfrak{O}_W$ be the structure sheaf of W and set $\mathfrak{O}_* = \mathfrak{O}_W \mid X$, then the pair $(X, \mathfrak{O}_*)$ forms a generalized complex space X_*. Clearly X_* depends only on, and in fact determines, the germ of embedding defined by $X \subset W$.

Let $\mathfrak{I} \subset \mathfrak{O}_W$ be the ideal sheaf of X. Then $\mathfrak{O}_W/\mathfrak{I} = \mathfrak{O}$ is a sheaf supported on X and is, in fact, the structure sheaf of X. For each $\mu \geqq 0$, we let $\mathfrak{O}^{(\mu)} = \mathfrak{O}_W/\mathfrak{I}^{\mu+1}$. The pair $(X, \mathfrak{O}^{(\mu)}) = X_\mu$ forms again a generalized complex space which may be thought of as the neighborhood of order μ, or *μ-neighborhood,* of $X \subset W$. By the *stable neighborhood* of $X \subset W$, we mean the μ-neighborhood for large μ.

There are obvious inclusions $X_\mu \to X_{\mu+\nu}$ $(\nu \geqq 0)$ which lead to maps $\mathfrak{O}^{(\mu+\nu)} \to \mathfrak{O}^{(\mu)}$. The inverse limit sheaf $\hat{\mathfrak{O}} = \lim_{\leftarrow \mu} \mathfrak{O}^{(\mu)}$ then exists, and the

pair $(X, \hat{\mathcal{O}})$ forms again a complex space called the *formal neighborhood* determined by $X \subset W$.

2. Extension problem for analytic objects. Given $X \subset W$, we have in mind extending certain "analytic objects" from X to W. By an *analytic object,* we mean something along the following lines: (i) an analytic vector bundle $\boldsymbol{E} \to X$, (ii) an element $\sigma \in H^q(X, \Omega(\boldsymbol{F}))$ where $\boldsymbol{F} \to X$ is the restriction of a holomorphic vector bundle $\boldsymbol{F}^{\#} \to W$; or (iii) a holomorphic mapping $f: X \to Z$ for some complex manifold Z. To *extend an analytic object* α on X means to find α_* on X_* such that $\alpha_* \mid X = \alpha$; to *formally extend* α means that there exists $\hat{\alpha}$ on $\hat{X}$ with $\hat{\alpha} \mid X = \alpha$. Finally, the *stable extension problem* is the following: Given α on X, can we find a $\mu_0 = \mu_0(\alpha)$ such that, for any germ of embedding $X \subset W'$ for which $X'_\mu \cong X_\mu$ and where there exists α'_μ on X'_μ with $\alpha'_\mu \mid X = \alpha$ $(\mu \geqq \mu_0)$, there exists α'_* on X'_* with $\alpha'_* \mid X = \alpha$? Similarly, we may speak of the *formal stable extension problem.* Throughout, the adjective "stable" should be thought of as meaning "finitely many obstructions."

Now whenever the extension problem is defined, we may speak of the *uniqueness problem* (if α_* and β_* are extensions of α, then does $\alpha_* = \beta_*$?) and the *formal uniqueness problem.* More generally, we may speak of the *stable uniqueness problem* (if $\alpha_* \mid X_\mu = \beta_* \mid X_\mu$, then does $\alpha_* = \beta_*$ provided that $\mu \geqq \mu_0$?) and the *formal stable uniqueness problem.*

After this list of definitions, it is perhaps well to state the general problem which has served as a guideline: Given an algebraic variety W and a subvariety $X \subset W$, how much can we say about W on the basis of the information given by X_μ for finite μ?

3. The obstructions to the formal extension problem. Given $X \subset W$, there is a sequence of inclusions $X \subset X_1 \subset \cdots \subset X_\mu \subset X_{\mu+1} \subset \cdots$. Given then an analytic object α on X, we may try to extend it step by step; if this can be done, then we have solved the formal extension problem for α. If we have found an extension $\alpha_{\mu-1}$ of α to $X_{\mu-1}$, then there will be a cohomology class $\omega(\alpha_{\mu-1})$ which will be the μ-*th* obstruction to extending α. If all $\omega(\alpha_{\mu-1}) = 0$, then we may formally extend α.

Before stating explicitly where $\omega(\alpha_{\mu-1})$ belongs, we introduce the following notations: $\boldsymbol{N} \to X$ is the *normal bundle* of $X \subset W$, $\boldsymbol{N}^*$ is the dual bundle, and $\boldsymbol{N}^{*\mu}$ is the μ-th symmetric product of $\boldsymbol{N}^*$. For any holomorphic vector bundle $\boldsymbol{F} \to X$, $\boldsymbol{F}(\mu) = \boldsymbol{F} \otimes \boldsymbol{N}^{*\mu}$. Finally, $\boldsymbol{T} = \boldsymbol{T}(W) \mid X$, $\boldsymbol{T}(W)$ being the tangent bundle of W. We have the usual exact bundle sequence

$$0 \to \boldsymbol{T}(X) \to \boldsymbol{T} \to \boldsymbol{N} \to 0. \tag{1.1}$$

PROPOSITION 1.1. *If α is a holomorphic vector bundle $\boldsymbol{E} \to X$, then* $\omega(\alpha_{\mu-1}) \in H^2(X, \Omega(\mathrm{Hom}(\boldsymbol{E}, \boldsymbol{E})(\mu)))$.

PROPOSITION 1.2. *If α is an element in $H^q(X, \Omega(\boldsymbol{F}))$, then* $\omega(\alpha_{\mu-1}) \in H^{q+1}(X, \Omega(\boldsymbol{F}(\mu)))$.

PROPOSITION 1.3. *If α is a holomorphic mapping $f: X \to Z$, then* $\omega(\alpha_{\mu-1}) \in H^1(X, \boldsymbol{T}^{\#}(\mu)))$ *where* $\boldsymbol{T}^{\#} = f^{-1}(\boldsymbol{T}(Z))$.

Proof of Proposition 1.2. Over X we have

$$0 \to \Omega(\boldsymbol{N}^{*\mu}) \to \mathfrak{O}^{(\mu)} \to \mathfrak{O}^{(\mu-1)} \to 0 \tag{1.2}$$

which immediately gives

$$0 \to \Omega(\boldsymbol{F}(\mu)) \to \mathfrak{O}^{(\mu)}(\boldsymbol{F}^{\#}) \to \mathfrak{O}^{(\mu-1)}(\boldsymbol{F}^{\#}) \to 0 \tag{1.3}$$

where $\mathfrak{O}^{(\mu)}(\boldsymbol{F}^{\#}) = \mathfrak{O}^{(\mu)} \otimes_{\mathfrak{O}_*} \Omega(\boldsymbol{F}^{\#})$ are the sections up to order μ of $\boldsymbol{F}^{\#}$ along X. The cohomology sequence of (1.3) gives

$$H^q(\mathfrak{O}^{(\mu)}(\boldsymbol{F}^{\#})) \to H^q(\mathfrak{O}^{(\mu-1)}(\boldsymbol{F}^{\#})) \xrightarrow{\delta} H^{q+1}(\Omega(\boldsymbol{F}(\mu)))\,;$$

clearly, $\alpha_{\mu-1} \in H^q(\mathfrak{O}^{(\mu-1)}(\boldsymbol{F}^{\#}))$ and the obstruction to extending $\alpha_{\mu-1}$ is $\delta(\alpha_{\mu-1}) \in H^{q+1}(\Omega(\boldsymbol{F}(\mu)))$.

Proof of Proposition 1.3. We first claim that it will suffice to assume that $f: X \to Z$ is an embedding. Indeed, replace Z by $Z' = W \times Z$ and f by f' where $f'(x) = (x, f(x))$. If we can find an extension F' of f', then $F'(w) = (g(w), F(w))$ and F will be an extension of f. Furthermore, $f'^{-1}(\boldsymbol{T}(Z')) = \boldsymbol{T} \oplus \boldsymbol{T}^{\#}$ and so, if Proposition 1.3 holds for embeddings, then the obstructions to extending F' will lie in $H^1(\Omega(\boldsymbol{T}(\mu))) \oplus H^1(\Omega(\boldsymbol{T}^{\#}(\mu)))$. Finally, if we require that the first factor g of F' should be the identity, $g(w) = w$, then the obstructions to finding F will clearly lie in $H^1(\Omega(\boldsymbol{T}^{\#}(\mu)))$. Thus we may suppose that f is an embedding and we may even require that the extension F of f should be an embedding of W.

Suppose then that we have $f_{\mu-1}: X_{\mu-1} \to Z$ and assume, for simplicity,* that $\mu \geqq 2$. We first choose a coordinate covering $\{U_\alpha\}$ of W by polycylinders with coordinates $(z_\alpha, w_\alpha) = (z_\alpha^1, \cdots, z_\alpha^n; w_\alpha^1, \cdots, w_\alpha^r)$ in U_α such that $U_\alpha \cap X = \{w_\alpha = 0\}$. Then $f_{\mu-1} \mid U_\alpha = f_\alpha$ gives an embedding of U_α into Z, and so we may cover Z by polycylinders V_α with coordinates $(z_\alpha, w_\alpha, \xi_\alpha)$ in V_α such that $f_\alpha(z_\alpha, w_\alpha) = (z_\alpha, w_\alpha, 0)$. Now

* The case $\mu = 1$ is given in 1.5, section (C) below.

$$(f_\alpha \circ f_\beta^{-1})(z_\beta, w_\beta) = (z_\alpha(z_\beta, w_\beta),\ w_\alpha(z_\beta, w_\beta), \xi_\alpha(z_\beta, w_\beta))$$

is a power series of the form

$$\sum_{\mu_1+\cdots+\mu_r=\mu} \Lambda_{\alpha\beta\mu_1\cdots\mu_r}(z_\beta) w_\beta^{\mu_1} \cdot \cdots \cdot w_\beta^{\mu_r} + [\mu+1]$$

where the $\Lambda_{\alpha\beta\mu\cdots\mu_r}(z_\beta)$ are vector valued functions and $[\mu+1]$ vanishes to order $\mu+1$ in w_β. Then $\Lambda = \{\Lambda_{\alpha\beta_{\mu_1\cdots\mu_r}}\}$ gives a 1-cochain with values in

$$\Omega(\boldsymbol{T}(Z) \mid f(X)) \otimes \mathfrak{I}^\mu/\mathfrak{I}^{\mu+1} \cong \Omega(\boldsymbol{T}^\#(\mu)).$$

It is, of course, straightforward to check that $\delta\Lambda = 0$; that $\Lambda - \Lambda' = \delta\lambda$ if, by a different coordinatization, we arrive at another 1-cochain Λ'; and that, if $\Lambda = \delta\Phi$, then we may lift $f_{\mu-1}$ to $f_\mu: X_\mu \to Z$. Clearly then $\omega(f_{\mu-1}) = \Lambda \in H^1(\Omega(\boldsymbol{T}^\#(\mu)))$.

We shall omit the proof of Proposition 1.1.

4. The obstructions to uniqueness. From § 1.3 above, it is fairly clear that the obstructions to uniqueness are cohomology classes in dimension one less than the obstructions to extension. However, uniqueness must be interpreted in an individual manner for each type of analytic object. For cohomology classes, the meaning of uniqueness of extension is obvious; for a holomorphic mapping $f: X \to Z$, uniqueness means that the power series expansion of an extension $F: W \to Z$ is uniquely determined. For bundles, to say that two extensions $\boldsymbol{E}^\# \to W$ and $\boldsymbol{E}^{\#'} \to W$ of $\boldsymbol{E} \to W$ are the same means that there is a bundle equivalence

$$\begin{array}{ccc} G: \boldsymbol{E}^\# & \to & \boldsymbol{E}^{\#'} \\ \downarrow & & \downarrow \\ W & \to & W. \end{array}$$

PROPOSITION 1.4. *Referring to Propositions* 1.1-1.3, *the obstructions to uniqueness for the extension problem are classes in* $H^1(\Omega(\mathrm{Hom}(\boldsymbol{E}, \boldsymbol{E})(\mu)))$, $H^q(\Omega(\boldsymbol{F}(\mu)))$, *and* $H^0(\Omega(\boldsymbol{T}^\#(\mu)))$ *respectively.*

5. Same special cases.

(a) Line bundles. We use the notation $\Omega(\mu) = \Omega(\boldsymbol{N}^{*\mu})$. The μ-th obstruction to extending a line bundle $\boldsymbol{E} \to X$ is an element in $H^2(X, \Omega(\mu))$. In order to check this directly, we let $\mathfrak{O}_X$, $\mathfrak{O}_W$ be the structure sheaves of X, W and $\mathfrak{O}_X{}^*$, $\mathfrak{O}_W{}^*$ the corresponding sheaves of units. Denote by $\mathfrak{I}^* \subset \mathfrak{O}_W{}^*$ the germs of the form $\exp(2\pi\sqrt{-1}\, f)$ where $f \in \mathfrak{I}$. Then, considering $X \subset W$ as a germ of embedding, $\exp: \mathfrak{I} \to \mathfrak{I}^*$ is an isomorphism, and we get a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \to & \mathfrak{I} & \to & \mathfrak{I}^* & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \boldsymbol{Z} & \to & \mathfrak{O}_W & \to & \mathfrak{O}_W^* \mid X & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \boldsymbol{Z} & \to & \mathfrak{O}_X & \to & \mathfrak{O}_X^* & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The extension problem for line bundles is to lift $\xi \in H^1(\mathfrak{O}_X^*)$ to $H^1(\mathfrak{O}_W^* \mid X)$. From the exact cohomology diagram of (1.4), the obstruction to lifting ξ is in $H^2(\mathfrak{I}^*)$. Now $\mathfrak{I}$ has a *decreasing filtration* by the subsheaves $\mathfrak{I}^\mu$ $(\mu \geqq 1)$, and the *associated graded sheaf* is $\sum_{\mu \geqq 1} \Omega(\mu)$. Thus, the μ-th obstruction to extending ξ is in $H^2(\Omega(\mu))$.

Slightly more generally, we have

PROPOSITION 1.5. *Let Z be a complex manifold and let $Y \subset Z$ be either a complex submanifold or an open subset. Assume that we have the sequences*

$$(1.5) \qquad \begin{cases} 0 \to H^2(Z, \mathfrak{O}_Z) \to H^2(Y, \mathfrak{O}_Y) \\ \quad H^2(Z, \boldsymbol{Z}) \to H^2(Y, \boldsymbol{Z}) \to 0 \\ \quad H^1(Z, \mathfrak{O}_Z) \to H^1(Y, \mathfrak{O}_Y) \to 0. \end{cases}$$

Then every line bundle on Y extends to Z.

Proof. This follows from the exact diagram

$$(1.6) \qquad \begin{array}{ccccccc}
H^1(Z, \mathfrak{O}_Z) & \to & H^1(Z, \mathfrak{O}_Z^*) & \to & H^2(Z, \boldsymbol{Z}) & \to & H^2(Z, \mathfrak{O}_Z) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1(Y, \mathfrak{O}_Y) & \to & H^1(Y, \mathfrak{O}_Y^*) & \to & H^2(Y, \boldsymbol{Z}) & \to & H^2(Y, \mathfrak{O}_Y).
\end{array}$$

(b) Transverse foliation. An embedding $X \subset W$ is called *transversely foliated* if, through each $x \in X$, there passes a complex subvariety, of dimension complementary to X, $S_x \subset W$ such that $S_x \cap X = \{x\}$ and $S_x \cap S_{x'} = \phi$ for $x \neq x'$.

PROPOSITION 1.6. *The obstructions to transversely foliating $X \subset W$ are classes in $H^1(X, \Omega(\boldsymbol{T}(X)(\mu)))$.*

Proof. $X \subset W$ is transversely foliated if, and only if, there exists a holomorphic map $F: W \to X$ such that $F \mid X =$ identity. The obstructions to finding such an F are in $H^1(\Omega(\boldsymbol{T}(X)(\mu)))$ since, in this case, $\boldsymbol{T}^\# = \boldsymbol{T}(X)$.

Remark. The first obstruction to transversely foliating $X \subset W$ is the obstruction to holomorphically splitting the exact bundle sequence (1.1).

This latter obstruction is well-known to be a class in $H^1(X, \Omega(\mathrm{Hom}(\boldsymbol{N}, \boldsymbol{T}(X)))$. This checks Proposition 1.6 at the first level.

(c) Rigidity of holomorphic embeddings. The *rigidity question* for germs of holomorphic embeddings is simply the following: Is the germ of embedding defined by $\phi: X \to W$ determined by a finite neighborhood X_μ? More precisely, does there exist a μ_0 such that, given $\phi: X \to W$, $\phi': X \to W'$, and an equivalence $f_\mu: X_\mu \to X'_\mu$ $(\mu \geqq \mu_0)$, there exists an equivalence of germs $f_*: X_* \to X'_*$? From Proposition 1.3 we get

PROPOSITION 1.7. *If there exists a μ_0 such that $H^1(X, \Omega(\boldsymbol{T}(\mu))) = 0$ for $\mu \geqq \mu_0$, then the germ of embedding defined by $X \subset W$ is formally rigid. If furthermore $H^0(X, \Omega(\boldsymbol{T}(\mu))) = 0$ for $\mu \geqq \mu_0$, then any formal equivalence $\hat{f}: \hat{X} \to \hat{X}'$ is uniquely determined by $f_{\mu_0} = \hat{f} \mid X_{\mu_0}$.*

Remark. Suppose that we have embeddings $\phi: X \to W$, $\phi': X \to W'$ with normal bundles $\boldsymbol{N}$, $\boldsymbol{N}'$ respectively. Assume that we have a bundle isomorphism

$$\begin{array}{ccc} \Psi: \boldsymbol{N} & \to & \boldsymbol{N}' \\ & \downarrow \quad & \downarrow \\ \Psi: X & \to & X. \end{array}$$

We want to know when Ψ is induced by an equivalence on 1-neighborhoods $\Lambda: X_1 \to X'_1$. Thus we wish to fill in the middle column in the diagram

$$\begin{array}{ccccccccc} & & & & \overset{\alpha}{\longrightarrow} & & \overset{\beta}{\longrightarrow} & & \\ 0 & \to & \boldsymbol{T}(X) & & \boldsymbol{T}' & & \boldsymbol{N}' & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & \Psi_* & \Lambda_1 & \sigma & & \Psi & & \\ & & \uparrow & \overset{\gamma}{\nearrow} & \uparrow & & \uparrow & & \\ 0 & \to & \boldsymbol{T}(X) & & \boldsymbol{T} & & \boldsymbol{N} & \to & 0. \end{array} \tag{1.7}$$

From (1.7) we get a diagram

$$\begin{array}{ccccccccc} & & & & & & 0 & & 0 \\ & & & & & & \downarrow & & \downarrow \\ & & & & & & H^0(\Omega(\mathrm{Hom}(\boldsymbol{N}, \boldsymbol{T}'))) & \longrightarrow & H^0(\Omega(\mathrm{Hom}(\boldsymbol{N}, \boldsymbol{N}'))) \\ & & & & & & \downarrow & & \downarrow \sigma \\ & & & & & & H^0(\Omega(\mathrm{Hom}(\boldsymbol{T}, \boldsymbol{T}'))) & \overset{\beta}{\longrightarrow} & H^0(\Omega(\mathrm{Hom}(\boldsymbol{T}, \boldsymbol{N}'))) \\ & & & & & & \downarrow \gamma & & \downarrow \gamma \\ 0 & \to & H^0(\Omega(\mathrm{Hom}(\boldsymbol{T}(X), \boldsymbol{T}(X)))) & \overset{\alpha}{\longrightarrow} & H^0(\Omega(\mathrm{Hom}(\boldsymbol{T}(X), \boldsymbol{T}'))) & \overset{\beta}{\longrightarrow} & H^0(\Omega(\mathrm{Hom}(\boldsymbol{T}(X), \boldsymbol{N}'))) & & \\ & & & & \downarrow \rho & & & & \\ & & & & H^1(\Omega(\mathrm{Hom}(\boldsymbol{N}, \boldsymbol{T}'))). & & & & \end{array} \tag{1.8}$$

The obstruction to finding $\Lambda \in H^0(\Omega(\mathrm{Hom}(\boldsymbol{T}, \boldsymbol{T}')))$ such that $\gamma(\Lambda) = \alpha(\Psi_*)$ in (1.8) is given by $\rho\alpha(\Psi_*) \in H^1(\Omega(\mathrm{Hom}(\boldsymbol{N}, \boldsymbol{T}')))$. If this obstruction vanishes, then we may find Λ with $\beta(\Lambda) = \sigma(\Psi)$ since $\gamma\beta(\Lambda) = \beta\gamma(\Lambda) = \beta\alpha(\Psi_*) = 0$. Any such Λ is unique if $H^0(\Omega(\mathrm{Hom}(\boldsymbol{N}, \boldsymbol{T}'))) = 0$.

This suffices to check Proposition 1.3 at the first level.

II. The Normal Bundle and the Extension Problem.

1. Some implications of the extension and uniqueness problems for cohomology. Let $X \subset W$ represent a germ of embedding, and let $n = \dim X$, $n + r = \dim W$. For non-negative integers s, t we introduce the following condition:

$*(s, t)$: For any holomorphic vector bundle $\boldsymbol{F}^\# \to W$, the stable extension problem for $H^q(X, \Omega(\boldsymbol{F}))$ has a solution for $0 \leqq q < s-1$, $n - t < q$. Furthermore, we have stable uniqueness for $0 \leqq q < s$, $n - t + 1 < q$.

Remark. If $*(s, t)$ is satisfied, then

$$\dim H^q(W, \Omega(\boldsymbol{F}^\#)) < \infty \text{ for } 0 \leqq q < s,\ q > n - t + 1. \tag{2.1}$$

Proposition 2.1. *Assume $*(s, t)$ is satisfied. Then there exists a μ_0 such that*

$$H^q(X, \Omega(\boldsymbol{F}(\mu))) = 0 \text{ for } 0 \leqq q < s-1,\ q > n - t + 1, \mu \geqq \mu_0. \tag{2.2}$$

Proof. To say that the stable extension problem has a solution means that, for $\mu \geqq \mu_0(\boldsymbol{F})$, we have

$$H^q(W, \Omega(\boldsymbol{F}^\#)) \to H^q(\mathfrak{O}^{(\mu)}(\boldsymbol{F}^\#)) \to 0; \tag{2.3}$$

stable uniqueness means that, for $\mu \geqq \mu_0$, we have

$$0 \to H^q(W, \Omega(\boldsymbol{F}^\#)) \to H^q(\mathfrak{O}^{(\mu)}(\boldsymbol{F}^\#)). \tag{2.4}$$

From the exact sheaf diagram

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ & & \Omega(\boldsymbol{F}(\mu+1)) & & \\ & & \downarrow & & \\ \Omega(\boldsymbol{F}^\#) & \to & \mathfrak{O}^{(\mu+1)}(\boldsymbol{F}^\#) & \to & 0 \\ \| & & \downarrow & & \\ \Omega(\boldsymbol{F}^\#) & \to & \mathfrak{O}^{(\mu)}(\boldsymbol{F}^\#) & \to & 0 \\ & & \downarrow & & \\ & & 0, & & \end{array} \tag{2.5}$$

we get a cohomology diagram

$$
\begin{array}{ccccc}
0 \to H^q(\Omega(\boldsymbol{F}^{\#})) & \to & H^q(\mathfrak{D}^{(\mu+1)}(\boldsymbol{F}^{\#})) & \to & 0 \\
\| & & \downarrow & & \\
0 \to H^q(\Omega(\boldsymbol{F}^{\#})) & \to & H^q(\mathfrak{D}^{(\mu)}(\boldsymbol{F}^{\#})) & \to & 0 \\
 & & \downarrow & & \\
 & & H^{q+1}(\Omega(\boldsymbol{F}(\mu+1))) & & \\
 & & \downarrow & & \\
0 \to H^{q+1}(\Omega(\boldsymbol{F}^{\#})) & \to & H^{q+1}(\mathfrak{D}^{(\mu+1)}(\boldsymbol{F}^{\#})) & \to & 0 \\
\| & & \downarrow & & \\
0 \to H^{q+1}(\Omega(\boldsymbol{F}^{\#})) & \to & H^{q+1}(\mathfrak{D}^{(\mu)}(\boldsymbol{F}^{\#})) & \to & 0
\end{array}
\tag{2.6}
$$

which, by $*(s,t)$, (2.3), and (2.4), is valid for $0 \leqq q+1 < s-1$, $q > n-t+1$. It follows that $H^q(X, \Omega(\boldsymbol{F}(\mu))) = 0$ for $\mu \geqq \mu_0$ and $0 \leqq q < s-1$, $q > n-t+1$.

PROPOSITION 2.2. *Assume that there exists* $\mu_0(\boldsymbol{F})$ *such that*

$$
H^q(X, \Omega(\boldsymbol{F}(\mu))) = 0 \quad \text{for } \mu \geqq \mu_0,\ 0 \leqq q < s,\ q > n-t. \tag{2.7}
$$

Then $*(s,t)$ *holds formally.*

Proof. This follows from the exact sequence

$$
H^q(\Omega(\boldsymbol{F}(\mu+1))) \to H^q(\mathfrak{D}^{(\mu+1)}(\boldsymbol{F}^{\#})) \to H^q(\mathfrak{D}^{(\mu)}(\boldsymbol{F}^{\#})) \to H^{q+1}(\Omega(\boldsymbol{F}(\mu+1)))
$$

since the formal stable extension problem means that

$$
H^q(\mathfrak{D}^{(\mu+1)}(\boldsymbol{F}^{\#})) \to H^q(\mathfrak{D}^{(\mu)}(\boldsymbol{F}^{\#})) \to 0 \qquad (\mu \geqq \mu_0), \tag{2.8}
$$

whereas formal stable uniqueness means that

$$
0 \to H^q(\mathfrak{D}^{(\mu+1)}(\boldsymbol{F}^{\#})) \to H^q(\mathfrak{D}^{(\mu)}(\boldsymbol{F}^{\#})) \qquad (\mu \geqq \mu_0). \tag{2.9}
$$

Remark. Conditions (2.1) are by no means independent of those of the form (2.2). For example: Let $\boldsymbol{E} \xrightarrow{\pi} X$ be a holomorphic vector bundle and embed $X \subset \boldsymbol{E}$ as zero cross-section. Let W be a small tubular neighborhood of $X \subset \boldsymbol{E}$ and let $\boldsymbol{F}^{\#} = \pi^{-1}(\boldsymbol{F}) \mid W$ for any holomorphic bundle $\boldsymbol{F} \to X$. We observe here that the normal bundle of $X \subset W$ is just $\boldsymbol{E}$.

LEMMA 2.1. $\sum_{\mu=0}^{\infty} \dim H^q(X, \Omega(\boldsymbol{F}(\mu))) \leqq \dim H^q(W, \Omega(\boldsymbol{F}^{\#}))$.

This is easy to verify using local coordinates. We may in fact sharpen it to the following:

$$
H^q(\mathfrak{D}^{(\mu)}(\boldsymbol{F}^{\#})) \cong \bigoplus_{0 \leqq \nu \leqq \mu} H^q(X, \Omega(\boldsymbol{F}(\nu))). \tag{2.10}
$$

The content of this section may be summarized as follows: The solution

of the stable extension and uniqueness problems for $X \subset W$ imply: (i) a *vanishing theorem* involving the normal bundle $\boldsymbol{N}$ of $X \subset W$, and (ii) a *finite dimensionality theorem* for cohomology groups $H^q(W, \Omega(\boldsymbol{F}^{\#}))$. Conversely, (i) gives at least the formal stable extension and uniqueness problems. Part of our program will be to formulate (i) in such a fashion that it gives the actual extension and uniqueness problems (and thence (ii)).

2. Positive and negative holomorphic bundles. It is well-known what it means for a line bundle $\boldsymbol{E} \to X$ to be positive, and recently there have been some generalizations of this to general holomorphic bundles (cf. [6] and [8]). For our purposes, we shall need the two notions of *weak positivity* and *positivity* for a holomorphic bundle $\boldsymbol{E} \to X$; the precise definitions are given in VI. 5 below and we shall only list some relevant properties here: (i) $\boldsymbol{E} \to X$ is *weakly negative* (*negative*) if, and only if, $\boldsymbol{E}^* \to X$ is weakly positive (positive); (ii) if $\boldsymbol{E} \to X$ is weakly positive, then, for some μ_0, the bundles $\boldsymbol{E}^{\mu}(\mu \geqq \mu_0)$ are positive; (iii) for line bundles, weak positivity = positivity; and (iv) if $\boldsymbol{E} \to X$ is weakly positive and if $\boldsymbol{F} \to X$ is a bundle, then

$$H^q(X, \Omega(\boldsymbol{F} \otimes \boldsymbol{E}^{\mu})) = 0 \text{ for } \mu \geqq \mu_0(\boldsymbol{F}), \qquad q > 0. \tag{2.11}$$

More generally, given positive integers s, t with $s + t \leqq n$, we shall define (VI. 5) what it means for $\boldsymbol{E} \to X$ to have *signature* (s, t); the main properties are: (i) if $\boldsymbol{E} \to X$ has signature (s, t), then $\boldsymbol{E}^* \to X$ has signature (t, s); (ii) signature $(n, 0) \equiv$ weak positivity; and (iii) if $\boldsymbol{E} \to X$ has signature (s, t) and if $\boldsymbol{F} \to X$ is any bundle, then

$$H^q(X, \Omega(\boldsymbol{F} \otimes \boldsymbol{E}^{*\mu})) = 0 \text{ for } \mu \geqq \mu_0(\boldsymbol{F}),\ 0 \leqq q < s \tag{2.12}$$

and $q > n - t$.

An example might clarify what it means for $\boldsymbol{E} \to X$ to have signature (s, t). Let Y, Z be algebraic manifolds of dimensions s, t respectively, let $\boldsymbol{F} \to Y$ be a positive line bundle and $\boldsymbol{G} \to Z$ a negative line bundle. Then $\boldsymbol{E} = \boldsymbol{F} \otimes \boldsymbol{G}$ has signature (s, t) over $X = Y \times Z$. Observe that, for large μ, $H^q(Y, \Omega(\boldsymbol{F}^{*\mu})) = 0$ for $q < s$ and $H^q(Z, \Omega(\boldsymbol{G}^{*\mu})) = 0$ for $q > 0$. By Künneth, $H^q(X, \Omega(\boldsymbol{E}^{*\mu})) = 0$ for $0 \leqq q < s$ and $q > n - t = s$.

From 2.12 and Propositions 1.1-1.4, we have

PROPOSITION 2.3. *Let $X \subset W$ define a germ of embedding whose normal bundle $\boldsymbol{N}$ has signature (s, t). Then:*

(i) *The statement $*(s, t)$ in* § II.1 *holds formally;*

(ii) *The rigidity problem is formally true provided that either* $s \geqq 2$ *or* $t = n$. *In the former case, we have uniqueness*;

(iii) *The extension problem for holomorphic bundles is formally true if either* $s \geqq 3$ *or* $t \geqq n-1$. *In the former case we have formal uniqueness always, and we have formal uniqueness in the latter case if* $t = n$.

Assume now that the normal bundle $\boldsymbol{N}$ of $X \subset W$ has signature (s, t) and let $\boldsymbol{F}^{\#} \to W$ be a holomorphic vector bundle. Set $\Omega_{\mu}(\boldsymbol{F}^{\#}) = \mathfrak{I}^{\mu+1} \cdot \Omega(\boldsymbol{F}^{\#})$ so that we have

$$0 \to \Omega_{\mu}(\boldsymbol{F}^{\#}) \to \Omega(\boldsymbol{F}^{\#}) \to \mathfrak{Q}^{(\mu)}(\boldsymbol{F}^{\#}) \to 0; \tag{2.13}$$

$$0 \to \Omega_{\mu+\nu}(\boldsymbol{F}^{\#}) \to \Omega_{\mu}(\boldsymbol{F}^{\#}) \quad (\nu \geqq 0); \text{ and} \tag{2.14}$$

$$0 \to \Omega_{\mu+1}(\boldsymbol{F}^{\#}) \to \Omega_{\mu}(\boldsymbol{F}^{\#}) \to \Omega(\boldsymbol{F}(\mu+1)) \to 0. \tag{2.15}$$

PROPOSITION 2.4. *For* $\mu \geqq \mu_0$ *and all* $\nu \geqq 0$ *the sequence* (2.14) *induces an isomorphism*

$$0 \to H^q(W, \Omega_{\mu+\nu}(\boldsymbol{F}^{\#})) \to H^q(W, \Omega_{\mu}(\boldsymbol{F}^{\#})) \to 0 \tag{2.16}$$

for $0 \leqq q < s$, $q > n - t + 1$.

Proof. This follows inductively on ν from the exact cohomology sequence of (2.15) together with (2.12).

PROPOSITION 2.5. *If, for large* μ,

$$H^q(W, \Omega_{\mu}(\boldsymbol{F}^{\#})) = 0 \text{ for } 0 \leqq q < s,\ q > n - t + 1,$$

then $*(s, t)$ *holds.*

Proof. This follows from (2.3), (2.4), and the exact cohomology sequence of (2.13).

Definition. If the conclusion (2.16) of Proposition 2.4 holds, then we shall say that the group $H^q(W, \Omega_{\mu}(\boldsymbol{F}^{\#}))$ *vanishes to infinite order* along X.

By Proposition 2.5, the *actual* existence theorem for the extension and uniqueness problems in cohomology reduces to showing that cohomology which vanishes to infinite order along X is necessarily zero. This seems to be a standard procedure for deducing existence theorems from formal statements.

3. Statement of the main results.

THEOREM I. *Let* $X \subset W$ *define a germ of embedding whose normal bundle* $\boldsymbol{N}$ *has signature* (s, t) *with* $s + t = n$. *Then, for any holomorphic*

vector bundle $\boldsymbol{F}^{\#} \to W$, *the stable extension problem for* $H^q(X, \Omega(\boldsymbol{F}))$ *has a solution for* $0 \leqq q < s-1$, $q > n-t$. *Furthermore, stable uniqueness holds for* $0 \leqq q < s$, $q > n-t+1$.

We shall use Theorem I to discuss the rigidity question and extension problem for holomorphic bundles in certain special cases. In order to state the results, we need the notion of a sufficiently positive bundle.

Definition. Let $\boldsymbol{E} \to X$ be a positive holomorphic vector bundle. Associated to X there is a finite number $\boldsymbol{F}_1, \cdots, \boldsymbol{F}_k$ of holomorphic bundles, and $\boldsymbol{E}$ is defined to be *sufficiently positive* if all the bundles $\boldsymbol{E} \otimes \boldsymbol{F}_i$ $(1 \leqq i \leqq k)$ are positive.

Remark. For a positive (but *not* in general weakly positive) bundle $\boldsymbol{E} \to X$, we have

$$H^q(X, \Omega(\boldsymbol{E}^{*\mu})) = 0 \qquad \text{for } q > 0, \mu > 0. \tag{2.17}$$

THEOREM II. *Let* $X \subset W$ *define a germ of embedding whose normal bundle* $\boldsymbol{N} \to X$ *is sufficiently positive.*

(i) *The germ of embedding is rigid provided that* $\dim X \geqq 3$.

(ii) *The extension problem for holomorphic bundles holds of* $\dim X \geqq 3$.

(iii) *Stable uniqueness holds in* (i) *and* (ii).

Remark. The hypothesis of "sufficient positivity" should be thought of as describing a class $\mathcal{S}$ of holomorphic vector bundles over X such that Theorem II is true for $X \subset W$ whenever $\boldsymbol{N} \in \mathcal{S}$. For codimension 1, $\mathcal{S}$ = all positive line bundles over X; this is simply because passing from the linear system $|\boldsymbol{L}|$ to $|\mu \boldsymbol{L}|$ $(\boldsymbol{L} = [X])$ corresponds to replacing $\boldsymbol{N} \to X$ by $\boldsymbol{N}^{\mu}$ which will be sufficiently positive for μ large. For codimension $r > 1$, there is no such (obvious) way of making a subvariety more positive and this is basically the reason for the hypothesis of sufficient positivity.

We ashall prove Theorem I by proving

THEOREM III. *With the notations and assumptions of Theorem* I, *we have*

$$H^q(W, \Omega_{\mu}(\boldsymbol{F}^{\#})) = 0 \text{ for } \mu \geqq \mu_0,\ 0 \leqq q < s \text{ or } q > n-t+1. \tag{2.18}$$

We now observe

PROPOSITION 2.6. *Let* $X \subset W$ *have positive normal bundle* $\boldsymbol{N}$ *and assume that* $\dim X \geqq 3$. *Then, assuming* (2.18), *the extension problem for a line bundle* $\boldsymbol{E} \to X$ *has unique solution.*

Proof. We shall verify (1.5). The middle line there is clear, and we have the exact sequence

$$H^1(W, \mathfrak{O}_W) \to H^1(X, \mathfrak{O}_X) \to H^2(W, \mathfrak{F}) \to H^2(W, \mathfrak{O}_W) \to H^2(X, \mathfrak{O}_X).$$

It will suffice to prove that $H^2(W, \mathfrak{F}) = 0$. From the sequences

$$0 \to \mathfrak{F}^{\mu+1} \to \mathfrak{F}^{\mu} \to \Omega(\mu) \to 0,$$

we get $0 \to H^2(W, \mathfrak{F}^{\mu+1}) \to H^2(W, \mathfrak{F}^{\mu}) \to 0$ since $H^q(X, \Omega(\mu)) = 0$ for $\mu \geqq 1$, $q \leqq 2 \leqq \dim X - 1$. This then gives $0 \to H^2(W, \mathfrak{F}^{\mu}) \to H^2(W, \mathfrak{F}) \to 0$ for all $\mu \geqq 1$ which, by (2.18), gives $H^2(W, \mathfrak{F}) = 0$, Q.E.D.

4. Outline of the proof of Theorem III. We give now an outline of the proof of Theorem III; the details and relevant calculations will be supplied in §§ V-VIII below. Thus let $X \subset W$ be an embedding of codimension r and where the normal bundle $\boldsymbol{N}$ has signature (s, t) with $s + t = n = \dim X$. By blowing up $X \subset W$, we obtain a hypersurface $\tilde{X} \subset \tilde{W}$ (§ V.1) and the normal bundle $\tilde{\boldsymbol{N}}$ of $\tilde{X} \subset \tilde{W}$ will have signature $(s, t + r - 1)$ (VI.7). Furthermore, it is at least plausible that, in order to examine cohomology vanishing on $X \subset W$, we may lift everything up and look at cohomology vanishing on $\tilde{X} \subset \tilde{W}$ (V.3).

We remark here that, even if $X \subset W$ has a positive normal bundle, then the normal bundle of $\tilde{X} \subset \tilde{W}$ will be of mixed signature, and this is the essential difficulty in the higher codimension case. In fact, if $X \subset W$ is a hypersurface whose normal bundle has either signature $(n, 0)$ or $(0, n)$, then Theorem III is easy to prove directly (IV.1 and [6], § 4).

Thus we may assume that $r = 1$. In this case, the sheaf

$$\Omega_{\mu}(\boldsymbol{F}^{\#}) = \Omega(\boldsymbol{F}^{\#} \otimes \boldsymbol{L}^{*\mu}) = \Omega(\boldsymbol{F}^{\#}(\mu)) \qquad (\boldsymbol{L} = [X]) \tag{2.19}$$

is locally free and we may hope to use the theory of harmonic integrals on open manifolds as developed by J. Kohn (VIII.1 and [15]) to study the groups $H^q(W, \Omega(\boldsymbol{F}^{\#}(\mu)))$.

Now in order to apply this harmonic integral theory, we must have an exhaustion $W = \bigcup_{0<c<c_0} W_c$ of W by nice relatively compact subdomains $W_c = \{w \in W : r(w) < c\}$ for some non-negative function r on W. Furthermore, for some $\epsilon > 0$, $\epsilon < c_0$, $\partial\bar{\partial} r$ in $W - W_{\epsilon}$ should have signature $(t + 1, s)$. Now, since $\boldsymbol{N}$ has signature (s, t) with $s + t = n$, such an r exists for the embedding $X \subset \boldsymbol{N}$ as zero cross-section (VI.5) and, by an approximation argument, r exists for $X \subset W$ provided that W is small enough (VIII.3).

If now $\partial\bar{\partial} r$ had signature $(t + 1, s)$ everywhere in W, then the trivial

bundle $\boldsymbol{l} \to W$ would have signature $(t+1, s)$ and it is fairly easy to conclude then that (VII. 4)

$$H^q(W, \Omega(\boldsymbol{E}^\#)) = 0 \quad \text{for } 0 \leqq q < s, q > n - t \tag{2.20}$$

for *every* holomorphic bundle $\boldsymbol{E}^\# \to W$. This is the so-called complete case and has been discussed in [2].

Now our function r is certainly *not* complete since $\partial\bar{\partial} r \mid X = \partial\bar{\partial}(r \mid X) = 0$. Thus (2.20) will not hold in general—all we can conclude is that

$$\dim H^q(W, \Omega(\boldsymbol{E}^\#)) < \infty \qquad 0 \leqq q < s, q > n - t. \tag{2.21}$$

However, it is perhaps not unreasonable to expect that (2.20) will hold provided that we can find in $\boldsymbol{E}^\# \to W$ a metric whose curvature form picks up the slack caused by the degeneracy of $\partial\bar{\partial} r$. In fact, this is the whole point: For μ sufficiently large, $\boldsymbol{F}^\#(\mu) = \boldsymbol{F}^\# \otimes \boldsymbol{L}^{*\mu}$ will have signature $(t, s+1)$ (VIII. 1) and furthermore the curvature form in $\boldsymbol{F}^\#(\mu)$ will have an s-dimensional negative eigenspace close to the s-dimensional negative eigenspace of $\partial\bar{\partial} r$ on $W - W_\epsilon$, and similarly for the t-dimensional positive spaces. (Observe here that there is a loss of 1-dimensional between (2.18) and (2.20).)

Thus, if one accepts this argument (which is almost obvious geometrically) and combines it with inequalities obtained from two integral formulae on open manifolds (VII. 3 and VII. 6), then Theorem III follows easily from the Bochner vanishing technique together with Kohn's results about representing cohomology by harmonic forms with boundary conditions.

III. Positive Embeddings and the Proof of Theorem II.

1. A simple example of rigidity. We shall prove a simple example of the rigidity theorem which will, however, contain the main ideas of the general argument to be given below. Let $X = P_n$ $(n \geqq 3)$ and let $\boldsymbol{H} \to X$ be the hyperplane bundle; $\boldsymbol{H} = [h]$ where $h \subset P_n$ is a hyperplane. Suppose now that we have an embedding $X \subset W$ with normal bundle $\boldsymbol{H}$.

LEMMA 3.1. $H^1(X, \Omega(\boldsymbol{T}(\mu))) = 0$ *for* $\mu > 0$.

The proof of this will be given in IV. 2. (i) below.

Thus we see that the germ of embedding defined by $X \subset W$ should be equivalent to the germ defined by embedding $X \subset \boldsymbol{H}$ as zero cross-section, and we shall use Theorem I to prove that this is the case. Note that this example is "dual" to the rigidity of quadratic transformations (cf. the appendix to [13]).

Set $\boldsymbol{L} = [X]$ so that $\mathfrak{I} \cong \Omega(\boldsymbol{L}^*)$ and $\boldsymbol{L} \mid X = \boldsymbol{H}$. We have the exact sheaf sequence

$$0 \to \mathfrak{O}_W \to \Omega(\boldsymbol{L}) \to \Omega_X(\boldsymbol{H}) \to 0 \tag{3.1}$$

LEMMA 3.2. $H^1(W, \mathfrak{O}_W) = 0$.

Proof. Since $H^1(X, \mathfrak{O}_X) = 0$, this will follow from $H^1(W, \mathfrak{I}) = 0$. But $H^1(W, \mathfrak{I}) = 0$ by Theorem I and since $\dim X \geqq 3$ (cf. the proof of Proposition 2.6). Q.E.D.

From (3.1) we get then

$$0 \to \boldsymbol{C} \to H^0(W, \Omega(\boldsymbol{L})) \xrightarrow{\rho} H^0(X, \Omega(\boldsymbol{H})) \to 0. \tag{3.2}$$

Thus $\dim H^0(W, \Omega(\boldsymbol{L})) = n + 2$ and we choose a basis $\xi_0, \cdots, \xi_{n+1}$ of $H^0(W, \Omega(\boldsymbol{L}))$ so that $\rho(\xi_{n+1}) = 0$. If we now define $\Phi: W \to P_{n+1}$ by

$$\Phi(w) = [\xi_0(w), \cdots, \xi_n(w), \xi_{n+1}(w)],$$

then an elementary consideration using the complete (since $H^1(X, \Omega(\boldsymbol{H})) = 0$) continuous system of $X \subset W$ (cf. III.2 and III.3) shows that Φ is an embedding and $\Phi(X) = P_n \subset P_{n+1}$. Thus $X \subset W$ defines the same germ as the usual embedding $P_n \subset P_{n+1}$.

On the other hand, let $S = \{\xi \in P_{n+1}: \xi_0 = \cdots = \xi_n = 0\}$. Then $S = \text{pt.}$ and $P_{n+1} - S$ is a neighborhood W' of $P_n \subset P_{n+1}$. Furthermore, we may define $\pi: W' \to P_n$ by $\pi[\xi_0, \cdots, \xi_n, \xi_{n+1}] = [\xi_0, \cdots, \xi_n]$ and then W' is a line bundle over P_n, and in fact $W' \cong \boldsymbol{H}$ as line bundles over P_n. Thus $X \subset W$ and $X \subset \boldsymbol{H}$ define equivalent germs of embeddings.

Remark. The compactification of $\boldsymbol{H}$ by adding on S to obtain a closed variety is a general phenomenon (cf. IV.1).

2. On the theory of continuous systems. Grauert has shown, if the normal bundle $\boldsymbol{N}$ of X in W is negative, then we may assume that W is *strongly-pseudo-convex*, that $X \subset W$ is the only non-trivial compact subvariety, and that there exists a Stein space Y and a proper holomorphic mapping $\pi: W \to Y$ such that $\pi(X) = y_0$ is a point and $\pi: W - X \cong Y - \{y_0\}$.

Now in case $\boldsymbol{N}$ is positive it is again true that we may assume that W is *strongly pseudo-concave*,* but we need better information than this alone. Namely, we shall need the fact that, if $\boldsymbol{N}$ is sufficiently positive, then the *characteristic system* of X in W is complete and we may write $W = \{X_t\}_{t \in B}$ where the X_t form an analytic family of compact submanifolds of W which are small deformations of X in W.

* Cf. VIII 3.

To be more specific, let $X \subset W$ be a germ of embedding with normal bundle $\boldsymbol{N}$. Let $\rho_x: H^0(X, \Omega(\boldsymbol{N})) \to \boldsymbol{N}_x$ be the restriction mapping and set $\delta(x) = \dim \rho_x\{H^0(X, \Omega(\boldsymbol{N}))\}$. Let $B \subset \boldsymbol{C}^m$ be an open neighborhood of the origin with linear coordinates $t = (t^1, \cdots, t^m)$. In [12] Kodaira has defined the concept of an *analytic family* $\{\boldsymbol{V}, B\}$ *of compact submanifolds* of W parametrized by B. We write $\boldsymbol{V} = \{X_t\}_{t \in B}$ with $X_0 = X$. By a *deformation* $\{\boldsymbol{V} \xrightarrow{\tilde{\omega}} B\}$ of X, with parameter space B, we mean a complex manifold $\boldsymbol{V}$ together with an onto projection $\tilde{\omega}: \boldsymbol{V} \to B$ which is everywhere of maximal rank and where $\tilde{\omega}^{-1}(0) \cong X$.

THEOREM 3.1. *Assume* $H^1(X, \Omega(\boldsymbol{N})) = 0$. *Then there exists*:

(i) *a neighborhood* B *of the origin in* $H^0(X, \Omega(\boldsymbol{N}))$;

(ii) *a family* $\mathfrak{V} = \{X_t\}_{t \in B}$ *of compact analytic submanifolds of* W;

(iii) *a deformation* $\{\boldsymbol{V} \xrightarrow{\tilde{\omega}} B\}$; *and*

(iv) *a holomorphic mapping* $F: \boldsymbol{V} \to W$ *such that* $F(\tilde{\omega}^{-1}(t)) = X_t$ *and such that rank of* F *at* $x \in X \subset V$ *is* $\delta(x) + \dim X$.

Proof. This result will be a consequence of the main theorem in [12]; we shall use the proof as an opportunity to establish some notation for later use. Let $\dim X = n$, $\dim W = n + r$, $\dim H^0(X, \Omega(\boldsymbol{N})) = m$, and agree on the ranges of indices $1 \leqq i, j \leqq n$; $1 \leqq a, b \leqq r$; and $1 \leqq \rho, \sigma \leqq m$. We choose a covering $\{U_\alpha\}$ of W by open sets such that, if $V_\alpha = U_\alpha \cap X$, then $U_\alpha \cap U_\beta = \emptyset \Longleftrightarrow V_\alpha \cap V_\beta = \emptyset$. We assume that we have in U_α holomorphic coordinates $(z_\alpha, w_\alpha) = (z_\alpha^1, \cdots, z_\alpha^n; w_\alpha^1, \cdots, w_\alpha^r)$ such that $U_\alpha \cap X$ is given by $\{w_\alpha = 0\}$. If we set $|z_\alpha| \max_{1 \leqq i \leqq n} |z_\alpha^i|$, $|w_\alpha| \max_{1 \leqq a \leqq r} |w_\alpha^a|$, we may suppose that U_α is the unit polycylinder $|z_\alpha| < 1$, $|w_\alpha| < 1$.

In $U_\alpha \cap U_\beta$ we have

$$\begin{cases} z_\alpha = f_{\alpha\beta}(z_\beta, w_\beta) \\ w_\alpha = g_{\alpha\beta}(z_\beta, w_\beta) \end{cases}; \quad g_{\alpha\beta}(z_\beta, 0) = 0.$$

The normal bundle $\boldsymbol{N}$ has transition functions given by the $r \times r$ matrix $N_{\alpha\beta}(z_\beta) = \{N_{\alpha\beta}(z_\beta)_b{}^a\}$ where

$$N_{\alpha\beta}(z_\beta)_b{}^a = \left\{ \frac{\partial g_{\alpha\beta}{}^a(z_\beta, w_\beta)}{\partial w_\beta{}^b} \right\}_{w=0}$$

We choose a basis $\sigma_1, \cdots, \sigma_m$ of $H^0(X, \Omega(\boldsymbol{N}))$. Then $\sigma_\rho \mid U_\alpha = \sigma_{\rho,\alpha}$ is column vector ${}^t(\sigma_{\rho,\alpha}{}^1, \cdots, \sigma_{\rho,\alpha}{}^r) = {}^t\{\sigma_{\rho,\alpha}{}^a\}$ * and satisfies $\sigma_{\rho,\alpha} = N_{\alpha\beta}\sigma_{\rho,\beta}$ in

* ${}^t(\#)$ means transpose of the row vector $(\#)$.

$V_\alpha \cap V_\beta$. A general point in $H^0(X, \Omega(\boldsymbol{N})) \cong \boldsymbol{C}^m$ will be written as $t = \sum_{\rho=1}^{m} t^\rho \sigma_\rho$. By Kodaira's theorem [12], there exists a neighborhood B of the origin in $\boldsymbol{C}^m$ and a family of vector-valued holomorphic functions $\phi_\alpha(z_\alpha, t) : V_\alpha \times B \to \boldsymbol{C}^r$ which satisfy the following conditions:

$$\phi_\alpha(z_\alpha, 0) = 0 \text{ and } |\phi_\alpha(z_\alpha, t)| < 1; \tag{3.3}$$

$$\frac{\partial \phi_\alpha(z_\alpha, t)}{\partial t^\rho}\Big]_{t=0} = \sigma_{\rho,\alpha}(z_\alpha); \tag{3.4}$$

$$\phi_\alpha(z_\alpha, t) = g_{\alpha\beta}(z_\beta, \phi_\beta(z_\beta, t)); \text{ and} \tag{3.5}$$

$$z_\alpha = f_{\alpha\beta}(z_\beta, \phi_\beta(z_{\beta,t})). \tag{3.6}$$

If we define $X_t \subset W$ by $X_t \cap U_\alpha = \{(z_\alpha, w_\alpha) : w_\alpha = \phi_\alpha(z_\alpha, t)\}$, then by (3.3), (3.5), and (3.6) X_t is a compact sub-manifold of W and $X_0 = X$. By (3.4), we may symbolically write $\frac{\partial X_t}{\partial t^\rho}\Big]_{t=0} = \sigma_\rho$. The collection $\{X_t\}_{t \in B}$ clearly gives an analytic family $\mathfrak{X}$ of compact submanifolds of W.

We construct an abstract complex manifold $\boldsymbol{V}$ as follows: Let $T_\alpha = V_\alpha \times B$. Then $\boldsymbol{V}$ is formed from the union $\bigcup_\alpha T_\alpha$ by identifying (z_α, t) with (z_β, t') if, and only if, $t = t'$ and $z_\alpha = f_{\alpha\beta}(z_\beta, \phi_\beta(z_\beta, t))$. From (3.5) and (3.6) it follows that this is an equivalence relation, and so $\boldsymbol{V}$ is a complex manifold. Furthermore, if we define $\tilde{\omega}: \boldsymbol{V} \to B$ by $\omega(z_\alpha, t) = t$ in T_α, $\tilde{\omega}$ is clearly well-defined and has maximal rank everywhere. Obviously then $\{\boldsymbol{V} \xrightarrow{\tilde{\omega}} B\}$ is a deformation of X.

We define $F: \boldsymbol{V} \to W$ as follows: $F_\alpha = F \mid T_\alpha$ is given by $F_\alpha(z_\alpha, t) = (z_\alpha, \phi_\alpha(z_\alpha, t)) \in U_\alpha$ by (3.3). It follows from (3.5) and (3.6) that $F_\alpha = F_\beta$ in $T_\alpha \cap T_\beta$, and thus F is globally defined. The Jacobian matrix $J(F_\alpha)$ of F_α, when evaluated on $X \cap T_\alpha$, is given by the $(n+m) \times (n+r)$ matrix $J(F_\alpha(z_\alpha)) = \begin{pmatrix} \boldsymbol{I}^n & * \\ 0 & Q(z_\alpha) \end{pmatrix}$ where $\boldsymbol{I}^n$ is the $n \times n$ unit matrix and $Q(z_\alpha)$ is the $m \times r$ matrix with entries

$$Q(z_\alpha)_\rho{}^a = \frac{\partial \phi_\alpha{}^a(z_\alpha, t)}{\partial t^\rho}\Big]_{t=0} = \sigma_{\rho,\alpha}{}^a(z_\alpha). \tag{3.7}$$

For fixed $z_\alpha{}^0 \in V_\alpha$, the vectors $\sigma_{1,\alpha}(z_\alpha{}^0), \cdots, \sigma_{m,\alpha}(z_\alpha{}^0)$ span a subspace of $\boldsymbol{C}^r$ of dimension $\delta(z_\alpha{}^0)$, and thus the rank of $J(F_\alpha(z_\alpha{}^0))$ is $\delta(z_\alpha{}^0)$. Q.E.D.

3. Positive embeddings and continuous systems. From Theorem 3.1 we obtain:

PROPOSITION 3.1. *If* $H^1(X, \Omega(\boldsymbol{N})) = 0$ *and if the restriction mapping* $\rho_x: H^0(X, \Omega(N)) \to \boldsymbol{N}_x$ *is everywhere surjective, then* $F: \boldsymbol{V} \to W$ *is onto and we may write* $W = \{X_t\}_{t \in B}$.

Remark. In [7] we assumed that, if $\boldsymbol{N}$ is trivial, then W is a space of deformations of X. This is not so, as the example below shows, but we do have:

PROPOSITION. *If* $H^1(X, \mathfrak{O}) = 0$ *and if* $\boldsymbol{N}$ *is trivial, then* W *is a space of deformations of* X.

Proof. $\Omega(\boldsymbol{N}) \cong \underbrace{\mathfrak{O} \oplus \cdots \oplus \mathfrak{O}}_{r}$ since $\boldsymbol{N}$ is trivial. Then $H^1(X, \Omega(\boldsymbol{N})) = 0$ and $\rho_x: H^0(X, \Omega(\boldsymbol{N})) \to \boldsymbol{N}_x$ is onto for all x. Consequently, $F: \boldsymbol{V} \to W$ is biregular in a neighborhood of X and so W is a space of deformations of X.

Example. We show that, if $H^1(X, \mathfrak{O}) \neq 0$, then there exists an embedding $X \subset W$ in which $\boldsymbol{N}$ is trivial but where no neighborhood of X in W is a deformation space of X. Let $h_{\alpha\beta}: V_\alpha \cap V_\beta \to \boldsymbol{C}$ be a 1-cocycle giving a non-zero element in $H^1(X, \mathfrak{O})$. For each complex number t, $|t| < \epsilon$, define a new complex manifold W_t from the union $\bigcup_\alpha \{V_\alpha \times \boldsymbol{C}\}$ by identifying (z_α, ξ_α) with (z_β, ξ_β) if, and only if, $z_\alpha = z_\beta$ and $\xi_\alpha = \exp(t h_{\alpha\beta}(z_\beta)) \cdot \xi_\beta$. Then W_t is a line bundle over X which, for ϵ small, has a non-identically zero cross-section if, and only if, $t = 0$. Let $W = \bigcup_{|t|<\epsilon} W_t$ and embed $X \subset W_0 \subset W$ as zero cross-section. Clearly, no neighborhood of X in W is a deformation space of X. On the other hand, by (6.2) the normal bundle $\boldsymbol{N}$ has transition functions

$$\left.\begin{pmatrix} \dfrac{\partial \xi_\alpha}{\partial \xi_\beta} & \dfrac{\partial \xi_\alpha}{\partial t} \\ \dfrac{\partial t}{\partial \xi_\beta} & \dfrac{\partial t}{\partial t} \end{pmatrix}\right]_{t=0=\xi} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Remark. This example is similar to Zappa's example of an incomplete continuous system. Let X, $\{h_{\alpha\beta}\} \in H^1(X, \mathfrak{O})$ be as above, and let $\boldsymbol{E} \to X$ be the bundle with fibre $\boldsymbol{C}^2$ and transition functions $g_{\alpha\beta} = \begin{pmatrix} 1 & h_{\alpha\beta} \\ 0 & 1 \end{pmatrix}$. If $P(\boldsymbol{E}) \xrightarrow{\pi} X$ is the associated projective bundle, then $P(\boldsymbol{E})$ has a covering by open sets $U_\alpha = \pi^{-1}(V_\alpha)$ with coordinates $(z_\alpha; [\xi_\alpha{}^0, \xi_\alpha{}^1])$ in U_α and the transition rules $\xi_\alpha{}^0 = \xi_\beta{}^0 + h_{\alpha\beta}(z_\beta)\xi_\beta{}^1$, $\xi_\alpha{}^1 = \xi_\beta{}^1$. Thus the cross-section S at infinity given by $S \cap U_\alpha = \{\xi_\alpha{}^1 = 0\}$ is defined, and the normal bundle

$\boldsymbol{N} \to S$ of $S \subset P(\boldsymbol{E})$ is trivial. Consequently, $H^0(S, \Omega(\boldsymbol{N})) \neq 0$ but the continuous system generated by $S \subset P(\boldsymbol{E})$ is empty. Indeed, if there exists a family $\{S_t\}$ with $S_0 = S$, then $S_t \cap S = \emptyset$ for small t (since $\boldsymbol{N}$ is trivial) and $\pi: S_t \to X$ is biregular. Consequently, there exists $\sigma: X \to P(\boldsymbol{E})$ with $\sigma(X) \cap S = \emptyset$. If $\sigma \mid V_\alpha = [\sigma_\alpha{}^0(z), \sigma_\alpha{}^1(z)]$, then $\tau_\alpha(z) = \sigma_\alpha{}^0(z)\sigma_\alpha{}^1(z)^{-1}$ satisfies $\tau_\alpha - \tau_\beta = h_{\alpha\beta}$ which is a contradiction.

Now consider the sequences

$$0 \to \boldsymbol{N}(x, x') \to H^0(X, \Omega(\boldsymbol{N})) \xrightarrow{\rho_{x,x'}} \boldsymbol{N}_x \oplus \boldsymbol{N}_{x'} \qquad (x \neq x') \tag{3.8}$$

$$\boldsymbol{N}(x) \xrightarrow{\sigma_x} \boldsymbol{m}_x/\boldsymbol{m}_x{}^2 \otimes \boldsymbol{N}_x \tag{3.9}$$

where $\boldsymbol{N}(x) = \ker \rho_x: H^0(X, \Omega(\boldsymbol{N})) \to \boldsymbol{N}_x$ and $\boldsymbol{m}_x \subset \mathfrak{O}_x$ is the maximal ideal at x.

PROPOSITION 3.2. *Assume $H^1(X, \Omega(\boldsymbol{N})) = 0$ and that $\rho_{x,x'}$ is everywhere surjective. Then, given $x \in X$ and $w \in W$, there exists an $X_t \subset W$ which passes through x and w.*

PROPOSITION 3.3. *Assume $H^1(X, \Omega(\boldsymbol{N})) = 0$ and that σ_x is everywhere surjective. Then, given $x \in X$ and $\xi \in \boldsymbol{T}_x(W)$, there exists an $X_t \subset W$ such that $x \in X_t$ and $\xi \in \boldsymbol{T}_x(X_t)$.*

Both Propositions follow from Theorem 3.1. We observe that, if $\boldsymbol{N} \to X$ is sufficiently positive, then $H^1(X, \Omega(\boldsymbol{N})) = 0$ and both $\rho_{x,x'}$ and σ_x are evereywhere onto (cf. [8]). In fact, the class $\mathcal{S}$ of sufficiently positive bundles over X (cf. the footnote to Theorem II) will consist of those holomorphic bundles $\boldsymbol{N} \to X$ for which $\rho_{x,x'}$ and σ_x are everywhere surjective, for which $\boldsymbol{N} \otimes \boldsymbol{K}^*$ is positive, and for which the bundles $\boldsymbol{L}^* \otimes \boldsymbol{N}^\mu$ $(\mu \geqq 1)$ are all positive for some ample line bundle $\boldsymbol{L} \to X$. An embedding $X \subset W$ for which $\boldsymbol{N} \in \mathcal{S}$ will be called a *positive embedding*.

Let $\boldsymbol{L} \to X$ be an ample line bundle so that $|\boldsymbol{L}|$ gives a projective embedding $\lambda: X \to P_s (s + 1 = \dim H^0(X, \Omega(\boldsymbol{L})))$.

PROPOSITION 3.4. *If $X \subset W$ is a positive embedding and if the extension problems for $\boldsymbol{L}$ and $H^0(X, \Omega(\boldsymbol{L}))$ can be solved, then λ extends to a projective embedding $\Lambda: W \to P_s$.*

Proof. Let $\boldsymbol{L}^\#$ be an extension to W of $\boldsymbol{L}$. Since $H^0(W, \Omega_\mu(\boldsymbol{L}^\#)) = 0$ for large μ, $\dim H^0(W, \Omega(\boldsymbol{L}^\#)) = S + 1 < \infty$. By assumption we also have $H^0(W, \Omega(\boldsymbol{L}^\#)) \to H^0(X, \Omega(\boldsymbol{L})) \to 0$. Thus λ extends to a mapping $\Lambda: W \to P_S$. On the other hand, we clearly have that $\Lambda_t = \Lambda \mid X_t$ is an embedding (for t

small). Let $x \in X$ and $\xi \in \boldsymbol{T}_x(W)$. By Proposition 3.3 there exists X_t such that $x \in X_t$ and $\xi \in \boldsymbol{T}_x(X_t)$. Then $\Lambda_*(\xi) = (\Lambda_t)_*(\xi) \neq 0$, and so Λ is of maximal rank in a neighborhood of X. Q. E. D.

COROLLARY. *Let $X \subset W$ be a positive embedding. Then, assuming Theorem* I, *we have that there exists a projective embedding $\Lambda: W \to P_N$, provided that* $\dim X \geqq 3$.

Proof. Let $\boldsymbol{L} \to X$ be an ample line bundle for which $\boldsymbol{L}^* \otimes \boldsymbol{N}^\mu$ is positive for $\mu > 0$. Then $H^2(X, \Omega(\mu)) = 0 = H^1(X, \Omega(L(\mu)))$ for $\mu > 0$. By Theorem I we can then solve the extension problems for $\boldsymbol{L}$ and $H^0(X, \Omega(\boldsymbol{L}))$. Q. E. D.

In case $X \subset W$ is a hypersurface, we need only assume that $\dim X \geqq 2$ since the line bundle $\boldsymbol{L} = [X]$ is already an extension of the positive bundle $\boldsymbol{N} = \boldsymbol{L} \mid X$. In fact, we have

PROPOSITION 3.5 [18]. *Let $\boldsymbol{E}^\# \to W$ be a holomorphic vector bundle and $\boldsymbol{E} = \boldsymbol{E}^\# \mid W$. Then, for $\mu \geqq \mu_0(\boldsymbol{E})$, the mapping*

$$H^0(W, \Omega(\boldsymbol{E}^\# \otimes \boldsymbol{L}^\mu)) \to H^0(X, \Omega(\boldsymbol{E} \otimes \boldsymbol{N}^\mu))$$

is onto.

Proof. Since $\mathfrak{I} \cong \Omega(\boldsymbol{L}^*)$, we have

$$0 \to \Omega(\boldsymbol{E}^\# \otimes \boldsymbol{L}^{\mu-1}) \to \Omega(\boldsymbol{E}^\# \otimes \boldsymbol{L}^\mu) \to \Omega_X(\boldsymbol{E} \otimes \boldsymbol{N}^\mu) \to 0 \tag{3.10}$$

for all μ. We choose ν_0 so that $H^1(X, \Omega(\boldsymbol{E} \otimes \boldsymbol{N}^\mu)) = 0$ for $\mu \geqq \nu_0$. From the exact cohomology sequences of (3.10), we get

$$H^1(W, \Omega(\boldsymbol{E}^\# \otimes \boldsymbol{L}^{\nu_0})) \to H^1(W, \Omega(\boldsymbol{E}^\# \otimes \boldsymbol{L}^{\nu_0+\mu}))) \to 0$$

for all $\mu \geqq 0$. On the other hand, we shall see in VII.1 and VIII.3 that $\dim H^1(W, \Omega(\boldsymbol{E}^\# \otimes \boldsymbol{L}^{\nu_0})) < \infty$. Thus, for large enough μ, the mappings

$$H^1(W, \Omega(\boldsymbol{E}^\# \otimes \boldsymbol{L}^{\mu-1})) \xrightarrow{\approx} H^1(W, \Omega((\boldsymbol{E}^\# \otimes \boldsymbol{L}^\mu))$$

are all isomorphisms. The Proposition now follows from this plus the exact cohomology sequence of (3.10).

4. Proof of Theorem II from Theorem I.

Proof of (i). Let $X \subset W$ be a positive germ of embedding. Then there exists a μ_0 such that, if $X \subset W'$ defines a germ of embedding for which we have an equivalence $f_\mu: X_\mu' \to X_\mu$, then there is a unique formal extension $\hat{f}: \hat{X}' \to \hat{X}$ of f_μ. We want to prove that $\hat{f}$ converges.

By the Corollary to Proposition 3.4, there exists an ample line bundle $\boldsymbol{L}_* \to W_*$ which gives a projective embedding $\Lambda: X_* \to P_S$.

Now we assert that, if we are given *two* projective embeddings $\Lambda: X_* \to P_s$, and $\Lambda^\#: X_* \to P_s$ such that $\Lambda \mid X_\mu = \Lambda^\# \mid X_\mu$ for μ sufficiently large, it follows that $\Lambda = \Lambda^\#$ (i.e. $\Lambda(W) = \Lambda^\#(W)$). Indeed, let Λ be given by sections $\sigma_0, \cdots, \sigma_s$ of $\boldsymbol{L}_*$ and $\Lambda^\#$ by sections $\tau_0, \cdots, \tau_s$ of $\boldsymbol{L}_*{}^\#$. Then, since $\boldsymbol{L}_* \mid X_\mu \cong \boldsymbol{L}_*{}^\# \mid X_\mu$, there exists a (unique) bundle equivalence $\phi: \boldsymbol{L}_* \to \boldsymbol{L}_*{}^\#$ and furthermore $\phi(\sigma_j) \mid X_\mu = \tau_j \mid X_\mu$. By the uniqueness of the power series expansions, it follows that $\phi(\sigma_j) = \tau_j$ as sections of $\boldsymbol{L}_*{}^\#$. Thus $\Lambda = \Lambda^\#$.

Let now $\hat{\boldsymbol{L}}$ be the formal line bundle on $\hat{X}$ determined by $\boldsymbol{L}_*$, and set $\hat{\boldsymbol{L}}' = \hat{f}^{-1}(\hat{\boldsymbol{L}})$.

Let μ be sufficiently large: Set $\boldsymbol{L}'_\mu = \hat{\boldsymbol{L}}' \mid X'_\mu$ and let ${}_\mu\sigma_0, \cdots, {}_\mu\sigma_s$ be the sections of $\hat{\boldsymbol{L}}'_\mu$ over X'_μ determined by $\hat{f}$. By Theorem I, we may find a bundle $\boldsymbol{L}'_* \to X'_*$ and sections $\sigma'_0, \cdots, \sigma'_s$ of L'_* such that $\boldsymbol{L}_*{}' \mid X_\mu = \boldsymbol{L}'_\mu$ and $\sigma'_j \mid \boldsymbol{L}'_\mu = {}_\mu\sigma_j$. Furthermore, the sections $\sigma'_0, \cdots, \sigma'_s$ give a projective embedding $\Lambda': X'_* \to P_s$.

Suppose now that we look at $\boldsymbol{L}'_{\mu+1} = \boldsymbol{L}' \mid X'_{\mu+1}$ and ${}_{\mu+1}\sigma_0, \cdots, {}_{\mu+1}\sigma_s$ as the sections of $\boldsymbol{L}'_{\mu+1}$ determined by $\hat{f}$. By Theorem I, we may find a bundle $\boldsymbol{L}_*{}^\# \to X'_*$ and sections $\tau'_0, \cdots, \tau'_s$ of $\boldsymbol{L}_*{}^\#$ such that $\boldsymbol{L}_*{}^\# \mid X'_{\mu+1} = \boldsymbol{L}'_{\mu+1}$ and $\tau_j \mid X_{\mu+1} = {}_{\mu+1}\sigma_j$. If then $\Lambda^\#: X'_* \to P_s$ is the corresponding projective embedding, it follows from the above remark that $\Lambda' = \Lambda^\#$. Thus, for every μ, $\hat{f} \mid X'_\mu = \Lambda' \mid X'_\mu$ and so $\hat{f} = \Lambda': W' \to \Lambda'(W) \subset P_s$. This says precisely that $\hat{f}$ is induced by an actual equivalence of germs of embeddings. Q.E.D.

Proof of (ii). Let $\boldsymbol{E} \to X$ be a holomorphic vector bundle and suppose that we have an extension $\boldsymbol{E}_\mu \to X_\mu$, where μ is sufficiently large. Then there exists a formal bundle $\hat{\boldsymbol{E}} \to \hat{X}$ such that $\hat{\boldsymbol{E}} \mid X_\mu = \boldsymbol{E}_\mu$. We want to prove that $\hat{\boldsymbol{E}}$ converges.

Let $\boldsymbol{L} \to X$ be the ample line bundle which has an ample extension $\boldsymbol{L}_* \to X_*$. For N sufficiently large, $\boldsymbol{E} \otimes \boldsymbol{L}^N$ will be ample, and we thus get an embedding $\sigma_\mu: X_\mu \to G(m,r) \to P_{m'}$, for some m and m', such that $\sigma_\mu^{-1}(\boldsymbol{E}(m,r)) = \boldsymbol{E}_\mu \otimes \boldsymbol{L}_\mu{}^N$. Here $G(m,r)$ is the Grassman variety of m-planes in $\boldsymbol{C}^{m+r}$ and $\boldsymbol{E}(m,r) \to G(m,r)$ is the universal bundle, with fibre $\boldsymbol{C}^r$, over $G(m,r)$. If we can extend σ_μ to a mapping $\sigma_*: X_* \to G(m,r)$, then we may set $\boldsymbol{E}_* = \sigma_*{}^{-1}(\boldsymbol{E}(m,r)) \otimes \boldsymbol{L}_*{}^{-N}$ and $\boldsymbol{E}_*$ is an extension to X_* of $\boldsymbol{E}_\mu \to X_\mu$.

By the same reasoning as above, we may extend σ_μ to a projective embedding $\tilde{\sigma}_*: X_* \to P_{m'}$. On the other hand, $\tilde{\sigma}_*(X)_\mu = \sigma_\mu(X) \subset G(m,r) \subset P_{m'}$ and the normal bundle $\boldsymbol{N}$ of X in X_* is a sub-bundle or the normal bundle of X in $G(m,r)$. We then consider the complete continuous system

$\{X_t\}$ of X in $G(m,r)$ corresponding to the sections of $\boldsymbol{N}$. The family $\{X_t\}$ gives a local sub-manifold $W' \subset G(m,r)$ and $X \subset W'$.* Furthermore, since $\tilde{\sigma}_* \mid X_\mu = \sigma_\mu$, their germs of embeddings $X \subset W'$ and $X \subset W$ are equivalent up to order μ. Thus, if μ is sufficiently large, they are equivalent by a mapping $g: W \to W'$. If we set $\sigma_* = g \cdot \tilde{\sigma}_*$, then $\sigma_*: W \to G(m,r)$ is our required mapping.

Proof of (iii). Suppose that $\boldsymbol{E}_* \to X_*$ and $\tilde{\boldsymbol{E}}_* \to X_*$ are vector bundles such that $\boldsymbol{E}_* \mid X_\mu \cong \tilde{\boldsymbol{E}}_* \mid X_\mu$ for μ large. This equivalence is given by an element $\gamma_\mu \in H^0(X, \Omega(\mathrm{Hom}(\boldsymbol{E}_* \mid X_\mu, \tilde{\boldsymbol{E}}_* \mid X_\mu)))$. By Theorem I we may find an extension $\gamma_* \in H^0(X, \Omega(\mathrm{Hom}(\boldsymbol{E}_*, \tilde{\boldsymbol{E}}_*)))$ of γ_μ, and clearly γ_* gives a bundle equivalence between $\boldsymbol{E}_* \to X_*$ and $\tilde{\boldsymbol{E}}_* \to X_*$. Q. E. D.

IV. Some Examples and Remarks about Positive Embeddings.

1. Proof of Theorem I for positively embedded hypersurfaces. Let $X \subset W$ be a positive embedding of codimension 1. By Propositions 3.5 and 3.4 (neither of which use Theorem I), there exists a projective embedding $\Phi: W \to P_N$ given by $|\boldsymbol{L}^{\mu_0}|$ $(\boldsymbol{L} = [X])$ for μ_0 large enough. On the other hand, by VIII. 2 and VIII. 3, there exists $W^\# \subset W$ and a real non-negative function r defined on W such that $\partial\bar{\partial} r$ has, in $W - W^\#$, n negative eigenvalues. Here $n = \dim X$ and we assume that $n \geqq 2$. Thus $\dim H^0(W, \Omega(\boldsymbol{E})) < \infty$ for any line bundle $\boldsymbol{E} \to W$. Then, by the argument of Siegel (cf. [1] for the details) we find that the field $\mathfrak{M}$ of meromorphic functions on W is an algebraic function field of transcendence degree less than or equal to $n+1 = \dim W$. Let $V \subset P_N$ be the smallest algebraic variety containing $\Phi(W)$ and $\mathfrak{R}$ the field of rational functions on V. Then the restriction mapping $\mathfrak{R} \to \mathfrak{M}$ is obviously an injection and so $\dim V = \deg \mathfrak{R} \leqq \deg \mathfrak{M} \leqq \dim W$; i. e. $\dim V = \dim W$. Thus V is an algebraic compactification of $\Phi(W) \subset P_N$. Furthermore, by construction, $\mu_0 \Phi(X) \subset V$ is a hyperplane section.

Let $\boldsymbol{L} \to V$ be the line bundle $[\Phi(X)]$. For $\boldsymbol{E} \to V$ any holomorphic bundle, set $\boldsymbol{E}(\mu) = \boldsymbol{E} \otimes \boldsymbol{L}^{*\mu}$. By Theorems A and B for algebraic varieties, we get

$$H^q(V, \Omega(\boldsymbol{E}(\mu))) = 0 \text{ for } 0 \leqq q < n+1, \mu \geqq \mu_0(\boldsymbol{E}). \tag{4.1}$$

Now set $Z = V - \Phi(W')$ where $W^\# \subset W' \subset W$. If Ψ is the family of all closed sets in Z, we have an exact cohomology sequence

* In fact, W' is the image of a complex manifold V' under a holomorphic mapping $F': V' \to G(m,r)$ such that F'_* has everywhere constant rank (cf. Thtorem 3.1).

$$(4.2)\qquad H_\Psi^q(Z,\Omega(\boldsymbol{E}(\mu)))\to H^q(V,\Omega(\boldsymbol{E}(\mu)))\to H^q(W',\Omega(\boldsymbol{E}(\mu)))\to H_\Psi^{q+1}(Z,\Omega(\boldsymbol{E}(\mu)))\to\cdots.$$

By [3], § 20, $H_\Psi^q(Z,\Omega(\boldsymbol{E}(\mu)))=0$ for $0\leqq q<n+1$ and then (4.1) and (4.2) give

$$(4.3)\qquad H^q(W',\Omega(\boldsymbol{E}(\mu)))=0 \text{ for } 0\leqq q<n, \mu\geqq\mu_0(\boldsymbol{E}).$$

Now (4.3) gives Theorem I, at least for those bundles $\boldsymbol{E}\to W$ which extend to V.*

COROLLARY. *Theorem II holds for positive embeddings of codimension* 1.

Proof. We must show that line bundles extend from $\Phi(W)$ to V. But this follows from Proposition (1.5).

2. Rigidity of some special embeddings. Let $\phi: X\to W$ and $\phi': X\to W'$ be positive embeddings. Assuming Theorem I and that $\dim X\geqq 3$, we may find projective embeddings $\Phi: W\to P_N$, $\Phi': W'\to P_{N'}$. Now the same argument as in § 4.1 above shows that there exist algebraic compactifications $V\subset P_N$, $V'\subset P_{N'}$ of $\Phi(W)$, $\Phi'(W')$ respectively. Thus $\Phi(W)\subset V$, $\dim\Phi(W)=\dim V$, and the restriction of rational functions from V to $\Phi(W)$ is an isomorphism of function fields. Consequently, any equivalence of germs $F: W\to W'$ will induce a meromorphic mapping $\Lambda: V\to V'$. Now it sometimes happens that Λ will automatically be bi-holomorphic (e.g. if $V=V'$ and V is a minimal variety such as P_n, an abelian variety, or an algebraic manifold uniformized by a bounded domain.* In this situation one may discuss, by direct geometric methods the rigidity problem. We give two examples of this:

(i) Projective embeddings. Let $P=P_n$ and let $\phi: X\to P$, $\phi': X\to P$ be two positive embeddings.

PROPOSITION 4.1. *Suppose that we have an equivalence between* 1*-neighborhoods* $h: X'_1\to X_1$ *such that* $h\mid X$ *is homotopic to the identity. Then there exists a projective transformation* $g\in SL(N+1,\boldsymbol{C})$ *such that* $g\circ\phi'=\phi$ *and* $g\mid X'_1=h$.

Proof. Let $\boldsymbol{H}\to P$ be the hyperplane bundle. We recall the exact sequence

$$(4.4)\qquad 0\to\boldsymbol{l}\to(\boldsymbol{H})^{N+1}\to\boldsymbol{T}(P)\to 0$$

* We remark that $H^{n+1}(W',\Omega(\boldsymbol{F}))=0$ for any holomorphic bundle $\boldsymbol{F}\to W'$.

* The point here is that every automorphism of the function field induces one of the variety.

where $\boldsymbol{l}$ is the trivial bundle and $(\boldsymbol{H})^{N+1} = \underbrace{\boldsymbol{H} \oplus \cdots \oplus \boldsymbol{H}}_{N+1}$.

Now $\Omega(\boldsymbol{T}(P)^*) = \Omega^1(P) =$ the germs of holomorphic 1-forms on P, and (4.4) is uniquely given by $\boldsymbol{l}$, $\boldsymbol{T}(P)$, and the extension elsment $\omega \in H^1(P, \Omega^1(P))$. Clearly ω is the fundamental form on P.

Let $\boldsymbol{T} = \boldsymbol{T}(P) \mid \phi(X)$, $\boldsymbol{T}' = \boldsymbol{T}(P) \mid \phi'(X)$ and denote by $\boldsymbol{N}$, $\boldsymbol{N}'$ the respective normal bundles of $\phi(X)$, $\phi'(X)$ in P. Then $h: X'_1 \to X_1$ is given by a bundle mapping $h: \boldsymbol{T}' \to \boldsymbol{T}$ which takes

$$0 \to \boldsymbol{T}(X) \to \boldsymbol{T}' \to \boldsymbol{N}' \to 0 \tag{4.5)'}$$

onto

$$0 \to \boldsymbol{T}(X) \to \boldsymbol{T} \to \boldsymbol{N} \to 0. \tag{4.5}$$

Write f for the automorphism of X which h induces.

Letting $\boldsymbol{L} = \boldsymbol{H} \mid \phi(X), \boldsymbol{L}' = \boldsymbol{H} \mid \phi'(X)$, we want to show that h extends to a mapping of

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & & & \boldsymbol{l} & & \\ & & & & & & \downarrow & & \\ & & & & & & (\boldsymbol{L}')^{N+1} & & \\ & & & & & & \downarrow & & \\ 0 & \to & \boldsymbol{T}(X) & \to & \boldsymbol{T}' & \to & \boldsymbol{N}' & \to & 0 \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array} \tag{4.6)'}$$

onto

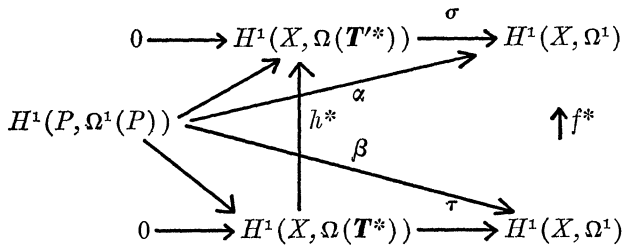
$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & & & \boldsymbol{l} & & \\ & & & & & & \downarrow & & \\ & & & & & & (\boldsymbol{L})^{N+1} & & \\ & & & & & & \downarrow & & \\ 0 & \to & \boldsymbol{T}(X) & \to & \boldsymbol{T} & \to & \boldsymbol{N} & \to & 0 \\ & & & & & & \downarrow & & \\ & & & & & & 0. & & \end{array} \tag{4.6}$$

Now (4.6)' is given by $\gamma' \in H^1(X, \Omega(\boldsymbol{T}'^*))$, (4.6) is given by $\gamma \in H^1(X, \Omega(\boldsymbol{T}^*))$, and h will extend to a mapping of (4.6)' onto (4.6) if $h^*(\gamma) = \gamma'$ where $h^*: H^1(X, \Omega(\boldsymbol{T}^*)) \to H^1(X, \Omega(\boldsymbol{T}'^*))$ is the cohomology map.

Since $\boldsymbol{N}$ and $\boldsymbol{N}'$ are positive and $\dim X \geqq 2$,

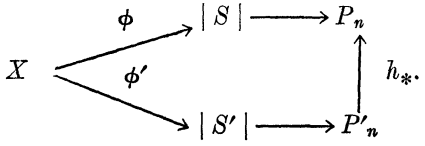
$$H^1(X, \Omega(\boldsymbol{N}^*)) = 0 = H^1(X, \Omega(\boldsymbol{N}'^*)).$$

Letting $\Omega^1 = \Omega(\boldsymbol{T}(X)^*)$, we have



Because $\sigma(\gamma') = \alpha(\omega)$ and $\tau(\gamma) = \beta(\omega)$, to prove that $h^*(\gamma) = \gamma'$ it will suffice to show that $f^*\beta(\omega) = \alpha(\omega)$. But this is true since $f\beta \sim \alpha$ and X is Kähler. Thus h induces a bundle mapping $h: \boldsymbol{L}' \to \boldsymbol{L}$ such that $h((4.6)') = (4.6)$.

The embedding $\phi: X \to P$ is given by a linear subspace $S \subset H^0(X, \Omega(\boldsymbol{L}))$; ϕ' is given by $S' \subset H^0(X, \Omega(\boldsymbol{L}'))$. To prove Proposition 4.1, we must show that $h_*(S') = S$ where $h_*: H^0(\Omega(\boldsymbol{L}')) \to H^0(\Omega(\boldsymbol{L}))$ is the cohomology map. Let $n+1 = \dim H^0(\Omega(\boldsymbol{L}))$. There are inclusions $P \subset |\boldsymbol{L}| \cong P_n$, $P \subset |\boldsymbol{L}'| \cong P'_n$. Furthermore, the following diagram commutes:



Finally, $h_*(|S'|) \subset P_n$ is a linear subspace which osculates along $\phi(X)$ to $|S|$ with at least first order contact. Thus $h_*(|S'|) = |S|$. Q. E. D.

Remark. As promised, we give now a proof of Lemma 3.1. We have the exact bundle sequence

$$0 \to \boldsymbol{T}(P_n)(\mu) \to \boldsymbol{T}(\mu) \to \boldsymbol{H}^{*\mu-1} = 0 \qquad (\mu \geqq 1)$$

and, since $H^1(P_n, \Omega(\boldsymbol{H}^{*\mu-1})) = 0$ $(\mu \geqq 1)$, it will suffice to show

$$H^1(P_n, \Omega(\boldsymbol{T}(P_n) \otimes \boldsymbol{H}^{*\mu})) = 0$$

for $\mu \geqq 1$. From (4.4) we have

$$0 \to \boldsymbol{H}^{*\mu} \to (\boldsymbol{H}^{*\mu-1})^{n+1} \to \boldsymbol{T}(P_n)(\mu) \to 0. \tag{4.7}$$

From the exact cohomology sequence of (4.7) it follows that

$$H^1(X, \Omega(\boldsymbol{T}(P_n)(\mu))) = 0 \qquad (\mu > 0),$$

provided that $n > 2$.

Q. E. D.

(ii) Embeddings in abelian varieties. Let X be a compact algebraic manifold. If we fix a point $x_0 \in X$, then there is defined an Abelian variety A_X, the *Albanese variety* of X, and regular mapping $\psi: X \to A_X$ which has the following universal property: Given any complex torus T and a mapping $\xi: X \to T$, there exists $t_\xi \in T$ and a homomorphism $\theta: A_X \to T$ such that $\xi(x) = \theta\psi(x) + t_\xi$ $(x \in X)$. We say that ξ is normalized if $\xi(x_0)$ is the identity in T.

LEMMA 4.1. *Let A be an abelian variety and $\phi: X \to A$ a positive normalized embedding. Then A_X is a finite covering torus of A and θ is the projection homomorphism.*

Proof. Letting $\boldsymbol{T} = \boldsymbol{T}(A) \mid \phi(X)$ and $\Omega^1 = \Omega(\boldsymbol{T}(X)^*)$, we get from $0 \to \boldsymbol{T}(X) \to \boldsymbol{T} \to \boldsymbol{N} \to 0$ the sequence

$$0 \to H^0(X, \Omega(\boldsymbol{T}^*)) \xrightarrow{\gamma} H^0(X, \Omega^1) \to 0. \tag{4.8}$$

Now $\boldsymbol{T}_e(A_X)^* \cong H^0(X, \Omega^1)$, $\boldsymbol{T}_e(A)^* \cong H^0(A, \Omega^1(A)) \cong H^0(X, \Omega(\boldsymbol{T}^*))$, and $\gamma: \boldsymbol{T}_e(A)^* \to \boldsymbol{T}_e(A_X)^*$ is just the transpose of θ_*, the differential of θ at the identity. The Lemma then follows from (4.8). Q.E.D.

From Lemma 4.1 we may easily prove

PROPOSITION 4.2. *Let A be an abelian variety. Then a positive embedding $\phi: X \to A$ $(\dim X \geqq 2)$ is uniquely determined, up to an automorphism of A, by its normal bundle.*

3. On ample hypersurfaces. We discuss now some applications of the rigidity theorem; the main one of these is

THEOREM 4.1. *Let X be an algebraic manifold of dimension $n \geqq 2$. Then there exists an integer $\mu_0 = \mu_0(X)$ such that the following is true: If W, W' are any two nonsingular varieties for which we have embeddings $\phi: X \to W$, $\phi': X \to W'$ as hyperplane section, and if we have an equivalence $X_\mu \cong X'_\mu$ for $\mu \geqq \mu_0$, then there exists a biregular mapping $\Phi: W \to W'$ such that*

$$\begin{array}{ccc} & & W \\ & \phi\nearrow & \\ X & & \downarrow\Phi \\ & \phi'\searrow & \\ & & W' \end{array}$$

commutes.

Remark. Briefly we may say that an algebraic variety of dimension $\geqq 3$ is determined by a finite neighborhood of a hyperplane section.

Let X be an algebraic variety of dimension $n \geqq 2$ and W a non-singular variety of dimension $n+1$. We say that an embedding $\phi: X \to W$ is *ample* if, and only if, the line bundle $\boldsymbol{L}_\phi = [\phi(X)]$ on W is positive. Equivalently, $|\boldsymbol{L}_\phi{}^\mu|$ for large μ should give a projective embedding. Clearly, if ϕ is ample and if $C \subset W$ is any curve, then $\phi(X) \cap C \neq \emptyset$. Also the normal bundle $\boldsymbol{N}_\phi$ to $\phi(X) \subset W$ is positive. Conversely, we have

PROPOSITION 4.3. *Let $\phi: X \to W$ be an embedding such that $\boldsymbol{N}_\phi$ is positive and such that $\phi(X) \cap C \neq \emptyset$ for any curve $C \subset W$. Then ϕ is ample.*

Proof. By the same argument as in IV.1, there exists a neighborhood U of $\phi(X) \subset W$ and a μ_0 such that:

(i) For $\mu \geqq \mu_0$, we have
$$H^0(U, \Omega(\boldsymbol{L}_\phi{}^\mu)) \to H^0(X, \Omega(\boldsymbol{N}_\phi{}^\mu)) \to 0;$$

(ii) $|\boldsymbol{L}_\phi{}^{\mu_0}|$ gives a projective embedding $\Phi: U \to P_N$.

By VIII.3 below, we may further assume:

(iii) The E. E. Levi form in a neighborhood of ∂U has everywhere n positive eigenvalues. By (iii) we see that, for any holomorphic bundle $\boldsymbol{E} \to W$, the restriction mapping gives an isomorphism
$$0 \to H^q(W, \Omega(\boldsymbol{E})) \to H^q(U, \Omega(\boldsymbol{E})) \to 0$$
for $q < n-1$. Since $n \geqq 2$, we get, in particular,
$$0 \to H^0(W, \Omega(\boldsymbol{E})) \to H^0(U, \Omega(\boldsymbol{E})) \to 0.$$
Thus we can extend Φ to a holomorphic mapping $\Phi: W \to P_N$.

Consider now the differential Φ_* of $\Phi: W \to P_N$. Since $\Phi(X) \cap C \neq \emptyset$ for any curve C, it follows that Φ_* is singular at finitely many points. But then it follows easily that $\boldsymbol{L}_\phi \to W$ is positive. Q. E. D.

COROLLARY 4.1. *The following are equivalent:*

(i) *$\phi: X \to W$ is ample;*

(ii) *$\boldsymbol{N}_\phi \to X$ is positive and $\phi(X) \cdot C > 0$ for every irreducible curve C.*

(iii) *There exists a positive bundle $\boldsymbol{L} \to W$ and $\sigma \in H^0(W, \Omega(\boldsymbol{L}))$ such that $\phi(X) = \{w \in W: \sigma(w) = 0\}$.*

Remark. Proposition 4.3 is closely related to a theorem of Nakai [16] which reads:

(*) Let $X \subset W$ be a hypersurface such that $X^r \cdot Y > 0$ for any sub-variety $Y \subset W$ of dimension $r < n+1$. Then X is ample.

In order to prove (*) from the Corollary above, we introduce the seemingly more general statement

(#) Let $\boldsymbol{E} \to W$ be a line bundle such that $(\boldsymbol{E})^r \cdot Y > 0$ for any r-dimensional sub-variety $Y \subset W$. Then $\boldsymbol{E}$ is positive.

Remark. $(\boldsymbol{E})^r \cdot Y$ is defined in the obvious way using the Chern class of $\boldsymbol{E}$.

In order to prove (*), using the Corollary above it will suffice to show that the normal bundle $\boldsymbol{N} \to X$ is positive. But this follows from (#) in one dimension lower, since for $Y \subset X$ an r-dimensional variety, $(\boldsymbol{N})^r \cdot Y = X^r \cdot Y > 0$ because of $\boldsymbol{N} = [X] \mid X$. On the other hand, (*) is true for $n = 1$ since $\deg \boldsymbol{N} = X \cdot X > 0$. Thus, by induction, we may assume that (*) holds in dimension n, (#) holds in dimension $n-1$, and show that (#) holds in dimension n. The argument for this is not difficult and will be omitted.

Proof of Theorem 4.1. Assuming $X_\mu \cong X'_\mu$ for $\mu \geqq \mu_0$, the integer for the rigidity theorem, we find neighborhoods U, U' of $\phi(X)$, $\phi'(X)$ in W, W' and a biholomorphic map $\Phi: U \to U'$ such that

$$\begin{array}{ccc} & & U \\ & \phi \nearrow & \\ X & & \downarrow \Phi \\ & \phi' \searrow & \\ & & U' \end{array}$$

commutes. Furthermore, we may assume that U and U' are embedded in a projective spaces of the same dimension by $|\boldsymbol{L}_{\phi}{}^{\mu_0}|$ and $|\boldsymbol{L}_{\phi'}{}^{\mu_0}|$ respectively. But then Φ extends to a meromorphic mapping $\Phi: W \to W'$ and the argument of Proposition 4.3 shows that Φ is everywhere defined. The same holds for Φ^{-1} and thus $\Phi: W \to W'$ is biregular. Q.E.D.

4. A partial generalization to higher codimension. Let $G(m,r)$ be the Grassmann variety of m-planes in $\boldsymbol{C}^{m+r}$. Over $G(m,r)$ there is the universal bundle $\boldsymbol{F}(m,r)$ whose fibre over $\xi \in G(m,r)$ is just the m-dimensional vector space ξ. There is the usual exact sequence

$$0 \to \boldsymbol{F}(m,r) \to \boldsymbol{C}^{m+r} \to \boldsymbol{E}(m,r) \to 0 \tag{4.9}$$

where $\boldsymbol{C}^{m+r}$ is the trivial bundle $G(m,r) \times \boldsymbol{C}^{m+r}$. We remark that $\dim G(m,r) = mr$ and that

$$\boldsymbol{T}(G(m,r)) \cong \mathrm{Hom}(\boldsymbol{F}(m,r), \boldsymbol{E}(m,r)). \tag{4.10}$$

One main difficulty in generalizing IV.3 above is the following:

PROPOSITION 4.4 [8]. *$\boldsymbol{E}(m,r) \to G(m,r)$ is weakly positive if, and only if, $r=1$.*

At the moment, we shall thus settle for a generalization of the implication (iii) $\Rightarrow$ (ii) in Corollary 4.1. Let then $\boldsymbol{E} \to W$ be a holomorphic bundle with fibre $\boldsymbol{C}^r$ over the compact variety W of dimension $n+r$ $(r \geqq 1)$. Let further $\dim H^0(W, \Omega(\boldsymbol{E})) = m+r$ $(m \geqq 0)$.

Definition. We shall say that $\boldsymbol{E} \to W$ is *ample* if: (i) the global sections in $H^0(W, \Omega(\boldsymbol{E}))$ generate every fibre $\boldsymbol{E}_w$; and (ii) if $w_0 \in W$ and if $s_1, \cdots, s_m \in H^0(W, \Omega(\boldsymbol{E}))$ are a basis for the sections which vanish at w_0, then the differentials $(ds_a)(w_0)$ $(1 \leqq a \leqq m)$ should span $\boldsymbol{E}_{w_0} \otimes \boldsymbol{T}_{w_0}(W)^*$.

Remark. If $\boldsymbol{E} \to W$ is ample, then it is weakly positive (VI.5). If, conversely, $\boldsymbol{E} \to W$ is weakly positive, then $\boldsymbol{E}^\mu \to W$ is ample for $\mu \geqq \mu_0$.

By definition, if $\boldsymbol{E} \to W$ is ample, then $\dim H^0(W, \Omega(\boldsymbol{E})) \geqq r + r(\dim W)$. Thus $\boldsymbol{E}(m,r)$ is not ample if $r > 1$. On the other hand, $\boldsymbol{T}(P_n)$ is ample (since the projective group acts transitively on the tangent directions to P_n); observe that $\dim H^0({}_n, \boldsymbol{T}(P_n)) = n^2 + 2n \geqq n^2 + n$.

If $\boldsymbol{E} \to W$ is ample, we have the exact sequences

$$0 \to \boldsymbol{F} \to W \times H^0(W, \Omega(\boldsymbol{E})) \xrightarrow{\rho} \boldsymbol{E} \to 0 \tag{4.11}$$

$$\boldsymbol{F} \xrightarrow{\sigma} \boldsymbol{E} \otimes \boldsymbol{T}(W)^* \to 0 \tag{4.12}$$

where $\rho(w,s) = s(w) \in \boldsymbol{E}_w$ for $w \in W$, $s \in H^0(W, \Omega(\boldsymbol{E}))$ and σ is given by $\sigma_w(s) = (ds)(w)$ since $\boldsymbol{F}_w \cong$ sections $s \in H^0(W, \Omega(\boldsymbol{E}))$ with $s(w) = 0$.

Now the mapping $w \to \boldsymbol{F}_w \subset H^0(W, \Omega(\boldsymbol{E}))$ defines $\phi: W \to G(m,r)$ such that $\phi^{-1}(\boldsymbol{E}(m,r)) = \boldsymbol{E}$ and $\phi^{-1}(\boldsymbol{F}(m,r)) = \boldsymbol{F}$. The differential ϕ_* of ϕ is a global section of

$$\mathrm{Hom}(\boldsymbol{T}(W), \phi^{-1}(\boldsymbol{T}(G(m,r)))) = \mathrm{Hom}(\boldsymbol{T}(W), \mathrm{Hom}(\boldsymbol{F}, \boldsymbol{E}))$$

by (4.10). Thus $\phi^* \in \mathrm{Hom}(\mathrm{Hom}(\boldsymbol{E}, \boldsymbol{F}), \boldsymbol{T}(W)^*)$. On the other hand, there is a natural isomorphism

$$\mathrm{Hom}(\mathrm{Hom}(\boldsymbol{E}, \boldsymbol{F}), \boldsymbol{T}(W)^*) \cong \mathrm{Hom}(\boldsymbol{F}, \boldsymbol{E} \otimes \boldsymbol{T}(W)^*)$$

and it is easy to see that, under this isomorphism, ϕ^* corresponds to σ given in (4.12). That is, for all $\lambda \in \boldsymbol{T}(W)$, $e^* \in \boldsymbol{E}^*$, $f \in \boldsymbol{F}$, we have

$$\langle \lambda, \phi^*(e^* \otimes f) \rangle = \langle e^* \otimes \lambda, \sigma(f) \rangle.$$

In particular, if σ is onto, then ϕ^* is onto but the converse is *not* true unless $r=1$. Thus $\phi: W \to G(m,r)$ is an embedding but this does *not* make $\phi^{-1}(\boldsymbol{E}(m,r))$ ample unless $r=1$. In fact, we can prove

PROPOSITION 4.5. *$\phi^{-1}(\boldsymbol{E}(m,r))$ is ample if $\phi: W \to G(m,r)$ is an embedding and if the holomorphic second fundamental form of $\phi(W) \subset G(m,r)$ (cf. VI.3) is non-degenerate.*

Thus, in order that $\phi^{-1}(\boldsymbol{E}(m,r))$ *be ample,* $\phi(W)$ must be sufficiently twisted in $G(m,r)$.

THEOREM 4.2. *Let $s \in H^0(W, \Omega(\boldsymbol{E}))$ and set $X = \{w \in W : s(w) = 0\}$. Then, assuming $\boldsymbol{E} \to W$ is ample, $X \cdot Y > 0$ for every irreducible variety $Y \subset W$ of dimension r.*

Remarks. (i) The normal bundle N of $X \subset W$ is given by $\boldsymbol{N} = E \mid X$. Thus $\boldsymbol{N} \to X$ has whatever positivity is present in $\boldsymbol{E} \to W$. In particular, $\boldsymbol{N}$ is weakly positive.

(ii) For $r=1$, this includes the implications (iii) $\Rightarrow$ (ii) of Corollary 4.1. Indeed, $\boldsymbol{L}^\mu \to X$ is ample for large μ, and $\phi(X) \cdot C = 1/\mu(\boldsymbol{L}^\mu) \cdot C > 0$.

(iii) Theorem 4.2 does *not* follow from the fact that $w \to \boldsymbol{F}_w$ gives an embedding $\phi: W \to G(m,r)$. In fact, Theorem 4.2 is false for $\boldsymbol{E}(m,r) \to G(m,r)$.

(*Proof.* For $a = (a_1, \cdots, a_r)$ with $0 \leqq a_1 < \cdots < a_r \leqq m+r$, the (generic) Schubert variety $\Omega_a \subset G(m,r)$ is defined ([10], § XIV.2); $\dim \Omega_a = \sum_{i=1}^{r} a_i - \frac{1}{2}r(r-1)$. If $b = (b_1, \cdots, b_r)$ is such that $\dim \Omega_a + \dim \Omega_b = mr$, then $\Omega_a \cdot \Omega_b \geqq 0$ and $\Omega_a \cdot \Omega_b > 0$ if, and only if, $a_i + b_{r-i} \geqq m+r$ $(1 \leqq i \leqq r)$. For $0 < s \leqq r$, we let Ω_s be the Schubert variety with the symbol $(\overbrace{m, \cdots, m+s-1}^{s}, \overbrace{m+s+1, \cdots, m+r}^{r-s})$. Then $\dim \Omega_s = mr - s$, and in fact Ω_s is dual to $c_s(\boldsymbol{E})$. Now we have $\Omega_r \cdot \Omega_b = \int_{\Omega_b} c_r(\boldsymbol{E}) = 0$ if say $b = (0, 2, 3, \cdots, r-1, r+1)$ (for the notations see the proof of Theorem 4.2). At the last step we have, of course, used that $r > 1$.)

Proof of Theorem 4.2. Let $c(\boldsymbol{E}) = c_0 + \cdots + c_r$ $(c_i \in H^{2i}(W, \boldsymbol{Z}))$ be the total Chern class of $\boldsymbol{E} \to W$. Then, if $y \in H_{2r}(W, \boldsymbol{Z})$ is the homology class of Y,

$$X \cdot Y = \langle c_r, y \rangle \tag{4.13}$$

where $\langle\ ,\ \rangle: H^{2r}(W,\boldsymbol{Z})\otimes H_{2r}(W,\boldsymbol{Z})\to\boldsymbol{Z}$ is the usual pairing. In order to prove Theorem 4.1, we shall prove the more general

THEOREM 4.2. *Let $Y\subset W$ be a irreducible variety of dimension q, $0<q\leqq r$. Then, with the above notation,*

$$\langle c_q, y\rangle > 0. \tag{4.14}$$

Now E fits in the exact bundle sequence

$$0\to\boldsymbol{F}\to\boldsymbol{C}^{m+r}\xrightarrow{\rho}\boldsymbol{E}\to 0 \tag{4.15}$$

where $\boldsymbol{C}^{m+r}$ is the trivial bundle $W\times H^0(W,\Omega(\boldsymbol{E}))$. The flat metric in the middle term induces then a metric in $\boldsymbol{E}$ and there are then defined real (q,q) forms θ_q on W with $d\theta_q=0$ and such that θ_q represents the de Rham class of c_q. Then

$$\langle c_q, y\rangle = \int_Y \theta_q \tag{4.16}$$

and it will suffice to show that $\theta_q \mid Y$ gives a volume element. This is a local differential-geometric question, and we shall use the concepts and notations developed in § VI below.

The curvature form Θ in $\boldsymbol{E}$ is locally written as $\Theta=\{\Theta_{\rho\sigma}\}$ where

$$\Theta_{\rho\sigma}=\sum\Theta_{\rho\sigma i\bar{j}}\omega^i\wedge\bar{\omega}^j \qquad (1\leqq\rho,\sigma,\tau\leqq r;\ 1\leqq i,j\leqq n+r).$$

Also, the θ_q are uniquely defined by (cf. [5]):

$$\sum_{t=0}^{r}\theta_q t^{r-q}=\det\left(t\boldsymbol{I}^r+\frac{\sqrt{-1}}{2\pi}\Theta\right) \tag{4.17}$$

where $\boldsymbol{I}^r$ is the $r\times r$ unit matrix. Thus, if $S(q)=$ permutation group on q letters,

$$\theta_q=\left(\frac{\sqrt{-1}}{2\pi}\right)^q\sum_{1\leqq\rho_1<\cdots<\rho_q\leqq r}\ \sum_{\pi\in S(q)}(-1)^\pi \Theta_{\rho_1\pi(\rho_1)}\wedge\cdots\wedge\Theta_{\rho_q\pi(\rho_q)} \tag{4.18}$$

Now, by VII.3 below, Θ is given as $\Theta={}^tb\wedge\bar{b}$ where $b=\{b_{a\rho}\}$ with

$$b_{a\rho}=\sum_i b_{a\rho i}\omega^i \qquad (1\leqq a\leqq m)$$

is a global holomorphic section of $\mathrm{Hom}(\boldsymbol{E},\boldsymbol{T}(W)^*\otimes\boldsymbol{F})$ called the second fundamental form of $\boldsymbol{F}$ in $\boldsymbol{C}^{m+r}$ (cf. VI.3 where this terminology is justified). In the present case, b is just ϕ^* where $\phi: W\to G(m,r)$ is the embedding of W in the Grassmannian such that $\phi^{-1}(4.9)=(4.15)$. Explicitly,

$$\Theta_{\rho\sigma ij} = \sum_{a=1}^{m} b_{a\rho i}\bar{b}_{a\sigma j}, \tag{4.19}$$

Given any point $w_0 \in W$, we may find local holomorphic coordinates $z = (z^1, \cdots, z^{n+r})$ centered at w_0, a trivialization of $\boldsymbol{E}$ near w_0, and generators $s_1, \cdots, s_m$ for $\boldsymbol{F}_{w_0} \subset H^0(W, \Omega(\boldsymbol{E}))$ such that

$$b_{a\rho i}(w_0) = \frac{\partial s_a{}^{\rho}}{\partial z^i}(w_0). \tag{4.20}$$

Furthermore, if $Y \subset W$ is a non-singular subvariety of dimension q and passing through w_0, we may assume that Y is locally defined by $z^j = 0$ $(q < j \leqq n+r)$ and, since $\boldsymbol{E} \to W$ is ample, we may choose our frames so that

$$b_{1\alpha\beta}(w_0) = \delta_{\beta}{}^{\alpha} \qquad (1 \leqq \alpha, \beta \leqq q). \tag{4.21}$$

LEMMA 4.3.

$$\theta_q = \frac{(-1)^{\frac{q(q-1)}{2}}}{q!}\left(\frac{\sqrt{-1}}{2\pi}\right)^q \sum_{\substack{\rho=(\rho_1,\cdots,\rho_q)\\ a=(a_1,\cdots,a_q)}} B_{\rho a} \wedge \bar{B}_{\rho a} \tag{4.22}$$

where $B_{\rho a} = \sum_{\pi \in S(q)} b_{\pi(a_1)\rho_1} \wedge \cdots \wedge b_{\pi(a_q)\rho_q}$ $(1 \leqq \rho_1 < \cdots < \rho_q \leqq r)$ and where $a = (a_1, \cdots, a_q)$ runs over *all* q-tuples of integers selected from $(1, \cdots, m)$.

Proof. $\sum_{\pi \in S(q)} (-1)^{\pi} \Theta_{\rho_1\pi(\rho_1)} \wedge \cdots \wedge \Theta_{\rho_q\pi(\rho_q)} =$ (by (4.19))

$$\sum_{\substack{a=(a_1,\cdots,a_q)\\ \pi \in S(q)}} (-1)^{\pi} b_{a_1\rho_1} \wedge \bar{b}_{a_1\pi(\rho_1)} \wedge \cdots \wedge b_{a_q\rho_q} \wedge \bar{b}_{a_q\pi(\rho_q)}$$

$$= (-1)^{\frac{q(q-1)}{2}} \sum_{a,\pi} (-1)^{\pi} b_{a_1\rho_1} \wedge \cdots \wedge b_{a_q\rho_q} \wedge \bar{b}_{a_1\pi(\rho_1)} \wedge \cdots \wedge \bar{b}_{a_q\pi(\rho_q)}$$

$$= (-1)^{\frac{q(q-1)}{2}} \sum_{a,\pi} b_{a_1\rho_1} \wedge \cdots \wedge b_{a_q\rho_q} \wedge \bar{b}_{\pi^{-1}a_1\rho_1} \wedge \cdots \wedge \bar{b}_{\pi^{-1}(a_q)\rho_q}$$

$$= \frac{(-1)^{\frac{q(q-1)}{2}}}{q!} \sum_{a} B_{\rho a} \wedge \bar{B}_{\rho a}.$$ Combining this with (4.18), we get

$$\theta_q = \frac{(-1)^{\frac{q(q-1)}{2}}}{q!}\left(\frac{\sqrt{-1}}{2\pi}\right)^q \sum_{\rho,a} B_{\rho a} \wedge \bar{B}_{\rho a},$$

which proves the Lemma.

We may now complete the proof of Theorem 4.2′. Let Y pass through w_0 and choose local frames $\omega^1, \cdots, \omega^{n+r}$ such that $\omega^j = 0$ on Y for $q < j$

$\leqq n+r$. Furthermore, choose a local frame for $\boldsymbol{E}$ so that (4.21) is satisfied. Now let $a_0=(1,\cdots,1)$ and $\rho_0=(1,\cdots,q)$. Then

$$B_{\rho_0 a_0}(w_0) = q!(-1)^{\frac{q(q-1)}{2}}\ \omega^1\wedge\cdots\wedge\omega^q$$

and so, by (4.22)

$$(\theta_q \mid Y)(w_0) = (-1)^{\frac{q(q-1)}{2}}\left(\frac{\sqrt{-1}}{2\pi}\right)^q \omega^1\wedge\cdots\wedge\omega^q\wedge\bar{\omega}^1\wedge\cdots\wedge\bar{\omega}^q + (\text{non-negative terms}).$$

Thus $\theta_q \mid Y$ is a volume element. Q. E. D.

Remark. We can generalize Theorem 4.2′ a little as follows: Let $Y\subset W$ be an irreducible subvariety of dimension s, and let $q_1,\cdots,q_t$, $1\leqq q_\mu\leqq r$, satisfy $\sum_{\mu=1}^{t} q_\mu = s$. Then we have

THEOREM 4.3. $\langle c_{q_1}\cdots c_{q_t}, y\rangle > 0$.

Proof. For simplicity, we assume that $t=2$. Thus let p, q satisfy $1\leqq p,q\leqq r$ and $p+q=s$. We agree on the ranges of indices $1\leqq\alpha,\beta\leqq p$; $p+1\leqq\mu,\nu\leqq p+q$. Then, by (4.22),

$$\theta_p\wedge\theta_q = \frac{(-1)^{\frac{1}{2}(p(p-1)+q(q-1))}}{p!q!}\left(\frac{\sqrt{-1}}{2\pi}\right)^{p+q}\sum_{\substack{\rho=(\rho_1,\cdots,\rho_p)\\ a=(a_1,\cdots,a_p)\\ \sigma=(\sigma_1,\cdots,\sigma_q)\\ d=(d_1,\cdots,d_q)}} \{B_{\rho a}\wedge\bar{B}_{\rho a}\wedge B_{\sigma d}\wedge\bar{B}_{\sigma d}\}$$

$$= \frac{(-1)^{\frac{1}{2}(p+q)(p+q-1)}}{p!q!}\left(\frac{\sqrt{-1}}{2\pi}\right)^{p+q}\sum_{\substack{\rho,\sigma\\ a,d}} B_{\rho a}\wedge B_{\sigma d}\wedge\overline{B_{\rho a}\wedge B_{\sigma d}}.$$

As above, we let Y pass through w, and be given locally by $z^j=0$ $(s<j\leqq n+r)$. Furthermore, we choose our local $\boldsymbol{E}$-frame so that $b_{1\rho j}(w_0)=0$ for $\rho>p, j>p$, $b_{1\alpha\beta}(w_0)=\delta_\beta^\alpha$; and $b_{2\rho j}(w_0)=0$ for $0\leqq j<p, \rho>q$, $j>p+q$, $b_{2\mu-p\nu}(w_0)=\delta_\nu^\mu$. It is possible to do this since $\boldsymbol{E}\to W$ is ample. Now let $\omega^1,\cdots,\omega^{n+r}$ be chosen so that $\omega^j(w_0)=dz^j$ and $\omega^j=0$ on Y for $j>p+q$. Then, if $\rho_0=(1,\cdots,p)$, $\sigma_0=(p+1,\cdots,p+q)$, $a_0=(1,\cdots,1)$, $d_0=(2,\cdots,2)$,

$$\frac{1}{p!q!}(B_{\rho_0 a_0}\wedge B_{\sigma_0 d_0}\mid Y)(w_0) = (-1)^{\frac{1}{2}(p+q)(p+q-1)}\ \omega^1\wedge\cdots\wedge\omega^p\wedge\omega^{p+1}\wedge\cdots\wedge\omega^{p+q}.$$

Thus $\theta_p \wedge \theta_q \mid Y$ is a volume element. Q.E.D.

Appendix to § IV: Numerically Positive Bundles. Let W be an algebraic manifold of dimension n.

Definition. Let ϕ be a real closed form of type (s,s) on W. Then we write $\phi > 0$ to mean that $\int_Y \phi > 0$ for all subvarieties $Y \subset W$ of dimension s.

Now let $\boldsymbol{E} \to W$ be a holomorphic bundle with total Chern class $c(\boldsymbol{E}) = \sum_{i=0}^{r} c_i$ $(c_i \in H^{2i}(W, \boldsymbol{Z}))$. We consider vectors $I = (i_1, \cdots, i_r)$ of non-negative integers and set $c_I = {c_1}^{i_1} \cdots {c_r}^{i_r}$, $|I| = i_1 + 2i_2 + \cdots + ri_r$. Then Theorem 4.3 may be rephrased as follows: If $\boldsymbol{E} \to W$ is ample, then

$$c_I > 0 \text{ for all } I \text{ with } |I| \leqq n. \tag{A.1}$$

As remarked in IV.3, (A.1) characterizes positive line bundles but for general vector bundles there are additional more subtle numerical inequalities which hold for ample bundles. After discussing these in §§ A.1 and A.2, we shall in § A.3 outline a proof of the Theorem that, if all the numerical inequalities for $\boldsymbol{E} \to W$ are satisfied, then the bundles $\boldsymbol{E}^{\mu}$ are ample for $\mu \geqq \mu_0$.

A.1. The Cone of Positive Polynomials for Ample Bundles. As an example of the additional numerical relations which hold for general bundles, we show

PROPOSITION A.1. *If $\boldsymbol{E} \to W$ is ample, then*

$$c_1^2 - 2c_2 > 0. \tag{A.2}$$

Proof. We use the notations from IV.4. Thus

$$\theta_1 = \frac{\sqrt{-1}}{2\pi} \sum_{\substack{1 \leqq a \leqq m \\ 1 \leqq \rho \leqq r}} b_{a\rho} \wedge \bar{b}_{a\rho}$$

and

$$\theta_2 = -\tfrac{1}{2}\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \sum_{\substack{\rho < \sigma \\ 1 \leqq a, d \leqq m}} b_{a\rho} \wedge b_{d\sigma} \wedge (\overline{b_{a\rho} \wedge b_{d\sigma} + b_{d\rho} \wedge b_{a\sigma}}).$$

Consequently,

$$\begin{aligned} {\theta_1}^2 - 2\theta_2 &= -\left(\frac{\sqrt{-1}}{2\pi}\right)^2 \{\sum b_{a\rho} \wedge b_{d\sigma} \wedge \bar{b}_{a\rho} \wedge \bar{b}_{d\sigma} - (\sum_{\rho < \sigma} b_{a\rho} \wedge b_{d\sigma} \\ &\quad \wedge \bar{b}_{a\rho} \wedge \bar{b}_{d\sigma} + b_{a\rho} \wedge b_{d\sigma} \wedge \bar{b}_{d\rho} \wedge \bar{b}_{a\sigma})\}. \end{aligned}$$

Now

$$0 \leqq \tfrac{1}{2}\sum_{a,d}(b_{a\rho}\wedge b_{d\sigma}-b_{d\rho}\wedge b_{a\sigma})\wedge\overline{(b_{a\rho}\wedge b_{d\sigma}-b_{d\rho}\wedge b_{a\sigma})}$$
$$=\sum_{a,d} b_{a\rho}\wedge b_{d\sigma}\wedge \bar{b}_{a\rho}\wedge \bar{b}_{d\sigma}-\sum_{a,d} b_{a\rho}\wedge b_{d\sigma}\wedge \bar{b}_{d\rho}\wedge \bar{b}_{a\sigma}.$$

Adding this last inequality to $\theta_1^2-2\theta_2$, we get

$$\theta_1^2-2\theta_2$$
$$\geqq -\left(\frac{\sqrt{-1}}{2\pi}\right)^2\{\sum_{\rho,\sigma} b_{a\rho}\wedge b_{d\sigma}\wedge \bar{b}_{a\rho}\wedge \bar{b}_{d\sigma}-2\sum_{\rho<\sigma} b_{a\rho}\wedge b_{d\sigma}\wedge \bar{b}_{a\rho}\wedge \bar{b}_{d\sigma}\}$$
$$=-\left(\frac{\sqrt{-1}}{2\pi}\right)^2\sum b_{a\rho}\wedge b_{d\rho}\wedge \bar{b}_{a\rho}\wedge \bar{b}_{d\rho}.$$

As in the proof of Theorem 4.3, we find easily that, if $\boldsymbol{E}\to W$ is ample, then

$$-\left(\frac{\sqrt{-1}}{2\pi}\right)^2\sum b_{a\rho}\wedge b_{d\rho}\wedge \bar{b}_{a\rho}\wedge \bar{b}_{d\rho}>0.$$

Q. E. D.

To get an idea what Proposition A.1 means, consider the case where $\boldsymbol{E}=\boldsymbol{F}\oplus\boldsymbol{G}$ is a sum of positive line bundles $\boldsymbol{F}$, $\boldsymbol{G}$ with respective Chern classes f, g. Then $c_1^2-2c_2=(f+g)^2-2fg=f^2+g^2>0$.

In this particular case with $\boldsymbol{E}=\boldsymbol{F}\oplus\boldsymbol{G}$, we have $c_1^2-\beta c_2>c_1^2-4c_2=(f-g)^2\geqq 0$ provided that $0\leqq\beta<4$. However this stronger type of relation will not hold in general. In fact, one may find an ample bundle $\boldsymbol{E}$ for which we have an exact sequence $0\to\boldsymbol{F}\to\boldsymbol{E}\to\boldsymbol{G}\to 0$ such that:

(i) $\boldsymbol{G}\to W$ is positive;

(ii) $f\geqq 0$ and $f^2=0$; and

(iii) $fg=g^2$.

Then $c_1^2-3c_2=(f+g)^2-3fg=-fg+g^2=0$.

This all suggests the following construction: Let $x_1,\cdots,x_r$ be indeterminants (over $\boldsymbol{Q}$) and write formally

$$1+x_1t+\cdots+x_rt^r=(1+\gamma_1t)\cdots(1+\gamma_rt).$$

We consider the ring $\boldsymbol{R}$ of polynomials

$$P(x_1,\cdots,x_r)=\sum_I p_I x^I$$
$$(I=(i_1,\cdots,i_r),\ x^I=x_1^{i_1}\cdots x_r^{i_r},\ p_I\in\boldsymbol{Q}),$$

and we grade $\boldsymbol{R}$ by assigning to x^I the weight $|I|=i_1+2i_2+\cdots+ri_r$. Then $\boldsymbol{R}=\bigoplus_{s\geqq 0}\boldsymbol{R}^s$ and $\boldsymbol{R}^s\cdot\boldsymbol{R}^t\subseteq\boldsymbol{R}^{s+t}$. We want to define the *positive cone*

$\mathbf{\Pi} \subset \boldsymbol{R}$ and this is done as follows: An element $P \in \boldsymbol{R}^s$ may be uniquely written as a homogeneous symetric polynomial $\sigma(\gamma_1, \cdots, \gamma_r)$ of degree s:

$$P(x_1, \cdots, x_r) = \sigma(\gamma_1, \cdots, \gamma_r) = \sum_{|J|=s} \sigma_J \gamma^J$$

$$(J = (j_1, \cdots, j_r), \gamma^J = {\gamma_1}^{j_1} \cdots {\gamma_r}^{j_r}, |J| = j_1 + \cdots + j_r).$$

We say then that P *is positive*, written $P > 0$, if $P \neq 0$ and all $\sigma_J \geqq 0$. Let $\mathbf{\Pi}^s \subset \boldsymbol{R}^s$ be the set of positive polynomials, $\mathbf{\Pi} = \oplus\, \mathbf{\Pi}^s$ the *positive cone.*

Definition. We say that $\boldsymbol{E} \to W$ *lies in the position cone* if $P_s(c_1, \cdots, c_r) > 0$ for all $P_s \in \mathbf{\Pi}^s$ $(0 \leqq s \leqq n)$.

Remarks and Examples. (i) If $r = 1$, $\mathbf{\Pi}^s =$ all polynomials λx^s with $\lambda > 0$. Thus, if $\boldsymbol{E} \to W$ is a line bundle which lies in the positive cone, then $\boldsymbol{E}$ is positive and viceversa.

(ii) $\mathbf{\Pi}^1 =$ polynomials λx_1 with $\lambda > 0$, $\mathbf{\Pi}^2 =$ polynomials $\alpha {x_1}^2 + \beta x_2$ with $\alpha \geqq 0$, $\beta + 2\alpha \geqq 0$, and where at least one of the inequalities is strict.

(iii) $\mathbf{\Pi}^s \cdot \mathbf{\Pi}^t \subset \mathbf{\Pi}^{s+t}$. Therefore $x^I \in \mathbf{\Pi}^{|I|}$ since

$$x^I = {x_1}^{i_1} \cdots {x_r}^{i_r} \text{ and } x_q = \sum_{1 \leqq j_1 < \cdots < j_q \leqq r} \gamma_{j_1} \cdots \gamma_{j_q} \in \mathbf{\Pi}^q.$$

(iv) Given r, there is for each s an important distinguished element $\Phi_s \in \boldsymbol{R}^s$ which arises from the positive symmetric polynomial

$$\sigma_s(\gamma_1, \cdots, \gamma_r) = \sum_{j_1 + \cdots + j_r = s} {\gamma_1}^{j_1} \cdots {\gamma_r}^{j_r}. \tag{A.3}$$

For example,

$$\begin{cases} \Phi_0(x_1, \cdots, x_r) = 1 \\ \Phi_1(x_1, \cdots, x_r) = x_1 \\ \Phi_2(x_1, \cdots, x_r) = {x_1}^2 - x_2 \\ \Phi_3(x_1, \cdots, x_r) = {x_1}^3 - 2x_1 x_2 + x_3, \text{ etc.} \end{cases} \tag{A.4}$$

An interpretation of the polynomials Φ_s will be given in Lemma A.1 in § A.3 below.

(v) The same sort of calculations as in the proofs of Theorems 4.2, 4.3 and Proposition A.1 gives us

THEOREM A.1. *If $\boldsymbol{E} \to W$ is ample, then $\boldsymbol{E}$ lies in the positive cone.*

(vi) The condition that $\boldsymbol{E} \to W$ lie in the positive cone is, in some respects, rather weak. For example, let X, Y be non-singular curves and $\boldsymbol{G} \to X$, $\boldsymbol{H} \to Y$ positive line bundles. Set $W = X \times Y$, $\boldsymbol{F} = \boldsymbol{G} \otimes \boldsymbol{H}$, $\boldsymbol{E} = \boldsymbol{F} \oplus \boldsymbol{H}$. Then every symmetric power $\boldsymbol{E}^\mu$ contains $\boldsymbol{H}^\mu$ as a direct sum-

mand, so that no power of E is ample. On the other hand, $\boldsymbol{E}$ is in the positive cone since $\alpha c_1^2 + \beta c_2 = \alpha(f+h)^2 + \beta fh = (2\alpha+\beta)gh > 0$.

A. 2. Numerically Positive Bundles. The example (vi) in § A. 1 shows that there are still further relations which should be present for an ample bundle $\boldsymbol{E} \to W$. To see what these should be, we consider the following two fundamental properties of an ample (or positive, or weakly positive) bundle: (1) If $\boldsymbol{E} \to W$ is ample and if $Y \subset W$ is a subvariety, then $\boldsymbol{E}_Y = \boldsymbol{E} \mid Y$ is ample; and (2) if $\boldsymbol{E} \to W$ is ample, then any quotient bundle $\boldsymbol{Q}$ of $\boldsymbol{E}$ is ample.

Definition. Let $\boldsymbol{E} \to W$ be a holomorphic vector bundle. Then we shall say that $\boldsymbol{E}$ is *numerically positive* if, for any subvariety $Y \subset W$ and any quotient bundle $\boldsymbol{Q}$ of $\boldsymbol{E}_Y$, $\boldsymbol{Q}$ lies in the positive cone.

In particular, the bundle in example (vi) of A. 1 is not numerically positive since $\boldsymbol{H}$ is a quotient bundle of $\boldsymbol{E}$ and $\boldsymbol{H} \mid X \times$ (pt.) is trivial. In fact, our main result is

THEOREM IV. *Let $\boldsymbol{E} \to W$ be a holomorphic vector bundle. Then, if $\boldsymbol{E}$ is positive, the bundles $\boldsymbol{E}^\mu$ are numerically positive if $\mu \geqq \mu_0$. Conversely, if $\boldsymbol{E}$ is numerically positive, then the bundles $\boldsymbol{E}^\nu$ are positive for $\nu \geqq \nu_0$.*

Before giving the proof of Theorem IV in § A. 3 below, we make some preliminary remarks. First, if $\boldsymbol{E} \to W$ is positive, then the bundles $\boldsymbol{E}^\mu$ ($\mu \geqq \mu_0$) are ample for $\mu \geqq \mu_0$ so that one-half of the Theorem follows from Theorem A. 1. Also, the bundles $\boldsymbol{E}^\nu$ ($\nu \geqq \nu_0$) are all positive if, and only if, the following vanishing theorem holds: For $\mathcal{S} \to W$ a coherent analytic sheaf, we have

$$H^q(W, \mathcal{S} \otimes \Omega(\boldsymbol{E}^\nu)) = 0 \text{ for } q \geqq 1 \text{ and } \nu \geqq \nu_0(\mathcal{S}). \tag{A. 5}$$

To prove the other half of Theorem IV, we shall thus prove (A. 5).

A. 3. Proof of Theorem IV.

(i) Let $\boldsymbol{E}^* \to W$ be the dual bundle to $\boldsymbol{E}$ and $P(\boldsymbol{E}^*) \xrightarrow{\pi} W$ the associated bundle of projective spaces. Over $P(\boldsymbol{E}^*)$ there is a tautological line bundle $\boldsymbol{L}^*$ whose fibre $\boldsymbol{L}^*_\xi$ ($\xi \in P(\boldsymbol{E}^*)$) is the line $[\xi] \subset \boldsymbol{E}^*_{\pi(\xi)}$. There is an obvious mapping $\psi\colon \boldsymbol{L}^* \to \pi^{-1}(\boldsymbol{E}^*)$ and we define $\phi\colon \pi^{-1}(\boldsymbol{E}) \to \boldsymbol{L}$ by

$$\langle \phi(\xi, e), \lambda^* \rangle = \langle \psi(\xi, \lambda^*), e \rangle \qquad (\xi \in P(\boldsymbol{E}^*),$$

$e \in \boldsymbol{E}_{\pi(\xi)}, \lambda^* \in \boldsymbol{L}^*_\xi$). Then ϕ induces $\phi_\mu\colon \pi^{-1}(\boldsymbol{E}^\mu) \to \boldsymbol{L}^\mu$ and, for any coherent sheaf $\mathcal{S} \to W$, the cohomology mappings

$$\text{(A. 6)}\qquad H^q(W, \mathfrak{S}\otimes\Omega(\boldsymbol{E}^\mu)) \to H^q(P(\boldsymbol{E}^*), \pi^{-1}(\mathfrak{S}\otimes\Omega(\boldsymbol{E}^\mu))) \xrightarrow{\phi_\mu} H^q(P(\boldsymbol{E}^*), \pi^{-1}(\mathfrak{S})\otimes\Omega(\boldsymbol{L}^\mu))$$

are all isomorphisms. Thus, by (A. 5), in order to prove Theorem IV it will suffice to show

PROPOSITION A. 2. *If* $\boldsymbol{E}\to W$ is numerically positive, then $\boldsymbol{L}\to P(\boldsymbol{E}^*)$ is numerically positive.

(ii) We suppose that we have in $\boldsymbol{E}\to W$ an Hermitian metric and let θ_q $(0\leqq q\leqq r)$ be the real (q,q) form which represents $c_q(\boldsymbol{E})\in H^{2q}(W,\boldsymbol{Z})$ (cf. (4. 17)). Then there is naturally induced in $\boldsymbol{L}\to P(\boldsymbol{E}^*)$ a metric and we denote by θ the corresponding $(1,1)$ form on $P(\boldsymbol{E}^*)$ which represents $c_1(\boldsymbol{L})=\omega$.

LEMMA A. 1. $\pi_*(\theta^{s+r-1})=\Phi_s(\theta_1,\cdots,\theta_r)+$ (*exact form*) *where* π_* *is integration over the fibre and where* $\Phi_s\in \boldsymbol{\Pi}^s$ *is given by* (A. 3).

Proof. Over $P(\boldsymbol{E}^*)$ *we have the exact bundle sequence*

$$\text{(A. 7)}\qquad 0\to \boldsymbol{F}\to \pi^{-1}(\boldsymbol{E})\to \boldsymbol{L}\to 0.$$

Therefore, if $c(\boldsymbol{F})=d_0+\cdots+d_{r-1}$, we have

$$\text{(A. 8)}\qquad c_q=d_q+d_{q-1}\omega \qquad (0\leqq q\leqq r).$$

By (A. 8), the alternating sum $\sum_{q=0}^{r}(-1)^q c_q\omega^{r-q}$ telescopes to zero, and we get the identity $\omega^r=\sum_{q>0}(-1)^{q+1}c_q\omega^{r-q}$. More generally, we have

$$\text{(A. 9)}\qquad \omega^{s+r}=\sum_{r\geqq q>0}(-1)^{q+1}c_q\omega^{s+r-q}.$$

Since $\theta^{r-1}\mid \pi^{-1}(w)$ is a normalized volume, $\pi_*(\theta^{r-1})$ is the constant function 1 on W. Inductively we assume that

$$\text{(A. 10)}\qquad \pi_*(\theta^{s+r-t})=\Phi_{s+1-t}(\theta_1,\cdots,\theta_r)+\text{(exact form)}\quad (1\leqq t\leqq s+r)$$

and we shall show that

$$\text{(A. 11)}\qquad \pi_*(\theta^{s+r})=\Phi_{s+1}(\theta_1,\cdots,\theta_q)+\text{(exact form)}.$$

Since (by (A. 9))

$$\theta^{s+r}=\sum_{0<q\leqq r}(-1)^{q+1}\theta^{s+r-q}\wedge\pi^*(\theta_q)+\text{(exact form)},$$

$$\pi_*(\theta^{s+r})=\sum_{0<q\leqq r}(-1)^{q+1}\pi_*(\theta^{s+r-q})\wedge\theta_q+\text{(exact form)}.$$

By (A.10), $\pi_*(\theta^{s+r}) = \sum_{0<q\leqq r} (-1)^{q+1}\theta_q \wedge \Phi_{s+1-q}(\theta_1, \cdots, \theta_q) +$ (exact form)

and where $\Phi_s = 0$ for $s < 0$. Thus what we have to prove is

$$\Phi_{s+1}(x_1, \cdots, x_r) = \sum_{0\leqq q<r} (-1)^q x_{q+1}\Phi_{s-q}(x_1, \cdots, x_r). \tag{A.12}$$

By (A.3) and induction, (A.12) amounts to the identity

$$\sum_{i_1+\cdots+i_r=s+1} \gamma_1{}^{i_1} \cdots \gamma_r{}^{i_r} = \sum (-1)^q \sum_{1\leqq j_1<\cdots<j_{q+1}\leqq r} \gamma_{j_1} \cdots \gamma_{j_{q+1}} \sum_{k_1+\cdots+k_r=s-q} \gamma_1{}^{k_1} \cdots \gamma_r{}^{k_r}. \tag{A.13}$$

We now use the notations

$$I = (i_1, \cdots, i_r),\ K = (k_1, \cdots, k_r),\ |I| = i_1 + \cdots + i_r,$$
$$\gamma^I = \gamma_1{}^{i_1} \cdots \gamma_r{}^{i_r}.$$

Also, A, B, C will stand for r-tuples $(a_1, \cdots, a_r)$ with $a_j = 0$ or 1. With this understood, (A.13) becomes

$$\gamma^I = \sum_{A+K=I} (-1)^{|A|-1}\gamma^A\gamma^K. \tag{A.14}$$

Given $I = (i_1, \cdots, i_r)$, we define $A_I = (a_1, \cdots, a_r)$ by $a_j = 0$ if $i_j = 0$, $a_j = 1$ if $i_j > 0$. Then

$$\sum_{A+K=I} (-1)^{|A|-1}\gamma^A\gamma^K = \Big(\sum_{B+C=A_I} (-1)^{|B|-1}\gamma^B\gamma^C\Big)\gamma^{I-A_I}. \tag{A.15}$$

Since $\gamma^I = \gamma^{A_I}\gamma^{I-A}$, by (A.14) and (A.15) it will suffice to prove

$$\gamma^{A_I} = \sum_{B+C=A_I} (-1)^{|B|-1}\gamma^B\gamma^C.$$

By changing the notation, this becomes the equation

$$\gamma_1 \cdots \gamma_t = \sum_i \gamma_i(\gamma_1 \cdots \hat{\gamma}_i \cdots \gamma_t) - \sum_{i<j} \gamma_i\gamma_j(\gamma_1 \cdots \hat{\gamma}_i \cdots \hat{\gamma}_j \cdots \gamma_t) + \cdots + (-1)^{t-1}\gamma_1 \cdots \gamma_t. \tag{A.16}$$

Now (A.16) is easy to check and this proves (A.11).

(iii) We pause to give an application of Lemma A.1.

For $Y \subset W$ a subvariety and $\boldsymbol{F} \to Y$ a line bundle, set

$$\chi(Y, \boldsymbol{F}(\mu)) = \sum_{0\leqq q\leqq \dim Y} (-1)^q H^q(Y, \Omega(\boldsymbol{F} \otimes \boldsymbol{E}^\mu)).$$

Furthermore we set $\chi(Y, \mu) = \chi(Y, \boldsymbol{l}(\mu))$ where $\boldsymbol{l}$ is the trivial bundle.

PROPOSITION A.3. *$\chi(Y, \mu)$ is a polynomial in μ with leading coefficient*

$$\text{(A.17)}\qquad \frac{\mu^{s+r-1}}{(s+r-1)!}\int_Y \Phi_s(\theta_1,\cdots,\theta_q)$$

Proof. Set $Y_\pi = \pi^{-1}(Y) \subset P(\boldsymbol{E}^*)$ and

$$\chi(Y_\pi,\mu) = \sum_{q=0}^{s+r-1}(-1)^q \dim H^q(Y_\pi, \Omega(\boldsymbol{L}^\mu)).$$

By (A.6) with $\mathfrak{S} = \mathfrak{O}_Y$, $\chi(Y,\mu) = \chi(Y_\pi,\mu)$. By the Riemann-Roch theorem ([9], § 20.4), $X(Y_\pi,\mu)$ is a polynomial in μ with leading coefficient

$$\text{(A.18)}\qquad \frac{\mu^{s+r-1}}{(s+r-1)!}\langle \omega^{s+r-1}, Y_\pi\rangle = \frac{\mu^{s+r-1}}{(s+r-1)!}\int_{Y_\pi}\theta^{s+r-1}.$$

But $\int_Y \theta^{s+r-1} = \int_Y \pi_*(\theta^{s+r-1})$ and then (A.17) follows from this, (A.18), and (A.11).

(iv) *Definition.* A connected non-singular subvariety $Z \subset P(\boldsymbol{E}^*)$ is said to be *fibered over* $Y \subset W$ if $\pi(Z) = Y$, if $\pi \mid Z$ has constant rank, and if the varieties $Z \cap \pi^{-1}(y) \subset P(\boldsymbol{E}^*_y)$ all have degree 1.

LEMMA A.2. *The subvarieties $Z \subset P(\boldsymbol{E}^*)$ which are fibered over $Y \subset W$ are in a one-to-one correspondence with the quotient bundles $\boldsymbol{Q}_Z$ of $\boldsymbol{E}_Y = \boldsymbol{E} \mid Y$.*

Proof. A point $\xi \in P(\boldsymbol{E}^*)$ is given by a line $[\xi] \subset \boldsymbol{E}^*_{\pi(\xi)}$. Let Z be fibered over Y and let $y \in Y$. We set $\boldsymbol{F}_y = \{e \in \boldsymbol{E}_y : \langle[\xi], e\rangle = 0 \text{ for all } \xi \in Z \cap \pi^{-1}(y)\}$. Then $\bigcup_{y\in Y} \boldsymbol{F}_y = \boldsymbol{F}_Z$ form a vector bundle over Y which is clearly a sub-bundle of $\boldsymbol{E}_Y$. We then set $\boldsymbol{Q}_Z = \boldsymbol{E}_Y/\boldsymbol{F}_Z$. Since this process is reversible, we are done.

Remark. In a natural way, $Z \cong P(\boldsymbol{Q}^*_Z)$ and $\boldsymbol{L} \mid Z$ corresponds to the dual of the tautological line bundle $\boldsymbol{L}^*_Z$ over $P(\boldsymbol{Q}^*_Z)$.

(v) LEMMA A.3. *If $Z \subset P(\boldsymbol{E}^*)$ is fibered over $Y \subset W$ and if $t = \dim Z$, then, if $\boldsymbol{E} \to W$ is numerically positive,*

$$\text{(A.19)}\qquad \int_{\boldsymbol{Z}} \theta^t > 0.$$

Proof. Let ψ be a real $(1,1)$ form on Z which represents $c_1(\boldsymbol{L}_Z)$. Assume that $s = \dim Y$ so that $\boldsymbol{Q}_Z$ has fibre dimension $t-s+1$ and write $c(\boldsymbol{Q}_Z) = h_0 + \cdots + h_{t-s+1}$. Then, by the Remark following Lemma A.2 and Lemma A.1,

$$\text{(A.20)}\qquad \int_Z \theta^t = \int_{\boldsymbol{Z}} \psi^t = \int_Y \Phi_s(h_0,\cdots,h_{t-s+1}).$$

Since $\boldsymbol{E} \to W$ is numerically positive, $\boldsymbol{Q}_Z \to Y$ is in the positive cone and

$$\int_Y \Phi_s(h_0, \cdots, h_{t-s+1}) > 0. \quad \text{Q.E.D.}$$

(vi) Now in order to prove Proposition A.2 we must show that, for any irreducible $Z \subset P(\boldsymbol{E}^*)$ of dimension t,

$$\int_Z \theta^t > 0. \tag{A.21}$$

For those Z which are fibered over some $Y \subset W$, (A.19) gives (A.21). In fact, Lemmas A.2 and A.3 contain the essential idea of the general proof but they need to be extended a little. In order to illustrate what has to be done, we prove a special case of (A.21).

LEMMA A.4. (A.21) *holds if* $t = 1$.

Proof. If $\pi(Z) =$ point we are done so that we may assume that Z is an m-sheeted branched covering over $Y = \pi(Z)$. Let $\Delta \subset Z$ be the branch locus and $D = \pi(\Delta)$. For $y \in Y$, let $z_1(y), \cdots, z_m(y)$ be the (not necessarily distinct) points in $\pi^{-1}(y) \cap Z$ and set $\boldsymbol{L}^{\#}{}_y = \boldsymbol{L}_{z_1(y)} \otimes \cdots \otimes \boldsymbol{L}_{z_m(y)}$. Then $\boldsymbol{L}^{\#} = \bigcup_{y \in Y} \boldsymbol{L}^{\#}{}_y$ is a line bundle over Y and there is an obvious mapping $\phi: \boldsymbol{E}_Y{}^m \to \boldsymbol{L}^{\#} \to 0$. Since $\boldsymbol{E} \to W$ is numerically positive, $\int_Y \psi > 0$ where $\psi = c_1(\boldsymbol{L}^{\#})$. Now $\int_Z \theta = \int_Y \pi_* \theta = \int_Z \pi^*(\pi_* \theta)$. But

$$(\pi_* \theta)(y) = \frac{1}{m} \sum_{i=1}^{m} \theta(z_i(y)) \quad \text{and so} \quad \pi^*(\pi_* \theta) = \frac{1}{m} c_1(\pi^{-1} \boldsymbol{L}^{\#})$$

since $\theta = c_1(\boldsymbol{L})$. Thus $\int_Z \theta = \frac{1}{m} \int_Z \pi^* \psi > 0$. Q.E.D.

(vii) For a variety V, let $\boldsymbol{H}_*(V) = \sum \boldsymbol{H}_q(V)$ be the rational algebraic homology ring: $\boldsymbol{H}_q(V) \subset \boldsymbol{H}_{2q}(V, \boldsymbol{Q})$ is the subspace over $\boldsymbol{Q}$ generated by the algebraic cycles. Let $\gamma_1, \cdots, \gamma_k$ be a minimal set of effective * generators for $\boldsymbol{H}_*(W)$ and set $\xi_\alpha = \pi^{-1}(\gamma_\alpha)$ $(1 \leqq \alpha \leqq k)$. If ω is an algebraic cycle on $P(\boldsymbol{E}^*)$ dual to $c_1(\boldsymbol{L})$, then $\boldsymbol{H}_*(P(\boldsymbol{E}^*))$ is generated by the products $\phi_{j\alpha} = \omega^j \cdot \xi_\alpha$ with the relation $\sum_{q=0}^{r} (-1)^q \omega^{r-q} \cdot \pi^{-1}(c_q) = 0$ $(c_q \in \boldsymbol{H}_q(W))$.

LEMMA A.6. ω *may be chosen to be an effective cycle.*

Proof. We must show that $H^0(P(\boldsymbol{E}^*), \Omega(\boldsymbol{L}^\mu)) \neq 0$ for μ sufficiently

* i.e. $\gamma_j = \sum \eta_{j\gamma} Y_\gamma$ where Y_γ are irreducible subvarieties and $\eta_{j\gamma} > 0$.

large, i.e. $H^0(W,\Omega(\boldsymbol{E}^\mu))\neq 0$ for μ large. Let $H\subset W$ be a hyperplane section. By induction on the dimension of W, we may assume that the bundles $\boldsymbol{E}^{\nu}{}_H$ ($\nu\geqq\nu_0, \boldsymbol{E}_H=\boldsymbol{E}\mid H$) are ample. Set $\boldsymbol{D}=[H]$ so that $\Omega(\boldsymbol{D}^*)$ is the ideal sheaf of $H\subset W$. Then we may find λ_0 so that

$$H^q(H,\Omega(\boldsymbol{D}^\sigma{}_H\otimes\boldsymbol{E}^\lambda{}_H))=0$$

for $\sigma\geqq 0$, $\lambda\geqq\lambda_0$, $0<q$. From the exact cohomology sequences of

$$0\to\Omega(\boldsymbol{D}^{\sigma-1}\otimes\boldsymbol{E}^\lambda)\to\Omega(\boldsymbol{D}^\sigma\otimes\boldsymbol{E}^\lambda)\to\Omega(\boldsymbol{D}^\sigma{}_H\otimes\boldsymbol{E}^\lambda{}_H)\to 0$$

we get $H^q(W,\Omega(\boldsymbol{D}^\sigma\otimes\boldsymbol{E}^\lambda))=0$ for $\sigma\geqq 0$, $\lambda\geqq\lambda_0$, $q>1$ and

$$H^1(W,\Omega(\boldsymbol{D}^{\sigma-1}\otimes\boldsymbol{E}^\lambda))\to H^1(W,\Omega(\boldsymbol{D}^\sigma\otimes\boldsymbol{E}^\lambda))\to 0$$

for $\sigma\geqq 0$, $\lambda\geqq\lambda_0$. Since $\chi(W,\lambda)=\sum(-1)^q\dim H^q(W,\Omega(\boldsymbol{E}^\lambda))$ is a polynomial in λ with leading coefficient $c\lambda^{n+r-1}$ ($c>0$), it follows that $\dim H^0(W,\Omega(\boldsymbol{E}^\lambda))\to\infty$ as $\lambda\to\infty$. This proves the lemma.

(viii) We shall call a variety $Z\subset P(\boldsymbol{E}^*)$ *linear* if, for each $w\in W$, the irreducible components of $Z\cap\pi^{-1}(w)$ are each linear subspaces of $P(\boldsymbol{E}^*{}_w)$. Since $\phi_{j\alpha}=\omega^j\cdot\pi^{-1}(\gamma_\alpha)$ ω is effective, and $\omega\mid P(\boldsymbol{E}^*{}_w)$ is a hyperplane section, it follows that the varieties $\phi_{j\alpha}$ ($0\leqq j\leqq r-1$) are linear. The same is true of any variety $Y_\pi\cdot\omega^j$ ($0\leqq j\leqq r-1$), $Y\subset W$, $Y_\pi=\pi^{-1}(Y)$.

Let now $Z\subset P(\boldsymbol{E}^*)$ be irreducible of dimension $t-1$ and suppose that $Y=\pi(Z)$ is of dimension s. Then $Z\cdot\pi^{-1}(y)$ ($y\in Y$ generic) is rationally equivalent to $\nu(Z)\lambda_y{}^{t-s}$ where $\nu(Z)>0$ and $\lambda_y{}^{t-s}$ is a linear subspace of dimension $t-s$ in $P(\boldsymbol{E}^*{}_y)$. Suppose that $Y\sim\sum\rho_\alpha\gamma_\alpha$ so that $Y_\pi\sim\sum\rho_\alpha\xi_\alpha$. Then $Z\sim\nu(Z)\sum\rho_\alpha\phi_{(r-1)-(t-s)}\sim\nu(Z)\omega^{(r-1)-(t-s)}\cdot Y_\pi$. The conclusion is

LEMMA A.7. *It will suffice to verify* (A.21) *for linear varieties* $Z\subset P(\boldsymbol{E}^*)$.

LEMMA A.8. (A.21) *is true for linear varieties.*

Proof. Let $Z\subset P(\boldsymbol{E}^*)$ be irreducible and linear. We shall assume that $Z_y=Z\cdot P(\boldsymbol{E}^*{}_y)$ is connected ($y\in Y=\pi(Z)$), the general case may be deduced from this special case by using branched coverings and symmetric products as in Lemma A.5. Now Z_y spans a linear subspace $\boldsymbol{F}_y\subset\boldsymbol{E}^*{}_y$ and $\boldsymbol{F}=\bigcup_{y\in Y}\boldsymbol{F}_y$ is a so-called quasi-linear space [6]. This $\boldsymbol{F}$ defines (in the usual manner) a coherent quotient sheaf $\mathfrak{Q}$ of $\Omega(\boldsymbol{E}_Y)$ ($\mathfrak{Q}=\Omega(\boldsymbol{E}_Y)/\mathfrak{S}$ where the sections of $\mathfrak{S}\subset\Omega(\boldsymbol{E})$ are those sections s of $\Omega(\boldsymbol{E}_Y)$ such that $\langle\boldsymbol{F}_y,s(y)\rangle=0$ for all y in the domain of s). The proof now may be completed by

using the Riemann-Roch and directly generalizing the argument of Lemma A.3 (which applied to the case where $\mathfrak{Q}$ is locally free).

V. Positive Embeddings and Monoidal Transformations.

1. Monoidal transformations. Let $X \subset W$ be an embedding of a complex manifold X in a complex manifold W. Suppose that $\dim X = n$, $\operatorname{codim} X = r > 1$, and that $\boldsymbol{N} \xrightarrow{\pi} X$ is the normal bundle. By replacing each x in X by $P(\boldsymbol{N}_x)$, the projective normal space to X at x, we obtain a new manifold $\tilde{W}$, the *monoidal transform* of W along X. There is a hypersurface $\tilde{X} \subset \tilde{W}$ and a proper mapping $\tilde{\omega}: \tilde{W} \to W$ such that: (i) $\tilde{\omega}: \tilde{W} - \tilde{X} \cong W - X$; (ii) $\tilde{X}$ may be identified with $P(\boldsymbol{N})$ and $\tilde{\omega} \mid \tilde{X}$ is the map induced by π; and (iii) if $\boldsymbol{L} = [\tilde{X}]$, then $\boldsymbol{L} \mid P(\boldsymbol{N}_x) = \boldsymbol{H}^*$ where $\boldsymbol{H}$ is the hyperplane bundle.

Thus, e.g., if $\boldsymbol{E} \xrightarrow{\pi} X$ is a holomorphic vector bundle, we may embed $X \subset \boldsymbol{E}$ as zero cross section. Furthermore, $\boldsymbol{C}^*$ acts on $\boldsymbol{E} - X$ and gives rise to a line bundle $\boldsymbol{L}(\boldsymbol{E}) \to P(\boldsymbol{E})$. Using then the above notation, $\tilde{W} = \boldsymbol{L}(\boldsymbol{E})$, $\tilde{X} = P(\boldsymbol{E})$ embedded in $\tilde{W}$ as zero-section, and $\tilde{W} - \tilde{X}$ is naturally identified with $W - X$.

In order to describe monoidal transformations locally, we may assume that $W \subset \boldsymbol{C}^{n+r}$ is the unit polycylinder with coordinates $(z_1, \cdots, z_n; w_1, \cdots, w_r)$ and that $X \subset W$ is given by $w_1 = \cdots = w_r = 0$. Let P_{r-1} have homogeneous coordinates $[t_0, \cdots, t_r]$. Then $\tilde{W} \subset W \times P_{r-1}$ is given as

$$\{(z, w, t) : w_\rho t_\sigma - t_\rho w_\sigma = 0 ; 1 \leqq \rho, \sigma \leqq r\}.$$

The mapping $\tilde{\omega}: \tilde{W} \to W$ is induced by the projection $W \times P_{r-1} \to W$ and $\tilde{X} = \tilde{\omega}^{-1}(X) \cong X \times P_{r-1}$.

Finally, we shall yield to custom and write $\boldsymbol{E}^{-1}$ for $\boldsymbol{E}^*$ whenever $\boldsymbol{E}$ is a line bundle.

2. Construction of a blowing down map. We shall discuss the following question: Let $X \subset W$ be complex submanifold of codimension 1 and set $\boldsymbol{L} = [X]$. Suppose that $X \xrightarrow{\pi} Y$ is a fibre bundle with fibre $\pi^{-1}(y) = P_y \cong P_{r-1}$ and that $\boldsymbol{L} \mid P_y = \boldsymbol{H}^{-1}$. We should then like to show that there is a manifold $Z \supset Y$ such that W is the monoidal transform of Z along Y and where X is the inverse image of Y, the mapping $X \xrightarrow{\pi} Y$ being induced by the projection $\tilde{\omega}: W \to Z$. In this case we say that we can *blow down W along X*.

We shall prove that it is possible to blow down W along X in case W is compact and with the assumption that the (hypothetical) normal bundle of Y in Z is positive. More precisely, we let $\boldsymbol{L} \mid X = \boldsymbol{N}$ and assume that $\boldsymbol{N}$ has signature $(n, (r-1))$ (cf. § VI. 3 below) where $n+r-1 = \dim X$. This assumption is an analytic way of expressing the geometric hypothesis that *no further blowing down of Z along Y can occur.*

THEOREM. *Let W be a closed algebraic variety and $X \subset W$ a subvariety which satisfies the above assumption. Then we can blow down W along X to a non-singular algebraic variety Z.*

Proof. (a) Let $\boldsymbol{F} \to W$ be a positive line bundle. Then $\boldsymbol{F} \mid P_y = \boldsymbol{H}^l$ and we assume that $l > 2$. Set $\boldsymbol{E} = \boldsymbol{F} \otimes \boldsymbol{L}^l \mid X$ and denote by $R^q\pi(\Omega(\boldsymbol{E}))$ the q-th Leray sheaf of the pair $(\Omega(\boldsymbol{E}); X \xrightarrow{\pi} Y)$.

LEMMA 5.1. *There exists a line bundle $\boldsymbol{G} \to Y$ such that $\pi^{-1}(\boldsymbol{G}) = \boldsymbol{E}$.*

Proof. Let $\{U_\alpha\}$ be a covering of Y by small polycylinders. Then we may find isomorphisms

$$\phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times P_{r-1} \text{ * and } \phi_\alpha\phi_\beta^{-1} : U_\beta \times P_{r-1} \to U_\alpha \times P_{r-1}$$

will be written as $\phi_\alpha\phi_\beta^{-1}(u_\beta, p_\beta) = (f_{\alpha\beta}(u_\beta), g_{\alpha\beta}(u_\beta)p_\beta)$ where $g_{\alpha\beta}(u_\beta)$ is a holomorphic family of projective transformations. Since $\boldsymbol{E} \mid P_y$ is trivial, it follows that $\boldsymbol{E} \mid \pi^{-1}(U_\alpha)$ is trivial and so we may choose an isomorphism

$$\psi_\alpha : \boldsymbol{E} \mid \pi^{-1}(U_\alpha) \longrightarrow \pi^{-1}(U_\alpha) \times \boldsymbol{C}.$$

Then we have

$$\eta_\alpha : \boldsymbol{E} \mid \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times P_{r-1} \times \boldsymbol{C}$$

where $\eta_\alpha = (\phi_\alpha, 1) \circ \psi_\alpha$. Clearly $\rho_{\alpha\beta} = \eta_\alpha \circ \eta_\beta^{-1}$ is given as

$$\rho_{\alpha\beta}(u_\beta, p_\beta, \xi_\beta) = (f_{\alpha\beta}(u_\beta), g_{\alpha\beta}(u_\beta)p_\beta, \gamma_{\alpha\beta}(u_\beta)\xi_\beta)$$

where $\gamma_{\alpha\beta}(u_\beta)$ is a no-where zero holomorphic function. It is then clear that, if we define $\boldsymbol{G} \to Y$ to be the line bundle given by the system of transition functions, $\{\gamma_{\alpha\beta}(u_\beta)\}$; then $\pi^{-1}(\boldsymbol{G}) = \boldsymbol{E}$.

(b) We recall now the formula

$$R_\pi{}^q(\mathfrak{S} \otimes_{\mathfrak{O}_X} \pi^{-1}\mathfrak{T}) \cong R_\pi{}^q(\mathfrak{S}) \otimes_{\mathfrak{O}_Y} \mathfrak{T} \tag{5.1}$$

where $\mathfrak{S} \to X$ and $\mathfrak{T} \to Y$ are locally free coherent sheaves.

* We remark that any fibre space with projective space as fibre is a fibre bundle.

LEMMA 5.2. *For any $\mathfrak{T} \to Y$ as above and any integer μ, there is a canonical isomorphism*

$$H^q(Y, \Omega(\boldsymbol{G}^\mu) \otimes \mathfrak{T}) \cong H^q(X, \Omega(\boldsymbol{E}^\mu) \otimes \pi^{-1}(\mathfrak{T}))$$

Proof. By Leray's theorem, there exists a spectral sequence $\{E_r\}$ such that $E_\infty \Rightarrow H^*(X, \Omega(\boldsymbol{E}^\mu) \otimes \pi^{-1}(\mathfrak{T}))$ and

$$E_2^{p,q} = H^p(Y, R_\pi{}^q(\Omega(\boldsymbol{E}^\mu) \otimes \pi^{-1}(\mathfrak{T}))).$$

Since $\boldsymbol{E} = \pi^{-1}(\boldsymbol{G})$,

$$R_\pi{}^q(\Omega(\boldsymbol{E}^\mu) \otimes_{\mathfrak{O}_X} \pi^{-1}(\mathfrak{T})) \cong R_\pi{}^q(\mathfrak{O}_X) \otimes_{\mathfrak{O}_Y} \Omega(\boldsymbol{G}^\mu) \otimes_{\mathfrak{O}_Y} \mathfrak{T}.$$

But clearly $R_\pi{}^q(\mathfrak{O}_X) = 0$ for $q > 0$ and $R_0{}^q(\mathfrak{O}_X) \cong \mathfrak{O}_Y$. Q. E. D.

(c) We now prove the

LEMMA 5.3. *$\boldsymbol{G} \to Y$ is a positive line bundle.*

Proof. By well-known techniques, it will suffice to show that for *any* $\mathfrak{T} \to Y$ a locally free coherent sheaf, $H^q(Y, \Omega(\boldsymbol{G}^{-\mu}) \otimes \mathfrak{T}) = 0$ for $0 \leqq q < n$ and $\mu \geqq \mu_0(\mathfrak{T})$. By the Lemma in (b), we may show that $H^q(X, \Omega(\boldsymbol{E}^{-\mu}) \otimes \mathfrak{S}) = 0$ for $0 \leqq q < n$ and $\mu \geqq \mu_0(\mathfrak{S})$, where $\mathfrak{S} \to X$ is coherent and locally free.

Now $\boldsymbol{E}^{-1} = (\boldsymbol{F} \mid X)^{-1} \otimes \boldsymbol{N}^{-l}$. Since $\boldsymbol{N}^{-l}$ has signature $((r-1), n)$ and $\boldsymbol{F}^{-1} \mid X$ has signature $(0, n+r-1))$, it follows that, at every point $x \in X$, the curvature form for $\boldsymbol{E}^{-1}$ has signature $((r-1-\alpha(x)), (n+\alpha(x)))$ where $\alpha(x) \geqq 0$. By the vanishing theorems (cf. § VII. 3 below) it follows that $H^q(X, \Omega(\boldsymbol{E}^{-\mu}) \otimes \mathfrak{S}) = 0$ for $0 \leqq q < n$ and $\mu \geqq \mu_0(\mathfrak{S})$. Q. E. D.

(d) By replacing $\boldsymbol{G}$ by a multiple $\boldsymbol{G}^k$ if necessary, we may assume that $|\boldsymbol{G}|$ gives a projective embedding $\Psi: Y \to P_s$. In doing this we replace $\boldsymbol{F}$ by $\boldsymbol{F}^k$ and l by kl.

From the lemma in (b) there is an isomorphism $H^0(Y, \Omega(\boldsymbol{G})) \cong H^0(X, \Omega(\boldsymbol{E}))$ and thus a diagram

$$\begin{array}{ccc} X & & \\ & \searrow^{\Lambda} & \\ {\scriptstyle\pi}\downarrow & & P_s \\ & \nearrow_{\Psi} & \\ Y & & \end{array}$$

where Λ is given by $|\boldsymbol{E}|$. We want to extend Λ to $\Phi: W \to P_N \supset P_s$ so that Φ is biregular on $W - X$. In order to do this, we prove

LEMMA 5.4. $H^1(X, \Omega(\boldsymbol{E}\otimes\boldsymbol{N}^{-t}))=0$ *for* $0\leqq t\leqq l$.

Proof. Set $\boldsymbol{E}_t=\boldsymbol{F}\otimes\boldsymbol{L}^l\otimes\boldsymbol{L}^{-t}\mid X$. Then $\boldsymbol{E}_0=\boldsymbol{E}$ and $\boldsymbol{E}_l=\boldsymbol{F}\mid X$. Let ω on Y be the curvature form for $\boldsymbol{G}\to Y$ (cf. § VI. 5 below) and ψ on X the form for $\boldsymbol{F}\mid X$. Then ψ gives a positive definite Kähler metric $\sum\omega_i\bar{\omega}_i$ on X and ω lifted to X is real and positive semi-definite. Thus we may choose the $\omega_1,\cdots,\omega_{n+r-1}$ so that $\omega=\frac{\sqrt{-1}}{2}\sum c_i\omega_i\wedge\bar{\omega}_i$ where $c_i\geqq 0$. Since $\boldsymbol{E}_l=\boldsymbol{E}_0\otimes\boldsymbol{N}^{-l}$, the curvature form ϕ for $\boldsymbol{N}$ is given by $\psi=\omega-l\phi$ or $\phi=\frac{\sqrt{-1}}{2}\sum\left(\frac{c_i-1}{l}\right)\omega_i\wedge\bar{\omega}_i$. Thus the curvature form ψ_t for $\boldsymbol{E}_t=\boldsymbol{E}_0\otimes\boldsymbol{N}^{-t}$ is given by $\psi_t=\omega-t\phi=\frac{\sqrt{-1}}{2}\sum\left(\frac{t}{l}+\left(\frac{l-t}{l}\right)c_i\right)\omega_i\wedge\bar{\omega}_i$. Consequently, for $0<t\leqq l$, $\psi_t>0$ and so $\boldsymbol{E}_t$ is positive.

We consider now the canonical bundle $\boldsymbol{K}$ of X. By using the bundle along the fibres, we see that $\boldsymbol{K}=\boldsymbol{K}_Y\otimes\boldsymbol{N}^r$ ($\boldsymbol{K}_Y=$ line bundle on Y). We want $\boldsymbol{E}_t\otimes\boldsymbol{K}^{-1}$ to be positive for $0<t\leqq l$; by replacing $\boldsymbol{G}$ by $\boldsymbol{G}\otimes\boldsymbol{K}_Y^{-1}$, it will suffice to have $\boldsymbol{E}_t\otimes\boldsymbol{N}^{-r}>0$ for $0<t\leqq l$.

Now we may assume that $0\leqq c_i\leqq 1$, because replacing $\boldsymbol{F}$ by $\boldsymbol{F}^2$ changes ω_i into $\omega_i=\sqrt{2}\omega_i$ and c_i into $c_i'=\frac{c_i}{2}$ The curvature form ω_t for $\boldsymbol{E}_t\otimes\boldsymbol{N}^{-r}$ is

$$\omega_t=\frac{\sqrt{-1}}{2}\sum\left\{\left[\frac{t}{l}+\left(\frac{l-t}{l}\right)c_i\right]-\left[\frac{r}{l}\,(c_i-1)\right]\right\}\omega_i\wedge\bar{\omega}_i.$$

Thus

$$\omega_t\geqq\frac{\sqrt{-1}}{2}\sum\left[\frac{t}{l}+\left(\frac{l-t}{l}\right)c_i\right]\omega_i\wedge\bar{\omega}_i>0 \text{ for } 0<t\leqq l.$$

It follows that $H^1(X,\Omega(\boldsymbol{E}_t))=0$ for $0<t\leqq l$. For $t=0$, we may assume that $H^1(X,\Omega(\boldsymbol{E}_0))=H^1(Y,\Omega(\boldsymbol{G}))=0$. Q. E. D.

(e) Set $\boldsymbol{F}_t=\boldsymbol{F}\otimes\boldsymbol{L}^{l-t}$ so that $\boldsymbol{F}_t\mid X=\boldsymbol{E}_t$.

For $0\leqq t<l$ we have the exact sheaf sequences

$$0\to\Omega(\boldsymbol{F}_{t+1})\xrightarrow{\sigma}\Omega(\boldsymbol{F}_t)\xrightarrow{\rho}\Omega(\boldsymbol{E}_t)\to 0 \tag{5.2}$$

where σ is multiplication by the local equation of $X\subset W$. By the lemma in (d) we get $H^1(W,\Omega(\boldsymbol{F}_{t+1}))\to H^1(W,\Omega(\boldsymbol{F}_t))\to 0$ for $0\leqq t<l$. But for $t=l-1$, $\boldsymbol{F}_{t+1}=\boldsymbol{F}$ and so $H^1(W,\Omega(\boldsymbol{F}_l))=0$. It follows that $H^1(W,\Omega(\boldsymbol{F}_t))=0$ for $0\leqq t<l$.

From the exact cohomology sequences of (5.2), we have then the array:

$$(5.3)\qquad \begin{cases} 0 \to H^0(\Omega(\boldsymbol{F}_1)) \xrightarrow{\sigma} H^0(\Omega(\boldsymbol{F}_0)) \xrightarrow{\rho} H^0(\Omega(\boldsymbol{E}_0)) \to 0 \\ 0 \to H^0(\Omega(\boldsymbol{F}_2)) \xrightarrow{\sigma} H^0(\Omega(\boldsymbol{F}_1)) \xrightarrow{\rho} H^0(\Omega(\boldsymbol{E}_1)) \to 0 \\ \qquad\qquad \vdots \\ 0 \to H^0(\Omega(\boldsymbol{F}_l)) \xrightarrow{\sigma} H^0(\Omega(\boldsymbol{F}_{l-1})) \xrightarrow{\rho} H^0(\Omega(\boldsymbol{E}_{l-1})) \to 0. \end{cases}$$

If now $\phi_0, \cdots, \phi_s$ is a basis for $H^0(\Omega(\boldsymbol{E}))$; $\psi_0, \cdots, \psi_t$ is a basis for $H^0(\Omega(\boldsymbol{E}_1))$; $\cdots$; $\gamma_0, \cdots \gamma_\mu$ a basis for $H^0(\Omega(\boldsymbol{F}))$ ($\boldsymbol{F}_l = \boldsymbol{F}$ and $\boldsymbol{E}_0 = \boldsymbol{E}$), then

$$(*)\qquad \phi_0, \cdots, \phi_s; \sigma\psi_1, \cdots, \sigma\psi_t; \cdots; \sigma^l\gamma_0, \cdots, \sigma^l\gamma_\mu$$

will be part of a basis for $H^0(\Omega(\boldsymbol{F}_0))$. Taking the elements in (*) as homogeneous coordinates we get a mapping $\Phi: W \to P_N$. Since $\sigma = 0$ on X, $\Phi \mid X = \Lambda$; on $W - X$, σ is non-zero so there Φ is given by

$$[\sigma^{-l}\phi_0, \cdots, \sigma^{-l}\phi_s; \cdots; \cdots; \gamma_0, \cdots, \gamma_l].$$

Thus Φ is biregular on $W - X$.

(f) Set $Z = \Phi(W) \subset P_N$. Since $Y \cong \Phi(X)$, there is an embedding $Y \subset Z$ such that: (i) $\Phi(X) = Y$; (ii) $\Phi \mid X = \pi$; and (iii) $\Phi: W - X \cong Z - Y$. We want to show that Z is non-singular and that W is the monoidal transform of Z along Y. In order to do this, we assume that $\boldsymbol{G}$ is such that the following holds: For any fibre $P_y \subset X$, if $\mathfrak{I}_y$ is the ideal sheaf of P_y, then, for $q > 0$, $H^q(Y, R\pi^0(\Omega(\boldsymbol{N}^{-1}) \otimes \mathfrak{I}_y) \otimes \Omega(\boldsymbol{G})) = 0$. Indeed, if our original $\boldsymbol{E} \to Y$ does not work, then we may, as before, pass to a suitable power.

LEMMA 5.5. *With the above notations, we have that* $H^1(X, \mathfrak{I}_y \cdot \Omega(\boldsymbol{E}_1)) = 0$ *for all* $y \in Y$.

Proof. If we can show that $R\pi^q(\mathfrak{I}_y \cdot \Omega(\boldsymbol{E}_1)) = 0$ for $q > 0$, then, by the Leray spectral sequence, we have

$$H^1(X, \mathfrak{I}_y \cdot \Omega(\boldsymbol{E}_1)) \cong H^1(Y, R\pi^0(\mathfrak{I}_y \cdot \Omega(\boldsymbol{E}_1))).$$

But $\boldsymbol{E}_1 = \boldsymbol{E}_0 \otimes \boldsymbol{N}^{-1} = \pi^{-1}(\boldsymbol{G}) \otimes \boldsymbol{N}^{-1}$ and thus

$$R\pi^0(\mathfrak{I}_y \cdot \Omega(\boldsymbol{E}_1)) \cong R\pi^0(\mathfrak{I}_y \cdot \Omega(\boldsymbol{N}^{-1})) \otimes \Omega(\boldsymbol{G}).$$

Because of $H^q(Y, R\pi^0(\Omega(\boldsymbol{N}^{-1}) \otimes \mathfrak{I}_y) \otimes \Omega(\boldsymbol{G})) = 0$ for $q > 0$, we are done.

Since $\boldsymbol{E}_1 \mid \boldsymbol{P}_{y'} = \boldsymbol{H}$, we clearly have $R_\pi{}^q(\mathfrak{F}_y \cdot \Omega(\boldsymbol{E}_1))_{y'} = 0$ for $q > 0$ and $y' \neq y$. Furthermore, $R_\pi{}^q(\mathfrak{F}_y \cdot \Omega(\boldsymbol{E}_1))_y \cong \boldsymbol{m}_y \hat{\otimes} H^q(P_y, \Omega(\boldsymbol{H}))$ for any q and where $\boldsymbol{m}_y \subset \mathfrak{O}_{Y,y}$ is the maximal ideal. This proves the lemma.

(g) Now we can complete the proof. Let $P \subset X$ be a general fibre of $X \xrightarrow{\pi} Y$ and $\mathfrak{F} \subset \mathfrak{O}_X$ the ideal sheaf of P. We have the sequence

$$0 \to \mathfrak{F} \cdot \Omega(\boldsymbol{E}_1) \to \Omega(\boldsymbol{E}_1) \to \Omega_P(\boldsymbol{H}) \to 0$$

and, by the Lemma in (f) we get

$$\begin{array}{ccccc} H^0(W, \Omega(\boldsymbol{F}_1)) & \xrightarrow{\rho} & H^0(X, \Omega(\boldsymbol{E}_1)) & \to & 0 \\ & \searrow^{\tilde{\rho}} & \downarrow & & \\ & & H^0(P, \Omega(\boldsymbol{H}_1)) & & \\ & & \downarrow & & \\ & & 0. & & \end{array}$$

Let $y_0 \in Y \subset Z$ be a fixed point and set $P = \pi^{-1}(y_0)$. Denote by $U(y_0) \subset W$ a small tubular neighborhood around P. Then we may choose elements $\xi_0, \cdots, \xi_n; \eta_0, \cdots, \eta_r$ from (*) such that: (i) $\xi_0, \cdots, \xi_n \in H^0(W, \Omega(\boldsymbol{F}_0))$ and $\rho(\xi_0), \cdots, \rho(\xi_n) \in H^0(X, \Omega(\boldsymbol{E}_0)) \cong H^0(Y, \Omega(\boldsymbol{G}))$ embed $\pi(X \cap \mathrm{U}(y_0))$ (ii) $\eta_1, \cdots, \eta_r \in H^1(W, \Omega(\boldsymbol{F}_1))$ and $\tilde{\rho}(\eta_1), \cdots, \tilde{\rho}(\eta_r) \in H^0(P_y, \Omega(\boldsymbol{H}))$ give a in a projective space. Also, we may assume that $\xi_0(w) \neq 0$ for $w \in U(y_0)$; basis for $y \in \pi(U(y_0) \cap X)$. Then, by rearranging the elements in (*), we may assume that $\Phi \mid U(y_0)$ has the form

$$\Phi(w) = [\xi_0(w), \cdots, \xi_n(w); \sigma\eta_1(w), \cdots, \sigma\eta_r(w); \cdots].$$

We set $\zeta_i(w) = \xi_i(w)/\xi_0(w)$ $(1 \leqq i \leqq n)$ and $\lambda_\alpha(w) = \dfrac{\sigma\eta_\alpha(w)}{\xi_0(w)}$ $(1 \leqq \alpha \leqq r)$. Then the ζ_i and λ_α are holomorphic functions in $U(y_0)$ and $\lambda_\alpha \mid X = 0$.

Let $V \subset \boldsymbol{C}^{n+r}$ be a polycylinder with coordinates $(\zeta_1, \cdots, \zeta_n; \lambda_1, \cdots, \lambda_r)$, and let P_{r-1} have homogeneous coordinates $[t_1, \cdots, t_r]$. Define $Q(V, P_{r-1}) \subset V \times P_{r-1}$ by

$$Q(V, P_{r-1}) = \{(\zeta, \lambda, t) : \lambda_\alpha t_\beta - \lambda_\beta t_\alpha = 0; 1 \leqq \alpha, \beta \leqq r\}.$$

Then define $F: U(y_0) \to Q(V, P_{r-1})$ by

$$F(w) = (\zeta_1(w), \cdots, \zeta_n(w); \lambda_1(w), \cdots, \lambda_r(w); [\eta_1(w), \cdots, \eta_r(w)]).$$

It is easy to check, using Hartog's theorem, that F is biregular on some tubular neighborhood $U \subset U(y_0)$ of $\pi^{-1}(y_0)$. Q. E. D.

3. Reduction to the codimension one case. Let $X \subset W$ be a germ of embedding of a compact, complex manifold X in a complex manifold W. Denote by $\tilde{W}$ the monoidal transform of W along X, let $\tilde{\omega}\colon \tilde{W} \to W$ be the blowing down map, and set $\tilde{X} = \tilde{\omega}^{-1}(X)$. Finally let $\mathfrak{I} \subset \mathfrak{O}_W$ be the ideal sheaf of X and $\tilde{\mathfrak{I}} \subset \mathfrak{O}_{\tilde{W}}$ the ideal sheaf of $\tilde{X}$. The same spectral sequence argument as used in 5.2 just above gives:

PROPOSITION 5.1. *Let $\boldsymbol{E} \to W$ be a holomorphic vector bundle and $\tilde{\boldsymbol{E}} = \tilde{\omega}^{-1}(\boldsymbol{E})$. Then, if $H^q(\tilde{W}, \tilde{\mathfrak{I}}^\mu \cdot \Omega(\tilde{\boldsymbol{E}})) = 0$ for $0 \leqq q < n$ and $\mu \geqq \mu_0$, it follows that $H^q(W, \mathfrak{I}^\mu \cdot \Omega(\boldsymbol{E})) = 0$ for $0 \leqq q < n$ and $\mu \geqq \mu_0$.*

Remark. In case the normal bundle $\boldsymbol{N}$ of $X \subset W$ is positive, this Proposition says that: (i) we can work on W with positive bundles and non-locally free sheaves; or (ii) we can lift to W and work with locally free sheaves but with bundles of mixed type.

VI. Differential-Geometric Foundations.

1. Local Hermitian geometry. Let X be a complex manifold on which we have an Hermitian metric. Locally we may choose a family $\omega^1, \cdots, \omega^n$ of $(1,0)$ forms such that $ds^2 = \sum \omega^i \bar{\omega}^i$. The *associated 2-form* is then $\gamma = \frac{\sqrt{-1}}{2} \sum \omega^i \wedge \bar{\omega}^i$.

PROPOSITION 6.1. *There exist (locally) a unique set of 1-forms $\phi_j{}^i$ which satisfy*

$$\phi_j{}^i + \bar{\phi}_i{}^j = 0 \tag{6.1}$$

$$d\omega^i = \sum \phi_j{}^i \wedge \omega^j + \tfrac{1}{2} \sum T_{jk}{}^i \omega^j \wedge \omega^k \qquad (T_{jk}{}^i + T_{kj}{}^i = 0). \tag{6.2}$$

Proof. We choose local holomorphic coordinates $z^1, \cdots, z^n$ and write $\omega^i = \sum a_j{}^i dz^j$. In matrix form, this becomes $\omega = a\,dz$ where $\omega = {}^t(\omega^1, \cdots, \omega^n)$ and $dz = {}^t(dz^1, \cdots, dz^n)$. If we define h by $h_{ij} = (dz^i, dz^j)$, then ${}^t\bar{h} = h$ and $ah{}^t\bar{a} = I$.

Now let $\phi = a(\partial h \cdot h^{-1})a^{-1} + da \cdot a^{-1}$. Then

$$d\omega = da \wedge dz = \phi \wedge \omega - a(\partial h \cdot h^{-1})a^{-1} \wedge \omega. \tag{6.3}$$

Because of (6.3), it will suffice to show that $\phi + {}^t\bar{\phi} = 0$. Given $x_0 \in X$, we may assume that our coordinate system has been chosen so that $h(x_0) = I = a(x_0)$. Then at x_0,

$$\phi + {}^t\bar{\phi} = \partial h + da + \bar{\partial}{}^t\bar{h} + d{}^t\bar{a} = da + dh + d{}^t\bar{a} = d(ah{}^t\bar{a})\,(x_0) = 0.$$

It is easy to verify that ϕ is unique. Q. E. D.

We write locally

$$\phi_j{}^i = \sum \Lambda_{jk}{}^i \omega^k + \sum \Lambda_{j\bar{k}}{}^i \bar{\omega}^k \tag{6.4}$$

where, by (6.1),

$$\Lambda_{j\bar{k}}{}^i + \bar{\Lambda}_{i\bar{k}}{}^j = 0. \tag{6.5}$$

Now

$$d\gamma = \frac{\sqrt{-1}}{2}\, d({}^t\omega \wedge \bar{\omega})$$

$$= \frac{\sqrt{-1}}{2}\{{}^t(\phi \wedge \omega + T) \wedge \bar{\omega} - {}^t\omega \wedge (\overline{\phi \wedge \omega + T})\} = \sqrt{-1}\,\mathrm{Re}({}^tT \wedge \bar{\omega})$$

and do $d\gamma = 0$ if, and only if, $T = 0$. This T is the so-called torsion tensor which vanishes if, and only if, the metric is Kähler.

We shall write a $(0, q)$ form Ψ on X as $\Psi = \frac{1}{q!}\sum_I \psi_{\bar{I}} \bar{\omega}^I$ where $\bar{I} = (\bar{\imath}_1, \cdots, \bar{\imath}_q)$ and $\bar{\omega}^I = \bar{\omega}^{i_1} \wedge \cdots \wedge \bar{\omega}^{i_q}$. The covariant derivatives $\psi_{\bar{I};j}$ and $\psi_{\bar{I};\bar{\jmath}}$ are then defined by

$$\sum \psi_{\bar{I};j}\omega^j + \sum \psi_{\bar{I};\bar{\jmath}}\bar{\omega}^j = d\psi_{\bar{I}} - q \sum \bar{\phi}_{i_1}{}^j \psi_{\bar{\jmath}\bar{\imath}_2 \cdots \bar{\imath}_q}. \tag{6.6}$$

The operator $\bar{\partial}\colon A^q \to A^{q+1}$ is defined as usual and satisfies $\bar{\partial}^2 = 0$. We also define $\bar{\delta}\colon A^q \to A^{q-1}$ by

$$\int_X \langle \bar{\delta}\psi, \eta\rangle = \int_X \langle \psi, \bar{\partial}\eta\rangle \qquad \text{(support } \eta \text{ compact)} \tag{6.7}$$

Then $\bar{\delta}^2 = 0$ and we have the Laplacian $\square$ is given by

$$\square = \bar{\partial}\bar{\delta} + \bar{\delta}\bar{\partial}. \tag{6.8}$$

To obtain a local expression for $\square$, we define $T\colon A^q \to A^{q+1}$ by

$$T(\psi) = \frac{1}{(q-1)!}\sum \bar{T}_{jk}{}^i \psi_{\bar{\imath}\,\bar{\imath}_2 \cdots \bar{\imath}_q} \bar{\omega}^j \wedge \bar{\omega}^k \wedge \bar{\omega}^{i_2} \wedge \cdots \wedge \bar{\omega}^{i_q} \tag{6.9}$$

and $T^{\#}\colon A^q \to A^{q-1}$ by

$$\langle T^{\#}\psi, \eta\rangle = \langle \psi, T\eta\rangle \qquad (\psi \in A^q, \eta \in A^{q-1}). \tag{6.10}$$

Finally, we let $\Phi_j{}^i = d\phi_j{}^i - \sum \phi_k{}^i \wedge \phi_j{}^k$ be curvature (which is of type $(1,1)$, $\Phi_{i\bar{\jmath}} = \sum \Phi_{ik\bar{\jmath}}{}^k$ the Ricci tensor, and define $\Phi\colon A^q \to A^q$ by

$$\Phi(\psi) = \frac{1}{(q-1)!}\sum \Phi_{i\bar{\jmath}} \psi_{\bar{\imath}\,\bar{\imath}_2 \cdots \bar{\imath}_q} \bar{\omega}^j \wedge \bar{\omega}^{i_2} \wedge \cdots \wedge \bar{\omega}^{i_q}. \tag{6.11}$$

Then a calculation gives

$$\Box\psi = \frac{1}{q!}\sum \psi_{\bar{I};\bar{J};j}\bar{\omega}^I + [T,\bar{\delta}](\psi) + [T^{\#},\bar{\partial}](\psi) - [T,T^{\#}](\psi) + \Phi(\psi) \tag{6.12}$$

where $[A,B](\psi) = A(B(\psi)) + B(A(\psi))$. (For example, it follows from (6.12) that $\Box = \frac{1}{2}\Delta(\Delta = d\delta + \delta d)$ if, and only if, $T = 0$. This is because $[T,T^{\#}] = 0$ if, and only if, $T = 0$.)

Now suppose that W' is a Hermitian manifold and that $W \Subset W'$ is a *finite submanifold.* Thus ∂W is a smooth, connected, compact real hypersurface. We may define a real function r in a neighborhood of ∂W such that (i) $|dr| = 1$, and (ii) $r < 0$ defines $W \subset W'$. Then $\partial\bar{\partial}r \mid \boldsymbol{T}(\partial W)$ gives the *E. E. Levi form* L of W in W'. Locally, in a neighborhood of a boundary point, we may choose $\omega^1, \cdots, \omega^n$ such that $\omega^n = \partial r$. By (6.2) and (6.5) we get

$$-\bar{\partial}\omega^n = \sum \Lambda_{j\bar{k}}{}^n \omega^j \wedge \bar{\omega}^k \tag{6.13}$$

so that the matrix $\{\Lambda_{j\bar{k}}{}^n\}_{1\leq j,k\leq n-1}$ gives L. For $\psi \in A^q$, we set

$$L(\psi,\psi) = (L\psi,\psi) = \frac{1}{(q-1)!}\sum_{j,k=1}^{n-1} \Lambda_{j\bar{k}}{}^n \psi_{\bar{J}\bar{i}_2\cdots\bar{i}_q}\bar{\psi}_{\bar{k}\bar{i}_2\cdots\bar{i}_q}. \tag{6.14}$$

2. Holomorphic vector bundles. Let $\boldsymbol{E} \to X$ be a holomorphic vector bundle and $P \xrightarrow{\pi} X$ the holomorphic principal bundle. A connexion in $\boldsymbol{E}$ is given by a matrix-valued one-form $\theta = \{\theta^\rho{}_\sigma\}$ on P which satisfies the usual conditions. Given θ, we write $\theta = \theta' + \theta''$ where $\theta'(\theta'')$ is the $(1,0)$ (resp. $(0,1)$) part of θ; we say that θ defines a *complex connexion* if $\theta'' = 0$. As usual, a connexion θ induces a covariant differential

$$D: A^{p,q}(\boldsymbol{E}) \to A^{p+1,q}(\boldsymbol{E}) \oplus A^{p,q+1}(\boldsymbol{E})$$

and we write $D = D' + D''$ for the type decomposition. A complex connexion is one for which $D'' = \bar{\partial}$.

Suppose now that we have in $\boldsymbol{E} \to X$ an Hermitian metric. The following is known:

PROPOSITION 6.2. *There exists a unique complex connexion θ which is compatible with the metric in $\boldsymbol{E}$. Furthermore, the curvature Θ of θ is of type $(1,1)$.*

Proof. P is the manifold of holomorphic $\boldsymbol{E}$-frames $(e_1, \cdots, e_r)$ and we define on P an Hermitian matrix h by $h_{\rho\sigma} = (e_\rho, e_\sigma)$. Now D and θ are

related by $De_\rho = \sum \theta_\rho{}^\sigma e_\sigma$ and to say that θ is compatible with the metric means that $d(e, e') = (De, e') + (e, De')$ for sections e, e' of $\boldsymbol{E}$. Thus θ is a solution of the system

$$\begin{cases} dh_{\rho\sigma} = \sum \theta_\rho{}^\tau h_{\tau\sigma} + \sum h_{\rho\tau}\bar{\theta}_\sigma{}^\tau \\ \theta' = 0. \end{cases} \tag{6.15}$$

The system (6.15) has the unique solution

$$\theta_\sigma{}^\rho = \sum \partial h_{\sigma\tau}(h^{-1})_{\tau\rho}. \tag{6.16}$$

Remarks. (i) If $P^\# \subset P$ is the real sub-bundle of unitary frames, $\theta \mid P^\#$ is given by one-forms $\theta_\sigma{}^\rho$ which satisfy $\theta_\sigma{}^\rho + \bar{\theta}_\rho{}^\sigma = 0$; then

$$\Theta_\sigma{}^\rho = d\theta_\sigma{}^\rho - \sum \theta_\tau{}^\rho \wedge \theta_\sigma{}^\tau;$$

and

$$\Theta_\sigma{}^\rho + \bar{\Theta}_\rho{}^\sigma = 0 \tag{6.17}$$

(ii) To interpret the second statement in Proposition 6.2, we suppose that $\boldsymbol{E} \to X$ is a *differentiable* bundle with fibre $\boldsymbol{C}^r$ and that θ on P gives a connexion for $\boldsymbol{E}$. If $\omega^1, \cdots, \omega^n$ are a local $(1,0)$ co-frame on X, then the differential system generated by the $\theta_\sigma{}^\rho$ and ω^i gives an almost-complex structure J^θ on P.

PROPOSITION. *If the curvature Θ of θ is of type $(1,1)$, then J^θ is integrable.*

Proof. Since $d\theta_\sigma{}^\rho - \sum \theta_\tau{}^\rho \wedge \theta_\sigma{}^\tau = \Theta_\sigma{}^\rho$, we see that, if Θ is of type $(1,1)$, then $d\theta_\sigma{}^\rho$ is in the exterior ideal generated by $\{\theta_\sigma{}^\rho; \omega^i\}$.

Example. Suppose that X is a compact Kähler manifold and that $\gamma \in H^2(X, \boldsymbol{Z})$ is a cohomology class of type $(1,1)$. Then there exists a differentiable line bundle (with fibre $\boldsymbol{C}$) $\boldsymbol{E} \to X$ such that $c_1(\boldsymbol{E}) = \gamma$. Let γ be a real form of type $(1,1)$ which represents γ and let θ be a connexion for $\boldsymbol{E}$. Then $\frac{\sqrt{-1}}{2\pi} d\theta = \gamma + d\eta$ where η is a global one-form on X. Thus $\theta - \frac{2\pi}{\sqrt{-1}}\eta$ is a connexion in $\boldsymbol{E}$ whose curvature γ is of type $(1,1)$ and so $\boldsymbol{E}$ is a holomorphic bundle. This is a theorem of Kodaira-Spencer [14].

(iii) *Example.* Let $\boldsymbol{E} \xrightarrow{\pi} X$ be an Hermitian holomorphic vector bundle and suppose that we have on X an Hermitian metric ds^2. Then $\boldsymbol{E}^\# = \pi^{-1}(\boldsymbol{E})$ is a holomorphic bundle over $\boldsymbol{E}$ and there is a *canonical section* $\sigma \in H^0(\boldsymbol{E}, \Omega(\boldsymbol{E}^\#))$ defined by $\sigma(z, \xi) = (z, \xi; \xi)$ $(z \in X, \xi \in \boldsymbol{E}_z)$. We define

on $\boldsymbol{E}$ an Hermitian metric $\phi = \pi^*(ds^2) + (D\sigma, D\sigma)$ where D is the covariant differential associated to the metric connexion θ.

PROPOSITION. *ϕ is a Kähler metric if, and only if, ds^2 is Kähler and the curvature Θ of θ is zero.*

Proof. Assume ds^2 is Kähler and denote by $\langle D\sigma, D\sigma\rangle$ the exterior form associated to $(D\sigma, D\sigma)$. Then

$$d\langle D\sigma, D\sigma\rangle = \langle D^2\sigma, D\sigma\rangle + \langle D\sigma, D^2\sigma\rangle = 2Re\langle\Theta\sigma, D\sigma\rangle = 0$$

if, and only if, $\Theta = 0$. Q. E. D.

Suppose that $\boldsymbol{E} \to X$ is a holomorphic Hermitian vector bundle over the Hermitian manifold X, and let $e_1, \cdots, e_r; \omega^1, \cdots, \omega^n$ be local unitary frames for $\boldsymbol{E}, \boldsymbol{T}^*$ respectively. Then $\phi \in A^q(\boldsymbol{E})$ is written as $\phi = \frac{1}{q!}\sum \phi_{\bar{I}}{}^{\rho}\bar{\omega}^I \otimes e_\rho$. If $\theta = (\theta_\sigma{}^\rho)$ is the connexion form of the metric connexion, then the covariant derivatives of ϕ are given by

$$\phi_{\bar{I}|j}{}^{\rho} = \phi_{\bar{I};j}{}^{\rho} + \sum \theta_{\sigma j}{}^{\rho}\phi_{\bar{I}}{}^{\sigma} \qquad (\theta_\sigma{}^\rho = \sum \theta_{\sigma i}{}^{\rho}\omega^i + \sum \theta_{\sigma\bar{j}}{}^{\rho}\bar{\omega}^j); \tag{6.18}$$

$$\phi_{\bar{I}|\bar{j}}{}^{\rho} = \phi_{\bar{I};\bar{j}}{}^{\rho}. \tag{6.19}$$

The operator $\vartheta: A^q(\boldsymbol{E}) \to A^{q-1}(\boldsymbol{E})$ is defined by

$$(\vartheta\phi)^\rho = \frac{(-1)^q}{(q-1)!}\sum \phi_{\bar{\imath}\cdots\bar{\imath}_{q-1}\bar{j}}{}^{|j}{}^{\rho}\bar{\omega}^{i_1} \wedge \cdots \wedge \bar{\omega}^{i_{q-1}} - T^{\#}(\phi). \tag{6.20}$$

Then $\vartheta^2 = 0$ and the Laplacian $\square = \bar{\partial}\vartheta + \vartheta\bar{\partial}$ is given as

$$(\square\phi)^\rho = \frac{1}{q!}\sum \phi_{\bar{I}|\bar{J}|j}{}^{\rho}\bar{\omega}^I + Q(\nabla\phi) + R(\bar{\nabla}\phi) + S(\phi) + \Theta(\phi) \tag{6.21}$$

where

$$Q(\nabla\phi) = [T, \vartheta](\phi); \quad R(\bar{\nabla}\phi) = [T^{\#}, \bar{\partial}](\phi);$$

$$S(\phi) = -[T, T^{\#}](\phi) + \Phi(\phi);$$

and

$$\Theta(\phi)^\rho = \frac{1}{(q-1)!}\sum \Theta_{\sigma i\bar{j}}{}^{\rho}\phi_{\bar{\imath}\,\bar{\imath}_2\cdots\bar{\imath}_q}{}^{\sigma}\bar{\omega}^j \wedge \bar{\omega}^{i_2} \wedge \cdots \wedge \bar{\omega}^{i_q}. \tag{6.22}$$

3. Induced connexions in sub- and -quotient bundles. Let $\boldsymbol{E} \to X$ be a holomorphic Hermitian vector bundle and suppose given an exact sequence of holomorphic bundles

$$0 \to \boldsymbol{S} \to \boldsymbol{E} \to \boldsymbol{Q} \to 0. \tag{6.23}$$

Let $r =$ fibre dimension of $\boldsymbol{E}$, $s =$ fibre dimension of $\boldsymbol{S}$, and we agree an the

ranges of indices $1 \leqq \rho, \sigma \leqq r$; $1 \leqq \alpha, \beta \leqq s$; $s+1 \leqq i, j \leqq r$. Also we denote by $B \subset P^{\#}$ the unitary frames $(e_1, \cdots, e_r)$ for $\boldsymbol{E}$ such that $e_1, \cdots, e_s$ lie in $\boldsymbol{S}$. Then the metric connexion $\theta = \{\theta^\rho{}_\sigma\}$ when restricted to B, defines connexions $\theta_S = \{\theta_\beta{}^\alpha\}$ and $\theta_Q = \{\theta_j{}^i\}$ in $\boldsymbol{S}$ and $\boldsymbol{Q}$ respectively. Thus, for $\xi = \sum \xi^\alpha e_\alpha \in A^0(\boldsymbol{S})$, $\eta = \sum \eta^i e_i \in A^0(\boldsymbol{Q})$, $D_S\xi = \sum (d\xi^\alpha + \theta_\beta{}^\alpha \xi^\beta) e_\alpha$ and $D_Q\eta = \sum (d\eta^i + \theta_j{}^i \eta^j) e_i$.

LEMMA 6.2. $D - D_S \in A^{1,0}(\mathrm{Hom}(\boldsymbol{S}, \boldsymbol{Q}))$.

Proof. For $\xi \in A^0(\boldsymbol{S})$ as above, $((D - D_S)\xi = \sum \theta_\alpha{}^i \xi^\alpha e_i$ and so $D - D_S$ is a 1-form with values in $\mathrm{Hom}(\boldsymbol{S}, \boldsymbol{Q})$. For $\zeta \in \boldsymbol{S}_{x_0}$, there exists a holomorphic section $\xi \in \Omega(\boldsymbol{S})_{x_0}$ with $\xi(x_0) = \zeta$. Then $(D''\xi)(x_0) = 0$ which gives $0 = (\sum \theta_\alpha{}^{i\prime\prime} \xi^\alpha e_i)(x_0) = \sum \theta_\alpha{}^{i\prime\prime} \zeta^\alpha e_i = (D - D_S)''\zeta$ and so $(D - D_S)'' = 0$; i.e. $D - D_S \in A^{1,0}(\mathrm{Hom}(\boldsymbol{S}, \boldsymbol{Q}))$. Q.E.D.

Definition. The operator $b = D - D_S \in A^{1,0}(\mathrm{Hom}(\boldsymbol{S}, \boldsymbol{Q}))$ is defined to be the *second fundamental form* of $\boldsymbol{S}$ in $\boldsymbol{E}$.

Remark. We shall give below two intepretations of b.

From $\theta_\alpha{}^{i\prime\prime} = 0$ and $\theta_\alpha{}^i + \bar{\theta}_i{}^\alpha = 0$, we get $\theta_i{}^{\alpha\prime} = 0$. Thus

$$c = D - D_Q \in A^{0,1}(\mathrm{Hom}(\boldsymbol{Q}, \boldsymbol{S}))$$

and is given by $c = -{}^t\bar{b}$.

PROPOSITION 6.3. *θ_S gives the metric connexion of $\boldsymbol{S}$ with the induced metric.*

Proof. Since $\theta_\beta{}^\alpha + \bar{\theta}_\alpha{}^\beta = 0$ we need only verify that $D_S'' = \bar{\partial}$. By Lemma 6.2, $D_S'' = D'' = \bar{\partial}$. Q.E.D.

The curvature Θ_S of θ_S is given by

$$\begin{aligned}(\Theta_S)_\beta{}^\alpha = d\theta_\beta{}^\alpha - \sum \theta_\gamma{}^\alpha \wedge \theta_\beta{}^\gamma &= \Theta_\beta{}^\alpha + \sum \theta_i{}^\alpha \wedge \theta_\beta{}^i \\ &= \Theta_\beta{}^\alpha - \sum \theta_i{}^\alpha \wedge \bar{\theta}_i{}^\beta, \text{ i.e.}\end{aligned}$$

$$\Theta_S = \Theta - b \wedge {}^t\bar{b}. \tag{6.24}$$

Similarly,

$$\Theta_Q = \Theta + {}^tb \wedge \bar{b}. \tag{6.25}$$

Examples. (i) Suppose that $\boldsymbol{E}$ is the trivial bundle and take the flat metric connexion. Then, in the notation of § IV.4, there is a holomorphic mapping $f: X \to G(s, r-s)$ such that $f^{-1}(\boldsymbol{E}(s, r-s)) = \boldsymbol{Q}$ and $b \in \mathrm{Hom}(\boldsymbol{T}(X), \mathrm{Hom}(\boldsymbol{S}, \boldsymbol{Q}))$ is the differential f_* of f. In this case, $\Theta = 0$ and

$$\Theta_{\boldsymbol{Q}} = {}^t b \wedge \bar{b}. \tag{6.26}$$

(ii) The metric in $\boldsymbol{E}$ defines a C^∞ splitting of (6.23) which is given by $\gamma: \boldsymbol{Q} \to \boldsymbol{E}$. Then $\bar{\partial}\gamma \in A^{0,1}(\mathrm{Hom}(\boldsymbol{Q}, \boldsymbol{S}))$ and represents in

$$H^1(X, \Omega(\mathrm{Hom}(\boldsymbol{Q}, \boldsymbol{S})))$$

the cohomology extension class defined by (6.23).

PROPOSITION. $c \equiv \bar{\partial}\gamma$.

Proof. If $e_1^*, \cdots, e_r^*$ is dual to $e_1, \cdots, e_r$, then $\gamma = \sum e_i \otimes e_i^*$. Thus

$$\begin{aligned}\bar{\partial}\gamma &= D''\gamma \\ &= \sum \theta_i^{j\prime\prime} e_j \otimes e_i^* + \sum \theta_i^{\alpha\prime\prime} e_\alpha \otimes e_i^* - \sum \theta_k^{i\prime\prime} e_i \otimes e_k^* - \sum \theta_\alpha^{i\prime\prime} e_i \otimes e_\alpha^* \\ &= \sum \theta_i^{\alpha\prime\prime} e_\alpha \otimes e_i^* = \mathrm{c}. \quad \text{Q. E. D.}\end{aligned}$$

(iii) In example (ii) we have given a cohomological interpretation of the second fundamental form of $\boldsymbol{S}$ in $\boldsymbol{E}$, and we shall now give a geometric interpretation which will justify the terminology. Let W be an Hermitian manifold of dimension r and $X \subset W$ a complex submanifold of dimension s. Let $\boldsymbol{T} = \boldsymbol{T}(W) \mid X$ so that, over X, we have

$$0 \to \boldsymbol{T}(X) \to \boldsymbol{T} \to \boldsymbol{N} \to 0 \tag{6.27}$$

where $\boldsymbol{N}$ is the normal bundle of X in W. Along X we choose co-frames $\omega^1, \cdots, \omega^r$ for W such that $\omega^{s+1} = 0, \cdots, \omega^r = 0$ on X. Let $\phi = (\phi_\sigma{}^\rho)$ be the Hermitian connexion on W (§ VI. 1) which is also the metric connexion in $\boldsymbol{T}(W)$. If $R \subset B$ is the bundle of unitary frames for X, then R is defined by $\omega^i = 0$ $(i = s+1, \cdots, r)$ and, on R, $\phi_\alpha{}^i = \sum A_{\alpha\beta}{}^i \omega^\beta + \sum C_{\alpha\bar{\beta}}{}^i \bar{\omega}^\beta$. But, on R, $0 = d\omega^i = \sum \phi_\alpha{}^i \wedge \omega^\alpha + \sum T_{\alpha\beta}{}^i \omega^\alpha \wedge \omega^\beta$ (by (6.2)) and it follows that $C_{\alpha\bar{\beta}}{}^i \equiv 0$ and $\phi_\alpha{}^i = \sum A_{\alpha\beta}{}^i \omega^\beta$. If the metric is Kähler $(T = 0)$, then we even get $A_{\alpha\beta}{}^i = A_{\beta\alpha}{}^i$. The quadratic form

$$\pi = \sum A_{\alpha\beta}{}^i e_i \otimes \omega^\alpha \otimes \omega^\beta \tag{6.28}$$

has values in $\boldsymbol{N}$ and, by direct analogy with the Riemannian case, may be called the *second fundamental form of X in W*. Clearly π is also the second fundamental form of $\boldsymbol{T}(X)$ in $\boldsymbol{T}$.

4. Comparison of Hermitian forms. Let X be an Hermitian manifold and suppose we are given a real $(1,1)$ form $A = \frac{\sqrt{-1}}{2} \sum a_{ij} \omega^i \wedge \bar{\omega}^j$ $(a_{ij} + \bar{a}_{ji} = 0)$. Then A defines an endomorphism A of $\boldsymbol{T}(X)$ by the rule

$(Ae_i, e_j) = a_{ij}$ where $e_1, \cdots, e_n$ is dual to $\omega^1, \cdots, \omega^n$. Since $A = {}^t\bar{A}$, we may choose our frames $e_1, \cdots, e_n$ so that $A = \frac{\sqrt{-1}}{2} \sum \lambda_i \omega^i \wedge \bar{\omega}^i$ where the λ_j are real.

Definition. A has signature (s, t) if, at each point, s of the λ_j are positive and t are negative.

In this case we may assume that

$$(6.29) \qquad A = \frac{\sqrt{-1}}{2} \{ \sum_{1 \leqq i \leqq s} \lambda_i \omega^i \wedge \bar{\omega}^i - \sum_{n-t<j \leqq n} \lambda_j \omega^j \wedge \bar{\omega}^j \} \qquad (\lambda_k > 0).$$

The point operators $L: A^{p,q} \to A^{p+1,q+1}$ and $\Lambda: A^{p,q} \to A^{p-1,q-1}$ are given by $L(\eta) = \gamma \wedge \eta$ and $(\Lambda\eta, \psi) = (\eta, L\psi)$ where $\gamma = \frac{\sqrt{-1}}{2} \sum \omega^i \wedge \bar{\omega}^i$. Suppose now that $A = \frac{\sqrt{-1}}{2} \sum a_{ij} \omega^i \wedge \bar{\omega}^j$ is a real $(1,1)$ form and define the following pointwise Hermitian forms:

$$(6.30) \qquad A(\eta, \eta) = \langle A\eta, \eta \rangle = \frac{1}{(q-1)!} \sum a_{ij} \eta_{\bar{i}\bar{k}_2 \cdots \bar{k}_q} \bar{\eta}_{\bar{j}\bar{k} \cdots \bar{k}_q}$$

$$(6.31) \qquad A^{\#}(\eta, \eta) = \langle \Lambda A \wedge \eta, \eta \rangle.$$

LEMMA 6.3.

$$(6.32) \qquad A^{\#}(\eta, \eta) = (\mathrm{Trace}\, A) \mid \eta \mid^2 - A(\eta, \eta) = A(*\bar{\eta}, *\bar{\eta}).$$

Proof. Write $A = \frac{\sqrt{-1}}{2} \sum_{m=1}^{n} \lambda_m \omega^m \wedge \bar{\omega}^m$. Then, for $L \subset [1, \cdots, n]$, $A \wedge \bar{\omega}^L = \frac{\sqrt{-1}}{2} \sum_{m \notin L} \lambda_m \omega^m \wedge \bar{\omega}^m \wedge \bar{\omega}^L$ and $\Lambda A \wedge \bar{\omega}^L = (\sum_{m \notin L} \lambda_m) \bar{\omega}^L$. Now

$$\langle \Lambda A \wedge \bar{\omega}^L, \bar{\omega}^{L'} \rangle = 0 \text{ for } L \neq L'$$

and

$$\begin{aligned} &\langle \Lambda A \wedge \bar{\omega}^L, \bar{\omega}^L \rangle \\ &= (\sum_{m \notin L} \lambda_m) \mid \bar{\omega}^L \mid^2 = (\sum_{m=1}^{n} \lambda_m) \mid \bar{\omega}^L \mid^2 - (\sum_{m \in L} \lambda_m) \mid \bar{\omega}^L \mid^2 \\ &= (\mathrm{Trace}\, A) \mid \bar{\omega}^L \mid^2 - A(\bar{\omega}^L, \bar{\omega}^L). \end{aligned}$$

Since $A(\bar{\omega}^L, \bar{\omega}^{L'}) = 0$ for $L \neq L'$, this proves the first part of (6.32). The second part may be checked directly.

PROPOSITION 6.4. *Assume that A has signature (s, t). Then there*

exist Hermitian metrics on X such that $A(\eta,\eta)$ is negative definite on A^q for $q > n-t$ and $A^{\#}(\eta,\eta)$ is positive definite on A^q for $q < s$.

Proof. Write $A = \frac{\sqrt{-1}}{2}\sum_{k=1}^{n}\lambda_k\omega^k \wedge \bar{\omega}^k$. Then, for positive real functions $c_1, \cdots, c_n$, the Hermitian form $\sum c_i\omega^i\bar{\omega}^i$ is well-defined provided that $c_i(x) = c_j(x)$ whenever $\lambda_i(x) = \lambda_j(x)$. Thus, given a constant $c > 0$, we may find a new metric $ds^2 = \sum \phi^i\bar{\phi}^i$ on X such that

$$A = \frac{\sqrt{-1}}{2}\{c\sum_{1\leqq i\leqq s}\phi^i \wedge \bar{\phi}^i - \frac{1}{c}\sum_{n-t<j\leqq n}\phi^j \wedge \bar{\phi}^j\}. \tag{6.33}$$

We consider increasing sets of indices $I = (i_1, \cdots, i_\alpha)$, $J = (j_1, \cdots, j_\beta)$, $K = (k_1, \cdots, k_\gamma)$ with

$$I \subset [1, \cdots, s],\ J \subset [s+1, \cdots, n-t],\ K \subset [n-t+1, \cdots, n].$$

If $\bar{\phi}^{IJK} = \bar{\phi}^I \wedge \bar{\phi}^J \wedge \bar{\phi}^K$, then $A(\bar{\phi}^{IJK}, \bar{\phi}^{I'J'K'}) = 0$ if $IJK \neq I'J'K'$ and

$$A(\bar{\phi}^{IJK}, \bar{\phi}^{IJK}) = (\alpha c - \frac{\gamma}{c})\,|\,\bar{\phi}^{IJK}\,|^2.$$

If now $\alpha+\beta+\gamma = q$, then $\gamma \geqq q-(n-t)$ so that, if $q > (n-t)$, we may choose $0 < c < q^{\frac{1}{2}}$ and then $\alpha c - \frac{\gamma}{c} < 0$. Thus $A(\eta,\eta)$ is negative definite for $q > n-t$.

If, on the other hand, $q > n-s$ and $c > q^{\frac{1}{2}}$ then $A(\eta,\eta)$ is positive definite on A^q. Thus, if $n-q > n-s$, $A(*\bar{\eta}, *\bar{\eta})$ will be positive definite for $\eta \in A^q$. By (6.32), if $q < s$ and $\eta \in A^q$, $A^{\#}(\eta,\eta) = A(*\bar{\eta}, *\bar{\eta})$ will be positive definite as required. Q. E. D.

Now let $\boldsymbol{E} \to X$ be a holomorphic Hermitian vector bundle. Then $A \in A^{1,1}(\mathrm{Hom}(\boldsymbol{E},\boldsymbol{E}))$ is said to be Hermitian if $\langle Ae, e'\rangle + \langle e, Ae'\rangle = 0$ for all $e, e' \in \boldsymbol{E}$. Locally we may choose orthonormal frames and write A as $A = \sum A_{\sigma i j}{}^{\rho} e_\rho \otimes e_\sigma^* \otimes \omega^i \wedge \bar{\omega}^j$, and to say that A is Hermitian means that

$$A_{\sigma i j}{}^{\rho} = \bar{A}_{\rho j i}{}^{\sigma} \tag{6.34}$$

By (6.17), the curvature Θ of the metric connexion is Hermitian.

We now define the following Hermitian forms for $\eta \in A^q(\boldsymbol{E})$:

$$A(\eta,\eta) = (A\eta,\eta) = \frac{1}{(q-1)!}\sum A_{\sigma i j}{}^{\rho}\eta_{\bar{i}\bar{k}_1 \cdots k_{q-1}{}^{\sigma}}\bar{\eta}_{j\bar{k}_1 \cdots \bar{k}_{q-1}{}^{\rho}} \tag{6.35}$$

$$A^{\#}(\eta,\eta) = (\Lambda A \wedge \eta, \eta) = \frac{1}{q!}\sum \langle \Lambda(A_{\sigma i j}{}^{\rho}\omega^i \wedge \omega^j \wedge \eta), \eta\rangle. \tag{6.36}$$

Observe that (6.36) is compatible with (6.22) and (6.14).

For $e \in \boldsymbol{E}$, $A\{e\} = \frac{\sqrt{-1}}{2}\langle Ae, e\rangle$ is a real $(1,1)$ form and we say that A has signature (s,t) if, for every $e \neq 0$, $A\{e\}$ has signature (s,t). The geometric meaning of this will be given in Proposition 6.5 below.

In the same way as we proved Lemma 6.3, we can show: For $\eta \in A^q(\boldsymbol{E})$,

$$A^{\#}(\eta,\eta) = (\mathrm{Trace}\, A)(\eta,\eta) - A(\eta,\eta) = A(\#\eta, \#\eta) \tag{6.37}$$

where $(\mathrm{Trace}\, A)(\eta,\eta) = \frac{1}{q!}\sum A_{\sigma\bar{\rho}}\eta_{\bar{I}}{}^{\sigma}\overline{\eta_{\bar{I}\rho}}$ and $\#: A^q(\boldsymbol{E}) \to A^{n-q}(\boldsymbol{E}^*)$ is the usual anti-isomorphism.

5. Positive and negative bundles. Let X be a complex Hermitian manifold and $\boldsymbol{E} \to X$ a holomorphic Hermitian vector bundle. The curvature $\Theta = \{\sum \Theta_{\sigma i j}{}^{\rho}\omega^i \wedge \bar{\omega}^j\} \in A^{1,1}(\mathrm{Hom}(\boldsymbol{E},\boldsymbol{E}))$ is an Hermitian operator and $\boldsymbol{E}$ is defined to be *weakly positive* if Θ has signature $(n,0)$. We say that $\boldsymbol{E}$ is *weakly negative* if $\boldsymbol{E}^*$ is weakly positive; this is the same as saying that Θ has signature $(0,n)$. More generally, we will say that $\boldsymbol{E}$ *has signature* (s,t) if Θ has signature (s,t).

Examples and Remarks. (i) $\boldsymbol{E}$ has signature (s,t) if, and only if, $\boldsymbol{E}^*$ has signature (t,s).

(ii) The condition that $\boldsymbol{E}$ be weakly positive is that the form

$$\Theta(\xi\otimes\eta, \xi\otimes\eta) = \sum \Theta_{\sigma i j}{}^{\rho}\xi^{\sigma}\bar{\xi}^{\rho}\eta^i\bar{\eta}^j \tag{6.38}$$

on *split tensors* $\xi\otimes\eta$ should be positive definite. We say that $\boldsymbol{E}\to X$ is positive if the forms $\Theta^{\#}(\eta,\eta)$ $(\eta \in A^q(\boldsymbol{E}), 0 \leqq q \leqq n-1)$ are positive definite on all tensors. In [8] it is proven that, if $\boldsymbol{E}$ is weakly positive and if X is compact, then high symetric powers $\boldsymbol{E}^{\mu}(\mu \geqq \mu_0)$ are positive. For example, $\boldsymbol{T}(P_n)$ is weakly positive but *not* positive for $n > 1$.

(iii) Suppose that we have a short exact sequence (6.23). Then the second fundamental form $b \in \mathrm{Hom}(\boldsymbol{T}(X)\otimes\boldsymbol{S}, \boldsymbol{Q})$ and ${}^t b \in \mathrm{Hom}(\boldsymbol{T}(X)\otimes\boldsymbol{Q}, \boldsymbol{S})$ are defined. For $\eta \in \boldsymbol{T}(X)$, $\xi \in \boldsymbol{S}$, $\zeta \in \boldsymbol{Q}$, we have by (6.24), (6.25)

$$\Theta_{\boldsymbol{S}}(\xi\otimes\eta, \xi\otimes\eta) = \Theta(\xi\otimes\eta, \xi\otimes\eta) - |\, b(\xi\otimes\eta)\,|^2 \tag{6.39}$$

$$\Theta_{\boldsymbol{Q}}(\zeta\otimes\eta, \zeta\otimes\eta) = \Theta(\zeta\otimes\eta, \zeta\otimes\eta) + |\, {}^t b(\zeta\otimes\eta)\,|^2. \tag{6.40}$$

Thus a sub-bundle of a weakly negative bundle is weakly negative and a quotient bundle of a weakly positive bundle is weakly positive.

In case $\boldsymbol{E}$ is the trivial bundle and $f: X \to G(s, r-s)$ is the associated mapping (§ VI. 3, example 1), it is not true that f_* injective implies that $\boldsymbol{Q}$

is weakly positive (because, e.g., $\boldsymbol{E}(s, r-s) \to G(s, r-s)$ is not weakly positive if $r-s>1$). It is, however, true that if $\boldsymbol{Q} \to X$ is ample (in the sense of § IV. 4), then it is weakly positive. (*Proof*: Under the isomorphism $\mathrm{Hom}(\boldsymbol{T}(X) \otimes \boldsymbol{S}, \boldsymbol{Q}) \cong \mathrm{Hom}(\boldsymbol{S}, \boldsymbol{Q} \otimes \boldsymbol{T}(X)^*)$, b defines an element $\beta \in \mathrm{Hom}(\boldsymbol{S}, \boldsymbol{Q} \otimes \boldsymbol{T}(X)^*)$ which is, by definition, surjective. Let now $\gamma_1, \cdots, \gamma_s$ be an orthonormal basis for $\boldsymbol{S}_x$ and let $\zeta \in \boldsymbol{Q}_x$, $\eta \in \boldsymbol{T}(X)^*_x$. Then

$$\Theta_{\boldsymbol{Q}}(\zeta \otimes \eta, \zeta \otimes \eta) = \sum_{\alpha=1}^{s} |\langle \beta(\gamma_\alpha), \zeta \otimes \eta \rangle|^2. \tag{6.41}$$

The assertion now follows from (6.41).)

(iv) Suppose that $\boldsymbol{E}$ and $\boldsymbol{F}$ are holomorphic Hermitian vector bundles and let $e_1, \cdots, e_r; f_1, \cdots, f_s$ be bases for $\boldsymbol{E}_x$, $\boldsymbol{F}_x$ $(x \in X)$. If

$$\phi = \sum \phi^{\rho\alpha} e_\rho \otimes f_\alpha \in (\boldsymbol{E} \otimes \boldsymbol{F})_x,$$

we define $\phi_\alpha \in \boldsymbol{E}_x$ by $\phi_\alpha = \sum \phi^{\rho\alpha} e_\rho$ and $\phi_\rho \in \boldsymbol{F}_x$ by $\phi_\rho = \sum \phi^{\rho\alpha} f_\alpha$. Then, for $\eta \in \boldsymbol{T}(X)_x$,

$$\Theta_{\boldsymbol{E}\otimes\boldsymbol{F}}(\phi \otimes \eta, \phi \otimes \eta) = \sum_\alpha \Theta_{\boldsymbol{E}}(\phi_\alpha \otimes \eta, \phi_\alpha \otimes \eta) + \sum_\rho \Theta_{\boldsymbol{F}}(\phi_\rho \otimes \eta, \phi_\rho \otimes \eta). \tag{6.42}$$

From (6.42) it follows that the tensor product of weakly positive bundles is again weakly positive.

(v) (cf. [3], § 23) Let $\boldsymbol{E} \xrightarrow{\pi} X$ be a holomorphic Hermitian vector bundle and define a real function ϕ on $\boldsymbol{E}$ by $\phi(x, \xi) = |\xi|_x^2 (\xi \in \boldsymbol{E}_x)$. Let $W \subset \boldsymbol{E}$ be defined by $\{(x, \xi) : \phi(x, \xi) < 1\}$. Then, if $\boldsymbol{E}$ has signature (s, t), the E. E. Levi form of ∂W has signature $(t+r-1, s)$. (*Proof*: Let $x_0 \in X$ be fixed and let $z = (z^1, \cdots, z^n)$ be holomorphic coordinates in a neighborhood U of x_0. If we choose a trivialization $\pi^{-1}(U) \cong U \times \boldsymbol{C}^r$, then the metric is given by an Hermitian matrix $h = (h_{\rho\sigma}(z))$ and we may assume that $h_{\rho\sigma}(o) = \delta_\sigma^\rho$, $dh((o)) = 0$. Let $e_0 = (x_0, \xi_0) \in \partial W \cap \boldsymbol{E}_{x_0}$. Since $\phi(z, \xi) = \sum h^\rho{}_\sigma \xi^\rho \bar{\xi}^\sigma$,

$$(\partial\bar{\partial}\phi)(e_0) = \sum d\xi^\rho \wedge d\bar{\xi}^\rho - \sum \Theta_{\sigma i \bar{\jmath}}{}^\rho \xi^\sigma \bar{\xi}^\rho \, dz^i \wedge d\bar{z}^j \tag{6.43}$$

and $\boldsymbol{T}_{e_0}(\partial W)$ is given by the vectors $\gamma = (\zeta^1, \cdots, \zeta^r; \eta^1, \cdots, \eta^n)$ which satisfy

$$\sum \xi_0^\rho \bar{\zeta}^\rho = 0 \tag{6.44}$$

thus $\partial\bar{\partial}\phi$ has, on $\boldsymbol{T}_{e_0}(\partial W)$, signature $(t+r-1, s)$.)

(v) Finally, let $P(\boldsymbol{E}^*) \to X$ be the bundle of projective spaces asso-

ciated to $\boldsymbol{E}^* \to X$ and $\boldsymbol{L} \to P(\boldsymbol{E}^*)$ the hyperplane bundle (cf. § A. 3). Then the computation in [8], § 9 gives

PROPOSITION 6.5. *If* $\boldsymbol{E} \to X$ has signature (s, t), then $\boldsymbol{L} \to P(\boldsymbol{E}^*)$, *with the induced metric, has signature* $(s + r - 1, t)$.

VII. Integral Formulae and Vanishing Theorems for Complex Manifold.

1. Harmonic integrals on open manifolds [15]. Let $W \subset W'$ be a finite complex manifold and $\boldsymbol{E} \to W$ a holomorphic vector bundle. We assume that we have Hermitian metrics in $\boldsymbol{E}$ and on W and that these metrics are *finite* in the sense that they extend to metrics in a neighborhood of W in W'. Denote by $\boldsymbol{A}^q(\boldsymbol{E})$ the space of $\boldsymbol{E}$-valued $(0, q)$ forms which are C^∞ on $\bar{W}$ $(= W \cup \partial W)$. If $\phi, \psi \in \boldsymbol{A}^q(\boldsymbol{E})$, we let

$$(7.1) \qquad (\phi, \psi) = \int_W \langle \phi, \psi \rangle d\mu = \int_W {}^t\phi \wedge h * \bar{\psi}$$

where h is the metric in $\boldsymbol{E}$. Stoke's formula gives

$$(7.2) \qquad (\bar{\partial}\phi, \psi) - (\phi, \vartheta\psi) = \int_{\partial W} {}^t\phi \wedge h * \bar{\psi} \qquad (\phi \in \boldsymbol{A}^{q-1}(\boldsymbol{E}), \psi \in \boldsymbol{A}^q(\boldsymbol{E})).$$

We let $\boldsymbol{D}^q(\boldsymbol{E}) \subset \boldsymbol{A}^q(\boldsymbol{E})$ be the domain of the adjoint ϑ of $\bar{\partial}$ (ϑ is given by (6.20)). By (7.2),

$$(7.3) \qquad \boldsymbol{D}^q(\boldsymbol{E}) = \{\psi \in \boldsymbol{A}^q(\boldsymbol{E}) : {}^t\phi \wedge h * \bar{\psi} \mid \partial W = 0 \text{ for all } \phi \in \boldsymbol{A}^{q-1}(\boldsymbol{E})\}.$$

For $\phi, \psi \in \boldsymbol{D}^q(\boldsymbol{E})$, we set

$$(7.4) \qquad \boldsymbol{D}(\phi, \psi) = (\bar{\partial}\phi, \bar{\partial}\psi) + (\vartheta\phi, \vartheta\psi) + (\phi, \psi);$$

$$(7.5) \qquad \boldsymbol{D}(\phi)^2 = \boldsymbol{D}(\phi, \phi);$$

and we let $\boldsymbol{H}^q(\boldsymbol{E}) = \{\phi \in \boldsymbol{D}^q(\boldsymbol{E}) : \bar{\partial}\phi = \vartheta\phi = 0\}$.

THEOREM [15]. *Assume that, for some* q, *we can find* $c > 0$ *such that, for all* $\phi \in \boldsymbol{D}^q(\boldsymbol{E})$, *we have*

$$(7.6) \qquad \boldsymbol{D}(\phi)^2 \geqq c \int_{\partial W} |\phi|^2.$$

Then $\dim \boldsymbol{H}^q(\boldsymbol{E}) < \infty$ *and there exists an isomorphism*

$$H^q(W,\Omega(\boldsymbol{E}))\cong \boldsymbol{H}^q(\boldsymbol{E}). \tag{7.7}$$

2. Statement of the vanishing theorems. Let $\boldsymbol{E}\to W$ be as in VII.1 and let $\boldsymbol{L}\to W$ be a line bundle in which we have a finite metric. Set $\boldsymbol{E}(\mu)=\boldsymbol{E}\otimes\boldsymbol{L}^\mu$ and let $\Theta(\mu)\in \boldsymbol{A}^{1,1}(\mathrm{Hom}(\boldsymbol{E}(\mu),\boldsymbol{E}(\mu)))$ be the curvature form in $\boldsymbol{E}(\mu)$. Our main result is

THEOREM VI. *There exists $c>0$, which is independent of μ, such that for $\phi\in \boldsymbol{D}^q(\boldsymbol{E}(\mu))$ we have*

$$\boldsymbol{D}(\phi,\phi)\geqq c\{-\int_W \Theta(\mu)^{\#}(\phi,\phi)-\int_{\partial W} L^{\#}(\phi,\phi)\}\qquad (0\leqq q<n) \tag{7.8}$$

$$\boldsymbol{D}(\phi,\phi)\geqq c\{\int_W \Theta(\mu)(\phi,\phi)+\int_{\partial W} L(\phi,\phi)\}\quad (0<q\leqq n). \tag{7.9}$$

(*Remark.* The Hermitian forms $\Theta(\mu)(\phi,\phi)$, $\Theta(\mu)^{\#}(\phi,\phi)$, $L(\phi,\phi)$, and $L^{\#}(\phi,\phi)$ are those discussed in § VI.4.)

From Theorem VI we shall prove

THEOREM 7.1. *Suppose that there exist integers s, t and positive constants μ_0, λ such that, for all $\phi\in \boldsymbol{D}^q(\boldsymbol{E}(\mu))$ with $\mu\geqq\mu_0$, we have*

$$\begin{cases}\Theta(\mu)^{\#}(\phi,\phi)\leqq-\lambda\mu\,|\,\phi\,|^2 \text{ and}\\ L^{\#}(\phi,\phi)\leqq-\lambda\,|\,\phi\,|^2 \quad \text{for } 0\leqq q<s;\ \text{and}\end{cases} \tag{7.10}$$

$$\begin{cases}\Theta(\mu)(\phi,\phi)\geqq\lambda\mu\,|\,\phi\,|^2 \text{ and}\\ L(\phi,\phi)\geqq\lambda\,|\,\phi\,|^2 \quad \text{for } n-t<q\leqq n.\end{cases} \tag{7.11}$$

Then there exists a ν_0 such that

$$H^q(W,\Omega(\boldsymbol{E}(\mu)))=0 \text{ for } \mu\geqq\nu_0,\ 0\leqq q<s,\ n-t<q\leqq n. \tag{7.12}$$

Proof of Theorem 7.1 from Theorem IV. From (7.8) and (7.10) we get, for $\phi\in \boldsymbol{D}^q(\boldsymbol{E})$ $0\leqq q<s$, the estimate

$$\boldsymbol{D}(\phi)^2\geqq c\lambda\mu\int_W |\,\phi\,|^2+c\lambda\int_{\partial W}|\,\phi\,|^2. \tag{7.13}$$

From (7.13) and (7.6), we have $H^q(W,\Omega(\boldsymbol{E}(\mu)))\cong\boldsymbol{H}^q(\boldsymbol{E}(\mu))$ and, for $\phi\in\boldsymbol{H}^q(\boldsymbol{E}(\mu))$, we get $(c\lambda\mu-1)\int_W|\,\phi\,|^2\leqq 0$. Thus, for $\mu>\frac{1}{c\lambda}$, $\boldsymbol{H}^q(\boldsymbol{E}(\mu))=0$ and $H^q(W,\Omega(\boldsymbol{E}(\mu)))=0$ for $\mu>\frac{1}{c\lambda}$, $0\leqq q<s$. From (7.9) and

(7.11) we similarly find $H^q(W, \Omega(\boldsymbol{E}(\mu))) = 0$ for $\mu \geqq \frac{1}{c\lambda}$, $n - t < q \leqq n$. Q. E. D.

The rest of Section VII will be devoted to proving Theorem VI.

3. A special integral formula. We shall prove:

THEOREM 7.2. *For* $\phi \in \boldsymbol{D}^q(\boldsymbol{E})$, we have

$$(\bar\partial\phi, \bar\partial\phi) + (\vartheta\phi, \vartheta\phi) = (\bar\nabla\phi, \bar\nabla\phi) + \int_W \Theta(\phi,\phi) + \int_W \Phi(\phi,\phi) + \int_{\partial W} L(\phi,\phi) + \int_W \Sigma(\phi,\phi) \tag{7.14}$$

where

$$\Sigma(\phi,\phi) = 2Re\langle[\bar\partial, T^\#]\phi, \phi\rangle - \langle[T, T^\#]\phi, \phi\rangle. \tag{7.15}$$

Proof. This is a routine calculation and we shall only give the relevant formulas. First we have

$$(\bar\partial - T)(\phi) = \frac{(-1)^q}{(q+1)!}\sum_J (\phi_{\bar J_1 \cdots \bar J_q \mid \bar J_{q+1}} - \phi_{\bar J_{q+1}\bar J_2 \cdots \bar J_q \mid \bar J_1} - \cdots - \phi_{\bar J_1 \cdots \bar J_{q-1}\bar J_{q+1} \mid \bar J_q})\bar\omega^J \tag{7.16}$$

and

$$(\vartheta - T^\#)\phi) = \frac{1}{(q-1)!}\sum_{I,j} \phi_{\bar i_1 \cdots \bar i_{q-1}\bar j \mid \bar J_q}\bar\omega^I. \tag{7.17}$$

Set $\xi = \frac{1}{(q-1)!}\sum \phi_{\bar J_1 \cdots \bar J_{q-1}\bar J^\rho \mid j}\bar\phi_{\bar J_1 \cdots \bar J_{q-1}\bar k^\rho}\omega^k$. Then ξ is a globally defined $(1,0)$ form and

$$-\delta\xi = \langle(\vartheta - T^\#)\phi, (\vartheta - T^\#)\phi\rangle + \frac{1}{(q-1)!}\sum \phi_{\bar J_1 \cdots \bar J_{q-1}\bar J^\rho \mid j \mid \bar k}\bar\phi_{\bar J_1 \cdots \bar J_{q-1}\bar k^\rho}. \tag{7.18}$$

By (7.18) and Green's theorem we have

$$((\vartheta - T^\#)\phi, (\vartheta - T^\#)\phi) = -\frac{1}{(q-1)!}\int_W \phi_{\bar J_1 \cdots \bar J_{q-1}\bar J^\rho \mid j \mid \bar k}\bar\phi_{\bar J_1 \cdots \bar J_{q-1}\bar k} + \int_{\partial W} *\bar\xi. \tag{7.19}$$

If, in a neighborhood of ∂W, we choose $\omega^1, \cdots, \omega^n$ as in VI.1, then the boundary conditions (7.3) become

$$\phi_{I^\rho} \mid \partial W = 0 \text{ whenever } n \in I. \tag{7.20}$$

Thus, on ∂W, $*\xi = \frac{1}{(q-1)!} \Sigma (\phi_{\bar{J}_1 \cdots \bar{J}_{q-1}\bar{J}^\rho} \mid {}_j\phi_{\bar{J}_1 \cdots \bar{J}_{q-1}n^\rho}) *\omega^n = 0$. The Ricci identity gives

$$\begin{aligned} \frac{1}{(q-1)!}\{\Sigma \phi_{\bar{J}_1 \cdots \bar{J}_{q-1}\bar{J}} \mid \bar{k} \mid {}_j{}^\rho \bar{\phi}_{\bar{J}_1 \cdots \bar{J}_{q-1}\bar{k}^\rho} - \phi_{\bar{J}_1 \cdots \bar{J}_{q-1}\bar{J}} \mid j \mid \bar{k}^\rho \bar{\phi}_{\bar{J}_1 \cdots \bar{J}_{q-1}\bar{k}^\rho} \\ = \Theta(\phi, \phi) + \Phi(\phi, \phi). \end{aligned} \tag{7.21}$$

Combining (7.19) and (7.21) gives

$$\begin{aligned} ((\vartheta - T^\#)\phi, (\vartheta - T^\#)\phi) = \int_W \Theta(\phi, \phi) + \int_W \Phi(\phi, \phi) \\ - \frac{1}{(q-1)!} \int_W \{\Sigma \phi_{\bar{J}_1 \cdots \bar{J}_{q-1}\bar{J}^\rho} \mid \bar{k} \mid {}_j\bar{\phi}_{\bar{J}_1 \cdots \bar{J}_{q-1}\bar{k}^\rho}\}. \end{aligned} \tag{7.22}$$

Now let $\eta = \frac{1}{(q-1)!} \phi_{\bar{J}_1 \cdots \bar{J}_{q-1}\bar{J}} \mid \bar{k}^\rho \bar{\phi}_{\bar{J}_1 \cdots \bar{J}_{q-1}\bar{k}^\rho \bar{\omega}^j}$. Then η is a $(0,1)$ form on W and

$$\begin{aligned} \delta\eta = -\frac{1}{(q-1)!}\{\Sigma \phi_{\bar{J}_1 \cdots \bar{J}_{q-1}\bar{J}} \mid \bar{k} \mid {}_j{}^\rho \bar{\phi}_{\bar{J}_1 \cdots \bar{J}_{q-1}\bar{k}^\rho} \\ + \phi_{\bar{J}_1 \cdots \bar{J}_{q-1}\bar{J}} \mid \bar{k}^\rho \bar{\phi}_{\bar{J}_1 \cdots \bar{J}_{q-1}\bar{k}} \mid \bar{J}^\rho\}. \end{aligned} \tag{7.23}$$

From the algebraic identity (using (7.16))

$$\begin{aligned} -\frac{1}{(q-1)!} \Sigma \phi_{\bar{J}_1 \cdots \bar{J}_{q-1}\bar{J}} \mid \bar{k}^\rho \bar{\phi}_{\bar{J}_1 \cdots \bar{J}_{q-1}\bar{k}} \mid \bar{J}^\rho = -\langle (\bar{\partial} - T)\phi, (\bar{\partial} - T)\phi \rangle \\ + \langle \bar{\nabla}\phi, \bar{\nabla}\phi \rangle \end{aligned}$$

and (7.22), (7.23), we get

$$\begin{aligned} ((\vartheta - T^\#)\phi, (\vartheta - T^\#)\phi + ((\bar{\partial} - T)\phi, (\bar{\partial} - T)\phi) \\ = (\bar{\nabla}\phi, \bar{\nabla}\phi) + \int_W \Theta(\phi, \phi) + \int_W \Phi(\phi, \phi) + \int_{\partial W} *\eta. \end{aligned} \tag{7.24}$$

Thus, in order to prove (7.14), we must show

$$*\eta \mid \partial W = L(\phi, \phi)\, d\sigma \qquad (d\sigma = \text{area element on } \partial W). \tag{7.25}$$

Now

$$*\eta \mid \partial W = \frac{1}{(q-1)!} \sum_{k<n} \phi_{\bar{J}_1 \cdots \bar{J}_{q-1}\bar{n}} \mid \bar{k}^\rho \bar{\phi}_{\bar{J}_1 \cdots \bar{J}_{q-1}\bar{k}^\rho} *\omega^n.$$

By (6.6),

$$\phi_{\bar{j}_1\cdots\bar{j}_{q-1}\bar{n}} \mid \bar{k}^\rho = e_{\bar{k}}\cdot\phi_{\bar{j}_1\cdots\bar{j}_{q-1}\bar{n}^\rho} - \sum \bar{\Lambda}_{nk^j}\phi_{\bar{j}_1\cdots\bar{j}_{q-1}\bar{j}^\rho}$$

$$-\sum_{s<q}\bar{\Lambda}_{j_s k^j}\phi_{\bar{j}_1\cdots(\bar{j})_s\cdots\bar{j}_{q-1}\bar{n}}.$$

Since $e_1,\cdots,e_{n-1}$ are tangent to ∂W, and $\phi_{\bar{j}_1\cdots\bar{j}_{q-1}\bar{n}^\rho} \mid \partial W = 0$, we have an ∂W the identity

$$\sum \phi_{\bar{j}_1\cdots\bar{j}_{q-1}\bar{n}} \mid \bar{k}^\rho\bar{\phi}_{\bar{j}_1\cdots\bar{j}_{q-1}\bar{k}^\rho} = -\sum_{j,k<n}\Lambda_{nk^j}\phi_{\bar{j}_1\cdots\bar{j}_{q-1}\bar{j}^\rho}\bar{\phi}_{\bar{j}_1\cdots\bar{j}_{q-1}\bar{k}^\rho}.$$

From (6.5) and (6.14) it follows that $*\eta \mid \partial W = L(\phi,\phi)\,d\sigma$. Q.E.D.

Proof of (7.9) *in Theorem VI.* First of all, we may find $\alpha > 0$ such that, for all $\epsilon > 0$, we have

$$\mid 2Re\int_W \langle[T^\#,\bar{\partial}]\phi,\phi\rangle \mid \leqq \alpha\epsilon(\bar{\nabla}\phi,\bar{\nabla}\phi) + \frac{\alpha}{\epsilon}(\phi,\phi)$$

and

$$\mid\int_W \Phi(\phi,\phi)\mid + \mid\int_W \langle[T,T^\#]\phi,\phi\rangle \mid \leqq \alpha(\phi,\phi).$$

Combining these inequalities with (7.14), (7.15) gives

$$(\bar{\partial}\phi,\bar{\partial}\phi) + (\vartheta\phi,\vartheta\phi) \geqq (1-\alpha\epsilon)(\bar{\nabla}\phi,\bar{\nabla}\phi) + \int_W \Theta(\phi,\phi)$$

$$+\int_{\partial W} L(\phi,\phi) - (\frac{\alpha}{\epsilon}+\alpha)(\phi,\phi).$$

If we choose $\epsilon < 1/\alpha$ and $c < \dfrac{\epsilon}{\alpha+\alpha\epsilon}$, (7.9) is satisfied. Indeed, it is clear from our computations that the constant c in (7.9) may be chosen to be independent of μ.

4. Some remarks and applications.

(i) Compact manifolds. Let X be a compact Hermitian manifold and $\boldsymbol{E}\to X$ a holomorphic vector bundle in which we have an Hermitian metric.

PROPOSITION 7.1. *If, for some q, the Hermitian form*

$$\Theta(\phi,\phi) + \Phi(\phi,\phi) - \mid T\phi\mid^2 - \mid T^\#\phi\mid^2 \tag{7.26}$$

is positive definite on all of X ($\phi \in A^q(\boldsymbol{E})$), then $H^q(X,\Omega(\boldsymbol{E})) = 0$.

Proof. $H^q(X, \Omega(\boldsymbol{E})) \cong \boldsymbol{H}^q(\boldsymbol{E})$ and, for $\phi \in \boldsymbol{H}^q(\boldsymbol{E})$,

$$\int_X \{\Theta(\phi, \phi) + \Phi(\phi, \phi) - |T\phi|^2 - |T^{\#}\phi|^2\} \leqq 0$$

by Theorem 6.2. Thus, if $\phi \in \boldsymbol{H}^q(\boldsymbol{E})$ and if (7.26) is positive definite, we must have $\phi \equiv 0$.

Now $\Theta(\phi, \phi) + \Phi(\phi, \phi) = \Theta_{\boldsymbol{E}\otimes\boldsymbol{K}^*}(\phi, \phi)$ where $\boldsymbol{K}$ is the canonical bundle. Thus (7.26) is

$$\Theta_{\boldsymbol{E}\otimes\boldsymbol{K}^*}(\phi, \phi) - \{|T\phi|^2 + |T^{\#}\phi|^2\} \tag{7.27}$$

((7.27) shows clearly the effect on the curvature of changing the metric and perhaps adding torsion.) From (7.27) and Proposition 7.1, we get the following:

a. If X is Kähler and $\boldsymbol{E} \otimes \boldsymbol{K}^*$ is positive, then $H^q(X, \Omega(\boldsymbol{E})) = 0$ for $q > 0$ (Kodaira vanishing theorem).

b. If X is Kähler and if $\boldsymbol{E}$ has signature $(0, t)$, then $H^q(X, \Omega(\boldsymbol{E})) = 0$ for $q < t$ (for $t = n$, this reduces to *a*).

c. If $\boldsymbol{E} \to X$ has signature (s, t), then, for $\mu \geqq \mu_0$,

$$H^q(X, \Omega(\boldsymbol{E}^\mu)) = 0 \text{ for } 0 \leqq q < t,\ n - s < q \leqq n. \tag{7.28}$$

(This is the vanishing theorem of [3], § 23.)

Proof. In case $\boldsymbol{E}$ is a line bundle, we use Proposition 6.4 and choose a metric on X such that $\Theta(\phi, \phi)$ is positive definite on $A^q(\boldsymbol{E})$ for $n - s < q \leqq n$. Then, for μ sufficiently large,

$$\Theta_{\boldsymbol{E}^\mu\otimes\boldsymbol{K}^*}(\phi, \phi) - \{|T\phi|^2 + |T^{\#}\phi|^2\}$$

will be positive definite for $n - s < q \leqq n$ and we may use Proposition 7.1 plus Serre duality to prove *c.* for line bundles. The general case follows from the case of line bundles, Proposition 6.5, and the fact that

$$H^q(X, \Omega(\boldsymbol{E}^\mu) \cong H^q(P(\boldsymbol{E}^*), \Omega(\boldsymbol{L}^\mu)).$$

(ii) Open manifolds. Let $W \subset W'$ be a finite complex manifold. We say that W is *t-complete* if there exists a real function r on W' such that $\{r < 0\}$ defines $W \subset W'$ and $\partial\bar{\partial}r$ has everywhere t positive eigenvalues.

PROPOSITION 7.2. *If W is t-complete and $\boldsymbol{E} \to W$ is any holomorphic vector bundle, then*

$$H^q(W, \Omega(\boldsymbol{E})) = 0 \text{ for } q > n - t. \tag{7.29}$$

Proof. If $\boldsymbol{L} \to W$ is the trivial line bundle, then e^{-r} is a metric in $\boldsymbol{L}$ whose curvature is $\bar\partial\partial \log e^{-r} = \bar\partial\partial r$. Thus (7.11) will hold for large μ, and so $0 = H^q(W, \Omega(\boldsymbol{E}(\mu))) = H^q(W, \Omega(\boldsymbol{E}))$ for $n - t < q \leqq n$.

Remark. For $t = n$ this is closely related to Cartan's Theorem B (for locally free sheaves). A more general form of (7.29) has been given by Andreotti-Vesentini [2].

5. An integral formula for functions on complex manifolds. The integral formula (7.14) was based entirely on a formula in Riemannian geometry (Bochner-Yano [4]) and, in fact, the precise analogue of (7.14) (boundary values included and with the second fundamental form of $\partial W \subset W'$ replacing the Levi form) has been in existence for some years (cf. [11]). The next formula we shall give has no Riemannian analogue and is based on the existence of an $(n-1, n-1)$ form Ω on W which has the property

$$(\partial - \bar\partial)\Omega \mid \partial W = 2(\operatorname{Trace} L)\sigma. \tag{7.30}$$

For example, in $\boldsymbol{C}^n$ with coordinates $z = (z^1, \cdots, z^n)$,

$$\Omega = (-1)^{\frac{n(n+1)}{2}} \left(\frac{\sqrt{-1}}{2}\right)^n \sum \{ \frac{(-1)^{i+j} z^i \bar z^j}{\sum |z^k|^2} - \delta_j{}^i \}$$
$$(dz^1 \cdots \hat{dz^i} \cdots dz^n d\bar z^1 \cdots \hat{d\bar z^j} \cdots d\bar z^n).$$

(we shall omit the exterior product symbols in this paragraph). For simplicity, we shall assume that the Hermitian metric on W' is without torsion. In the general case, the terms with torsion can be absorbed in estimates as in the proof of (7.9) given in § VII. 3 above. Let $\gamma = \frac{\sqrt{-1}}{2} \sum \omega^i \wedge \bar\omega^i$ be the fundamental form of the Kähler metric and consider real non-negative functions τ on W' which are equal to one in a neighborhood of ∂W. For each such τ, we define the $(n-1, n-1)$ form Ω_τ by

$$\Omega_\tau = \gamma^{n-1} - \tau * (\partial r \wedge \bar\partial r) \tag{7.31}$$

THEOREM 7.3. *For g a C^∞ function on $\bar W$, we have*

$$\int_W g\, \bar\partial\partial \Omega_\tau + \int_W (\partial\bar\partial g) \wedge \Omega_\tau = \int_{\partial W} g (\operatorname{Trace} L)\sigma. \tag{7.32}$$

We begin with some preliminary computations. If

$$\lambda = (-1)^{\frac{n(n-1)}{2}} \left(\frac{\sqrt{-1}}{2}\right)^n,$$

then the volume on W is given by $\Psi = \lambda\omega^1 \wedge \cdots \wedge \omega^n \wedge \bar{\omega}^1 \wedge \cdots \wedge \bar{\omega}^n$. We let

$$\Psi_k = (-1)^{k-1}\lambda(\omega^1 \cdots \hat{\omega}^k \cdots \omega^n \cdot \bar{\omega}^1 \cdots \bar{\omega}^n)$$

and

$$\Psi_{\bar{k}} = (-1)^{n+k-1}\lambda(\omega^1 \cdots \omega^n \cdot \bar{\omega}^1 \cdots \hat{\bar{\omega}}^k \cdots \bar{\omega}^n).$$

Then $\Psi_k = *\bar{\omega}^k$ and $\Psi_{\bar{k}} = *\omega^k$. Also we set

$$\Psi_{i\bar{j}} = (-1)^{n+i+j}\lambda(\omega^1 \cdots \hat{\omega}^i \cdots \omega^n \bar{\omega}^1 \cdots \hat{\bar{\omega}}^j \cdots \bar{\omega}^n).$$

Since $|dr|^2 = |\omega^n|^2 + |\bar{\omega}^n|^2 = 2$, $\sigma = (\frac{1}{2}*dr) \mid \partial W$. Thus

$$\sigma = \tfrac{1}{2}(\Psi_{\bar{n}} + \Psi_n) \mid \partial W = \Psi_{\bar{n}} \mid \partial W$$

since $(\omega^n + \bar{\omega}^n) \mid \partial W = 0$. Using the fact that the metric has no torsion, a computation based on (6.2), (6.4), (6.5) gives

$$d\Psi_j = (\sum \bar{\Lambda}_{ii}{}^j)\Psi \tag{7.33}$$

$$d\Psi_{\bar{j}} = (\sum \Lambda_{i\bar{i}}{}^j)\Psi. \tag{7.34}$$

Since $\gamma^{n-1} = (-1)^{n-1}(\sum_{k=1}^{n} \Psi_{k\bar{k}})$ we find that, in a neighborhood of ∂W,

$$\Omega_\tau = (-1)^{n-1}(\sum_{k<n} \Psi_{k\bar{k}}). \tag{7.35}$$

LEMMA 7.1. $(\partial - \bar{\partial})\Omega_\tau \mid \partial W = +2(\text{Trace } L)\sigma$.

Proof. Since $\omega^n \wedge \bar{\omega}^n \mid \partial W = 0$, we shall neglect terms containing $\omega^n \wedge \bar{\omega}^n$. Then

$$\begin{aligned}(-1)^{n-1}\partial\Psi_{k\bar{k}} &= (-1)^n(-1)^{2n-3}\lambda(\omega^1 \cdots \hat{\omega}^k \cdots \omega^n\bar{\omega}^1 \cdots \hat{\bar{\omega}}^k \cdots \partial\bar{\omega}^n) \\ &= (-1)^{n+1}\lambda\bar{\Lambda}_{k\bar{k}}{}^n(\omega^1 \cdots \hat{\omega}^k \cdots \omega^n\bar{\omega}^1 \cdots \hat{\bar{\omega}}^k \cdots \bar{\omega}^{n-1}\omega^k\bar{\omega}^k) \\ &= \lambda\Lambda_{k\bar{k}}{}^n(\omega^1 \cdots \omega^n\bar{\omega}^1 \cdots \bar{\omega}^{n-1}) = \Lambda_{k\bar{k}}{}^n\Psi_{\bar{n}}.\end{aligned}$$

Similarly, $-(1)^{n-1}\bar{\partial}\Psi_{k\bar{k}}{}^n = \Lambda_{k\bar{k}}{}^n\Psi_{\bar{n}}$. Thus

$$(\partial - \bar{\partial})\Omega \mid \partial W = 2(\text{Trace } L)\Psi_{\bar{n}} \mid \partial W = 2(\text{Trace } L)\sigma.$$

We can now complete the proof of (7.32). Lemma 7.1 and Stokes theorem give

$$+\int_{\partial W} g(\mathrm{Trace}\, L)\sigma = \tfrac{1}{2}\int_W d\{g(\partial - \bar\partial)\Omega_\tau\}. \tag{7.36}$$

Now

$$\begin{aligned} d\{g(\partial-\bar\partial)\Omega_\tau\} &= 2g\bar\partial\partial\Omega_\tau + (dg)\wedge(\partial-\bar\partial)\Omega_\tau \\ &= 2g\bar\partial\partial\Omega_\tau + d\{(\partial g)\wedge\Omega_\tau\} - d\{(\bar\partial g)\wedge\Omega_\tau + 2\partial\bar\partial g\wedge\Omega_\tau\}. \end{aligned}$$

By (7.36) and Stokes theorem again (since $\Omega_\tau \mid \partial W = 0$), we get

$$+\int_{\partial W} g(\mathrm{Trace}\, L) = \int_W g\bar\partial\partial\Omega_\tau + \int_W \partial\bar\partial g\wedge\Omega_\tau. \quad \text{Q. E. D.}$$

We now define operators $\bar\partial_\tau$, ∂_τ, and Δ_τ by

$$\bar\partial_\tau f = \bar\partial f - \tau\langle\bar\partial f, \bar\partial r\rangle\bar\partial r, \qquad \partial_\tau f = \partial f - \tau\langle\partial f, \partial r\rangle\partial r,$$

$$\Delta_\tau f = \langle\partial\bar\partial f, \gamma - \tau\partial r\wedge\bar\partial r\rangle.$$

At those points where $\tau = 0$, $\bar\partial_\tau f = \bar\partial f$, $\partial_\tau f = \partial f$, and $\Delta_\tau f = \Delta f$. From (7.32) we get, upon setting $g = |f|^2$ and $\Phi_\tau = \bar\partial\partial\Omega_\tau$,

$$\begin{aligned} &\int_W |f|^2\Phi_\tau + \int_W |\partial_\tau f|^2 + 2Re\int_W \Delta_\tau f\cdot\bar f \\ &\qquad = \int_W |\bar\partial_\tau f|^2 + \int_{\partial W} |f|^2(\mathrm{Trace}\, L). \end{aligned} \tag{7.37}$$

In particular, if f is holomorphic, then

$$\int_W |f|^2\Phi_\tau + \int_W |\partial_\tau f|^2 = \int_{\partial W} |f|^2(\mathrm{Trace}\, L) \tag{7.38}$$

which is, for us, an important formula. For example, suppose that L has one negative eigenvalue and choose a metric so that $\mathrm{Trace}\, L < 0$. Then we have $\int_W |f|^2\Phi_\tau + \int_W |\partial_\tau f|^2 \leqq 0$. One may prove that this inequality implies that f is constant.

Example. We shall give an integral formula in $\boldsymbol{C}^n$ which is similar to, but does not quite follow from, (7.38). Let $B =$ unit ball in $\boldsymbol{C}^n$, $S = \partial B$. Let

$$\omega = \lambda\Big(\sum_{k=1}^{n}(-1)^{n+1}(\cdots d\hat{z}^k\cdots d\hat{\bar{z}}^k\cdots)$$

(here $(\cdots d\hat{z}^k\cdots d\hat{\bar{z}}^k\cdots)$

$$= (dz^1\wedge\cdots\wedge d\hat{z}^k\wedge\cdots\wedge d\bar{z}^1\wedge\cdots\wedge d\hat{\bar{z}}^k\wedge\cdots\wedge d\bar{z}^n))$$

and set

$$\Omega = \omega + \lambda\sum(-1)^{i+n+j}z^i\bar{z}^j(\cdots d\hat{z}^i\cdots d\hat{\bar{z}}^j\cdots).$$

Then $(\partial-\bar{\partial})\Omega = \lambda\{\sum(-1)^{i-1}z^i(\cdots d\hat{z}^i\cdots) + \sum(-1)^{n+j-1}\bar{z}^j(\cdots d\hat{\bar{z}}^j\cdots)\}$ so that $d(\partial-\bar{\partial})\Omega = 2n\Psi$ where Ψ is the volume element. We claim that $\Omega \mid S = 0$. Since Ω is invariant under the unitary group acting on S, it will suffice to show that $(\Omega \mid S)(x_0) = 0$ where $x_0 = (1, 0, \cdots, 0)$. On S, $\sum \bar{z}^i dz^i + \sum z^j d\bar{z}^j = 0$ so that, at x_0 $dz^1 + d\bar{z}^1 = 0$. But then $(\omega \mid S)(x_0) = \lambda(-1)^{n+1}(d\hat{z}^1\cdots d\hat{\bar{z}}^1\cdots)$ and so $(\Omega \mid S)(x_0) = 0$. Similarly we find that $(\partial-\bar{\partial})\Omega \mid S = 2\sigma$ where σ is the area element on S. Using Stoke's theorem twice as in the proof of Theorem 7.3, we find the formula

$$\int_B \{\sum\left|\frac{\partial f}{\partial z^k}\right|^2 - \left|\sum z^k\frac{\partial f}{\partial z^k}\right|^2\} + n\int_B |f|^2 = \int_S |f|^2. \tag{7.39}$$

For example, on the unit disc writing $f(z) = \sum a_n z^n$, (7.39) reduces to the identity

$$2\pi\{\sum\frac{n^2|a_n|^2}{2n} - \sum\frac{n^2|a_n|^2}{2n+2} + \sum\frac{|a_n|^2}{2n+2}\} = \pi\{\sum|a_n|^2\}.$$

6. Proof of Theorem VI. Let $\boldsymbol{F}\to W$ be a holomorphic vector bundle and let $\phi\in\boldsymbol{D}^q(\boldsymbol{F})$. In the same way as we defined $\bar{\partial}_\tau f$ and $\partial_\tau f$ above, we may now define $\bar{\nabla}^\tau\phi$ and $\bar{\nabla}^\tau\phi$. Writing $\bar{\nabla}\phi = \bar{\nabla}^\tau\phi + \bar{\nabla}^\eta\phi$ ($\bar{\nabla}^\eta$ = covariant differential in the normal direction), we have

$$|\bar{\nabla}\phi|^2 = |\bar{\nabla}^\tau\phi|^2 + |\bar{\nabla}^\eta\phi|^2. \tag{7.40}$$

Also, we define Trace $\Theta^\tau(\phi,\phi) = *\{\frac{1}{q!}\sum\Theta_{\sigma i\bar{j}\rho}\phi_{\bar{I}}{}^\sigma\bar{\phi}_{\bar{I}}{}^\rho\omega^i\wedge\bar{\omega}^j\wedge\Omega_\tau\}$. At those points where $\tau = 0$, Trace $\Theta^\tau(\phi,\phi) =$ Trace $\Theta(\phi,\phi)$ as given in (6.37). Our proof of (7.8) is based on the following:

THEOREM 7.4. *If the metric is without torsion, then*

$$(\bar{\partial}\phi,\bar{\partial}\phi) + (\vartheta\phi,\vartheta\phi) \geqq (\bar{\nabla}^\eta\phi,\bar{\nabla}^\eta\phi) + (\nabla^\tau\phi,\nabla^\tau\phi)$$

$$-\int_W \Theta^{\#}(\phi,\phi) - \int_{\partial W} L^{\#}(\phi,\phi) + \int_W \Xi(\phi,\phi) \tag{7.41}$$

where

$$\begin{aligned}\Xi(\phi,\phi) &= \Phi(\phi,\phi) + |\phi|^2 |\Phi_\tau|^2 \\ &\quad + \{\mathrm{Trace}\,\Theta(\phi,\phi) - \mathrm{Trace}\,\Theta^\tau(\phi,\phi)\}\end{aligned} \tag{7.42}$$

Proof. We shall prove (7.41) by combining (7.14) and (7.32) when $g = |\phi|^2$. Now

$$\begin{aligned}\partial\bar\partial |\phi|^2 = \frac{1}{q!}\{&2Re \sum \phi_{\bar I^\rho \mid \bar k \mid j} \bar\phi_{\bar I^\rho} \\ &+ \sum \phi_{\bar I \mid j^\rho} \bar\phi_{\bar I \mid k^\rho} - \sum \phi_{\bar I \mid \bar k^\rho} \bar\phi_{\bar I \mid \bar j k} \\ &- \sum \Theta_{\sigma j k^\rho} \phi_{\bar I^\sigma} \bar\phi_{\bar I^\rho}\}\omega^j \wedge \omega^k.\end{aligned}$$

Thus

$$\begin{aligned}\partial\bar\partial |\phi|^2 \wedge \Omega_\tau = \{&|\nabla^\tau \phi|^2 - |\bar\nabla^\tau \phi|^2 \\ &- \mathrm{Trace}\,\Theta^\tau(\phi,\phi) + 2Re\langle \nabla_k{}^\tau \bar\nabla_k{}^\tau \phi, \phi\rangle\}\Psi.\end{aligned} \tag{7.43}$$

Because the operator ∇^τ involves no normal derivatives, we may integrate by parts to find

$$\int_W \{2Re \sum \langle \nabla_k{}^\tau \bar\nabla_k{}^\tau \phi, \phi\rangle\} \leqq 0. \tag{7.44}$$

From (7.43) and (7.44) we get

$$\begin{aligned}\int_W \partial\bar\partial |\phi|^2 \wedge \Omega_\tau \geqq \int_W |\nabla^\tau \phi|^2 - \int_W |\bar\nabla^\tau \phi|^2 \\ - \int_W \mathrm{Trace}\,\Theta^\tau(\phi,\phi).\end{aligned} \tag{7.45}$$

We now use (7.32) in the form

$$\int_W |\phi|^2 \Phi_\tau - \int_{\partial W} (\mathrm{Trace}\, L)|\phi|^2 = -\int_W \partial\bar\partial |\phi|^2 \wedge \Omega_\tau. \tag{7.46}$$

Combining (7.45) and (7.46) gives

$$\begin{aligned}\int_W |\bar\nabla^\tau \phi|^2 \geqq \int_W |\nabla^\tau \phi|^2 - \int_W \mathrm{Trace}\,\Theta^\tau(\phi,\phi) \\ - \int_{\partial W} (\mathrm{Trace}\, L)|\phi|^2 + \int_W |\phi|^2 \Phi_\tau.\end{aligned} \tag{7.47}$$

$$(\bar\partial\phi, \bar\partial\phi) + (\vartheta\phi, \vartheta\phi) = (\bar\nabla^\tau\phi, \bar\nabla^\tau\phi) + (\bar\nabla^\eta\phi, \bar\nabla^\eta\phi) + \int_W \Theta(\phi,\phi) + \int_{\partial W} L(\phi,\phi). \tag{7.48}$$

Then (7.48) and (7.47) taken together give (7.41). Q.E.D.

Now in case the metric has torsion, we must, at least, add the term

$2Re\int_W \langle[\bar\partial, T^\#]\phi, \phi\rangle - \int_W \langle[T, T^\#]\phi, \phi\rangle$ to the right hand side of (7.41).

Since $\phi \in \boldsymbol{D}^q(\boldsymbol{F})$

$$\int_W \langle[\bar\partial, T^\#]\phi, \phi\rangle = \int_W \langle[\bar\partial_\eta, T^\#]\phi, \phi\rangle + \int_W \langle[\vartheta_\tau, T]\phi, \phi\rangle$$

and we then get an estimate

$$|\, 2Re\int_W \langle[\bar\partial, T^\#]\phi, \phi\rangle\,| \leqq \epsilon\beta(\bar\nabla^\eta\phi, \bar\nabla^\eta\phi) + \epsilon\beta(\nabla^\tau\phi, \nabla^\tau\phi) + \frac{\beta}{\epsilon}(\phi,\phi).$$

Thus any such terms involving the torsion can be absorbed in the right hand side of (7.41) and we find an estimate

$$(\bar\partial\phi, \bar\partial\phi) + (\vartheta\phi, \vartheta\phi) \geqq -\int_W \Theta^\#(\phi,\phi) - \int_{\partial W} L^\#(\phi,\phi) - \alpha\int_W |\,\phi\,|^2 + \int_W \{\mathrm{Trace}\,\Theta(\phi,\phi) - \mathrm{Trace}\,\Theta^\tau(\phi,\phi)\} \tag{7.49}$$

where $\alpha > 0$ is a large constant not depending on the bundle $\boldsymbol{F}$. Consequently, in order to prove (7.8) and with it Theorem VI, it will suffice to show that, with an appropriate choice of metrics in $\boldsymbol{E}$ and $\boldsymbol{L}$, we may have

$$\mathrm{Trace}\,\Theta(\mu)(\phi,\phi) - \mathrm{Trace}\,\Theta(\mu)^\tau(\phi,\phi) \geqq 0 \tag{7.50}$$

for all ϕ. Now $\mathrm{Trace}\,\Theta(\mu)(\phi,\phi) - \mathrm{Trace}\,\Theta^\tau(\phi,\phi) =$

$$\frac{\tau}{q!}\sum \Theta_{\sigma n\bar n^\rho}(\mu)\phi_{\bar I^\sigma}\bar\phi_{\bar I^\rho}. \tag{7.51}$$

On the other hand, for any function ψ on W, we may modify the metric in $\boldsymbol{L}$ by a factor $e^{-\psi}$ and this adds to the curvature a term which is essentially $\partial\bar\partial\psi$. From (7.51), in order to have (7.50) satisfied we must choose ψ with $\psi_{|n|n} = \frac{1}{\tau}\langle\partial\bar\partial\psi, \partial r \wedge \bar\partial r\rangle > 0$. It is easy to check that any function ψ which is equal to r in a small open neighborhood of $\partial W \subset W'$ will do.

(We are doing nothing more here than using the principle that for any open complex manifold $W \subset W'$ of dimension $n > 1$, we may choose a function r such that $r < 0$ defines $W \subset W'$ and such that $\partial\bar{\partial}\psi$ has a positive normal part. This is closely related to the fact that, for any bundle $\boldsymbol{F} \to W$, $H^n(W, \Omega(\boldsymbol{F})) = 0$.)

VIII. Properties of Tubular Neighborhoods and the Proof of Theorem III. The main result of this section is:

THEOREM VII. *Let $X \subset W$ be a germ of embedding whose normal bundle $\boldsymbol{N}$ has signature (s, t) $(s + t = n = \dim X)$. Suppose furthermore that X is a hyperpsurface and set $\boldsymbol{L} = [X]$ so that $\Omega(\boldsymbol{L}^*) = \mathfrak{I}$, the ideal sheaf of X. Then there exists an embedding $W \subset W'$, Hermitian metrics on W' and in $\boldsymbol{L} \to W'$, and a function r on W' with $W = \{r < 0\}$ such that:*

(i) *$\Theta^{\#}(\phi, \phi)$ is negative definite for $\phi \in \boldsymbol{A}^q(\boldsymbol{L})$ and $L^{\#}(\phi, \phi)$ is negative definite for $\phi \in \boldsymbol{D}^q(\boldsymbol{L}^*)$, $0 \leqq q < s$;*

(ii) *$\Theta(\phi, \phi)$ is positive definite for $\phi \in \boldsymbol{A}^q(\boldsymbol{L}^*)$ and $L(\phi, \phi)$ is positive definite on $\boldsymbol{D}^q(\boldsymbol{L}^*)$, $n - t < q \leqq n + 1$.*

Proof of Theorem III. By Theorem VII we see that (7.10) and (7.11) are satisfied and so, by Theorem 7.1, we have

$$H^q(W, \Omega(\boldsymbol{E} \otimes \boldsymbol{L}^{*\mu})) = 0 \text{ for } \mu \geqq \mu_0(\boldsymbol{E}), \quad 0 \leqq q < s, n - t < q \leqq n + 1. \tag{8.1}$$

Because of Proposition 5.1, (8.1) gives Theorem III.

1. Metrics in line bundles. We retain the notation and assumptions of Theorem VII.

PROPOSITION 8.1. *For $W \supset X$ sufficiently small, there exists in $\boldsymbol{L}^* \to W$ an Hermitian metric whose curvature has signature $(t, s + 1)$.*

Proof. Relative to a suitable coordinate covering $\{U_i\}$ of W, $X \cap U_i$ is given by a local equation $f_i = 0$. The transition functions of $\boldsymbol{L}^*$ are $g_{ij} = f_j/f_i$ and a metric in $\boldsymbol{L}^*$ is given by a family of positive functions ψ_i in U_i which satisfy $\psi_j = |g_{ij}|^2 \psi_i$ in $U_i \cap U_j$. We may choose ψ such that $\psi \mid X$ is the metric ϕ in $\boldsymbol{N}^* = \boldsymbol{L}^* \mid X$ whose curvature has signature (t, s). Thus $\psi_i \mid U_i = \phi_i + \rho\tau_i$ where ρ is the distance from X.

We fix $x_0 \in X \cap U_i$ and change holomorphic coordinates in U_i so that

$(d\psi_i)(x_0) = 0$. Since $(d\psi_i)(x_0) = (d\phi_i + \tau_i d\rho)(x_0)$, and since $d\phi_i$ is tangent to X whereas $d\rho$ is normal, it follows that $(d\phi_i)(x_0) = 0 = \tau_i(x_0)$. At x_0, $\bar\partial\partial\psi_i = \bar\partial\partial\phi_i + (\bar\partial\rho \wedge \partial\tau_i - \partial\rho \wedge \bar\partial\tau_i)$. Since $X \subset W$ is a complex submanifold, $\bar\partial\rho$ and $\partial\rho$ are both zero on $\boldsymbol{T}_{x_0}(X)$. Thus $\bar\partial\partial\psi_i \mid \boldsymbol{T}_{x_0}(X) = \bar\partial\partial\phi_i \mid \boldsymbol{T}_{x_0}(X)$ and this implies that, for W sufficiently small, the curvature form for ψ has signature at least (t, s).

Now we observe that, for any $m \geqq 0$, a new metric in $\boldsymbol{L}^*$ is given by functions $\psi_i(m) = \psi_i + m \mid f_i \mid^2$. Furthermore, $\psi_i(m) \mid X \cap U_i = \phi_i$ and, at x_0 chosen as above, $\bar\partial\partial\psi_i(m) = \bar\partial\partial\psi_i - m df_i \wedge \overline{df_i}$. We claim that, for m sufficiently large, $\bar\partial\partial\psi_i$ will have signature $(t, s+1)$.

To prove this, we choose holomorphic coordinates $z_1, \cdots, z_n$; w for W around x_0 such that: (i) $f_i = w$ and $z_1, \cdots, z_n$ are coordinates along X; (ii) $d\psi_i(x_0) = 0$; and (iii)

$$\bar\partial\partial\phi_i(x_0) = -\sum_{1 \leqq \alpha \leqq s} c_\alpha dz^\alpha \wedge d\bar z^\alpha + \sum_{n-t < \rho \leqq n} c_\rho dz^\rho \wedge d\bar z^\rho$$

where $c_\alpha > 0$, $c_\rho > 0$. Let $\boldsymbol{S} \subset \boldsymbol{T}_{x_0}(W)$ be spanned by $\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^s}; \frac{\partial}{\partial w}$. Then $\zeta \in \boldsymbol{S}$ is an $(s+1)$-triple $\zeta = (\lambda_1, \cdots, \lambda_s; \xi)$ and

$$\langle \bar\partial\partial\psi_i(m), \zeta \wedge \bar\zeta \rangle = -\sum_\alpha c_\alpha \mid \lambda_\alpha \mid^2 - m \mid \xi \mid^2 + \sum_\alpha (\rho_\alpha \lambda_\alpha \xi + \overline{\rho_\alpha \lambda_\alpha \xi})$$

for some s-tuple $\rho = (\rho_1, \cdots, \rho_s)$. Now, for $\epsilon > 0$,

$$\mid (\rho_\alpha \lambda_\alpha \xi + \rho_\alpha \lambda_\alpha \xi) \mid \leqq 2\epsilon \mid \lambda_\alpha \mid^2 + \frac{1}{2\epsilon} \mid \rho_\alpha \mid^2 \mid \xi \mid^2$$

and so

$$\langle \bar\partial\partial\psi_i(m), \zeta \wedge \zeta \rangle \leqq \sum (-c_\alpha + 2\epsilon) \mid \lambda_\alpha \mid^2 + \left(-m + \sum \frac{\mid \rho_\alpha \mid^2}{2\epsilon}\right) \mid \xi \mid^2.$$

If we first choose $2\epsilon < \min_{1 \leqq \alpha \leqq s} c_\alpha$ and then $m > \sum \frac{\mid \rho_\alpha \mid^2}{2\epsilon}$, it follows that $\partial\partial\psi_i(m)(x_0)$ has signature $(t, s+1)$. The Proposition follows by compactness.

COROLLARY. *If $X \subset W$ is a hypersurface with positive normal bundle, then $[X] \to W$ is positive for W sufficiently small.*

2. Continuous systems and the pseudo-characteristics of tubular neighborhoods. Let $X \subset W$ be an embedding and suppose that $\dim X = n$, $\dim W = n + r$. If the normal bundle $\boldsymbol{N} \to X$ of X in W is sufficiently positive

(§ II. 3), then the maximal continuous system $\{X_t : t \in \Sigma\}$ of X in W (§ III. 2) fills out W and it is geometrically reasonable to expect that, by interpreting some of the X_t as pseudo-characteristics of a suitable tubular neighborhood T of $X \subset W$, the E. E. Levi form on ∂T will have signature $(r-1, n)$. This section will be devoted to making this precise; in § VIII. 3 below we shall give an analytical proof of a more general fact.

(a) Suppose that $W \subset W'$ is a finite open complex manifold defined by $\{\phi < 0\}$ where ϕ is a smooth real function on W' and where $d\phi \neq 0$ in a neighborhood of ∂W. Let $m = \dim W$ and assume that $\partial\bar{\partial}\phi$ has everywhere $m-s$ positive eigenvalues in a neighborhood of ∂W. Denote by $\boldsymbol{T}^+ \subset \boldsymbol{T}(W)$ the bundle of positive eigenspaces for $\partial\bar{\partial}\phi$.

LEMMA 8.1. *Suppose that, at* $w_0 \in \partial W$, $\boldsymbol{T}^+_{w_0} \subset \boldsymbol{T}_{w_0}(\partial W)$. *Then there exists (locally) holomorphic functions* $\psi_1, \cdots, \psi_s$ *such that, in*

$$S = \{\psi_1 = \cdots = \psi_s = 0\},\ \boldsymbol{T}_{w_0}(S) \subset \boldsymbol{T}_{w_0}(\partial W),\ S \cap \partial W = \{w_0\},$$

and S *and* ∂W *have first order contact at* w_0.

Proof. Choose holomorphic coordinates $z^1, \cdots, z^m$ centered at w_0 and such that

$$\partial\bar{\partial}\phi(w_0) = \sum_{1 \leqq i \leqq m-s} dz^i \wedge d\bar{z}^i + \sum_{m-s < j \leqq m} \gamma_j dz^j \wedge d\bar{z}^j \qquad (\gamma_j \leqq 0).$$

Since $\boldsymbol{T}^+_{w_0} \subset \boldsymbol{T}_{w_0}(\partial W)$, we have

$$\phi(z) = \sum_{m-s < j \leqq m} \alpha_j z^j + \sum_{m-s < j \leqq m} \overline{\alpha_j z^j} + \sum \beta_{ij} z^i z^j + \overline{\sum \beta_{ij} z^i z^j}$$
$$+ \sum_{1 \leqq i \leqq m-s} z^i \bar{z}^i + \sum_{m-s < j \leqq m} \gamma_j z^j \bar{z}^j + [3]$$

where $\beta_{ij} = \beta_{ji}$ and [3] means terms of order at least three. Also, we may assume that $\alpha_m \neq 0$. Now set

$$\psi_\alpha = z_{m-s+\alpha},\ 1 \leqq \alpha < s,\quad \psi_s = z_m + \frac{1}{\alpha_m} \sum_{1 \leqq i, j \leqq m-s} \beta_{ij} z^i z^j.$$

Then $\phi \mid S = \sum_{1 \leqq i, j \leqq m-s} z^i \bar{z}^i + [3]$ and so $\phi(z) > 0$ for $z \in S - \{w_0\}$. Furthermore, $\boldsymbol{T}_{w_0}(S) = \boldsymbol{T}^+_{w_0} \subset \boldsymbol{T}_{w_0}(\partial W)$ and it is clear that ∂W and S have only first order contact at w_0.

(b) We call S a *local* $(m-s)$*-dimensional pseudo-characteristic* for W at w_0.

LEMMA 8.2. *Suppose that, at each $w_0 \in \partial W$, there is a local $(m-s)$-dimensional pseudo-characteristic. Then the Levi form on ∂W has $m-s$ positive eigenvalues.*

Proof. Let $S = \{\psi_1 = \cdots = \psi_s = 0\}$ be a local pseudo-characteristic at w_0. We may choose local coordinates $z^1, \cdots, z^m$ so that $\psi_\alpha = z^\alpha$ $(1 \leqq \alpha \leqq s)$.

Then $\boldsymbol{T}_{w_0}(S)$ is spanned by the vectors $\xi = \sum_{s+1 \leqq \rho \leqq m} \xi^\rho \dfrac{\partial}{\partial z^\rho}$. Since $\boldsymbol{T}_{w_0}(S) \subset \boldsymbol{T}_{w_0}(\partial W)$,

$$\phi(z) = \sum_{1 \leqq \alpha \leqq s} (\gamma_\alpha z^\alpha + \overline{\gamma_\alpha z^\alpha}) + \sum_{i,j} (b_{ij} z^i z^j + \overline{b_{ij} z^i z^j} + \sum_{i,j} c_{i\bar{j}} z^i \bar{z}^j + [3]$$
$$(b_{ij} = b_{ji},\ c_{i\bar{j}} = \bar{c}_{j\bar{i}}).$$

Thus

$$\phi \mid S = \sum_{s+1 \leqq \rho, \sigma \leqq m} (b_{\rho\sigma} z^\rho z^\sigma + \overline{b_{\rho\sigma} z^\rho z^\sigma}) + \sum_{s+1 \leqq \rho, \sigma \leqq m} c_{\rho\bar{\sigma}} z^\rho \bar{z}^\sigma + [3].$$

By a linear change of coordinates, we may assume $c_{\rho\bar{\sigma}} = \delta_\sigma{}^\rho \lambda_\rho$. Since $S \cap \partial W = \{w_0\}$, ϕ has the origin as a local maximum and so $\lambda_\rho \geqq 0$. Suppose say that $\lambda_m = 0$ and let Λ be the complex line $\{(0, \cdots, 0, \lambda)\}$. Then $\Lambda \subset S$ and $\phi \mid \Lambda = b\lambda^2 + \overline{b\lambda^2} + [3]$. We may choose λ_0 so that $b\lambda_0{}^2 + \overline{b\lambda_0{}^2} = 2Re\, b\lambda_0{}^2 = 0$, and then, on the real line $\Lambda_0 = \{(0, \cdots, 0, t\lambda_0)\}$ $(t \in \boldsymbol{R})$, $\phi \mid \Lambda_0 = [3]$. Thus S and ∂W osculate to second order along Λ_0 which is contrary to assumption and so all $\lambda_\rho > 0$ $(\rho = s+1, \cdots, m)$. Q.E.D.

(c) Let $\boldsymbol{E} \xrightarrow{\pi} X$ be a holomorphic vector bundle with fibre $\boldsymbol{C}^s$ and assume that $\dim X = n$.

LEMMA 8.3. *Assume that $\boldsymbol{E}$ is weakly positive and let $W \subset \boldsymbol{E}$ be the unit tubular neighborhood. Then W satisfies the conditions of Lemma* 8.1 *and therefore admits local pseudo-characteristics of dimension n.*

The proof of this Lemma follows from § VI. 5, example (v), (6.43), and (6.44).

(d) Assume that $\boldsymbol{E} \to X$ is sufficiently positive. Then, given $w_0 \in \partial W$ ($W =$ unit tubular neighborhood), there exists a neighborhood U of w_0 in $\boldsymbol{E}$ and $\sigma \in H^0(X, \Omega(\boldsymbol{E}))$ such that $\sigma(X) \cap U$ is a local pseudo-characteristic for W at w_0.

Proof. We require that $\boldsymbol{E}$ be positive enough that $H^1(X, \boldsymbol{m}_x{}^3 \cdot \Omega(\boldsymbol{E})) = 0$ $(x \in X)$ where $\boldsymbol{m}_x$ is the ideal sheaf of $\{x\} \subset X$. This implies that

$$(8.2) \qquad H^0(X, \Omega(\boldsymbol{E})) \xrightarrow{\rho_x} \Omega(\boldsymbol{E})_x/\boldsymbol{m}_x^3 \cdot \Omega(\boldsymbol{E})_x \to 0 \qquad (x \in X).$$

Referring to the proof of Lemma 8.1 as applied to the case at hand, we see that S is defined by an $\boldsymbol{E}_{\pi(w_0)}$-valued quadratic polynomial in the local coordinates centered at $\pi(w_0)$; i.e. S is defined by an element $\psi \in \Omega(\boldsymbol{E})_x/\boldsymbol{m}_x^3 \cdot \Omega(\boldsymbol{E})_x$. By (8.2), there exists $\sigma \in H^0(X, \Omega(\boldsymbol{E}))$ with $\rho_x(\sigma) = \psi$. Q.E.D.

Definition. We let $X_\sigma = \sigma(X)$ and call X_σ a *global pseudo characteristic* for W at w_0.

(e) PROPOSITION 8.2. *Let $X \subset W$ be an embedding with sufficiently positive normal bundle $\boldsymbol{N} \to X$. Then there exists a neighborhood T of X in W such that the E. E. Levi form has on ∂T signature $(r-1, n)$.*

Proof. We assume that $\boldsymbol{N} \to X$ is positive enough so that every point $z_0 \in Z$ (= unit neighborhood of $X \in \boldsymbol{N}$) admits a global pseudo-characteristic $X_\sigma (\sigma \in H^0(X, \Omega(\boldsymbol{N}))$. Let $\Sigma \subset H^0(X, \Omega(\boldsymbol{N}))$ be the set $\{\sigma \in H^0(X, \Omega(\boldsymbol{N})) : X_\sigma$ is a pseudo-characteristic at some $z_0 \in \partial Z\}$. Then denote by $T_\delta \subset W$ the set which is enveloped by the varieties $\{X_\sigma\}_{\sigma \in (\delta\Sigma)}$. (By this, we mean that we consider the mappings $F: X \times \Sigma \to \boldsymbol{N}$, $G: X \times \Sigma \to W$ and set $T_\delta = G(F^{-1}(Z_\delta))$.) Then T_δ is a neighborhood of X in W and an easy approximation argument (cf. § VIII. 3 below) shows that the varieties X_σ $(\sigma \in (\delta\Sigma))$ will, for δ small enough, give a set of global pseudo-characteristics for T_δ.

3. Another proof of the concavity, convexity of tubular neighborhoods. We follow the notations of § VIII. 1 above. Let $\sigma \in H^0(W, \Omega(\boldsymbol{L}))$ be the canonical section given by $\sigma \mid U_i = f_i$ and, for $\tau \in \boldsymbol{R}^+$, we set $\sigma_\tau = \tau\sigma$. Then $X \subset W$ is given by $\{w \in W : \sigma(w) = 0\}$. Assume that we are given a metric γ in $\boldsymbol{N} \to X$ whose curvature has signature (s, t) $(s + t = n = \dim X)$. Then γ defines in $\boldsymbol{N}$ a real non-negative function ϕ such that $\bar{\partial}\partial\phi$ has signature $(s, t+1)$ on $\boldsymbol{N} \to X$ (cf. § VI. 5, example (v)). By a partition of unity, we extend ϕ to a metric function ψ defined on $\boldsymbol{L} \supset \boldsymbol{N}$ (cf. § VIII. 1) and we define ϕ_τ on W by $\phi_\tau = \psi \circ \sigma_\tau$.

PROPOSITION 8.3. *There exists constants τ, c_1, c_2 such that*: (i) $0 < c_1 < c_2$ *and the locus* $\{w \in W : \phi_\tau(w) < c_2\}$ *defines an open submanifold* $W' \subset W$; *and* (ii) *if* $W'' = \{w : \phi_\tau(w) < c_1\}$, *then on* $W' - W''$, $\partial\bar{\partial}\phi_\tau$ *has signature* $(t+1, s)$.

Proof. We choose a covering of $\boldsymbol{L}$ by neighborhoods $V_i = \overline{\pi}^1(U_i)$ ($\pi\colon \boldsymbol{L} \to W$ being the projection). If (z_i, w_i) $(z_i = (z_i^1, \cdots, z_i^n)$ and $X \cap U_i = \{w_i = 0\})$ are coordinates in U_i then in V_i we have coordinates (z_i, w_i, ξ_i) with $\xi_i = (w_i/w_j)\xi_j$ in $V_i \cap V_j$. On V_i we write

$$\psi(z_i, w_i, \xi_i) = \phi(z_i, \xi_i) + \rho(z_i, w_i)\eta(z_i, w_i, \xi_i)$$

where ρ is a non-negative function on W which defines the distance to X. Since ψ comes from a metric, it is locally divisible by $|\xi_i|^2$ and we have

$$\psi(z_i, w_i, \xi_i) = (\phi_i(z_i) + \rho(z_i, w_i)\eta_i(z_i, w_i))\,|\xi_i|^2.$$

Now $\sigma_\tau(W) \cap V_i$ is given by $\xi_i - \tau w_i = 0$ and thus

$$\phi_\tau \mid U_i = \{\phi_i(z_i) + \rho(z_i, w_i)\eta_i(z_i, w_i)\}\tau^2 \mid w_i|^2.$$

Then

$$\bar{\partial}\partial\phi_\tau = (\bar{\partial}\partial\phi_i)\tau^2 \mid w_i|^2 - \tau^2\phi_i\, dw_i \wedge d\bar{w}_i + \tau^2 \mid w_i|^2\{\bar{\partial}\partial\eta_i\rho + \eta_i\bar{\partial}\partial\rho + \bar{\partial}r \wedge \partial\eta_i - \partial r \wedge \bar{\partial}\eta_i\}.$$

Since ρ, $\partial\rho$, $\bar{\partial}\rho$, $\bar{\partial}\partial\rho$ are all zero along X, we may find a constant $c > 0$ such that, if $W_c = \{w\colon \phi_1(w) < c\}$, then there is at every point $w \in W_c$ an n-dimensional subspace $\boldsymbol{S}_w \subset \boldsymbol{T}_w(W)$ on which

$$(\bar{\partial}\partial\phi_i + \{\bar{\partial}\partial\eta_i\rho + \eta_i\bar{\partial}\partial\rho + \bar{\partial}\rho \wedge \partial\eta_i - \partial\rho \wedge \bar{\partial}\eta_i\})$$

will have signature (s, t). Furthermore, by taking if necessary a smaller c, $\dfrac{\partial}{\partial w_i}$ will be normal to $\boldsymbol{S}_w$. Then the same argument as in § VIII. 1 shows that we may find a large $\tau \gg 0$ and $c' < c$ such that $\bar{\partial}\partial\phi_\tau$ has in $W_c - W_{c'}$ the signature $(s, t+1)$. Q. E. D.

Corollary (Grauert [6]). *If $\boldsymbol{N} \to X$ is negative, then there exists a strongly pseudo-convex neighborhood of X in W.*

4. Proof of Theorem VII. We first choose a metric in $\boldsymbol{L}^* \xrightarrow{\pi} W$ whose curvature has signature $(t, s+1)$ according to Proposition 8.1. By Proposition 6.4 we may choose Hermitian metrics on $W' \supset W$ such that $\Theta^{\#}(\eta, \eta)$ is negative definite for $\eta \in A^q(\boldsymbol{L}^*)$, $0 \leqq q < s$ and $\Theta(\eta, \eta)$ is positive definite for $\eta \in A^q(\boldsymbol{L}^*)$, $n + 1 - t < q \leqq n + 1$.

Now let $W \supset X$ be chosen sufficiently small and let ϕ_τ be as in § VIII. 3.

Then, by the explicit construction, we see that for all $w \in W - X$ there exists an n-dimensional subspace $\boldsymbol{S}_w \subset \boldsymbol{T}_w(W)$ on which Θ and $\partial\bar{\partial}\phi_\tau$ each have signature (t, s). Furthermore we may choose subspaces $\boldsymbol{S}_w^+$, $\boldsymbol{S}_w^-$ such that $\boldsymbol{S}_w^+ + \boldsymbol{S}_w^- = \boldsymbol{S}_w$ and such that Θ, $\partial\bar{\partial}\phi_\tau$ are on $\boldsymbol{S}^+$ positive definite and are negative definite on $\boldsymbol{S}^-$. Finally, for W still sufficiently small, $\boldsymbol{S}_w$ will be close to $\boldsymbol{T}_w(\partial W)$. It then follows that, with the chosen Hermitian metrics, $L^{\#}(\eta, \eta)$ will be negative definite for $\eta \in \boldsymbol{D}^q(\boldsymbol{L}^*)$, $0 \leqq q < s$ and $L(\eta, \eta)$ will be positive definite for $\eta \in \boldsymbol{D}^q(\boldsymbol{L}^*)$, $n - t < q$ (we use here the boundary conditions as expressed by (7.20)). Finally, without effecting any of our choices, we may multiply the metric in $\boldsymbol{L}^*$ by $\bar{e}^{\lambda\phi_\tau}$ $(\lambda >> 0)$ so that the new metric curvature will have signature $(t+1, s)$ and $\Theta(\eta, \eta)$ will be positive definite for $\eta \in A^q(\boldsymbol{L}^*)$, $n - t < q$. This proves Theorem VII.

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