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Final Remark. A very fine bibliography is to be found in [K].

EXTERIOR DIFFERENTIAL SYSTEMS

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Introduction

This is the first part of a paper where we will give an exposition of the theory of exterior differential systems, as founded by Elie Cartan and developed by Erich Kähler. We hope the paper will give an introduction to an important but still fairly inaccessible subject.

The second part will include, among others, the topics: prolongation, the Cartan-Kuranishi theorem, applications to partial differential equations and differential geometry. In writing the paper we will not be content with a treatment of an existing theory. Our principal aim is to make the formalism available for further development, such as the non-analytic case and the application to global problems, where we believe it will be useful.

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1. Algebraic Preliminaries

Let V be a real vector space of dimension n and V^* its dual space. An element $x \in V$ is called a vector and an element $x' \in V^*$ a covector. V and V^* have a "pairing"

$$(1) \quad \langle x, x' \rangle, \quad x \in V, \quad x' \in V^*,$$

which is a real number and is linear in each of the arguments x, x' .

Over V there is the exterior or Grassmann algebra, which is a graded algebra:

$$(2) \quad \Lambda(V) = \Lambda^0(V) \oplus \Lambda^1(V) \oplus \dots \oplus \Lambda^n(V), \quad \Lambda^0(V) = \mathbb{R}, \quad \Lambda^1(V) = V.$$

An element $\xi \in \Lambda^p(V)$ is called a multivector of degree p . Multiplication in $\Lambda(V)$, to be denoted by \wedge , is associative, distributive and satisfies the relation

$$(3) \quad \xi \wedge \eta = (-1)^{pq} \eta \wedge \xi, \quad \xi \in \Lambda^p(V), \quad \eta \in \Lambda^q(V).$$

The multivector ξ is called decomposable, if it can be written

$$(4) \quad \xi = x_1 \wedge \dots \wedge x_p, \quad x_i \in V.$$

$\xi \neq 0$, if and only if x_1, \dots, x_p are linearly independent, in which case ξ defines the p -dimensional subspace W of V , spanned by the

x 's. The decomposable multivector ξ is a homogeneous coordinate of W , called its Grassmann coordinate; it is defined up to a non-zero factor.

In the same way there is over V^* the exterior algebra

$$(5) \quad \Lambda(V^*) = \Lambda^0(V^*) \oplus \Lambda^1(V^*) \oplus \dots \oplus \Lambda^n(V^*),$$

$$\Lambda^0(V^*) = \mathbb{R}, \quad \Lambda^1(V^*) = V^*.$$

An element of $\Lambda^p(V^*)$ is called a form of degree p or simply a p -form. If $\alpha \in \Lambda^p(V^*)$, we denote by $\Pi_p \alpha \in \Lambda^p(V^*)$ its component in $\Lambda^p(V^*)$.

Let e_i be a base of V and e'^k its dual base, so that

$$(6) \quad \langle e_i, e'^k \rangle = \delta_i^k, \quad 1 \leq i, k \leq n.$$

Then an element $\xi \in \Lambda^p(V)$ can be written

$$(7) \quad \xi = \frac{1}{p!} \sum a^{i_1 \dots i_p} e_{i_1} \wedge \dots \wedge e_{i_p}$$

and an element $\alpha \in \Lambda^p(V^*)$ as

$$(8) \quad \alpha = \frac{1}{p!} \sum b_{i_1 \dots i_p} e'^{i_1} \wedge \dots \wedge e'^{i_p}.$$

In (7) and (8) the coefficients $a^{i_1 \dots i_p}$ and $b_{i_1 \dots i_p}$ are supposed to be anti-symmetric in any two of their indices, so that they are well-defined. The "pairing" of $\Lambda^p(V)$ and $\Lambda^p(V^*)$ is given by

$$(9) \quad \langle \xi, \alpha \rangle = \frac{1}{p!} \sum a^{i_1 \dots i_p} p_{b_{i_1 \dots i_p}}.$$

It is independent of choice of base.

A pairing of $\Lambda(V)$ and $\Lambda(V^*)$ is then defined by distributivity and by the requirement that

$$(10) \quad \langle \xi, \alpha \rangle = 0, \quad \xi \in \Lambda^p(V), \quad \alpha \in \Lambda^q(V^*), \quad p \neq q.$$

Each of the exterior algebras $\Lambda(V)$ and $\Lambda(V^*)$ is a graded algebra. We define

$$(11) \quad \widetilde{\xi} = (-1)^p \xi, \quad \text{for } \xi \in \Lambda^p(V).$$

An endomorphism f of the additive structure of $\Lambda(V)$ is called a derivation if it satisfies the conditions:

- (1) $f\widetilde{\xi} = \widetilde{f\xi}$,
- (2) $f(\xi \wedge \eta) = f(\xi) \wedge \eta + \xi \wedge f(\eta)$.

Such an endomorphism is called an anti-derivation if:

- (1) $f\widetilde{\xi} = -\widetilde{f\xi}$,
- (2) $f(\xi \wedge \eta) = f(\xi) \wedge \eta + \widetilde{\xi} \wedge f(\eta)$.

It is said to be of degree k if

$$f(\Lambda^p) \subset \Lambda^{p+k}.$$

Given $x \in V$, we define the exterior product

$$e(x): \Lambda(V) \rightarrow \Lambda(V)$$

by

$$(12) \quad e(x)\xi = x \wedge \xi, \quad \xi \in \Lambda(V).$$

By the relation

$$(13) \quad \langle \xi, i(x)\alpha \rangle = \langle e(x)\xi, \alpha \rangle, \quad \xi \in \Lambda(V), \quad \alpha \in \Lambda^*(V),$$

the "adjoint operator of $e(x)$ ":

$$i(x): \Lambda(V^*) \rightarrow \Lambda(V^*)$$

is defined. $i(x)$ is an anti-derivation of degree -1 ; it is called the interior product.

Exercise. If ξ and α are given by (7) and (8) respectively and if

$$x = \sum x^i e_i,$$

give the expressions of $e(x)\xi$ and $i(x)\alpha$ in terms of the basis.

We will consider ideals in $\Lambda(V^*)$. A subring $I \subset \Lambda(V^*)$ is called an ideal if:

(1) $\alpha \in I$ implies that its homogeneous components $\Pi_p \alpha \in I$, $0 \leq p \leq n$.

(2) $\alpha \in I$ implies $\alpha \wedge \beta \in I$, $\forall \beta \in \Lambda(V^*)$. It follows that $\beta \wedge \alpha \in I$, $\forall \beta \in \Lambda(V^*)$.

A minimal set of generators of I can be described by the following consideration: Given an ideal $I \subset \Lambda(V^*)$, all the $x \in V$ such that $i(x)I \subset I$ (i.e., $i(x)\alpha \in I$, $\forall \alpha \in I$) form a subspace $A_I \subset V$, to be called the associated space of I . Its annihilator $A_I^\perp \subset V^*$, defined to consist of all $x' \in V^*$ such that

$$\langle x, x' \rangle = 0, \quad \forall x \in A_I,$$

will be called the dual associated space of I . Then we have:

Theorem. As an ideal, I has a system of generators consisting of the elements of $\Lambda(A_I^\perp)$.

Proof. Let A_I be spanned by e_{r+1}, \dots, e_n , so that A_I^\perp is spanned by e'^1, \dots, e'^r . Let $\{G_i\}$ be a set of generators of I . Without loss of generality we assume that each G_i (of degree ≥ 1) is homogeneous. Suppose G_1 be linear, i.e.,

$$G_1 = \sum_i a_i e'^i, \quad 1 \leq i \leq n.$$

By hypothesis we have

$$i(e_\lambda)G_1 = a_\lambda \in I, \quad r+1 \leq \lambda \leq n.$$

Hence $a_\lambda = 0$ and

$$G_1 = a_1 e'^1 + \dots + a_r e'^r \in A_I^\perp.$$

In other words, the generators of degree 1 belong to A_I^\perp .

Suppose that the generators of degree $\leq k$ belong to $\Lambda(A_I^\perp)$. Let G be a generator of degree $k+1$. We put

$$G' = G - e'^{r+1} \wedge (i(e_{r+1})G).$$

The second term belongs to I (since $e_{r+1} \in A_I$ and I is an ideal).

Moreover, it is generated by elements of degree $\leq k$. On the other hand, $i(e_{r+1})$ being an anti-derivation, we have

$$i(e_{r+1})G' = +e'^{r+1} \wedge (i(e_{r+1})^2 G) = 0.$$

This proves that G' does not contain e'^{r+1} . Continuing this construction, we can replace G by an element \tilde{G} which does not contain e'^{r+1}, \dots, e'^n and hence belongs to $\Lambda(A_I^\perp)$. Thus the induction is complete.

2. Cauchy characteristics

Let M be a manifold of dimension n , with the local coordinates x^1, \dots, x^n . An exterior differential form (later to be called simply a form) of degree p has locally the expression

$$(1) \quad \alpha = \frac{1}{p!} \sum_{i_1, \dots, i_p} a_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}, \quad 1 \leq i_1, \dots, i_p \leq n,$$

where the coefficients are smooth functions and are anti-symmetric in any two of the indices. An exterior differential system is a system of equations

$$(2) \quad \alpha_A = 0, \quad 1 \leq A \leq N,$$

where α_A are forms which are generally of different degrees.

An integral manifold of (2) is a submanifold $f: V \rightarrow M$ such that

$$(3) \quad f^* \alpha_A = 0, \quad 1 \leq A \leq N.$$

Since (3) implies

$$df^* \alpha_A = f^* d\alpha_A = 0,$$

the problem remains the same, if we add to (2) the equations obtained by exterior differentiation.

More generally, the forms of M generate a graded ring. A subring I is called a differential ideal, if: (1) $\alpha \in I$ implies that each homogeneous component of α belongs to I ; (2) $\alpha \in I$ implies that $\beta \wedge \alpha \in I$ for any form β . The differential ideal is called closed if $\alpha \in I$ implies $d\alpha \in I$. An integral manifold of I is a submanifold $f: V \rightarrow M$ such that

$$(4) \quad f^* \alpha = 0, \quad \forall \alpha \in I.$$

The fundamental problem in exterior differential systems is to study the integral manifolds of a closed differential ideal. We shall use interchangeably the terms differential ideal or differential system.

Any system of ordinary or partial differential equations can be expressed as an exterior differential system. For example, the equation of the first order

$$(5) \quad F(x^i, z, \frac{\partial z}{\partial x^i}) = 0, \quad 1 \leq i \leq n$$

is equivalent to the exterior differential system

$$(5a) \quad \begin{aligned} F(x^i, z, p_i) &= 0, \\ dF &= 0, \\ dz - \sum_i p_i dx^i &= 0, \\ \sum_i dx^i \wedge dp_i &= 0 \end{aligned}$$

in the $(2n+1)$ -dimensional space (x^i, z, p_i) , and the classical

integration problem for (5) is to search for n -dimensional integral manifolds on which

$$dx^1 \wedge \dots \wedge dx^n \neq 0.$$

This example shows a disadvantage of exterior differential systems, namely, the large number of equations. We believe it cannot be avoided for a proper understanding of the equation. At the end of this section we will show how this formulation leads to the characteristics.

Perhaps the simplest exterior differential system is the one where the closed differential ideal I is generated by forms of degree one. Let the generators be

$$(6) \quad \alpha^1, \dots, \alpha^{n-r},$$

which we suppose to be linearly independent. The condition that I is closed gives

$$(F) \quad d\alpha^i \equiv 0, \text{ mod } \alpha^1, \dots, \alpha^{n-r}, \quad 1 \leq i \leq n-r.$$

This condition (F) is called the Frobenius condition. A differential system

$$(6a) \quad \alpha^1 = \dots = \alpha^{n-r} = 0$$

satisfying (F) is called completely integrable.

Geometrically the α 's span at every point $x \in M$ a subspace W_x of dimension $n-r$ in the cotangent space T_x^* or, what is the same, a subspace W_x^1 of dimension r in the tangent space T_x . Following Chevalley, the data $W_x^1 \subset T_x$, $x \in M$, is called a distribution. Notice that the condition (F) is intrinsic, i.e., independent of local coordinates, and is also invariant under a linear change of the α 's.

The fundamental theorem on completely integrable systems is:

Theorem 2.1 (Frobenius). Let I be a closed differential ideal having as generators the linearly independent forms $\alpha^1, \dots, \alpha^{n-r}$ of degree one. In a sufficiently small neighborhood there is a coordinate system y^1, \dots, y^n such that I is generated by dy^{r+1}, \dots, dy^n .

Proof. We will prove the theorem by induction on r . Let $r = 1$. Then the subspace $W_x^1 \subset T_x$, $x \in M$, is of dimension 1. By a well-known theorem we can choose coordinates y^1, \dots, y^n , such that W_x^1 is spanned by the vector $\frac{\partial}{\partial y^1}$; then W_x is spanned by dy^2, \dots, dy^n . The latter clearly form a set of generators of I . Notice that in this case the condition (F) is void.

Suppose $r \geq 2$ and the theorem be true for $r-1$. Let x^i , $1 \leq i \leq n$, be local coordinates such that

$$\alpha^1, \dots, \alpha^{n-r}, dx^r$$

are linearly independent. The differential system defined by these

$n - r + 1$ forms also satisfies the condition (F). By the induction hypothesis there are coordinates y^i so that

$$dy^{r+1}, \dots, dy^n, dy^r$$

are a set of generators of the corresponding differential ideal. It follows that dx^r is a linear combination of these forms or that x^r is a function of y^r, \dots, y^n . Without loss of generality we suppose

$$\frac{\partial x^r}{\partial y^r} \neq 0.$$

Then $\alpha^1, \dots, \alpha^{n-r}$ differ from dy^{r+1}, \dots, dy^n by a non-singular linear transformation mod dx^r . We can choose as new generators of I

$$\alpha'^i = dy^{r+i} + p^i dx^r, \quad 1 \leq i \leq n-r,$$

and the condition (F) remains satisfied. This gives

$$d\alpha'^i = dp^i \wedge dx^r = \sum_{1 \leq \lambda \leq r-1} \frac{\partial p^i}{\partial y^\lambda} dy^\lambda \wedge dx^r \equiv 0, \quad \text{mod } \alpha'^1, \dots, \alpha'^{n-r}.$$

It follows that

$$\frac{\partial p^i}{\partial y^\lambda} = 0, \quad 1 \leq i \leq n-r, \quad 1 \leq \lambda \leq r-1,$$

which means that p^i are functions of y^r, \dots, y^n . Hence in the y -coordinates we are studying a system of $n-r$ forms of degree one

involving only the coordinates y^r, \dots, y^n . This reduces to the situation settled at the beginning of this proof. Hence the induction is complete.

The theorem gives a "normal form" of a completely integrable system, i.e., the system can be written locally as

$$(7) \quad dy^{r+1} = \dots = dy^n = 0$$

in a suitable coordinate system. The maximal integral manifolds are

$$(7a) \quad y^{r+1} = \text{const}, \dots, y^n = \text{const},$$

and are therefore of dimension r . We say that the system defines a foliation, of dimension r and codimension $n-r$, of which the submanifolds (7a) are the leaves.

The condition (F) has a formulation in terms of vector fields, which is also useful. We add to the forms (6) r forms $\alpha^{n-r+1}, \dots, \alpha^n$, so that $\alpha^i, 1 \leq i \leq n$, are linearly independent. Then we have

$$(8) \quad d\alpha^i = \frac{1}{2} \sum_{j,k} c_{jk}^i \alpha^j \wedge \alpha^k, \quad 1 \leq i, j, k \leq n, \quad c_{jk}^i + c_{kj}^i = 0.$$

The condition (F) can be expressed as

$$(9) \quad c_{pq}^a = 0, \quad 1 \leq a \leq n-r, \quad n-r+1 \leq p, q \leq n.$$

Let f be a smooth function. The equation

$$(10) \quad df = \sum_i (X_i f) \alpha^i$$

defines n operators or vector fields X_i , which form a dual base to α^i . Exterior differentiation of (10) gives

$$\frac{1}{2} \sum \{ (X_k X_j - X_j X_k) f \} \alpha^k \wedge \alpha^j + \sum_i (X_i f) d\alpha^i = 0.$$

Substituting (8) into this equation, we get

$$(11) \quad [X_k, X_j]f = (X_k X_j - X_j X_k)f = - \sum_i c_{kj}^i X_i f.$$

Equation (11) is the dual version of (8). The vectors X_{n-r+1}, \dots, X_n span at each point $x \in M$ the subspace W_x^\perp of the distribution. Hence the condition (F) or (9) can be expressed as follows: Let a distribution in M be defined by the subspaces $W_x^\perp \subset T_x$, $\dim W_x^\perp = r$. The condition (F) says that, for any two vector fields X, Y , such that $X_x, Y_x \in W_x^\perp$, their bracket $[X, Y]_x \in W_x^\perp$.

Given a closed differential ideal I , we ask whether there is a local coordinate system so that I will be generated by forms involving a smaller number of the coordinates. This question is completely answered by the notion of associated space discussed in §1.

We recall that at $x \in M$,

$$(12) \quad (A_I)_x = \{ \xi \in T_x \mid i(\xi)I \subset I \}.$$

A vector field ξ such that $\xi_x \in (A_I)_x$ is called a Cauchy characteristic

vector field of I .

Lemma 2.2. If ξ, η are Cauchy characteristic vector fields of a closed differential ideal I , so is their bracket $[\xi, \eta]$.

Let L_ξ be the Lie derivative defined by ξ . It is well-known that

$$(13) \quad L_\xi = di(\xi) + i(\xi)d.$$

Since I is closed, we have $dI \subset I$. If ξ is a characteristic vector field, we have $i(\xi)I \subset I$. By (13) it follows that $L_\xi I \subset I$. The lemma follows from the identity

$$(14) \quad [L_\xi, i(\eta)] \stackrel{\text{def}}{=} L_\xi i(\eta) - i(\eta)L_\xi = i([\xi, \eta]),$$

which is valid for any two vector fields ξ, η .

To prove (14) we observe that L_ξ is a derivation and $i(\eta)$ is an anti-derivation, so that $[L_\xi, i(\eta)]$ is an anti-derivation. It therefore suffices to verify (14) when the two sides act on functions f and differentials df . Clearly, when acted on f , both sides give zero. When acted on df , we have

$$\begin{aligned} [L_\xi, i(\eta)]df &= L_\xi(\eta f) - i(\eta)d(\xi f) = [\xi, \eta]f \\ &= i([\xi, \eta])df. \end{aligned}$$

This proves (14).

Theorem 2.3. Let I be a closed differential ideal whose dual associated space A_I^1 has constant dimension $s = n - r$. Then there is a neighborhood in which there are coordinates $(x^1, \dots, x^r; y^1, \dots, y^s)$ such that I has a set of generators that are forms in y^1, \dots, y^s .

Proof: By lemma 2.2 the differential system defined by A_I^1 (or, what is the same, the distribution defined by A_I) is completely integrable. We may choose coordinates so that the foliation so defined is given by

$$(x^1, \dots, x^r; y^1, \dots, y^s) + (y^1, \dots, y^s).$$

For $t = (t_1, \dots, t_r)$ we set $\phi_t = \exp(t_1 \frac{\partial}{\partial x^1} + \dots + t_r \frac{\partial}{\partial x^r})$. This

gives an action of a neighborhood of the origin in (x^1, \dots, x^r) -space on M (a germ of \mathbb{R}^r -action if you like), and since $L_{\frac{\partial}{\partial x^\alpha}}(I) \subseteq I$

($1 \leq \alpha \leq r$) it follows that

$$\phi_t^*(I) \subseteq I.$$

We choose a point $p \in M$ where $\dim I(p)$ is maximal. It may be assumed that p is the origin in our coordinate system, and we choose a minimal set of algebraic generators $\theta^1, \dots, \theta^u$ for the spaces $I(0, y)$ for $|y^\mu| < \epsilon$, $1 \leq \mu \leq s$. Thus each form θ^ν defines a form in $AT_{(0,y)}^*(M)$

and these generate $I(0, y)$. Using ϕ_t we may uniquely extend the θ^ν to forms in $AT_{(x,y)}^*(M)$ for small x^α . In this way we obtain generators $\theta^\nu = \theta^\nu(y, dx, dy)$ for I that depend on y^μ , dy^μ , and dx^α but do not depend on x^α . We will prove that we may choose the θ^ν so as not to depend on dx^α .

The proof is by induction on the degree k of θ^ν . When $k = 1$ we have

$$0 = i(\frac{\partial}{\partial x^\alpha})\theta^\nu$$

which gives the claim in this case. Assume that the generators of I in degree $\leq k$ do not contain dx^α and let θ^ν be a new generator in degree $k + 1$. We may assume that θ^ν involves dx^1, \dots, dx^s but does not involve dx^{s+1}, \dots, dx^r (in case $s = r$ this statement is vacuous). With the additional index range $1 \leq \alpha \leq s - 1$ write

$$\theta^\nu = dx^s \wedge \eta + \psi$$

where $\eta = \eta(dx^\alpha, y^\mu, dy^\mu)$ and $\psi = \psi(dx^\alpha, y^\mu, dy^\mu)$ do not involve dx^s . Then

$$i(\frac{\partial}{\partial x^s})\theta^\nu = \eta$$

is an element of I in degree k . Since θ^ν is assumed to be a new generator in degree $k + 1$ we must have $\psi \neq 0$. We then replace θ^ν by $\theta^\nu - dx^s \wedge \eta = \psi$, and thereby eliminate dx^s from θ^ν . An obvious induction then completes the proof of Theorem 2.3.

Definition 2.4. The leaves defined by the distribution A_x are called the Cauchy characteristics.

We will apply this theorem to the equation (5). Equations (5a) can be written

$$\begin{aligned} F(x^i, z, p_i) &= 0 \\ dz - \sum p_i dx^i &= 0, \\ (15) \quad \sum_i (F_{x^i} + F_{z p_i}) dx^i + \sum_i F_{p_i} dp_i &= 0, \\ \sum_i dx^i \wedge dp_i &= 0. \end{aligned}$$

The differential ideal I is generated by the left-hand members of (15) and is closed.

To determine the associated space A_I consider the vector

$$(16) \quad \xi = \sum u^i \frac{\partial}{\partial x^i} + u \frac{\partial}{\partial z} + \sum v_i \frac{\partial}{\partial p_i}$$

and express the condition that the interior product $i(\xi)$ keeps I stable. This gives

$$u - \sum p_i u^i = 0,$$

$$(17) \quad \sum (F_{x^i} + F_{z p_i}) u^i + \sum F_{p_i} v_i = 0,$$

$$\sum u^i dp_i - \sum v_i dx^i = 0.$$

Comparing the last equation of (17) with the third equation of (15), we get

$$(18a) \quad u^i = \lambda F_{p_i}, \quad v_i = -\lambda (F_{x^i} + F_{z p_i}),$$

and the first equation of (17) then gives

$$(18b) \quad u = \lambda \sum_i p_i F_{p_i}.$$

The parameter λ being arbitrary, equations (18a) and (18b) show that $\dim A_I = 1$, i.e., the characteristic vectors at each point form a one-dimensional space. The characteristic curves in the space (x^i, z, p_i) , or characteristic strips in the classical terminology, are the integral curves of the differential system

$$(19) \quad \frac{dx^i}{F_{p_i}} = - \frac{dp_i}{F_{x^i} + F_{z p_i}} = \frac{dz}{\sum p_i F_{p_i}}.$$

These are the equations of Charpit and Lagrange. To construct an integral manifold of dimension n it suffices to take a "non-characteristic" ** $(n-1)$ -dimensional integral manifold and draw the characteristic strips through its points. Putting it in another way, an n -dimensional integral manifold is generated by characteristic strips.

*"Non-characteristic means transverse to the Cauchy characteristic vector field.

We wish to apply the Cauchy characteristics to prove the following global theorem:

Theorem 2.5. Consider the differential equation

$$(20) \quad \sum_1^n \left(\frac{\partial z}{\partial x^i} \right)^2 = 1, \quad 1 \leq i \leq n.$$

If $z = z(x^1, \dots, x^n)$ is a solution for all points $(x^1, \dots, x^n) \in E^n$ ($= n$ -dimensional euclidean space), then z is a linear function in x^1 , i.e.,

$$(21) \quad z = \sum_1^n a_i x^i + b,$$

when a_i, b are constants satisfying $\sum_1^n a_i^2 = 1$.

Proof. We will denote by E^{n+1} the space of (x^1, \dots, x^n, z) , so that the solution can be interpreted as a graph Σ in E^{n+1} . We will also identify E^n with the hyperplane $z = 0$. Our hypothesis says that Σ has a one-one projection to E^n . For the equation (20) the denominators in the middle term of (19) are zero, so that the Cauchy characteristics satisfy

$$(22) \quad p_i = \text{const.}$$

The equations (19) can be integrated and the Cauchy characteristic curves, when projected to E^{n+1} , are the straight lines

$$(23) \quad x^i = x_0^i + p_i t, \quad z = z_0 + t,$$

where x_0^i, z_0 are constants. The graph Σ must have the property that it is generated by the "Cauchy lines" (23), whose projections in E^n form a foliation of E^n .

Our theorem for $n = 2$ follows immediately. For the only foliation of E^2 by straight lines is given by a family of parallel lines, say

$$x^1 = x_0^1 + \cos \theta t, \quad x^2 = x_0^2 + \sin \theta t, \quad \theta = \text{const.},$$

and we have

$$z = \cos \theta x^1 + \sin \theta x^2 + c, \quad c = \text{const.}$$

For any n we consider the level sets $\Sigma_h = \Sigma \cap \{z = h\}$, defined by

$$(24) \quad z(x^1, \dots, x^n) = h.$$

Because of (20), Σ_h is a regular hypersurface. At every point of Σ , The Cauchy line through it cuts Σ_h orthogonally. Let

$$(25) \quad \forall z = \left(\frac{\partial z}{\partial x^1}, \dots, \frac{\partial z}{\partial x^n} \right) \in E^n.$$

The map $\phi: \sum_0 \times \mathbb{R} \rightarrow E^n$ given by

$$(26) \quad (x, t) \mapsto (x + tVz), \quad t \in \mathbb{R}, \quad x = (x^1, \dots, x^n) \in \sum_0$$

maps each $\{x_0\} \times \mathbb{R}$ into the projection of a Cauchy line in E^n .

Therefore it must be a diffeomorphism. It follows that \sum_0 is connected. We will show that \sum_0 must be a hyperplane in E^n .

To do this, we compute the Jacobian of the map ϕ and use the fact that it is never zero. \sum_0 being a regular hypersurface on E^n , we use a local orthonormal frame field $(x, e_1, \dots, e_{n-1}, e_n)$, $x \in \sum_0$, where $e_n = Vz$ is the unit normal at x . On \sum_0 let ω_α be the dual coframe to e_β , $1 \leq \alpha, \beta \leq n-1$. Then we have

$$(27) \quad \begin{aligned} dx &= \sum \omega_\alpha \otimes e_\alpha, \\ de_n &= -\sum h_{\alpha\beta} \omega_\beta \otimes e_\alpha, \quad h_{\alpha\beta} = h_{\beta\alpha}, \\ &\quad 1 \leq \alpha, \beta \leq n-1, \end{aligned}$$

so that

$$(28) \quad II = -(dx, de_n) = \sum h_{\alpha\beta} \omega_\alpha \omega_\beta$$

is the second fundamental form of \sum_0 . The condition $II = 0$, or $h_{\alpha\beta} = 0$, characterizes \sum_0 to be a hyperplane.

From (27) we have

$$(29) \quad d(x + te_n) = \sum (\delta_{\alpha\beta} - th_{\alpha\beta}) \omega_\beta \otimes e_\alpha + dt \otimes e_n.$$

Hence the pull-back (under ϕ) of the volume element of E^n is, up to a non-zero factor,

$$(30) \quad \det(\delta_{\alpha\beta} - th_{\alpha\beta}) \omega_1 \wedge \dots \wedge \omega_{n-1} \wedge dt.$$

It follows that

$$(31) \quad \det(\delta_{\alpha\beta} - th_{\alpha\beta}) \neq 0, \quad t \in \mathbb{R}.$$

The matrix $(h_{\alpha\beta})$ being a symmetric matrix, this is possible only when all the eigenvalues of $(h_{\alpha\beta})$ are zero, i.e., $h_{\alpha\beta} = 0$. Therefore \sum_0 is a hyperplane and its unit normal vector Vz is a constant. From this the conclusion (21) follows.

3. Pfaffian systems

Another simple exterior differential system is one which consists of one equation

$$(1) \quad \alpha = 0,$$

where α is a form of degree 1. This problem was studied by Pfaff in 1814-15. The corresponding closed differential ideal I has the generators $\alpha, d\alpha$. The integer r defined by

$$(2) \quad \alpha \wedge (d\alpha)^r \neq 0, \alpha \wedge (d\alpha)^{r+1} = 0$$

is called the rank of the equation (1). It is invariant under the change

$$(3) \quad \alpha \rightarrow a\alpha, \quad a \neq 0.$$

Putting it in a different way, the two-form $d\alpha$, mod α , has an even rank $2r$.

The study of the integral manifolds of (1) is completely cleared up by the formulation of a "normal form", as given by the

Theorem 3.1. In a neighborhood suppose the equation (1) has a constant rank r . Then there exists a coordinate system w^1, \dots, w^n , possibly in a smaller neighborhood such that the equation becomes

$$(4) \quad dw^1 + w^2 dw^3 + \dots + w^{2r} dw^{2r+1} = 0.$$

Proof. For $r = 0$ condition (2) is the Frobenius condition and the theorem becomes the Frobenius Theorem.

By induction suppose that the theorem is true for $r-1$. Let I be the ideal generated by α and $d\alpha$. The dual associated space A_I^1 has dimension $2r+1$. By Theorem 2.3 there are coordinates x^1, \dots, x^n such that, by multiplying by a factor if necessary, α is a form in x^1, \dots, x^{2r+1} . There is no loss of generality in assuming $n = 2r+1$.

Let J be the closed differential ideal generated by $d\alpha$. Since $(d\alpha)^r \neq 0$, the dual associated space A_J^1 has dimension $2r$, and A_J has dimension one. Hence J is generated by a 2-form ϕ in $2r$ variables y^1, \dots, y^{2r} , and $d\alpha$ differs from it by a factor:

$$d\alpha = a\phi, \quad a \neq 0.$$

We have

$$(d\alpha)^r = a^r \phi^r \neq 0,$$

so that

$$\phi^r = b(y) dy^1 \wedge \dots \wedge dy^{2r}, \quad b(y) \neq 0.$$

The fact that $(d\alpha)^r$ is closed gives

$$da \wedge dy^1 \wedge \dots \wedge dy^{2r} = 0,$$

implying that α is a function of the y 's. It follows that $d\alpha$ is a form in y^1, \dots, y^{2r} .

Since $d\alpha$ is itself closed, there is a non-zero one-form β in y^1, \dots, y^{2r} such that

$$d\alpha = d\beta.$$

Being in a $2r$ -dimensional space, the form $d\beta$, mod β , cannot have rank $2r$, and hence must have the rank $2r - 2$. In other words, the pfaffian equation $\beta = 0$ has rank $r - 1$.

To the equation $\beta = 0$ we apply the induction hypothesis and write it as

$$dz^1 + z^2 dz^3 + \dots + z^{2r-2} dz^{2r-1} = 0,$$

so that β itself becomes

$$\beta = u(dz^1 + z^2 dz^3 + \dots + z^{2r-2} dz^{2r-1}).$$

Since

$$d(\alpha - \beta) = 0,$$

there is a function v such that

$$\alpha = dv + \beta.$$

By an obvious change of notation we write

$$\alpha = dw^1 + w^2 dw^3 + \dots + w^{2r} dw^{2r+1}.$$

Since

$$\alpha \wedge (d\alpha)^r \neq 0,$$

the functions w^1, \dots, w^{2r+1} are independent and can be extended to a full coordinate system.

From the normal form (4) we see that the general maximal integral manifolds are of dimension r and are given by

$$(5) \quad w^1 = f(w^3, w^5, \dots, w^{2r+1}),$$

where f is an arbitrary function.

Other integral manifolds are, for instance, given by

$$(6) \quad w^1 = f(w^3, \dots, w^{2s+1}), \quad s < r$$

$$w^{2t+1} = \text{const}, \quad w^{2t} \text{ arbitrary},$$

$$s + 1 \leq t \leq r.$$

Theorem 3.3. Let α be a one-form. Then α has the normal form

$$(10) \alpha = y^0 dy^1 + \dots + y^{2r} dy^{2r+1}, \text{ if } r+1 = s;$$

$$(11) \alpha = dy^1 + y^2 dy^3 + \dots + y^{2r} dy^{2r+1}, \text{ if } r = s.$$

In these expressions, the y 's are independent functions and are therefore parts of a local coordinate system.

Proof. Let I be the closed differential ideal generated by α and $d\alpha$. By Theorem 3.1 there are coordinates y^1, \dots, y^n in a neighborhood such that

$$\alpha = u(dy^1 + y^2 dy^3 + \dots + y^{2r} dy^{2r+1}).$$

A change of notation allows us to write

$$\alpha = z^0 dy^1 + z^2 dy^3 + \dots + z^{2r} dy^{2r+1}.$$

Then

$$(d\alpha)^{r+1} = c dz^0 \wedge dy^1 \wedge dz^2 \wedge dy^3 \wedge \dots \wedge dz^{2r} \wedge dy^{2r+1}, \quad c = \text{const. } c \neq 0.$$

If $s = r+1$, this is $\neq 0$, and the functions z^0, z^2, \dots, z^{2r} , $y^1, y^3, \dots, y^{2r+1}$ are independent. This proves the normal form (10).

Consider next the case $r = s$. Then $d\alpha$ is a two-form of rank $2r$.

By Theorem 3.2 we can write

$$\begin{aligned} d\alpha &= dw^1 \wedge dw^2 + \dots + dw^{2r-1} \wedge dw^{2r} \\ &= d(w^1 dw^2 + \dots + w^{2r-1} dw^{2r}). \end{aligned}$$

Hence the form

$$\alpha = (w^1 dw^2 + \dots + w^{2r-1} dw^{2r})$$

is closed, and is equal to dv . A change of notation gives (11).

A manifold of dimension $2r+1$ provided with a one-form α , defined up to a factor, such that

$$(12) \quad \alpha \wedge (d\alpha)^r \neq 0,$$

is called a contact manifold. An example is the projectivized cotangent bundle of a manifold, whose points are the non-zero one-forms defined up to a factor. A manifold of dimension $2r$ provided with a closed two-form of maximum rank $2r$ is called a symplectic manifold. Both contact manifolds and symplectic manifolds play a fundamental role in theoretical mechanics. Unlike Riemannian manifolds their local properties are fairly simple.

A Pfaffian system is given by

$$(13) \quad \alpha^1 = \dots = \alpha^r = 0,$$

where the α 's are one-forms and are supposed to be linearly independent. At a point $x \in M$ the α 's span an r -dimensional subspace $W_x \subset T_x^*$, which in turn determines its annihilator $W_x^\perp \subset T_x$. Thus a Pfaffian system is geometrically a subbundle. In the above we treated the classical case $r = 1$, in which case the major invariant is the rank. In the case $r = n - 1$ the Frobenius condition is always satisfied and the system has a normal form given by Frobenius theorem. The local invariants in the general case are very complicated. For a recent work, cf. [1].

4. The Cartan-Kähler Theorem.

Let M be a manifold of dimension n and I a closed differential ideal. Let I_p be the set of all elements of I , which are homogeneous of degree p . A submanifold $f: V \rightarrow M$, $\dim V = p$, is called an integral manifold if

$$(1) \quad f^* \alpha = 0, \quad \forall \alpha \in I.$$

By transitivity a submanifold of an integral manifold is an integral manifold. We suppose that the equations

$$\alpha = 0, \quad \alpha \in I_0$$

define a submanifold of M . By restricting to this submanifold, we will suppose in the following that I_0 is empty.

If $v \in V$ and $T_v V$ is the tangent space to V at v , then

$$(2) \quad \langle f_* T_v V, \alpha \rangle = \langle T_v V, f^* \alpha \rangle = 0, \quad \forall \alpha \in I_p.$$

In general, if $x \in M$ and E^p is a p -dimensional subspace of the tangent space T_x at x , we say that (x, E^p) is a p -dimensional integral element of I if

$$(3) \quad \langle E^p, \alpha \rangle = 0, \quad \forall \alpha \in I_p.$$

Lemma 4.1. Let E^q be a q -dimensional subspace of E^p , $q \leq p$. If (x, E^p) is an integral element of I , so is (x, E^q) .

Proof. We choose dual bases e_i, e^{*k} , $1 \leq i, k \leq n$, in T_x, T_x^* so that

$$\langle e_i, e^{*k} \rangle = \delta_i^k$$

and such that $e_1 \wedge \dots \wedge e_q, e_1 \wedge \dots \wedge e_p$ define E^q, E^p respectively. The hypothesis can be expressed by

$$\langle e_1 \wedge \dots \wedge e_p, \alpha \rangle = 0, \quad \forall \alpha \in I_p.$$

Let $\beta \in I_q$. It suffices to prove that

$$\beta \equiv 0, \text{ mod } e^{*q+1}, \dots, e^{*n}.$$

Suppose the contrary. There exists a form γ of degree $p - q$ such that

$$\beta \wedge \gamma \equiv c e^{*1} \wedge \dots \wedge e^{*p}, \quad c \neq 0, \text{ mod } e^{*p+1}, \dots, e^{*n}.$$

Since I is an ideal, $\beta \wedge \gamma \in I_p$. Clearly

$$\langle e_1 \wedge \dots \wedge e_p, \beta \wedge \gamma \rangle = c \neq 0,$$

and we have a contradiction.

Theorem 4.2. Let I be a closed differential ideal on a manifold M . Let $f: V \rightarrow M$, $\dim V = p$, be a submanifold. Then V is an integral manifold if and only if $f_* T_x V$, $x \in V$, are p -dimensional integral elements. It follows that V is an integral manifold if and only if (1) is satisfied for all $\alpha \in I_p$.

Proof. The "only if" part follows from (2). To prove the converse, take $\alpha \in I$ and suppose α be homogeneous. If $\deg \alpha > p$, $f^* \alpha = 0$ for dimension reasons. If $\deg \alpha = p$, $f^* \alpha = 0$ by hypothesis. Finally, let $\deg \alpha = q < p$. By Lemma 4.1, $f^* \alpha = 0$ on any q -dimensional submanifold of V . This is possible only when $f^* \alpha = 0$ on V .

In view of this theorem the problem of finding integral manifolds of I can be geometrically interpreted as "piecing together" the integral elements into a submanifold.

Given an integral element (x, E^{p-1}) of dimension $p - 1$, the first step toward the construction of a p -dimensional integral manifold is to find a vector $\xi \in T_x$ such that $(x, E^{p-1} \wedge \xi)$ is an integral element of dimension p . (Here we identify E^{p-1} with its coordinate $(p-1)$ -vector, necessarily decomposable and defined up to a non-zero factor.) The condition on ξ is

(4)

$$\alpha(x, E^{p-1}, \xi) \stackrel{\text{def}}{=} \langle E^{p-1} \wedge \xi, \alpha \rangle = \pm \langle E^{p-1}, i(\xi)\alpha \rangle = 0, \quad \forall \alpha \in I_p.$$

This is a system of linear homogeneous equations in ξ and is satisfied

whenever $\xi \in E^{p-1}$. (In classical terminology (4) is called "polarization".) The ξ 's satisfying (4) form a linear subspace of T_x , containing E^{p-1} . We will call it the polar space of E^{p-1} and denote it by $L(E^{p-1})$. Its dimension will be denoted by $r_p + p$, $r_p \geq -1$; the notation is so chosen that there is no (resp. a unique) p -dimensional integral element through E^{p-1} when $r_p = -1$ (resp. $= 0$). The integer r_p depends on (x, E^{p-1}) . To determine an integral element E^p through E^{p-1} it suffices to take in T_x a subspace Y of dimension $p - r_p$ through E^{p-1} but otherwise in general position with $H(E^{p-1})$; E^p is then the intersection of Y and $H(E^{p-1})$.

Consider the Grassmann bundle $G_{p-1}(M) \rightarrow M$, whose fibre at each point $x \in M$ is the Grassmann manifold of all $(p-1)$ -dimensional subspaces of the tangent space T_x . We have

$$(5) \quad \dim G_{p-1}(M) = n + (p-1)(n-p+1).$$

The condition for (x, E^{p-1}) to be an integral element is

$$(6) \quad \beta(x, E^{p-1}) \stackrel{\text{def}}{=} \langle E^{p-1}, \beta \rangle = 0, \quad \forall \beta \in I_{p-1}.$$

(E^{p-1} , as a $(p-1)$ -vector, is defined up to a factor, but the condition (6) is well-defined.) From now on and throughout this section we suppose all data to be real or complex analytic. It follows that the $(p-1)$ -dimensional integral elements form an analytic subvariety of $G_{p-1}(M)$.

Definition 4.3. An integral element (x_0, E_0^{p-1}) is called Kähler regular or K-regular if:

1) There exist $\beta^1, \dots, \beta^s \in I_{p-1}$ such that the subvariety $V_{p-1}(I)$ of $(p-1)$ -dimensional integral elements on $G_{p-1}(M)$ is defined in a neighborhood of (p_0, E_0^{p-1}) by

$$(7) \quad \beta^\sigma(x, E^{p-1}) = 0, \quad 1 \leq \sigma \leq s,$$

whose differentials $d\beta^\sigma$ are linearly independent.

2) r_p is constant in a neighborhood of (x_0, E_0^{p-1}) on V_{p-1} .

It follows that V_{p-1} is a manifold in a neighborhood of a K-regular integral element (x_0, E_0^{p-1}) . Moreover, any analytic function f on $G_{p-1}(M)$, which vanishes on V_{p-1} , can be written, in a neighborhood of (x_0, E_0^{p-1}) , as

$$(8) \quad f = g_1 \beta^1(x, E^{p-1}) + \dots + g_s \beta^s(x, E^{p-1}),$$

where the g 's are analytic functions.

An integral element is called K-singular if it is not K-regular. An integral manifold whose tangent spaces are K-regular integral elements is called K-regular.

Lemma 4.4. Let I be an analytic differential ideal, $V_{p-1} \subset G_{p-1}(M)$ be the variety of its $(p-1)$ -dimensional integral elements, and $(x_0, E_0^{p-1}) \in V_{p-1}$ be K-regular. In a neighborhood of (x_0, E_0^{p-1}) on V_{p-1} there are $n - r_p - p$ linearly independent functions in ξ among the $\alpha(x, E^{p-1}, \xi)$ defined in (4), say $\alpha^t(x, E^{p-1}, \xi)$, $1 \leq t \leq n - r_p - p$. To any $\alpha \in I_p$ there exist analytic functions h_t on $G_{p-1}(M)$ and $g_\sigma(x, E^{p-1}, \xi)$ on $G_{p-1}(M) \times R^n$, the latter being linear in $\xi \in R^n$, such that

$$(9) \quad \alpha(x, E^{p-1}, \xi) - \sum_t h_t(x, E^{p-1}) \alpha^t(x, E^{p-1}, \xi) = \sum_{1 \leq \sigma \leq s} g_\sigma(x, E^{p-1}, \xi) \beta^\sigma(x, E^{p-1}),$$

where β^σ are defined in Definition 4.3.

Proof. When considered as linear functions in ξ , the $\alpha(x, E^{p-1}, \xi)$ and $\alpha^t(x, E^{p-1}, \xi)$, $1 \leq t \leq n - r_p - p$ have a matrix of coefficients which are analytic functions on $G_{p-1}(M)$. Restricted to V_{p-1} , the matrix has rank $n - r_p - p$. Hence there are $(n - r_p - p) \times (n - r_p - p)$ minors μ_t of this matrix such that

$$\mu(x, E^{p-1}) \alpha(x, E^{p-1}, \xi) + \sum_t \mu_t(x, E^{p-1}) \alpha^t(x, E^{p-1}, \xi)$$

is a linear function in ξ which restricts to zero on V_{p-1} and such that $\mu(x, E^{p-1})|_{V_{p-1}} \neq 0$. The lemma follows by using the remark following Definition 4.3.

The Cartan-Kähler theorem gives a construction of integral manifolds by induction on dimension. It is a local theorem and is a natural generalization of the Cauchy-Kowalewsky theorem. We now give the statements of both theorems.

Theorem 4.5. (Cauchy-Kowalewsky).

Let z_1, \dots, z_m be functions in n independent variables x^1, \dots, x^n . Consider the system of partial differential equations

$$(10) \quad \frac{\partial z_i}{\partial x^n} = f_i(x^1, \dots, x^n; z_1, \dots, z_m; \frac{\partial z_1}{\partial x^1}, \dots, \frac{\partial z_1}{\partial x^{n-1}}), \quad 1 \leq i, j \leq m$$

where the functions f_i are analytic in a neighborhood of the values

$$(11) \quad x^\alpha = x_0^\alpha, \quad z_i = z_i^0, \quad \frac{\partial z_1}{\partial x^k} = q_{jk}^0, \quad 1 \leq \alpha \leq n, \quad 1 \leq k \leq n-1.$$

Let $\phi_i(x^1, \dots, x^{n-1})$ be m functions, which are analytic in a neighborhood of $(x_0^1, \dots, x_0^{n-1})$ and satisfy the initial conditions

$$(12) \quad \phi_i(x_0^1, \dots, x_0^{n-1}) = z_i^0, \quad \left(\frac{\partial \phi_i}{\partial x^k} \right)_0 = q_{ik}^0.$$

Then the system (5) has a uniquely determined system of solutions

$$(13) \quad z_i = \phi_i(x^1, \dots, x^n)$$

satisfying the initial conditions

$$(14) \quad \phi_i(x^1, \dots, x^{n-1}, x_0^n) = \phi_i(x^1, \dots, x^{n-1}).$$

Theorem 4.6 (Cartan-Kähler)

On an analytic manifold M of dimension n let I be a closed differential ideal. Let N^{p-1} be a K -regular $(p-1)$ -dimensional integral manifold and let (x_0, E_0^{p-1}) be an integral element of N^{p-1} .

Let Y be an $(n-r_p)$ -dimensional submanifold such that

$$(1) \quad N^{p-1} \subset Y;$$

(2) At x_0 the tangent space $T_{x_0} Y$, of dimension $n - r_p$, contains exactly one p -dimensional integral element (x_0, E_0^p) through (x_0, E_0^{p-1}) .

Then in a neighborhood of x_0 there is a uniquely determined integral manifold N^p tangent to E_0^p at x_0 such that

$$(15) \quad N^{p-1} \subset N^p \subset Y.$$

We will not give a proof of Theorem 4.4, which is classical. We proceed to prove Theorem 4.5.

We write $r = r_p$. By changing coordinates if necessary, we suppose Y to be defined by

$$(16) \quad x^{n-r+1} = \dots = x^n = 0.$$

The integral manifold $N^{p-1} \subset Y$ can then be supposed to be given by the additional equations

$$(17) \quad x^p = 0, \quad x^i = \phi^i(x^1, \dots, x^{p-1}), \quad p+1 \leq i \leq n-r.$$

We take x_0 to be the origin and suppose E_0^{p-1} and E_0^p to be defined by

$$\left(\frac{\partial}{\partial x^1} \wedge \dots \wedge \frac{\partial}{\partial x^{p-1}} \right)_{x_0} \quad \text{and} \quad \left(\frac{\partial}{\partial x^1} \wedge \dots \wedge \frac{\partial}{\partial x^p} \right)_{x_0}$$

respectively. The latter is the uniquely determined p -dimensional integral element containing E_0^{p-1} and tangent to Y at x_0 . The integral manifold $N^p \subset Y$ to be constructed can therefore be defined by

$$(18) \quad x^i = \phi^i(x^1, \dots, x^p), \quad p+1 \leq i \leq n-r,$$

satisfying the initial conditions

$$(19) \quad \phi^i(x^1, \dots, x^{p-1}, 0) = \phi^i(x^1, \dots, x^{p-1}), \quad p+1 \leq i \leq n-r.$$

The unknown functions $\phi^i(x^1, \dots, x^p)$ are to annihilate all $\alpha \in I_p$. Let

$$(20) \quad \alpha = \frac{1}{p!} \sum a_{i_1 \dots i_p} (x) dx^{i_1} \wedge \dots \wedge dx^{i_p}, \quad 1 \leq i_1, \dots, i_p \leq n,$$

where the coefficients are analytic functions which are anti-symmetric in their indices. Let

$$(21) \quad E^{p-1} = \frac{1}{(p-1)!} \sum \frac{\partial(x^{i_1}, \dots, x^{i_{p-1}})}{\partial(x^1, \dots, x^{p-1})} \frac{\partial}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{i_{p-1}}},$$

$$(22) \quad \xi = \sum \frac{\partial x^i}{\partial x^p} \frac{\partial}{\partial x^i}.$$

Here we regard x^1, \dots, x^p as independent variables and x^i , $p+1 \leq i \leq n$, as their functions given by (16) and (18). Then the equations in question can be written

$$(23) \quad \alpha(x, E^{p-1}, \xi) = \frac{1}{(p-1)!} \sum a_{i_1 \dots i_p} \frac{\partial(x^{i_1}, \dots, x^{i_{p-1}})}{\partial(x^1, \dots, x^{p-1})} \frac{\partial x^p}{\partial x^{i_p}} = 0,$$

$$\alpha \in I_p.$$

Since (x_0, E_0^{p-1}) is a regular integral element, it has a neighborhood in $V_{p-1} \subset G_{p-1}(M)$, on which $n-r-p$ of the $\alpha(x, E^{p-1}, \xi)$ are linearly independent. Call α^t , $1 \leq t \leq n-r-p$, the corresponding p -forms and denote their coefficients by $a_{i_1 \dots i_p}^t$. The corresponding equations can be written

$$(24) \quad \sum_{j=p+1}^{n-r} \sum_{i_1, \dots, i_{p-1}} a_{i_1 \dots i_{p-1} j}^t \frac{\partial(x^{i_1}, \dots, x^{i_{p-1}})}{\partial(x^1, \dots, x^{p-1})} \frac{\partial x^j}{\partial x^p} = - \sum a_{i_1 \dots i_{p-1} p}^t \frac{\partial(x^{i_1}, \dots, x^{i_{p-1}})}{\partial(x^1, \dots, x^{p-1})}.$$

This is a system of $n-r-p$ linear equations in the same number of unknowns $\partial x^j / \partial x^p$, $p+1 \leq j \leq n-r$. At (x_0, E_0^{p-1}) it has by hypothesis the unique solution $(\partial x^j / \partial x^p)_0 = 0$. Hence in a neighborhood of (x_0, E_0^{p-1}) in $G_{p-1}(M)$ the determinant of the coefficients of the linear system is not zero. The system can be solved to give

$$(25) \quad \frac{\partial x^j}{\partial x^p} = F^j(x^1, \dots, x^{n-r}, \frac{\partial(x^{i_1}, \dots, x^{i_{p-1}})}{\partial(x^1, \dots, x^{p-1})}),$$

$$p+1 \leq j \leq n-r, \quad 1 \leq i_1, \dots, i_{p-1} \leq n-r.$$

This is a Cauchy-Kowalewsky system in a neighborhood of (x_0, E_0^{p-1}) . By Theorem 4.5 there exists a uniquely determined submanifold N^p defined by (18) and satisfying (15).

It remains to show that N^p is an integral manifold of I , i.e., that it annihilates all the forms $\alpha \in I_p$. By construction it annihilates α^t , $1 \leq t \leq n-r-p$. The crux of the matter is to show that the tangent elements (x, E^{p-1}) of N^p , with E^{p-1} given by (21), are $(p-1)$ -dimensional integral elements.

We restrict to the submanifold N^p on which x^1, \dots, x^p are the coordinates. The β^σ in Definition 4.3 can be written

$$(26) \quad \beta^\sigma = \sum_{1 \leq l \leq p} (-1)^{l-1} B_l^\sigma dx^1 \wedge \dots \wedge \widehat{dx^l} \wedge \dots \wedge dx^p, \\ 1 \leq \sigma \leq s.$$

We can write

$$(27) \quad E^{p-1} = \frac{\partial}{\partial x^1} \wedge \dots \wedge \frac{\partial}{\partial x^{p-1}},$$

so that

$$(28) \quad \beta^\sigma(x, E^{p-1}) = \frac{(-1)^{p-1}}{(p-1)!} B_p^\sigma = B^\sigma, \text{ say.}$$

Let $\alpha \in I_p$. On N^p we can write

$$(29) \quad \alpha = A dx^1 \wedge \dots \wedge dx^p.$$

By (9) in Lemma 4.4, since the α^t , $1 \leq t \leq n-r-p$, vanish on N^p , we have

$$(30) \quad A = \sum_{1 \leq \sigma \leq s} g_\sigma B^\sigma,$$

where g_σ are analytic functions in a neighborhood of x_0 .

Among the forms in I_p are $d\beta^\sigma$ (because I is closed) and $dx^l \wedge \beta^\sigma$, $1 \leq l \leq p$ (because I is an ideal). We find

$$(31) \quad d\beta^\sigma = \left(\frac{\partial B_1^\sigma}{\partial x^1} + \dots + \frac{\partial B_p^\sigma}{\partial x^p} \right) dx^1 \wedge \dots \wedge dx^p \quad 1 \leq \sigma \leq s, \\ dx^l \wedge \beta^\sigma = B_l^\sigma dx^1 \wedge \dots \wedge dx^p \quad 1 \leq l \leq p.$$

By (30) their coefficients are linear combinations of B^1, \dots, B^s .

Expressing this fact, we get a Cauchy-Kowalewsky system

$$(32) \quad \frac{\partial B^\sigma}{\partial x^p} = \text{linear combination of } B^1, \frac{\partial B^1}{\partial x^1}, \dots, \frac{\partial B^1}{\partial x^{p-1}} \\ 1 \leq \sigma, \tau \leq s$$

where B^σ satisfy the initial conditions

$$(33) \quad B^\sigma(x^1, \dots, x^{p-1}, 0) = 0.$$

Hence $B^\sigma = 0$ and it follows by (30) that any $\alpha \in I_p$ restricts to zero on N^p . This proves Theorem 4.6.

In the proof we can define Y by the equations

$$(34) \quad x^\mu = \psi^\mu(x^1, \dots, x^p), \quad n-r+1 \leq \mu \leq n.$$

Hence the general solution depends on r_p functions in p variables.

The Cartan-Kähler theorem permits us to construct integral manifolds by induction on dimension. For applications the notion of Cartan regularity plays an important role. It is given by:

Definition 4.7. An integral element (x_0, E_0^{p-1}) is called Cartan regular or C-regular if:

1) It contains a C-regular integral element (x_0, E_0^{p-2}) . (This condition is empty for $p = 1$).

2) r_p is constant in a neighborhood of it on V_{p-1} .

If only the condition 1) is satisfied, we say that (x_0, E_0^{p-1}) is C-ordinary.

It follows that a C-ordinary integral element (x_0, E_0^{p-1}) is the end-element of a nested sequence of integral elements

$$(35) \quad x_0 \in E_0^1 \subset E_0^2 \subset \dots \subset E_0^{p-2} \subset E_0^{p-1},$$

of which (x_0, E_0^k) , $1 \leq k \leq p-2$, is C-regular. (35) is called a regular integral flag. By a successive application of the Cartan-Kähler theorem 4.6, we conclude that if (x_0, E_0^{p-1}) is C-ordinary there is an integral manifold N^{p-1} through x_0 and tangent to E_0^{p-1} .

The relation between K-regularity and C-regularity is clarified by the following theorem and example.

Theorem 4.8. If an integral element (x_0, E_0^{p-1}) is C-regular, then it is also K-regular.

In the example following this theorem, we will show that the converse is not true, that is, K-regularity is more general than C-regularity.

Proof. For simplicity we will assume $I_0 = \{0\}$. Then every point of M is integral and, for $p = 1$, C-regularity clearly coincides with K-regularity.

In general, our hypothesis says that we have the regular integral flag (35). Suppose E_0^q be spanned by e_1^0, \dots, e_q^0 , $q \leq p-1$. We extend these into a frame field e_i , $1 \leq i \leq n$, in a neighborhood of x_0 . It follows by continuity that in a neighborhood of x_0 , the E^q spanned by e_1, \dots, e_q , $q \leq p-1$, also form a regular integral flag

$$(36) \quad x \in E^1 \subset \dots \subset E^{p-1}.$$

Let ω^j , $1 \leq j \leq n$, be one-forms which are the dual coframe, so that

$$(37) \quad (e_i, \omega^j) = \delta_i^j, \quad 1 \leq i, j \leq n.$$

An element (x, E^{p-1}) (not necessarily integral) near (x_0, E_0^{p-1}) will be spanned by the vectors

$$(38) \quad e_q = \sum_r \ell_q^r e_r, \quad 1 \leq q \leq p-1, \quad p \leq r \leq n.$$

Then the local coordinates x^i of x and the ℓ_q^r will form a local coordinate system on $G_{p-1}(M)$.

Let

$$(39) \quad \alpha_q^t, \quad 1 \leq t \leq n - r_q - q,$$

be the q -forms which define the polar space $H(E^{q-1})$. Then near (x_0, E_0^{p-1}) the variety $V_{p-1}(I)$ of C -regular $(p-1)$ -dimensional integral elements consists exactly of those (x, E^{p-1}) which annihilate the $(p-1)$ -forms

$$(40) \quad \alpha_1^t \wedge \omega^2 \wedge \dots \wedge \omega^{p-1}, \quad \alpha_2^t \wedge \omega^3 \wedge \dots \wedge \omega^{p-1}, \quad \dots, \quad \alpha_{p-1}^t,$$

The latter constitute a basis of I_{p-1} . By expressing the condition that the E^{p-1} spanned by the vectors in (38) annihilate these forms, we get equations which are linear in ℓ_q^r for each q and define $V_{p-1}(I)$ as a regular submanifold in $G_{p-1}(M)$. This proves that (x_0, E_0^{p-1}) is K -regular.

Remark. These equations for $V_{p-1}(I)$ are clearly independent. Their number is

$$\sum_{1 \leq i \leq p-1} (n - r_i - i) = (p-1)n - \sum_i r_i - \frac{1}{2} p(p-1).$$

Since the fiber of $G_{p-1}(M)$ is of dimension $(p-1)(n-p+1)$, it follows that

$$\dim V_{p-1}(I) = n + \sum_{1 \leq i \leq p-1} r_i - \frac{1}{2} (p-1)(p-2).$$

In the integral flag $E_0^{q-1} \subset E_0^q$, if a vector spans an integral element with E_0^q , it does so with E_0^{q-1} . Hence $H(E_0^q) \subset H(E_0^{q-1})$, and we have

$$r_{q+1} + q + 1 \leq r_q + q.$$

We introduce the integers

$$(41) \quad s_q = r_q - r_{q+1} - 1 \geq 0, \quad 1 \leq q \leq p-1.$$

Corollary 4.9. Let the integral element (x_0, E_0^{p-1}) be C -regular. Near it the variety $V_{p-1}(I)$ of integral elements has the dimension

$$(42) \quad \begin{aligned} \dim V_{p-1}(I) &= n + \sum_{1 \leq i \leq p-1} r_i - \frac{1}{2} (p-1)(p-2) \\ &= n + s_1 + 2s_2 + \dots + (p-2)s_{p-2} + (p-1)r_{p-1}. \end{aligned}$$

Example. In R^5 consider the coframe $\omega^1, \omega^2, \alpha, \beta^1, \beta^2$, satisfying the equations

$$(43) \quad \begin{aligned} d\omega^1 &= \alpha \wedge \beta^1, \quad d\omega^2 = \alpha \wedge \beta^2, \\ d\beta^1 &= d\beta^2 = 0, \quad d\alpha = \beta^1 \wedge \beta^2. \end{aligned}$$

Let I be the differential system generated by $\{\omega^1, \omega^2, d\omega^1, d\omega^2\}$. It is clearly closed. It has only one two-dimensional integral element E^2 , i.e., $\omega^1 = \omega^2 = \alpha = 0$. Hence it is K-regular.

On the other hand, the system J generated by $\{\omega^1, \omega^2, \alpha\}$ is not closed. To make it so, we should add the form $\beta^1 \wedge \beta^2$. Thus it cannot have two-dimensional integral manifolds. Hence E^2 is not C-ordinary.

5. Isometric imbedding of Riemannian manifolds; molding surfaces

This section will be concerned with the local isometric imbedding of a real analytic Riemannian manifold into an Euclidean space. The main theorem is:

Theorem 5.1. (Schläfli-Cartan) A real-analytic Riemannian manifold of dimension n can be locally imbedded in an Euclidean space E^N of dimension $N = n(n+1)/2$.

Let M be a Riemannian manifold of dimension n . In local coordinates x^i , suppose that the Riemannian metric is

$$(1) \quad ds^2 = \sum_{i,k} g_{ik} dx^i dx^k, \quad 1 \leq i, j, k \leq n,$$

where $g_{ik} = g_{ki}$ are analytic functions of x^j . The isometric imbedding problem is to find functions $y^A(x^1, \dots, x^n)$ such that

$$(2) \quad ds^2 = \sum_A (dy^A)^2, \quad 1 \leq A \leq N.$$

In other words, the functions y^A are to satisfy the differential equations

$$(3) \quad \sum_A \frac{\partial y^A}{\partial x^i} \frac{\partial y^A}{\partial x^k} = g_{ik}.$$

The solution of the problem depends on an understanding of the geometry.

We will treat the case $n = 2$, postponing the general case to a later occasion. Over M let P be the principal bundle of orthonormal frames (x, e_1, e_2) , where $x \in M$ and e_1, e_2 are mutually perpendicular unit vectors at x . Let ω_1, ω_2 be the coframe dual to e_1, e_2 . Then the metric on M is

$$(4) \quad ds^2 = \omega_1^2 + \omega_2^2.$$

There exists a uniquely determined one-form ω_{12} in P , the connection form, so that the structure equations

$$(5) \quad \begin{aligned} d\omega_1 &= \omega_{12} \wedge \omega_2, \quad d\omega_2 = -\omega_1 \wedge \omega_{12}, \\ d\omega_{12} &= -K\omega_1 \wedge \omega_2 \end{aligned}$$

are fulfilled. K is a function on M and is the Gaussian curvature.

Similarly, in the three-dimensional Euclidean space E^3 consider the space \tilde{P} of all orthonormal frames $(y, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$. It is a six-dimensional manifold and can be identified with the space of all rigid motions of E^3 . By the equations

$$(6) \quad \begin{aligned} dy &= \sum \tilde{\omega}_i \tilde{e}_i, \\ d\tilde{e}_i &= \sum \tilde{\omega}_{ij} \tilde{e}_j, \quad 1 \leq i, j, k \leq 3, \\ \tilde{\omega}_{ij} + \tilde{\omega}_{ji} &= 0, \end{aligned}$$

we introduce the forms $\tilde{\omega}_i, \tilde{\omega}_{ij}$. Since

$$d(dy) = d(d\tilde{e}_i) = 0,$$

we have the structure equations in E^3 :

$$(7) \quad \begin{aligned} d\tilde{\omega}_i &= \sum \tilde{\omega}_j \wedge \tilde{\omega}_{ji}, \\ d\tilde{\omega}_{ij} &= \sum \tilde{\omega}_{ik} \wedge \tilde{\omega}_{kj}. \end{aligned}$$

The metric in E^3 is

$$(8) \quad d\tilde{s}^2 = \tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{\omega}_3^2.$$

An isometric immersion is a mapping $f: M \rightarrow E^3$ such that $f^*d\tilde{s}^2 = ds^2$. It gives rise to the following diagram of mappings:

$$(9) \quad \begin{array}{ccc} P & \xrightarrow{\tilde{y}} & \tilde{P} \\ \pi \downarrow & & \downarrow \tilde{\pi} \\ M & \xrightarrow{y} & E^3 \end{array}$$

Here π and $\tilde{\pi}$ are the respective projections assigning to a frame its origin, y is the isometric immersion, and \tilde{y} sends the orthonormal frame (x, e_1, e_2) to the frame $(y(x), y_*(e_1), y_*(e_2), \tilde{e}_3)$, where y_* is the mapping induced by y on tangent vectors and $\tilde{e}_3 = y_*(e_1) \times y_*(e_2)$, the latter being the vector product. \tilde{y} sends orthonormal frames into orthonormal frames because y is an isometric immersion. The diagram (9) is clearly commutative. It leads to the differential system

$$(10) \quad \tilde{\omega}_1 - \omega_1 = \tilde{\omega}_2 - \omega_2 = \tilde{\omega}_3 = 0$$

in the 9-dimensional space $P \times \tilde{P}$. Clearly, a solution of (10), satisfying $\omega_1 \wedge \omega_2 \neq 0$, gives an isometric immersion of M in E^3 .

The system (10) is not closed. Exterior differentiation of the first two equations of (10) gives, by the use of the structure equations (5) and (7),

$$(\tilde{\omega}_{12} - \omega_{12}) \wedge \omega_1 = (\tilde{\omega}_{12} - \omega_{12}) \wedge \omega_2 = 0.$$

Since $\omega_1 \wedge \omega_2 \neq 0$, this gives

$$(10a) \quad \tilde{\omega}_{12} - \omega_{12} = 0.$$

Geometrically this means that the isometry preserves the connection.

It follows that the system (10) should be "prolonged" to the following:

$$(11) \quad \tilde{\omega}_1 - \omega_1 = \tilde{\omega}_2 - \omega_2 = \tilde{\omega}_3 - \omega_3 = \tilde{\omega}_{12} - \omega_{12} = 0.$$

The exterior derivatives of the first two equations are now identically satisfied, as a consequence of the equations themselves, and the exterior derivatives of the last two equations give

$$(11a) \quad \begin{aligned} \omega_1 \wedge \tilde{\omega}_{13} + \omega_2 \wedge \tilde{\omega}_{23} &= 0, \\ \tilde{\omega}_{13} \wedge \tilde{\omega}_{23} - K\omega_1 \wedge \omega_2 &= 0. \end{aligned}$$

The isometric imbedding problem is thus reduced to the system consisting of the equations (11), (11a), which is closed.

We seek two-dimensional integral elements E^2 given by

$$(12) \quad \begin{aligned} \tilde{\omega}_{13} &= l_{11}\omega_1 + l_{12}\omega_2, \\ \tilde{\omega}_{23} &= l_{21}\omega_1 + l_{22}\omega_2, \end{aligned}$$

such that $\omega_2 = 0$ defines a regular one-dimensional integral element E^1 contained in it. Equations (11a) give

$$(13) \quad \begin{aligned} l_{12} &= l_{21} \\ l_{11}l_{22} - l_{12}^2 &= K. \end{aligned}$$

These equations determine l_{12}, l_{22} , if $l_{11} \neq 0$. In this case the integral element E^2 through E^1 is uniquely determined. We take an integral curve, which has the integral elements E^1 as tangents. Through it a two-dimensional solution of the system (11), (11a) is uniquely determined. This proves Theorem 5.1 for the case $n = 2$.

More precisely, given a curve C on M and a curve \tilde{C} in E^3 , we wish to construct an isometric imbedding of M in E^3 such that C goes into \tilde{C} . This requires that the mapping of C into \tilde{C} be an isometry, as expressed by the first equation of (11). The last equation of (11) means that C and \tilde{C} have the same geodesic curvature at corresponding points. The geodesic curvature of \tilde{C} is given by $\tilde{k} \sin \theta$, where \tilde{k} is the curvature and θ the angle which the principal normal of \tilde{C} makes with the surface normal. As a consequence we must have $\tilde{k} \geq |k_g|$ at corresponding points, k_g being the geodesic curvature of C . The equality of the geodesic curvatures at corresponding points of C and \tilde{C} gives $\sin \theta = k_g / \tilde{k}$, which in turn gives two determinations of the surface normal. Once a choice of the surface normal is made,

the surface through \tilde{C} and isometric to M is uniquely determined, assuming $k_{11} \neq 0$.

The condition $k_{11} \neq 0$ has also a simple geometric meaning. In fact,

$$\omega_1 \tilde{\omega}_{13} + \omega_2 \tilde{\omega}_{23} = k_{11} \omega_1^2 + 2k_{12} \omega_1 \omega_2 + k_{22} \omega_2^2$$

is the second fundamental form of the imbedded surface. The curve \tilde{C} , being defined by $\omega_2 = 0$, is an asymptotic curve if and only if $k_{11} = 0$. Hence the above imbedding theorem applies only to the case that \tilde{C} is nonasymptotic. In fact, if \tilde{C} is an asymptotic curve, its torsion is $\pm\sqrt{K}$ (Enneper's theorem), so that \tilde{C} is subject to more conditions. A corresponding isometric imbedding theorem is not contained in our general theory without a further prolongation of the differential system.

Intimately related to the problem of isometric imbedding is that of rigidity. The above discussion shows that surfaces in E^3 are locally not rigid, i.e., isometry does not imply congruence. It is natural to impose further conditions. In particular, we shall study the interesting problem of isometric surfaces such that the isometry preserves the lines of curvature. This study leads to the molding surfaces in a remarkable way.

Let M be a surface in E^3 . We shall stay away from its umbilics. Then at every point $x \in M$ we take the orthonormal frame (x, e_1, e_2, e_3) , where e_1, e_2 are along the principal directions and e_3 is the unit surface normal vector at x . This family of frames satisfies the equations

$$\begin{aligned} dx &= \omega_1 e_1 + \omega_2 e_2, \\ (14) \quad de_i &= \sum_j \omega_{ij} e_j, \\ \omega_{ij} + \omega_{ji} &= 0, \quad 1 \leq i, j, k \leq 3. \end{aligned}$$

As in (7), we have the structure equations

$$\begin{aligned} (15) \quad d\omega_1 &= \omega_{12} \wedge \omega_2, \quad d\omega_2 = \omega_1 \wedge \omega_{12}, \\ d\omega_{ij} &= \sum_k \omega_{ik} \wedge \omega_{kj}. \end{aligned}$$

The condition that e_1, e_2 are along principal directions is expressed by

$$(16) \quad \omega_{13} = a\omega_1, \quad \omega_{23} = c\omega_2,$$

so that $K = ac$ is the Gaussian curvature. We shall suppose $K \neq 0$.

Exterior differentiation of (16) and use of (15) give

$$\begin{aligned} (17) \quad da \wedge \omega_1 + (a - c)\omega_{12} \wedge \omega_2 &= 0, \\ dc \wedge \omega_2 + (a - c)\omega_{12} \wedge \omega_1 &= 0. \end{aligned}$$

Since the frame (x, e_1, e_2, e_3) is completely determined at x , we can write

$$(18) \quad \omega_{12} = h\omega_1 + k\omega_2.$$

We will use ω_1, ω_2 to express the differential of any function f on M , thus

$$(19) \quad df = f_1 \omega_1 + f_2 \omega_2,$$

so that f_1, f_2 are "the directional derivatives" of f . Using this notation, we have, from (17), (18),

$$(20) \quad \begin{aligned} a_2 &= (a - c)h, \\ c_1 &= (a - c)k. \end{aligned}$$

Let M^* be a surface isometric to M such that the isometry preserves the lines of curvature. Using asterisks to denote the quantities pertaining to M^* , we have

$$(21) \quad \begin{aligned} \omega_1^* &= \omega_1, \quad \omega_2^* = \omega_2, \quad \omega_3^* = 0, \quad \omega_{12}^* = \omega_{12}, \\ \omega_{13}^* &= t\omega_1, \quad \omega_{23}^* = \frac{c}{t} \omega_2. \end{aligned}$$

The last two equations follow from the fact that M^* has the same Gaussian curvature as M at corresponding points. Equation (20) gives, when applied to M^* ,

$$(22) \quad \begin{aligned} (ta)_2 &= (ta - \frac{c}{t})h, \\ (\frac{1}{t}c)_1 &= (ta - \frac{c}{t})k. \end{aligned}$$

Comparison of (20) and (22) gives

$$(23) \quad t_1 = t(1 - t^2)ac^{-1}k,$$

$$t_2 = -t^{-1}(1 - t^2)a^{-1}ch$$

or

$$(23a) \quad \frac{tdt}{1-t^2} = t^2 ac^{-1}k\omega_1 - a^{-1}ch\omega_2.$$

From now on we suppose $t^2 \neq 1$, discarding the trivial case that M^* is congruent or symmetric to M . We put

$$(24) \quad \begin{aligned} \pi_1 &= ac^{-1}k\omega_1, \\ \pi_2 &= a^{-1}ch\omega_2, \end{aligned}$$

so that (23a) can be written

$$(23b) \quad \frac{tdt}{1-t^2} = t^2 \pi_1 - \pi_2.$$

Exterior differentiation of (23b) gives

$$(25) \quad t^2(d\pi_1 - 2\pi_1 \wedge \pi_2) = d\pi_2 - 2\pi_2 \wedge \pi_1.$$

This equation, if not satisfied identically, completely determines t^2 . On substituting into (23), we get conditions on the surfaces M , to which there exist isometric but not congruent or symmetric surfaces preserving the lines of curvature. The latter are uniquely determined up

to position in space.

The most interesting case is when the equation (25) is identically satisfied, i.e., both sides of (25) are zero. We write

$$(26) \quad h' = a^{-1}h, \quad k' = c^{-1}k;$$

then (24) becomes

$$(27) \quad \pi_1 = k' \omega_{13}, \quad \pi_2 = h' \omega_{23},$$

and they are to satisfy the equations

$$(28) \quad d\pi_1 = d\pi_2 = 2\pi_1 \wedge \pi_2.$$

Substituting (27) into (28) and making use of the structure equations (15), we get

$$(29) \quad \begin{aligned} (dh' - h'k'\omega_{13}) \wedge \omega_{23} &= 0, \\ (dk' + h'k'\omega_{23}) \wedge \omega_{13} &= 0. \end{aligned}$$

We shall show that these equations imply $hk = 0$. It will be a remarkable piece of calculation; an important trick simplifying the calculation is to use ω_{13} , ω_{23} in place of ω_1 , ω_2 as the independent one-forms.

Equations (29) allow us to set

$$(30) \quad \begin{aligned} dh' &= h'k'\omega_{13} + q'\omega_{23}, \\ dk' &= p'\omega_{13} - h'k'\omega_{23}. \end{aligned}$$

On the other hand, equation (18) can be written

$$(31) \quad \omega_{12} = h'\omega_{13} + k'\omega_{23}.$$

By (15) we have

$$(32) \quad \begin{aligned} d\omega_{13} &= \omega_{12} \wedge \omega_{23} = h'\omega_{13} \wedge \omega_{23}, \\ d\omega_{23} &= -\omega_{12} \wedge \omega_{13} = -k'\omega_{13} \wedge \omega_{23}. \end{aligned}$$

Taking the exterior derivative of (31) and using (30), (32), we get

$$(33) \quad p' - q' + 1 + h'^2 + k'^2 = 0.$$

If h' and k' are considered as unknown functions, equations (30) and (33) give three relations between their directional derivatives. This primitive counting shows that the differential system is over-determined. To study our problem there is no other way but to examine the integrability conditions through differentiation of (30), (33). In this case the integrability conditions give a very simple conclusion.

We first record the formulas

$$(34) \quad \begin{aligned} d(h'k') &= (h'k'^2 + h'p')\omega_{13} + (-h'^2k' + k'q')\omega_{23}, \\ \frac{1}{2} d(h'^2 + k'^2) &= (h'^2k' + k'p')\omega_{13} + (-h'k'^2 + h'q')\omega_{23}, \end{aligned}$$

which follow from (30). Exterior differentiation of (30) gives

$$(35) \quad \begin{aligned} (dq' + 2h'^2 k' \omega_{13}) \wedge \omega_{23} &= 0, \\ (dp' + 2h' k'^2 \omega_{23}) \wedge \omega_{13} &= 0, \end{aligned}$$

which allow us to set

$$(36) \quad \begin{aligned} dp' &= p'' \omega_{13} - 2h' k'^2 \omega_{23}, \\ dq' &= -2h' k'^2 \omega_{13} + q'' \omega_{23}. \end{aligned}$$

Differentiation of (33) then gives

$$(37) \quad \begin{aligned} p'' &= 2k'(-2h'^2 - p'), \\ q'' &= 2h'(-2k'^2 + q'). \end{aligned}$$

From (33) we can set

$$(38) \quad \begin{aligned} p' &= u - \frac{1}{2} (1 + h'^2 + k'^2), \\ q' &= u + \frac{1}{2} (1 + h'^2 + k'^2), \end{aligned}$$

so that

$$(39) \quad 2u = p' + q'.$$

It follows from (36)-(39) that

$$(40) \quad \begin{aligned} du &= +k'(-u + \frac{1}{2} - \frac{5}{2} h'^2 + \frac{1}{2} k'^2) \omega_{13} \\ &\quad + h'(u + \frac{1}{2} + \frac{1}{2} h'^2 - \frac{5}{2} k'^2) \omega_{23}. \end{aligned}$$

Taking the exterior derivative of this equation, we get

$$(41) \quad h' k' = 0 \quad \text{or} \quad hk = 0,$$

as we have stated above.

We wish to describe these surfaces geometrically. Suppose $k = 0$. Then, by (30), (33)

$$p' = 0, \quad q' = 1 + h'^2.$$

It follows that the surfaces in question satisfy the equations

$$(42) \quad \begin{aligned} \omega_3 &= 0, \quad \omega_{13} = a\omega_1, \quad \omega_{23} = c\omega_2, \quad \omega_{12} = h\omega_1, \\ d(\frac{h}{a}) &= c(1 + \frac{h^2}{a^2})\omega_2, \\ \omega_1 \wedge da - h(a - c)\omega_1 \wedge \omega_2 &= 0, \\ \omega_2 \wedge dc &= 0. \end{aligned}$$

The last three equations are obtained by exterior differentiation of the three equations before them. Hence the differential system (42) is closed.

To describe these surfaces observe that

$$\omega_1 = 0, \quad (\text{resp } \omega_2 = 0)$$

defines a family of lines of curvature, to be denoted by Γ_2 (resp Γ_1).

Along a curve of Γ_2 , we have $\omega_{12} = 0$, so that these curves are geodesics. Writing $\omega_2 = ds$, we have, along a curve of Γ_2 ,

$$\frac{dx}{ds} = e_2, \quad \frac{de_2}{ds} = ce_3, \quad \frac{de_3}{ds} = -ce_2, \quad \frac{de_1}{ds} = 0.$$

Hence it is a plane curve with curvature c , the plane having the normal e_1 . The last equation of (42) says that dc is a multiple of ω_2 , which means that all the curves of Γ_2 are congruent to each other.

Since

$$de_1 = \omega_{12}e_2 + \omega_{13}e_3 = (he_2 + ae_3)\omega_1,$$

the intersection of two neighboring planes of the curves of Γ_2 is a line in the direction

$$e_1 \times (he_2 + ae_3) = -ae_2 + he_3.$$

By (14) and (42), we have

$$d(-e_2 + \frac{h}{a}e_3) = +\frac{h}{a}\omega_{23}(-e_2 + \frac{h}{a}e_3).$$

Hence this direction is fixed. It follows that the planes of the lines of curvature in Γ_2 are the tangent planes of a cylinder Z .

The curves of Γ_1 , being tangent to e_1 , are the orthogonal trajectories of the tangent planes of Z . Each line of curvature of Γ_1 is thus the locus of a point in a tangent plane of Z as the latter rolls about Z . The curves of Γ_2 are the orthogonal trajectories of those of Γ_1 . Each of them is therefore the position taken by a fixed curve on a tangent plane through the rolling.

The surfaces defined by (42) can be kinematically described as follows: Take a cylinder Z and a curve C on one of its tangent planes. The surface M is the locus described by C as the tangent plane rolls about Z . Such a surface is called a molding surface. It depends on two arbitrary functions in one variable, one defining the base curve of Z and the other the plane curve C .

On a molding surface the equation (23b) is completely integrable and has a solution t which depends on an arbitrary constant. We get in this way a non-trivial family of isometric surfaces preserving the lines of curvature (and in fact, all such families). The geometrical conclusion, reached after a lucky computation, can be stated in the following theorem:

Theorem 5.2. In the three-dimensional Euclidean space E^3 consider two pieces of surfaces M, M^* , such that: (a) their Gaussian curvature

$\neq 0$ and they have no umbilics; (b) they are connected by an isometry $f: M \rightarrow M^*$ preserving the lines of curvature. Then M and M^* are in general congruent or symmetric. There are surfaces M , for which the corresponding M^* is distinct relative to rigid motions. The molding surfaces, and only these, are such surfaces belonging to a continuous family of distinct surfaces, which are connected by isometries preserving the lines of curvature.

We observe that among the molding surfaces are the surfaces of revolution.

The computations in this case point to the unpredictable nature of the integrability conditions of an overdetermined system. In deriving such results the general theory of overdetermined systems does not seem to be very helpful.

6. Involution

Let I be a closed differential system on a manifold M and suppose that E^p is a p -dimensional integral element. We ask the question: When is there an integral manifold of I having E^p as a tangent space? In case E^p is a Cartan ordinary integral element and everything is real analytic, the existence of such an integral manifold is provided by the Cartan-Kähler theorem. However, it is clear that this is not necessary, and moreover sometimes there are additional conditions on E^p that must be satisfied. In particular we wish to mention at the beginning that involution corresponds to the Cauchy initial value problem being well posed, and prolongation is nothing more than the introduction of derivatives as new variables.

In (a) we introduce the important concepts of a differential system with independence condition (cf. (6.1)) and the property of involution for such systems (cf. (6.5)).

Although the definition of involution is easy, there are some hidden subtleties involving the variety of all integral elements, and these are taken up in (b). More importantly, it is clumsy to check from the definition when a system is in involution, and so also in (b) we give a numerical criterion called Cartan's test for involution (cf. (6.29)). In a subsequent paper we will show that the homological formulation of this test using Spencer cohomology is the central technical point in the proof of the Cartan-Kuranishi prolongation theorem.

In (c) we discuss the important class of Pfaffian systems in good form; for these Cartan's test assumes a formulation purely in terms of linear algebra.

Finally, in (d) we introduce the concept of a Pfaffian system in dual form and use this setting to discuss a single 2nd order P.D.E. in one unknown function.

We remark that we have chosen our notations and terminology so as to try and make accessible the beautiful examples at the end of [2].

Finally, we remark that for the remainder of these notes regular will be Cartan regular and ordinary will be Cartan ordinary, both as defined in §4. We shall also borrow some terminology from algebraic geometry, primarily the following: A Zariski open subset of a real analytic variety is the complement (assumed non-empty) of a closed subvariety. A real analytic variety V is irreducible if it is not the union of two non-empty subvarieties.

(a) We assume given a closed differential system I on a manifold M . It will simplify matters if we assume from the outset that all data are real analytic; once this case is understood the possible extensions to the C^∞ situation will be clear. Our considerations will be local, and therefore will only be valid in a neighborhood of a point.

Many problems require the existence of integral manifolds satisfying a transversality condition given by the following:

(6.1) Definition: A differential system with independence condition, denoted by (I, ω) , is given by a closed differential ideal together with a decomposable p -form ω that, modulo I , is well-defined and non-zero.

Explicitly, in each open set U_α of a covering $\{U_\alpha\}$ of M we should be given

$$\omega_\alpha = \omega_\alpha^1 \wedge \dots \wedge \omega_\alpha^p$$

such that

$$\omega_\alpha \equiv \omega_\beta \text{ modulo } I \text{ in } U_\alpha \cap U_\beta$$

and

$$\omega_\alpha(x) \notin I(x) \quad x \in U_\alpha.$$

We shall work in a neighborhood of a point and write

$$\omega = \omega^1 \wedge \dots \wedge \omega^p.$$

(6.2) Definitions: (i) An admissible integral element is a p-dimensional integral element for I on which ω is non-zero; (ii) an admissible integral manifold is given by a p-dimensional manifold N together with a maximal rank mapping

$$f: N \rightarrow M$$

such that each $f_*(T_y(N))$ ($y \in N$) is an admissible integral element (thus $f^*(I) = 0$ and $f^*\omega \neq 0$). Admissible integral manifolds will also be referred to as integral manifolds of (I, ω) .

(6.3) Example: Any P.D.E. system

$$F_\lambda(x^i, z^\alpha, \partial z^\alpha / \partial x^i, \dots, \partial^k z^\alpha / \partial x^i) = 0, \quad 1 \leq \lambda \leq k$$

may be written as a differential system with independence condition. For instance, in the 2nd order case we introduce variables

$$p_i^\alpha, p_{ij}^\alpha = p_{ji}^\alpha$$

and then the system is defined on the space of variables $(x^i, z^\alpha, p_i^\alpha, p_{ij}^\alpha)$ and is generated by the equations

$$\begin{cases} F_\lambda(x^i, z^\alpha, p_i^\alpha, p_{ij}^\alpha) = 0, & 1 \leq \lambda \leq k \\ dz^\alpha - p_i^\alpha dx^i = 0 \\ dp_i^\alpha - p_{ij}^\alpha dx^j = 0 \end{cases}$$

with the independence condition given by $\omega = dx^1 \wedge \dots \wedge dx^p$. An admissible integral manifold of the differential system with independence condition is the same as a local solution to the P.D.E. system.

(6.4) Example: The isometric embedding problem, discussed at the beginning of §5, is naturally formulated as finding an admissible integral manifold of a differential system with independence condition.

The following is one of the main concepts in the theory:

(6.5) Definition: The differential system with independence condition (I, ω) is in involution at $x \in M$ if there exists an ordinary, admissible integral element $E^p \subset T_x(M)$.

We sometimes say that (I, ω) is involutive at $x \in M$.

When (I, ω) is in involution, the Cartan-Kähler theorem may be applied to conclude that there exist local integral manifolds of (I, ω) passing through $x \in M$.

As will be explained in the next section the definition of being in involution contains some hidden subtleties. For the moment we emphasize

that the question of whether (I, ω) is involutive entails the following considerations:

- (i) The inadmissible integral elements form a subvariety $Z(\omega)$ of $G_p(M)$ (the notation stands for "zeroes of ω ");
- (ii) Among the variety $V_p(I)$ of p -dimensional integral elements, the ordinary ones, denoted $O_p(I)$, are either empty or a Zariski open set;
- (iii) Consequently, assuming the fibre of $O_p(I)$ over x is non-empty, (I, ω) fails to be in involution exactly when

$$O_p(I) \subseteq Z(\omega),$$

i.e., when the condition that $E^p \subset T_x(M)$ be ordinary forces a linear relation among the restrictions $\omega^i|_{E^p}$ (this is the definition given by Cartan [2]).

(6.6) Example: On a 6-dimensional manifold with basis

$\theta^1, \theta^2, \omega^1, \omega^2, \pi^1, \pi^2$ for the 1-forms, we consider a Pfaffian system $\theta^1 = \theta^2 = 0$ with independence condition $\omega^1 \wedge \omega^2 \neq 0$ and structure equations

$$d\theta^1 \equiv \pi^1 \wedge \omega^1 \mod I,$$

$$d\theta^2 \equiv \pi^1 \wedge \omega^2 \mod I.$$

We shall show that this system is not in involution.

For this we denote by $\partial/\partial\theta^1, \partial/\partial\theta^2, \partial/\partial\omega^1, \partial/\partial\omega^2, \partial/\partial\pi^1, \partial/\partial\pi^2$ the basis of tangent vectors dual to the above basis of forms. A general vector in the space $\theta^1 = \theta^2 = 0$ is

$$(6.7) \quad \xi = \xi^0 \frac{\partial}{\partial\pi^1} + \xi^1 \frac{\partial}{\partial\pi^2} + \xi^2 \frac{\partial}{\partial\omega^1} + \xi^3 \frac{\partial}{\partial\omega^2}.$$

Using self-evident notation, the polar equations

$$\langle d\theta^1, \xi \wedge \tilde{\xi} \rangle = 0 = \langle d\theta^2, \xi \wedge \tilde{\xi} \rangle$$

of the vector ξ in (6.7) are

$$(6.8) \quad \begin{cases} \xi^0 \tilde{\xi}^2 - \xi^2 \tilde{\xi}^0 = 0 \\ \xi^0 \tilde{\xi}^3 - \xi^3 \tilde{\xi}^0 = 0. \end{cases}$$

If $\xi^0 \neq 0$ their rank is 2.

On the other hand, any 2-plane E_0^2 on which $\theta^1 = \theta^2 = 0$ and $\omega^1 \wedge \omega^2 \neq 0$ is given by linear equations in the tangent spaces

$$\begin{cases} \pi^1 = \ell_1^1 \omega^1 + \ell_2^1 \omega^2, \\ \pi^2 = \ell_1^2 \omega^1 + \ell_2^2 \omega^2. \end{cases}$$

The condition that this 2-plane be integral is $\ell_1^1 = \ell_2^1 = 0$. Thus, any $E_0^1 \subset E_0^2$ will have a basis vector

$$\xi_0 = \xi^1 \frac{\partial}{\partial \pi^2} + \xi^2 \frac{\partial}{\partial \omega^1} + \xi^3 \frac{\partial}{\partial \omega^2}.$$

By (6.8) the rank of the polar equations of ξ_0 is 1. Consequently, there are no ordinary, admissible 2-planes and the system fails to be involutive.

We remark that with independence condition given by $\pi = \pi^1 \wedge \pi^2 \neq 0$ the system (I, π) is in involution.

(b) In this section we will clarify several foundational points concerning the variety of integral elements, introduce some numerical characters, and (most importantly) give Cartan's test for involution.

(6.9) Notations: (i) $V_p(I) \subset G_p(M)$ is the real analytic variety of p -dimensional integral elements of I , $O_p(I) \subset V_p(I)$ is the set of ordinary integral elements, and $R_p(I) \subset O_p(I)$ is the set of regular integral elements;

(ii) For X a Zariski open subset of $V_p(I)$, we denote by $F(X)$ the set of all flags $x \in E^1 \subset \dots \subset E^{p-1} \subset E^p$ where $E^p \in X$. In case $X \subset O_p(I)$ we denote by $F_R(X)$ the subset of such flags where x, E^1, \dots, E^{p-1} are all regular integral elements.

If $E_0^p \in O_p(I)$, then by definition there exists a flag $x_0 \in E_0^1 \subset \dots \subset E_0^{p-1} \subset E_0^p$ where E_0^k is regular for $0 \leq k \leq p-1$. If $E^p \in V_p(I)$ is an integral element close to E_0^p , then any flag $x \in E^1 \subset \dots \subset E^{p-1} \subset E^p$ close to above flag in E_0^p has E^k regular for $0 \leq k \leq p-1$. Moreover, E^p is regular in case E_0^p is. Finally, by choosing E^p to be a smooth point of $V_p(I)$ and to have $\dim H(E^p)$ minimal among integral elements close to E_0^p , we may assume that E^p is regular. These observations establish the first assertion in the following

(6.10) Proposition: (i) $O_p(I)$ and $R_p(I)$ are both Zariski open sets in $V_p(I)$ and $R_p(I)$ is dense in $O_p(I)$;

- (ii) if $X \subset O_p(I)$ is a Zariski open set, then $F_R(X)$ is a Zariski open dense subset of $F(X)$;
- (iii) in case I contains no functions, $O_p(I)$ is irreducible.

Proof: We will prove (iii) leaving the argument for (ii) to the reader.

The proof of (iii) is by induction on p . When $p = 0$, by our assumption that I contains no functions

$$O_0(I) = M$$

is irreducible. By induction we assume that $O_{p-1}(I)$ is irreducible. Then $R_{p-1}(I)$, $F(R_{p-1}(I))$, and (by (ii)) $F_R(R_{p-1}(I))$ are irreducible. If $O_p(I)$ is empty there is nothing to prove, so we assume that $O_p(I) \neq \emptyset$. We have

$$\begin{array}{ccc} & F_R(O_p(I)) & \\ \pi \swarrow & & \searrow \pi' \\ F_R(R_{p-1}(I)) & & O_p(I) \end{array}$$

where

$$\pi\{x \in E^1 \subset \dots \subset E^{p-1} \subset E^p\} = \{x \in E^1 \subset \dots \subset E^{p-1}\},$$

$$\pi'\{x \in E^1 \subset \dots \subset E^{p-1} \subset E^p\} = E^p.$$

The mapping π' is surjective, while π is dominant in the sense that its image contains a Zariski open subset of $F_R(R_{p-1}(I))$ (its image contains an ordinary open set, and hence must contain a Zariski open set).*

Now, and this is the main point, since $F_R(R_{p-1}(I))$ is irreducible and π is dominant we may speak of the general fibre of π . This general fibre is a projective space built on a vector space defined by real, linear equations (the polar equations of an $E^{p-1} \in R_{p-1}(I)$). Since projective spaces are irreducible it follows that $F_R(O_p(I))$ is irreducible. Then the same must be true of $\pi'(F_R(O_p(I))) = O_p(I)$. Q.E.D.

Remark: The point is that even though the equations of $V_p(I)$ may be very complicated, the ordinary integral elements are defined inductively by real linear equations.

(6.11) Corollary: If I is a real analytic differential system such that the image of the mapping

$$O_p(I) \rightarrow M$$

contains an open set, then exactly one irreducible component of the analytic variety $V_p(I)$ contains ordinary integral elements.

Proof: The hypothesis implies that $I_0 = \{0\}$ and hence that $O_p(I)$ is irreducible.

*An additional argument shows that π is surjective, but we won't need this fact.

As will be seen below we shall need to consider components of the real analytic variety $V_p(I)$ that may not contain ordinary integral elements. We give here a simple illustration of the possible reducibility of $V_p(I)$.

(6.12) Example. We consider the Pfaffian system on a F -dimensional manifold M generated by 1-forms $\theta^1, \theta^2, \theta^3$ with structure equations

$$d\theta^1 \equiv \pi^1 \wedge \omega^1 \mod I$$

$$d\theta^2 \equiv \pi^2 \wedge \omega^2 \mod I$$

$$d\theta^3 \equiv \pi^1 \wedge \pi^2 \mod I$$

where $\theta^\alpha, \pi^\varepsilon, \omega^i$ ($1 \leq \alpha \leq 3, 1 \leq \varepsilon \leq 2, 1 \leq i \leq 2$) give a basis for the forms on M . If we denote by

$$\frac{\partial}{\partial \theta^\alpha}, \frac{\partial}{\partial \pi^\varepsilon}, \frac{\partial}{\partial \omega^i}$$

the dual basis of tangent vectors, then any 2-plane on which $\theta^1 = \theta^2 = \theta^3 = 0$ has a basis

$$\xi = \xi^0 \frac{\partial}{\partial \pi^1} + \xi^1 \frac{\partial}{\partial \pi^2} + \xi^2 \frac{\partial}{\partial \omega^1} + \xi^3 \frac{\partial}{\partial \omega^2},$$

$$\tilde{\xi} = \tilde{\xi}^0 \frac{\partial}{\partial \pi^1} + \tilde{\xi}^1 \frac{\partial}{\partial \pi^2} + \tilde{\xi}^2 \frac{\partial}{\partial \omega^1} + \tilde{\xi}^3 \frac{\partial}{\partial \omega^2}.$$

The conditions that ξ and $\tilde{\xi}$ span an integral 2-plane are

$$(6.13) \quad \begin{aligned} \xi^0 \tilde{\xi}^2 - \tilde{\xi}^0 \xi^2 &= 0 \\ \xi^1 \tilde{\xi}^3 - \tilde{\xi}^1 \xi^3 &= 0 \\ \xi^0 \tilde{\xi}^1 - \tilde{\xi}^0 \xi^1 &= 0. \end{aligned}$$

To understand these equations we remark that the fibre of $G_2(M) \rightarrow M$ over any point is isomorphic to the Grassmannian $G_2(4)$ of 2-planes in \mathbb{R}^4 . Under the Plücker embedding

$$G_2(4) \subset \mathbb{P}^5$$

the 2-plane spanned by ξ and $\tilde{\xi}$ has homogeneous coordinates

$$\pi_{ab} = \xi^a \tilde{\xi}^b - \tilde{\xi}^a \xi^b, \quad 0 \leq a < b \leq 3.$$

It is well known that $G_2(4)$ is the quadric hypersurface defined by

$$(6.14) \quad \pi_{01} \pi_{23} - \pi_{02} \pi_{13} + \pi_{03} \pi_{12} = 0.*$$

Consequently, the equations (6.13) and (6.14) are

$$\begin{cases} \pi_{01} = \pi_{02} = \pi_{13} = 0 \\ \pi_{03} \pi_{12} = 0. \end{cases}$$

Since the last equation is reducible, it follows that the fibres of

* This is because $\pi \in \Lambda^2 \mathbb{R}^m$ is decomposable $\Leftrightarrow \pi \wedge \pi = 0$, and when $m = 4$ this is one equation.

$V_2(I) \rightarrow M$ each consist of a pair of intersecting lines in P^5 .

The same considerations also apply (and are simpler) to a differential system with independence condition (I, ω) when we restrict the defining equations for $V_p(I)$ to the open subset of admissible p-planes in $G_p(M)$.

(6.15) Example (continued): In the previous example with independence condition

$$\omega = \omega^1, \omega^2 \neq 0,$$

the admissible 2-planes are given by

$$\pi^1 = \alpha\omega^1 + \beta\omega^2,$$

$$\pi^2 = \gamma\omega^1 + \delta\omega^2.$$

The equations that this 2-plane be integral are

$$\begin{cases} \beta = \gamma = 0, \\ \alpha\delta = 0. \end{cases}$$

If we denote the variety of admissible 2-planes by $V_2(I, \omega)$, then each fibre of $V_2(I, \omega) \rightarrow M$ is isomorphic to the union of two coordinate axes in R^4 . In a little while we shall give a simple means of testing which of these two components contains an ordinary integral element.

We now define some numerical characters of I .

(6.16) Definitions: (i) In case $O_k(I)$ is non-empty we define $S_k = S_k(I)$ and $s_k = s_k(I)$

S_k = maximum rank of polar equations of $E^k \in O_k(I)$

$$S_k = S_{k-1} + s_k.$$

(ii) In case $O_{k-1}(I)$ is non-empty we define $\sigma_k = \sigma_k(I)$ by

$$s_0 + \dots + s_{k-1} + \sigma_k = n - k.$$

Remarks: Since, for $E^k \in V_k(I)$

$$E^k \subset H(E^k)$$

$$H(E^k) \subset H(E^{k-1}) \text{ for any } E^{k-1} \subset E^k,$$

by virtue of our respective assumptions $O_k(I) \neq \emptyset$, $O_{k-1}(I) \neq \emptyset$, we have for E_k regular

$$(6.17) \quad \begin{cases} k \leq \dim H(E^k) = n - S_k, \\ s_k \geq 0 \text{ and } \sigma_k \geq -1. \end{cases}$$

The σ_k 's are just the r_k 's of Sh; the s_k 's and σ_k 's are the notation frequently used by Cartan and will generally be followed for the remainder of these notes.

Now we let X be a Zariski open set in an irreducible component of $V_p(I)$. We define

$$G_k(X) \subset G_k(M)$$

to be the set of k -planes contained in some $E_0^p \in X$.

(6.18) Definitions: Assuming that $0 \leq k \leq p$, we define two sets of numerical characters

$$S_k(X), s_k(X), \sigma_k(X)$$

and

$$S'_k(X), s'_k(X), \sigma'_k(X)$$

associated to X as follows: (i)

$$S_k(X) = \left\{ \begin{array}{l} \text{maximum rank of the polar equations of} \\ \text{an } E^k \in V_k(I) \text{ that is close to an } E_0^k \in G_k(X) \end{array} \right\}$$

$$S_k(X) = S_{k-1}(X) + s_k(X)$$

$$S_{k-1}(X) + \sigma_k(X) = n - k;$$

and (ii)

$$S'_k(X) = \text{maximum rank of the polar equations of } E_0^k \in G_k(X)$$

$$S'_k(X) = S'_{k-1}(X) + s'_k(X),$$

$$S'_{k-1}(X) + \sigma'_k(X) = n - k.$$

Remarks: The important and somewhat subtle point is that, in the definition of the $S'_k(X)$, the polar equations are restricted to k -planes E_0^k contained in an $E_0^p \in X$. On the other hand, in the definition of the $S_k(X)$ the relevant polar equations allow E^k to vary in a neighborhood of $G_k(X)$.

We also observe that, since X is assumed to be irreducible we may compute the $s'_k(X)$ by taking first a general $E_0^p \in X$, and then determining the rank of the polar equations of a general $E_0^k \subset E_0^p$.

It is obvious that

$$S'_k(X) \leq S_k(X),$$

and we have the

(6.19) Proposition: The component X contains an ordinary integral element if, and only if,

$$s'_0(X) = s_0(X), \dots, s'_{p-1}(X) = s_{p-1}(X).$$

Proof: The condition $s'_k(X) = s_k(X)$, $0 \leq k \leq p-1$, is equivalent to saying that for a general flag in X

$$(6.20) \quad x_0 \in E_0^1 \subset \dots \subset E_0^{p-1} \subset E_0^p \in F(X)$$

the polar spaces $H(E^k)$ have constant dimension when $E^k \in V_k(I)$ varies

in a neighborhood of E_0^k . By the argument in Chapter 4 which shows that Cartan regularity implies Kähler regularity for an integral element, this implies that E_0^k is a smooth point of $V_k(I)$ around which the equations coming from I give a regular set of defining functions. In this case E_0^k is regular for $0 \leq k \leq p-1$, and consequently X contains an ordinary integral element. (The crucial point is that the above flag is taken to be general among flags in X . The hypothesis essentially says it is then general among all integral flags.)

Conversely, if X contains an ordinary integral element E_0^p , then considering the polar equations of a regular flag of E_0^p yields the desired equalities. Q.E.D.

(6.20) Example. We reconsider example (6.6) above now without the independence condition $\omega^1 \wedge \omega^2 \neq 0$. Over each point x of M the vectors ξ given by (6.7) on which $\theta^1 = \theta^2 = 0$ form an \mathbb{R}^4 , and so the 2-dimensional integral elements over x will be a subvariety of the Grassmannian $G_2(4)$. As in example (6.12) we use the Plücker embedding

$$G_2(4) \subset \mathbb{P}^5$$

under which $G_2(4)$ is the quadric (6.14) where

$$\pi_{ab} = \xi_{\xi}^{a\bar{b}} - \xi_{\xi}^{b\bar{a}}, \quad 0 \leq a < b \leq 3.$$

It follows from (6.8) that the equations of $V_2(I)$ are

$$\begin{cases} \pi_{02} = \pi_{03} = 0 \\ \pi_{01} \pi_{23} = 0. \end{cases}$$

Thus $V_2(I)$ decomposes into two irreducible components

$$X_1 = \{\pi_{01} = \pi_{02} = \pi_{03} = 0\}$$

$$X_2 = \{\pi_{23} = \pi_{02} = \pi_{03} = 0\},$$

each of which is a family of \mathbb{P}^2 's over M . For these two components we have

$$\begin{cases} s'_0(X_1) = 2, & s'_1(X_1) = 2 \\ s_0(X_1) = 2, & s_1(X_1) = 2, \end{cases}$$

$$\begin{cases} s'_0(X_2) = 2, & s'_1(X_2) = 1 \\ s_0(X_2) = 2, & s_1(X_2) = 2. \end{cases}$$

It follows that X_1 contains a dense open set of ordinary integral 2-planes whereas X_2 does not contain any (observe that X_1 and X_2 intersect in a family of integral 2-planes E^2 such that every $E^1 \subset E^2$ has $\dim H(E^1)$ larger than is generically the case).

We now have the following dimension count:

(6.21) Proposition: Assume that the image of

$$O_p(I) \rightarrow M$$

contains an open set. Then

$$\dim O_p(I) = n + s_1 + 2s_2 + \dots + (p-1)s_{p-1} + p\sigma_p.$$

Proof: By proposition (6.10) it will suffice to prove the same formula for $\dim R_p(I)$. We do this by induction on p .

The result is obvious for $p = 0$, and we assume it for $p - 1$.

Define the incidence correspondence

$$R_{p-1,p}(I) \subset R_{p-1}(I) \times R_p(I)$$

to be the set of pairs $E^{p-1} \subset E^p$ where E^{p-1} and E^p are both regular integral elements.

We have projection mappings

(6.22)

$$\begin{array}{ccc} & R_{p-1,p}(I) & \\ \pi_{p-1} \swarrow & & \searrow \pi_p \\ R_{p-1}(I) & & R_p(I) \end{array}$$

where

$$\pi_{p-1}\{E^{p-1} \subset E^p\} = E^{p-1}$$

$$\pi_p\{E^{p-1} \subset E^p\} = E^p.$$

Clearly π_p is surjective while, as in the proof of (6.10), π_{p-1} is dominant (again, it may be proved that π_{p-1} is surjective, but we don't need this). Thus by the induction assumption,

(6.23)

$$\dim(\text{Image } \pi_{p-1}) = n + s_1 + 2s_2 + \dots + (p-2)s_{p-2} + (p-1)\sigma_{p-1}$$

where

$$s_0 + s_1 + \dots + s_{p-2} + \sigma_{p-1} = n - p + 1.$$

The fibre $\pi_{p-1}^{-1}(E^{p-1})$ consists of all regular integral p -planes containing E^{p-1} , and is therefore an open set in the projective space

$$\mathbb{P}(H(E^{p-1})/E^{p-1}).$$

Using obvious dimension properties of (6.22)

$$(6.24) \quad \dim \pi_{p-1}^{-1}(E^{p-1}) = n - p - s_{p-1} \\ = \sigma_p ;$$

$$(6.25) \quad \dim R_{p-1,p}(I) = \dim R_{p-1}(I) + \dim \pi_{p-1}^{-1}(E^{p-1}) \\ = n + s_1 + \dots + (p-2)s_{p-2} + (p-1)s_{p-1} \\ + p\sigma_p + p - 1$$

where the second equality is a consequence of (6.23), (6.24), and

$$\sigma_{p-1} = \sigma_p + s_{p-1} + 1 ; \\ (6.26) \quad \dim R_p(I) = \dim R_{p-1,p}(I) - \dim \pi_p^{-1}(E^p) \\ = \dim R_{p-1,p}(I) - (p-1)$$

since $\pi_p^{-1}(E^p)$ is an open set in the projective space $\mathbb{P}(E^{p*})$ of hyperplanes in E^p . Comparing (6.25) and (6.26) gives the result. Q.E.D.

(6.27) Example (symplectic geometry): Suppose that ϕ is a closed 2-form of maximal rank p on a manifold of dimension $2p$. Let I be the differential system generated by ϕ . We will prove that

$$(6.28) \quad s_0 = 0, \quad s_1 = \dots = s_p = 1$$

(it is well-known that $V_{p+1}(I)$ is empty while, as will be seen below, $\dim O_p(I) = 2p + p(p+1)/2$).

It is clear that $s_0 = 0$ since I contains no 1-forms.

Let V be the vector space $T_x(M)$ and $\phi: V \times V \rightarrow \mathbb{R}$ the alternating bilinear form induced by $\phi(x)$. Let E^p be a null-plane of maximal dimension of ϕ , and choose linear forms $\omega^{p+1}, \dots, \omega^{2p} \in V^*$ so that E^p is defined by $\omega^{p+1} = \dots = \omega^{2p} = 0$. Elementary considerations show that there exist 1-forms $\omega^1, \dots, \omega^p$ such that

$$\phi = \omega^1 \wedge \omega^{p+1} + \dots + \omega^p \wedge \omega^{2p}.$$

Since $\phi^{2p} \neq 0$, the 1-forms $\omega^1, \dots, \omega^{2p}$ give a basis for V^* . It follows that every null plane for ϕ looks like the standard maximal null plane $x^{p+1} = \dots = x^{2p} = 0$ in \mathbb{R}^{2p} provided with the symplectic form $dx^1 \wedge dx^{p+1} + \dots + dx^p \wedge dx^{2p}$. If ξ_1, \dots, ξ_{2p} is the basis for V dual to $\omega^1, \dots, \omega^{2p}$ then, setting

$$E^k = \text{span}\{\xi^1, \dots, \xi^k\}$$

we may easily see that

$$E^1 \subset E^2 \subset \dots \subset E^{p-1} \subset E^p$$

gives a flag all of whose elements are regular (the point is that any flag looks like the standard one). The polar equations of E^k are

$$\omega^{p+1} = \dots = \omega^{p+k} = 0.$$

Thus, for $1 \leq k \leq p$

$$\begin{aligned} s_k &= k \\ &= s_1 + \dots + s_k, \end{aligned}$$

and this proves (6.28).

We note that since

$$s_1 + 2s_2 + \dots + ps_p = p(p+1)/2,$$

(6.21) is in agreement with (6.28) and the well-known dimension count for the variety of null-planes for Φ .

The main result of this section is the following

(6.29) Theorem (Cartan's test): Let X be a Zariski open set in an irreducible component of $V_p(I)$. Then

$$\dim X \leq n + s'_1(X) + 2s'_2(X) + \dots + ps'_p(X).$$

Moreover, equality holds if, and only if, X contains a dense open set of ordinary integral elements.

Proof: We use the notations

$$G_k(X) = \{E^k \in G_k(M) : E^k \subset E^p \text{ for some } E^p \in X\}$$

$$G_{k-1,k}(X) = \{E^{k-1} \subset E^k : E^k \in G_k(X)\}.$$

We then have a diagram (compare with (6.22))

$$(6.30) \quad \begin{array}{ccc} & G_{k-1,k}(X) & \\ \pi_{k-1} \swarrow & & \searrow \pi'_k \\ G_{k-1}(X) & & G_k(X) \end{array}$$

where π_{k-1} and π'_k are surjective. Since all varieties are irreducible we may compute dimensions by using general fibres of the maps in (6.30). Thus, for $E^{k-1} \in G_{k-1}(X)$ a general point,

$$\begin{aligned} (6.31) \quad \dim G_{k-1,k}(X) &= \dim G_{k-1}(X) + \dim \pi_{k-1}^{-1}(E^{k-1}) \\ &\leq \dim G_{k-1}(X) + \dim \mathbb{P}(H(E^{k-1})/E^{k-1}) \\ &= \dim G_{k-1}(X) + n - k - s'_{k-1}(X); \end{aligned}$$

and for $E^k \in G_k(X)$ a general point

$$\begin{aligned} (6.32) \quad \dim G_{k-1,k}(X) &= \dim G_k(X) + \dim \pi_k^{-1}(E^k) \\ &= \dim G_k(X) + k - 1. \end{aligned}$$

If we compare (6.31) and (6.32) with the inductive step in the proof of proposition (6.21), then the same computation that gives the dimension formula in that proposition gives the inequality in Cartan's test (we remark that, in case $X = \mathcal{O}_p(I)$, $G_{k-1,k}(X)$ contains $R_{k-1,k}(I)$ as a dense Zariski open subset, cf. (ii) in (6.10)).

(6.33) Observation: If we define an enlargement of $E^{k-1} \in V_{k-1}(I)$ to be an $E^k \in V_k(I)$ such that $E^{k-1} \subset E^k$, then strict inequality will hold in Cartan's test exactly when, for some k with $0 \leq k \leq p-1$, a general enlargement E^k of a general $E^{k-1} \in G_{k-1}(X)$ does not satisfy the condition that $E^k \subset E^p$ for some $E^p \in X$.

Returning to proof of proposition (6.29), if X contains an ordinary integral element then, by propositions (6.19) and (6.21), equality holds in Cartan's test.

Conversely, suppose that strict inequality holds in Cartan's test. Referring to observation (6.33), we let $k-1$ be the first time that a general enlargement of a general $E^{k-1} \in G_{k-1}(X)$ does not lie in $G_k(X)$. Then the diagram (6.30) cannot have a Zariski open set in common with the corresponding diagram for $\mathcal{O}_p(I)$. Consequently by proposition (6.10), X cannot contain any ordinary integral elements. Q.E.D.

Remark. The mechanism of the proof is this:

If

$$s'_0(X) = s_0, \dots, s'_{k-1}(X) = s_{k-1} \quad \text{but} \quad s'_k(X) < s_k$$

then (i) $G_0(X), \dots, G_{k-1}(X)$ will all have dense Zariski open sets in common with $G_0(\mathcal{O}_p(X)), \dots, G_{k-1}(\mathcal{O}_p(X))$; (ii) therefore, a general enlargement E^k of a general $E^{k-1} \in G_{k-1}(X)$ will be a general point $E^k \in G_k(\mathcal{O}_p(I))$, and (iii) the condition $s'_k(X) < s_k$ means that our general enlargement E^k will not lie in $G_k(X)$, and therefore at this stage the inequality in the dimension count becomes strict.

In practice Cartan's test will be used as follows:

(6.34) Given a differential system with independence condition (I, ω) , let X be an irreducible component of the analytic variety $V_p(I, \omega)$ of admissible, p -dimensional integral elements. Then, if we set $s'_p(X) = \sigma'_p(X)$,

$$\dim X \leq n + s'_1(X) + 2s'_2(X) + \dots + ps'_p(X)$$

with equality holding if, and only if, (I, ω) is in involution and a general point of X is an ordinary integral element.

The notation change $\sigma'_p(X) \rightarrow s'_p(X)$ is to conform to what the reader will find in Cartan [2] (only he does not specify the irreducible component of $V_k(I, \omega)$ in which he is working).

(6.35) Example. We reconsider example (6.20) with independence condition $\omega = \omega^1 \wedge \omega^2 \neq 0$. Then $V_2(I, \omega)$ is irreducible and is a Zariski open set X in X_1 . Thus

$$\begin{cases} n = 6, & s'_0(X) = 2, & s'_1(X) = 1, & s'_2(X) = 1 \\ \dim X = 6 + 2 = 8 \\ n + s'_1(X) + 2s'_2(X) = 9 > 8, \end{cases}$$

which confirms our computation (6.6) that (I, ω) fails to be involutive.

For later use, and also as an aid in reading [2], we want to explain the reduced polar equations. Let (I, ω) be a differential system with independence condition $\omega^1 \wedge \dots \wedge \omega^p \neq 0$. Complete the ω^i to a basis $\{\omega^i, \phi^\mu\}$ of the 1-forms. Admissible p-planes are given by

$$(6.36) \quad \phi^\mu = \ell^\mu_i \omega^i$$

(the ℓ^μ_i give coordinates on the open set of admissible p-planes in each $T_X(M)$). The condition that the p-plane (6.36) be integral is given by real analytic equations

$$(6.37) \quad F_\tau(x, \ell^\mu_i) = 0, \quad 1 \leq \tau \leq t.$$

Let X be a Zariski open subset of an irreducible component of the variety $V_p(I, \omega)$ given by (6.37). Given $E^p_0 \in X$ and a flag $x_0 \in E^1_0 \subset \dots \subset E^{p-1}_0 \subset E^p_0$, the polar equations of E^k_0 are linear equations of the form

$$(6.38) \quad L_{\lambda i} \alpha^i + L_{\lambda \mu} \beta^\mu = 0, \quad \lambda = 1, \dots, N_k,$$

where

$$\xi = \alpha^i \frac{\partial}{\partial \omega^i} + \beta^\mu \frac{\partial}{\partial \phi^\mu}$$

is a tangent vector (as before $\{\frac{\partial}{\partial \omega^i}, \frac{\partial}{\partial \phi^\mu}\}$ is the basis of tangent vectors dual to $\{\omega^i, \phi^\mu\}$).

(6.39) Definition: The reduced polar equations are

$$L_{\lambda \mu} \beta^\mu = 0, \quad \lambda = 1, \dots, N_k.$$

(6.40) Proposition: If $E^p_0 \in X$ is a general point and $x_0 \in E^1_0 \subset \dots \subset E^{p-1}_0 \subset E^p_0$ is a general flag, then the rank of the reduced polar equations of E^k_0 is equal to $s'_k(X)$.

Proof: In $\mathbb{R}^p \times \mathbb{R}^q$ with coordinates (α^i, β^μ) we consider the linear equations (6.38). As $\omega^1 \wedge \dots \wedge \omega^k \neq 0$ on E^k_0 we may assume that the solutions to (6.38) project onto \mathbb{R}^p . Then, for fixed $\{\alpha^i\}$ the equations (6.38), viewed as inhomogeneous equations in the β^μ , will have the same rank, $s'_k(X)$, as the full set of equations (6.38). But these inhomogeneous equations have the same rank as their corresponding homogeneous equations, which are just the reduced polar equations. Q.E.D.

(c) In this section we will introduce and discuss an important class of Pfaffian systems.

Suppose that (I, ω) is a Pfaffian system generated by linearly independent 1-forms $\theta^1, \dots, \theta^s$ together with their exterior derivatives and with independence condition $\omega = \omega^1 \wedge \dots \wedge \omega^p \neq 0$. By adding linearly independent 1-forms π^1, \dots, π^t we may complete the θ^α and ω^i to a basis (thus $n = s + p + t$). For the remainder of the paper we shall use the range of indices

$$\begin{cases} 1 \leq \alpha, \beta \leq s \\ 1 \leq i, j \leq p \\ 1 \leq \epsilon, \delta \leq t \end{cases}$$

The exterior derivatives of the θ^α are always of the form

$$(6.41) \quad d\theta^\alpha \equiv a_{\epsilon i}^\alpha \pi^\epsilon \wedge \omega^i + \frac{1}{2} c_{ij}^\alpha \omega^i \wedge \omega^j + \frac{1}{2} e_{\epsilon \delta}^\alpha \pi^\epsilon \wedge \pi^\delta \pmod{I}.$$

Throughout we make the convention that

$$c_{ij}^\alpha + c_{ji}^\alpha = 0 = e_{\epsilon \delta}^\alpha + e_{\delta \epsilon}^\alpha.$$

It is natural to ask if (6.41) may be simplified by a change of basis. Admissible changes of bases are given by invertible linear substitutions

$$(6.42) \quad \begin{cases} \tilde{\theta}^\alpha = A_\beta^\alpha \theta^\beta \\ \tilde{\omega}^j = B_j^i \omega^i + B_\alpha^i \theta^\alpha \\ \tilde{\pi}^\epsilon = C_\delta^\epsilon \pi^\delta + C_i^\epsilon \omega^i + C_\alpha^\epsilon \theta^\alpha \end{cases}$$

Under this change the coefficients in (6.41) undergo a complicated transformation, but the condition

$$(6.43) \quad e_{\epsilon \delta}^\alpha = 0$$

remains invariant.

(6.44) Definition: The Pfaffian system (I, ω) is said to be in good form if (6.43) is satisfied. In this case (6.41) simplifies to

$$(6.45) \quad d\theta^\alpha \equiv a_{\epsilon i}^\alpha \pi^\epsilon \wedge \omega^i + \frac{1}{2} c_{ij}^\alpha \omega^i \wedge \omega^j \pmod{I}.$$

As initial motivation for studying Pfaffian systems in good form we will give the equations that define the variety $V_p(I, \omega)$ of admissible integral elements. Any p -plane on which $\theta^1 = \dots = \theta^s = 0$ and $\omega^1 \wedge \dots \wedge \omega^p \neq 0$ is given by linear equations

$$(6.46) \quad \pi^\epsilon = l_i^\epsilon \omega^i.$$

By (6.41) the conditions that this be an integral p -plane are

$$(6.47) \quad e_{\epsilon \delta}^\alpha (l_i^\epsilon l_j^\delta - l_i^\delta l_j^\epsilon) + (a_{\epsilon j}^\alpha l_i^\epsilon - a_{\epsilon i}^\alpha l_j^\epsilon) + c_{ij}^\alpha = 0.$$

If $e_{\varepsilon 0}^{\alpha} \neq 0$ these are (inhomogeneous) quadratic equations and $V_p(I, \omega)$ may have the complexity of a general set of such equations. However, in case the Pfaffian system is in good form, (6.47) reduces to the set of affine linear equations

$$(6.48) \quad a_{\varepsilon j}^{\alpha} l_i^{\varepsilon} - a_{\varepsilon i}^{\alpha} l_j^{\varepsilon} + c_{ij}^{\alpha} = 0.$$

Assume that (I, ω) is in good form and that $V_p(I, \omega)$ surjects onto an open set in M . Restrict attention to this set. Assume also either that we are in the real analytic case or that we are in the C^{∞} case and the linear operator $(l_i^{\varepsilon}) \rightarrow (a_{\varepsilon j}^{\alpha} l_i^{\varepsilon} - a_{\varepsilon i}^{\alpha} l_j^{\varepsilon})$ has constant rank. Then we may choose a real analytic (respectively C^{∞}) solution $l_i^{\varepsilon}(x)$ to the equations (6.48) and replace π^{ε} by $\pi^{\varepsilon} - l_i^{\varepsilon}(x) \omega^i$. When this is done, (6.45) and (6.48) reduce to

$$(6.49) \quad d\theta^{\alpha} \equiv a_{\varepsilon i}^{\alpha} \pi^{\varepsilon} \wedge \omega^i \pmod{I}$$

$$(6.50) \quad a_{\varepsilon j}^{\alpha} l_i^{\varepsilon} - a_{\varepsilon i}^{\alpha} l_j^{\varepsilon} = 0$$

with the same $a_{\varepsilon i}^{\alpha}$ as in (6.45).

It is clear that, for Pfaffian systems in good form, $V_p(I, \omega)$ has none of the algebro-geometric complexities of the general case: over each $x \in M$ the solutions to (6.45) form an affine linear space. As will soon be explained there is a corresponding linear algebra test for when the system (I, ω) is involutive.

For each $x \in M$ we denote by $S(a)_x$ the space of solutions to the homogeneous equations (6.50) when $a_{\varepsilon i}^{\alpha} = a_{\varepsilon i}^{\alpha}(x)$. In $\Lambda(T_x^*(M))$ we imagine the algebraic ideal generated by the forms

$$\theta^{\alpha}(x), \quad a_{\varepsilon i}^{\alpha}(x) \pi^{\varepsilon}(x) \wedge \omega^i(x).$$

For this ideal we may consider the polar equations associated to the (homogeneous linear) variety of admissible integral p-planes, and we denote by $s'_1(x), \dots, s'_p(x)$ the characters given by (ii) in (6.18) associated to these polar equations. According to Cartan's test (6.29) (more precisely, according to its proof) we have

$$(6.51) \quad \dim S(a)_x \leq s'_1(x) + 2s'_2(x) + \dots + ps'_p(x).$$

(6.52) Definition: We shall say that $\|a_{\varepsilon i}^{\alpha}(x)\|$ forms an involutive tableau in case equality holds in (6.51).

Remark: This is a property of the coefficients $a_{\varepsilon i}^{\alpha}(x)$ in (6.45) at each point $x \in M$ (in particular it makes sense even if there are no admissible integral p-planes).

From now on we will assume that $\dim S(a)_x$ and $s'_1(x), \dots, s'_p(x)$ are constant (in the real analytic case this is always true on a Zariski open subset of M), and shall frequently omit reference to the point x in (6.51) and (6.52).

The following is merely a reformulation of (6.29):

(6.53) The Pfaffian system (I, ω) in good form is involutive at x if, and only if, the image of $V_p(I, \omega) \rightarrow M$ contains a neighborhood of x and $\|a_{\epsilon i}^\alpha(X)\|$ is an involutive tableau.

There is a useful numerical criterion for when $\|a_{\epsilon i}^\alpha\|$ forms an involutive tableau. By a linear substitution

$$\tilde{\omega}^i = B_j^i \omega^j$$

we may assume that, for E_0^p a general admissible p -plane given by (6.46), the subspaces $E_0^k \subset E_0^p$ defined by

$$\omega^{k+1} = \dots = \omega^p = 0$$

give a general flag. The reduced polar equations (6.39) for E_0^k are then

$$(6.54) \quad \begin{cases} a_{\epsilon 1}^\alpha \xi^\epsilon = 0 \\ \vdots \\ a_{\epsilon k}^\alpha \xi^\epsilon = 0. \end{cases}$$

Set

(6.55)

$$M_k = \begin{vmatrix} 1 & & 1 \\ a_{11} & \dots & a_{t1} \\ \vdots & & \vdots \\ a_{s1} & \dots & a_{t1} \\ \vdots & & \vdots \\ 1 & & 1 \\ a_{1k} & \dots & a_{tk} \\ \vdots & & \vdots \\ a_{sk} & \dots & a_{tk} \end{vmatrix}.$$

By Proposition (6.40) the equations (6.54) have rank

$$(6.56) \quad S'_k = \text{rank } M_k + s. *$$

(6.57) Define S'_k by (6.55), (6.56) and s'_1, \dots, s'_p by

$$\begin{cases} S'_k = S'_{k-1} + s'_k, & 1 \leq k \leq p-1 \\ S'_{p-1} + s'_p = n - p. \end{cases}$$

Define $S(a)$ to be the space of solutions of

$$a_{\epsilon i}^\alpha \xi_j^\epsilon = a_{\epsilon j}^\alpha \xi_i^\epsilon.$$

Then we have:

$$\dim S(a) \leq s'_1 + 2s'_2 + \dots + ps'_p$$

*Note that $s'_0 = s$ is the number of independent 1-forms θ^α .

with equality holding if, and only if, $\|a_{\epsilon i}^\alpha\|$ is an involutive tableau.

Finally, we will reformulate (6.57) in a coordinate free manner.

We introduce vector spaces V, W with respective bases $\left\{\frac{\partial}{\partial x^i}\right\}$, $\{y_\alpha\}$ and denote by $\{x^i\}$ the dual basis of V^* . Then we associate to the tableau $\|a_{\epsilon i}^\alpha\|$ the linear subspace

$$a \subset \text{Hom}(V, W) \cong W \otimes V^*$$

having the elements

$$(6.58) \quad z_\epsilon = a_{\epsilon i}^\alpha y_\alpha \otimes x^i$$

as generators. From (6.42) it is clear that a is invariantly attached to the tableau of the Pfaffian system (I, ω) in good form. It will be convenient to drop the " \otimes " and think of $W \otimes V^*$ as W -valued linear functions on V . For $0 \leq i \leq p$ we set

$$(6.59) \quad a_i = \{T \in a : \frac{\partial T}{\partial x^1} = \dots = \frac{\partial T}{\partial x^i} = 0\}.$$

Then

$$a = a_0 \supset a_1 \supset \dots \supset a_{p-1} \supset a_p = (0).$$

(6.60) Proposition: Assume that (I, ω) has no Cauchy characteristics and that $\{x^i\}$ is a general basis of V^* . Then

$$\dim S(a) \leq \dim a_0 + \dim a_1 + \dots + \dim a_{p-1}$$

with equality holding if, and only if, $\|a_{\epsilon i}^\alpha\|$ is an involutive tableau.

Proof. We will use two auxiliary results.

(6.61) Lemma: For $\{x^i\}$ a general basis of V^*

$$\dim a = \dim a_k + s'_1 + \dots + s'_k, \quad 0 \leq k \leq p-1.$$

Proof: We consider the map

$$m_k: a \rightarrow \underbrace{W \oplus \dots \oplus W}_k$$

defined by

$$m_k(T) = \underbrace{\frac{\partial T}{\partial x^1} \oplus \dots \oplus \frac{\partial T}{\partial x^k}}.$$

The kernel of m_k is a_k , and the matrix representing m_k is M_k in (6.55). Thus

$$\begin{aligned} s'_1 + \dots + s'_k &= \text{rank } M_k \\ &= \dim a - \dim a_k. \end{aligned}$$

(6.62) Lemma: If χ is the dimension of the space of Cauchy characteristic vectors of the form $\xi = \xi^\varepsilon \frac{\partial}{\partial \pi^\varepsilon}$, then

$$\dim a = t - \chi . *$$

Proof: The conditions (of which the first is automatic)

$$\begin{cases} i(\xi)\theta^\alpha = 0 \\ i(\xi)d\theta^\alpha \equiv 0 \pmod{I} \end{cases}$$

are equivalent to

$$a_{\varepsilon i}^\alpha \xi^\varepsilon = 0 .$$

Now define a linear map

$$\psi : \mathbb{R}^t \rightarrow w \otimes v^*$$

by

$$\psi(0, \dots, 0, 1, 0, \dots, 0) = a_{\varepsilon i}^\alpha y_\alpha \otimes x^i$$

ε^{th} position

and extend by linearity. Then

*The general form of (6.60) is

$$\dim S(a) \leq \dim a_0 + \dim a_1 + \dots + \dim a_{p-1} + p\chi$$

with equality holding if, and only if, $\|a_{\varepsilon i}^\alpha\|$ is an involutive tableau.

$$\text{Image } \psi = a$$

$$\dim \text{kernel } \psi = \chi$$

The lemma follows.

Returning to the proof of (6.60), we add up the result of lemma

(6.61) for $k = 0, \dots, p-1$

$$(6.63) \quad p \dim a = \dim a_0 + \dots + \dim a_{p-1} + (p-1)s'_1 + \dots + s'_{p-1} .$$

On the other hand, s'_p is defined (cf. (ii) in (6.18) and recall that $t = n - p - s_0$) by

$$s'_1 + \dots + s'_{p-1} + s'_p = t ,$$

and therefore by (6.51) and lemma (6.62)

$$\begin{aligned} \dim S(a) &\leq s'_1 + \dots + (p-1)s'_{p-1} + p(\dim a - s'_1 - \dots - s'_{p-1}) \\ &= p \dim a - ((p-1)s'_1 + \dots + s'_{p-1}) \end{aligned}$$

By (6.63) this gives

$$\dim S(a) \leq \dim a_0 + \dots + \dim a_{p-1}$$

with equality holding if, and only if, $\|a_{\varepsilon i}^\alpha\|$ is an involutive tableau. Q.E.D.

(d) We consider a Pfaffian system in good form (cf. (6.44)), which we write as

$$(6.64) \quad \begin{cases} \theta^\alpha = 0 \\ d\theta^\alpha \equiv a_{\epsilon i}^\alpha \pi^\epsilon \wedge \omega^i + \frac{1}{2} c_{ij}^\alpha \omega^i \wedge \omega^j \pmod{I} \end{cases}$$

with independence condition $\omega = \omega^1 \wedge \dots \wedge \omega^p \neq 0$. The useful concept of expressing (6.64) in dual form will be explained and the analogue of (6.57) given.

As in (6c) we may consider the tableau $\|a_{\epsilon i}^\alpha\|$ in (6.64) as intrinsically defining a subspace $a \subset V^* \otimes W$. For reasons to appear in the second part of these notes we give the

(6.65) Definition: The annihilator of a , denoted

$$b = a^\perp \subset V \otimes W^*,$$

will be called the symbol of (6.64). Using the notation from §6c we let

$$b^r = b_{\alpha}^{ri} \gamma_\alpha^* \otimes \frac{\partial}{\partial x^i}$$

be a basis for the symbol and we set

$$(6.66) \quad \pi_i^\alpha = a_{\epsilon i}^\alpha \pi^\epsilon + \frac{1}{2} c_{ij}^\alpha \omega^j.$$

The forms π_i^α are intrinsically defined modulo the span $\{\theta^\alpha, \omega^i\}$ by the tableau $\|a_{\epsilon i}^\alpha\|$, and (6.64) is

$$(6.67) \quad \begin{aligned} \theta^\alpha &= 0 \\ d\theta^\alpha &\equiv \pi_i^\alpha \wedge \omega^i \pmod{I} \\ b_\alpha^{ri} \pi_i^\alpha &\equiv C_j^r \omega^j \pmod{I} \end{aligned}$$

with the independence condition $\omega = \omega^1 \wedge \dots \wedge \omega^p \neq 0$ being understood.

(6.68) Definition: We shall say that (6.67) is the dual form of the Pfaffian system (6.64).

For the Pfaffian system (6.67) we consider the following matrix whose entries are 1-forms

$$(6.69) \quad \begin{vmatrix} \pi_1^1 & \dots & \pi_p^1 \\ \vdots & & \vdots \\ \pi_1^s & \dots & \pi_p^s \end{vmatrix} \pmod{\text{span}\{\theta^\alpha, \omega^i\}}$$

(working modulo the span $\{\theta^\alpha, \omega^i\}$ is equivalent to considering the π_i^α restricted to the subspace $\theta^\alpha = \omega^i = 0$).

(6.70) Proposition: Assuming that the basis $\{\omega^i\}$ has been chosen generically, S'_k is the number of linearly independent 1-forms in the first k columns of (6.69).

Proof: We assume that E_0^p is a general admissible p -plane and that $x_0 \in E_0^1 \subset \dots \subset E_0^{p-1} \subset E_0^p$ is a general flag where E_0^k is given by $\omega^{k+1}|_{E_0^p} = \dots = \omega^p|_{E_0^p} = 0$. The k^{th} set of reduced polar equations (6.39) is (cf. (6.54))

$$\begin{cases} a_{\epsilon 1}^{\alpha} \xi^{\epsilon} = 0 \\ \vdots \\ a_{\epsilon k}^{\alpha} \xi^{\epsilon} = 0. \end{cases}$$

By (6.66) this is

$$\begin{cases} \langle \pi_1^{\alpha}, \xi \rangle = 0 \\ \vdots \\ \langle \pi_k^{\alpha}, \xi \rangle = 0. \end{cases}$$

Q.E.D.

To formulate Cartan's test for the dual form we will use the following two results.

(6.71) Proposition: Assuming that the matrices $A^{\alpha}(x) = \|a_{\epsilon i}^{\alpha}(x)\|$ span a space of constant dimension, the following are equivalent:

- (i) the space $V_p(I, \omega)$ of admissible integral p -planes subjects onto M ;
- (ii) we may choose the π^{ϵ} so that $c_{ij}^{\alpha} = 0$ in (6.64); and
- (iii) we may choose the π_i^{α} so that $c_j^r = 0$ in (6.67).

Proof: We shall show that (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i).

Assuming (i) we can find $\ell_i^{\epsilon} = \ell_i^{\epsilon}(x)$ such that the equations

$$\pi^{\epsilon} = \ell_i^{\epsilon} \omega^i$$

define an integral p -plane. Making the substitution

$$\pi^{\epsilon} \rightarrow \pi^{\epsilon} - \ell_i^{\epsilon} \omega^i$$

then eliminates the c_{ij}^{α} 's in (6.64).

Assuming (ii) we define π_i^{α} by (6.66) and then the c_j^r 's are zero.

Finally we turn to the implication (iii) \Rightarrow (i). Since the 1-forms π_i^{α} are not linearly independent modulo I , we cannot in general say that the equations $\pi_i^{\alpha} = 0$ define a tangent p -plane to M (the point is that these equations may imply relations among the ω^i 's). The conditions that these equations do define a p -plane are

$$b_{\alpha \epsilon i}^{ri} a_{\epsilon i}^{\alpha} = 0 \Rightarrow b_{\alpha}^{ri} \pi_i^{\alpha} \equiv 0 \pmod{I},$$

and from this it is clear that (iii) \Rightarrow (i) . Q.E.D.

(6.72) Proposition: Assume that $C_j^r = 0$ in (6.67). Then the
admissible integral p-planes are given by

$$\begin{cases} \pi_i^\alpha = l_{ij}^\alpha \omega^j \\ l_{ij}^\alpha = l_{ji}^\alpha \\ b_\alpha^{ri} l_{ij}^\alpha = 0 . \end{cases}$$

Proof: The admissible integral p-planes are given by (6.46) where
(6.50) is satisfied. For these l_j^ϵ we have by (6.66)

$$\begin{aligned} \pi_i^\alpha &= a_{\epsilon i}^\alpha l_j^\epsilon \omega^j \\ &= l_{ij}^\alpha \omega^j \end{aligned}$$

where by (6.50)

$$\begin{aligned} l_{ij}^\alpha &= a_{\epsilon i}^\alpha l_j^\epsilon \\ &= l_{ji}^\alpha , \end{aligned}$$

and where

$$b_\alpha^{ri} l_{ij}^\alpha = 0$$

by the definition of $b = a^\perp$.

Conversely, given l_{ij}^α satisfying the conditions of the proposition,
since

$$b^\perp = (a^\perp)^\perp = a$$

we may write

$$l_{ij}^\alpha = a_{\epsilon i}^\alpha l_j^\epsilon .$$

The symmetry $l_{ij}^\alpha = l_{ji}^\alpha$ then implies (6.50). Q.E.D.

Combining the preceeding three propositions gives the following
analogue of (6.57):

(6.73) Define S'_k to be s plus the dimension of the span of the first
k columns in (6.69), and s'_1, \dots, s'_p by

$$\begin{cases} S'_k = S'_{k-1} + s'_k & 1 \leq k \leq p-1 \\ S'_{p-1} + s'_p = n - p . \end{cases}$$

Define

$$a^{(1)} \subset W \otimes \text{Sym}^2 V^*$$

to be the subspace spanned by the vectors

$$T = \ell_{ij}^{\alpha} y_{\alpha}^i x^j$$

where ℓ_{ij}^{α} satisfies the conditions of Proposition (6.72). Then

$$\dim a^{(1)} \leq s'_1 + 2s'_2 + \dots + ps'_p$$

with equality holding if, and only if, $\|a_{\epsilon i}^{\alpha}\|$ is an involutive tableau.

Remark: By proposition (6.72) there is an isomorphism

$$(6.74) \quad S(a) \cong a^{(1)};$$

the reason for the notation $a^{(1)}$ will appear in the sequel to these notes.

(6.75) Example: We shall discuss one 2nd order P.D.E.

$$(6.76) \quad F(x^i, z, \frac{\partial z}{\partial x^i}, \frac{\partial^2 z}{\partial x^i \partial x^j}) = 0.$$

Introducing variables p_i and $p_{ij} = p_{ji}$ this is equivalent to the differential system

$$(6.77) \quad \begin{cases} F(x^i, z, p_i, p_{ij}) = 0 \\ dz - p_i dx^i = 0 \\ dp_i - p_{ij} dx^j = 0 \end{cases}$$

with independence condition $dx^1 \wedge \dots \wedge dx^p \neq 0$. Under the assumption that, for general z, p_i, p_{ij} satisfying the first equation in (6.77), the symmetric matrix $\partial F / \partial p_{ij}$ has maximal rank, we shall show that (6.77) is an involutive system.

For this it is useful to introduce an analogue of moving frames, called adapted systems, for the Pfaffian system generated by

$$(6.78) \quad \begin{cases} dz - p_i dx^i = 0 \\ dp_i - p_{ij} dx^j = 0 \end{cases}$$

with independence condition $dx^1 \wedge \dots \wedge dx^p \neq 0$. The exterior derivatives of (6.78) give

$$\begin{cases} dp_i \wedge dx^i = 0 \\ dp_{ij} \wedge dx^j = 0. \end{cases}$$

We set

$$\begin{cases} \bar{\omega}^i = dx^i \\ \bar{\theta} = dz - p_i dx^i \\ \bar{\theta}_i = dp_i - p_{ij} dx^j \\ \bar{\pi}_{ij} = dp_{ij} = \bar{\pi}_{ji}. \end{cases}$$

The system (6.78) is now

$$(I) \quad \begin{cases} \bar{\theta} = 0 \\ \bar{\theta}_i = 0 \end{cases}$$

with structure equations

$$\begin{cases} d\bar{\theta} \equiv 0 \mod I \\ d\bar{\theta}_i \equiv -\bar{\pi}_{ij} \wedge \omega^j \mod I. \end{cases}$$

To give the definition of adapted systems we use the following notation:

For a collection of 1-forms $\alpha^i, \beta^v, \gamma^\sigma, \dots$ we set

$$\text{Flag}\{\alpha^i; \beta^v; \gamma^\sigma; \dots\} = \text{span}\{\alpha^i\} \subset \text{span}\{\alpha^i, \beta^v\} \subset \text{span}\{\alpha^i, \beta^v, \gamma^\sigma\} \subset \dots$$

(6.79) Definition: An adapted system is given by 1-forms $\theta, \theta_i, \omega^i, \pi_{ij}$ satisfying

$$\text{Flag}\{\theta; \theta_i; \omega^i; \pi_{ij}\} = \text{Flag}\{\bar{\theta}; \bar{\theta}_i; \bar{\omega}^i; \bar{\pi}_{ij}\}$$

$$(6.80) \quad \begin{cases} d\theta \equiv 0 \mod I \\ d\theta_i \equiv -\pi_{ij} \wedge \omega^j \mod I \end{cases}$$

(the first equation in (6.80) is automatic).

We remark that the two substitutions

$$(6.81) \quad \begin{cases} \theta = \bar{\theta} \\ \theta_i = \tilde{A}_i^j \bar{\theta}_j \mod I \\ \omega^i = A_j^{i-j} \mod I \\ \pi_{ij} = \tilde{A}_i^k \bar{\pi}_{kl} \tilde{A}_j^l \mod I \end{cases} \quad A = \tilde{A}^{-1}$$

$$(6.82) \quad \begin{cases} \theta = \bar{\theta}, \quad \theta_i = \bar{\theta}_i, \quad \omega^i \equiv \bar{\omega}^i \mod I \\ \pi_{ij} = \bar{\pi}_{ij} + B_{ijk} \omega^k \mod I, \quad B_{ijk} = B_{ikj} \end{cases}$$

both give adapted systems.

We will now show that the Pfaffian system (6.77) has an algebraic normal form in a suitable adapted system, and from this it will be easy to verify Cartan's test (6.73) for involution. Setting $dF = 0$ introduces one relation

$$(6.83) \quad Q^{ij} \bar{\pi}_{ij} + \bar{C}_i \bar{\omega}^i \equiv 0 \mod I$$

where $Q^{ij} = Q^{ji} = \partial F / \partial p_{ij}$. By our assumption, $\det Q^{ij} \neq 0$, and therefore we may make a substitution (6.81) so that (6.83) becomes

$$\varepsilon^i \pi_{ii} + C_i \omega^i \equiv 0 \pmod{I}.$$

By a substitution (6.82) we may eliminate the C_i . At this point we have the algebraic normal form

$$(6.84) \quad \varepsilon^i \pi_{ii} \equiv 0 \pmod{I}$$

for (6.77).

Our Pfaffian system $\theta = \theta_i = 0$ with structure equations (6.80) and symbol relation (6.84) is in dual form. The matrix (6.69) is

$$\begin{vmatrix} 0 & \dots & 0 \\ \pi_{11} & \dots & \pi_{1p} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \pi_{p1} & \dots & \pi_{pp} \end{vmatrix}.$$

Aside from the relations

$$\pi_{ij} = \pi_{ji}, \quad \varepsilon^i \pi_{ii} = 0$$

the 1-forms π_{ij} are linearly independent modulo $\text{span}\{\theta, \theta_i, \omega^i\}$.

It follows that (cf. (6.73))

$$(6.85) \quad s'_1 = p, \quad s'_2 = p-1, \quad \dots, \quad s'_{p-1} = 2, \quad s'_p = 0.$$

Thus

$$s'_1 + 2s'_2 + \dots + ps'_p = p(p+1)(p+2)/6 - p.$$

On the other hand, admissible p -dimensional integral elements on which $\theta = \theta^i = 0$ are given by

$$\pi_{ij} = \ell_{ijk} \omega^k$$

where

$$(6.86) \quad \ell_{jik} = \ell_{ijk} = \ell_{ikj}, \quad \varepsilon^i \ell_{iik} = 0,$$

If we denote by $a^{(1)}$ the space of ℓ_{ijk} satisfying (6.86), then

$$\begin{aligned} \dim a^{(1)} &= \dim \text{Sym}^3 \mathbb{R}^p - p \\ &= \binom{p+2}{p-1} - p \\ &= s'_1 + 2s'_2 + \dots + ps'_p. \end{aligned}$$

By Cartan's test the system is in involution.

By the Cartan-Kähler theorem, in the real analytic case a general solution of (6.76) depends on 2 functions of $(p-1)$ -variables. The simplest case of this is the wave equation

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$$

in \mathbb{R}^2 , whose general solution is

$$z(x, y) = u(x + y) + v(x - y)$$

where u and v are arbitrary functions of one variable.

To conclude this section we want to supplement this discussion with some standard P.D.E. concepts.*

(6.87) Definitions: (i) The (Monge) characteristics of $F(x^i, z, \frac{\partial z}{\partial x^i}, \frac{\partial^2 z}{\partial x^i \partial x^j})$ = 0 at a point (x^i, z, p_i, p_{ij}) satisfying $F(x^i, z, p_i, p_{ij}) = Q$ are given by all $\xi = (\xi^i) \in \mathbb{R}^p$ that satisfy

$$(\partial F / \partial p_{ij}) \xi^i \xi^j = 0.$$

(ii) If there are no Monge characteristics, then the equation is elliptic (at the point in question).

(iii) If $\det(\partial F / \partial p_{ij}) \neq 0$ and the equation is not elliptic, then it is hyperbolic.

It is easy to see that elliptic and hyperbolic systems have the respective algebraic normal forms

* In the sequel we will generalize these to arbitrary Pfaffian systems in good form.

$$\sum_{i=1}^p \pi_{ii} = 0$$

$$\sum_{i=1}^{2\ell} \pi_{i, \ell+i} + \sum_{j=2\ell+1}^p \pi_{jj} = 0.$$

The matrices Q^{ij} are, respectively, the identity and

$$\begin{pmatrix} 0 & I_\ell & 0 \\ I_\ell & 0 & 0 \\ 0 & 0 & I_{p-2\ell} \end{pmatrix}.$$

It is also easy to check that in the elliptic case every admissible integral p -plane E_0^p is regular, as is every flag $x_0 \in E_0^1 \subset \dots \subset E_p^{p-1} \subset E_0^p$, while this fails in the hyperbolic case. For example, the coordinate flag for the hyperbolic case in the above normal form has

$$s'_1 = p, \dots, s'_{\ell-1} = p - \ell + 1, s'_\ell = p - \ell - 1$$

in contrast to (6.85).

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The real Monge-Ampère equation
and affine flat structures

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§0. Introduction.

Many nonlinear partial differential equations arise naturally from geometric problems. The real and complex Monge-Ampère equation are some of them which do not admit a natural variational characterization.

For a function u defined on a domain in \mathbb{R}^n with coordinate (x^1, \dots, x^n) , a real Monge-Ampère equation is an equation of the form

$$\det \left(\frac{\partial^2 u}{\partial x^i \partial x^j} \right) = F(x, u, \nabla u).$$

The complex Monge-Ampère equation is an equation of the form

$$\det \left(\frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} \right) = F(x, u, \nabla u),$$

where u is a function on some domain in \mathbb{C}^n with complex analytic coordinate (z^1, \dots, z^n) . If $(\tilde{z}^1, \dots, \tilde{z}^n)$ is another set of complex analytic coordinate then

$$(0.1) \quad \det \left(\frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} \right) = \left| \frac{\partial \tilde{z}}{\partial z} \right|^2 \det \left(\frac{\partial^2 u}{\partial \tilde{z}^i \partial \bar{\tilde{z}}^j} \right).$$

Thus the complex Monge-Ampère operator is a natural partial

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