ON THE ANALYTIC FORM OF THE GEOMETRIC THEORY OF AUTOMORPHIC FORMS

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Had we but world enough, and time,...
Andrew Marvell, 1621–1678

Contents

Still another title—the unhappy and troubling experience of an elderly mathematician in the wilds of differential geometry! It has been made more unhappy, more troubling by the necessity to provide a translation two years after the Russian version was completed, just as my memory, not only of the paper but in general, begins to fail me! I did not think that would be necessary.

I. Introduction

The geometric theory of automorphic forms was introduced by Russian mathematicians, for example, Vladimir Drinfeld, and developed by the Russian-American school,[1] but I am dissatisfied with this theory. The contemporary arithmetical theory arose in the sixties of the twentieth century from four sources: from the revival by Siegel of the investigations of the nineteenth century; from the Hecke theory; from class field theory; from the theory of group representations, in which the names of Frobenius, Herman Weyl, and Harish-Chandra are important. In this arithmetic theory the Hecke eigenvalues are an irreplaceable element. Using these the extremely important L-functions are determined. These numbers are the eigenvalues of the Hecke operators. These operators are defined in the arithmetic theory but not in the geometric theory introduced by Gaitsgory or Frenkel.[2] The difficulty is that in the theory of the Russian-American school the eigenvectors are replaced by eigensheaves, the existence of which is difficult to establish, until now even impossible. Moreover the description of the classifying space in this theory presupposes concepts from the theory of sheaves and stacks and presupposes as well topological questions introduced in order to create a theory of classifying spaces that satisfies in large part the functorial demands of Grothendieck.

Their theory is important, but in my view it is not the theory that is necessary for expressing and proving the geometrical form of that which I call in the arithmetical theory functoriality and reciprocity. For a reason that I explain later it may be better to use the term duality, but functoriality is a consequence of duality. One of the aims of the arithmetic theory is to establish functoriality and, using this functoriality, to construct the automorphic galoisian group,[3] but in the geometric theory this group is already at hand. None the less it is necessary to show that a given group possesses the desired properties. What it is necessary to understand for the foundations of the geometric theory, introduced in this essay, is a general understanding from the sphere of sets, spaces and measures.

The purpose of this essay is to describe what seems to me a suitable analogue of the arithmetical theory and to establish this for an interesting, even striking although easily accessible case: the group GL(2) over an elliptic curve. However, there are two geometric theories, one over a finite field[4] and another, as in this essay, over the field of complex numbers. I examined neither [G] nor [L] with care, but it seems(!) to me that [L] supposes that for a field of functions over a finite field a theory that is compatible with the Rosetta Stone of André Weil[5]. In addition to that and, in my view very interesting, according to [L, Th. 0.1] for such a field the automorphic Galois group is isomorphic to $\mathrm{Gal}(\overline{F}/F)$. On the other hand, just as the theory of Paley-Wiener or the theory of Fourier transforms for the general Schwartz functions do not replace the L^2 theory of the Fourier transform on $\mathbf{R}$ or on $\mathbf{R}^n$, $n = 2, 3, \ldots$, the theory proposed in [G] does not replace an L^2 theory

[1][G] The report, *Progrès récents dans la théorie de Langlands géométrique,* Sém. Bourbaki, Janvier, 2016, written by Dennis Gaitsgory is an appropriate reference.

[2][F] *Lectures on the Langlands program and conformal field theory, Frontiers in number theory, physics, and geometry.*

[3]This was a fundamental misconception on my part. I was not aware of the importance of Tannakian categories as I wrote it.

[4][L] V. Lafforgue, *Chtoucas et programme de Langlands pour les corps de fonctions* (`https://arxiv.org/abs/1404.3998`)

[5][W] *De la métaphysique aux mathématiques,* Oeuvres scientifiques, vol. II pp. 408–412.

of Hecke operators. I believe that such a theory exists for each reductive group over an arbitrary Riemann surface (compact). Occasionally I refer to the general case but I have not thought seriously about it. The goal of this essay is the description of a theory for the group GL(2) over an elliptic curve, a theory that is already accessible. Although I can imagine a general spectral form of the geometric theory, I cannot imagine a theory that unites it with the theory described in [G]. As a final observation, the geometric theory is related in part to complex differential geometry, but this essay is written for specialists of the theory of automorphic forms, for whom this geometric theory may be unfamiliar. Because of such personal weaknesses I include some elementary explanations of the concepts in [AB]. These explanations often become lengthy digressions. Moreover, I also allow myself other digressions, some rather elementary, arising from the assumption that, like me, the majority of readers have little experience with differential geometry.

The conclusions are certainly limited but the appearance of the decisive sign where I least of all expected it confirmed my confidence in the correctness of my conviction. No more was necessary to persuade me.

The exposition is insufficiently detailed, but the topic is new and our understanding is incomplete, so that an elaborate explanation would be inappropriate.

Automorphic galoisian group. Since the principal purpose of this article, an introduction to a geometrical theory and a preliminary—perhaps temporary, perhaps definitive—construction of the relevant mathematical objects, I preferred to define this immediately. It presupposes the introduction of a group that is defined with a prescription for the transformation of a homomorphism from it to the group ${}^LG$ to an eigen conjugacy section. These sections are described below. These prescriptions may be complicated. For example, they may involve the Atiyah-Bott theorem. In any case we confirm a law of reciprocity for GL(2) and elliptic curves, at first listing eigenvalues of Hecke operators and then Yang-Mills connections. Not supposing that the present undertaking is in any form whatsoever comparable to that of Dedekind, I propose another title for this paper, *Was ist und was soll die geometrische Theorie der automorphen Formen?* ■[6]

Remarks about the nature and content of this essay. It is written for mathematicians familiar with the theory of automorphic forms over number fields, but who like me, with a meagre knowledge of differential geometry, in particular of vector bundles and connections. Thus substantial space is taken by simple or familiar concepts. A slight familiarity with them is inadequate, as I discovered on studying [A] and [AB]. The explanations of these authors are brief and smooth but rarely precise and rarely detailed, evidently because they suppose that the reader has some knowledge of the basic concepts. I thought that I possessed it, but initially it was insufficient for a precise understanding of their conclusions and even of their basic assertions. Thus I include in this essay all supplementary reflections that I felt were necessary or useful, but only in so far as they seemed necessary.

I add a simple but useful remark. In the development of this article there are three stages: a clear notion of eigenfunctions and eigenvalues of Hecke operators; a clear notion of a Yang-Mills connection; their comparison.

I arrived at an understanding of the necessary foundations of the theory only as I read the references and as I wrote the article, so that it began with an assertion whose proof was assured only slowly, when I arrived at an understanding of the relevant theory. My

[6] This square indicates the end of a digression.

initial concept of the theorem established included neither a precise assertion nor much understanding of the differential geometric foundations. Thus there was much that I could not foresee and I was not prepared. This led to redundancy and digressions, some elementary, that would be superfluous for a geometer but, perhaps, useful for specialists in areas related to the basic problem of this essay, namely reciprocity and functoriality in the theory of automorphic forms, and not only in one of its three aspects but in all, although this article is devoted exclusively to the geometric theory.

In the end I did not announce a theorem, although the reader could infer the relevant assertion. What is striking is the precise fit of the enumeration of the eigen classes of Hecke sections and the structure of the Yang-Mills theory. Although initially I did not have a precise memory of the counting in the classical theory (neither of the counting in Gauss nor of the later in, say, Hasse and I began to fear that I had not studied them adequately), it seemed to me that there was an unrecognized similarity between them and those in this essay. In both cases two, apparently, different objects were related by a third unrecognized clockwork. In this article this appears to be a consequence of the similarity between two rather complex, although concrete, countings, and in the very end the comparison could be completed only thanks to an astonishing detail that appears in both, although for different reasons. To hide this in the assertion of a theorem that in the end would only be a special case seemed clumsy.

The structure of this essay is simple. At the end of §III, the goals of functoriality in the geometric context are explained. In §IV I recall the classical theory of elliptic curves, on which we draw, in order to render the discussion as solid as possible. The following three sections are devoted to the introduction of Hecke operators and the enumeration of their eigenfunctions and eigenvalues but in the context of Hilbert spaces. After this in §VIII and §IX the necessary differential geometry is recalled, sometimes at a basic level. In §X and §XI a further discussion of the Yang-Mills theory leads to earlier enumerations in the Hecke theory and then easily to the desired comparison. The last section is brief. I allow myself many digressions, some rather elementary, on the assumption that the majority of my readers will, like me, have minimal experience with differential geometry.

Finally a word about the language. It is pertinent only to the original Russian version. When I inscribed myself in the university in Vancouver at the age of sixteen years, coming from the countryside I was initiated to a new world, not only that of mathematics. I also learned that there existed in the world a large collection of languages, some of which at that time were necessary for a successful career as a mathematician. Although slowly, I recognized that they offered more than just mathematics. Unfortunately for reasons that it is not necessary to describe in detail here the mathematical profession no longer offers this window to the world. Nevertheless the desire to write a paper in Russian remained. This paper was the last chance to do so. It turned out to be too late.

Reflecting on this article after it was completed and on its style I was troubled by the many digressions and wondered about their relation, but the reason became clear. It is related to the circumstance that the basic part of the labour was devoted to my efforts to understand concepts taken from [A] and some parts of [AB]. After the structure of Bun_G and the nature of Yang-Mills connections was understood, my unique idea was the introduction of Hecke operators as operators on a Hilbert space. This was simple although new and even somewhat revolutionary, for the attachment of mathematicians to sheaves was universal. There are no sheaves in this paper.

In a final conclusion I emphasize my dissatisfaction with the present exposition, but an adequate exposition demands a brief but complete general account of the Yang-Mills theory, as well as an understanding of the last paragraph of this article, in which the direct image of bundles and induced representations appear, but an understanding under general conditions. At present this may be just possible. I leave it as a problem to the reader. ■

II. The fundamental concepts of a general theory

We need first of all a reductive algebraic group G over a compact nonsingular complex curve M and its classifying space Bun_G. I shall explain the general hypothesis, that I shall explain principally for the group $G = \mathrm{GL}(2)$ over an elliptic curve M. This case is already striking. I shall use the conclusions of the paper of Atiyah.[7] It is possible that they are available also for the group $\mathrm{GL}(n)$.

Let F be the field of meromorphic functions on M. Let $\mathbf{A}_F$ be the algebra of adeles of the field F and F_x the local field at the point $x \in M$. The ring $\mathcal{O}_x$ is the ring of integral elements in F_x and $\mathcal{O} = \prod_x \mathcal{O}_x$. The relation

$$\mathrm{Bun}_G = G(F)\backslash G(\mathbf{A}_F)/G(\mathcal{O})$$

is familiar. It is established in ([F, §3]). I shall explain this below. We explain briefly later in what fashion Bun_G becomes a topological space although not a Hausdorff space and that it carries a local metric structure with metric μ. For $G = \mathrm{GL}(2)$ over an elliptic curve M both the structure and the metric are simple.

The Hecke operators are linear transformations of the space $L^2(\mu)$. The principal theme of this article is the collection of Hecke operators and their proper values. For each point $x \in M$ there exists a commutative Hecke algebra $\mathfrak{H}_x$. These algebras are commutative and commute with each other. Suppose $\Theta \in \mathfrak{H}_x$. Then the Hermitian conjugate operator $\widetilde{\Theta}$ is also a Hecke operator. Consequently there exists a corresponding spectral decomposition of the space $L^2(\mu)$ and the purpose of this essay—description of the eigenvalues and eigenfunctions of this decomposition. Although elliptic curves are the principal objects in this paper, I cannot resist some general remarks. If God wills I shall return to general curves later. In the following section I formulate general assumptions that I establish, at least in part, for elliptic curves. Each Hecke operator Θ determines a correspondence Θ. This correspondence is a subset of the set $\mathrm{Bun}_G \times \mathrm{Bun}_G$. This correspondence also carries a measure that is best described later.

III. Hypothesis

Although we establish this hypothesis only for curves of genus one and not for higher genus, and only for $\mathrm{GL}(2)$, it seems to me that it is also correct for $\mathrm{GL}(n)$, thanks to the article [A]. In general there are two levels: (i) $\mathrm{GL}(2)$ is replaced by another group; (ii) M is replaced by an arbitrary compact Riemann surface. I have not yet thought about this seriously. I am proposing however a convincing conjecture, but in order to confirm it for $g > 1$ it will be necessary to understand the complexity of Bun_G. There is of course one other level, the ramified theory, but I have not reflected on this.

It is known that the Hecke algebra of the group $\mathrm{GL}(2)$ or of an arbitrary reductive group—in this section we are speaking in this generality—is isomorphic to the ring of representations of the dual group G and that each homomorphism of this algebra into $\mathbf{C}$ is

[7] [A] *Vector bundles over an elliptic curve,* Proc. London Math. Soc. vol. 7, 1957

given by a semi-simple class θ in ${}^LG$. Consequently the eigenfunctions of all Hecke operators or better eigen sections correspond to functions whose values at the point $x \in M$ represent a semi-simple class $\{\theta(x)\}$. We call this the eigen conjugacy section. The structure of this set of sections is not clear. It is known that there is such a theory for all M and all G and we consider in this paragraph the general case.

In the theory of automorphic forms the concept of functoriality expresses the following: the set of sections or the set of those sections that belong to the L^2-theory are given by homomorphisms, or unitary homomorphisms, of the conjectural automorphic Galois group into ${}^LG$. It is supposed that the difference between the arithmetic and geometric theories is that the geometric theory can be described simply with the use of familiar concepts. In the arithmetic theory this is not so. In this section I describe a general hypothesis that I establish later, but for a particular case, and after some preparations.

A concept related to the automorphic Galois group is introduced in the paper of Atiyah-Bott.[8] This paper, together with the paper of Atiyah already mentioned, had a great influence on the present paper. The pertinent concept is the group $\Gamma_{\mathbf{R}}$ ([AB, Th. 6.7]). For curves of genus g this group is an extension of the central extension

$$1 \longrightarrow \mathbf{Z} \longrightarrow \Gamma \longrightarrow \pi_1(M) \longrightarrow 1\ , \quad \Gamma_{\mathbf{R}} = \mathbf{R} \times_{\mathbf{Z}} \Gamma. \tag{1.a}$$

The group Γ is generated by elements $A_1, \dots, A_g, B_1, \dots, B_g$ and $J = 1 \in \mathbf{Z}$ with one relation

$$A_1B_1A_1^{-1}B_1^{-1}A_2B_2A_2^{-1}B_2^{-1} \cdots A_gB_gA_g^{-1}B_g^{-1} = J, \quad J = 1 \in \mathbf{Z}. \tag{1.b}$$

In the following sections, where $g = 1$, it is understood that A_1, B_1 represent the loops $(0, 2\omega_1)$, $(0, 2\omega_2)$ in the elliptic curve M.

We are concerned only with representations ϕ of the group Γ for which the order of the elements $\phi(A_i)$, $\phi(B_i)$ and $\phi(J)$ are all finite. They are called admissible. We need groups for which $\widetilde{\Gamma} = \mathbf{Z} \times \Gamma$. It is possible that the order of the elements $\phi(z \times 1) \in \mathbf{U} \times 1$, $z \in \mathbf{Z}$, $\mathbf{U} = \{\, w \in \mathbf{C} \mid |w| = 1 \,\}$, $1 \in \Gamma$ is infinite. The term $\mathbf{Z}$ does not appear in [AB] because in that article the Chern class of the bundle is so given that Bun_G is connected, namely it is the connected component of the correct Bun_G. The imbedding of this $\mathbf{Z}$ in $\widetilde{\Gamma}$ is somewhat arbitrary. It is related to the choice of the section $A = A_0$ in [A, Th. 6], and §IV of the present article. Now[9]

$$A_1B_1A_1^{-1}B_1^{-1}A_2B_2A_2^{-1}B_2^{-1} \cdots A_gB_gA_g^{-1}B_g^{-1} = 0 \times J. \tag{1.c}$$

I refer to this new group as the automorphic Galois group. More precisely the automorphic Galois group Γ_{aut} is the product of $\mathbf{Z}$ with the inverse limit of all finite quotients of the group $\widetilde{\Gamma}$. In this group (on the level of each finite quotient of this group—it is this that is important) the order of the images $\phi(A_i)$, $\phi(B_i)$ and $\phi(J)$ is finite. For example, we first

[8][AB] *The Yang-Mills equations over Riemann surfaces,* Phil. Trans. Royal Soc. Lond., vol. 308, 1983

[9]It is useful to observe that

$$\alpha \begin{pmatrix} 0 & 0 & 0 \cdots\cdots & 0 & 1 \\ 1 & 0 & 0 \cdots\cdots & 0 & 0 \\ 0 & 1 & 0 \cdots\cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots\cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \cdots\cdots\cdots 0 \\ 0 & \alpha & 0 \cdots\cdots\cdots 0 \\ 0 & 0 & \alpha^2 \cdots\cdots\cdots 0 \\ \vdots & \vdots & \vdots \qquad \vdots \\ 0 & 0 & 0 \cdots\cdots \alpha^{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \cdots\cdots\cdots 0 \\ 0 & \alpha & 0 \cdots\cdots\cdots 0 \\ 0 & 0 & \alpha^2 \cdots\cdots\cdots 0 \\ \vdots & \vdots & \vdots \qquad \vdots \\ 0 & 0 & 0 \cdots\cdots \alpha^{n-1} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \cdots\cdots & 0 & 1 \\ 1 & 0 & 0 \cdots\cdots & 0 & 0 \\ 0 & 1 & 0 \cdots\cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots\cdots & 1 & 0 \end{pmatrix},$$

where $\alpha = \exp(2\pi i/n)$.

construct the intersection Γ_2 of all kernels of homomorphisms of the group Γ to groups with an order dividing 2, then Γ_{36}, Γ_{27000} and so on. Then

$$\Gamma_{\text{aut}} = \varprojlim \mathbf{Z}/n(k)\mathbf{Z} \times \varprojlim \Gamma/\Gamma_{n(k)}, \tag{1.d}$$

where $n(k) = (k!)^k$.

This annoying scrupulousness is necessary because the connections and the eigen conjugacy sections are mutually related but different. In particular the supplementary $\mathbf{Z}$ is necessary because Bun_G is not connected. As will be clear later, it is not in vain that $\Gamma_{\mathbf{R}}$ and Γ_{aut} are determined as modifications of the group Γ. Eigen conjugacy sections and the connections of Yang-Mills are closely related. I remark also that in Γ_{aut} the equation $J = 1$, $1 \in \mathbf{Z}$ is replaced by

$$J = 1, \qquad 1 \in \varinjlim_n \mathbf{Z}/n\mathbf{Z},$$

where $z \mapsto nz/m$, $m|n$ in an increasing sequence to the right.

Let ${}^LG_{\text{unit}}$ be a compact form of the group ${}^LG$. I suppose the following, because this supposition makes it possible to avoid the difficulties of endoscopy, namely that there exists a bijective correspondence of the set of eigen conjugate connections with the set of homomorphisms of the group Γ_{aut} to ${}^LG_{\text{unit}}$. The purpose of the present paper is to show that for the group GL(2), although it seems to me that the paper [A] allows one to prove this for GL(n). There is a difficulty here. I still do not know how to recognize the irreducible representations of the group Γ_{aut} in GL(n) if the dimension is greater than two, even for elliptic curves. It will be clear to the reader that I began to understand the papers [AB] and [A] only as I wrote this article. Initially my efforts were more modest, but it seems to me that the principal assertions of the geometric theory over $\mathbf{C}$ are amazingly simple, although not obvious. I do not know if it is necessary or even useful for readers of [G] or [L] to understand them.

The case of genus zero. If the genus is zero the equation (1.b) is senseless because it is necessary to determine Γ_{aut} differently. It seems to me that there is no choice but to suppose that it is equal to $\mathbf{Z}$. This is compatible with the assertion that for genus zero all sections are direct sums of linear sections, each of which is itself a power of a unique linear bundle of degree zero. This case is clearly excluded from the discussion in [AB]. It will be clear to the reader everywhere in this paper that it would be better written, both with respect to clarity as with respect to precision if I were more familiar with the Yang-Mills theory in particular and with differential geometry in general. ■

There are two important remarks. First of all, I consider only the unramified theory. Secondly, in the arithmetic theory the existence of an automorphic galoisian group is equivalent to functoriality.[10] The close relation between the theory of algebraic numbers and the theory of algebraic curves over $\mathbf{C}$ was described by Dedekind and Weber in the book [DW][11]. The theory of algebraic curves over a Galois field was added by André Weil [W].

The theories of the two papers [A] and [AB] are not well known, neither to me nor to the majority of my readers.

[10] This statement may or may not be correct. It is almost certainly incorrect as it stands. What in the arithmetic theory is a Tannakian category may be, and likely is, a group in the geometric theory, but this is no question to be discussed in general here. I am not up to it now, and perhaps never.

[11] [DW] R. Dedekind and H. Weber, *Theorie der algebraischen Funktionen einer Veränderlichen,* 1880

IV. Reduction theory for an elliptic curve

This important theory reached a final form over fields of algebraic numbers in the report of Borel and Harish-Chandra[12] but for Riemann surfaces it is not yet available. In the article [A] Lemma 4 is the beginning of this theory for arbitrary genus, but for $g = 1$ there is a complete theory for GL(2) and even for GL(n) in the same paper, which I explain primarily for GL(2) but without proofs. This theory is a final description of Bun_G, thus a description of all two-dimensional vector bundles. In order to establish it, it is necessary to establish a contemporary theory of connections, but I prefer to apply the theory of Weierstrass as it is presented in the book of Whittaker and Watson.[13] This is of course not necessary. It is rather a test of my understanding of the theory of Atiyah and Atiyah-Bott and a sign of my mathematical addictions. The conceptions in [AB] that reflect of course the contemporary view of complex differential geometry—and therefore of topology and complex analysis—are exceptionally elegant. But they are abstract and it is easy for a novice to overlook their complexity and delicacy, as often happened to me as I read their papers. I reached the conclusion that their concrete expression avoids misunderstanding. Thus if $L = 2\mathbf{Z}\omega_1 \oplus 2\mathbf{Z}\omega_2$, ω_1, $\omega_2 \in \mathbf{C}$, $\omega_1/\omega_2 \notin \mathbf{R}$, then the curve $M = \mathbf{C}/L$. In this essay a GL(n)-section is a matrix-valued meromorphic function[14] $M(z)$, $z \in \mathbf{C}$, such that $M(z+\lambda) = M(z)K_\lambda(z)$ for all $\lambda \in L$, where the matrix K_λ is holomorphic.

In the theory of Weierstrass the sigma-function $\sigma(z)$, $z \in \mathbf{C}$, plays a basic role. These are its properties: (i) it is holomorphic; (ii) its expansion in a power series is $\sigma(z) = z + \cdots$; (iii) $\sigma(z+2\omega_1) = -e^{2\eta_1(z+\omega_1)}\sigma(z)$, $\sigma(z+2\omega_2) = -e^{2\eta_2(z+\omega_2)}\sigma(z)$; (iv) $\eta_1\omega_2 - \eta_2\omega_1 = \pi i/2$; (v) if $\sigma(z) = 0$ then $z \in L$.

Let $a_1, \ldots, a_m, b_1, \ldots, b_n$ be points in $\mathbf{C}$. Then

$$\phi(z) = \frac{\sigma(z-a_1)\sigma(z-a_2)\cdots\sigma(z-a_m)}{\sigma(z-b_1)\sigma(z-b_2)\cdots\sigma(z-b_n)} \tag{2}$$

is a meromorphic function of $z \in \mathbf{C}$. In addition,

$$\begin{aligned}\phi(z+2\omega_1) &= \phi(z)(-1)^{m-n}e^{2\eta_1(m-n)(z+\omega_1)}e^{-2\eta_1\left\{\sum_{i=1}^m a_i - \sum_{j=1}^n b_j\right\}}\\ &= \phi(z)(-1)^{m-n}e^{2\eta_1(m-n)(z+\omega_1)}e^{-2\eta_1\theta},\end{aligned} \tag{2.a}$$

$$\begin{aligned}\phi(z+2\omega_2) &= \phi(z)(-1)^{m-n}e^{2\eta_2(m-n)(z+\omega_2)}e^{-2\eta_2\left\{\sum_{i=1}^m a_i - \sum_{j=1}^n b_j\right\}}\\ &= \phi(z)(-1)^{m-n}e^{2\eta_2(m-n)(z+\omega_2)}e^{-2\eta_2\theta},\end{aligned} \tag{2.b}$$

where $\theta = \sum_{i=1}^m a_i - \sum_{j=1}^n b_j$. The function ϕ is periodic only if $m = n$. In this case it is an ordinary function on $\mathbf{C}$ which covers the curve M. If $m = n$ and $\theta = 0$ then ϕ is a function on M. If σ is replaced by a function $\sigma'(z) = \sigma(z)\exp(\lambda z)$, $\lambda \in \mathbf{C}$, then the equation (iii) is replaced by the equations

$$\sigma'(z+2\omega_i) = -e^{2\eta_i+2\lambda\omega_i}\sigma'(z) = -e^{2\eta'_i}\sigma'(z), \quad \eta'_i = \eta_i + \lambda\omega_i. \tag{2.c}$$

That is these equations appear only as normalisations of the function σ or, if you like, σ and σ' determine different but equivalent linear sections. The equation (iv) does not change by this.

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[12][BH] Borel, Armand and Harish-Chandra, *Arithmetic subgroups of algebraic groups,* Bull. Amer. Math., 1961, v. 67, 579–583 and Borel, Armand and Harish-Chandra, Arithmetic subgroups of algebraic groups, Ann. of Math., 1962, v. 75, 485–535.

[13][WW] *A Course of Modern Analysis,* Camb. Univ. Press, 1958

[14]$z = x + iy$ serves as the coordinate in $\mathbf{C}$ and sometimes other purposes.

The equations (iii) determine the pasting and do not change the bundle. In the domain $2x\omega_1 + 2y\omega_2$, $0 \leqslant x, y < 1$ the function σ has only one zero. This is the reason for the equation (iv).[15] In general ϕ determines a linear section $\Lambda = \Lambda(a_1, \ldots, a_m, b_1, \ldots, b_n)$ for which it itself appears as a multi-valued section. Two sections Λ and $\Lambda' = \Lambda(a'_1, \ldots, a'_{m'}, b'_1, \ldots, b'_{n'})$ are isomorphic if and only if $m - n = m' - n'$, and

$$a_1 + \cdots + a_m - b_1 - \cdots - b_n = a'_1 + \cdots + a'_{m'} - b'_1 - \cdots - b_{n'} \mod 2\pi\omega_1\mathbf{Z} + 2\pi\omega_2\mathbf{Z}.$$

Then the degree of the section is $n - m$. In my view this description of a linear (one-dimensional) bundle is the clearest, but for bundles of larger dimension, like those in the paper of Atiyah, it is frequently necessary to use cohomological methods. The theorems and the lemmas of Atiyah are fastidious, perhaps because the sets with which one is concerned are also stacks, although they appear neither in the paper of Atiyah nor here. For him they are spaces and for us papers with local metrics and with a metric. Here I would like first to describe the space $\mathrm{Bun}_{\mathrm{GL}(2)}$ following Atiyah.

On an elliptic curve there are two forms of two-dimensional bundles, decomposable bundles, $\Phi = \Lambda_1 \oplus \Lambda_2$, and bundles of Atiyah type, thus the others. Let $\mathfrak{D}(m, n)$ be the set of $\Phi = \Lambda_1 \oplus \Lambda_2$ for which the degree $\deg \Lambda_1 = m$ of Λ_1 and the degree $\deg \Lambda_2 = n$. The set of bundles of Atiyah type is a union

$$\left\{ \bigcup_{m \in \mathbf{Z}} \mathfrak{A}(m, m) \right\} \cup \left\{ \bigcup_{m \in \mathbf{Z}} \mathfrak{A}(m+1, m) \right\}.$$

For general curves there are both decomposable bundles and indecomposable bundles. In the paper of Atiyah the sets of the latter are denoted $\mathfrak{E}(r, d)$ where r is the rank and d the degree. Before I describe the bundles I remark that $\Lambda_1 \oplus \Lambda_2$ is equivalent to $\Lambda'_1 \oplus \Lambda'_2$ if and only if $\{\Lambda_1, \Lambda_2\} = \{\Lambda'_1, \Lambda'_2\}$. Consequently $\mathfrak{D}(m, n)$ is a two-dimensional complex manifold. If $m = n$ there is singular curve for which $\Lambda_1 = \Lambda_2$.

In contrast to the arithmetic theory, reduction in the geometric theory is precise. That is the fundamental domain is described precisely. In the article of Atiyah (Lemma 3), as a first step and as a consequence of the Riemann-Roch theorem for bundles of larger dimension, it is shown that for a two-dimensional bundle over an elliptic curve there is a representative

$$\Theta = \begin{pmatrix} \Lambda_1 & * \\ 0 & \Lambda_2 \end{pmatrix}, \quad \deg \Lambda_2 \leqslant 2 + \deg \Lambda_1. \tag{3}$$

Although this is not necessary Atiyah prefers to assume the sufficiency of the sections of a given bundle, that is that for each point $x \in M$ the map $\Gamma(\Theta) \mapsto \Theta_x$ is surjective, where $\Gamma(\Theta)$ consists of the section Θ. For this it is sufficient to replace the bundle Θ by the bundle $\Theta' = \Lambda \otimes \Theta$, where Λ is the appropriate linear bundle. Each conclusion for Θ' is also a conclusion for Θ. In general, if a bundle of arbitrary dimension possesses sufficiently many sections, then it has an upper triangular representative. As a second Λ Atiyah asserts

[15] The correct condition for the divisor $(a_1, \ldots, a_n)$ to be equivalent to $(b_1, \ldots, b_n)$ is described by the equations

$$\theta = a_1 + \cdots + a_n - b_1 - \cdots - b_n \in 2\mathbf{Z}\omega_1 + 2\mathbf{Z}\omega_2.$$

That is, if this equation is valid, thus if there is a $\lambda \in \mathbf{C}$ such that the function $\exp(-\lambda z)\phi(z)$ is periodic with respect to $2\mathbf{Z}\omega_1 + 2\mathbf{Z}\omega_2$. For this it is necessary that $-2\lambda\omega_k - 2\eta_k\theta \in 2\pi i\mathbf{Z}$ for $k = 1, 2$. Let $\theta = 2\omega_1$. There are two numbers: for $k = 1$, $-2\lambda\omega_1 - 4\eta_1\omega_1$, and for $k = 2$, $-2\lambda\omega_2 - 4\eta_2\omega_1$. If $\lambda = -2\eta_1$ the first number is 0 and the second $4\eta_1\omega_2 - 4\eta_1\omega_2 = 2\pi i$. If $\theta = 2\omega_2$ the conclusion is similar. Such considerations are superfluous but comforting.

(Lemma 6′) that if the bundle Θ is indecomposable and if $\Gamma(\Theta) \neq 0$, then there it has a maximal decomposition

$$\Theta \simeq \begin{pmatrix} \Lambda_1 & * & *\cdots\cdots\cdot * \\ 0 & \Lambda_2 & *\cdots\cdots\cdot * \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0\cdots\cdots\Lambda_n \end{pmatrix}, \tag{4}$$

such that $\Lambda_i \geqslant \Lambda_1 \geqslant 1$, $\Gamma\operatorname{Hom}(\Lambda_1, \Lambda_i) \neq 0$, $i = 2, \ldots, n$ and $\Gamma(\Lambda_1) \neq 0$. At present $n = 2$ but it is useful to discuss the general case. This is again a consequence of the theorem of Riemann-Roch.

But the argument of Atiyah that follows is difficult and I want to discuss only those consequences that are important for us, sometimes postponing explanations, leaving out those that are not important for us. I use some outmoded concepts. There is in the theory of Weierstrass a second important function, this function is

$$\zeta(z) = \frac{d}{dz}\ln\sigma(z) = \frac{\sigma'(z)}{\sigma(z)}. \tag{2.d}$$

It satisfies an additive condition,

$$\zeta(z + 2\omega_1) = \zeta(z) + 2\eta_1, \quad \zeta(z + 2\omega_2) = \zeta(z) + 2\eta_2. \tag{2.e}$$

In the fundamental domain for the group $L \subset \mathbf{C}$ it has only one pole and this lies at the point 0. The matrix

$$M(z) = \begin{pmatrix} 1 & \zeta(z) \\ 0 & 1 \end{pmatrix} = \exp\begin{pmatrix} 0 & \zeta(z) \\ 0 & 0 \end{pmatrix} \tag{5}$$

satisfies a multiplicative condition,

$$M(z + 2\omega_i) = M(z)\begin{pmatrix} 1 & 2\eta_i \\ 0 & 1 \end{pmatrix}, \quad i = 1, 2.$$

Consequently it determines a GL(2)-bundle Π. If Λ is a linear bundle then $\Lambda \otimes \Pi$ is also a GL(2) bundle $\Lambda \otimes \Pi$ and $\Lambda' \otimes \Pi$ are equivalent only if Λ and Λ' are equivalent ([A], Th. 5). The degree $\deg(\Lambda \otimes \Pi)$ is equal to $2\deg\Lambda$. The set of such bundles with degree $2m$ is the set $\mathfrak{A}(m, m)$. A more general form of the definition (4) is presented in [A, Th. 5],

$$F_r = \exp\left(\begin{pmatrix} 0 & \zeta(z) & 0\cdots\cdots\cdots 0 \\ 0 & 0 & \zeta(z)\cdots\cdots\cdot 0 \\ \vdots & \vdots & \vdots \quad\quad \vdots \\ 0 & 0 & 0\cdots\cdots\cdot\zeta(z) \\ 0 & 0 & 0\cdots\cdots\cdots 0 \end{pmatrix}\right), \tag{6}$$

a matrix of order r. It will be necessary to show that F_r is indecomposable, but we consider only the group GL(2). It is possible to change the disposition of the poles; multiply on the left by

$$\begin{pmatrix} 1 & h(\cdot) \\ 0 & 1 \end{pmatrix},$$

where $h(\cdot)$ is a meromorphic function with poles at 0 and at another arbitrary point, for which, of course, the sum of the residues is zero.

To determine the set $\mathfrak{A}(m+1,m)$ is in one sense more difficult because there is no canonical choice but in another sense easier. It consists of

$$F(d) = \begin{pmatrix} 1 & \phi(z) \\ 0 & \sigma^{-1}(z-d) \end{pmatrix}, \quad \phi(z) = \frac{\sigma(z-a_1)\sigma(z-a_2)}{\sigma(z-b_1)\sigma(z-b_2)}, \tag{7}$$

where a_1, a_2, b_1, b_2 are fixed, $a_1 + a_2 = b_1 + b_2$, $a_1 - a_2 \neq b_1 - b_2$ but d is variable. This means that as a set $\mathfrak{A}(m,m) \simeq \mathfrak{A}(m,m+1) \simeq M$. It is possible that this description is somewhat arbitrary, but it is necessary to consider [A] in order to understand that it is unavoidable. I admit by the way that I found [A] difficult to understand.[16]

To describe the general conclusions of Atiyah ([A, Th. 6]) is useful and for the present essay necessary. He describes all indecomposable bundles M of dimension or rank r and degree d. It is useful to choose first a given line bundle $\Lambda_0 = \Lambda_{A_0}$ of degree one. This bundle is given by a chosen point A_0. Then a bundle of any degree d is determined: if $d = 0$ the bundle is trivial; if $d > 0$ then there is a section with a unique pole of degree d in the point A_0 but no zero; if $d < 0$ the zero is replaced by a pole. With these choices, for a given r the sets $\mathcal{E}(r,d)$, which we introduce, are all determined.

Suppose now that $d = ar + d'$, where $0 \leqslant d' < r$. If $a = 0$ then we pass directly to the second stage. If $a > 0$ then

$$N = A^a \otimes N',$$

where N' is an indecomposable bundle of dimension $r' = r$ and degree d'. Consequently we may pass to the second stage and suppose that $d' = d - ar < r' = r$. If $d' = 0$ this is the last stage but if $d' > 0$,

$$N' = \begin{pmatrix} I & * \\ 0 & N'' \end{pmatrix},$$

where I is the identity matrix of rank $s < r'$. Let r'' the rank of N''. Then r'' is less than r' and the degree $d'' = d'$. The precise form of the matrix $*$ is irrelevant. What is important is just that M' is indecomposable. Then the rank $r'' = r - s < r$. The initial multiplicity is novel. Continuing we arrive at the pair $(\widetilde{r}, \widetilde{d} = 0)$, $\widetilde{N} = F_{\widetilde{r}}$. For a bundle of higher degree it is possible to replace (7) by a matrix

$$\begin{pmatrix} 1 & 0 \cdots\cdots 0 & \phi(z) \\ 0 & 1 \cdots\cdots 0 & \phi(z) \\ \vdots & \vdots \qquad \vdots & \vdots \\ 0 & 0 \cdots\cdots 1 & \phi(z) \\ 0 & 0 \cdots\cdots 0 & \sigma^{-1}(z-d) \end{pmatrix} \tag{7'}$$

The set $\mathrm{Bun}_{\mathrm{GL}(r)}$ is a structure in which there are r steps.

The description by Atiyah of this transformation is different but very instructive. It is useful to describe it but with some supplementary details. Let $\mathcal{E}(r,d)$ be the set of indecomposable bundles of dimension r and degree d, and let h be the largest common divisor of r and d. Atiyah describes the map $\alpha_{r,d} : \mathcal{E}(h,0) \to \mathcal{E}(r,d)$ that is inverse to our transformation. Thus he constructs elements of $\mathcal{E}(r,d)$, and like us classifies them. The

[16] These words were written as I began to write the Russian version of this article. They remained correct even when I finished it two years later and remain so even now after another two years have passed, but I am growing old.

basic elements of his construction are: (i) $\alpha_{r,0} : F_r \to F_r$; (ii) $\alpha_{r,d+r}(E) : E \to A \otimes \alpha_{r,d}(E)$; (iii) if $0 < d < r$, then

$$\alpha_{r,d}(E) = \begin{pmatrix} 1 & 0 \cdots\cdots 0 & * \cdots\cdots\cdots\cdots\cdots\cdots * \\ 0 & 1 \cdots\cdots 0 & * \cdots\cdots\cdots\cdots\cdots\cdots * \\ \vdots & \vdots \qquad \vdots & * \cdots\cdots\cdots\cdots\cdots\cdots * \\ 0 & 0 \cdots\cdots 1 & * \cdots\cdots\cdots\cdots\cdots\cdots * \\ 0 & 0 \cdots\cdots 0 & \\ \vdots & \vdots \qquad \vdots & \alpha_{r-d,d}(E) \\ 0 & 0 \cdots\cdots 0 & \end{pmatrix}, \qquad E \in \mathcal{E}(h, 0). \tag{8}$$

This continues so long as $r' = r - nd < d = d' + r'$. Then we apply (ii) and continue. In this way for a given r and each d the set $\mathcal{E}(r, d)$ is identified with the elliptic curve M and all steps of the infinite ladder, $-\infty < d < \infty$, are almost the same. So far we have introduced insufficient structure in the set $\mathrm{Bun}_G = G(F)\backslash \mathrm{GL}(n, \mathbf{A}_F)/G(\mathcal{O})$. There is a topological structure but it is useless. A set in Bun_G is open if and only if its inverse image in $\mathrm{GL}(n, \mathbf{A}_F)/G(\mathcal{O})$ is open. None the less there exists a decomposition of Bun_G whose relation to its topology is largely not important. Namely every bundle is a direct sum of indecomposable bundles of dimension $r_1, \ldots, r_k$, $r_1 + \cdots + r_k = r$. It appears that for Hecke operators the set Bun_G consists of separate sets $\mathcal{D}(r_1, \ldots, r_k)$ according to the unordered set $\{r_1, \ldots, r_k\}$, which is determined by the conjugacy class of the Levi subgroup. So far the structure that we introduced in the set $\mathrm{Bun}_G = G(F)\backslash \mathrm{GL}(n, \mathbf{A}_F)/G(\mathcal{O})$ is insufficient. There is a topology but it is useless. A set in Bun_G is open if and only if its inverse image in $\mathrm{GL}(n, \mathbf{A}_F)/G(\mathcal{O})$ is open. None the less there exists a decomposition of Bun_G whose relation to its topology is in large part not important. Namely each bundle is a direct sum of indecomposable bundles of dimension $r_1, \ldots, r_k$, $r_1 + \cdots + r_k = r$. It appears that for Hecke operators, the set Bun_G consists of separate sets $\mathcal{D}(r_1, \ldots, r_k)$ according to the unordered set $\{r_1, \ldots, r_k\}$, that is determined by the conjugacy class of a Levi subgroup. Let $\mathcal{E}(r)$ be the set of indecomposable bundles of dimension r and $\widetilde{\mathcal{E}}_k(r)$ the symmetrized k-fold product of $\mathcal{E}(r)$ with itself. Then

$$\mathcal{D}(r_1, \ldots, r_k) = \widetilde{\mathcal{E}}_{k_1}(s_1) \times \cdots \times \widetilde{\mathcal{E}}_{k_\ell}(s_\ell),$$

where $\{r_1, \ldots, r_k\}$ formed from s_1 repeated k_1 times and so on. In essence $\mathcal{D}(r_1, \ldots, r_k)$ is the product of the sets

$$\bigcup_{d=-\infty}^{\infty} \mathcal{E}(r, d), \tag{9}$$

where r is given and $\mathcal{E}(r, d)$ is the set of indecomposable bundles of dimension r and degree d. As a topological space this is approximately $\mathbf{Z} \times M = \mathbf{Z} \times \mathbf{U} \times \mathbf{U}$, where $\mathbf{U} = \{\, z \in \mathbf{C} \mid |z| = 1 \,\}$. This is also a topological group and its group of characters is $\mathbf{U} \times \mathbf{Z} \times \mathbf{Z}$. It is possible to suppose that it also parametrizes (approximately) the eigenvalues of the Hecke operators. Keeping this in mind we turn to the hypothesis but first an encouraging remark. According to Theorem 7 in [A], if χ is a character of the group $\mathcal{E}(1, 0)$ then χ determines a function with values in $\mathbf{U}$ on each $\mathcal{E}(r, d)$ in the set (9). If $z \in \mathbf{U}$, then the second function is $\eta_z : N \to z^d$. It is therefore possible that there is a simple, rather uncomplicated, description of the full set of eigenfunctions of the Hecke operators. We shall give this for $r = 2$.

For an elliptic curve, $g = 1$ and the group Γ is generated by elements A, B, $ABA^{-1}B^{-1} = 1 \neq 0 \in \mathbf{Z}$. According to the conjecture, the eigenfunctions of the Hecke operators for $\mathrm{GL}(n)$ correspond to the representations of Γ_{aut} of dimension n. The parabolic eigenfunctions correspond to irreducible representations. Let ρ be such a representation. Then $\rho(1) = \zeta \in \mathbf{U}$ and

$$\rho(A)\rho(B)\rho(A)^{-1}\rho(B)^{-1} = \zeta I.$$

Since $\det(\zeta I) = 1$, ζ is a root of unity. Let k be its order. In addition $\rho(B)$ and $\zeta\rho(B)$ are similar matrices. Let $k|n$ be the order of $\rho(B)$. The simplest example is

$$A = \lambda \begin{pmatrix} 0 & 1 & 0 & 0 \cdots\cdots 0 \\ 0 & 0 & 1 & 0 \cdots\cdots 0 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & 0 \cdots\cdots 0 \end{pmatrix}, \quad B = \mu \begin{pmatrix} 1 & 0 & 0 \cdots\cdots\cdots 0 \\ 0 & \eta & 0 \cdots\cdots\cdots 0 \\ 0 & 0 & \eta^2 \cdots\cdots 0 \\ \vdots & \vdots & \vdots \quad\quad \vdots \\ \vdots & \vdots & \vdots \quad\quad \vdots \\ 0 & 0 & 0 \cdots\cdots \eta^{k-1} \end{pmatrix}.$$

However in this article I consider only $\mathrm{GL}(2)$.

In the following three sections I consider the Hecke operators for the group $\mathrm{GL}(2)$. The last section is the most important, the most remarkable but the first and the second sections contain necessary preparation.

V. The Hecke correspondence

For each point $x \in M$ there is a commutative algebra $\mathfrak{H}_x$, determined by correspondences and a measure. We begin with the group $\mathrm{GL}(2)$ and an elliptic curve and for this group and this curve the measure will be evident. Thus I postpone its general definition, hoping to find a future occasion to explain this. Similar concepts are introduced in [AB, §9]. The algebra $\mathfrak{H}_x$ is generated by two dual modules. One of these is easy. There is, of course, an obvious local coordinate z on $M = \mathbf{C}/L$, but in order to introduce Hecke it is necessary to specify a point $x \in M$ and to use an arbitrary coordinate z_x, $z_x(x) = 0$ because x is an essential parameter of the operator. For a given point there are two basic operators, two basic double cosets modulo $G(\mathcal{O}_x)$:

$$\Delta_1 = G(\mathcal{O}_x)\begin{pmatrix} z_x^{-1} & 0 \\ 0 & 1 \end{pmatrix} G(\mathcal{O}_x); \quad \Delta_2 = \begin{pmatrix} z_x^{-1} & 0 \\ 0 & z_x^{-1} \end{pmatrix} G(\mathcal{O}_x).$$

In order to determine the two corresponding operators it is necessary to introduce a metric μ on Bun_G or $G(\mathbf{A}_F)/G(\mathcal{O}_F)$. A general definition of the measure is difficult, but for the group $\mathrm{GL}(2)$ there will be an obvious choice, which we use as temporary definition. The second operator is simple

$$\Theta_2 = \Theta_{2,x} : f \to f',\ f'(g) = f\left(g\begin{pmatrix} z_x^{-1} & 0 \\ 0 & z_x^{-1} \end{pmatrix}\right),$$

but the first operator is

$$\Theta_1 = \Theta_{1,x} : f \to f',\ f'(g) = \int_{h \in g\Delta_1/G(\mathcal{O}_x)} f(h)\, dh. \tag{10}$$

The first integral is over a point and the measure is such that the measure of a point is equal to 1. We describe the domain of integration for the second operator Θ_2^*. This operator is not hermitian but if we add a correction Θ_2^* the algebra generated is commutative and hermitian closed.

We consider various cases: (a) $g \in \mathfrak{D}(m,n)$, $m-n \geqslant 2$; (b) $g \in \mathfrak{D}(m,n)$, $m-n=1$; (c) $g \in \mathfrak{D}(m,n)$, $m-n=0$; (d) $g \in \mathfrak{A}(m,n)$, $m-n=1$; (e) $g \in \mathfrak{A}(m,n)$, $m-n=0$. It is important to remark that the complex dimension is $\dim \mathfrak{D}(m,n) = 2$ and that $\dim \mathfrak{A}(m,n) = 1$. Then corresponding to these five possibilities the element h in (10) is to be found in

(a) $\mathfrak{D}(m+1,n) \cup \mathfrak{D}(m,n+1)$;
(b) $\mathfrak{D}(m+1,n) \cup \mathfrak{D}(m,n+1) \cup \mathfrak{A}(m,n+1)$, where $m=n+1$;
(c) $\mathfrak{D}(m+1,n) \cup \mathfrak{A}(m+1,n)$, where $m=n$;
(d) $\mathfrak{D}(m,n+1) \cup \mathfrak{A}(m,n+1)$ where $m=n+1$;
(e) $\mathfrak{D}(m+1,n) \cup \mathfrak{A}(m+1,n)$ where $m=n$.

The case $m=n$, $m=n\pm 1$ is underlined because it corresponds to a bundle of Atiyah type. There is another way to express this conclusion. If the support of the function f is in (a) $\mathfrak{D}(m,n)$, $m-n \geqslant 2$; (b) $\mathfrak{D}(m,n)$, $m-n=1$; (c) $\mathfrak{D}(m,n)$, $m-n=0$; (d) $\mathfrak{A}(m,n)$, $m-n=1$; (e) $\mathfrak{A}(m,n)$, $m-n=0$ then the support of the function f' in (10) lies correspondingly in (a) $\mathfrak{D}(m+1,n)\cup\mathfrak{D}(m,n+1)$; (b) $\mathfrak{D}(m+1,n)\cup\mathfrak{D}(m,n+1)\cup\mathfrak{A}(m,m)$; (c) $\mathfrak{D}(m+1,m)\cup\mathfrak{D}(m,m)\cup\mathfrak{A}(m+1,m)$; (d) $\mathfrak{D}(m,m)\cup\mathfrak{A}(m,m)$; (e) $\mathfrak{D}(m+1,m)\cup\mathfrak{A}(m+1,m)$. This is somewhat finicky but I am not completely certain.

I begin with the description of the representatives of the left-contiguous classes in

$$G(\mathcal{O}_x)gG(\mathcal{O}_x), \qquad g = \begin{pmatrix} z_x^{-1} & 0 \\ 0 & 1 \end{pmatrix}. \tag{11}$$

They are given by multiplying g on the left by the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}, \qquad a \in \mathbf{C}. \tag{12}$$

This gives

$$\begin{pmatrix} z_x^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} z_x^{-1}a & 1 \\ z_x^{-1} & 0 \end{pmatrix}, \qquad a \in \mathbf{C}. \tag{13}$$

It is necessary to recall that in these two equations $\mathbf{C} \subset \mathcal{O}_x$. If λ is a scalar, that is $\lambda \in \mathbf{I}_F = \mathbf{A}_F^\times$, and $f_1(g) = f(\lambda g)$, then $f_1'(g) = \Theta_1 f_1 = f'(\lambda g)$. This allows some simplifications in the calculations.

It is easy to describe the elements of $G(F_x)$ and even the elements of $G(F_x)/G(\mathcal{O}_x)$ because in principle it is necessary to offer an infinite set of coordinates, but it is possible to take the identity matrix almost anywhere. These coordinates I do not write explicitly.

If we begin with $g \in \mathfrak{D}(m,n)$ then we can take:

$$\begin{aligned} g &= \begin{pmatrix} z_u^{-m} & 0 \\ 0 & 1 \end{pmatrix}, & m \neq n = 0; \\ &= \begin{pmatrix} z_u/z_v & 0 \\ 0 & 1 \end{pmatrix}, & m = n = 0. \end{aligned}$$

because the Hecke operator commutes with multiplication on a linear bundle. We multiply g on the right by the matrix (13). If $m \neq 0$ this yields[17]

$$\begin{pmatrix} z_u^{-m} z_x^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \tag{14}$$

which belongs to $\mathfrak{D}(m+1, 0)$, and

$$\begin{pmatrix} z_u^{-m} z_x^{-1} a & z_u^{-m} \\ z_x^{-1} & 0 \end{pmatrix}. \tag{14$'$}$$

All choices of $a \neq 0$ yield equivalent bundles. If $a \neq 0$, the bundle belongs to $\mathfrak{D}(m, 1)$. We take $a = 1$ and rearrange the columns. Then (14′) is equal to

$$z_x^{-1} \begin{pmatrix} z_u^{-m} z_x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & z_x^{-1} \\ 0 & 1 \end{pmatrix}. \tag{14$''$}$$

If $f \in F$ we can multiply on the left by

$$\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}$$

in order to obtain

$$z_x^{-1} \begin{pmatrix} z_u^{-m} z_x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & z_x^{-1} + f z_u^m z_x^{-1} \\ 0 & 1 \end{pmatrix}. \tag{15}$$

There are three cases: $m \geqslant 2$, $m = 1$, $m = 0$. We described in the theory of Weierstrass all meromorphic functions on M. If $m \geqslant 2$, we can choose a function f such that $z_x^{-1} + f z_u^m z_x^{-1} \in \prod_x \mathcal{O}_x$ even if $u = x$. Consequently the bundle (15) is the decomposable bundle

$$\begin{pmatrix} z_u^{-m} & 0 \\ 0 & z_x^{-1} \end{pmatrix} \tag{15$'$}$$

in $\mathfrak{D}(m, 1)$. The case $m = 1$,

$$\begin{pmatrix} z_u^{-1} z_x^{-1} & z_u^{-1} \\ z_x^{-1} & 0 \end{pmatrix} \sim \begin{pmatrix} z_u^{-1} & z_u^{-1} z_x^{-1} \\ 0 & z_x^{-1} \end{pmatrix}, \tag{16}$$

and the case $m = 0$ are different. If $g_1,\ g_2 \in G(\mathbf{A}_F)$, then $g_1 \sim g_2$ expresses equality in Bun_G.

[17] The description of an element in Bun_G or even in $G(\mathbf{A})/G(\mathcal{O}_F)$ is difficult because a large number of coordinates are redundant. Moreover the representative of an element in Bun_G in $G(\mathbf{A})/G(\mathcal{O}_F)$ is not unique. Good will and the attention of the reader are necessary. For example, if $u \neq x$ then rather than (14) it is better to write

$$\begin{pmatrix} z_u^{-m} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_x^{-1} & 0 \\ 0 & 1 \end{pmatrix} \prod_{y \neq x, u} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} z_u^{-m} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_x^{-1} & 0 \\ 0 & 1 \end{pmatrix},$$

and instead of (14′)

$$\begin{pmatrix} z_u^{-m} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_x^{-1} a & 1 \\ z_x^{-1} & 0 \end{pmatrix}.$$

If $m=1$, let $u+x$ and $2v$ be linearly equivalent. We consider

$$z_v\begin{pmatrix} z_u^{-1} & z_u^{-1}z_x^{-1} \\ 0 & z_x^{-1}\end{pmatrix}. \tag{16$'$}$$

It is necessary to know when the bundle (16′) is isomorphic to (5). If $u=x=v$ then this is clear. Let $u\neq x$ and let $\Lambda_1=(z_v/z_u)$, $\Lambda_2=(z_v/z_x)$. Then $\mathrm{Hom}(\Lambda_1,\Lambda_2)=0$. According to [A, Lemma 5] this implies that the bundle (16′) is decomposable. It is possible to verify this directly because the matrix in (16′) is equal to

$$\begin{pmatrix} 1/2 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} z_u^{-1} & z_u^{-1} \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & z_x^{-1} \\ 0 & z_x^{-1}\end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1\end{pmatrix}\sim\begin{pmatrix} 1 & -1 \\ 0 & 1\end{pmatrix}\left\{\begin{pmatrix} z_u^{-1} & z_u^{-1} \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & z_x^{-1} \\ 0 & z_x^{-1}\end{pmatrix}\right\}$$

where the first matrix on the right lies in $\mathrm{GL}(2,F)$. This matrix lies in $\mathrm{GL}(2,\mathbf{A}_F)$ but outside the point $\{u,x\}$ it lies in $G(\mathcal{O}_w)$. We consider the factors in u and x separately:

$$\begin{pmatrix} 1 & -1 \\ 0 & 1\end{pmatrix}\begin{pmatrix} z_u^{-1} & z_u^{-1} \\ 0 & 1\end{pmatrix}=\begin{pmatrix} z_u^{-1} & z_u^{-1}-1 \\ 0 & 1\end{pmatrix}=\begin{pmatrix} z_u^{-1} & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 1-z_u \\ 0 & 1\end{pmatrix}\sim\begin{pmatrix} z_u^{-1} & 0 \\ 0 & 1\end{pmatrix}; \tag{16.a}$$

$$\begin{pmatrix} 1 & -1 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & z_x^{-1} \\ 0 & z_x^{-1}\end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & z_x^{-1}\end{pmatrix}. \tag{16.b}$$

If $m=0$ we obtain

$$\begin{pmatrix} z_uz_v^{-1}z_x^{-1} & 0 \\ 0 & 1\end{pmatrix},\ \begin{pmatrix} z_uz_v^{-1}z_x^{-1}a & z_uz_v^{-1} \\ z_x^{-1} & 0\end{pmatrix}. \tag{17}$$

The first bundle lies in $\mathfrak{D}(1,0)$. If $a=0$, the second bundle is also in $\mathfrak{D}(1,0)=\mathfrak{D}(0,1)$. The other cases, for which $a\neq 0$, yield a single bundle, given by

$$\begin{pmatrix} z_uz_v^{-1}z_x^{-1} & z_uz_v^{-1} \\ z_x^{-1} & 0\end{pmatrix}\sim\begin{pmatrix} z_uz_v^{-1} & z_uz_v^{-1}z_x^{-1} \\ 0 & z_x^{-1}\end{pmatrix}, \tag{17$'$}$$

because

$$\begin{pmatrix} \alpha a & \beta \\ \gamma & 0\end{pmatrix}=\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & 0\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & a^{-1}\end{pmatrix},\qquad a\in\mathbf{C}^\times.$$

If $u=v$,

$$\begin{pmatrix} z_x^{-1} & 1 \\ z_x^{-1} & 0\end{pmatrix}\sim\begin{pmatrix} 1 & z_x^{-1} \\ 0 & z_x^{-1}\end{pmatrix}\sim\begin{pmatrix} 1 & 0 \\ 0 & z_x^{-1}\end{pmatrix}.$$

If $u\neq v$, let $v-u\sim x-y$. There exists a function f with zeros in y, v and poles in x, u. This means that $\mathrm{Hom}(\Lambda_1,\Lambda_2)\neq 0$ if $\Lambda_1=(z_u/z_v)$, $\Lambda_2=(1/z_x)$ and that (17′) is indecomposable. $\Lambda_2=(1/z_x)$ and it is possible to conclude that (17′) is indecomposable.[18]

$$\begin{pmatrix} z_uz_v^{-1} & z_uz_v^{-1}z_x^{-1} \\ 0 & z_x^{-1}\end{pmatrix}\sim z_uz_v^{-1}\begin{pmatrix} 1 & z_x^{-1} \\ 0 & z_y^{-1}\end{pmatrix}. \tag{17$''$}$$

[18]Warning (April 2020): this statement is unclear and has to be examined at some other time.

If $1 \neq y$ there is a function $f \in F$ such that at x_1 it behaves like $f \sim 1/z_{x_1}$ but that at x its behaviour is given by $f \sim -1/z_x$, where f is holomorphic in the remaining points and $f(y) = 0$. We multiply on the left by the matrix

$$\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}$$

in order to obtain

$$z_u z_v^{-1} \begin{pmatrix} 1 & z_{x_1}^{-1} \\ 0 & z_y^{-1} \end{pmatrix} = z_u z_v^{-1} \Theta_y, \qquad x_1 \neq y. \tag{18}$$

The matrix Θ appears in (4). Another representative is the bundle

$$\Theta \sim \begin{pmatrix} 1 & z_w^{-2} \\ 0 & z_w^{-1} \end{pmatrix}, \qquad w \sim v + y - u \sim x. \tag{18$'$}$$

This equivalence is explained in [A]. I admit that my confidence in these calculations is limited. I am not comfortable with the set $G(F)\backslash G(\mathbf{A}_F)/G(\mathcal{O}_F)$.

In the case (d) we choose the representative (18$'$) of the sets $\mathfrak{A}(1,0)$ but with a change of notation, w becomes u. Multiplying by the matrix (13) we obtain

$$\begin{pmatrix} z_x^{-1} & z_u^{-2} \\ 0 & z_u^{-1} \end{pmatrix}, \begin{pmatrix} z_x^{-1}a + z_x^{-1}z_u^{-2} & 1 \\ z_u^{-1}z_x^{-1} & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & z_x^{-1}a + z_x^{-1}z_u^{-2} \\ 0 & z_u^{-1}z_x^{-1} \end{pmatrix}. \tag{19}$$

If $u = x$ the first bundle is equal to

$$z_x^{-1} \begin{pmatrix} 1 & z_x^{-1} \\ 0 & 1 \end{pmatrix}, \tag{19$'$}$$

which is of type $\mathfrak{A}(1,1)$. If $u \neq x$ we consider

$$\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_x^{-1} & z_u^{-2} \\ 0 & z_u^{-1} \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} z_x^{-1} & f z_u^{-1} + z_u^{-2} + c z_x^{-1} \\ 0 & z_u^{-1} \end{pmatrix}, \tag{19$''$}$$

where $c \in \mathbf{C}$ and $f \in F$. We choose f, with poles only in x and u, such that the pole in u is removed. Then it is also possible to remove the terms of lower order in the upper right entry in order to obtain

$$\begin{pmatrix} z_x^{-1} & 0 \\ 0 & z_u^{-1} \end{pmatrix} \tag{19$'''$}$$

of type $\mathfrak{D}(1,1)$.

For the matrices of second type in (19) there are several possibilities. Before we examine them it is necessary to explain the theory of Atiyah. This does not propose a clear recipe for deciding when a given bundle is decomposable or not. Consider for example the bundle

$$\Theta = \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}, \qquad d \in \mathbf{I}_F,\ b \in \mathbf{A}_F.$$

Let

$$\Theta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \Theta_2 = \big(d\big).$$

We suppose that $\deg(\Theta_2) > 0$ and that $\Theta \simeq \Lambda_1 \oplus \Lambda_2$ is decomposable. If $\Lambda_i \neq (1,0)^{\mathrm{tr}}$, $i = 1$, 2, then $\Lambda_i \to \Theta_2$ is surjective and $\deg \Lambda_i \geqslant \deg \Theta_2$. Consequently

$$\deg \Theta_2 = \deg \Lambda_1 + \deg \Lambda_2 \geqslant 2 \deg \Theta_2$$

and this is impossible. In other words, either Λ_1 or Λ_2 is equal to, rather equivalent to $(1,0)^{\mathrm{tr}}$.

Let $\Lambda_1 = (1,0)^{\mathrm{tr}}$. Then

$$\begin{pmatrix}\Lambda_1 & \Lambda_2\end{pmatrix} = \Theta g, \qquad g = \prod_{v \in M} g_v \in G(\mathcal{O}_F),\ g_v = \begin{pmatrix}1 & \beta_v \\ 0 & \delta_v\end{pmatrix},\ \beta_v \in \mathcal{O}_v,\ \delta_v \in \mathcal{O}_v^\times, \tag{20}$$

and

$$\Lambda_2 = \Lambda \otimes \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}, \qquad \phi_i \in F,\ i = 1, 2, \tag{21}$$

where Λ is a linear bundle and $(b + \beta_x d)/\delta_x d = (\phi_1/\phi_2)_x$ a meromorphic function of x in M. We shall use this conclusion in order to establish the indecomposability of various bundles. It appears to me somewhat clumsy and above all inappropriate if the genus $g > 1$, but there is presently no alternative.[19]

Consider the matrix

$$\Theta = \begin{pmatrix}1 & z_x^{-1} a + z_x^{-1} z_u^{-2} \\ 0 & z_u^{-1} z_x^{-1}\end{pmatrix} \tag{22}$$

in (19). If $a = 0$ then this bundle is equivalent to

$$\begin{pmatrix}1 & z_x^{-3} \\ 0 & z_x^{-2}\end{pmatrix},\ b = z_x^{-3}, d = z_x^{-2}, \quad u = x; \tag{23.a}$$

or

$$\begin{pmatrix}1 & z_x^{-1} z_u^{-2} \\ 0 & z_u^{-1} z_x^{-1}\end{pmatrix},\ b = z_x^{-1} z_u^{-2}, d = z_x^{-1} z_u^{-1}, \quad u \neq x. \tag{23.b}$$

where $(b + \beta_v d)/\delta_v d$, $v \in M$, has a single pole at the point u. But there is no such meromorphic function. Consequently (21) is indecomposable.

If $a \neq 0$ it is better, perhaps necessary to write (19) precisely as

$$\begin{pmatrix}1 & z_x^{-1} a + z_x^{-1} \\ 0 & z_x^{-1}\end{pmatrix} \begin{pmatrix}1 & z_u^{-2} \\ 0 & z_u^{-1}\end{pmatrix}. \tag{24}$$

But there is again for $(b + \beta_v d)/\delta_v d$ a single pole at the point u. Consequently (24) is indecomposable. According to [A] there are four indecomposable bundles with a given determinant of even degree. Thus if x is given, four u's yield a given bundle.

The case (e) is the easiest. Since Hecke operators commute with respect to tensor products with line bundles, it is sufficient for a given x to consider

$$\begin{pmatrix}1 & z_x^{-1} \\ 0 & 1\end{pmatrix} \in \mathfrak{A}(0,0). \tag{25}$$

[19] I comment here, when trying to compose an English version of the original Russian text, that my command of the mathematics was much greater two years ago than it is now! I thank Anthony Pulido for drawing my attention to some of the consequent confusion.

The transform of this point is given by two types of points:

$$\begin{pmatrix} z_x^{-1} & z_x^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} z_x^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} z_x^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in \mathfrak{D}(1,0);$$

and

$$\begin{pmatrix} z_x^{-1}a + z_x^{-2} & 1 \\ z_x^{-1} & 0 \end{pmatrix}.$$

But

$$\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z_x^{-1}a + z_x^{-2} & 1 \\ z_x^{-1} & 0 \end{pmatrix} \sim \begin{pmatrix} z_x^{-2} & 1 \\ z_x^{-1} & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & z_x^{-2} \\ 0 & z_x^{-1} \end{pmatrix} \in \mathfrak{A}(1,0).$$

Although the proofs look unsatisfactory, these conclusions appear convincing. We explain.

We first present the results of our reflections. In particular, we recognize that the dimension $\dim\big(g\Delta_1/G(\mathcal{O}_x)\big)$, although this is a subset of the set $G(\mathbf{A}_F)/G(\mathcal{O}_F)$ or of its image in Bun_G, may contain more than one element. Consequently the domain of integration in (9) is a finite set. In general, although I do not explain this here, the measure for this integral is determined by a Pfaffian form, but for a finite set it is so determined that the measure of each point is 1. Thus, we may postpone the introduction of the general theory of this form until later.

We consider five cases (a),...,(e). For each case we use the relation,

$$\begin{aligned} \Theta_i f_z(g) &= f'_z(g), \\ f_z(g) &= f(zg), \\ f'_z(g) &= f'(zg), \\ f'(g) &= \Theta_i f(g), \end{aligned} \qquad i = 1,2 \quad z \in \mathbf{A}_F^\times$$

That is the maps Θ_i and $f \to f_z$ commute.

(a) If $m - n \geqslant 2$ then the image of the matrix

$$\begin{pmatrix} z_u^{-m} & 0 \\ 0 & z_v^{-n} \end{pmatrix}$$

consists of two points

$$\begin{pmatrix} z_u^{-m} z_x^{-1} & 0 \\ 0 & z_v^{-n} \end{pmatrix}, \begin{pmatrix} z_u^{-m} & 0 \\ 0 & z_v^{-n} z_x^{-1} \end{pmatrix}. \tag{26}$$

(b) If $m - n = 1$ it consists of (26) and, according to (16′),

$$z_u^{-m}\begin{pmatrix} 1 & z_x^{-1} \\ 0 & 1 \end{pmatrix} \in \mathfrak{A}(m,m),$$

if the divisor $m \cdot u - n \cdot v$ is linearly equivalent to the divisor x.

(c) If $m - n = 0$ then the image consists again of (26) but now, according to (18),

$$z_u z_v^{-1}\begin{pmatrix} 1 & z_{x_1}^{-1} \\ 0 & z_y^{-1} \end{pmatrix}, \qquad v - u \sim x - y,$$

if $u \neq v$ and

$$g = z_v^{-k}\begin{pmatrix} z_u/z_v & 0 \\ 0 & 1 \end{pmatrix}\Theta.$$

The notation is somewhat modified with respect to (18). I remark that $u = v$ does not give more than (26).

(d) In (19) u runs over M and the appropriate domain is determined by Θ_1,

$$\begin{pmatrix} 1 & z_u^{-2} \\ 0 & z_u^{-1} \end{pmatrix},$$

and the domains, that is the domain of definition of the operator Θ_1 together with the domain of definition of $\mathfrak{A}(1,0)$ are given by (19). From the first elements of the set (19) we obtain first, from $u = x$, the point (19′) in $\mathfrak{A}(1,1)$. From the other u we obtain the points (19″) in $\mathcal{D}(1,1)$. At the first glance at (19′) it seems that there is only one missing point, $u = x$, but $\mathcal{D}(1,1)$ is two-dimensional! The second elements of the set (19) are indecomposable, with determinant $z_u^{-1}z_x^{-1}$. According to the paper [A] there are precisely four points in $\mathfrak{A}(1,1)$ with a given determinant. In $\mathfrak{A}(1,0)$ there is only one. That is the correspondence $\mathfrak{A}(1,0) \to A(1,1)$ is a correspondence of type $4 \to 1$.

(e) For a given x a general element of the set the definitions are given by matrices

$$z_u z_v^{-1}\begin{pmatrix} 1 & z_x^{-1} \\ 0 & 1 \end{pmatrix} \in \mathfrak{A}(0,0), \tag{27}$$

thus as a bundle (27) that is independent of x. The image is either

$$z_u z_v^{-1}\begin{pmatrix} z_x^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in \mathfrak{D}(1,0), \tag{27′}$$

or

$$z_u z_v^{-1}\begin{pmatrix} 1 & z_x^{-2} \\ 0 & z_x^{-1} \end{pmatrix} \in \mathfrak{A}(1,0). \tag{27″}$$

For $A(m,m)$, $m \neq 0$, $z_u z_v^{-1}$ is replaced by z_u^{-m}. The correspondence $\mathfrak{A}(0,0) \to \mathfrak{A}(1,0)$ is bijective.

VI. Hecke operators

Although Bun_G is difficult to define as a topological space even for the case $G = \mathrm{GL}(2)$ over an elliptic curve, it is simple as a differential geometric space, at least for this pair. We consider this case now. For it the Picard variety is given as a product of two circles with $\mathbf{Z}$. According to the paper [A] the set Bun_G is given as the union

$$\mathfrak{D} = \left\{\bigcup_{m>n} \mathfrak{D}(m,n)\right\} \cup \left\{\bigcup_{m} \mathfrak{D}(m,m)\right\},$$

therefore as the symmetric product of $\mathrm{Pic}(M)$ with itself, and

$$\mathfrak{A} = \cdots \cup \mathfrak{A}(-1,-1) \cup \mathfrak{A}(-1,0) \cup \mathfrak{A}(0,0) \cup \mathfrak{A}(1,0) \cup \mathfrak{A}(1,1) \cup \mathfrak{A}(2,1) \cdots$$

The terminology here is somewhat arbitrary. We shall introduce a metric on this space later; it will be essentially a product of Haar measures, thus a Lebesgue measure μ. The Hecke operators form an algebra of bounded commuting operators on $L^2(\mu)$ closed with respect to Hermitian conjugacy.

I stress that $\dim \mathfrak{A} = 1$, although $\dim \mathfrak{D} = 2$, that $L^2(\mu) = L^2(\mu, \mathfrak{D}) \oplus L^2(\mu, \mathfrak{A})$ and that continuous functions with compact support are dense in $L^2(\mu)$. It is amazing and initially a cause of anxiety that these two spaces are invariant with respect to Hecke operators. I explain. This is evident for $\Theta_{2,x}$. The operator $\Theta_{1,x}$ may be defined as

$$\begin{pmatrix} DD & DA \\ AD & AA \end{pmatrix}.$$

It is necessary to establish that $DA = 0$, $AD = 0$, that is that they are zero as operators from $L^2(\mathfrak{A})$ to $L^2(\mathfrak{D})$ and, in the same way, from $L^2(\mathfrak{D})$ to $L^2(\mathfrak{A})$. These two equations express what I call the dominance of the diagonal blocks. It is possible that the principle of which they are examples is generally valid. If so, that is the principal conclusion of this paper.

According to (10), with $\Theta_{1,x}$ it is necessary to consider first;

(i) for a given $g \in \mathfrak{D}(m,n)$ the set $g\Delta_1/G(\mathcal{O}_x) \cap \mathfrak{A}$;

and then

(ii) for a given $g \in \mathfrak{A}(m,n)$ the set $g\Delta_1/G(\mathcal{O}_x) \cap \mathfrak{D}$.

These connections determine or limit the carrier f' in (10). Thus they define the support, perhaps not small, of the function (10) in this or that set. The kernel of the operator $\Delta_{1,x}$—a form of line or column matrix that parametrizes the set $\mathfrak{D} \cup \mathfrak{A}$. Consequently this kernel is formed by four blocks, diagonal blocks $(\mathfrak{D}, \mathfrak{D})$, $(\mathfrak{A}, \mathfrak{A})$ and off-diagonal blocks $(\mathfrak{D}, \mathfrak{A})$, $(\mathfrak{A}, \mathfrak{D})$. Our present concern is the off-diagonal blocks.

In the first case, for which the correspondence $\mathfrak{D} \to A$ carries $\mathfrak{D}$ to $\mathfrak{A}$ and transfers functions from $\mathfrak{A}$ to $\mathfrak{D}$, the set of appropriate elements in the block is empty if $m - n \neq 0$, ± 1. If $m = 1$, $n = 0$ this is essentially $(16')$ if $u = x$. More precisely, the set consists of

$$\Lambda \begin{pmatrix} 1 & z_x^{-1} \\ 0 & 1 \end{pmatrix},$$

where Λ is an arbitrary linear bundle. Thus if f is continuous with support in $\mathfrak{A}(n,n)$, the function f' in (10) as a function on $\mathfrak{D}$ is carried by

$$\left\{ \Lambda \begin{pmatrix} z_x^{-1} & 0 \\ 0 & 1 \end{pmatrix} \middle| \Lambda \text{ linear bundle} \right\}.$$

That is, its carrier is a subset of this set. But such a function as a function in $L^2(\mathrm{Bun}_G)$ is the null function, because this set as a complex manifold is one-dimensional while $\mathfrak{D}$ is two-dimensional. The general case $m = n + 1$ is exactly the same. However this argument is unsatisfactory. The block AD is equal to 0 for two reasons: (i) there is no natural way of restricting an L^2-function to a space of lower dimension; the Hecke operators are hermitian. Consequently it is decreed: this block is zero. But this yields the correct conclusions and a convincing theory. It represents also the essential conclusion of the following remarks.

If $m = n$ we suppose that $m = n = 0$; then the class of the bundle $(17'')$ is determined by the point y, because $v - u \sim x - y$ and x is given. That is the image of the set $\mathfrak{A}(0,0)$ and generally the image of $\mathfrak{A}(m,m)$ in $\mathfrak{D}$ is one-dimensional. Consequently $DA = 0$.

In the second case it is obligatory that $m - n = 0$, ± 1. If $m = 1$, $n = 0$ the intersection $g\Delta_1/G(\mathcal{O}_x) \cap \mathfrak{D}$ is given by $(19''')$ with an arbitrary $u \in X$. That is, the dimension of the intersection is 1. Consequently, it cannot carry a non-trivial L^2 function. If $m = n = 0$ the very same conclusion is a consequence of the equation $(27')$.

It is possible that this independence of $L^2(\mu,\mathfrak{D})$ and $L^2(\mu,\mathfrak{A})$ is what distinguishes the L^2 theory from the sheaf theory in [G].

VII. Eigenvalues and eigenfunctions of Hecke operators

Before we consider the spectrum of the Hecke operators, we turn to the ladder $\mathcal{E}(r,d)$, $d=-\infty,\dots,\infty$ for $r=2$. We do not consider the case of general r. Both the specific case and the general case are considered in [A]. For $r=2$, the ladder is

$$\dots,\mathfrak{A}(-1,-1),\mathfrak{A}(0,-1),\mathfrak{A}(0,0),\mathfrak{A}(1,0),\mathfrak{A}(1,1),\mathfrak{A}(2,1),\dots \tag{28}$$

If Λ is a linear bundle of degree 1 then $\Theta\mapsto\Lambda\otimes\Theta$ is a homomorphism such that

$$\mathfrak{A}(j,j)\mapsto\mathfrak{A}(j+1,j+1)$$

and

$$\mathfrak{A}(j,j-1)\mapsto\mathfrak{A}(j+1,j),$$

but the Hecke operator Θ_1 is such that

$$\mathfrak{A}(j,j)\mapsto\mathfrak{A}(j+1,j),\quad \mathfrak{A}(j,j-1)\mapsto\mathfrak{A}(j,j).$$

I begin with a correction or better with a more precise form of the theorem [A, Th. 6] and of the following theorems with a correction or, rather, a more precise form of the theorem [A, Th. 6] and the following theorems, that is $\deg A=1$, where A is the linear bundle of this theorem. Let $\mathrm{Pic}_n(M)$ be the set of linear bundles of degree n. I prefer to formulate [A, Lemma 16, Th. 5,6,7], above all Th. 6, in the following form. There is a bijective map $\mathfrak{A}\leftrightarrow\mathrm{Pic}(M)$ such that:

(i) if Λ_1 is a linear bundle and $\Theta\leftrightarrow\Lambda$ then $\Lambda_1\cdot\Theta\leftrightarrow\Lambda_1\cdot\Lambda$;
(ii) the bundle F_2 in Th. 5, or in the definition (6), corresponds to the trivial bundle;
(iii) if Θ is given by (4) and $\Lambda=(\sigma^{-1})$ then $\Theta\leftrightarrow\Lambda$.

Consequently, we can identify $\mathfrak{A}$ with $\mathrm{Pic}(M)$, but this identification is artificial, and $\mathfrak{D}$ with the symmetric power of $\mathrm{Pic}(M)$ with itself. In addition to that each unordered pair (χ_1,χ_2) of unitary characters defines a function

$$\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\to\chi_1(\alpha)\chi_2(\beta)+\chi_2(\alpha)\chi_1(\beta)\qquad \alpha,\beta\in\mathrm{Pic}(M) \tag{29}$$

on $\mathfrak{D}$. We show that these functions yield all eigenfunctions of the Hecke operators, in the sense of this theory, with support $\mathfrak{D}$ and that the eigen class of the function (29) at the point x is

$$\begin{pmatrix}\chi_1^{-1}(x) & 0\\ 0 & \chi_2^{-1}(x)\end{pmatrix} \tag{30}$$

for all x in M. It is sufficient to show that all these functions are eigenfunctions, because it is clear that they yield the spectral decomposition. I recall that the eigenfunction determines for each point x a semi-simple, even unitary, eigen class in $\mathrm{GL}(2,\mathbf{C})$, namely the determinant with Θ_2, but this is obvious, and the trace with Θ_1.

We confirm this first for $\mathfrak{D}$, postponing consideration of the set $\mathfrak{A}$. There are again two essential cases:

$$g=\begin{pmatrix}z_u^{-m} & 0\\ 0 & 1\end{pmatrix},\ m\neq n=0;\quad g=\begin{pmatrix}z_u/z_v & 0\\ 0 & 1\end{pmatrix},\ m=n=0.$$

I remark that z_u^{-m} and z_u/z_v represent the divisors $m\cdot u$ and $v-u$. From (14) and (14″) we conclude that if $m\neq 0,\ \pm 1$ the function multiplied by $\chi_1(z_x^{-1})+\chi_2(z_x^{-1})$ in the point g, thus

$$\chi_1(\alpha)\chi_2(\beta)+\chi_2(\alpha)\chi_1(\beta), \tag{30.a}$$

is replaced by the sum[20]

$$\chi_1(\alpha)\chi_1(z_x^{-1})\chi_2(\beta)+\chi_2(\alpha)\chi_2(z_x^{-1})\chi_1(\beta)+\chi_1(\alpha)\chi_2(\beta)\chi_2(z_x^{-1})+\chi_2(\alpha)\chi_1(\beta)\chi_2(z_x^{-1}),$$

which is equal to the product

$$\big\{\chi_1(\alpha)\chi_2(\beta)+\chi_2(\alpha)\chi_1(\beta)\big\}\big\{\chi_1(z_x^{-1})+\chi_2(z_x^{-1})\big\}.$$

If $m=1$ such a conclusion follows from the equations (14) and (16.a) with (16.b), rather from (25). For $m=0$ the argument is the same. It is evident that the function (30.a) yields a full set of eigenfunctions on $\mathfrak{D}$.

We now consider $\mathfrak{A}$. The combinatorial analysis of this case is somewhat complicated. The eigenfunction is determined by the eigen conjugacy class, thus for each point in M it is necessary to introduce two eigen numbers. For $\mathfrak{D}$ we use two independent characters, but for $\mathfrak{A}$ two independent characters are not available.

Let $\mathfrak{L}$ be the set of linear bundles and $\mathfrak{L}_m$ those whose degree is equal to m. Then, according to [A], $\mathfrak{A}(m,m)=\{\Lambda\otimes F_2\}$ and $\mathfrak{A}(m+1,m)$ is the set $\big\{\Lambda\otimes F(d)\big\}$, where $\Lambda\in\mathfrak{L}_m$, and where $F(d)$ is one of the matrices (7), given but none the less arbitrary. In addition $\Lambda_1\otimes F(d)\sim\Lambda_2\otimes F(d)$ if and only if $\Lambda_1^2\sim\Lambda_2^2$. It will be better to explain the consequences of this immediately.

Lemma 1. *The correspondence $\Delta_1:\mathfrak{A}(j,j)\to\mathfrak{A}(j+1,j)$ is four-to-one, but the correspondence $\Delta_1:\mathfrak{A}(j,j-1)\to\mathfrak{A}(j,j)$ is one-to-four.*

This is a consequence of three circumstances: (i) Δ_1 commutes with multiplication by a linear bundle; (ii) on $A(j,j)$ the determinant is four-to-one; (iii) on $A(j,j+1)$ the determinant is mutually single-valued. The first assertion is obvious and the other is established in [A, Th. 7]. For example, even if in the matrix

$$\begin{pmatrix} z_x^{-1} & z_u^{-2}\\ 0 & z_u^{-1}\end{pmatrix}=\begin{pmatrix} z_x^{-1} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & z_u^{-2}\\ 0 & z_u^{-1}\end{pmatrix}$$

the class of the second factor is not changed by the replacement

$$\Lambda\begin{pmatrix} 1 & z_u^{-2}\\ 0 & z_u^{-1}\end{pmatrix},\quad \Lambda^2\sim 1,$$

the class of the second factor in (19′) does change if Λ is not trivial. The best way of parametrizing the elements in $\mathfrak{A}$ is the following:
(i) first, one parameter is the degree of the determinant; (ii) If the degree is odd then this determinant itself represents the second parameter; (iii) if the degree is even, then this element is equal to ΛF_2, where F_2 is given by (6), and this second parameter is Λ. Consequently the parameter is given by the degree and a linear bundle, and the first element is superfluous.

[20]Unfortunately the notation, $\chi(z_x^{-1})=\chi^{-1}(x)$ is bad. I hope that this is acceptable! The choice was made in (30).

The first operator of Hecke Θ_1 increases the degree by one. The determinant is multiplied by $\Lambda = (z_x^{-1})$. According to the lemma, on $\mathfrak{A}(m, m+1)$ whose image is given by the Hecke correspondence in the ratio $1:4$, and on $\mathfrak{A}(m,m)$ where the ratio is $4:1$.

A given character χ of the group $\mathrm{Pic}(M)$ is determined by a function f'_χ on the set

$$\mathfrak{A}_{\mathrm{even}} = \bigcup_{m=-\infty}^{\infty} \mathfrak{A}(m,m) = \left\{ \Lambda \otimes F_2 \;\middle|\; \Lambda \in \mathrm{Pic}(M) \right\}, \tag{31}$$

namely, $f'_\chi : \Lambda \otimes F_2 \mapsto \chi(\Lambda)$. If χ is trivial on $\left\{ \Lambda \in \mathfrak{A}_{\mathrm{even}} \;\middle|\; \Lambda^2 = 1 \right\}$ then there is a $\widetilde{\chi}$ such that $\chi(\Lambda) = \widetilde{\chi}(\Lambda^2)$. Let the function f'_χ be an extension of f_χ to $\mathfrak{A}$ such that $f'_\chi(\Lambda) = \widetilde{\chi}(\det \Lambda)$. If χ is not such, then f'_χ is equal to zero on $\mathfrak{A}_{\mathrm{odd}}$, the supplementary set to $\mathfrak{A}_{\mathrm{even}}$ in $\mathfrak{A}$,

$$\mathfrak{A}_{\mathrm{odd}} = \bigcup_{m=-\infty}^{\infty} \mathfrak{A}(m,m+1). \tag{31.a}$$

These functions form a full set of eigenfunctions of the Hecke operators with support in $\mathfrak{A}$. More precisely, $\mathfrak{S}_0$ the space of functions on $\mathfrak{A}$ is formed from four subspaces. The first $\mathfrak{S}_0$ consists of those functions f such that $f(\Lambda \otimes \Theta) = f(\Theta)$ if $\deg \Theta$ is even and $\Lambda^2 = 1$, but if $\deg \Theta$ is odd there is no such condition. The function is arbitrary on $\mathfrak{A}_{\mathrm{odd}}$, but continuous or square-integrable according to the circumstances. There are three more subspaces, $\mathfrak{S}_i$, $i = 1, 2, 3$. Let χ_i, $i = 1, 2, 3$ be the non-trivial characters of the group $\mathrm{Pic}_2(M) = \left\{\Lambda \in \mathrm{Pic}(M)\right\}$, where $\Lambda^2 = 1$. I observe that the order of this group is four and that the square of each $\Lambda \in \mathrm{Pic}_2(M)$ is trivial. Let $\mathfrak{S}_i$ be the set of those functions f for which $f(\Lambda\Theta) = \chi_i(\Lambda) f(\Theta)$, $\Lambda \in \mathrm{Pic}_2(M)$ and $f(\Theta) = 0$ if $\deg \Theta$ is odd. For the theorem of completeness, it is necessary to demand that the square, rather the absolute value, of these functions be integrable, but sometimes other conditions are appropriate, for example, for the description of the eigenfunctions and eigen conjugacy classes of the Hecke operators. The following lemma is an immediate consequence of the first lemma. The third statement is a consequence of the equation (32.a)

Lemma 2. (a) *On each space* $\mathfrak{S}_i$*,* $i = 1, 2, 3$ *the operator* Θ_1 *is null.* (b) *Each space* $\mathfrak{S}_i$*,* $i = 0, 1, 2, 3$ *is invariant under* Θ_2*.* (c) *All the eigenfunctions appear in* $\mathfrak{A}$ *with multiplicity one.*

Consequently for $i = 1, 2, 3$ and all $x \in M$ the form of the eigen class is

$$\begin{pmatrix} \alpha_x & 0 \\ 0 & -\alpha_x \end{pmatrix}, \qquad \alpha \in \mathbf{C},\ |\alpha| = 1, \tag{32}$$

thus its trace is zero. Indeed, the eigenfunction f for $\mathfrak{S}_i$, $i = 1, 2, 3$ is such a function that $f = 0$ on $\mathfrak{A}_{\mathrm{odd}}$ and that $f(\Lambda\Theta) = \chi_i(\Lambda) f(\Theta)$ if $\Lambda \in \mathrm{Pic}_2(M)$. The full set of these eigenfunctions is formed by the restrictions to the set $\mathfrak{A}_{\mathrm{even}}$ of those functions $\Lambda F_2 \to \chi(\Lambda)$ for which the restriction of the character χ to $\mathrm{Pic}_2(M)$ is equal to χ_i. Consequently[21]

$$\alpha_x = \pm\sqrt{-f(z_x^{-1})}. \tag{32.a}$$

[21]When I first encountered this question, I was not aware of the complications, insignificant but important, arising from this description. The function f is single valued, but the function α_x does not appear to be so always. What we can and must do is to choose it to be such that it is single-valued on one of the three double coverings M' of the curve M. Then it is clear the eigenfunctions form a complete orthogonal basis. I hope that this brief explanation is adequate.

The sign does not affect the conjugacy class (32). We may choose the square root in a continuous fashion and then determine it globally up to a sign.[22] We shall return to a proof of the third assertion in §XI.

The remaining space is formed from functions of the determinant, $\Theta \to f(\det \Theta)$. Which of these functions is an eigenfunction of the Hecke operators? Let $\rho(\Lambda)$, $\Lambda \in \mathrm{Pic}(M)$, be equal to $1/2$ if $\deg \Lambda$ is even and 1 if $\deg \Lambda$ is odd. Then the eigenvalues $\Theta_{1,x}$ and $\Theta_{2,x}$ are given as $\rho\chi$, where χ is a character of $\mathrm{Pic}(M)$. The eigenvalues $\Theta_{1,x}$ and $\Theta_{2,x}$ are equal to $2\chi(x)$, $\chi^2(x)$, and the eigen conjugacy class by the matrix

$$\begin{pmatrix} \chi(x) & 0 \\ 0 & \chi(x) \end{pmatrix}. \tag{33}$$

The combinatorics here are somewhat unusual. I underline that it appears that the multiplicity of each conjugacy class in $L^2(\mathfrak{A})$ is one. But these supplementary conjugacy classes appear also in $L^2(\mathfrak{D})$. This appears to me as somewhat astonishing.

There is one point that is necessary, but there are others to underline as useful to remember. The set of characters of the group $\mathrm{Pic}(M)$ is a union of sets of pairs $\{\chi, \eta\chi\}$, where $\eta(\Lambda) = (-1)^{\deg \Lambda}$. Each pair is associated to an eigen class.

I remark also, that it is extremely difficult to distinguish eigenfunctions, eigen points and eigenvalues. There is an arbitrary choice, unclear in the first, but not in the others. It is important to distinguish their properties, but not easy.

Lemma 3. *This construction is injective.*

The trace of the matrix (32) is 0 for all x. For the trace of classes lying in the principle series, thus attached to a pair of characters of $\mathrm{Pic}(M)$, this is impossible. Thus it is sufficient to consider the classes attached to $\mathfrak{S}_i$, $i = 1, 2, 3$. Then the trace is uniformly zero. Consequently only the determinant is relevant here. But according to the equation (32.a) the determinant is equal to $f^2(z_x)$. But the admissible functions are such that, if f_1, f_2 is admissible and $f_1 = \pm f_2$ everywhere, then $f_1 = \epsilon f_2$ everywhere with a constant ϵ.

The final theory will be a theory for reductive groups. Thus it is possible that the group GL(2) leads to confusion. We therefore consider briefly the group SL(2), for which the parabolic spectrum is finite. It is probable that this is correct for all reductive groups. We did not consider this group and its Hecke operators, but $\mathrm{Bun}_{\mathrm{SL}(2)} \subset \mathrm{Bun}_{\mathrm{GL}(2)}$ and their eigenfunctions are obtained by restriction. I have still not verified that the appearance of dominant diagonal blocks[23] is justified for multiplication, for all Hecke operators, and for SL(2), in particular for operators determined by the matrix

$$\begin{pmatrix} z_x^{-1} & 0 \\ 0 & z_x \end{pmatrix}.$$

Nevertheless I use them. The restriction of each space $\mathfrak{S}_i$, $i = 0, 1, 2, 3$, to SL(2) is one-dimensional. Thus the parabolic spectrum—rather the parameters for the parabolic spectrum—of the group SL(2) consists of four pieces, each parametrized by $\mathbf{Z}^2$ (or classes).[24] One of these pieces (or classes) is also parabolic. I am inclined to think that for all

[22] It is extremely difficult to distinguish eigenfunctions, eigen classes and eigenvalues. There is an arbitrary choice, unclear for the first but not for the second.

[23] Thus the equality $AD = DA = 0$.

[24] In the original Russian text a linguistic confusion was introduced. I wrote четырехразмерен rather than объединение четырёх одномерных классов, то есть.

semi-simple groups the parabolic spectrum is finite, but I still do not know how to count it, at least not in general.

Before we turn to the fundamental question there are some simple remarks. If two characters χ_1, χ_2 are such that the coupled functions $f_1 = f_2$, $f_i : \Lambda F_2 \to \chi_i(\Lambda)$ yield the same eigen conjugacy class (31.a), thus $\chi_1(z_x^-) = \pm\chi_2(x^{-1})$ everywhere, then $f_1 = f_2$. In addition $\Lambda \to \det \Lambda F_2$ yields a natural covering of degree four. It is perhaps unimportant, but for $i = 1, 2, 3$ the periods of the conjugacy class (32) are ω_1 and ω_2.

Periods and functions. Before we conclude this section, I would like to make some evident small observations, because what confuses me is in the final analysis related to important parts of the theory that are usually overlooked by the perceptive authors of the paper [AB]. The curve M is given by the lattice $L = 2\omega_1\mathbf{Z} + 2\omega_2\mathbf{Z}$, thus its points are given by $\mathbf{C}/L$. But the functions, connections, and so on are given as $\mathbf{C}/\widetilde{L}$, where $\widetilde{L} = 2\eta_1\mathbf{Z} + 2\eta_2\mathbf{Z}$. But we meet above not only these parameters but also those that are given by $\mathbf{C}/2\widetilde{L}$ because there is a two-dimensional determinant that appears in §IX for other reasons. This last possibility was not mentioned in [AB].

VIII. Connectivity and curvature

The theorem of Atiyah-Bott is described in the following section, but in order to establish this theorem they need some fundamental results taken from global differential geometry that were unknown to me. It is possible that readers will be aware of them, but more frequently they will know no more than I. I prefer therefore to explain them here, but only those that are necessary. The matter is such that we need to be familiar not only with the basic definitions of differential geometry but need also some experience with their use.

In [AB], before describing the theorem, the authors introduce the bundle Q that I want to describe now for the curve M. In the words of [AB] "let $Q \to M$ be a $\mathbf{U}(1)$-bundle with Chern class 1 endowed with a fixed harmonic or Yang-Mills connection. If we normalize the metric on M so that it has total volume 1 the curvature[25] of this harmonic connection on Q is $-2\pi i\omega$, where ω is the volume form on M. The universal covering $\widetilde{M} \to M$ is of course a flat $\pi_1(M)$-bundle, so that the fibre product $Q \times_M \widetilde{M}$ is a $\mathbf{U}(1) \times \pi_1(M)$-bundle over M with connection still having curvature $-2\pi i\omega$. In particular this connection A is a Yang-Mills connection…" The assertion that the Chern class is equal to 1 means that the section has only one zero, rather than a pole. Although these assumptions and these assertions are clear and familiar to a geometer, this is not so for me. For me and for some other readers it will be simple to misunderstand. I thus describe the concepts in detail, in

[25] Here there is an implicit supposition or condition about which I shall later make an observation. The introduction of the factor $\pi_1(M)$ in [AB, 6.5] affects the group of the bundle but not its Lie algebra. Consequently the concept of curvature is a little modified. Nevertheless the presence of the group will be important. This will be explained in more detail later. I needed some time in order to understand this important construction. For the moment I confine myself to a single remark. If $G = \mathbf{U}$, $\rho|\mathbf{Z}$ is necessarily equal to unity. If, however, $g = 1$, and also if $g > 1$, but here we consider particular cases, we may then modify the value of $\rho(A)$ and $\rho(B)$ as we please. A need for this will appear in section X.

It appears, to my great astonishment, that in the present article only bundles $\mathbf{U}(1) = \mathbf{U}(1)^n$ of degree 0, thus $n = 0$, are important, thus the trivial bundle with a constant metric. I understood this only after serious reflection on the questions of the article and on the conclusions of [AB].

particular the bundle Q and the connection A.[26] But where can we find this connection. At first it was difficult to understand.

It is an explanatory difficulty. It is not a question of one or two definitions. In [AB] it is supposed that the reader has some knowledge of differential geometry. It seems to me that in the present article it is ill-advised to do so. For me and for the reader it is necessary to explain some concepts in more detail in order to arrive at a genuine familiarity with them and their possibilities. The concept of a Yang-Mills connection, whose consequences are complicated, is in itself not easy to understand. I begin with some material from the article [AB], but only for elliptic curves.

Although it is not necessary, I prefer a description in the context of a theory better known to me, the Weierstrass theory. In essence a connection is given locally by the function σ of (2.b), but a complete description is complicated. It is also necessary to derive from it the Yang-Mills connection. But the Yang-Mills condition, which I shall describe later, is determined in addition to that not only by the metric on the bundle but also by the metric on M. This is already a secondary question.

It is possible that the reader is also unfamiliar with complex differential geometry, as I am. I therefore propose, for him(her) some fundamental definitions, that appear in [AB] and with which geometers are familiar. As far as I know, curvature, which is so important for the present article, is absent in the Russian-American theory, although it is possible that in it the Yang-Mills equation, as a physical theory, makes its appearance in the background. This is essential for our goals, because the theorem of Atiyah-Bott is necessary, although secondary. It seems to me as well that it is useful and even necessary not to confuse the geometrical theory of automorphic forms with conformal field theory or the theory of gauge fields. I first explain briefly the relation of the theory in [AB] to the eigenvalues of Hecke operators. Some concepts, related to curvature and connections, will be explained by the following example.

The core of the matter is that the eigenvalues of the Hecke operators are given by functions on M whose value at a point $x \in M$ lie in the hermitian component of the Lie algebra of the group ${}^LG$. If $G = \mathrm{GL}(2)$, whose compact form is the unitary group $\mathbf{U}(2)$, ${}^LG = G$, the Lie algebra $\mathfrak{g}$ of the group G is the direct sum $\mathbf{U} \oplus \mathfrak{h}$, where $\mathfrak{h} = i\mathbf{U}$ is the space of two-dimensional hermitian matrices. On the other hand there are connections whose values lie in ${}^L\mathfrak{g} = \mathfrak{g}$, but[27] we are largely concerned with those whose values lie in ${}^L\mathbf{U}$. We show that the eigenvalues[28] of Hecke operators are given by the integral of a Yang-Mills connection with appropriate initial conditions and multiplication by $i = \sqrt{-1}$. I suppose of course that such an assertion is generally valid, but even for GL(2) and an elliptic curve there is still much to understand and much to explain. This correspondence is a consequence of the relation of the fine points of these two sets.

The concept of a Yang-Mills connection is determined only for unitary connections. In any case I consider only such connections. The following explanations are taken from [AB, §3,4,5,6], although the notation is slightly modified. Let P be a bundle for the group

[26][T] For the basic definitions, the book *Differential Geometry* of Clifford Taubes is a useful supplementary reference.

[27]I admit that the circumstance, that the tangent space of an n-dimensional complex bundle on a one-dimensional complex curve is $4n$-dimensional, always confuses me.

[28]It is best to recall that the local Hecke algebra is isomorphic to the ring of representations of ${}^LG$ and that the character of this ring corresponding to $\gamma \in {}^LG$ is given by the trace of $\pi \mapsto \operatorname{tr} \pi(\gamma)$. Normally γ is unitary, but the trace of such a γ has no distinguishing properties.

$\mathbf{U} = \mathbf{U}(2)$, that is for its Lie algebra $\mathfrak{U}(2)$. The connection A is a splitting of the sequence

$$0 \longrightarrow T_F P \longrightarrow \mathrm{TP} \longrightarrow \pi^{-1}\,\mathrm{TM} \longrightarrow 0\ , \tag{34}$$

where T denotes a tangent space and ${}_FP$ a fibre of the bundle. Moreover, it is necessary that the splitting be invariant relative to $\mathbf{U}$! Thus the connection is determined by $\omega_A : \mathrm{TP} \to T_F P$ together with the simple necessary properties, which are evident. Thus it is possible to lift a vector field X from TM to TP. Let $X \to \widetilde{X}$. Then $F_A(X,Y) = \omega_A[\widetilde{X},\widetilde{Y}]$ is a measure of the curvature of the connection. As a function of the pair $\{X,Y\}$, F_A is a differential form whose values are in the Lie algebra $\mathbf{U}$. Generally a vector bundle is attached to a bundle P and each representation of $\mathfrak{g}$ or $\mathbf{U}$—the first of the latter derived from the second. It is obvious that F_A is a section of the bundle determined by the tensor product of a two-dimensional form with this attached form. Fortunately $*F_A$ is simpler. It is a section of the bundle P and this section is invariant.

This remark is important for the proof of a theorem in [AB, §6, Th. 6.7]. Thus it is necessary to understand it. Since $F(A) \in \Omega^2\big(M;\mathrm{ad}(P)\big)$, $*F_A \in \Omega^0\big(M,\mathrm{ad}(P)\big)$, thus $*F_A$ is a function, whose values in the fibre lie in $\mathrm{ad}(P) = \mathbf{U}$. Since A is invariant $*F(pg) = \mathrm{Ad}\, g^{-1} * F(p)$. Thus its conjugacy class is constant within the fibre. Its global constancy is a consequence of the Yang-Mills condition, which we explain below. I remark that the condition of Yang-Mills, although important for the theorem of Atiyah-Bott, is not at first relevant.

There is here a possibility for confusion. The construction of the bundle Q as a Yang-Mills bundle is necessary, but its precise construction with constant curvature is not necessary. Nevertheless, it is useful for comparison with the theory of Hecke operators.

We are considering connections on a Riemann surface and they possess particular properties.[29] It is difficult to remember that M in [AB] is above all a real manifold. Thus the Hodge star[30]

$$* : \Omega^1_M(\mathbf{C}) = \Omega^1_M \otimes \mathbf{C} = \Omega^{1,0}(M) \oplus \Omega^{0,1}(M) \to \Omega^1_M(\mathbf{C})$$

is such that $* = -i$ on $\Omega^{1,0}$ and $* = i$ on $\Omega^{0,1}$. We chose a metric, a constant metric, such that the corresponding complex structure is given by the customary $d' = \partial/\partial z = d/dx + id/dy$, $d'' = d/dx - id/dy$. With this construction we transform a real space to a complex space. We can repeat this with the usual decomposition, creating d'_A and d''_A from $\mathbf{U}$-connection dA. Since this paper is not addressed to differential geometers, I explain. Attached to the connection A and all associated decompositions there is a covariant derivative and its conjugate differential operator.

I recall that the foregoing is correct also for bundles

$$\Omega^1_{\mathbf{C}}\big(M,\mathrm{ad}(P)\big) = \Omega^{1,0}\big(M,\mathrm{ad}(P)\big) \oplus \Omega^{0,1}\big(M,\mathrm{ad}(P)\big). \tag{35}$$

[29] [AB, §4,5] or [GH] *Principles of Algebraic Geometry,* p. 72, P. Griffiths, J. Harris.

[30] As I remarked several times, I learned differential geometry as I was writing this paper. Thus definitions were sometimes given prematurely, incorrectly or incompletely. For example, for the construction of [GH] the metric on M is irrelevant. I shall return to this question later.

These spaces are eigenspaces of the operator $\star$. Let d_A be the covariant derivative attached to A.

$$\begin{array}{ccc}
\Omega^{1,0}\big(M, \mathrm{ad}(P)\big) & \xrightarrow{d''_A} & \Omega^2_{\mathbf{C}}\big(M; \mathrm{ad}(P)\big) \\
{\scriptstyle d'_A}\Big\uparrow & & \Big\uparrow{\scriptstyle d'_A} \\
\Omega^0_{\mathbf{C}}\big(M, \mathrm{ad}(P)\big) & \xrightarrow{d''_A} & \Omega^{0,1}\big(M; \mathrm{ad}(P)\big)
\end{array} \tag{35.a}$$

There is a construction inverse to this diagram that is given in [GH]. It is useful to underline the different relation of the constructions in [AB] and [GH] to this diagram—in [AB] of the metric together with the unitary connection to the holomorphic structure; but in [GH] of the metric together with the holomorphic structure to the unitary connection.

I no doubt am explaining in too much detail but it is important to understand the basic definitions. These I did not adequately explain because I did not understand them well enough initially. In [AB] it is shown that the lower arrow in (35.a) determines a complex-analytic structure on the bundle. On the other hand, in [GH, p. 72] it is explained how a vector bundle with a complex-analytic structure and a hermitian metric determines a connection. In the present discussion, this means that the diagram (35.a), in which the connection is only implicit, it is given by two different sets. I arrived at an understanding of this equivalence only slowly. I repeat that for me the present article is an occasion to learn some complex differential geometry. This theory yields [AB, Th. 6.7], which suggests the definition of the automorphic galoisian group. We explain later the construction of this metric in a particular case.

I would first like to explain briefly the relation of [AB] and [GH] to the diagram (35.a). In [AB] the metric is given and limits the connection, which gives the arrows; in [GH] the metric is also given but with the arrow $d''_A = \overline{\partial}$, and together they determine the connection. Therefore if the dimension of the bundle is equal to one and if there is a holomorphic section, and if, of course, the section is local, then it and its variable length determine, at least locally, a connection that, together with the metric or if we so prefer with a modified metric, is compatible with the connection. I stress that there are two metrics, one on M and another on the bundle. At the moment we are concerned with the metric on the connection. This connection is independent of the holomorphic section and is therefore determined globally.

My construction of the bundle is related to the theory of Weierstrass. This is unnecessary and even clumsy, at least in the beginning, but for me it puts it in a more familiar and more convincing context. At first everything was unfamiliar to me: the construction of a connection when the metric on the bundle is given [GH, p. 73]; curvature; the close relation of a Yang-Mills connection with constancy.

We obtain an appropriate bundle Q for [AB, Th. 6.7] from the trivial bundle, if we allow poles and zeros. I wrote these words several months ago (now two years) but I did not understand them correctly, causing myself considerable confusion.[31] They imply that there is no canonical expression for the section in a neighbourhood of zero. The function ϕ is chosen with a pole of order one, for example σ^{-1} or ζ and the actual section ψ, the one with a pole, is written as $\psi(\cdot) = f(\cdot)\phi(\cdot)$. The section f is not canonical because ϕ is not canonical. The function σ^{-1} is better because it can be used everywhere in $\mathbf{C}$.

[31]It was unexpectedly difficult for me to understand the construction of the sheaf Q.

In order to use the lemma in [GH][32] that yields the dimension, we need a hermitian metric on Q. Since the dimension of Q is equal to 1, it is sufficient to introduce a positive function $s(z) > 0$ such that[33]

(36.a)
$$s(z)\big|\sigma(z)\big|^{-2} = s(z+2\omega_i)\big|\sigma(z+2\omega_i)\big|^{-2} = s(z+2\omega_i)\exp\Big(-4\operatorname{Re}\big(\eta_i(z+\omega_i)\big)\Big)\big|\sigma(z)\big|^{-2}.$$

Before we show what such a function is, I remark the following. Let $g(\cdot)$ be a possibly non-holomorphic section of a bundle.[34] Then $f(z) = g(z)\sigma(z)$ is everywhere finite and the metric is defined at Q:

$$\big(g(z), g(z)\big) = \big|f(z)\big|^2 s^{-1}(z). \tag{36.b}$$

I have no other basis for its introduction, only that it serves the purpose.

The equation (36.a) is equivalent to

$$s(z+2\omega_i) = s(z)\exp\Big(4\operatorname{Re}\big(\eta_i(z+\omega_i)\big)\Big). \tag{36.c}$$

Consequently,

$$\begin{aligned}
s(z+2\omega_1+2\omega_2) &= s(z+2\omega_1)\exp\Big(4\operatorname{Re}\big(\eta_2(z+\omega_1+\omega_2)\big)\Big)\\
&= s(z)\exp\Big(4\operatorname{Re}\big(\eta_1(z+\omega_1)\big)\Big)\exp\Big(4\operatorname{Re}\big(\eta_2(z+\omega_1+\omega_2)\big)\Big);\\
s(z+2\omega_2+2\omega_1) &= s(z+\omega_2)\exp\Big(4\operatorname{Re}\big(\eta_1(z+\omega_2+\omega_1)\big)\Big)\\
&= s(z)\exp\Big(4\operatorname{Re}\big(\eta_2(z+\omega_2)\big)\Big)\exp\Big(4\operatorname{Re}\big(\eta_1(z+\omega_2+\omega_1)\big)\Big).
\end{aligned}$$

This is possible only if

$$\exp\big(4\operatorname{Re}(\eta_2\omega_1)\big) = \exp\big(4\operatorname{Re}(\eta_1\omega_2)\big), \tag{36.d}$$

[32]This lemma is not necessary for the determination of the connection because it yields a known function, but this outline is also an occasion to learn some differential geometry. Besides that I began with this lemma and only after much thought recognized that it was not completely appropriate for the needs of the article. In particular, I did not recognize the advantages, perhaps the necessity, of the introduction of the function $s(\cdot)$.

[33]The reader will have to pardon me, but for questions of differential geometry there are two metrics that it is necessary to consider, one on the base and one on the fibre. The curvature reflects both one and the other. Thus the curvature is determined by the difference of rapidity of rotation in the fibre generated by the movement in the base. On the base, because of its metric, there is at each point a standard area, so that a variable real number is equivalent to a changing square and conversely. Yang-Mills connections, at least if the dimension is equal to one, are such that their curvature is constant, thus that it satisfies the equation $\star F = 0$. It was difficult for me to understand how this could be possible for the connection Q. One purpose, an initial purpose, of this section is to understand this, but I quickly recognized that there were many more things that I did not understand.

The construction that is explained in [GH] is independent of the metric on M, but the curvature depends on it. On the other hand, the Yang-Mills connection is such that the curvature is constant. It is not clear to me how the metric on the fibre affects the curvature. I proposed two examples, one more attractive than the other. I remark that the final goal, the comparison of two unearthly things, Hecke conjugacy classes, which vary from point to point, and integrals of Yang-Mills connections, the nature of which is explained to some extent in [AB]. For elliptic curves the difficulties are relatively simple.

It turns out that the construction in [GH] is not exactly what we need, but it offers an initial understanding of the methods of constructing connections with prescribed properties.

[34]Thus if $\lambda \in L$ and g is defined at the point λ, then the function $(z-\lambda)g(z)$ is finite at the point λ.

which is a consequence of the equation $2\eta_1\omega_2 - 2\eta_2\omega_1 = \pi i$ [WW, p. 446].

We first determine s and after that recall the argument in [GH]. Let $z = 2a\omega_1 + 2b\omega_2$, a, $b \in \mathbf{R}$. For the present goal a, b are appropriate real variables.[35] Let

$$s(z) = s(a,b) = \exp(\alpha a^2 + \beta ab + \gamma b^2 + \delta a + \epsilon b). \tag{36.e}$$

Then according to (36.c) the relevant equations are

$$\begin{aligned} s(a+1,b) &= s(a,b)\exp\bigl(8\operatorname{Re}(\eta_1\omega_1 a) + 8\operatorname{Re}(\eta_1\omega_2 b) + 4\operatorname{Re}(\eta_1\omega_1)\bigr),\\ s(a,b+1) &= s(a,b)\exp\bigl(8\operatorname{Re}(\eta_2\omega_2 b) + 8\operatorname{Re}(\eta_2\omega_1 a) + 4\operatorname{Re}(\eta_2\omega_2)\bigr); \end{aligned} \tag{36.f}$$

but according to (36.e) the real (rather than complex) equations are

$$\begin{aligned} s(a+1,b) &= s(a,b)\exp(2\alpha a + \alpha + \beta b + \delta),\\ s(a,b+1) &= s(a,b)\exp(\beta a + 2\gamma b + \gamma + \epsilon). \end{aligned} \tag{36.g}$$

A comparison of these equations yields α, γ, δ, ϵ without difficulty, but for β there are two determinations, which yield the same value because of (36.d). The values of these five numbers are:

$$\alpha = 4\operatorname{Re}(\eta_1\omega_1), \quad \beta = 4\operatorname{Re}(\eta_1\omega_2) = 4\operatorname{Re}(\eta_2\omega_1), \quad \gamma = 4\operatorname{Re}(\eta_2\omega_2), \quad \delta = \epsilon = 0. \tag{36.h}$$

At this stage we have already chosen a metric on the fibre, as in [GH]. Later we shall choose as a supplementary experiment a different one. However we depart now from the method in [GH], in which the holomorphic structure guarantees a supplementary condition (thus $D'' = \overline{\partial}$ [GH, p. 73]) and introduce a right-angular movement of the connection. I return later to this argument in the context of an attempt to understand adequately the concept of curvature, in so far as it appears in the Yang-Mills theory.

The curve M, as a real manifold, is two-dimensional and the connection moves in $\mathbf{C}$, which is also two-dimensional. Consequently this connection is determined infinitesimally by four real parameters. The condition of unitarity reduces this by two, which are purely imaginary. We shall return to these two parameters later, because they yield the curvature. Globally we have now a complex line-bundle with metric. I repeat. If $z_1 - z_2 \in L$, then $\bigl(z_1, \sigma(z_1)^{-1}\bigr)$, $\bigl(z_2, \sigma(z_2)^{-1}\bigr)$ represent the same point in the bundle only if $\sigma(z_1) = \sigma(z_2)$ and this is improbable. If $f \in \mathbf{C}$, $z \in \mathbf{C}$, and $\lambda \in L$, then (z, f) and $(z+\lambda, f)$ represent the same point in the bundle. To repeat, the principal object is the function $\sigma(\cdot)$ but it is not single-valued. None the less it determines the metric on the fibres and the complex structure of the bundle. This metric is $f(\cdot) \to s^{-1/2}(\cdot)\bigl|\sigma(\cdot)\bigr|\bigl|f(\cdot)\bigr|$.

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[35] For the following calculations I remark that

$$z\overline{\omega}_2 - \overline{z}\omega_2 = 2a(\omega_1\overline{\omega}_2 - \overline{\omega}_1\omega_2); \quad z\overline{\omega}_1 - \overline{z}\omega_1 = 2b(\omega_2\overline{\omega}_1 - \overline{\omega}_2\omega_1).$$

Consequently the linear relations of the linear variables a, b to the variables z, $\overline{z}$ are given by

$$a = \frac{1}{2}\frac{z\overline{\omega}_2 - \overline{z}\omega_2}{\omega_1\overline{\omega}_2 - \overline{\omega}_1\omega_2}, \quad b = \frac{1}{2}\frac{z\overline{\omega}_1 - \overline{z}\omega_1}{\omega_2\overline{\omega}_1 - \overline{\omega}_2\omega_1}, \tag{37}$$

and

$$\frac{\partial a}{\partial z} = \frac{1}{2}\frac{\overline{\omega}_2}{\omega_1\overline{\omega}_2 - \overline{\omega}_1\omega_2}, \quad \frac{\partial b}{\partial z} = \frac{1}{2}\frac{\overline{\omega}_1}{\omega_2\overline{\omega}_1 - \overline{\omega}_2\omega_1}, \qquad \frac{\partial a}{\partial \overline{z}} = \frac{1}{2}\frac{-\omega_2}{\omega_1\overline{\omega}_2 - \overline{\omega}_1\omega_2}, \quad \frac{\partial b}{\partial \overline{z}} = \frac{1}{2}\frac{-\omega_1}{\omega_2\overline{\omega}_1 - \overline{\omega}_2\omega_1}.$$

For clarity we remove from these expressions the common denominator $\eta = \omega_1\overline{\omega}_2 - \overline{\omega}_1\omega_2$, in order to obtain

$$\frac{\partial a}{\partial z} = \frac{\overline{\omega}_2}{2\eta}, \quad \frac{\partial b}{\partial z} = -\frac{\overline{\omega}_1}{2\eta}, \qquad \frac{\partial a}{\partial \overline{z}} = -\frac{\omega_2}{2\eta}, \quad \frac{\partial b}{\partial \overline{z}} = \frac{\omega_1}{2\eta}. \tag{37.a}$$

I add that, up to a factor ± 1, the number η is twice the area of the parallelogram with sides ω_1, ω_2.

As I wrote this article I recognized that there were several basic mathematical concepts of which my understanding was insufficient or even mistaken. I explain this in the following lines, but without drawing particular attention to my misunderstandings, for example of Poincaré's lemma. Indeed, that a non-trivial differential equation $\overline{\partial} f = 0$ is such that the set of its solutions is closed with respect to multiplication is, when one reflects, an extraordinarily unusual property, whose oddness and unexpectedness I had for years never recognized, although its importance is evident. Thus the construction of Q, the existence of which was evident for Atiyah and Bott, filled some gaps in my imagination.

Although I explain the construction in [GH], returning to this treatment in the context of an attempt to understand adequately the concept of curvature, in so far as it appears in the Yang-Mills theory, I prefer to begin with a simple description and construction of appropriate one-dimensional connections. Usually and preferably the movement is given in logarithmic form $\eta_1 = \exp(\rho_1 + i\theta_1)$, where ρ_1 and θ_1 are real functions. The function ρ is completely determined by the metric. The connection is given by the expression

$$\frac{\eta_1'}{\eta_1} = \rho_1'(\cdot) + i\theta_1'(\cdot).$$

It is possible to determine η_1 in relation to the given section, which may be σ^{-1}, thus to determine

$$\eta_1(\cdot) = \eta(\cdot) - \frac{d\sigma}{\sigma}, \quad \eta = \rho + i\theta. \tag{37.b}$$

The notation here is such that the second member, although with a minus sign, is the connection! I observe that it is a connection whose curvature is equal to zero, thus its integral σ^{-1} is uniquely determined locally.

In so far as the connection is unitary,

$$s^{-1}(\cdot)\big|\sigma(\cdot)\big|^2 \exp(2\rho) = \text{constant}.$$

Consequently, $d\rho(\cdot)$ is uniquely determined. It is given by the equation

$$2\frac{d\rho}{\rho} = \frac{ds}{s} - \frac{d\sigma}{\sigma} - \frac{d\overline{\sigma}}{\overline{\sigma}}.$$

However this is not the ρ in which we are interested; this is θ. Although I did not recognize this for a long time, there is an obvious choice, the imaginary analogue of the equations (36.g) and (36.h),

$$\frac{ds}{s} = (2\alpha a + \beta b)\, da + (\beta a + 2\gamma b)\, db,$$

thus[36]

$$(2\widetilde{\alpha} a + \widetilde{\beta}_1 b)\, da + (\widetilde{\beta}_2 a + 2\widetilde{\gamma} b)\, db, \tag{36.i}$$

where $\widetilde{\alpha} = 4\,\mathrm{Im}(\eta_1\omega_1)$, $\widetilde{\beta}_1 = 4\,\mathrm{Im}(\eta_1\omega_2)$, $\widetilde{\beta}_2 = 4\,\mathrm{Im}(\eta_2\omega_1)$, $\widetilde{\gamma} = 4\,\mathrm{Im}(\eta_2\omega_2)$. The location of the two numbers is to a significant degree arbitrary. They may be exchanged. The result is only a change of sign. Those who are familiar with curvature will immediately recognize that the curvature so obtained is $\widetilde{\beta}_1 - \widetilde{\beta}_2 = 2\pi$. It is necessary, however, to recognize that this is the curvature relative to the coordinates (a, b) and that relative to these coordinates the area of the fundamental domain is equal to 1. Thus these calculations are compatible

[36]The notation leads to confusion. The notation a and b appears ambiguous, but da and db appear to be the differentials at the initial point, a point for which there is no notation.

with the assertions in [AB]. I underline first that this conclusion presupposes a particular choice of metric on M, namely a metric that is linearly invariant and, secondly, that the choice θ, although natural, is also arbitrary.

We introduced a modification of the connection rather accidentally, but this was a very important change that gave us a choice of the bundle Q, that introduced on p. 560 of [AB]. Since the curvature of the connection is constant and the metric on M invariant relative to transport, this modified connection is of Yang-Mills type. I admit that I am at this point somewhat undecided, because the affirmation demands an explanation of the equation [AB, 6.1], namely for the given connection and for the metric considered the curvature $\star F$ is constant—it seems that a constant curvature is a property of pairs, a connection and a metric. In addition, as I shall remark again, rather timidly, in another place in this essay, the action of d_A is for ordinary functions a transfer, so that $\star F$ does not change in relation to it. This is equation [AB, 6.1]. I admit that this explanation is a little forced, but it is confirmed by the assertion (6.1) in [AB].

In order to convince oneself that our argument is correct, we have to repeat the calculations in (36.a)–(36.h), but in logarithmic form and only for the imaginary part. It is therefore evident. Nevertheless in order to reassure myself and the reader I make the calculation. As I proceed, as I continue to attempt to understand [AB] adequately, in order to apply it to the geometric theory of automorphic forms, I grow less constrained in the treatment of fundamental differential geometric concepts. None the less, I continue to feel insufficient confidence during the seemingly arbitrary placing of poles or zeros by pasting around a circle. I cannot say that I understand this completely.

For example, a change of the real part of a connection, multiplicatively as in (36.a) or logarithmically, does not change the curvature in so far as it is determined by the logarithmic derivative of a function defined on the covering $\mathbf{C}$ of the curve M. We may do the same for the imaginary part, for which the curvature is more important. For a GL(1)-bundle over a curve, the relation of two local parameters z_1 and z_2 is given by an exponential function

$$\frac{z_1}{z_2} = \exp(c + di), \qquad c, d \in \mathbf{R}, \tag{36.j}$$

where c and d are real and d is locally single-valued up to a constant in $2\pi\mathbf{Z}$. The number c is uniquely determined. Simply expressed, in the exponent there is an additive imprecision. It lies in $2\pi\mathbf{Z}$.

A pause for reflection. The introduction of zeros and poles appears to be particular to the theory of linear bundles on curves. It is best to ask oneself what is actually happening. I acknowledge that on the whole, and in particular for GL(1), it is difficult for me to remember that a connection is given by a logarithmic derivative. It is that the discussion in the preceding section is valid also for two divisors with zeros and poles under the condition that the order of the zeros and poles is everywhere the same, rather for the quotient of two functions representing one and the same divisor. The previous discussion (36.j) was for one pole or one zero. I understand that it is fastidious. It is however one thing for an experienced geometer to consider the details, another for an elderly interloper. A reader may ask himself what is troubling me. The question is this, "why is the introduction of zeros and poles in a unitary bundle related to a complex bundle a well-defined operation?" The answer lies in the relation (36.j)! ■

What I propose in (36.i) amounts to this, to reproduce the sequence (36.a)–(36.j) for the purely imaginary part of a connection, rather than the purely real part. In this way the formulas for $\widetilde{\alpha}$, $\widetilde{\beta}_1$, $\widetilde{\beta}_2$, $\widetilde{\gamma}$ are sufficiently clear.

Unnecessary or premature explanations. None the less they are appropriate. This is rather a brief digression whose significance for this paper will not be clear until we come to the equation (53). It anticipates and partially coincides with the following explanations. As I already observed in another place in this text or as I observe below, the principal conclusion is a mutually single-valued mapping between two sets, and the elements of these sets are themselves sets, so that it is important to choose the simplest representatives. More than anything else, one of these sets is determined only after we have chosen some other defining parameters. This is particularly important for the Yang-Mills connections, for which it is necessary to choose the connection Q with Chern class 1 as well as a metric on M and on Q. I did this and came unwillingly—rather as a piece of good fortune than by good management—to a constant connection as above and then, as shown below, to the exponential functions (53), which we can compare with the conclusions in VII. The case $G = \mathrm{GL}(1)$ is of course special because the conjugacy classes have a single representative, but for GL(2), which is different, this is no longer the case. As we shall see in XI, a comparison is none the less possible.

In §IX we refer to the comments of [AB] about linear bundles, especially those of degree 0. For elliptic curves they are particularly simple. They are given by $\Lambda_0\Lambda_1^{-1}$, where Λ_0 is a given linear bundle of degree one, and Λ_1 a translation of it. We may also suppose, simply to be definite, that A_0 and Λ_0 are associated to 0 in L. We may also simply transfer the point 0 to $z \in \mathbf{C}$ in order to obtain Λ_1. The curvature is a product—this is the difference of the curvature of linear bundles, one of which is the transfer of the other. The connection associated with the product $\Lambda_0\Lambda_1^{-1}$ is the difference of two connections of the form (36.i), thus

$$\Big(2\widetilde{\alpha}(a_0 - a_1) + \widetilde{\beta}_1(b_0 - b_1)\Big)\, da + \Big(\widetilde{\beta}_2(a_0 - a_1) + 2\widetilde{\gamma}(b_0 - b_1)\Big)\, db. \tag{36.k}$$

The coefficients are now all constant, so that the curvature is equal to zero. The two-dimensional vector

$$\begin{pmatrix} a_0 - a_1 \\ b_0 - b_1 \end{pmatrix} \in \mathbf{R}^2 \tag{36.ℓ}$$

is arbitrary but

$$\begin{pmatrix} 2\widetilde{\alpha} & \widetilde{\beta}_1 \\ \widetilde{\beta}_2 & 2\widetilde{\gamma} \end{pmatrix} \tag{36.m}$$

is given. There is a formula for $\widetilde{\alpha}$, $\widetilde{\beta}_1$, $\widetilde{\beta}_2$, $\widetilde{\gamma}$ as functions ω_1, ω_2 because there exist formulas for η_i, $i = 1, 2$, [WW, pp. 445–446]. Nevertheless, I do not know how the determinant of this matrix behaves, for example, where it vanishes, but this does not seem to be an appropriate goal for the present paper. We return to this question later, but without a fully established answer to it. It is possible that it is not obvious to the reader that, choosing the metric on the fibre determined by $s(\cdot)$ and the connection determined by the equation (36.i), I for some time did not recognize that I had chosen the metric and the connection introduced in [AB, p. 560]. The search for a connection with constant curvature occupied my thoughts. At first I did not understand that constant curvature with the standard metric on $M = \mathbf{C}/L$ assumes, by definition, that the connection is a connection of Yang-Mills. The goal was simplicity. This simplicity renders the final comparison with the conclusion §VIII much simpler. I draw the attention of the reader to the circumstance that the constancy of the Yang-Mills connection in (36.k) is not that which appears later in (53).[37]

This last appears as a trivial linear bundle on M, the first on a bundle in which a zero and a pole have been introduced. It seems that the first is not related to our purposes. I mention it only in order to place our discussion in an appropriate light. There is a confusion of concepts, as seems an inevitable side of complex differential geometry, that it is perhaps useful to show, that a transition from complex connections to unitary connections on a unitary bundle or, more correctly, to an attached real line bundle signifies that otherwise embarrassing poles disappear.

On the other hand, I am not certain that my understanding of unitary connections is adequate. I would like to bring that here to the consideration of the reader. We are speaking of the transformation of locally meromorphic sections of a complex bundle into continuous sections of a real, that is a unitary, bundle that is related with it, that is determined by the imaginary part of its logarithm, thus its polar part. Locally we write a meromorphic function as $az^k \exp\bigl(\varphi(z)\bigr)$, $k \in \mathbf{Z}$, where $a = |a| \exp(i\eta) \neq 0$ where $\eta \in \mathbf{R}$ is a real constant, $\varphi(0) = 0$, $z = r\exp(i\theta)$, $r \geqslant 0$. The unitary connection is defined by this function and given as $ik\eta + i\,\mathrm{Im}\,\varphi(z)$.

This is sufficiently clear but the nature of a unitary connection that is so defined is difficult to discover. It is determined by a gluing as is the initial meromorphic bundle, but it is torn in the centre. The patches have a small central disc with outer patchings, whose local determination in the region immediately surrounding the patch is not difficult to picture. This, however, affects the displacement along the purely real numbers, in so far as the introduction of the (imaginary) logarithm demands substitution of the circle of an exponential function by the line of its logarithm. This occurs on the boundary of the small circle, but only on the boundary. In the centre of the circle some secret remains, where everything is torn apart.

There is still one peripheral remark related to bundles like $\Lambda = \Lambda_0\Lambda_1^{-1}$. We imagine that M is a sea and that the connection describes a flow. For example, the flow Λ is a combination of Λ_0 and Λ_1^{-1}. The first has a special property, it is a whirlpool; the second also has a special property, for it is also a whirlpool, but in the opposite sense. It is even possible that they have different strengths. If they have the same strength, then they will be mutually compensated as they move together. I have difficulty imagining this, both geometrically and physically, but the physical analogue makes it easier. I propose it to those mathematicians who like me have trouble understanding the complexity of bundles and connections. ■

[37]On the contrary they are constant with curvature equal to zero and a linear integral, as in (53). Thus they turn out to be Yang-Mills connections, that is they satisfy the equation of [AB, 6.1]. I offer this proposal, in order to make clear how much the theory presented or proposed in this article depends on details and how slowly I understood it. In passing I remark one more circumstance, whose significance I did not understand immediately. The bundle Q in [AB] coincides with the bundle A in [A], which in the present article became A_0. The degree of A_0^n, which is encountered in the description of connections of Yang-Mills type, corresponds to the coefficient $u \mapsto u^n$, $u \in \mathbf{U}(1)$ in the corresponding homomorphism from $\Gamma_{\mathbf{R}}$ to GL(1), [AB, (6.6)]. As a supplementary remark to the formalism, the linear bundle Q turns out to be a bundle of type Yang-Mills because the equation [AB, (6.1)] is additive with respect to tensor products.

But that which is important for us is that the curvature is constant and not equal to zero. The section determined by the function σ^{-1} has, apparently, no curvature, but this is not so. The curvature is hidden in the pole at the point 0. The comparison of eigen conjugacy classes with the integrals of Yang-Mills connections seems more important. These objects are determined by their representations, which are taken to some degree arbitrarily. It is difficult to explain the purpose of this article, the comparison of eigen conjugacy classes with Yang-Mills connections and with representations of the automorphic galoisian group. At least, the difficulty appears to be a consequence of two, even three, factors. The conjugacy class is described with the choice of a representative, which appears somewhat arbitrary. Moreover, the class changes from point to point; in addition the concept of a Yang-Mills connection depends on the choice of a metric on M. It is preferable to choose those that render the nature of the conclusions most clearly. I prefer to present now the consequences of an unsuccessful choice of the metric and the connection. There are, of course, two metrics on M and on the fibre.[38]

An extended digression. This digression is somewhat lengthy. This is because it is impossible to understand the nature of a connection without a clear understanding of the relation between connections and functions. They may be real-valued, purely imaginary or complex-valued. The theory for them is the same, although in the third case we are dealing with contour integrals for which the curvature is zero. It is easiest to think of real-valued functions.

The initial calculations[39] for the connections determined by (37.b) appear in (2.e). The domain—the fundamental domain is given by $\{\, a\omega_1 + b\omega_1 \mid -1 \leqslant a, b \leqslant 1 \,\}$. Since the set L does not overlap its boundary, we can calculate the integral of this connection along the boundary without supplementary information. For reasons demanding a lengthy explanation, which we postpone, only the imaginary part of the expression (37.a) is significant. According to (2.d) this is the imaginary part of $\zeta(z)\,\partial z$. The integral along the boundary is the sum of

$$\begin{aligned} \int_{(-\omega_1,-\omega_2)}^{\omega_1,-\omega_2} \zeta(z)\,dz + \int_{(\omega_1,\omega_2)}^{-\omega_1,\omega_2} \zeta(z)\,dz &= \int_{(-\omega_1,-\omega_2)}^{\omega_1,-\omega_2} \big(\zeta(z) - \zeta(z+2\omega_2)\big)\,dz \\ &= -\int_{(-\omega_1,-\omega_2)}^{\omega_1,-\omega_2} 2\eta_2\,dz = 4\omega_1\eta_2 \end{aligned} \tag{38.a}$$

[38]The reader will notice my anxiety as I attempt to acquire an unknown differential geometry. I once again draw his—or her—attention to one equation, whose particular form is important for this article. This is the Yang-Mills equation [AB, 6.1], thus $d_A \star F(A) = 0$, but for a linear bundle on a curve such that the bundle is $\mathbf{U}(1)$. Recall [AB, p. 548] that $F(A) \in \Omega^2\big(M, \mathrm{ad}(P)\big)$. Since the representation $\mathrm{ad}(\mathbf{U})$ is trivial $\Omega^2\big(M, \mathrm{ad}(P)\big) = \Omega^2(M)$ and $\star F(A)$ lies in a trivial bundle where the connection is trivial. Thus the connection that we just now constructed is a Yang-Mills connection. Although I still do not understand the many consequences of this concept, I assert that this remark is fundamental for this essay! We are dealing with a linear bundle, thus with an abelian group G. This is an anticipation of a general argument in [AB, p. 560] that is so important for this paper. Repeating myself, I underline that in our circumstances, together with a linear bundle and a uniform metric, a sign of a Yang-Mills equation is constant curvature. The value of the curvature determines the Yang-Mills class of the bundle.

[39]Returning to the paper some two years after having written the Russian version, I have made some small corrections to the labelling and the exposition, about both of which I begin to be uncertain. I have not tried to correct, in this sense, the Russian version, which has not attracted many readers.

and

$$\int_{(\omega_1,-\omega_2)}^{\omega_1,\omega_2}\zeta(z)\,dz+\int_{(-\omega_1,\omega_2)}^{-\omega_1,-\omega_2}\zeta(z)\,dz=\int_{(-\omega_1,\omega_2)}^{-\omega_1,-\omega_2}\big(\zeta(z)-\zeta(z+2\omega_1)\big)\,dz$$
$$=-\int_{(-\omega_1,\omega_2)}^{-\omega_1,-\omega_2}2\eta_1\,dz=-4\omega_2\eta_1. \tag{38.b}$$

Consequently this sum is equal to $4\omega_1\eta_2-4\omega_2\eta_1=-2\pi i$, as is explained in [WW], where this equation is justified by arguments from the theory of holomorphic functions of one variable. This calculation is clearly related with those, that lead to the expression (36.i) but their relation, at the first glance is both incomprehensible and unclear. First of all, I remark that according to (2.d) the function $\zeta(\cdot)$ is the logarithmic derivative of the function $\sigma(\cdot)$, so that the integral of the function $\zeta(\cdot)$ is given by the expression $\ln\sigma(\cdot)$, which is not single-valued.

However, we created a difficulty, mixing two different operations or two different concepts: the integral of a complex function of a complex variable and the curvature, that affects the integral of a real function of two real variables.

I recognize that, although I am elderly I have never sufficiently understood the concept of curvature. This article was necessarily an occasion to do so. Since it was written principally as a contribution to the theory of automorphic forms, I without hesitation added some fundamental concepts to my purview. The change of context was formal. I replaced the Lie algebra of $\mathbf{U}(1)$ by the Lie algebra $\mathbf{R}$ and the domain $\mathbf{C}$ by the domain $\mathbf{R}+\mathbf{R}$. This revealed to me what appeared to be the essence of the notion of curvature. A connection is a differential form $\alpha(x,y)\,dx+\beta(x,y)\,dy$. The question is how can

$$\int_{(u_1,v_1)}^{(u_2,v_2)}\big\{\alpha(x,y)\,dx+\beta(x,y)\,dy\big\} \tag{39}$$

not be independent of the path of integration. In other words, how can an integral over a closed curve C not be equal to zero.

Let, in this brief digression, ϵ be a small, even infinitesimal, number. We cover the plane with a lattice $\{m\epsilon,n\epsilon\}$, m, $n\in\mathbf{Z}$ and take the union X of those cells B that find themselves in the interior of the domain defined by a (simple) closed curve Y. This interior is (approximately) a union

$$\int_Y f(x,y)\,dx\,dy=\sum_{B\subset X}\int_B f(x,y)\,dx\,dy+O(\epsilon). \tag{39.a}$$

Let $f(x,y)=d\alpha/dy-d\beta/dx$. This will be the curvature. The boundary of each B is given by four ordered edges in counter-clockwise order $e_1(B)$, $f_2(B)$, $e_2(B)$, $f_1(B)$, where for example e_1 is given as $\big\{(a,b),(a+\epsilon,b)\big\}$. Leaving errors aside, we obtain from the sum in (39.a) the sum

$$\sum_{B\subset X}\left\{\int_{e_1(B)}\alpha\,dx+\int_{e_2(B)}\alpha\,dx+\int_{f_2(B)}\beta\,dy+\int_{f_1(B)}\beta\,dy\right\}.$$

This is equal to

$$
\begin{aligned}
&\sum_{B\subset X}\left\{\int_{\widetilde{e}_1(B)}\{\alpha(x,y)-\alpha(x,y+\epsilon)\}\,dx+\int_{\widetilde{f}_1(B)}\{-\beta(x,y)+\beta(x+\epsilon,y)\}\,dy\right\}\\
&=\sum_{B\subset X}\left\{-\epsilon\int_{\widetilde{e}_1(B)}\frac{d\alpha}{dy}\,dy+\epsilon\int_{\widetilde{f}_1}\frac{d\beta}{dx}\,dx\right\}=\sum_{B\subset X}\int_B\left\{\frac{d\beta}{dx}-\frac{d\alpha}{dy}\right\}dx\,dy \qquad (39.b)\\
&=\int_Y\left\{\frac{d\beta}{dx}-\frac{d\alpha}{dy}\right\}dx\,dy+O(\epsilon)
\end{aligned}
$$

where, in order to be clear, I introduced two axial ribs $\widetilde{e}_1=e_1$, $\widetilde{f}_1=-f_1$. This means that (39) does not depend on the curvature if the curvature is

$$\frac{d\beta}{dx}-\frac{d\alpha}{dy}=0. \qquad (39.c)$$

Expressing the left side simply, the difference of movement on opposite horizontal sides is of order $O(\epsilon^2)$. If the curvature is equal to zero, this difference is $O(\epsilon^3)$. This examination raises two questions. To what degree does the curvature depend on the metric and on the coordinates. I am not certain that these questions are important. At the moment they are not relevant. What is important is that the integral of the curvature over the interior of a simple closed curve is equal to the integral of the connection along the same curve. In any case the remark following the expression (36.i) is justified to some degree simply as an assertion taken from the article [AB].

We have still not introduced a metric on M. Consequently we are not able to speak of a Yang-Mills connection. Nevertheless we have already introduced an example of a metric with constant curvature. This is a favourable sign. I wanted to establish this with the help of a geometric argument and in such a way that it appeared as an assertion about curvature. But I was convinced that the point $0\in L$ did not contribute anything, and thus, neglecting this, I was confused more than once. I searched during several weeks for my mistake, which turned out to be rather evident. In the following pages, as help to the reader, I try to explain to myself the source of my mistakes, in themselves instructive.

I have referred many times to a lemma in [GH], which proposes the construction of a connection with this or that property, but missed the possibilities of an obvious fact, that the logarithmic derivative of a function that is nowhere zero and that may be positive, or with absolute value equal to one, or complex leads to a connection, even a connection related to the present discussion. This was not intended. It was the action of an innocent, stunned by the complexity of the subject. I recognized the possibilities that were offered, but only vaguely. I did not have the good sense to explain them clearly. This means that it is useful to understand the lemma in [GH] but not obligatory for our purposes. It is difficult to understand that I did not fully understand the assertion that zero curvature, apart from some subtleties, turns out to be both a necessary and a sufficient condition for a connection to determine a function.

At the same time I consider the inverse construction given in [GH, p. 72]. On the complex or on a Riemann surface, there are three forms of one-dimensional connections: one-dimensional real connection, purely imaginary one-dimensional connection, complex connection, that is for example, $f(\cdot)\,dz$ or $f(\cdot)\,d\bar{z}$. Each of these appears as a function on a variable one-dimensional complex space, thus on a tangent space. The domain of definition of the first two is one-dimensional but for the third it is two-dimensional. The construction in [GH] yields a connection of the third type. It is useful to reflect on it. Locally the connection prescribes a direction in a two-dimensional fibre. Thus locally or at a point it is given by four numbers, because it assigns to each of two vectors another vector, but the condition $D'' = \overline{\partial}$ ([GH], p. 73) reduces this number to two. A clever argument that uses their second condition, that is the condition of compatibility with the metric, but then determines the connection completely.

There is, however, a function that corresponds to our conditions. It is $\sigma^{-1} = \epsilon \exp(i\theta)$, where $\epsilon > 0$, although it is not single-valued. Its logarithmic derivative determines the connection. For this argument choice of metric on the fibres is important. I chose $f(\cdot) \to s^{-1/2}(\cdot)\big|f(\cdot)\big|$, which is not the one that we just now used, although it is similar. It has an essential inadequacy because it yields a metric that is not determined in points of the set L. This is not important presently, but it is the source of phenomena with which we shall have to deal. During the preceding explanation it was possible not to consider the points in L. They were not in essence particular. This is no longer so. I want to explain that with this connection the points in L evoke difficulties. The connection and the metric on the fibres remain to be defined. Thus the metric on the fibres must be changed. As a consequence the connection ceases to be a Yang-Mills connection, a concept that remains to be defined. But this is easy. That which is unpleasant is that in order to create a Yang-Mills connection it is necessary to introduce an artificial and unpleasant metric on the base M or an artificial connection. This is instructive for those who want to consider the general theory.

We recall the present circumstances. We consider functions $f(\cdot)$ on the complex plane periodic in relation to the lattice L that allow a first-order pole in points of this lattice. At first the metric on the fibre was $f(\cdot) \to \big|f(\cdot)\big|$ but this is unsatisfactory in L because of the possible poles. It is necessary to modify[40] it in a neighbourhood of L.

For this[41] we multiply $s(z)$ with a smooth periodic function $m(z) = m\big(|z-\lambda|\big)$, $\lambda \in L$, such that $m(z) = |z-\lambda|^2$ if $|z-\lambda|$, $\lambda \in L$, is very small, for example $|z-\lambda| < \delta/2$, that $m(z) > 0$ if $z \notin L$ and $\delta/2 < |z-\lambda| < \delta$ for some $\lambda \in L$, and that $m(z) = 1$ if $|z-\lambda| \geqslant \delta$ for all $\lambda \in L$. If $\lambda \in L$ it is better to suppose that $\lambda = 0$. This is a change of metric that yields a section with a pole whose order is equal to one and with bounded length in a neighbourhood of zero. With this construction σ^{-1} is a local section of the bundle with finite length. Another interpretation is that this is a local section after multiplication by

[40]It is necessary to insert here a supplementary remark. The function σ^{-1} determines a connection on a holomorphic bundle and, as we explain, there is a metric on this bundle. According to a lemma in [GH] the holomorphic structure together with the metric determine a connection. This connection is a Yang-Mills connection but only with respect to an artificial metric on M. This example is instructive.

[41]I attempt to use now not only the method but also the notation from [GH], which is customary but clever. For example, I distinguish ∂ and $d = \partial + \overline{\partial}$. None the less $d\sigma = \partial\sigma$! I observe also that $d\sigma$ is a complex differential that we can integrate along curves in $\mathbf{C}$, thus dz is paired with $1/dt$ and the curve is given by $z = z(t) \in \mathbf{C}$, $t \in \mathbf{R}$. Contemporary differential geometric language is smooth and comprehensive, but it often does not reveal the key question. I remark also that the operator $\partial/\partial z$ transforms a real function to a complex function. I repeat, 'complex differential geometry is foreign to me.' With the notation in [GH], which we otherwise do not use,

$$\theta = \frac{1}{s(z)}\frac{\partial s(z)}{\partial z}\,dz - \frac{d\sigma(z)}{\sigma(z)} = \frac{\partial s}{s(z)} - \frac{\sigma'(z)}{\sigma(z)}\,dz, \qquad \frac{\sigma'(z)}{\sigma(z)}\,dz = \frac{\partial\sigma}{\sigma(z)}, \tag{40}$$

because σ is holomorphic.

$\sigma(\cdot)$, but this is not necessary for us. It is first necessary to introduce in the construction of the section, in its infinitely small form, a supplementary additive term [GH, p. 73]

$$\frac{1}{m(z)}\frac{\partial m}{\partial z}\partial z = \frac{1}{m(z)}\partial m = \frac{\partial m}{m(z)}. \tag{41}$$

It is possible to calculate it independently. The given calculation remains correct in the domain in which the coefficient $m(\cdot)$ is equal to one, but outside this domain there is a supplementary additive term. What I did not initially understand was that the contribution of the expression (40) was zero and that the contribution (41) was the essential one. The purpose of the large digression of this section was to acquire some understanding of the concepts and calculations implicit in the concept of curvature and necessary for an understanding of the connection A. The principal theme is the relation between the integral of a connection along the boundary of a given domain and the integral of its curvature over the domain.

I want to show this with the help of geometric arguments in such a way that it is clearly an assertion about curvature. I was, however, convinced that the point $0 \in L$ contributed nothing and thus, ignoring it, I obtained the value 0 more than once. For weeks I searched for the mistake because such a mistake can easily be the result of an incorrect sign, namely the Cauchy-Riemann equations can contribute to the confusion.

Let[42] $\sigma = \epsilon e^{i\theta}$, $\epsilon \geqslant 0$, $\theta \in \mathbf{R}$. Then $\ln\sigma = \ln\epsilon + i\theta$ is holomorphic. Consequently

$$\frac{1}{\epsilon}\frac{\partial \epsilon}{\partial x} = \frac{\partial \ln\epsilon}{\partial x} = \frac{\partial\theta}{\partial y}; \quad \frac{1}{\epsilon}\frac{\partial \epsilon}{\partial y} = \frac{\partial \ln\epsilon}{\partial y} = -\frac{\partial\theta}{\partial x}.$$

In addition

$$\frac{\partial^2 \ln\epsilon}{\partial x^2} = \frac{\partial}{\partial x}\left\{\frac{\partial\theta}{\partial y}\right\} = \frac{\partial}{\partial y}\left\{\frac{\partial\theta}{\partial x}\right\} = -\frac{\partial}{\partial y}\left\{\frac{\partial\ln\epsilon}{\partial y}\right\} = -\frac{\partial^2\ln\epsilon}{\partial y^2}; \quad \frac{\partial^2\ln\epsilon}{\partial x^2} + \frac{\partial^2\ln\epsilon}{\partial y^2} = 0.$$

Thus $\ln\epsilon$—and also θ—are harmonic functions. I re-examine this fundamental relation partly because a long time has passed since I read the book of Weyl, but also because the sign is so important for the calculation of the curvature. It seems to me possible that I may have mistakenly arrived at $\partial^2\epsilon/\partial x^2 + \partial^2\epsilon/\partial y^2$ and not at $\partial^2\epsilon/\partial x^2 - \partial^2\epsilon/\partial y^2$. With this in mind, I searched for a long time for such a mistake but in vain. Finally, recognizing that the curvature lay in the behaviour at the singular point 0, I quickly understood the correct determination feeling finally that I was rather stupid.

Indeed, this article gave me an occasion to reflect on the concept of curvature, something that I had never done earlier. But before we pass to curvature, I would like to confess that there is a fundamental, even elementary, assertion whose significance I did not understand. This is the equation

$$\frac{d}{dy}\left\{\frac{df}{dx}\right\} = \frac{d}{dx}\left\{\frac{df}{dy}\right\} \tag{42}$$

for functions of two variables. Indeed, it is possible that as a result of the capriciousness of my education I accepted it without a proof. In any case, a one-dimensional unitary connection is given locally by the determination of the angular change φ, which changes the velocity with a linear movement and which is itself linear. Thus the change φ in the case of

[42] The sign of θ is modified.

a movement (dx, dy) is given by $\alpha\,dx + \beta\,dy$. It is, however, that this movement does not yield a function on integrating it because for such a function it would be necessary that

$$\frac{d\alpha}{dy} = \frac{d}{dy}\left\{\frac{df}{dx}\right\} = \frac{d}{dx}\left\{\frac{df}{dy}\right\} = \frac{d\beta}{dy}$$

but ordinarily for a given connection

$$\frac{d\alpha}{dy} - \frac{d\beta}{dx} \neq 0. \tag{42.a}$$

The left side of this inequality yields the definition of curvature. This simple inequality implies that the curvature is frequently not zero. But we needed a long sequence of reflections before arriving at (42.a).

Since the expression (42.a) is linear, we can calculate the contributions of the expressions (40) and (41) separately. Before we begin there is a necessary remark. As determined by us, each fibre of the bundle is identified with **C**. Thus, at an infinitesimal level, the movement of the connection is given by a single-valued determination of a sum of a real movement and an imaginary movement. The real movement is determined in a single-valued way by the metrical invariance. Its purpose is to maintain a constant length. Thus only the imaginary movement is pertinent.

It is possibly useful for us to continue the discussion of curvature with some brief explanations, before we begin the calculations necessary for the true purposes. My understanding will be more certain if it is reflected in simple calculations.

We begin with the imaginary part of the expression (40), that is with the imaginary part of the expression $\partial\sigma/\partial z$, which is equal to $\partial \ln\sigma = \zeta(z)\,dz$. Only the imaginary part of this expression is relevant:[43]

$$\partial \ln\sigma = \frac{1}{2}\left\{\frac{\partial \ln\epsilon}{\partial x} - i\frac{\partial \ln\epsilon}{\partial y}\right\}dz + \frac{i}{2}\left\{\frac{\partial\theta}{\partial x} - \frac{i}{2}\frac{\partial\theta}{\partial y}\right\}d\bar{z},$$

where θ is an angular variable. The imaginary part of this expression is $i = \sqrt{-1}$ times twice

$$\frac{1}{2}\frac{\partial \ln\epsilon}{\partial x}dy - \frac{1}{2}\frac{\partial \ln\epsilon}{\partial y}dx + \frac{1}{2}\frac{\partial\theta}{\partial x}dx - \frac{1}{2}\frac{\partial\theta}{\partial y}dy = \frac{1}{2}\left\{\left(\frac{\partial\theta}{\partial x} - \frac{\partial \ln\epsilon}{\partial y}\right)dx + \left(\frac{\partial \ln\epsilon}{\partial x} - \frac{\partial\theta}{\partial y}\right)dy\right\}.$$

Consequently the curvature is equal to half of the expression

$$-\frac{\partial^2\theta}{\partial y\,\partial x} + \frac{\partial^2\theta}{\partial x\,\partial y} + \frac{\partial^2 \ln\epsilon}{\partial x^2} + \frac{\partial^2 \ln\epsilon}{\partial y^2}. \tag{43}$$

It is clear that the sum of the first two terms is equal to zero; the sum of the second two terms is also equal to zero because of the Cauchy-Riemann equations. Although I am still

[43]In order to be confident that I understand the notation of the book [GH], I calculate

$$\begin{aligned} dz\cdot\frac{\partial}{\partial z} + d\bar{z}\cdot\frac{\partial}{\partial\bar{z}} &= \frac{1}{2}dz\cdot\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) + \frac{1}{2}d\bar{z}\cdot\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \\ &= \frac{1}{2}(dx + i\,dy)\cdot\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) + \frac{1}{2}(dx - i\,dy)\cdot\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \\ &= dx\cdot\frac{\partial}{\partial x} + dy\cdot\frac{\partial}{\partial y}. \end{aligned}$$

uneasy about my calculations, they at least yield a conclusion that is compatible with the accepted theory.

The last and the decisive calculation of curvature is the contribution of the variable r, which is given by (41). I want first, simply to improve my understanding, to describe the rotationally symmetric connection. It is given by the infinitesimal rotation and smooth even at $r = 0$

$$g^{-1}\,dg = \alpha(x, y)\,dx + \beta(x, y)\,dy.$$

I want this connection to be rotationally symmetric, thus invariant with respect to

$$(x, y) \to (\cos\varphi x - \sin\varphi y, \sin\varphi x + \cos\varphi y)$$

or

$$\alpha(x, y) \to \alpha(x, y)\cos\varphi x + \beta(x, y)\sin\varphi x, \quad \beta(x, y) \to -\alpha\sin\varphi x + \beta\cos\varphi x,$$

which is equal to

$$\begin{pmatrix}\alpha(x, y) & \beta(x, y)\end{pmatrix} \to \begin{pmatrix}\alpha(x, y) & \beta(x, y)\end{pmatrix}\begin{pmatrix}\cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi\end{pmatrix}.$$

Consequently

$$\begin{pmatrix}\alpha(x, y) & \beta(x, y)\end{pmatrix} = \begin{pmatrix}a & b\end{pmatrix}\begin{pmatrix}\cos\psi & \sin\psi \\ -\sin\psi & \cos\psi\end{pmatrix}, \tag{44}$$

where a and b are functions of r and $(x, y) = r(\cos\psi, \sin\psi)$. We calculate the curvature.

$$\begin{aligned}\frac{d\beta}{dx} &= \frac{da}{dr}\frac{dr}{dx}\sin\psi + a\cos\psi\frac{d\psi}{dx} + \frac{db}{dr}\frac{dr}{dx}\cos\psi - b\sin\psi\frac{d\psi}{dx}\\ &= \frac{da}{dr}\cos\psi\sin\psi - a\cos\psi\sin\psi/r + \frac{db}{dr}\cos^2\psi + b\sin^2\psi/r.\\ \frac{d\alpha}{dy} &= \frac{da}{dr}\frac{dr}{dy}\cos\psi - a\sin\psi\frac{d\psi}{dy} - \frac{db}{dr}\frac{dr}{dy}\sin\psi - b\cos\psi\frac{d\psi}{dy}\\ &= \frac{da}{dr}\sin\psi\cos\psi - a\sin\psi\cos\psi/r - \frac{db}{dr}\sin^2 y - b\cos^2\psi/r\end{aligned}$$

The difference is equal to

$$0 + 0 + \frac{db}{dr} + b/r = \frac{db}{dr} + b/r = \frac{1}{r}\frac{d(br)}{dr}. \tag{45}$$

Thus the curvature does not depend on the coefficient a.

According to the equations in [GH, p. 72], the supplementary factor $m(\cdot)$ introduces an additive term (41). The curvature is calculated as in (43), but the first two terms are absent and $\ln\epsilon$ is replaced by the function $\ln m(\cdot)$. But in contrast to the term $\ln\epsilon$, the term $\ln m(\cdot)$ is not a harmonic function. Disregarding for the moment the 1/2, implicit in (41), we calculate

$$\frac{\partial\ln m}{\partial x} = \frac{1}{m(x, y)}\frac{dm}{dr}\frac{\partial r}{\partial x} = \frac{1}{m(x, y)}\frac{x}{r}\frac{dm}{dr}, \quad \frac{\partial\ln m}{\partial y} = \frac{1}{m(x, y)}\frac{dm}{dr}\frac{\partial r}{\partial y} = \frac{1}{m(x, y)}\frac{y}{r}\frac{dm}{dr}$$

Continuing and differentiating another time, we find

$$\frac{\partial^2 \ln m}{\partial x^2} = -\frac{1}{m^2(x,y)}\left(\frac{x}{r}\frac{dm}{dr}\right)^2 + \frac{1}{m(x,y)}\frac{x^2}{r^2}\frac{d^2m}{dr^2} + \frac{1}{m(x,y)}\left\{\frac{1}{r} - \frac{x^2}{r^3}\right\}\frac{dm}{dr}$$

$$\frac{\partial^2 \ln m}{\partial y^2} = -\frac{1}{m^2(x,y)}\left(\frac{y}{r}\frac{dm}{dr}\right)^2 + \frac{1}{m(x,y)}\frac{y^2}{r^2}\frac{d^2m}{dr^2} + \frac{1}{m(x,y)}\left\{\frac{1}{r} - \frac{y^2}{r^3}\right\}\frac{dm}{dr}$$

The sum is equal to

$$-\frac{1}{m^2(x,y)}\left(\frac{dm}{dr}\right)^2 + \frac{1}{m(x,y)}\frac{d^2m}{dr^2} + \frac{1}{m(x,y)}\frac{1}{r}\frac{dm}{dr} \tag{46}$$

and the product of this expression with the function r and with the forgotten factor $1/2$ is equal to the derivative df/dr if

$$f = \frac{r}{2m}\frac{dm}{dr}. \tag{46.a}$$

The function $m(x,y) = m(r)$, $r = \sqrt{x^2+y^2}$. The value $f(0)$ is uniquely determined and equal to 2. If $z > \delta$ the value $f(z) = 1$. Consequently the integral of the curvature is

$$\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{\partial^2 \ln m}{\partial x^2} + \frac{\partial^2 \ln m}{\partial y^2}\right\}dx\,dy = \pi\int_0^{\infty}\frac{df}{dr} = -2\pi. \tag{47}$$

With this calculation we confirm that as far as the curvature is concerned our connection is the same as the connection in [AB, p. 560]. It is possible that there is a change of sign but that is not important.

There are still two secondary objects that it is necessary to explain, the first in order to compare the definition of curvature described above with the second definition, which is given in [AB], and the second, in order to explain in these simple contexts the concept of a Yang-Mills connection. Although this might be important in other cases, in particular if the dimension is greater than one, the concept of a Yang-Mills connection scarcely has an essential significance in the given case. Indeed, the purpose of our brief explanation is to make this clear. Nevertheless it is better to accustom oneself to the conditions of [AB, p. 560]. Indeed it is necessary to choose a metric on M relative to which the given connection is a Yang-Mills connection. It is this that I initially thought and that I explain now. It is instructive to explain the different aspects of this concept singly, as they appear.

We may now use the definition of this connection given in [AB]. The result is two definitions, that derived from (39.c) and the one found in [AB], but the two definitions are hardly different. We consider the one found in [AB] because the construction in [AB, §7] is so important for us. It is necessary to explain carefully concepts that we use. Indeed, to read [AB] was a pleasure but it was difficult to understand. The deductions were too easy, too casual yet difficult, not for the two authors but for me, who was unfamiliar with the theory of connections. Perhaps this is also true for the reader. If I do not add some calculations, I do not understand correctly. Perhaps this is also true of the reader. I, indeed, misunderstood several times. Thus I introduced the formula for $F(A)$ [AB, §3] for the simple case in which we are interested. The determination of $F(A)$ is important for the assertion [AB, Th. 6.7]. Thus it is important for us. Besides that it is related to the Yang-Mills theory.

It is possible that my explanation is not clear, but we pass from a metric to a unitary connection. We can now use the definition of the curvature of this connection given in [AB]. The result of the two definitions, that derived from (39.c) and that in [AB], are the same. However we consider that in [AB] because the construction in [AB, §7] is so important for us. It is necessary to explain carefully the concepts we use.

The calculations are local, the fibre one-dimensional, thus $\mathbf{U}(1)$ or $\mathbf{R}$ and the base an open set in $\mathbf{C}$ with coordinates x, y. The appropriate definitions are in [AB, §3]. It is sufficient to consider two tangent vector fields $X = \frac{d}{dx}$, $Y = \frac{d}{dy}$. The connection assigns to each vector field or direction on the basis a horizontal vector field in such a way that

$$
\begin{aligned}
X \mapsto \widetilde{X} &= \left\{\frac{d}{dx}, a(x,y)\frac{d}{d\theta_1}\right\}, \quad a(x,y) = \frac{\partial\theta}{\partial x}, \\
Y \mapsto \widetilde{Y} &= \left\{\frac{d}{dy}, b(x,y)\frac{d}{d\theta_1}\right\}, \quad b(x,y) = \frac{\partial\theta}{\partial y},
\end{aligned}
\tag{48}
$$

where θ_1, which is not connected with θ, is the coordinate on the fibre and $d/d\theta_1$ is only a constant tangent vector on the fibre. Moreover $a(x,y)$, $b(x,y)$ are essentially parameters that are determined by the function $\theta = \operatorname{Im}\log\sigma$. I repeat that $\widetilde{X}$, $\widetilde{Y}$ are displacements determined by the connection.

We begin the calculation now, but it is useful to explain the guiding principle because there is a sequence of particular circumstances that yield the conclusion. First the conditions of Cauchy-Riemann and then the various relations in the theory of Weierstrass. The consistency of mathematics is striking.

The second member in brackets yields the representation ω_A [AB, §3] and determines the curvature, which is given by

$$
\widetilde{X}\widetilde{Y} - \widetilde{Y}\widetilde{X}.
\tag{48.a}
$$

We may replace the pair $\{X, Y\}$ by the pair $\{fX, gY\}$ but as observed in [AB] it is easy to verify that $[fX_1, gY_1] = fg[X_1, Y_1]$ for all $\{X_1, Y_1\}$. Consequently it is sufficient to consider the chosen pair. A connection is determined by functions $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$. In contrast to the authors of [AB] I use the customary coordinates for $\mathbf{U}(1)$ and $\mathbf{U}(1)$.

Although this is a fastidiousness that the authors of [AB] did not need, I calculate first $F_A(X,Y)$ and then $\int_M F(A)$. We begin with the equation

$$
\begin{aligned}
\widetilde{X}\widetilde{Y} &= \frac{d^2}{dx\,dy} + \frac{db}{dx}\frac{d}{d\theta_1} + a(x,y)b(x,y)\frac{d^2}{d\theta_1^2}; \\
\widetilde{Y}\widetilde{X} &= \frac{d^2}{dy\,dx} + \frac{da}{dy}\frac{d}{d\theta_1} + b(x,y)a(x,y)\frac{d^2}{d\theta_1^2}.
\end{aligned}
$$

in which the terms

$$
a(x,y)\frac{d^2}{dy\,d\theta_1}, \quad b(x,y)\frac{d^2}{dx\,d\theta_1}
$$

are absent because $d/d\theta_1$ is independent of x and y. Indeed, regardless of the notation $d/d\theta_1$ is a vector in $\mathbf{U}(1)$—at least I hope so. Consequently

$$
\widetilde{X}\widetilde{Y} - \widetilde{Y}\widetilde{X} = \left\{\frac{db}{dx} - \frac{da}{dy}\right\}\frac{d}{d\theta_1}
$$

and the curvature is given by

$$F_A(X,Y) = \left\{\frac{db}{dx} - \frac{da}{dy}\right\} dx \wedge dy = \frac{db}{dx}\, dx \wedge dy + \frac{da}{dy}\, dy \wedge dx. \tag{48.b}$$

This is the notation of [AB], where the two-form $(X,Y) \to F_A(X,Y)$ is written as $F(A)$.

Allow me to add a word about the Yang-Mills theory. In order to determine a Yang-Mills connection or the Hodge star it is necessary to choose a metric on M. This metric (met) given, the Hodge star is determined as in [T] by the equation

$$v \wedge \star v' = (v, v')\, \mathrm{vol}_M, \tag{49}$$

where (v, v') is given by a metric and vol_M is a differential form. Our construction is such that it does to refer to a metric. Consequently it is free. It is, however, necessary to choose it correctly. The condition for this is given as the equation (6.1) in [AB, p. 559],

$$d_A \star F(A) = 0. \tag{49.a}$$

Thus the application of the Hodge star to the curvature $F(A)$ yields a section constant relative to dA. In [AB] there are the notations A or dA for a connection but the notation in [GH] is D. This curvature is given. Until now the metric was not significant but now it has to be determined such that this condition is satisfied. The Hodge star is given by the metric on M.[44]

The curve M is given as a factor space $\mathbf{C}/L$. We begin with the customary coordinates (x, y) on $\mathbf{C}$ and $\mathbf{C}/L$ and with the customary metric $\sqrt{x^2 + y^2}$. In the domain where $|z - \lambda| \geqslant \delta$ for all $\lambda \in L$ we do not change this. In this domain the curvature is equal to zero, $\star dx = dy$, $\star dy = -dx$, and (49.a) is valid. Where $|z - \lambda| < \delta/2$ the curvature is given by (43), which is equal to

$$-\frac{1}{r^4} \cdot 4r^2 + \frac{1}{r^2} \cdot 2 + \frac{1}{r^2} \cdot \frac{1}{r} \cdot 2r = 0.$$

Consequently we do not change on $\mathbf{C}$ or on $\mathbf{C}/L$ in this region. In the ring $\delta/2 \leqslant |\lambda| \leqslant \delta$, we modify the metric by the factor $g(\cdot)$ which is rotational symmetric, thus $g = g(r)$ and is constant inside and outside the ring and is smooth. I observe that we ignore the terms of the connection that are given by (40) because they contribute nothing to the curvature. I was a little careless earlier. More precisely the curvature $F(A)$ is not real, it is purely imaginary. But this is not important. The significance of (49.a) is that the sum of the movements d_A and $F(A)$ cancel each other with the multiplication $\star$, that is with the infinitesimal movement. This determines the volume, thus the area, in the ring $\delta/2 \leqslant |z| \leqslant \delta$. Outside the ring we multiply the area by a constant if necessary.

I observe that the Hodge star is defined in [T] only for particular bundles but the definition is more generally valid. A particular feature of (49) is that v, v' lie in the fibre but vol_M is related to the tangent space. Although I described at some length this example

[44] The following relations have of course a basic significance for the understanding of the Yang-Mills equations and, as they are presented, are related to some difficult concepts, but for the present we are dealing only with algebraic curves. Thus, $\star F$ is simply a real-valued function, so that a section is not an arbitrarily chosen bundle that is to be determined but a trivial line bundle. Consequently the equation implies that $\star F$ is constant, but we chose on an elliptic curve a constant metric with constant curvature, so that this is equivalent to the demand that F be constant. The metric is relevant for forms of degree one or even two, but not for the trivial bundle, where $\star F$ is defined. It is possible that in regard to this question I repeat myself or have repeated myself. But this was initially a fine distinction that was not clear to me.

of a Yang-Mills connection I am not certain this conception is important for a theory of automorphic forms in which the basic space is of complex dimension one. The previous section in which the initial assertion concerning the existence of a Yang-Mills connection is confirmed is preparation for the following one.

A difficulty arises but it is better to postpone its resolution until the point at which it becomes clear. Nevertheless we can describe it immediately. Below in the theorem I use the concept of a Yang-Mills connection, thus a metric has been chosen. Then I compare the consequences with the consequences of section VII, that is with the description of the eigen conjugacy classes. But such a class is given by one of its representatives. At random we choose an appropriate representative. On the other hand the choice of metric, thus the choice of connection, does not seem pertinent. Although we are saved by the general theory and the mastery by both authors of this theory, we can fortunately work in the context of a particular metric. It is better to say that our choice of a metric on the bundle is not good and that it was necessary to introduce some arbitrary corrections. This was clumsy, as we explain below. ■

Supplementary explanation and correction. This section is devoted to an explanation of the construction of the indispensable bundle $\mathbf{Q}$. In order to simplify my efforts, I introduced a suitable metric in a digression. But for us the principal assertion in [AB] is Th. 6.7, the theme of which is Yang-Mills connections. Their definition is such that the metric on M has a decisive influence on their properties. This is particularly so for elliptic curves, for which there exists a constant metric. This constancy is destroyed by the introduction of the point Q. The function σ^{-1} is a multiple-valued section of the bundle $\mathbf{Q}$.

I already explained this, but for me differential geometry is a labyrinth, that I entered because the similarity of the two theories, arithmetic and geometric, is so striking. Incidentally a question arises that I did not attempt to answer. Namely, in the present article complex differential geometry, complex curves, and the Yang-Mills theory related to them are discussed. I do not know whether the Yang-Mills theory for manifolds of higher dimension is relevant to the theory of automorphic forms. It seems unlikely, but I am not yet familiar with either the first or the second of these two theories, nor with another theory, namely differential geometry, nor with the Yang-Mills theory in higher dimensions. As an aside, it seems to me that in [AB] the principal particularity of the theory is its linearity.[45]

This linearity does not appear in our calculations, for which we introduce a somewhat arbitrary factor $m(\cdot)$ but without a change of the given connection. It manifested itself with a change of the connection. We shall make a better choice below. ■

IX. The Theorem of Atiyah-Bott[46]

The proof will be more important than the theorem, but we begin with the theorem. In order to state the theorem it will be necessary to introduce an appropriate $\Gamma_{\mathbf{R}}$-connection.

[45]This appears only with the accidental coincidence of the metrics on the fibre and on the curve M. One affects the construction taken in part from [GH], the other one's understanding of a Yang-Mills connection. The corresponding choice was made in this essay.

[46]It is possible that the reader has noticed that there is an essential improved understanding between this section and the following. This section is in need of a reworking, but I prefer leaving it as it stands for two reasons. The first is laziness and, perhaps more convincing, the second is that the possible reader may share my initial ignorance. There is also my conviction that, once provided with the following information, it will be easier to wait until the theory is, as a whole, better understood.

Then for each homomorphism[47] $\rho : \Gamma_{\mathbf{R}} \to G$ there is an induced G-connection A_ρ. The statement of the theorem is simple ([AB, Th. 6.7])

Theorem. *The transformation $\rho \to A_\rho$ determines a mutually single-valued mapping between classes of conjugate homomorphisms $\rho : \Gamma_{\mathbf{R}} \to G$ and classes of equivalent Yang-Mills connections on M.*

Apart from the explanatory digressions in the preceding chapter this is the first appearance of a connection and as a result—of differential geometry, in the argument of this article and the reason for the preceding explanations. I do not want to give a complete proof of this theorem, but complex differential geometry is a highly developed domain, whose understanding demands experience and understanding. It is easy to misunderstand the reflections of experienced geometers—and the authors of [AB] are certainly experienced—because of an inadequate understanding of essential ideas such as curvature. For me it was so, but many errors were corrected with the help of the classical theory of elliptic curves[48]

There are so many sides to the theory that I initially misunderstood, that I repeatedly examined what I had written. For example, neither the group G in the theorem nor $\Gamma_{\mathbf{R}}$ are necessarily closed. Nevertheless for the proof in [AB] it is assumed that the group G is connected. It seems to me that it might be useful to add some words about the general case. Indeed it seems to me that their proof entails a transfer to a group of lower dimension that may be disconnected. I explain the difficulty although for us it is not weighty. In [AB, p. 561] the authors suppose that $G_X = G$, but it is possible that G_X is disconnected even if G is connected. Consequently the move to the group $H \times S$ was hasty. However, as I explain below, this is not so, although there is a subtlety.

The bulk of the difficulty was a result of my inadequate understanding of the argument. The beginning phrase '$\widetilde{M} \to M$ is of course a flat $\pi_1(M)$-bundle' seemed to me to be empty words and the significance and truth of the phrase 'A is a Yang-Mills connection' seemed clear. The phrase 'of course' is a temptation for an inexperienced reader, who has not carefully thought about the words he is reading. As a consequence he has not fully understood them. These words are not empty. The fundamental group is defined by loops emerging from a point. Consequently they are well-defined locally but not globally. They are determined globally up to an isomorphism defined by the route from one point to another. Thus the definition of the fundamental group is delicate. Is it rather a bundle or also a bundle? In any case, as I slowly appreciated, this is a relation of a $\Gamma_{\mathbf{R}}$ connection to a G connection. This was a question of inexperience. The change is so smooth. In addition I did not adequately understand the simplest examples, for example the logarithmic function on $\mathbf{C}^\times \subset \mathbf{C}$ with the trivial bundle in which a pole is introduced at the point 0. It is important to understand that neither $\Gamma_{\mathbf{R}}$ nor G are necessarily connected.

[47]Indeed in this expression it is necessary to replace the group G by the group ${}^LG$, but if $G = \mathrm{GL}(n)$ then $G = {}^LG$. The authors of the paper [AB] are not, of course, familiar with the notion of L-group, but if $G = \mathrm{GL}(n)$ then $G = {}^LG$. There is an exact sequence

$$1 \longrightarrow \mathbf{Z} \longrightarrow \Gamma_{\mathbf{R}} \longrightarrow \mathbf{U}(1) \times \pi_1(M) \longrightarrow 1 \quad .$$

Since the groups $\Gamma_{\mathbf{R}}$ and $\mathbf{U}(1) \times \pi_1$ are not compact, the following construction is not completely evident.

[48]This paragraph and the first few of the following paragraphs were written as I was attempting to confirm a suspicion or a hope that the theorem of Atiyah-Bott was one of the keys to reciprocity in the geometric theory. I decided to keep them. They reflect my difficulties with [AB] from which the notation is taken.

In any case as soon as we understand the connection between M, $\widetilde{M}$, $\Gamma_{\mathbf{R}}$ and the bundle associated with them we understand how to pass from the homomorphism $\Gamma_{\mathbf{R}} \to G$ to the Yang-Mills G-bundle. It is necessary to confirm that in this way we obtain each G-bundle. It is clear from the definition that this construction gives a connection.

The following lines are meant for readers who like me are inexperienced geometers, but in order to understand the argument of [AB, p. 560/561] it is necessary to understand the constructions in their simplest form. For example, rather than inserting a pole or zero, we choose a constant point $n \in \mathbf{Z}$ and place, in the vicinity of the given point a simple function $f(re^{i\theta})$ with values in $\mathbf{U}(1)$, but glued to a function g given by the equation $f(re^{i\theta})\exp(in\theta) = g(re^{i\theta})$. If $n \neq 0$, the equation is valid only in a wedge. It is not possible that the 'function' or 'section' is defined in a complete neighbourhood of the point. This determination is suitable for the construction of the $\mathbf{U}(1)$-bundle on [AB, p. 560]. The construction of a $\pi_1(M)$-bundle from a representation of the group $\pi_1(M)$ is known. These remarks allow us to follow the construction on pp. 560–561.

The argument in the direction from homomorphism to connection is almost formal. From connection to homomorphism the first step is to use a definition, principally the constancy of the function $\star F$, that is the Yang-Mills condition, in order to determine $X \in \mathfrak{g}$, the Lie algebra.[49] Thanks to the conditions described in the footnote[50] the group $G_X = G$, introduced at the beginning of the page [AB, p. 562], is connected. This clearly is understood indistinctly in the discussion on that page. The existence of the point X and the group G_X is necessary for the description of the connection on this page. On that page there is also a resolution of the difficulty that troubled me at the time of my reflections on these questions. That is the Hecke eigenvalue in a given point is a class of conjugate elements that is given by determining one element of the class. Which one? There is still something that it is necessary to understand. But I would at first like consider the classification[51] [AB, (6.12)] for the group GL(2).

It seems to me that it is useful to describe now the possible forms of the group $\Gamma_{\mathbf{R}}$, at least for the irreducible unitary representations of GL(2). The group $\Gamma_{\mathbf{R}} = \mathbf{R} \times_{\mathbf{Z}} \Gamma$ is determined by A, B, $1 \in \mathbf{R}$ and $ABA^{-1}B^{-1} = J = 1 \in \mathbf{R}$, in which 1 is not the unit element. If $G = \mathrm{GL}(1)$, ${}^LG = \mathrm{GL}(1)$ and the representations $\rho : \Gamma_{\mathbf{R}} \to \mathrm{GL}(1)$ are easy to describe. They are given by the relations: $A \to \alpha \in \mathbf{U}(1)$, $B \to \beta \in \mathbf{U}(1)$, $x \in \mathbf{R}/\mathbf{Z}$, $x \to \chi(x) \in \mathbf{U}(1)$, where χ is a character of the group $\mathbf{R}$. If $G = \mathrm{GL}(2)$, ${}^LG = \mathrm{GL}(2)$, $\det\rho(J) = 1$, J central and there are two possibilities:

$$\rho(A) = \begin{pmatrix} \chi_1(A) & 0 \\ 0 & \chi_2(A) \end{pmatrix}, \quad \rho(B) = \begin{pmatrix} \chi_1(B) & 0 \\ 0 & \chi_2(B) \end{pmatrix}, \quad \rho(J) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{50.a}$$

[49] On the one hand we introduce the L-group, on the other we do not attempt to use or describe its formal properties, tied to the twisting and the broadening of poles. The groups GL(1) or GL(2) are in no sense twisted. The argument on [AB, pp. 561–562] is sometimes unclear so that I want my assumptions and intentions to be clear. (Added to this version: unfortunately, two years later they are not!)

[50] According to [K] A. Knapp, *Lie groups Beyond an Introduction,* Second Edition, Birkhauser, Boston, 2002, Corollary 4.51, thus "In a compact connected Lie group, the centralizer of a torus is connected."

[51] **Caution.** Topologists are not like the rest of us. What is clear to them is difficult for us. I would like to insert two propositions from [AB, p. 561] in order to be certain that I understand them properly. 'Any G-bundle P with connection induces a $\overline{G}$-bundle $\overline{P}$ with connection. Conversely if $\overline{P}$ lifts to P then P is unique and inherits a connection from that of $\overline{P}$. Thus, the group $\overline{G}$ is a group of quotients of the group G by a finite subgroup. Thus, if $\overline{g} \in \overline{G}$ let $G_{\overline{g}} = \{\, g \in G \mid g \to \overline{g} \,\}$.

where $\chi_1(\cdot)$, $\chi_2(\cdot)$ are two continuous unitary characters of the group $\mathbf{Z} \times \Gamma$, determined by the group of linear equivalent divisors:

$$\rho(A) = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad \rho(B) = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}, \quad \rho(J) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{50.b}$$

a, $b \in \mathbf{C}^\times$, $|a| = |b| = 1$. In addition it is necessary that the connection determines ρ on $\mathbf{R}$ and $\alpha = \det \rho$ is derived from ρ.

It is now necessary to recognize the basic confusion in my reflections on eigenfunctions and Hecke connections. The connection between the first and the second is indirectly determined by two groups; $\Gamma_{\mathbf{R}}$ and Γ_{aut}. It is not necessary to introduce intermediate connections. Distracted by my initial ignorance of connections, I lost all my time in attempts to understand them. But this was not a complete loss. We do understand arithmetic reciprocity in the broad sense and the close connection between the Yang-Mills theory and geometric reciprocity cannot be completely accidental, so the time was not lost. In addition to that, although I frequently forget the nature of the relation between these two sets, they are direct! The eigen conjugacy sections are determined as integrals of connections.

As I explained, I understand that the existence of Q is known to those familiar with the theory of bundles, but it was not so for me and I preferred, at least in part, a description in the context of the familiar Weierstrass theory. At the beginning I had difficulty understanding how a fibre space and a connection were given locally by a zeta-function as in (2.e). I also did not understand the necessary modifications of the metric on M and on the bundle. I did everything possible in order to present the necessary explanations in the preceding chapter, although, as the reader possibly noticed, I sometimes had difficulty absorbing the first paragraphs of the section [AB, §4]. So in this section we can consider the proof of the theorem of Atiyah-Bott. It would also be prudent for the reader to examine the paper [AB, p. 560], where the equation $d_A \star F = 0$ is important.

The mysterious statement or transition on page 561—this is an unclear and incautious supposition during the passage from line 5 to line 6 that $G_X = G$ is connected. This is only a careless mistake, but it will be necessary to examine a particular case and to describe the necessary changes.

At this point it is necessary to explain carefully the difference between the present conditions, that is the conditions of the following section **IX** and the conditions of [AB]. We want to describe the eigen conjugacy classes and eigenfunctions of the Hecke operators. This is a function on Bun_G, which has a finite number of connected components. Consequently it is necessary to calculate on a connected conjugacy class. This is reflected in the factor $\mathbf{Z}$ in the equation (1.d). On the other hand, from the connected family given by the theorem of Atiyah-Bott only those connections for which the integral is single-valued are relevant. This will be discussed in the following section. We consider now the argument of the paper [AB].

There is, however, still one difficulty. It will be necessary for us to compare the description of Atiyah-Bott, which presupposes a given class of Yang-Mills connections, that is a given metric on M, with our description of Hecke eigenfunctions. Rather, we have to convince ourselves that the description of Atiyah-Bott does not depend on the metric, thus on the unclear dependence on it of the concept of a Yang-Mills connection. It is also necessary to understand how it depends on the choice of Q. In the preceding paragraph I tried to explain this with examples. It seems to me that this suffices. Such fastidiousness is necessary because geometers have both experience and an ability to assimilate various realizations of the same concept but I lack this experience. But it is better that I first explain the proof of the Atiyah-Bott theorem and then consider the various realizations. For the reader it is undoubtedly better to leave them aside for the moment. As I explained, for now preliminary explanations suffice.

Before I continue the discussion of the Atiyah-Bott proof, I would like to confess that I did not understand correctly, at least not adequately, what I was trying to prove. I continue the introduction of two sets of parameters whose relation to each other is rather complicated. Both are given by representations of the group ${}^L G$, which in this article is usually GL(2), but GL(2), PGL(2) and SL(2) are similar. For the theorem of Atiyah-Bott this group is $\Gamma_{\mathbf{R}}$; for the Hecke theory this group is Γ_{aut}. Thus they are closely related. Their representations are parametrized by similar objects: for the Atiyah-Bott theorem this is the Yang-Mills connection, for the Hecke theory this is the group Γ_{aut}. They are therefore closely related. Their representations are parametrized by similar objects: for the theorem of Atiyah-Bott these are the Yang-Mills connections, for the Hecke theory these are the eigen conjugacy sections. It is necessary to suppose that this entails a connection between sets of parameters. The obvious construction is the integral of a connection, but this requires an initial value, thus a supplementary parameter. More precisely, an initial value is necessary on one connected component. This is also, as we shall see, not entirely correct.

My original goal was to create the beginning of a theory of the automorphic galoisian group in the context of the geometric theory. It seemed to me that it was to be found in [AB, Th. 6.7], but it turned out that not only was an understanding of curvature necessary but also of variational calculus, although in a modest form. It was also necessary to amend the theory in so far as this was appropriate in our context.

Our description of the Hecke eigenfunctions and eigen classes for an elliptic curve showed that on each component of the set Bun_G they are given by matrices of coefficients that are given by exponential functions whose exponents are linear, that is each component is given as $\mathbf{C}/L$ and the lifted exponent is linear. Thus they are given as the integral of a flat connection. That is, by a happy coincidence by a Yang-Mills connection. Linearity appears because in M a group structure is implicit.[52] Thus in general we expect a Jacobian or a discrete accumulation of products of such sets. But I have neither the time nor the courage to speculate on the general case.

It is extremely difficult to separate eigenfunctions, eigen points and eigenvalues. There is an arbitrary choice, unclear for the first, but not in the others.[53]

I acknowledge also that the question of parameters was somewhat troubling. From the point of view of Hecke operators, additive or linear parameters on M are similar, especially for Θ_2. From the point of view of the harmonic theory there are two possibilities: first metric, then the connection as in [AB]; first connection, then the metric as in this article, at least sometimes. They can lead to different conceptions of a harmonic connection. I chose the second but in [AB] the first is used. Thus the question of equivalence arises. I leave it aside. This question and other similar questions are implicit in [AB] where they are not discussed. It is necessary to suppose that the necessary proofs are not difficult. Indeed the question for them is so easy that I overlooked its explanation of, for example,

[52] It is not necessary to take this assertion seriously.

[53] Added in translation—this statement is unclear, but the idea is not for we are dealing with points in conjugacy classes. The first are somewhat arbitrary and the second variable.

the equation (6.10) $F(A) = X \otimes \omega$, $X \in \mathfrak{g}$. Here I allow myself a supposition. In the Hecke theory this constancy is an expression of the influence of the Jacobian and in the Atiyah-Bott theory it reflects the Yang-Mills condition.

A recollection of the structure of the set Bun_G**.** At first, some secondary questions arise, first of all, the eigenfunctions of the Hecke operators, which are functions on Bun_G and this space is not connected. Consequently they are not given everywhere by an integral. We chose, however, in §IV a linear bundle $\Lambda_0 = \Lambda_{A_0}$, which allows us to identify the set of irreducible bundles whose rank, that is dimension, is equal to r and which are of degree $d + rn$ with the set of bundles of rank r and degree $M \leftrightarrow M \otimes A_0^n$ [A, Th. 6]. For $d = 2$, the case we are considering, this is related to $\mathfrak{A}_{\text{even}}$, $\mathfrak{A}_{\text{odd}}$ in §5. When we consider the consequences of this, we understand that for the group GL(2) it is necessary first of all to distinguish two types of eigenfunctions, the carriers of which, strange as it is, have no comment elements. The first kind correspond to the direct sum of two linear bundles. The second kind are described in §VII. Our intention in this section is to explain the theorem of Atiyah-Bott.[54] However we can pass to a covering of the curve M. The combinatorics of [AB] demand, I suspect considerable attention. For example for $G = \mathrm{GL}(1)$ and M an elliptic curve, a one-dimensional representation is arbitrary on A and B, but necessarily equal to 1 on J in (6.6) and, consequently, on $\mathbf{Z}$, but this is not so for representations of larger dimensions. It seems to me that the authors explain this in (6.12) on p. 561, but it is possible that this is not immediately clear.

This entails not only a change of the assertion but also a distinction between $\Gamma_{\mathbf{R}}$ or Γ and Γ_{aut}. The second set is related to Bun_G for which there is a parameter, that is degree, which is related to multiplication by $\Lambda_0 = \Lambda_{A_0}$. This allows, for each eigenfunction f and each number $\alpha \in \mathbf{U}(1)$, the introduction of another function $N \mapsto f(N)\alpha^{\deg N}$. This is the factor $\mathbf{Z}$ in (1.d) which monitors α. This factor is absent in (1.a).

Leaving this consideration aside, there is a second difference between connections and eigen conjugacy classes, beside those that are obvious. First of all, the values of the first lie in the Lie algebra (of the group ${}^LG$!), but the values of the second lie in the Lie group of ${}^LG$, but if $G = \mathrm{GL}(2)$, $G = {}^LG$. Therefore for the comparison not only the integration but also the value.[55]

We return to this question in the following section, but it is first necessary to understand the argument in [AB, §6], thus the proof of Theorem 6.7 and the necessary changes. We begin with the equation (49.a) and its significance, not forgetting that we are not concerned with linear bundles alone, that is only with bundles whose dimension is one. It expresses the relation of the connection to the metric, on M and on the bundle. We already saw how to pass from the connection to the metric. The authors remark that this connection, whose group is $\Gamma_{\mathbf{R}}$ [AB, p. 596, (6.5)], is not compact.

[54]**Insignificant oversight.** In their proof there are two insignificant omissions, as is clear already from the example GL(2). But it is possible that I misunderstood the fourth line in [AB, p. 561].

[55]This is given by the values at A_0 of a character, determined on the supplementary factor $\mathbf{Z}$ in (1.d). Moreover, it is necessary that the integral, rather its conjugacy class, be single-valued. This is the reason that I introduce the groups Γ/Γ_n in (1.d).

We already[56] considered the equation $d_A \star F = 0$ as an equation from which F is determined but with a different purpose. I find the argument in [AB] marvellous. However, the more one thinks about this the more the concept in its simplicity becomes clear, at least for some hours. I suppose that for manifolds of higher dimension it is really complicated.

There is an important detail of the definition in [AB, p. 560/561]. In the beginning, [AB, p. 526], the group G is compact but not necessarily connected. This is important because $G_X = G$ [AB, p. 561], so that the first group is also not necessarily connected. However it is unexpectedly and inappropriately taken to be connected in the following section. It seems to me that in the case (50.b) G_X is necessarily connected. Thus in [AB] there is a small mistake. I believe that it is an essential mistake that it is possible to correct by passing to a finite covering of M that for (50.b) is also an elliptic curve. We shall return to this.

First of all $F \to \star F$ turns the curvature $F \in \Omega^2(M, \operatorname{ad}\mathfrak{g})$ into the function $\star F$ that is equivariant in relation to the action of G (more precisely of ${}^L G$). The authors come first [AB, p. 560] to the conclusion that $\star F = \star F(A)$ and that $A = A_\rho$ is constant on a horizontal curve. This allows them to transfer the group structure from G to G_X, the stabilizer of the point X in $\star F = \star F(A)$. The point X is an element chosen arbitrarily from the orbit of the points $\star F$. Let $P_X = \star F^{-1}\{X\}$. Then $P_X/G_X = M$. Consequently $F = F(A) = X \otimes \omega$ where ω is a volume. Since $X \in \mathfrak{g}$, it is possible that its exponential integral, whose value lies in G and which is determined by initial conditions, yields the eigen (conjugate) connection of the Hecke operator. We explain this in the following section but only for elliptic curves. It is probable the theory is in general similar. But I want to explain the proof on [AB, p. 561] for elliptic curves and the group GL(2).

At first we continue with the discussion of their proof, taking $G = G_X$. This is an important simplification, but it is necessary to understand first the statement, 'The Yang-Mills connection A_ρ defined by a homomorphism $\rho : \Gamma_{\mathbf{R}} \to G$ has curvature $X_\rho \otimes \omega$ where X_ρ is the element of the Lie algebra $\mathfrak{g}$ of G defined by $d\rho : \mathbf{R} \to \mathfrak{g}$.' I remark that in the preceding section the dimension of the bundle was equal to one. It is now arbitrary. Consequently the expression (48.b) is now a skew-symmetric form whose values lie in $\mathfrak{g}$. Therefore [AB, p. 561] $\rho(\Gamma_{\mathbf{R}})$ commutes with $X = X_\rho$ and $\rho : \Gamma_{\mathbf{R}} \to G_X = G$.

We are concerned with the theory in [AB] only in connection with automorphic forms, but it is useful to first describe the conclusions for the group GL(2) [AB, p. 561]. Let $H = \mathrm{GL}(1)$, $S = \mathrm{SL}(2)$, $G = \mathrm{GL}(2)$. There is a homomorphism $H \times S \to G$ with kernel $D = \{\pm 1\}$. Let $\overline{G} = G/D$, $\overline{H} = H/D$, $\overline{S} = S/D$. Thus $\overline{G} = \overline{H} \times \overline{S}$. Hidden in the discussion [AB, p. 611] is something similar to the relation of the discrete series to a compact Cartan subgroup. It appears in the form of a lifting of a connection and its integrals that reminds one of the general relation between Cartan subgroups and the classification of representations or of automorphic forms. However, from the point of view, taken in the article [AB], unclearly or unconsciously, this means a transfer to a double covering, that takes the group G to the group G_X. In other words, either $G = G_X$ or it is necessary to pass to a uniquely determined double covering by an elliptic curve before introducing the description that appears in [AB, p. 561]. This is the doubling that appears in [50.a/50.b]. In this way the

[56] From a geometric point of view our examples are very simple! It is useful to recall that the notation Hodge star is a transposition of a function and a two-form. Moreover it is possible to broaden its definition to all vector bundles.

possibility of three parameters for one and the same connection appears. This has to be verified.[57]

The relation of the description (50.a) with direct sums of one-dimensional connections is clear. In the contrary case there is a uniquely determined quadratic covering of the curve M on which G is replaced by its connected component. After this step we can return to the description [AB, (6.12)]. The image β in (6.12) is the projective image of the map ρ in (50.b). However at this stage it is necessary to return to the assumption appearing a few pages earlier, 'Conversely if $\overline{P}$ lifts to $P, \ldots$' I remark that I correct their slips only for the particular cases considered and not in general. That I leave to others. I add also that there are three possible quadratic coverings of M and all three are admissible.

The case (50.a) is of less interest. It is related to the continuous spectrum. It seems to me that it is now necessary to stop and think about what we want to show. Above all, we consider only the group $G = \mathrm{GL}(2)$ or groups closely related with it, SL(2) or PGL(2). My first rough suggestion was a correspondence between eigenvalues or eigen conjugacy sections of Hecke operators and Yang-Mills connections. However, the correspondence is not quite that, The first obvious reason is that connections are determined on connected sets, but eigen conjugate sections are determined on disconnected sets. The second reason is that their parameters are different. The correct assertion is as follows, that the first is obtained from the second in two steps, an integration from which we retain only some results and simple multiplicative relations of different sets. We explain this in the following two sections.

Before we begin the necessary explanations in the following limits, it is best to form a clear, although also approximate notion of the form of the parameters for the two sets, Bun_G and the sets of eigenfunctions of the Hecke operators. The component parts of the sets Bun_G are the sets $\mathbf{Z}$, $\mathbf{R}$, $\mathbf{U} = \mathbf{U}(1) = \mathbf{R}/\mathbf{Z}$. If $G = \mathrm{GL}(1)$, Bun_G is topologically $\mathbf{Z} \times \mathbf{U} \times \mathbf{U}$. If $G = \mathrm{GL}(2)$, Bun_G then approximately—the first component consists of pairs, in which the order is irrelevant, but I am not concerned with the transition to the particular: this is not relevant to a coarse description—the union of two sets—

$$\mathbf{Z} \times \mathbf{U} \times \mathbf{U} \times \mathbf{Z} \times \mathbf{U} \times \mathbf{U}$$

and

$$\mathbf{Z} \times \mathbf{U} \times \mathbf{U}.$$

Dually, for GL(2) the set of Hecke eigenfunctions, which we considered in §VII, is also the union of two sets, but their coarse description is

$$\mathbf{U} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{U} \times \mathbf{Z} \times \mathbf{Z}$$

and

$$\mathbf{U} \times \mathbf{Z} \times \mathbf{Z}.$$

These rough remarks will trouble the reader, but this is not the choice of the connections, as we studied them in the preceding sections, but the choice of integrals taken from them, which we introduce in the following section and which correspond to the Hecke functions.

[57]Indeed there are three. I was very confused for a very long time by the eigen sections of the form $\mathfrak{D}$ and their relations.

There is also a supplementary parameter, dual by the degree of a sheaf not directly related to the connection. It is the source of the factor $\mathbf{Z}$ in (1.d).[58]

The following suggestion is all that remains of a series of inappropriate pages. This is the imprint of the desperation that sometimes overcame me as I wrote this article. I confess that for me differential geometry is difficult because there are so many equivalences under changes of coordinates or metrics.

Reading these rough identifications, we begin to understand the connection between the two factors in (1.d). However, beginning the following chapter, I remarked that, absorbed by the particular case of an elliptic curve, I did not observe the particularities appearing for the projective line. They are worthy of a brief explanation. If the genus is zero, the definition on [AB, p. 559] is possible only if the empty product is equal to 1. Consequently

$$\prod_1^g [A_i, B_i] = 1,$$

but this 1 lies in the multiplicative group and for cohomological reasons it is glued to $0 \in \mathbf{R}$. Thus $\Gamma_{\mathbf{R}} = \mathbf{R}$. According to the equation [AB, 6.12], for each $n \in \mathbf{Z}$ there exists a single Yang-Mills connection of degree n and dimension one. Before we pass to the following chapter, it is perhaps worthwhile to describe them, if only to become more familiar with the concept of a connection. The projective line, thus $\mathbf{C}$ together with ∞ or $\{\,(z,1) \mid z \in \mathbf{C}\,\} \cup \{1,0\}$, is a factor space of the group $\mathbf{U}(2)$. Better, it is $\mathbf{C} \times \mathbf{C} - (0,0)$ modulo $(z_1, z_2) \to (z_1 z, z_2 z)$, $z \in \mathbf{C}^\times$ or $\{\,(z,1) \mid z \in \mathbf{C}\,\} \cup \{1,0\}$, thus $\mathbf{C}$ together with ∞. An invariant metric, thus invariant in relation to $\mathbf{U}(2)$, is given by the expression

$$\frac{dx^2 + dy^2}{(1 + x^2 + y^2)^2}, \tag{51}$$

$z = x + iy$, or its square root. The notation is not clear. The expression (dx, dy) is a tangent vector. We use this metric for the determination of the concept of a Yang-Mills metric. This choice of metric yields also the Hodge star. The customary volume is given in particular by

$$\frac{dx \wedge dy}{(1 + x^2 + y^2)^2}$$

and with this choice

$$\star dx = -\frac{dy}{(1 + x^2 + y^2)^2}, \quad \star dy = \frac{dx}{(1 + x^2 + y^2)^2}. \tag{51.a}$$

We use this metric in order to determine the metric Q on the Riemann sphere. The following calculations show that it is invariant under the action of the unitary group

$$z \to \frac{az + b}{cz + d}.$$

Thus in this case the fundamental group is trivial and $\widetilde{M} = M$. This case is of little interest.

Let

$$z_1 = \frac{az + b}{cz + d}, \quad dz_1 = \left\{ \frac{a}{cz + d} - \frac{(az + b)c}{(cz + d)^2} \right\} dz = \frac{ad - bc}{(cz + d)^2}\, dz, \tag{51.b}$$

[58]Added by the translator, thus by the author, two and more years after writing the paper. As I discovered during the translation, it is impossible to appreciate the sense of these words or even of the next few paragraphs without having read the paper to the end. Even then, it will not be easy!

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

is unitary. We verify that the transformation (1.b) preserves the metric and the volume, thus

$$\frac{dx^2+dy^2}{(1+x^2+y^2)^2} = \frac{dx_1^2+dy_1^2}{(1+x_1^2+y_1^2)^2} \tag{51.c}$$

and

$$\frac{dz\,d\overline{z}}{(1+x^2+y^2)^2} = \frac{dz_1\,d\overline{z}_1}{(1+x_1^2+y_1^2)^2},$$

but these two equations are not one and the same.

$$\begin{aligned} dz_1\,d\overline{z}_1 &= \frac{|ad-bc|^2}{|cz+d|^4}\,dz\,d\overline{z} = \frac{dz\,d\overline{z}}{|cz+d|^4}, \\ 1+x_1^2+y_1^2 = 1+\frac{|az+b|^2}{|cz+d|^2} &= \frac{|cz+d|^2+|az+b|^2}{|cz+d|^2} = \frac{1+|z|^2}{|cz+d|^2} \end{aligned} \tag{51.d}$$

because the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is unitary. Consequently the equation (51.c) is valid.

After this careful and possibly superfluous calculation, I would like to entertain some brief general reflections, because there are simple features of the theory of bundles, that perplex me. The authors of the article do not introduce the bundle Q when the genus is zero because for the projective line there is only one Yang-Mills connection, the trivial Yang-Mills connection. None the less, there is a bundle with a pole at ∞, even more than one. We consider them, but in relation to the coordinates r, θ, supposing that at infinity they have a simple pole, that is with flat coordinates they are given around infinity by the expression $f(1/z)z^{-1}$, $f(0)\neq 0$. The basic particularity of its behaviour is given by the imaginary part of the logarithm $\log 1/z$, that is by $-\theta$, which is not defined at $r=0$.

In other words, my earlier description from the point of view of vortices was unsatisfactory. Nothing particular, like a whirlpool, appears. We introduced a completely new subject into the theory, the behaviour of which close to, but not at, a given point is given by a whole number m which measures the change $2\pi m$ of θ as we move around the central point. It can be considered only as a new and different peculiarity, not literally but graphically or conceptually. This influences the neighbourhood of the point because this leads to a change of level as in a circular ladder. We do not return there where we started.

In the present case we may attempt to use the coordinates r, θ in the whole plane and consider the connection, moving in a circumference of unchanging radius relative to the point $r=0$. We meet difficulties and this is the heart of the matter, thus the reason that there are no interesting connections on the projective line. When we pass to $r=0$, we cannot close the connection. We clash at $r=0$ with the impossible choice of an angular movement because of the discontinuity. This is a topological limit. A complete angular movement around a closed curve cannot be modified without an abrupt change in the nature of the connection. For example, its curvature cannot remain zero. Rather, in order to compensate for some vague or impossible introduction of the curvature of the singularity, it is necessary to introduce yet something else. This might be a full passage to a covering, in which the closed curve ceases to be closed and the limitations imposed by the curvature disappear. I needed some time not only in order to compose the present paper but also to evaluate it. This decision was not possible (not relevant?) for the projective line.

Thus, for it at infinity it would have the form $a\theta + br$, where a is a non-zero constant, say 1. As a novice, I thought it was very important not to depart far from the connections determined by holomorphic connections, that is that the concept of a connection depended on the theory of holomorphic or meromorphic functions. This was thoughtless. Holomorphic connections are important, especially the imaginary or angular part of the logarithm has a large independent geometric interest and, on passage to the logarithm and the imaginary part it becomes simply a real connection. The curvature is determined by it. The initial reference to [GH] is, to some degree, a reflection of my earlier belief in the priority of the holomorphic theory.

What is the difference between the complex sphere and an elliptic curve. Why can we introduce a non-trivial connection on the second and not on the first. At present, perhaps, the question does not evoke much concern, but it is worthwhile to consider it briefly. On an elliptic curve we could take a simple pole in one point and extend it to a meromorphic function on the whole curve without any supplementary pole or zero. The function σ^{-1} is such a function. Then the imaginary part of the logarithm yields us a connection with one pole, a connection that we can then modify in such a fashion that the curvature is constant, but this last step was only for the sake of elegance. There were also earlier steps that were significant.

We paid a price for this extension, whose value was perhaps unclear with the function σ, namely it occurred to us to pass to a covering, to the complex plane, in order to be in a condition in which we could integrate obtaining a single-valued function. This is possible because an elliptic curve, unlike the sphere, is not simply-connected. It is easy to imagine the function obtained. An elliptic curve is covered by the plane and in such a way that it is a union of parallelograms and the integral of a connection is uniform, uniformly increasing and decreasing from parallelogram to parallelogram, but at the same time remaining continuous. For the Riemann sphere, which is simply connected there is no covering to which one can pass. The proximity of any point, for example, the point at infinity, slowly closes up. There is no escape. This structure is immediately visible on the plane that covers an elliptic curve. For curves of higher genus my intuition is inadequate.

But why do we not allow ourselves two poles? This is a good question. All that I can reply is that I was strongly influenced by [AB] and its introduction of Q. The readers have to answer this question for themselves. This may be significant for the development of the Hecke theory.

In the present article we, even hypothetically, do not consider ramified eigenfunctions of the Hecke operators. They have to appear as unramified eigenfunctions on a ramified covering. This is the principle of functoriality. Thus, if a general unramified theory exists and if functoriality is valid in the context of geometric automorphic forms the ramified theory may be investigated by means of ramified covers resulting from a multiplicity in the base. In contrast to the arithmetic case, coverings of curves are relatively accessible, or so I would believe! ■

Each Yang-Mills connection can choose an eigen section and an eigen metric.[59] Thus, it is possible to propose a connection only if the bundle is considered but the following choice of metric on the bundle may be accommodated to various goals, for example, so that in relation to it the given connection is a Yang-Mills connection. We have already met this earlier. I also remark that for us Yang-Mills connections are only intermediate concepts. The final correspondence is a correspondence of conjugate sections and representations of the group Γ_{aut}.

I add a few more words about the Riemann sphere. Now for each $k \in \mathbf{Z}$ we take a trivial bundle with a pole of arbitrary degree k at infinity. If $k < 0$ this is not a pole but a zero. For the present example $G = \mathrm{GL}(1)$. If $n = 0$ the appropriate Yang-Mills connection is trivial. Consequently, it is flat with respect to any metric. As earlier, I choose a rotationally invariant metric, $d\theta^2 + m(r)\,dr^2$. However we now consider planes, implicitly adding ∞ and not $\mathbf{C}/L$. A connection is given as in (41) and there is no supplementary term. Thus the calculation leading from (41) to (47) may be repeated, although there is a change of direction and, consequently, it is difficult to take into account the change of sign and orientation. The result is $2\pi k$ if $m(z) = |z|^{2k}$ in a neighbourhood of the point ∞ and $m(e^{i\theta}z) = m(z) > 0$, $z \in \mathbf{C}$. If we want to obtain a Yang-Mills connection the last step is to modify the metric on the projective line. As in the earlier example it remains free.

So much arbitrariness is frightening. However, I want to underline that in the end the connection appears to be only intermediate. The comparison will be between eigen conjugacy classes and homomorphisms $\Gamma_{\mathrm{aut}} \to {}^LG$. Allow me to attempt to understand the heart of the matter. Above all, a determination of a Yang-Mills connection presupposes that a metric is given both on M and on the bundle. Then it is possible to assign to each homomorphism ρ of the group Γ_{aut} to ${}^LG$ a homomorphism of $\varprojlim \Gamma/\Gamma_{n(k)}$ to ${}^LG$ and therefore also a homomorphism of Γ to ${}^LG$. This last homomorphism yields according to [AB, Th. 6.7] a Yang-Mills connection. Choosing a suitable initial value, we can integrate this connection. The result is a function whose values lie in ${}^LG$. Undoubtedly this function depends on the choice of metric, that on M and that on the bundle. However the function itself is not significant. What is important is only the conjugacy class, which varies from point to point. I did not verify that the class, at a given point, provided by the possibilities of this paper, does not depend on the two metrics used in its determination. I had neither the strength nor the time.

Before I return to the application of the theory in [AB], I want to discuss briefly the case that M is given by a projective line, which was not considered by us because it was too simple, thanks to the theorem of Grothendieck. It is better, however, to consider first and briefly the assertions in [AB, Th. 6.7], from which curves of genus zero are excluded, but only to assure ourselves that we understand them. In the proof both $\mathbf{U}(1)$ and $M(1)$ appear and both influence the connections that yield the theory. For our goals the influence of the first is not important. This means that we are dealing with one-dimensional representations, arbitrary on A and B and very simple on the inverse image of the group $\mathbf{U}(1)$.

On the preceding pages there was only a brief attempt to familiarize ourselves with the theorem [AB, Th. 6.7] and its consequences. There is still, however, one question that it is easy to consider. It is possible that the restriction of $\rho : \Gamma_{\mathbf{R}} \to G$ to $\mathbf{R}$ is such that $\rho(J) \neq 1$. It is frequently a root of unity.

[59] We suppose that in this assertion there is a study of equivalence classes of connections together with metrics that I have not undertaken!

X. Automorphic galoisian group and integrable connections

It is not the Yang-Mills connections themselves and their defining integrals that correspond to the homomorphisms of the automorphic galoisian group to ${}^LG$. Initial conditions for the integrals lie in the unitary form of the group ${}^LG$, which for us is usually the group GL(2) and only incidentally GL(1) or a modification of the group GL(2) such as SL(2) and PGL(2). Both the connections and the integrals are restricted to connected sets. These may be bundles whose degree is zero.[60] Consequently it is necessary to begin with them but it is appropriate to explain first the next step. The bundle Λ_0 was introduced but §IV and a tensor product $N \to \Lambda_0 \otimes N$ yields isomorphisms $\mathfrak{D}(m,n) \to \mathfrak{D}(m+1,n+1)$, $\mathfrak{A}(m,m) \to \mathfrak{A}(m+1,m+1)$, $\mathfrak{A}(m+1,m) \to \mathfrak{A}(m+2,m+1)$. The group Γ_{aut} is given in (1.d) as a product. Consequently an arbitrary irreducible representation of this group is given as a product of representations of the group $\mathbf{Z}$: $1 \in \mathbf{Z} \to \epsilon \in \mathbf{C}$, $|\epsilon| = 1$ and of the group $\varinjlim \Gamma/\Gamma_{n(k)}$. There is no sense in considering the same eigenfunctions with support in $\mathfrak{D}$. They correspond to unordered direct sums of two one-dimensional representations. For one-dimensional representations it is possible to replace $\varinjlim_n \Gamma/\Gamma_{n(k)}$ by the group $\varinjlim \mathbf{Z}/n\mathbf{Z} \times \varinjlim \mathbf{Z}/n\mathbf{Z}$. This yields what is necessary. Apart from the parameter ϵ and the function $n \in \mathbf{Z} \to \epsilon^n$, this yields an exponential function $\exp(ak + b\ell)$, k, $\ell \in \mathbf{Z}$ on M with parameters a, b given in (36.e), which yield a full orthonormal system on Bun_G if $G = \mathrm{GL}(1)$. Recall that $\mathrm{Bun}_G \simeq \mathbf{Z} \times M$, although this identification is somewhat arbitrary. I remark that a general identification of such a sort for all groups entails a structure on a full set of eigenvalues, considered for all groups simultaneously. This does not appear to lack interest. I remark also that the purpose of the groups G (${}^LG!$) in [AB, Th. 6.7] differs somewhat from that which is appropriate here. The exponential function $\exp(ak + b\ell)$ is a definite integral of the connection $(a,b) \to ak + b\ell$. It is easy to forget, although it is not appropriate to forget, that in §III and §IV there is an arbitrary but inevitable choice of the fixed point $A_0 \in M$ and on the trivial linear bundle.

Exponential functions with linear exponents appear naturally in our discussion of Hecke eigenvalues, whose definition presupposes a metric neither on M nor on the fibres of an arbitrary bundle. On the other hand, both appear in the determination of a Yang-Mills connection, for which the metric on M is particularly important. It appears in the determination of Q in the definition on p. 560 of [AB] and in the construction, in the previous section, of the connection ω on Q of a particularly simple form, that for which the curvature is constant. It is now proposed to the reader to study [AB, p. 560] taking as we do, thanks to the calculations of the preceding paragraph, the metric on M that is derived from the metric on $\mathbf{C}$, itself invariant with respect to translations, in such a manner that $\star$, $F = F(A)$ and ω are all constant.[61] Consequently the connection mentioned above is also

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[60] We have not yet revealed our principal goal, the description of eigen conjugacy classes. These are functions on M, the values of which are conjugacy classes. They are given by functions of points in M, which in principle are indistinguishable. These functions are determined by integrals with initial conditions. Since the function, at least, the conjugacy class determined by an integration and initial conditions, must be single-valued, the choice of admissible initial conditions may be extremely small. For a bundle whose dimension is larger than one, the concept of an initial condition is subtle. The continuous parameter appears as a parameter, dual to $z \in \mathbf{Z}$, which is the parameter of the connected components. One point Λ_0^0 is the trivial bundle, independent of Λ_0.

[61] I underline that in the previous section it was not simple to construct on the bundle Q a connection with constant curvature. We use it now. Neither in the particular case of an elliptic curve, nor in general, do I try to understand how the curvature changes with the connection. On the whole, it is still difficult for me

constant and, as above, the exponent of its integral is an exponential function with linear exponent. It is namely this that allows a comparison of the results of the two sections VII and IX.

A Guess. This is the suggestion "... we can restrict ourselves... to the case $G_X = G$." on [AB, p. 561]. This is the guess that can be assessed easily.

The difficulty for the curve M, the case with which we are dealing is that the presence of A and B in the Poincaré group are forgotten. This case must be typical. We may pass to a finite covering for which the description in [AB] is correct. I add that this case is for us the most interesting. It is better to consider GL(1) before GL(2). Then (50.a) is replaced by

$$\rho(A) = \big(\chi(A)\big), \quad \rho(B) = \big(\chi(B)\big), \quad \rho(J) = \big(1\big), \tag{52}$$

and (50.b) does not appear. The supplementary parameters $\chi(x)$, $x \in \mathbf{R}/\mathbf{Z}$, $\chi(x) \in \mathbf{U}(1)$ replace $X \in \mathbf{R}$, $x \in \mathbf{R}$, because the first is an exponential integral of the function iX, thus $\exp(ixX)$. There are still many functions of the second order. Only those parameters of the second kind, the integrals of which give single-valued functions on M are relevant.

For the group GL(1) the eigenfunctions (thus variable conjugacy classes) are given by the characters χ of the Picard group, which give, above all, their values $\chi(A_0)$ at the distinguished point A_0, thus by a character of the group $\mathbf{Z}$ in (1.d). Besides that, a character of the subgroup given by the divisors of degree zero, thus points of the group $M = \mathbf{C}/L$ itself. Such characters are given by the logarithmic integral of linear functions.

For eigen classes of the form $\mathfrak{D}$, the relation between eigen conjugacy classes and eigenfunctions is explained by the formulas (30) and (30.a) together with the intervening explanations.

Confession and uneasiness. We begin to apply the conclusions of [AB, Th. 6.7] I learned about the Yang-Mills equations as I wrote this paper, because this is important for the understanding of its ideas, but my experience of working with it and with differential geometry is limited. Now we attempt to understand its simplest properties. For $G = \mathrm{GL}(1)$ homomorphisms from the group $\Gamma_{\mathbf{R}}$, [AB, 6.6], to $\mathbf{U}(1)$ decompose as a product $\mathbf{U}(1) \to \mathbf{U}(1)$ and $\pi_1(M) \to \mathbf{U}(1)$. As a simple but important instance we consider the argument on p. 560 in [AB] with the group $G = \mathrm{Lie}\,\mathbf{U}(1)$, $\mathrm{Lie}\, G = \mathfrak{g} = \mathbf{R}$. Then $X \in \mathbf{R}$ is constant. None the less, there is an important question that, because it was clear for them, the authors did not explain clearly, but it is somewhat difficult for me to explain. I am accustomed to deal with reductive algebraic groups and their compact forms, for example $\mathbf{U}(1)$, but there is no such limit on the group G in the formula $\rho : \Gamma_{\mathbf{R}} \to G$ on [AB, p. 560]. It is possible to allow $G = \mathbf{R}$. The two groups have one and the same Lie algebra. It is only that the second choice proposes more choices. More precisely, together with the theorem [AB, Th. 6.7] this guarantees that the integrand in the exponents of the equation (53) represents a Yang-Mills connection, but not without further discussion. The possibility of taking $G = \mathbf{R}$ is already mentioned in the footnote 'Insignificant oversight.'

The following equation (53) is the first step in our comparison of Yang-Mills connections, on one hand, and Hecke eigenfunctions, on the other, but this is a delicate comparison. The eigenvalues are given as functions of a parameter in Bun_G with values in ${}^LG$, which for us is

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to provide myself with a concept of connection, although I have convinced myself of its importance for the geometrical theory. I add here, because it is necessary to add it somewhere, that with the metric that we chose a Yang-Mills connection on a bundle of degree 0 is constant. The relevant exposition demands much more experience than I have and many supplementary pages.

either GL(1) or GL(2), but this parameter is only a conjugacy class of semi-simple elements in the group. Thus there is a great deal of ambiguity in the choice of representatives. On the other hand, The concept of a Yang-Mills connection requires a metric both on M and on the fibres. Fortunately, for an elliptic curve there is a very small class of natural metric on $M \subset \mathrm{Pic}(M)$ and they are invariant with respect to translations. Our choice is somewhat arbitrary. Moreover, this concept demands a choice of a linear bundle Q and a connection on it. Moreover in order to establish a bijection between Yang-Mills connections and homomorphisms $\rho : \Gamma_{\mathbf{R}} \to G$ [AB, p. 560] it is necessary to choose a metric on a given linear bundle Q, as well as $\rho : \Gamma_{\mathbf{R}} \to G$. Then the correspondence depends on all these parameters, perhaps weakly, perhaps strongly but it is necessary that in the end we arrive at a clearly determined bijection between eigen conjugacy sections for a given group and homomorphisms of the automorphic galoisian group to ${}^LG$. I observe in particular that we must understand the dependence on Q. The given choice, leading to a constant curvature was made before I recognized its importance, at least for this paper. It is easy to forget the large part of arbitrariness in the constructions of the correspondences of this paper. ■

Rather than to begin directly with the case GL(2), with which we are occupied, we study the group GL(1), for which the conclusions are simpler and the fastidiousness less necessary. First of all, the eigenvalue, thus the conjugacy class, is pointwise a single number. In addition, the section is unambiguous, thus a function and these functions are easily calculated. They are characters of the Picard group. This group is the product of the group $\{\, \Lambda_0^n \mid n \in \mathbf{Z} \,\} \simeq \mathbf{Z}$, which is the first factor in (1.d) with the group of divisors $A - A_0$, $A \in M$, thus with the group $M = \mathbf{C}/L$ itself. The group of continuous characters of this group is given by pairs of whole numbers k, $\ell \in \mathbf{Z}$, as in (53) below. For our purposes it is necessary to transform these characters into pairs of complex numbers of absolute value one, in order that we can attach[62] to each homomorphism from $\mathbf{C}/L$ to $\mathbf{U}$ a character of the group $\varprojlim \Gamma/\Gamma_{n(k)}$. If $k = \ell = 0$, this homomorphism is equal to the trivial homomorphism $A \to 1$, $B \to 1$. The vector (k, ℓ) is, of course, subject to a unimodular transformation because the basis of the lattice L is not uniquely determined. Thus if the parameter is not equal to 0, we may suppose that after a unimodular transformation it has the form $(k', 0)$, $k' \neq 0$. Although we are dealing with something very simple,[63] the best of all is that it presents itself in full clarity. Otherwise there would be confusion, at least in my brain. The linear function $k'a'$ determined by the new coordinates (a', b'), thus by those which are determined by the new basis as (a, b) in (36.e) by the initial basis $(2\omega_1, 2\omega_2)$, is clearly defined and does not depend on the choice of the modified coordinates. It remains for us to explain the possible choices.

$$\begin{pmatrix} a & b \end{pmatrix} = \begin{pmatrix} a' & b' \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad \alpha, \beta, \gamma, \delta \in \mathbf{Z},$$

[62]This transformation has a strange form, a part of the numerator becomes the denominator, which puzzles me and may puzzle the reader. Thus, although this is a simple question, I prefer to describe it in detail in order to convince myself, if not the reader. There are two incompatible notations, that in [A] and that in [AB]. The meaning of the symbol [A] changes. Here we suddenly change the notation from the first to the second.

[63]but very important!

where $\alpha\delta - \beta\gamma = 1$, $a' = k'$, $b' = 0$. The matrix is determined up to the factor

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \qquad x \in \mathbf{Z}.$$

A change of basis entails a change of generators,

$$A \to A' = A^\alpha B^\beta, \quad B \to B' = A^\gamma B^\delta.$$

In so far as we are trying to determine a one-dimensional representation of the second factor of the group Γ_{aut}, we in essence are dealing with the generators A' and B' of an abelian group and a representation $A' \to \exp(2\pi i/k)$, $B' \to 1$. Speaking frankly, this action troubles me.

However we can return and express it as a linear function of the initial coordinates. It also determines a one-dimensional representation of the group $\varprojlim \Gamma/\Gamma_{n(k)}$ in (1.d), $A' \to \exp(2\pi i/k')$, $B' \to 1$. In so far as we are dealing with a one-dimensional representation, we may also determine this representation as $A \to \exp(2\pi i a/k')$ and $B \to \exp(2\pi i b/k')$ simply by an expression of A' and B' modulo the group generated by the commutators. This possibility has the strange peculiarity that a part of the numerator becomes the denominator, which troubles me and may trouble the reader.[64]

I hope that it is now obvious that what remains to explain for the group GL(1) appears to be the parametrisation of the restrictions of conjugate eigen connections to the set $\{ A - A_0 \mid A \in M \}$, whose connected component is $\mathrm{Bun}_{\mathrm{GL}(1)}$. For myself, just as for the reader, I observe that for linear bundles most questions, perhaps even all, can lead to questions for bundles of degree 0, by taking the tensor product with Λ_0. This is not so for bundles of a larger dimension.

In the following equation (53) the exponent is an integral of a connection on a curve from z_0 to z,

$$\{z_0, z\} \mapsto \exp\left\{4\pi i \int_0^1 (ka + \ell b)\, d\theta\right\} = \exp\bigl\{4\pi i(ka + \ell b)\bigr\}, \tag{53}$$

where a, b are coordinates as in (36.e) of the point $z - z_0 \sim A - A_0 = AA_0^{-1}$, k, $\ell \in \mathbf{Z}$ and $0 \leqslant \theta \leqslant 1$. It is easy, at least was easy for me, to overlook the determination of $\pi_1(M)$ in the definition $\Gamma_{\mathbf{R}}$ or the difference between $G = \mathbf{U}$ and $G = \mathbf{R}^\times$ in the relation $\rho : \Gamma_{\mathbf{R}} \to G$ on [AB, p. 560]. If $G = \mathbf{R}$, then the whole numbers k and ℓ are determined by the restriction of ρ to π_M, thus on its inverse image. It is not appropriate to forget that for the complete determination of the Hecke eigenvalues we must show in full the eigenfunction and, therefore determine its value at A_0. This is given by the image of $1 \in \mathbf{Z}$, an element of the first factor in (1.d).

The whole numbers k, ℓ are freely chosen. The purpose of this condition, thus that k, $\ell \in \mathbf{Z}$, is that the integrand in (53) be a single-valued function of the variable $z \in \mathbf{C}/L$. The simple form of the integrated function is a consequence of the circumstance that the connection that appears in the integrand is a Yang-Mills connection. These connections themselves are consequences of the carefully chosen metric on the base and on the bundle Q. I explain! The choice of metric on M was not complicated but the choice of metric on the fibres was inspired without a clear goal. The subsequent choice of a connection was a

[64] On translating these statements, whose validity I do not doubt, from the Russian to the English, their obviousness is difficult to recapture.

reasoned guess. I was already convinced or understood that Yang-Mills connections are frequently constant.[65]

Recall that the integral (53) is definite. For us, what is important is that (53) determines a character, that is a representation of the second group in the product (1.d). At any time it is possible to replace the second factor in $\mathbf{Z} \times \varprojlim \Gamma/\Gamma_{n(k)}$ by the group $\varprojlim_n \mathbf{Z}/n\mathbf{Z} \times \varprojlim_n \mathbf{Z}/n\mathbf{Z}$ because GL(1) is abelian. For this reason k and ℓ must be integral.[66]

In the discussion on the pages [AB, pp. 559/560] some fundamental understanding of differential geometry is understood. We may consider it to be self-evident, but I would like nevertheless that the reader be aware of it. It is simply a repetition of the connection between zero curvature and well-defined integrals of a connection but in multiply-connected domains. Its first consequence is the equation (6.10) and in particular the assertion [AB, p. 561, ll. 7–9] ‘Now line-bundles with harmonic connection… can be uniquely expressed as $Q^k \otimes L_0 \ldots$ where L_0 is flat.’ The word harmonic has many meanings and the best way to understand what the authors meant is to study the assertion of their Th. 6.7 and its applications. First, in so far as I understand, there is for them no difference between the words ‘harmonic’ and ‘Yang-Mills.’

The preceding assertion taken from [AB] evokes in me some confusion because my knowledge of elliptic curves appears superficial. Without preparation I cannot show that multiplication by a linear bundle of degree zero is equivalent to the imposition of a divisor of degree zero, or if a point is fixed, as for our Q, so that the curve[67] becomes a group and multiplication (or addition) simply the composition of points. This answers the question after (36.m) that remained open earlier.

We may, in particular, use the assertion from [AB], in order to amplify our discussion of the equation (36.1). These remarks, of course, are not entirely satisfactory, because they presuppose a complete understanding of the assertion and its proof, but it is clear that this would take us too far afield. Besides, it is clear that we are dealing with questions that are known to specialists. In any case in the above statement it is supposed that the identification that it describes entails a relation to the tensor product of linear bundles.

The transition from Γ_{aut} to $\Gamma_{\mathbf{R}}$. We apply [AB, Th. 6.7] to GL(1) bundles and connections, studying the homomorphism $\Gamma_{\mathbf{R}} \to \mathrm{GL}(1)$. It is clear that $ABA^{-1}B^{-1} \mapsto 1 \in \mathrm{GL}(1)$. Thus $z \in \mathbf{Z} \mapsto \exp(2\pi i n z)$, $n \in \mathbf{Z}$. The images of A, B are equal to $\exp(2\pi i\alpha)$, $\exp(2\pi i\beta)$, where α, β are at the present moment not defined. We now interrupt ourselves in order to think. We are searching for representations of Γ_{aut} related to eigen conjugacy classes, at the moment in GL(1). A one-dimensional representation of Γ_{aut} is given by a representation of $\mathbf{Z}$ in GL(1), $z \in \mathbf{Z} \to \exp(2\pi i n z)$, and two numbers α, β in $\mathbf{Q}/\mathbf{Z}$. These two numbers appeared as α/n and β/n in the discussion preceding formula (53). Unfortunately, I could not maintain a consistent notation. Thus two sets of parameters are the same. It remains to describe the connections, determined, according to [AB, Th. 6.7], by these parameters, that is by these representations of $\Gamma_{\mathbf{R}}$ in GL(1). They are given as products of representations of $\mathbf{U}(1)$ with a representation of the group $\pi_1(M)$. A representation of the group $\mathbf{U}(1)$ is

[65] More precisely, the importance of our choice was aesthetic. The final comparison of the homomorphisms $\Gamma_{\text{aut}} \to {}^L G$ with eigen conjugate sections expresses this simply.

[66] We chose the factor 2 from [WW], but it is sometimes a hindrance. There is another circumstance, which appears with the study of bundles. Linear bundles of degree one, which we considered with such care in the preceding section, do not appear in the Hecke theory for the group GL(1).

[67] supposed to be a curve of genus one

given by an integer. This appears as the degree of the basic linear bundle Q as a factor of the bundle that is associated with the given representation. The second factor is a flat linear bundle, determined by the representation $\pi_1(M)$, in a direct way by the fundamental group without the bundle Q, by two parameters α, β. However we do not need all pairs, only those that yield a periodic function. For the two groups this is a question of periodic conjugacy classes. ■

The reader will observe that the group $\mathbf{U}(1)$ in [AB, 6.6] is not yet an important part of our discussion. This is because those one-dimensional representations of the group $\Gamma_{\mathbf{R}}$ that are appropriate in the given context have a trivial inverse image in this group. I stress that the Yang-Mills theory appears to be the principal factor in our discussion but we need only a minor part of this theory. Moreover and above all, we are dealing with the theory for curves. Secondly, the restriction of the homomorphism $\rho : \Gamma_{\mathbf{R}} \mapsto G$ to $\mathbf{U}(1)$ or its inverse in $\Gamma_{\mathbf{R}}$ have a particular form. It is such, that $\mathbf{Z} \to 1$. Other possibilities arise for GL(2), which is for us the principal case.

However the theory for this group will be deduced from the theory for GL(1), using, above all, this theory together with induced representations in the context of the group Γ_{aut} on one hand and the group $\Gamma_{\mathbf{R}}$ on the other.

For $G = \mathrm{GL}(1)$, we choose a particular class of representations ρ, and then a subset of Yang-Mills connections related to these representations. For $G = \mathrm{GL}(2)$ the argument is similar. If the representation is reducible, we consider the two components separately, leading to a pair of connections and to a pair of functions as in (53) with similar conditions. They will correspond to eigenfunctions of type $\mathfrak{D}$. Thus let ρ be irreducible. This supposes that $\rho : ABA^{-1}B^{-1} \to -1 \in \mathrm{GL}(2)$ and that, with appropriate coordinates,

$$A \to \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}, \quad B \to \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix}, \qquad \alpha, \beta \in \mathbf{U}(1), \tag{54}$$

and thus that

$$J = 1 \in \mathbf{Z} \subset \mathbf{R} \to \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{54.a}$$

In order to understand a connection that is determined by this representation it is best to pass to an unramified double covering M' of the curve M. There are three of them. Which we choose is only a question of convenience or notation. They yield similar results but the implicit multiplicity, as we shall see, is real.[68] Suppose, for example, that it is determined by the subgroup generated by $\{A, B^2\}$.

Yet another digression. Although we need only particular cases of the following calculations, it is better to understand that which is generally true and generally important for other groups. We also began to study the formal relations between Γ_{aut} on the one hand and $\Gamma_{\mathbf{R}}$ on the other, as well as between Γ_{aut} and conjugacy classes on one hand and $\Gamma_{\mathbf{R}}$ and its relation with Yang-Mills on the other. In particular, for $m = 1$ we fully understand the relation. Although it is not our intention to consider an arbitrary group other than GL(1) and GL(2), it is best to include some general observations about the eigenvalues of $\rho(A)$ and $\rho(B)$ for irreducible representations of dimension n of the group $\Gamma_{\mathbf{R}}$. Each

[68] I do not know, how this manifests itself for other groups. For our group it appears, but only at the very end, to be a false assertion. I left it in order to remind myself of the uncertainty in which I remained for a long time.

such representation is given by a representation of the group Γ ([AB, p. 559]) combined with a compatible representation of the group $\mathbf{Z}$ ([AB, (6.5)]). The basic obstacle to the theory for $\mathrm{GL}(n)$ does not lie in the theory of Yang-Mills connections but in the theory of Hecke operators, which is still inaccessible for $\mathrm{GL}(n)$. Let $m > 0$, and the greatest common divisor $(m, n) = 1$.

$$A = \alpha \begin{pmatrix} 1 & 0 & 0 \cdots\cdots\cdots 0 & & 0 \\ 0 & \exp(2\pi i m/n) & 0 \cdots\cdots\cdots 0 & & 0 \\ \vdots & \vdots & \vdots \quad \vdots & \exp\big(2\pi i m(n-2)/n\big) & 0 \\ 0 & 0 & 0 \cdots\cdots\cdots 0 & & \exp\big(2\pi i (n-1)m/n\big) \end{pmatrix},$$

(55)

$$B = \beta \begin{pmatrix} 0 & 0 & 0 \cdots\cdots 0 & 1 \\ 1 & 0 & 0 \cdots\cdots 0 & 0 \\ \vdots & \vdots & \vdots \quad \vdots & \\ 0 & 0 & 0 \quad \vdots \quad 0 & 0 \\ 0 & 0 & 0 \quad \vdots \quad 1 & 0 \end{pmatrix},$$

so that $ABA^{-1}B^{-1} = J$ is equal to $\exp(2\pi i m/n)$ multiplied with the unit. This expression, of course, not uniquely determined, since only the conjugacy class is determined and expressing the conjugacy class of the connection by powers of the matrices, we multiply A and B independently with an arbitrary nth root of unity. These general observations are important and it is useful to recall them, but they are not directly appropriate. ■

We chose M'. I immediately introduce, at first in a clumsy form, because I am inexperienced, then in a general form, a two-dimensional connection. The restriction of $\rho : \Gamma_{\mathbf{R}} \to \mathrm{GL}(2)$ to the corresponding subgroup $\Gamma'_{\mathbf{R}}$ yields then a direct sum and an associated bundle, the existence of which is assured by the preceding considerations, is also a direct sum. Then we consider a linear bundle on the covering M', chosen from one or the other of its components.

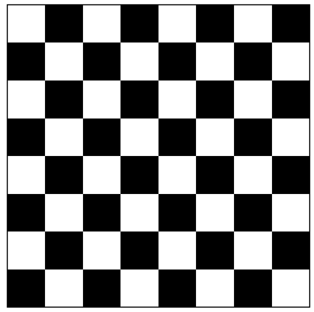
We turn now to the description of two-dimensional bundles, which was interrupted by these accidental remarks. It is sufficiently clear what happens with the projection of a linear bundle on M' to a two-dimensional bundle on M. What is necessary is a clear description and a verification that the result does not depend[69] on the three possible choices of the covering M'. I apologize to the reader for such a detailed explanation, but for my personal edification or education, this is very useful.

It is appropriate to represent the fundamental domain Δ for M as a parallelogram, placed more or less vertically, from the point of view, of the reader, in the complex plane. For M' this would be the parallelogram together with its translation upwards. Indeed, I wanted to examine the construction carefully, explaining that, for example, how the result depends on the choice of the generators A and B and on the choice of the subgroup given by $\{A, B^2\}$ rather than that given by $\{A^2, B\}$ in the definition of M'. In order to include a simple diagram, I propose that A be horizontal and B vertical, and that the fundamental domain be given by a square.[70]

[69]As already observed, on the contrary, it does depend on the choice.

[70]There are three choices. For one, the fundamental domain is given by the union of the squares at $(0, 0)$ and $(1, 0)$, for another the union of the squares at $(0, 0)$ and $(0, 1)$, and for a third the union of the squares at

(56)



The one-dimensional bundle on M' is such that the connection on it is also a one-dimensional bundle or connection on $\mathbf{C}$. Then we take the direct sum of this bundle and its translation by $2\omega_2$. This yields a two-dimensional bundle or connection on M.

But it is better to think like a topologist. If I were a topologist, I would say: "Take the direct image of this line bundle relative to the double covering $M' \to M$, in order to obtain a two-dimensional vector bundle on M." This yields the desired outcome. Indeed, although in order to convince myself about this, I had to make an effort, it was all only a reformulation of the assertion on [AB, p. 560]. We continue the discussion, asking the reader not to forget about the three possible double coverings.

Before we continue, we have to examine the influence of the transition to M' on the exact sequence [AB, 6.1]. The point is that, as we already know, J changes and becomes

(57) $$J' = AB^2A^{-1}B^{-2} = (ABA^{-1})^2B^{-2} = (JB)^2B^{-2} = J^2.$$

On this covering the connection is a direct sum, whose components are different. Consequently its construction is clear. The question reduces to the previous case, but with $\mathbf{Z}$ replaced by $\mathbf{Z}' = 2\mathbf{Z}$ and $\mathbf{U}(1)$ replaced by its double covering $\mathbf{U}'(1)$. At the same time the group $\pi_1(M)$ is replaced by a subgroup of index two. Thus $\Gamma'_{\mathbf{R}}$ itself is a subgroup of the group $\Gamma_{\mathbf{R}}$ of index two. The group $\mathbf{U}$ also changes. Earlier it was $\mathbf{Z}\backslash\mathbf{R}$, but now it is $\mathbf{Z}'\backslash\mathbf{R}$ and we allow representations that on $\mathbf{Z}'\backslash\mathbf{Z}$ are the unique non-trivial character of this group. They determine, according to the previous explanation, above all a Yang-Mills connection of dimension one on M' and, secondly, according to the description of the above construction, a Yang-Mills connection on M of dimension two. The first is related to a representation of $\Gamma'_{\mathbf{R}}$ of dimension one. The second is related to the induced representation of the group $\Gamma_{\mathbf{R}}$, and this has dimension two. It is irreducible because $ABA^{-1}B^{-1} \to J$ and $J \to -1$.

I do not know how this manifests itself for other groups but for the present example it is useful not only to consider the phenomenon from the point of view of the pair of groups $\Gamma_{\mathbf{R}}$ and $\Gamma'_{\mathbf{R}}$ but also from the point of view of the pair Γ_{aut} and Γ'_{aut}, one pair geometrical, the other arithmetical. For the first pair, for which the corresponding definition appears on [AB, p. 560], it is evident that the restriction of the representation ρ to the group $\Gamma'_{\mathbf{R}}$ becomes the direct sum of two different representations because the image $\rho(A) \in \mathrm{GL}(2)$ has two different eigenvalues ± 1.[71]

$(0,0)$ and $(1,1)$. They correspond to three non-zero elements in $L/2L$, namely $(1,0)$, $(0,1)$ and $(1,1)$. In the diagram, offered by Anthony Pulido, it is this last choice that is offered. It would be best of all to present it with another choice of fundamental domain, the square rotated by 45 degrees.

[71] As was observed, the relation $G_X = G_Y$ appearing on the upper part of the following page of the article [AB] is not necessarily well-founded.

A question of conscience. There are important remarks about GL(1)-bundles that I forgot to explain, although the necessary preparation had been made. Above all, for $G = \mathrm{GL}(1)$ the homomorphisms of ρ [AB, p. 560] are equal to 1 on $z = J = 1 \in \mathbf{Z}$. This is simply all that we need. For the group $G = \mathrm{GL}(2)$ this will be another matter. Indeed, for our purposes, for the theory of automorphic forms it is equal to 1 on $\mathbf{U}(1)$ if $G = \mathrm{GL}(1)$. This means, that they are given by two numbers in $\mathbf{U}(1)$. The existence of the related bundles has to be deduced either from the phrase on [AB, p. 561] cited earlier or from our discussion (36.k). However, at a first glance, we are dealing with differing concepts of flatness. First of all, there exist many-valued functions, in which changes are determined by homomorphisms of the Galois group to $\mathbf{U}(1)$, and secondly linear flat functions in which the slopes are given by two real numbers, one for each side of the fundamental parallelogram. Thus they are essentially the same. It is the second form that will be preferred in (53).

It is also necessary not to lose sight of the condition [AB, 6.6] and the definition of ρ. If the restriction of ρ to the inverse image $\mathbf{U}(1)$ is trivial, then for the determination of the connection attached to it we may simply replace Q by the trivial one-dimensional bundle because the construction understood in the words 'Given any homomorphism $\rho : \Gamma_{\mathbf{R}} \to G$ we then get an induced G-connection,' assumes a division by the kernel of the mapping ρ, thus $Q \times_M \widetilde{M}$ is replaced by $1 \times_M \widetilde{M} = \widetilde{M}$ or rather $\mathbf{U}(1) \times_M \widetilde{M}$. Such a connection is necessarily constant. In our particular case the constancy of the kernel and the metric mean that the powers of $\mathbf{U}(1)$ yield a flat connection.[72]

That which becomes transparent, when we begin to think about Hecke eigenvalues of the form $\mathfrak{A}$, is the similarity between $\Gamma_{\mathbf{R}}$ and the automorphic Galois group, which is so important for the theory of automorphic forms and representation theory.[73] This leads, in the same way as in this theory, to some functoriality in the Yang-Mills, but it also leads to the possibility of direct images, and this I have not seen in the customary automorphic theory or in the local representation theory. At least, if I saw it I did not recognize it. Perhaps this is a change of base? None the less, in particular for the theory for the groups GL(1) and GL(2) that we are about to describe, there is a hint on something known. That which appears in the geometric theory and not in the arithmetic theory is induced representations and direct images. The fundamental concept is $\Gamma_{\mathbf{R}}$, which I call the Atiyah-Bott group. It is related to the group Γ_{aut}, with which we are already familiar. The theory on M' manifests itself as a part of the theory on M. As is known from the arithmetic theory—rather as is expected from the arithmetic theory and from the geometric theory—a series of important deductions in this theory may be expressed at present from the point of view of the automorphic galoisian group, although we are far from a general understanding of this group.

The group $\Gamma_{\mathbf{R}}$ appears to be the analogue of the automorphic Galois group in the sense that its homomorphisms to the group G determine the Yang-Mills connections with values in G. However the automorphic Galois group has in addition a local form. It is possible that some readers are familiar neither with the global nor the local form.[74] The group

[72]The purpose of $\mathbf{U}(1)$ and its representation are not clear in this paper, in which only GL(1) and GL(2) are considered. It would be clearer if we examined groups of higher rank. Rather they, the purposes of this article, manifest themselves only at the end of this article.

[73]It is best to deal with these assertions carefully. The geometric theory and the arithmetic theory are different.

[74]One local form appears in the article [L2] *On the Classification of Irreducible Representations of Real Algebraic Groups,* Math. Surveys and Monographs vol. 31, 1988 as a Weil group. The local form of the group,

$\Gamma_{\mathbf{R}}$ does not have a local form. Rather we consider only its quotient, that determined by unramified representations, thus representations of the Hecke algebra. Consequently it is possible that this is the partial, local form, the simple local ${}^LG$, but it seems to me better to consider $\mathrm{GL}(n)$ before reaching any conclusions. It is possible that $\mathrm{GL}(n)$ is more complicated.[75]

In this way, we transport an automorphic form from M to M', as far as we understand this, related to the image of Γ'_{aut} in Γ_{aut}. In the theory of base change[76] we pass from one field to a larger field. The field of the curve M' is larger than the field M, so that we study here something similar. The taste is, however, different. Indeed, the movement is in the opposite sense, and this is confusing. M' appears as a covering of M and we are dealing with a direct image.[77] In the given case this corresponds to induction of a line bundle on M to a two-dimensional vector bundle on M'. We already described this in detail and simply. The goal is now to explain it as induction from a one-dimensional representation of Γ'_{aut} to a two-dimensional representation of the group Γ_{aut}. This, in its turn, can be expressed as a passage to a two-dimensional connection, that with integration, just as for $\mathrm{GL}(1)$, in order to determine the possible eigen conjugacy sections. Then this is to be compared in the following section with the Hecke classes described in §VII. ■

We pass from forms on M' to forms on M. On the other hand, in the theory of Yang-Mills, thus in a theory in which connections are the principal objects, there exists a direct image, and indeed a direct image from fibre to fibre. What is its relation to the homomorphisms of Atiyah-Bott and to the homomorphisms of the automorphic galoisian group. In this article we study a simple but instructive case of this question. There are many tiresome secondary questions that it is necessary to explain, but their presentation is necessarily disordered. The reader will have to excuse me.

The first matter, which is not clear to me, is the relation between A_0 or Λ_0 and the multiplier $\mathbf{U}(1)$ in (6.6). We also did not clearly resolve the difference between the theory of Yang-Mills and the theory of Hecke eigenfunctions. The first, on first glance, is for connected spaces, thus for connected components of the set Bun_G. The second is for all of Bun_G. For us, as for Atiyah, there is a fixed A_0 and a fixed Λ_0. The tensor product with a power of Λ_0 carries us from one component to another. If ϵ is an arbitrary complex number of absolute value 1, then we can extend the function from one connected component of Bun_G to another with the equation $f(\Lambda_0 \otimes \Lambda) = \epsilon f(x)$. This means that in the construction of the Hecke eigenfunctions or eigenvalues we may focus on some connected components of low degree. For $\mathrm{GL}(1)$ this degree is 0 and for $\mathrm{GL}(2)$ the degree is 0 and 1. It is possible that I forgot to say this because I was unaware of its significance, but the bundle Q of Atiyah-Bott also appears as the bundle A of Atiyah. That is, it is understood that they are

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in so far as we are considering only unramified representations, but for all base fields, thus fields of algebraic numbers or function fields, appears to be simply $\mathbf{Z}$, so that we are dealing with the image of 1 in $\mathbf{Z}$ and its conjugacy class.

[75] Some mnemonic comments. If $M' \to M$ is a smooth finite-dimensional covering, then the inverse image of the bundle on M is a bundle on M' with the same dimension. On the other hand, the dimension of the direct image of the bundle A on M' is equal to $[M' : M] \dim A$. The first construction is base change, but it is the second that is important at the moment.

[76] [L1] *Base change for* GL(2), Annals of Mathematics Studies, vol. 96 (1980).

[77] In the customary theory the group on the larger field is considered as a group on the smaller field. The relation to a direct image is not immediately seen. We return to this. The reader will observe a certain amount of repetition. This is because I only slowly come to understand what I am explaining.

equal (or, perhaps, one of them is inverse to the other, in which case our discussion has to be slightly modified.[78]

It seems that there is no possibility of avoiding the introduction of A_0. Unfortunately, this evidently presupposes a choice, in so far as this is possible, of A'_0 for each unramified covering M and in a compatible form. We may think of A_0 simply as a point in M and A'_0 as a point in M. In this way, we may suppose that if $M'' \to M' \to M$ then $A''_0 \to A'_0 \to A_0$. Thus we suppose that we have consistent choices of base points for all unramified coverings M, Thus we have a consistent possibility of transformations of linear bundles on any M' on bundles of degree 0. For a two-dimensional bundle the tensor product with a power of Λ_0 transforms it into a bundle of degree 0 or 1. Thus it is sufficient to consider these two possibilities. Of course, the choice A'_0 then determines Λ'_0.

A delicate question. In order to take a direct image the lattice L' must be a subset of the lattice L. In the given case, $2L \subset L' \subset L$. On the other hand, it appears that the parameters of linear functions associated with L' are associated only with those that are related to L, $L' \subsetneq L$. However, this is not so because [AB, Th. 6.7] allows the necessary freedom. Irreducible representations, which are the essential representations, are given by a glueing of representations of Γ and $\mathbf{R}$. ■

We have not yet described an example of the connections between representations of the group $\Gamma_{\mathbf{R}}$ and linear bundles. The time has come to do so. It is now clear that we need only consider bundles of degree zero. According to [AB] it will be given by a one-dimensional representation π_M and that the representation will be, above all, trivial on J, in so far as J is a commutator, and, secondly, that it will be trivial on $\mathbf{U}(1)$ if the degree is 0. In so far as we consider only elliptic curves, this means that it is given by the images $\exp(4ka\pi i)$, $\exp(4\ell b\pi i)$ as in (53). These two numbers lie in $\mathbf{U}(1)$, but are otherwise arbitrary. On the other hand, for those representations $\Gamma_{\mathbf{R}}$, which are related to the Hecke theory, the two numbers k, ℓ must be whole numbers. According to the discussion on p. 560 in [AB], in particular, the equation (6.10), the representation is linear on $\mathbf{R}$: $x \to \epsilon x$, $x \in \mathbf{R}$, and $\epsilon J = 1$, in so far as $ABA^{-1}B^{-1} = J \in \mathbf{R}$. This is the determination of the number ϵ. The numbers a and b lie in $\mathbf{R}$. The condition in (53), that they lie in $\mathbf{Z}$ is not imposed by the Yang-Mills condition but because of the relation to Hecke eigenvalues. A more interesting question seems to be the purpose of the direct image. We need only to concern ourselves with the case that M' is a double covering of M, because we are considering quadratic extensions of the field.

Some general remarks. Although our general theme is now the appearance of the group GL(2), it is possible that our conclusions will be more convincing, if we interrupt the discussion with some remarks about GL(n)-bundles for general $n > 0$. We consider only irreducible bundles. Although our theme in this article is chiefly GL(2), I would suggest that with the help of [A] the case of general n, if not simple, is at least worth reflection because it is accompanied by the discovery of all classes of eigen conjugacy sections for GL(n).

We have already begun to study the formal properties of the relation between $\Gamma_{\mathbf{R}}$ on one hand and Γ_{aut} on the other, as well as the relation between Γ_{aut} and conjugacy classes on the one hand and $\Gamma_{\mathbf{R}}$ and connections on the other. In particular, for $m = 1$ we completely understand all relations.

[78] I find it difficult to recall the precise definition of the Chern class.

These observations are related to other subordinate questions that, although they are also related to difficulties familiar from the general theory of automorphic forms, were not expected in the present essay. The function field of the curve M' is a quadratic extension of that of the curve M. Thus the L-group of the group GL(1) for M' is an extension of the L-group for M, namely $\mathrm{GL}(1, \mathbf{C})$ is replaced by $\mathbf{Z}_2 \times \mathrm{GL}(1, \mathbf{C})$, because the appropriate Galois group is $\mathbf{Z}_2$. This alteration has to be included in the corresponding diagrams [AB, 6.5] and [AB, 6.6]. More precisely, if we consider the geometric theory on the double covering $M' \to M$, then the Galois group $\mathrm{Gal}(M'/M)$ has to be included in the Galois group of the diagram (6.6), which is replaced for the group M' in such a way that $\pi_1(M)$ is replaced by $\pi_1(M')$, but $J \to 2 \in \mathbf{Z}$ by $J' \to 1 \in Z$, in such a way that $(J')^2 = J$. In this way the representation π' of the group $\Gamma'_{\mathbf{R}}$ is given by $z \to z$, $z \in \mathbf{U}(1)$ together with the given representation of the group $\pi_1(M')$. Thus, the corresponding representation $\Gamma'_{\mathbf{R}}$ is a bundle defined by the representation $\mathbf{U}(1) \times \pi_1(M')$. In other words the corresponding representations of $\Gamma_{\mathbf{R}}$ as well as of their restrictions to $\mathbf{Z}$ map J' to -1, J to 1. Thus the inclusion of G in the theory is somewhat subtle because the passage from bundles on M' and of the Yang-Mills connections related to them determined by the direct image is an extension of the one-dimensional representation of the group $\Gamma_{\mathbf{R}}$ to an induced two-dimensional representation of $\Gamma'_{\mathbf{R}}$. ■

As a second step we explain further the purpose of the bundle A_0 in the present circumstances, the eigen conjugacy sections for GL(2) on elliptic curves. They were studied in §VI and divided into two classes: of type $\mathfrak{A}$ and type $\mathfrak{D}$. Those whose type is $\mathfrak{D}$ were already considered and their relation to representations of Γ_{aut} was explained, although not to representations of the Atiyah-Bott group. We consider now eigenfunctions of the form $\mathfrak{A}$. They have a curious but appropriate property. They vanish on the set $\mathfrak{A}_{\mathrm{odd}}$. Thus they can be uniquely expressed as the tensor product of powers of linear bundles Λ_0 and bundles of degree 0 or 1. Thus these eigenfunctions apart from a character of the group $\mathbf{Z}$ are determined by their values on bundles in $\mathfrak{A}$ of degree 0 or 1. Namely, each bundle Θ of dimension two is the product of a linear bundle Λ and a bundle Θ' of degree 0 or 1. Namely, each bundle Θ of dimension two is the product of a line bundle Λ and a bundle Θ' of degree 0 or 1. Each eigen conjugate section f is such that $f(\Lambda \cdot \Theta) = \chi(\Lambda) f(\Theta)$, where χ is a character of the Picard group. In the present case, duality imposes itself but it demands some explanation.

We turn to the Yang-Mills theory, allowing ourselves some room. This theory, which appears at the level of connections, thus related to manifolds, in contrast to the Hecke theory, which is uniquely determined at every level. In other words, the Hecke operators move between degrees, but connections at different degrees are independent. They are however connected with each other by powers of the bundle A_0. Thus to each is assigned an integer. Thus the separate sets are identified with each other, in a somewhat arbitrary form, so that each function or connection on the bundles of degree 0 determines a function, corresponding to the connection, on all of Bun_G. In this way, a function on one of the manifolds can be transformed or translated to all of the rest, using this integer and the given character on $\mathbf{Z}$, the first factor in (1.d). Thus, we understand the purpose of the bundle A_0. We do not, however, understand it in the context of Yang-Mills connections. There is a difficulty. The tensor product with A_0 does not change the determinant of the bundle. It changes the degree. This degree is the dimension of the bundle. Thus, for example, for a bundle whose dimension is equal to two the factor is A_0^2, and this is for us a serious problem. Connections for even and odd degrees must be considered separately.

The group $\Gamma_{\mathbf{R}}$ is related not alone to the two factors (1.d) but also to products. The first factor $\mathbf{Z}$ is related to the factor $\mathbf{U}(1)$ in [AB, (6.6)], but also with products. The first factor $\mathbf{Z}$ is related to the multiplier $\mathbf{U}(1)$ in [AB, (6.6)]. From [AB, (6.6)] and the following explanation it is clear that we can replace a representation of the group $\Gamma_{\mathbf{R}}$ with the degree of a representation of $\mathbf{U}(1)$ itself, using the mapping $\Gamma_{\mathbf{R}} \to \mathbf{U}(1)$, and that this changes the degree by a product with the dimension of the initial representation. The conclusion is that, with our present assumptions, we need to examine only two-dimensional bundles of degree 0 or 1.

The comparison of Yang-Mills connections with Hecke conjugacy classes, with representations of $\Gamma_{\mathbf{R}}$, and with representations of Γ_{aut} turn out to be fastidious. Thus we stop explaining what is necessary and explain, rather, what we did. The appropriate comparison, a comparison of the set $\{\, \rho \otimes \sigma^n \mid n \in \mathbf{Z} \,\}$, where σ is a representation of $\mathbf{U}(1)$ in $\mathbf{U}(1)$ and ρ a representation of $\Gamma_{\mathbf{R}}$, because Bun_G is not connected and the definition of the group $\Gamma_{\mathbf{R}}$ in [AB] is adapted to connected curves. Recall that in (1.d) there is an implicit choice of A_0. The calculations leading to (56) lead one to think that for a comparison with representations of Γ_{aut} only those $\Gamma_{\mathbf{R}}$-representations for which J, which we may suppose is 1 in $\mathbf{R}$, has an image of finite order matter.[79]

The one-dimensional representations of $\Gamma_{\mathbf{R}}$ that correspond to one-dimensional bundles are necessarily trivial on $\mathbf{R}$ because of [AB, 6.5] and have already been examined. For them $J \to 1$. We recall that we are dealing with a series, infinite in both directions, of representations. For the central element of this series of connections and our choice of metric and of the connection Q, the connection is constant. The others oscillate regularly and all are periodic with a period dividing 1. There is no finite-dimensional representation of $\Gamma_{\mathbf{R}}$ for which the image of J is irrational.[80]

It remains now to understand the relation of the eigenfunctions of the form $\mathfrak{A}$ that are not also of the form $\mathfrak{D}$, on one side, and to two-dimensional irreducible representations of the group $\Gamma_{\mathbf{R}}$ or Γ_{aut} on the other side. Once again there is a condition of rationality, placed on the first. The three sets $\mathfrak{S}_i$ are different, namely the behaviour of each is determined by its relation to the three characters χ_i. They correspond to the three possible unramified quadratic extensions or the three coverings M' of the curve M. We consider one of them, let's say the one whose fundamental group is generated by A^2 and B.

We consider the quadratic covering M' of the curve M. What is the influence of such a change on the product of the Atiyah-Bott group $\Gamma'_{\mathbf{R}}$ with $\mathrm{Gal}(M'/M)$. This is sufficiently clear. The group $\pi_1(M')$ in [AB, 6.5] is contained in $\pi_1(M)$. It follows from this that J' for M', thus $J' \in \Gamma'_{\mathbf{R}}$ maps as $2J$, thus as J^2, to $\Gamma_{\mathbf{R}}$. This determines the mapping from $\Gamma'_{\mathbf{R}}$ to $\Gamma_{\mathbf{R}}$, $x \in \mathbf{R}$ is mapped to $2x$. It is obvious that the index of the set $\Gamma'_{\mathbf{R}}$ in $\Gamma_{\mathbf{R}}$ is equal to

[79] We may expect that the problems posed by functoriality in the geometric context similar to those which we consider here appear also in the arithmetic theory. In that theory they, undoubtedly, will be more complicated, including the simultaneous use of the trace formula and calculations that resemble those in the book of Hasse, *Klassenkörpertheorie,* although much more difficult. There are few mathematicians with the courage to consider these problems and I am not one of them.

[80] When it came to translating the Russian text into English, and thus when I had to some extent forgotten the details of the paper, it was very difficult to make sense of some of them, partly because I had forgotten the structure of the theory as a whole. I had found it best to envisage the functions as waves! Indeed, at first glance the Russian text in this part of the paper made absolutely no sense!

two. This means, thanks to the theorem of Atiyah-Bott, that a one-dimensional Yang-Mills bundle for M' yields a two-dimensional Yang-Mills bundle for M. Geometrically, the second is simply the direct image of the first, but none the less it is better to assimilate all these concepts and constructions, which may be as unfamiliar to the reader as they are to me, and to continue the study of their definitions. In particular, it is useful to reflect on the nature of the direct image in connection with the present considerations.

Recall that the author of this essay has some understanding of the theory of automorphic forms over algebraic number fields, over $\mathbf{C}$, and even over finite fields although we do not consider these in this article. The guiding principle is this, that Hecke eigenfunctions are parametrized, possibly with some minor changes, by homomorphisms of a hypothetical automorphic Galois group to the L-group. The structure is functorial, thus a homomorphism from the group ${}^LG_1$ to ${}^LG_2$ leads to a mapping from the Hecke eigen conjugacy classes of the group G_1 to those of the group G_2. There are many questions in connection with this, basically unresolved. In this article, we are dealing principally with GL(1) and GL(2).

Apology. There are many reasons for an explanation. It is possible that some may seem to be a repetition of others. It is so here. For me this is simply a confirmation of the functorial relations, in which the theory of automorphic forms, and thus other theories, similar to the Yang-Mills theory, lie. I ask the reader's forgiveness for any enthusiasm that he considers extreme. I cannot simultaneously grasp all consequences. ■

We presented the possible choice of this automorphic galoisian group Γ_{aut} by analogy with the theory of class fields. If we were dealing with automorphic forms over a number field a supposition would be that this group had as a quotient group the Galois group of the field considered, more precisely the inverse limit of the Galois groups of its finite-dimensional extensions. Here it is precisely the inverse limit group of the Galois groups of unramified finite Galois coverings of the curve M.

For a field of algebraic numbers it is conjectural, but even more is conjectured, that it counts (rather so defined that it counts) the L-functions of all (perhaps of all smooth) algebraic manifolds. That is, this group would be an important component part in the creation of a theory of L-functions for algebraic manifolds over number fields. We suggested above the group Γ_{aut}, which would play a similar role for the fields of functions over a curve M, although we excluded for now the possibility of ramification.

As we remarked, a Yang-Mills connection appears with a factor from a connected manifold, because any comparison is made between infinite series, from $-\infty$ to ∞, related to the connection and to a representation of $\Gamma_{\mathbf{R}}$. More than that, if the representation is finite-dimensional, then the image of J has to be of finite order. Finally, we are dealing with representations, in which the image of A and B are of finite order. If this is accepted, then the existence of a bijective relation between such families of representations and representations of Γ_{aut} is clear.

If one is not careful, the circumstance that the bundle of the form $\mathfrak{A}$ is two-dimensional although it is related to parameters that appear to be one-dimensional can lead to confusion.

What it is now necessary to explain, in so far as in this essay we consider only particular curves, this relation of induction of one-dimensional representations of the group $\Gamma'_{\mathbf{R}}$ to a two-dimensional representation of $\Gamma_{\mathbf{R}}$ and to an eigen section of the form $\mathfrak{A}$. The curve M' appears as one of three possible elliptic curves, covering M as double coverings. In order to be concrete, we suppose that this is such that the fundamental group is generated by A^2 and B. These three curves correspond to the three characters χ_i of the group $\mathrm{Pic}_2(M)$.

It is necessary to observe, once again, that J is replaced by $J' = J^2$, thus $J' = 2J = 2$ in $\mathbf{R}$. The basic relation is $ABA^{-1}B^{-1} = J$ or equivalently $BAB^{-1}A^{-1} = J^{-1}$. Consequently $ABA^{-1} = JB$ and $A^nBA^{-n} = J^nB$. The fundamental group $\pi_1(M')$, associated with the covering M', has index two in $\pi_1(M)$. Consequently $[\Gamma_{\mathbf{R}} : \Gamma'_{\mathbf{R}}] = 2$.

In order not to confuse myself and the reader, I consider the Hecke eigen sections only on the connected component of Bun_G, although they cannot be determined only for this component, but are Yang-Mills connections without any reference to the possibility of a tensor product with a representation of the group $\mathbf{U}(1)$. The necessary supplementary discussion is sufficiently clear. In order to be precise, if the representation of $\mathbf{R}$ in $\Gamma_{\mathbf{R}}$ is the trivial representation, then thanks to the preparation in §VII we are dealing with the trivial connection with a constant integral, but if the representation of $\mathbf{R} \in \Gamma_{\mathbf{R}}$ is not trivial, then we are dealing with a power of the connection Q [AB, p. 560] constructed in the well-defined manner of §VII.

There exists three classes of eigen sections, each of which corresponds to a subgroup of order two in $\mathbf{C}/L$ or in the lattice L'/L. The curve M' is determined as a covering $M' = \mathbf{C}/L'$, $L \supsetneq L' \supsetneq 2L$, of the curve $M = \mathbf{C}/L$, a covering of degree two. The direct image of a linear bundle on M' would then be a bundle of dimension two on the curve M. On the other hand, linear Yang-Mills connections on M' with single-valued integrals are parametrized, as we know, by characters of M', but we now know how to treat them. However, only those that are not characters of M, for which the conditions of periodicity are more demanding, are relevant now. Indeed, with more precision, they are parametrized by one-dimensional representations of the group $\Gamma_{\mathbf{R}}$ or, with even more precision, by one-dimensional representations that represent $\pi_1(M)$ in a finite set. We must show that the functions on $\mathfrak{A}$ with which we are dealing are parametrized by the one-dimensional Yang-Mills connections on M' already described and also by their direct images on M, where M' is a double covering of the curve M.[81] It is clearer that M' determines L' in L, where $L/L' = \mathbf{Z}/2\mathbf{Z}$, but we are working with functions on $\mathbf{C}/\widetilde{L}$, which appears as a factor of $\mathbf{C}/L$ and which is isomorphic to $\mathbf{C}/L'$.

In §VI we saw that eigen sections of the form $\mathfrak{A}$, that are not of type $\mathfrak{D}$, are given by characters of the group $\mathbf{C}/L'$ that are not functions on $\mathbf{C}/\widetilde{L}$, $\widetilde{L} = L/2$.

In the end we have also to take into account the possibility that two different line bundles have one and the same direct image but at the moment we ignore this possibility.

The circumstances are the following. There are parameters for M'. They determine first of all linear bundles on M' with connections and also one-dimensional representations of $\Gamma'_{\mathbf{R}}$. Attached to each of the linear bundles with a connection there is also its direct image, a two-dimensional vector bundle on M with a connection. Both the initial connection and its direct image are Yang-Mills connections. Each of them is related to representations, one of the group $\Gamma_{\mathbf{R}}$, another of the group $\Gamma'_{\mathbf{R}}$. We have to show that the first is induced from the second, so that the necessary compatibility is clear. There are compatibilities that we have to understand clearly. The formalism allows us to perform the test on linear bundles of degree 0 and on the resulting class of representatives of $\Gamma_{\mathbf{R}}$. One factor is the important circumstance that allows an appropriate reduction for two-dimensional representations. It is that the corresponding eigenfunctions are equal to 0 on $\mathfrak{A}_{\mathrm{odd}}$, so that on this set the Hecke eigen sections are not determined. This means that when examining the conjugacy

[81] In the end it will be necessary to consider the possibility that two different linear bundles have one and the same direct image, but so far we have not noticed this. One might examine [AB, Th. 6.7].

sections we can, by taking tensor products with powers of Λ_0, reduce our investigations to bundles of degree 0, as we shall do. Our uneasiness in regard to the Hecke eigen sections is related principally to their restriction to the bundles of degree 0 and their values on the others are tensor products with the powers of Λ_0. This is also in order, because the tensor products with powers of the bundle Q, which, although we constructed it explicitly for elliptic curves, appears to be a somewhat obscure object, not easily grasped. We attempted this in §V. The direct image of these bundles appears to be important and interesting, as do their relations to two-dimensional representations of the groups $\Gamma_{\mathbf{R}}$ or Γ_{aut}. For the first this is explained in [AB]. For the second it appears as a consequence of the relation between $\Gamma_{\mathbf{R}}$ and Γ_{aut}, namely between Γ and Γ_{aut}, where Γ is a subgroup of $\Gamma_{\mathbf{R}}$ that appears in [AB] and in (1.a). We explained above the significance in [AB] of $\Gamma_{\mathbf{R}}$ as a section of Γ. In fact, we postponed this, although not entirely, by the transition to bundles of degree zero.

Before we begin, allow me to explain how the reflections developed. We introduced earlier the correspondence between one-dimensional representations of $\Gamma_{\mathbf{R}}$ and one-dimensional Hecke eigen sections. This is possible for all elliptic curves, in particular M'. The group $\Gamma'_{\mathbf{R}}$ is embedded in $\Gamma_{\mathbf{R}}$ as a subgroup of index two, in such a way that J' is mapped to $2J$. The index 2 is given by $\left[\pi_1(M):\pi_1(M')\right]=2$.

The parameters of the Hecke eigen sections of the form $\mathfrak{D}$ that are not equal to those of type $\mathfrak{A}$ are related to one of the three possible coverings M'. For a given M', the parameters, which appear in §VI, turn out to be those characters of the group M' that are not restrictions of characters of M. The eigen sections are exponential integrals of Yang-Mills connections.

In addition, it is possible to combine a parameter that is a character of M' with an arbitrary character of the group $\mathbf{R}$ that is -1 on J and 1 on J' in order to determine a one-dimensional representation of $\Gamma'_{\mathbf{R}}$. This determines by induction a two-dimensional representation $\Gamma_{\mathbf{R}}$, which is necessarily irreducible, for the image of J is otherwise necessarily the unit matrix. On the other hand, the character of M' also determines in the theory for GL(1) described above a linear bundle on M' with a connection, the direct image of which is a two-dimensional bundle on M with a connection. The integral of this connection is then a Hecke conjugacy class. The question remains, 'Which representation or what character of $\mathbf{R}$ in $\Gamma_{\mathbf{R}}$ should we take?' The diagram [AB, 6.6] is replaced by

$$1\to\mathbf{Z}'\to\Gamma'_{\mathbf{R}}\to\mathbf{U}'(1)\times\pi_1(M')\to 1,\qquad \mathbf{Z}'=2\mathbf{Z},\ \pi_1(M)/\pi_1(M')=\mathbf{Z}_2,$$

and the sequence

$$1\longrightarrow\mathbf{Z}_2\longrightarrow\mathbf{U}'(1)\longrightarrow\mathbf{U}(1)\longrightarrow 1$$

is exact. The two groups of rotation of a circle are, of course, isomorphic.

I note that we already observed that the one-dimensional representation from which we induced is not trivial on J, since the following two-dimensional representation is necessarily irreducible. Each of these characters determines a connection on M' of Yang-Mills type. It is of degree 0 and its direct image on M will also be of degree 0 and of Yang-Mills type.[82] We want to convince ourselves that the connected two-dimensional union is of Yang-Mills

[82] We need now a general remark. Let $\widehat{M}$ be a finite Galois covering of the curve M. We can first choose $\widehat{Q}$ and then for each M', $M\subset M'\subset\widehat{M}$, take

$$Q'=\bigotimes_{\sigma\in\mathrm{Gal}(\widehat{M}/M')}\sigma\widehat{Q}.$$

type and that the integrated exponential of the union determines a connected eigen part. This is in some sense obvious but we still must try to sort things out in the confusion of definitions.[83]

Supplementary general remarks. The group $\Gamma_{\mathbf{R}}$ is almost a direct sum of products of the group $\mathbf{R}$ with the group Γ [AB, p. 559]. It is equal to this product divided by $\{ n \times J^{-n} \mid n \in \mathbf{Z} \}$. Keeping this in view, as well as the relation (55), I would like to consider the irreducible finite-dimensional representations of $\Gamma_{\mathbf{R}}$. These are of course simply tensor products of an irreducible representation $\rho(\cdot)$ of the group Γ, which we already discussed, with an appropriate character χ of the group $\mathbf{R}$, namely $\rho(J)^{-1}\chi(1)$ is equal to 1. Since we may extend each character of the group $\mathbf{Z}$ to a character of $\mathbf{R}$, it follows from the precise conclusions of [AB, 6.6] that each irreducible representation of the group $\Gamma_{\mathbf{R}}$ is a product of a representation ρ of the group $\mathbf{R}$ and a representation of the group $\mathbf{U}(1) \times \pi_1(M)$. We already treated the second factor earlier, rather its first factor. This means that we need only treat the irreducible representation $\pi_1(M)$, but we did this in (55). The factors α and β have less structural importance.

If we suppose that, no matter what else, the representation of the group, $\mathbf{R}$ in $\mathbf{U}(1)$ is simply $x \to \exp(2\pi i x a)$ and if we suppose, again as we may, that a is rational then we can write it as $a = b + c/d$, where b, c, d are integers, $0 \leqslant c/d < 1$, and where $c \geqslant 0$, $d > 0$ are mutually prime. What I want to do is to establish that the eigen connections of type $\mathfrak{A}$ that appear in §VII can be realized as direct images of integrals of Yang-Mills connections on quadratic coverings of M. As soon as one recognizes this possibility, the temptation arises to examine it in a more general context, in particular for coverings of higher degree. However this immediately opens far too many possibilities. I am completely content to leave this investigation to others, at least to those who are competent and honest!!

We must, however, first understand why Atiyah-Bott introduced the field $\mathbf{R}$ and $\Gamma_{\mathbf{R}}$ in [AB, 6.4]. It is clear that with our conditions[84], this is simply to allow an addition to the connection of an arbitrary imaginary term $i\theta$, $\theta \in \mathbf{R}$. For our goals it is suitable to limit ourselves to rational multipliers, because we want to introduce periodic connections. Then $\mathbf{R} \to \Gamma_{\mathbf{R}}$ in [AB, 6.5] is replaced by $\mathbf{Q} \to \Gamma_{\mathbf{Q}}$, thus by the image of J (and its multiples) in $\mathbf{Q}$.

More precisely, if we consider only continuous finite-dimensional representations $\pi_1(M)$, then according to (55) there exist three parameters: a, b and the image J, which is necessarily a root of unity. Moreover, for our purposes a and b will be roots of unity, if the representation is single-valued. This means that the representation is constructed in two parts. First of all the character of the group $\mathbf{R}$. The image of J is also necessarily a root of unity, if the representation is finite-dimensional. This means that the representation is constructed in two parts. First a character of the group $\mathbf{R}$ of the form $x \to \exp(iax)\exp(ibx)$ with $a \in \mathbf{Q}$, and $0 \leqslant a < 1$, and with $b \in \mathbf{Z}$ a whole number. At first $a + b$ appears to be an

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This is understood in the following discussion, in which $\widehat{M} = \mathbf{C}/2L$. The question arises whether this is compatible with our choices. (See [WW].)

[83] The present text is a rough translation into English of the Russian text. I confess that in the two years that passed between the writing of the original text and the writing of the translation, I lost to a large extent the vivid understanding of the material that I had acquired with six or seven years of reflection. Whatever defects the original might have, I understood clearly what I was saying or trying to say. This is no longer entirely so and it is sometimes obvious.

[84] Thus the representation $\Gamma_{\mathbf{R}} \to G$ is determined by the given representation $\Gamma_{\mathbf{R}} \to \pi_1(M)$ and $\pi_1(M) \to G$.

arbitrary real number but for our goals it is better that $a+b \in \mathbf{Q}$. Thus $\Gamma_{\mathbf{Q}}$ appears as an inverse limit of groups. It is possible to obtain the second of these characters of the group $\mathbf{R}$, $x \to \exp(ibx)$, by the composition of $\Gamma_{\mathbf{Q}} \to \mathbf{U}(1)$ (or $\Gamma_{\mathbf{R}} \to \mathbf{U}(1)$) with the degree of a single representation of $\mathbf{U}(1)$. This is a representation of $\Gamma_{\mathbf{R}}$ or, rather, of $\Gamma_{\mathbf{Q}}$, the inverse limit of finite subgroups in $\mathbf{U}(1)$, or rather of a representation of the inverse limit $\mathbf{U}(1)$ in $\Gamma_{\mathbf{Q}}$, which is also a group. They must be supplemented by an appropriate representation of $\pi_1(M)$, compatible with the image of J as in (55) and thus of finite order.[85]

The following step will appear initially to be inappropriate. If we have a sublattice L' of the lattice L in the plane and of index two, then we may choose such a basis of L that, relative to this basis $L' = \{abc, bd\}$, c, $d \in \mathbf{Z}$ arbitrary, a, $b \in \mathbf{Z}$, a, $b > 0$. For the present purposes, and namely for the group GL(2), a will be 2 and b will be 1. For other groups there will be more coordinates and more possibilities. At present we consider two elliptic curves, the curve M with which we began, its lattice L, a sublattice L', the associated covering M', itself covered by the curve defined by $2L$, which is equivalent to M itself. There are three possible coverings M' implicit in §VI.

Having explained this, we turn to the analysis of the preceding paragraph.[86] First of all, the restriction of the irreducible representation to $\mathbf{Z}$, the subgroup generated by the element J, is given by a root of unity $\exp(2\pi ia/k)$, where either $a = 0$ and k is irrelevant or $k > 0$ and $0 < a < k$. The greatest common divisor $(a, k) = 1$. Then the representation on $\mathbf{R}$ is given by the formula $x \to \exp\bigl(2\pi i(a/k + \ell)\bigr)x$, where ℓ is a whole number and $0 \leqslant a < k$. Then, as an integer, ℓ determines a character of the group $\mathbf{R}/\mathbf{Z}$ that, as seen, in particular, from [AB, 6.6], determines in turn a representation of $\mathbf{R}$. The difference between this representation of the group $\mathbf{R}$ and the initial representation appears as a representation of $\mathbf{R}$ given by the fraction a/k, which itself is given by its value at $J = 1 \in \mathbf{R}$, which is a root of unity. Observe that k is the length of a full period and a is the number of waves in the period.

We already observed that the tensor product with a representation of the group $\mathbf{U}(1)$ changes the order by a whole number. Thus we may suppose that $\ell = 0$. Thus four numbers are being considered, a, k and the two numbers α and β in (56), which will be groups of unity if we are examining $\Gamma_{\mathbf{Q}}$. More than that, the object of our considerations in this paper now appears to be only irreducible representations for GL(2), so that according to the remark, following (56) $a = 1$, $k = 2$. This case is, without a doubt, typical. We are given the curve M and a double covering M'. This covering was implicitly discussed above. More than that, we also discovered earlier, following the determinations in [AB], that the data that we have in hand, determine a one-dimensional representation of the group $\Gamma_{\mathbf{Q}}$. These data are a one-dimensional representation of $\mathbf{U}(1)$, thus, a whole number, together with a character of finite order of the group $\varinjlim_n \mathbf{Z}/n\mathbf{Z}$ and, finally the images, also of finite

[85]Added with the translation. I have already observed that what I clearly understood when writing the Russian version, which entailed some linguistic obscurity, is no longer so clear to me so that there is some mathematical obscurity of ideas with which I had no trouble two years ago. There are none the less signs of fatigue.

[86]It is possible that these evidently petty complications are the essence of non-abelian class field theory for the function field of an algebraic curve defined over $\mathbf{C}$. Thus it is better not to scorn them. Unfortunately, the construction, like my explanation, is clumsy and also repetitive. My understanding is complete but also somewhat blurry. I still cannot present it linearly. The presence of fine elementary calculations is, thanks to their similarity with the classical theory from Gauss to Hasse, promising. I add in 2020 that this statement may or may not have some truth, but it seems a little premature.

order, forming A', B' in $\pi_1(M')$, chosen for compatibility with the representation of M' as a quotient of $\mathbf{C}$ by the lattice L'. This character is given by[87]

$$x \to \exp(2\pi i a x), \qquad x \in \mathbf{Z}/k\mathbf{Z},$$

and compatible forms on all higher levels of the inverse limit.

Before we continue, I note that which, although evident, may be ignored—the restriction to irreducible two-dimensional representations of $\Gamma_{\mathbf{R}}$ or $\Gamma_{\mathbf{Q}}$ to the subgroup generated by $\pi_1(M)$ as in (55). Besides that $J = -I$. Thus, the single free parameter is $\chi : \mathbf{R} \to \mathbf{R}$, $\chi(J) = 1$.

I present this construction in a somewhat different way, so that it is clear. The first step in the construction of one-dimensional representations of $\Gamma'_{\mathbf{Q}}$, but also of $\Gamma_{\mathbf{R}}$, appear as a construction of a representation of $\mathbf{U}(1) \times \pi_1(\mathbf{U})$, and thus a simultaneous representation of the group[88] $\varinjlim \mathbf{Z}/n\mathbf{Z}$.

It is now necessary to explain the relation between $\Gamma'_{\mathbf{R}}$ and $\Gamma_{\mathbf{R}}$ or that between $\Gamma'_{\mathbf{Q}}$ and $\Gamma_{\mathbf{Q}}$. It is possible that this is all clear to the reader. It suffices to explain it for the second pair. The single difference, apart from the inequality $\pi_1(M') \subsetneq \pi_1(M)$ is the relation $J' = 2J$. Besides this $[\Gamma_{\mathbf{R}} : \Gamma'_{\mathbf{R}}] = 2$, but this is because M' is a double covering of M, thus $\big[\pi_1(M) : \pi_1(M')\big] = 2$. We know how to pass from a one-dimensional Yang-Mills connection on M to another on M. This is a direct image from a one-dimensional connection to a two-dimensional connection. What relation is there between the corresponding Yang-Mills groups and the related representations? The natural supposition is that one is induced from the other.

We first consider the imbedding of one in the other. For simplicity—the imbedding $\Gamma'_{\mathbf{R}} \subset \Gamma_{\mathbf{R}}$. This is clearly determined by the imbedding $aJ' \mapsto 2aJ$, $a \in \mathbf{R}$ and $\pi_1(M') \subset \pi_1(M)$. Thus, each one-dimensional representation of the group $\Gamma'_{\mathbf{R}}$ determines a two-dimensional representation of the group $\Gamma_{\mathbf{R}}$, necessarily irreducible, in so far as the image of J' equals 1, and the image of J is equal to -1. This is a condition on the restriction of the first representation to $\mathbf{R}$. It yields in the given circumstances $x \to \exp(\pi m x)$, m odd. The significance of this condition will become clear later. I stress that we are discussing here the key to this paper.

The last question, preparatory for us, consists of the following: is this induced representation a representation related to the direct image? We may also suppose in the present discussion, since this is a condition that is significant for us, that the condition in the last hypothesis of the preceding paragraph is fulfilled. That is, m is odd. On one hand, this is clear from initial conditions in [AB, p. 560]; on the other hand, this condition is rather brief. Nevertheless, I shall also be brief. This is not the place for a lengthy discussion of its nature. In particular, the factor $\mathbf{U}(1)$ does not play a role in the present conditions,[89] so that we are discussing the comparison of two bundles, one on M, the other on M', each determined by the same bundle on the universal covering of M. Consequently our assertion is tautological.

All that remains now it to compare this deduction with the conclusion §VII. This is the theme of the following section.

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[87]Unfortunately we have ceased to be consistent in our notation. The coordinates (x, y) here appear in (36.c) and in other places as (a, b). I am not capable of complete consistency.

[88]If the reader is somewhat confused by the various limits, so am I.

[89]The relevant $\rho : \Gamma_{\mathbf{R}} \to G$ on p. 560 is trivial on $\mathbf{U}(1)$.

Before all we have to understand clearly the relation between the diagram [AB, 6.5] and the corresponding diagram for M'. The second is inserted in the first with $\mathbf{Z}' = 2\mathbf{Z}$, so that $\mathbf{U}'(1)$ is a double covering of the group $\mathbf{U}(1)$. The group $\pi_1(M')$ is imbedded in $\pi_1(M)$.

We have already chosen the bundle Λ_0' of the group M' identified with the connected component of its Picard group Pic_0, and the characters of this group parametrize, as we saw, the eigen sections of the group GL(1) up to a supplementary factor, related to the supplementary factor in (1.d). The last we may boldly ignore. As we saw, these characters are also given by Yang-Mills connections, whose direct image on M is relevant. These direct images will be two-dimensional connections. These to a significant degree correspond by the definitions in [AB] to a representation of the group $\Gamma_{\mathbf{R}}$ induced from a representation of $\Gamma'_{\mathbf{R}}$. This is, in my view, a striking conclusion.

The possibility of a general theory appears, but we do not pursue that here. There is still something, equally striking, at least for those familiar with the work of Harish-Chandra. The curve M has three unramified coverings M'. We discovered four classes of GL(2)-bundles on M, one class associated with decomposable bundles, and each of the other three with one of the unramified quadratic extensions of the function field on M. The similarity with the spectral theory of semi-simple groups over the field of real numbers, even with conditions that are more general, is evident, but much remains to do.

Peripheral remarks. At the cost of overdoing the matter but for clarity, I again return to [AB, 6.1]. The expression $\star F(A)$ turns out to be simply a real-valued function and not a section of some more complicated bundle, d_A is simply the ordinary differential of this function. More than that, for the bundle Q we chose $F(A)$ and $\star F(A)$ constant, if A was Q or a power of Q. As a consequence, the connections introduced with the constructions preceding the definition ρ in [AB, p. 560], turn out to be flat, in a strong sense, since they yield the usual derivative on M, in this case the (x, y)-plane. This appears in (53). I remark this here, because it is necessary to explain that, even though this connection is very simple, it contains two free real parameters, the velocity in two independent direction. With the limitation that k, ℓ lie in $\mathbf{Z}$, and not in $\mathbf{R}$, we abandon the domain of connections and pass to the eigenvalues of Hecke operators, which for GL(1) are given by functions with values in $\mathbf{U}(1)$, and not functions whose values are conjugacy classes. We shall return to this. ■

Unexpected consequences that will be emended below. It seems that the introduction of the function $s(\cdot)$ will turn out to be the central discovery of this article, at least for me, because with it abstract concepts are accessible. First of all it leads to the definition (36.h) and then to (36.i), which, in turn, leads to the constant curvature (47). There is still another agreeable consequence. This is the connection to the natural curvature in the formula (47) and its relation to A_0 and Λ_0. I assert that replacing A_0 by A_0' leads to a constant (in relation to flat coordinates) change in the connection and, therefore, there is no change of curvature. It remains constant. More than that, the change in the connection itself is very simple. A constant term is added. ■

Still more remains to do, but it is best to stop and consider our position. In essence, when we, finally, return to the demonstration [AB, Th. 6.7], a great part of the preparation will turn out to be superfluous. However, at least for me, this was very useful. So far we have a complete view of the structure of linear Yang-Mills connections, and we find ourselves now on the edge of an understanding of the structure of Yang-Mills bundles of dimension two. This is of course only for elliptic curves. They appear as direct images of line bundles on one of three possible double coverings of M. The transition is from M and a representation of $\Gamma_{\mathbf{R}}$ to the covering M' and the representation ρ of its fundamental group $\Gamma'_{\mathbf{R}} \subset \Gamma_{\mathbf{R}}$.

In my view, even if it is not the view of the reader, there is some careless understanding that has to be corrected. During the discussion of the consequence (36.i) we introduced a connection on the bundle $\Lambda_0\Lambda_1^{-1}$, and the curvature of this connection was zero. In so far as the metric on M or on $\mathbf{C}$ is invariant under translation, this connection is (rather these connections are) Yang-Mills connections.

They are not the connections appearing in (53). These are simply constant connections on the trivial bundle. They are introduced in [AB, (6.12)] with H and β trivial, if we replace $\mathbf{U}(1) \times \pi_1(M)$ by $\mathbf{R}^\times$. If $G = \mathrm{GL}(1)$, $\overline{S} = \{1\}$. This is the correct assertion. It would seem that there is another careless, although minor, error. The word "flat" does not mean well-defined. It means well-defined on the universal covering. The reader with a limited familiarity with differential geometry should be aware that a flat connection is one whose integral defines a function, locally well-defined but, perhaps, globally multi-valued. Topologically, it distinguishes itself by changes among the leaves. Thus, for an elliptic curve it may be linear. There is still something else to keep in mind. The two groups $\mathbf{R}$ and $\mathbf{U}(1) \times \mathbf{Z}$ are very close. I excuse myself for the footnotes and digressions, but we are working in a complex context.

Useful but superfluous explanations. In this attempt to introduce or to create a theory of functoriality in the context of the simplest case but with almost no understanding of the relevant ideas, I avoided the very concepts that, inserting themselves on the path through ideas that, although not relevant in any strong sense, became in the final analysis the source of the solution. I made every effort to cast odd, false or unnecessary ideas away and to keep only relevant ones, but this turned out to be impossible, because they were inseparably attached in my head with the solution. However, the following lines are clearly unnecessary. I kept them in order to stress that the entire article was in need of a revision, but that it was better to wait for the creation of a general theory. What is finally striking in the resolution is the complete precision of the comparison between eigen sections and Yang-Mills connections. The diagram (56) is for amusement.

In the diagram (56) each square, whether black or white, represents by itself a simple covering of M. The union of a black square with the white square above it is a simple covering of M'. A one-dimensional Yang-Mills connection on M' is given by a constant, which itself determines a bounded representation ρ' of $\mathbf{Z}$, thus $z \to cz$, $c \in \mathbf{R}$. A one-dimensional (local) section $z \to f(z)$ of a bundle on M' is simply a function $f(\cdot)$ with values in $\mathbf{R}$. This determines (again locally) a section of a two-dimensional bundle on M. It is given by $z \to \big(f(z), f(z+B)\big)$, thus the value of f in the black square joined with its value in the white square lying above it. A brief reflection is necessary in order to understand the two-dimension gluing that this entails. The value in a given black square must change smoothly when we move in either direction, up or down, and left or right, and also diagonally, not to the neighbouring white square but to the black square. The same is true for movement from a white square to a white square. But there is still more. The movement includes in itself a change of two coordinates as well as a change of the tangent vector in a way given by the section. If we are dealing with a connection, this is a regular movement in a diagonal direction.[90] For this the black and white squares can be considered separately. It is however necessary to ensure that the result is smooth when we pass from squares of one type to squares of the other type. This, undoubtedly, will be so for constant connections and their integrals. As for the partitions, but not the connection, the behaviour in the centre of the black squares does not depend on the behaviour in the white squares. The corrections occur on the boundaries of the squares. I remark as well that A and B play the same role.[91] ■

There is yet another detail that should not be forgotten. In the transfer from M to M' we replace J by the element J' and, thus, we replace the group $\mathbf{U}(1)$ with another group, although it still turns out to be $\mathbf{U}(1)$. Just as for GL(1) the resolved connection yields a constant function, but now two constant functions, the order of which is irrelevant.

Finally, we discovered a two-dimensional form of the expression (53), but the question remains. What are the conditions of periodicity. The situation is that we want to represent the eigen conjugacy class as an integrated exponential function. For GL(1) the conjugacy class has a unique representative, but for GL(2) this is not so. We return briefly to the equation (53), but the question remains. What are the conditions of periodicity? The difficulty is that we want to represent the eigen conjugacy as an integrated exponential function. For GL(1) the conjugacy class has a unique representative, but for GL(2) this is not so. We return briefly to the equation (53), for which there are two conditions under discussion: periodicity and initial conditions. We already discussed periodicity. This is the integrality of the numbers k and ℓ. The initial conditions are introduced by means of the multiplication of (53) with an arbitrary element of $\mathbf{U}(1)$.

Initial conditions. Namely this element is determined by a character of the subgroup $\mathbf{Z}$ of the group Γ_{aut} in (1.d), thus on $1 \in \mathbf{Z}$. This $\mathbf{Z}$ consists of powers of an arbitrarily chosen $\Lambda_0 = \Lambda_{A_0}$, which itself determines the lower limit of integration in (53).

Here something arises that it is easy to lose sight of, the arbitrariness of our choice of Λ_0 or A_0. This is necessary but also ubiquitous. In (53) the moving point is $a\theta + b\theta$ and the connection provides the integrand. By itself this expression does not provide a conjugacy class in the point A_0. It is necessary to multiply it with the value of the conjugacy class at the point A_0. This can be arbitrarily chosen and is the reason for the supplemental factor $\mathbf{Z}$ in (1.d). There is something that it is easy to forget, when we arrive at the desired pair—the integral of a Yang-Mills connection on one hand, and the conjugacy class for GL(1) or the eigen conjugacy section for GL(2) on the other. Thus, we are not attempting to find the eigenfunction for which there exists an indeterminate constant, but a family of eigenvalues and this is not ambiguous.

For GL(2) the construction is more complicated. As already explained, in the geometric theory the eigenvalue varies from point to point, changing the conjugacy class, which also varies from point to point on M. For GL(2) this is a question of giving two numbers, the eigenvalues of the class, and a single ambiguity—this is the order in which they are given.

[90]This statement is unclear. I believe it is a reference to a movement in a four-dimensional space given by the base and the fibre.

[91]This explanation is not satisfying, for reasons related to the footnote 'Insignificant oversight,' namely, we were imprecise in connection with the pasting. We are dealing with flat connections and these we may integrate on a single-valued connection over $\widetilde{M}$. Moreover, a line joining a point in one of the squares of the diagram with another determines an element of the fundamental group and, therefore, a movement from the fibre at one end to the fibre at the other. This is the glueing understood here.

A clearer expression is that we are dealing with a conjugacy class of two-by-two hermitian matrices, and these determine its eigenvalues, both of which are real. However they do not have a determined order. Thus there is a collection of possibilities for presenting the spectral decomposition. The group $\pi_1(M)$ has three normal subgroups of index two, each of which contains only one of A, B, AB and we may pass to the covering determined by any choice of one of them. When we do this, the two-dimensional representation of the corresponding subgroup of the group G has two different irreducible components of dimension one. Thus we have in essence six possibilities for describing related Yang-Mills connections. This evoked in me considerable uncertainty and confusion. My description of the conclusion may evoke the same feelings in the reader. However it became familiar to me.[92]

For $G = \mathrm{GL}(1)$ the correspondence consists, on the one hand, of a Yang-Mills connection and the representation of $\Gamma_{\mathbf{R}}$ attached to it, on the one hand, and the corresponding eigen conjugacy class as well as the representation of Γ_{aut} attached to it on the other. We understand how this functions also for $G = \mathrm{GL}(2)$ in so far as the representations or the connections are direct sums of some corresponding one-dimensional objects. In the general case, general for GL(2), we consider only Yang-Mills connections, and they, according to [AB], are given by representations of $\Gamma_{\mathbf{R}}$. Besides that, as we already discovered for GL(1), for our purposes not all connections are relevant. This entails the restriction to a special choice of such representatives, those trivial on the inverse image of $\mathbf{U}(1)$ in $\Gamma_{\mathbf{R}}$. For GL(2) this is not so.

The analog of (53) will be an exponential function, whose exponent is the integral of a matrix. It may be diagonal in so far as the coordinates a and b are determined on the sides of the fundamental domain. The condition is that the integrals of the two diagonal elements are periodic except for a common sign.

We chose a metric on M invariant relative to the periods and the point A is given. At first it was $0 \in L$. We may replace it by any other point, thus a transfer of the point A_0, which yields a transfer of the function $s(\cdot)$ and also of the form (36.i). The first is arbitrary and determines the two others. The changes entail a modification of the Yang-Mills connection, but as we saw there is no change in the curvature, which is constant. The function $s(\cdot)$ itself becomes a function on the plane and not on M. As we described, the replacement of A_0' by A_0 entails a change of the related connection, although this change is not in its curvature. Its change, thus the difference between the connections related to A_0 and A_0', is flat, that is it has zero curvature. In so far as A_0 is given, the remark just made allows us to determine the Yang-Mills connection for the group GL(1) with integers k, ℓ and a linear bundle of degree zero. It is itself related to the difference $A_0' - A_0$. For the group GL(1) the homomorphism ρ in [AB, Th. 6.7] is necessarily such that $\mathbf{Z} \subset \Gamma_{\mathbf{R}}$. The restriction $\rho|\mathbf{U}(1)$ is given with a whole number and $\rho|\pi_1(M)$ is given by two numbers.

All this, perhaps, it may be best, at least if the language used is Russian to describe in connection with the notion of "строка." This does not really mean anything to me. A reader with a limited familiarity with differential geometry may know that a flat connection is such that its integral determines a function that is locally single-valued, but globally, perhaps, multi-valued. It distinguishes itself topologically as a shifting among loops. Thus for an elliptic curve it may be considered linear. Let $k = 0$, $L_0 = A \cdot A_0^{-1}$. A connection attached to $A_0 = 0 \in L$, has already been introduced, but it depends on A (or A_0') as well as a point

[92]Added in 2020—but is daily growing more uncertain.

in $\mathbf{C}$ rather than a point in $\mathbf{C}/L$. The calculations related to A_0 are also applicable for A. This is all simply a question of introducing new values for the coordinates. In particular, the curvature remains the same and it is constant. The new bundle is simply a translation related to A. As far as the change of coordinates, this is simply a translation of the initial data, thus simply the addition of constants to the two coordinates a and b in (36.i). These two connections determine a connection on the new L_0, whose curvature we can calculate. It is the difference of two curvatures, the curvatures for A and A_0. In other words, the constant in [AB, (6.10)] is arbitrary, although there is a decision that I find inappropriate and that evokes perplexity. This is to take it modulo 2π. The reader can decide for himself. Either he must, like me, simply correct these as signs of a hurried composition or he must take them seriously. As a consequence of the difference [AB, (6.10)] all these are Yang-Mills connections for the group GL(1).

In the introductory sentences there is a certain ambiguity. Two different connections may lead to isomorphic sections. For example, the Lie algebra of the two groups $\mathbf{R}$ and $\mathbf{U}(1)$ are the same, so that they may have equal connections but with different integrals. I usually take the first possibility.

Explanation. When are two Yang-Mills connections equivalent? This is only one of many questions arising during an examination of the conclusions in [AB, §6]. For example, in so far as the corresponding connections must be unitary, they, as we have seen, depend on the metric on the fibre and it is not invariant. This means, in particular for the connections that we are examining, that they vary linearly with the parameters a and b and, therefore, are not L invariant (36.i). This is difficult for me to understand. An integral modification of the parameters a and b changes the metric on the fibres but not the connection; consequently it is necessary to consider it as a modification of the data. Thus, the family stops being a torus and becomes a complex line. This is important for our statements and for [AB, Th. 6.7], which without this has no sense. The condition [AB, 6.1] is linear. ■

This assertion is almost incompatible with [AB, (6.10)] and with the assertion that a flat connection on the trivial bundle of Yang-Mills is of Yang-Mills type.

The error corrected or the devil is in the details.[93] This essay is a digression. This is because it was impossible for me to understand the material in [AB] on the first, second or even third reading. I had to return frequently to the sources, in order to understand fully the meaning of the authors' statements. If M is an elliptic curve then $\widetilde{M} = \mathbf{C}$. It is easy to imagine. It is also easy to imagine the group $\Gamma_{\mathbf{R}}$ and its representations in GL(1). It is also important to note that $\Gamma_{\mathbf{R}}$ has several automorphisms, so that some of these, evidently different, representations, may be equivalent. This would be the source of possible equalities, described in the discussion of the equation (36.k). The group $\Gamma_{\mathbf{R}}$ is generated by A, B and the group $\mathbf{R}$ of real numbers with the one relation $ABA^{-1}B^{-1} = \exp(2\pi \cdot i)$. We write the element x in $\mathbf{R}$ formally as $2\pi \cdot xi$. Some, possibly all, automorphisms of the

[93]In so far as there was much in [AB] that I did not completely understand and also one or two points that at the best were inadequately explained by the authors, different parts of this article were written at various steps of my struggle with the material, I resolved, partly because of indolence and partly because it might be better for those, who like me are studying the basics of differential geometry together with its application to the theory of automorphic forms, leave a part of the material in the disordered form in which I first understood it.

group are given by the equations $A \to \exp(i\lambda)A$, $B \to \exp(i\mu)B$, λ, $\mu \in \mathbf{R}$, and if $x \in \mathbf{R}$, $x \to x$.

All one-dimensional representations of the group $\Gamma_{\mathbf{R}}$ are such that $\exp(2\pi \cdot 1) \to 1 \in \mathbf{C}$. Thus, there is such an $m \in \mathbf{Z}$ that $\lambda \in \mathbf{R} \to \exp(2m\pi i\lambda)$, which is equal to 1 if $\lambda \in \mathbf{Z}$, but $A \to \alpha$, $B \to \beta$, $|\alpha| = |\beta| = 1$. These two numbers are arbitrary. At the present moment we are dealing only with linear bundles and only with linear bundles of degree zero. This is possible because we introduced a supplementary linear bundle that allows us to describe each linear bundle as a product of this linear bundle with itself several times and, in addition, a bundle of degree zero. Can this be done in another way. This is a question that at the movement it is best to set aside. At the moment our problem is to find the linear bundles associated with the homomorphisms just described. ■

Inappropriate hesitancies. The factor $s(\cdot)$ or its inverse element do not determine with the equation (36.b) the metric on M, they determine a metric on its universal covering. This means that a large number of our assertions are correct only for the universal covering. This is in order but it is necessary to recall it. The ambiguity is removed only when we pass to a connection related to the difference $A_0' - A_0$. Even so, we have obtained a great advantage. The Yang-Mills connections that are determined by a metric invariant under translations are given by two real parameters, determining a movement with constant speed in the plane. Thus we proceeded with the minor explanations related to (53). What happens? The factor A in $A \cdot A_0^{-1}$ changes but it is possible that it returns to its starting point in M but not in $\widetilde{M} = \mathbf{C}$. This is possible because $s(\cdot)$ is a function on $\widetilde{M}$ but not on M. ■

The incomprehensible becomes understandable. §VIII begins with five assumptions, the full significance of which was clear only to me, but not fully. In particular, I did not understand the sentence that followed them, 'Given any homomorphism $\rho : \Gamma_{\mathbf{R}} \to G$ we then get an induced G-connection A_ρ also satisfying the Yang-Mills equations...'. Up to that moment I had succeeded in coming to some understanding of the construction of the bundle Q and its consequences, but without a full understanding of its precise significance. Something, something special, had fully escaped me. This was the construction of a flat connection of dimension one. Rather, I understood this but not its consequences, at least not with their full significance. Besides this, beginning with the classical theory, thus the theory of Weierstrass and others, I did not know how far this still remained from the passage to unitary connections. I had not evaluated the distance, dividing it from the unitary theory. This question arose not so much for $\mathbf{U}(1)$ connections as for the related $\mathbf{R}$-bundles. I had difficulty imagining the consequences of the introduction of zeros or poles, as with the construction of Q. I am not a topologist! More than that, I introduced the extremely important connection Q very casually, with the definitions of $\widetilde{\alpha}$, $\widetilde{\beta}_1$, $\widetilde{\beta}_2$, $\widetilde{\gamma}$ underneath the formula (36.i). Now we have to examine it with more care if we want to appreciate the definition preceding [AB, Th. 6.7], in so far as this is the principal particularity of their efforts. Rather, I think that we are, finally, in a position to use this theory for a comparison of eigen conjugacy classes with Yang-Mills connections. ■

After all our efforts, in particular, after all my efforts, an unexpected question arises in connection with the application of [AB, Th. 6.7]. Do we take an arbitrary homomorphism $\rho : \Gamma_{\mathbf{R}} \to G$ or a limited class? In order to continue, I suppose that [AB, p. 559/560] is open before the reader. The difficulty is that we apply a theorem with a limited choice of ρ.

I am amazed, but I have often been amazed as I wrote this article. We take $G = \mathrm{GL}(1)$ as an additive group $i\mathbf{R}$ with a one-dimensional representation ρ, whose values are arbitrary on the representatives of A and B. On $\mathbf{R}$ their value is zero. In particular, the factor $\mathbf{U}(1)$ is not relevant here. The representation ρ is trivial on it. On the other hand, there is more freedom than I thought in the form of the G-homomorphism ρ. $\mathbf{U}$ is replaced by $\mathbf{R}^\times$.

Thus the exponents that appear in formula (53) of the Yang-Mills connections are limited by the condition that the exponential functions determined by their integrals are single-valued on M. With this it must be clear that the fundamental conclusion of this article is established for GL(1). However, as the final goal we hope to show that (1.d) on a given algebraic curve is the automorphic galoisian group. None the less, in this article we are concerned only (or primarily) with GL(2). There are two kinds of eigenfunctions: those whose support is $\mathfrak{D}$ and those whose support is $\mathfrak{A}$. The first correspond to an unordered pair of eigenfunctions for GL(1), whose parameter is given by (50.a). For those whose parameter is given by (50.b), the relation of the parameter to the eigenfunction is more complicated. Yang-Mills connections appear. The transition is: the homomorphism $\Gamma_{\mathrm{aut}} \to {}^L G$; the Yang-Mills connection; the choice of integral that gives the eigen conjugacy class is made. The conclusion is then compared with the results of Section VII.

It is necessary first to turn to the second doubtful assertion in [AB, p. 561]. There is no reason for $G_X = G$. By the way, to avoid confusion I recall that $\Gamma_{\mathbf{R}}$ and Γ_{aut} are different, although closely related groups.

For GL(2) there exists eigenfunctions of two forms in relation with the subset B on which they are defined: $\mathfrak{D}$ and $\mathfrak{A}$. For the first, the eigen class of the section is an unordered direct sum, the parameter of which is also a sum of two one-dimensional representations of the group Γ_{aut}. Some of these, of the second kind, are also related to the first kind. We can ask why they appear but we do not need to find for them new parameters. These are other functions with which we have to deal and each of them is related to one-dimensional representations of three subgroups of the group $\Gamma_{\mathbf{R}}$, determined by the inverse image of three subgroups of index two in π_M. We remark that the representation τ induced from the trivial representation of each of these subgroups of the group $\Gamma_{\mathbf{R}}$ is the same and their restriction to the given subgroup is the direct sum of the trivial representation and the unique non-trivial representation that is trivial on M^2. As a particular choice, we chose the subgroup generated by $A' = A$ and $B' = B^2$.

The passage to the covering group affects not only $\pi_1(M)$, which is replaced by the subgroup $\pi_1(M')$ of index two, but also the element J, which is replaced by $J' = 2J$. The group $\mathbf{R}$ however is not changed. Consequently $\Gamma'_{\mathbf{R}}$ turns out to be a subgroup of order two in $\Gamma_{\mathbf{R}}$. On the other hand, we began with a one-dimensional connection on M' and related to it a two-dimensional connection on M. At the same time, the one-dimensional connection is related to the one-dimensional representation of $\Gamma_{\mathbf{R}}$ and the two-dimensional connection is also related to the two-dimensional representation of $\Gamma_{\mathbf{R}}$. What we want to do is to verify that the two-dimensional connection is the one given by the theorem of Atiyah-Bott for two two-dimensional representations. Once again, the two-dimensional representations of $\Gamma_{\mathbf{R}}$ are not all relevant.

We now consider the discussion on [AB, p. 560], applying it to both groups $\Gamma_{\mathbf{R}}$ and $\Gamma'_{\mathbf{R}}$. The elements X in $\mathfrak{g}$ will be the same for $\Gamma_{\mathbf{R}}$ and $\Gamma'_{\mathbf{R}}$. Thus we pass to G_X for both of them, not asserting however that G_X is connected. The $\mathbf{U}(1) \times \pi_1(M)$-bundle on M turns out to be simply the direct image of the bundle $\mathbf{U}(1) \times \pi_1(M')$ on M, although our geometric description, perhaps, does not show this clearly. In so far as the composition of two direct images turns out to be again a direct image, our argument is complete.

Now only a minor matter remains. This is simply a question of leftovers. In 'The transition from Γ_{aut} to $\Gamma_{\mathbf{R}}$' we considered this passage for one-dimensional representations. The discussion is equally pertinent to the induced representations of two groups, for the field and for an extension of the field.

Another confession. For me, it is difficult to understand completely the theory of Yang-Mills. I never considered the general theory. Consequently it is necessary to recall constantly its consequences. My apologies. ■

As I frequently acknowledged, I only slowly understood and still only partially understand, the relation between eigen conjugacy classes and Yang-Mills connections. In particular, I was careless when I wrote (53), rather I did not explain how to choose the initial values. They are given by the powers of Λ_0, in particular by Λ_0^0, thus by the trivial linear bundle, where the value is 1 (perhaps rather 0). The value in other degrees is given as we explained in the section 'Initial conditions.' The definition is complicated but consistent. More than one initial condition is necessary because Bun_G is not connected. The eigen sections for GL(2) are not so easily described. In addition, even for GL(1), only those conditions that yield a single-valued result are allowed. It is also necessary to underline, that for a given group G the appropriate connection is a ${}^LG$-connection, better a ${}^L\mathfrak{g}$-connection or ${}^L\mathbf{U}_G$-connection, where ${}^L\mathbf{U}_G$ is a unitary form of the group ${}^LG$. Only the latter appears fully precise. The designation is not fully precise. In contrast to the group ${}^LG$, which is somewhat imprecise, because we may introduce a galoisian component. In addition, ${}^L\mathfrak{g}$ is ambiguous only in so far as it may be the Lie algebra of the group G or of its compact form.

In order to continue, I suppose that $G = \mathrm{GL}(1)$, $\mathrm{GL}(2)$ or, perhaps, $\mathrm{SL}(2)$. The fixed point A_0 and the constant linear bundle Λ_0 were chosen in §4, as in [A]. According to [A, Th. 5, Th. 6], after this choice, for $G = \mathrm{GL}(2)$, the connected components of Bun_G, represent for us a fundamental interest. They are given either by an unordered pair $\{(m, \Lambda_1), (n, \Lambda_2)\}$, or by (m, Λ), m_1, m_2, m lie in $\mathbf{Z}$. We know already that eigenfunctions are determined either by a set of points of the first kind or by a set of points of the second kind. One factor is $\alpha^m\beta^n$, α, $\beta \in \mathbf{C}^\times$, $|\alpha| = |\beta| = 1$ or is simply α^m.

The intention of this section is to persuade ourselves that Γ_{aut} has some convincing properties, at least for the groups GL(2) and SL(2) and elliptic curves. The presence of $\mathbf{Z}$ in the determination of the groups $\widetilde{\Gamma}$ and Γ_{aut} arises from the presence of the degree and its purpose is clear. We consider principally Γ and $\lim \mathbf{Z}/n(k)\mathbf{Z}$.

XI. Integrable connections and eigenfunctions of Hecke operators

The case $G = \mathrm{GL}(1)$ was treated in the paragraph 'Some general remarks.' The function[94] (53) may be considered as a function on all of M, taken as a connected component of Bun_G. Then they can be extended to all of Bun_G, adding a factor α for $\Lambda_0^n\Lambda$, if Λ_0^n is a

[94] There is an important question that puzzled me when I wrote this article and whose answer, discovered only at the end, is amazing. The eigen conjugacy sections, apparently just as for GL(1) are given by an integral of a Yang-Mills connection, but how are the initial conditions calculated? For GL(1) this is clear. We determine them with the degree of Λ_0. For GL(2) not only the bundle but also the initial conditions are given by the direct image of the section of the GL(1)-bundle. It is possible that something similar is true in general.

member of this component and $\alpha \in \mathbf{U}(1)$ is given. In this paragraph a parametrisation of the representations of Γ_{aut} is also described.

On the other hand, for GL(1) the set Bun_G is an abelian group with a connected component. It is possible to treat it as the direct product of $\mathbf{Z}$ with M. Thus, in this way, two functions, the one equal to the Hecke eigenfunctions and the one determined by its eigenvalues, reveal themselves as one and the same function on M. The eigenfunction, of course, is determined only up to a multiplicative constant.

But for $G = \mathrm{GL}(2)$ there is not a clearly determined element in ${}^LG$ at each point. Only a conjugacy class is available and this ambiguity makes itself clear. Two apparently different sections may belong to a single class. Thus at each point their value is only a conjugacy class in Bun_G. Recall that these are ordinarily, perhaps always, unitary elements in ${}^LG$.[95] We observe finally that the natural representative of a conjugacy class might be a smooth function on the plane, whose values at two points, differing by an element of L, may be conjugate but not equal. We already saw this in §VII, although only implicitly, because the sign in (32.a) is indeterminate. However, the function $f(\cdot)$ is uniquely determined on the particular covering M'. This is an M' for which there are three possibilities. We describe one; the others are similar. What happens is that the coefficient α_x in (32) may change in sign when we move through a period, but the conjugacy class does not change. Moreover, this may happen in three different directions, given by $L/2L$. Thus, as we added a supplementary factor[96]

$$\varprojlim \mathbf{Z}/n\mathbf{Z}$$

in (1.d), we may add it to $\Gamma_{\mathbf{Q}}$. This enables us to pass to all of $\mathfrak{A}$, once we understand $\mathfrak{A}(0,0)$. Thus, it suffices to study the following construction on $\mathfrak{A}(0,0)$. The logarithmic function or its derivative determine a one-dimensional connection on M' and its direct image is a two-dimensional connection on M. Both the one and the other are Yang-Mills connections. It is clear that the integral of this section is related to the function $f(\cdot)$.

It remains to discuss the connection between these bundles and representations of $\Gamma_{\mathbf{Q}}$ (or of $\Gamma_{\mathbf{R}}$) and $\Gamma'_{\mathbf{Q}}$ (or $\Gamma'_{\mathbf{R}}$). Recall that $[\Gamma_{\mathbf{R}} : \Gamma'_{\mathbf{R}}] = 2$ and that $J' = 2J$. Moreover, and this is the key to and the solution of this article, the character of $\mathbf{R} \subset G'_{\mathbf{R}}$, connected with the character χ determining $f(\cdot)$ in (32), is equal to -1 in the point J. We waver between Bun_G, thus a disconnected set and a connected subset, thus the set of bundles of degree zero. Some details are left to the readers.

I repeat something that we already noticed. A character of the group $\mathbf{R}$ may be described as

$$\exp(2\pi i\alpha x)\exp(2\pi i\beta x), \quad \alpha \in \mathbf{Z},\ 0 \leqslant \beta < 1,$$

where the first factor has the period 1, because its wavelength is equal to $1/\alpha$, but the second factor has a wavelength equal to a whole number, itself equal to the denominator of the fraction β. In the present case this is two, reflecting the group GL(2). Thus we know what to expect if $G = \mathrm{GL}(n)$. I do not know what to expect for other groups.

The situation is the following. Eigen sections of the form $\mathfrak{A}$ are related to the one-dimensional bundle on M' the value of which at the point $J = -1$. Indeed the wavelength

[95] This contrast between the phenomena familiar for a number field and those that we observe in this essay on the geometric theory for GL(2) astonish me.

[96] Although I very carefully chose A_0, A'_0 and the initial conditions in (53) I often forgot to draw my own attention and that of the reader to this. None the less they are present constantly during all of our discussion and remain similar, but not explicit and, apparently, not identical.

with respect to J is equal to 2.[97] It determines a one-dimensional representation of the group[98] $\Gamma'_{\mathbf{Q}}$. We already understand that the induced representation of the group $\Gamma_{\mathbf{Q}}$ is irreducible.

We discovered that the corresponding representations have two forms $\mathfrak{A}$ and $\mathfrak{D}$. The parameters of the latter are simply two unordered parameters for GL(1). Thus the ambiguity is also simple. For the first ambiguity is also possible, but it has a different appearance and I was confused for a long time. Of course, the representations of the form $\mathfrak{A}$ were studied in §VII and it is only necessary to turn to that. We discovered that the eigen sections were given by functions with values in ${}^LG$, which were periodic either with period L or with periods in a lattice lying between L and $L/2$, so that they could be linked to one of three double coverings M' of the curve M. They were discovered before I understood the corresponding parts of [AB, §6] sufficiently well and they were a major riddle for me.

Important recollections. There is something so simple that it does not seem worthwhile to recall it again, but for us it is very important to keep it in mind. The group $\mathbf{R} \subset \Gamma_{\mathbf{R}}$ is such that $J = 1 \in \mathbf{R}$. Even so, it is not necessary that for a given representation $J \to 1$. Nevertheless we consider only those characters χ of the group $\mathbf{R}$ such that $\chi(J)$ is a root of unity. Then $\chi(x)$, $x \in \mathbf{R}$, is such that $\chi(x) = \exp\bigl(2\pi i(k+\alpha)x\bigr)$, $k \in \mathbf{Z}$ and, if we consider $\Gamma_{\mathbf{Q}}$, then $\alpha \in \mathbf{Q}$, $0 \leqslant \alpha < 1$. The two characters $x \to \exp(2\pi ikx)$, $x \to \exp(2\pi i\alpha x)$ have altogether different purposes. The values of the first characters are 1 on $\mathbf{Z}J$. Consequently they are given on $\mathbf{R}$ as the inverse image of a character of the group $\mathbf{U}(1)$, thus $u \to u^k$. We already discussed these. The second factor is the value of J and we already discussed this in (55). I underline that the value of J is $1 \in \mathbf{R} \subset \Gamma_{\mathbf{R}}$ but it is possible that $\rho(J)$ is not the unit element. These reflections are important for us when we consider the double covering M' of the curve M, since then $J' = 2J$. ■

At the first glance the argument in §VI consists of the following: there exists four forms of eigen conjugacy sections, which are related to the type $\mathfrak{A}$; each section is given by a character χ on $\mathrm{Pic}(M)$ and its form is distinguished by its restriction to $\mathrm{Pic}_2(M) \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, which is either trivial or one of the characters χ_i of the group $\mathrm{Pic}_2(M)$. Those χ that are related to the trivial character of this group are equivalent to classes of the form $\mathfrak{D}$. This was established in §VI.

Those which are related to the other three characters are determined by functions $x \to \alpha_x^2$, as in (32), because only the conjugacy class of the matrix (32) is relevant. However, if two characters are equal on $\mathrm{Pic}_{\mathrm{even}}(M)$, then they are equal on $\mathrm{Pic}(M)$ up to a character of the group $\mathbf{Z}/2\mathbf{Z}$. Although we did not examine them, it is clear in §VI that this is relevant only for the restriction of the character to $\mathfrak{A}_{\mathrm{even}}$. This restriction to $\mathfrak{A}_{\mathrm{even}}$ is what determines the eigenfunction and the eigen class. Consequently, the three different classes of eigen conjugacy classes do not intersect.[99] I did not expect this. Some time passed before I remarked this important circumstance, and this invoked in me a serious confusion. It was rather difficult for me to explain the obvious consequence of the discussion that (32) and (32.a) evoked. If the squares of two continuous non-vanishing functions are equal everywhere then they themselves are equal up to a constant sign. Thus, if the functions

[97] Added in translation: this second sentence clarifies the first.

[98] The reader is to complete the determination of the supplementary factor.

[99] If two classes of the same type coincide, then we would have two characters χ_1 and χ_2 with periods in $L = 2L/2$, such that χ_1/χ_2 would be equal to ± 1 everywhere and consequently would be equal to 1 everywhere. This is impossible if the two characters are not equal.

are equal at some point of a connected set, then they are equal everywhere. From this it follows, that the three mutually exclusive forms discovered in §VI do not overlap.[100] Two such functions cannot be equal everywhere unless the coefficients m, n are equal. As a consequence, the eigen classes related to $\mathfrak{A}$ are all equal, even those that are also related to $\mathfrak{D}$. In particular, there exist three types of these that are related only to $\mathfrak{A}$, each of which is determined by a different non-trivial character of Pic_2 and, consequently, a different quadratic extension. The distinction is determined by the remainders of m and n modulo two. Whole numbers determine the exponential function in (32).[101]

We now have in the class of representations related to $\mathfrak{A}$ four classes, one of which is also related to the class $\mathfrak{D}$ and three related to three non-trivial characters of the group $\mathrm{Pic}_2(M)$. Each of these non-trivial characters determine with their kernel a quadratic covering M' of M. A pair, M' together with a character, determines a two-dimensional representation of the group $\Gamma_{\mathbf{R}}$. All that we need do is to show that these representations are different, but this is clear from the above discussion, and also that they contain in themselves all two-dimensional representations of $\Gamma'_{\mathbf{R}}$ of the appropriate form. But what is this? It is clear that the representation of $\Gamma_{\mathbf{R}}$ or of $\Gamma_{\mathbf{Q}}$ is irreducible only if the restrictions to Γ are irreducible. We understand this from the diagram (55). They are induced representations and $J \to -I$.

This is now completely clear. We have three types of M, just as in §VII, each of which is uniquely determined by the relation of M' and M. Each of them then determines, again uniquely, a character of M' of order two. This is namely the one that we needed in the preceding section, in order to determine the representation $\Gamma_{\mathbf{R}}$, and then $\Gamma_{\mathbf{Q}}$.

On the other hand, the direct image of a one-dimensional Yang-Mills connection is a two-dimensional Yang-Mills connection. This is an obvious consequence of the definitions in [AB, p. 560]. The construction of Atiyah-Bott clearly connects a two-dimensional representation of the group $\Gamma_{\mathbf{R}}$ (or of $\Gamma_{\mathbf{Q}}$), induced from a one-dimensional representation of $\Gamma'_{\mathbf{R}}$ (or of $\Gamma'_{\mathbf{Q}}$) to the direct image of a linear bundle related to it. It is necessary only to read carefully the first few lines of p. 560. We may take this conclusion as our theorem. This is unconditionally the conclusion of this article, but a very much more important statement is under discussion, namely a precise general theory, which we might consider as the basis of the geometric theory of automorphic forms—a theory parallel to the arithmetic theory, which itself has yet to be constructed.

I recall to the reader once again, but for the last time, that $J \to -1$ is a relation for a one-dimensional representation of the group $\Gamma'_{\mathbf{R}}$!

XII. On the possibility of a general theorem

The problem is clear. We need first of all a general form of the theory of Atiyah and with this it will be necessary to understand the Hecke operators. This seems to me accessible and very promising, but difficult. I have no definite proposals but it does seem to me that the Gauss-Bonnet for particular manifolds would be necessary, thus the study of Pfaffian forms on Bun_G. It seems to me that this alone would attract significant interest in connection

[100] Although I leave this conclusion in its present form, it would have been better to remark that the eigen conjugacy sections considered are determined by their determinant and that these are exponential functions with a linear exponent $2ma + 2nb$, $m, n \in \mathbf{Z}$.

[101] This latest paragraph is unnecessary and confusing, but it is correct.

with Hecke operators.[102] I do not know what differential geometric difficulties will manifest themselves in such an attempt. It is not clear whether a ramified theory could be considered a path to the study of coverings of the initial curve.

Confession. My attempt to write a mathematical article in Russian has no basis, except a wish to understand finally a language, with which I acquired some familiarity, although not a great deal, in my youth, in the beginning of my life as a mathematician, for which this was at the time regarded as necessary. This is no longer so. An incentive to the attempt was an invitation by Dmitri Lebedev to visit Moscow, which I accepted and for which I was and remain thankful.

I discovered that my knowledge was not deep. My native tongue is the English language and, according to my experience, the mastery or the attempt to master Russian or Turkish is completely different than a decision to learn French or German. A genuine composition, as this was, is better than remaining completely silent. Russian words, Russian nouns, Russian adjectives express the real world. This is a world whose nature I still do not understand. The present essay is better thanks to a conversation with Valentina Sergeev about the nature of Russian words, for example, 'скрыть.' Above all, it is better because Oktay Pashaev read the essay carefully from beginning to end and made innumerable suggestions for improvement.

He read the initial text, correcting my lexicology, my choice of nouns and verbs, from beginning to end, and made numerous suggestions for improvement.

If and when the article is published, if ever, it will be subject to further corrections,[103] but I, at least, now understand that with Russian as with Turkish, we are dealing with a structural complexity at a different level than that of languages, like German or French, closer to my own.

The article is a consequence of two impulses, first of all, an attempt to understand the nature of the geometric theory, to form for myself a clear notion of the difference between it and the arithmetic theory and, of course, of the similarities. Here, I believe, I was successful, although I proved little. The second goal was to improve seriously my knowledge of Russian. Here I had limited success. For me, Russian is at a completely different level than the two foreign languages, French and German, with which I am familiar. As I observed above, the Russian language is substantially more complex than I imagined, even more than Turkish, another language with which I have a limited ability acquired only with difficulty. Thus my efforts and the efforts of friends and acquaintances, who encouraged me as I wrote this article, had limited success. I am none the less pleased that, in spite of my age, I do not regret either the time or the effort I gave to it.

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[102] There is a preprint by Paolo Aluffi and Mark Goresky but I know of no other reference.

[103] I have, myself, noticed one curious redundancy caused by the difficulties of keeping the two texts, the initial one and its English counterpart, in my head simultaneously. I can no longer find it. So it remains as a curiosity for the reader.

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