Letter to B. Gross and J. Harris May 2011

Dear Dick and Joe,

Thanks for pointing out to me your old paper on real algebraic curves and ovals. The question of the behavior of your other invariants for the random curve is interesting and I expect that some of this can be examined by the methods that are used below. My aim in this letter is to clarify what I mean by a random curve as well as the comment that I made "the random curve is 4% Harnack".

Let $m \geq 1$, $n \geq 0$ and $W(m,n)$ be the real vector space of homogeneous polynomials of degree n in $m+1$ variables. The interest below is for m fixed and $n \to \infty$. If $V(f)$ is the zero set of f in $\mathbb{P}^m(\mathbb{R})$ and $N(f)$ the number of connected components of $V(f)$, then how big is $N(f)$ for a random $f \in W(m,n)$?

To define random we choose a Gaussian density (ensemble) on $W(m,n)$. Such a density with mean 0 corresponds to a choice of a real inner product $\langle\ ,\ \rangle$ on W (the correspondence being $d\mu = \exp(-\langle v, v\rangle)\, dm(v)$ with dm a Haar measure normalized so that $d\mu$ is a probability measure). There are two familiar choices for $\langle\ ,\ \rangle$ and we add a third which it appears has not been considered much (if at all) and which from many points of view is the richest and most natural.

(A) The standard Gaussian:

$$\langle f, g\rangle_S = \sum_{|J|=n} a_J b_J \qquad \text{——— (1)}$$

where

$$f(x) = \sum_{|J|=n} a_J x^J, \quad g(x) = \sum_{|J|=n} b_J x^J \qquad \text{——— (2)}$$

$$x^J = x_1^{j_1} x_2^{j_2} \cdots x_{m+1}^{j_{m+1}}, \quad |J| = j_1 + j_2 \cdots + j_{m+1}.$$

In this model the monomials x^J form an orthonormal basis and the random f is of the form

$$f(x) = \sum_{|J|=n} a_J x^J \qquad \text{——— (3)}$$

with the a_J's independent standard Gaussians. While this model is the first that one would think of, it has the feature of $V(f)$ being very small.

(B) The complex Fubini-Study study ensemble:

Let $\| \ \|$ be a hermitian norm on $\mathbb{C}^{m+1}$. An $f \in W(m,n)$ extends to a complex polynomial $f(z)$ and define $\langle \ , \ \rangle_{\mathbb{C}}$ on W by.

$$\langle f, g \rangle_{\mathbb{C}} := \int_{\mathbb{C}^{m+1}} f(z)\, \overline{g(z)}\, e^{-\|z\|^2} dm(z) \qquad \text{—— (4)}$$

$$= * \int_{\mathbb{P}^m(\mathbb{C})} f(\xi)\, \overline{g(\xi)}\, d\sigma(\xi) \qquad \text{—— (5)}$$

where σ is the volume form of the Fubini-Study metric on $\mathbb{P}^m(\mathbb{C})$ corresponding to $\| \ \|$. If $\|z\|^2 = \sum |z_j|^2$ is the standard hermitian form then up to a constant

$$\langle f, g \rangle_{\mathbb{C}} = \sum_{|J|=n} a_J b_J \binom{n}{J} \qquad \text{—— (5)}$$

for f, g as in (2). Thus again the monomials x^J are orthogonal but not orthonormal. For this ensemble density $\mu(\mathbb{C})$ on $W(m,n)$, the random function is

$$f(x) = \sum_{|J|=n} c_J x^J \qquad \text{—— (6)}$$

with c_J independent real Gaussians with mean zero and variance $\binom{n}{|J|} = \frac{n!}{j_1! j_2! \cdots j_{m+1}!}$.

(C) The real Fubini Study ensemble:

Let $\| \ \|$ be a Euclidian norm on $\mathbb{R}^{m+1}$ and define $\langle \ , \ \rangle_{\mathbb{R}}$ on $W(m,n)$ by

$$\langle f, g \rangle_{\mathbb{R}} = \int_{\mathbb{R}^{m+1}} f(x) g(x) e^{-\|x\|^2} dm(x)$$

$$= * \int_{\mathbb{P}^m(\mathbb{R})} f(\xi) g(\xi) d\sigma(\xi) \tag{7}$$

Where σ is the corresponding real Fubini-Study volume form on $\mathbb{P}^m(\mathbb{R})$. Denote by $\mu(\mathbb{R})$ the corresponding ensemble density on $W(m,n)$. This time the monomials x^J are not orthogonal. A diagonalization of $\langle \ , \ \rangle_{\mathbb{R}}$ on $W(m,n)$ can be achieved by decomposing the corresponding representation of the orthogonal group, $O(\| \ \|)$ into irreducibles which consist of spherical harmonics of various degree up to n.

Since our interest is in $V(f)$ and $N(f)$ which are independent of f up to scalars and also a linear change of basis for $\mathbb{R}^{m+1}$, we pass from W to PW and then to $\tilde{W} := PW/PGL_{m+1}$. For our ensemble density μ on W let $\nu = p^*(\mu)$ be its push forward to $PW(m,n)/PGL_{m+1}$ where p is the composition

$$\begin{tikzcd} W \arrow[r] \arrow[rr, bend right, "p"'] & PW \arrow[r] & PW/PGL_{m+1} \end{tikzcd} \tag{8}$$

One checks that the probability densities $\nu(\mathbb{C}) = p^*(\mu(\mathbb{C}))$ and $\nu(\mathbb{R}) = p^*(\mu(\mathbb{R}))$ don't depend on the choice of the hermitian norm (respectively Euclidean) $\| \ \|$ that we chose before. In this way we get those canonical complex and real Fubini–Study Gaussians $(\widetilde{W}(m,n), \nu(\mathbb{C}))$ and $(\widetilde{W}(m,n), \nu(\mathbb{R}))$ on the quotient spaces of forms $\widetilde{W}(m,n) := W(m,n)/PGL_{m+1}$.

The "Kac-Rice" formula allows one to compute the expected values of quantities associated with $V(f)$ as long as they are defined locally (see [Ka], [E-K], [B-S-Z]). For example one can compute the expected $(m-1)$ dimensional volume of $V(f)$ in $\mathbb{P}^m(\mathbb{R})$ ([K]). In the special case $m=1$ and f nonsingular $\mathrm{Vol}_{m-1}(V(f)) = N(f)$ and this allows one to study $N(f)$ in depth. One gets for $n \to \infty$

$$\left.\begin{aligned} &E_{\nu(S),\tilde{W}(1,n)}[N(f)] \sim \log n \\ &E_{\nu(\mathbb{C}),\tilde{W}(1,n)}[N(f)] = \sqrt{n} \\ &E_{\nu(\mathbb{R}),\tilde{W}(1,n)}[N(f)] \sim n/\sqrt{3} \end{aligned}\right\} \quad (9)$$

[There is a simple reason for these different behaviors, for $\nu(S)$ the real zeros of $\sum_{j=0}^{n} a_j x^j$ are all near $x = \pm 1$ and so there are very few of them, for $\nu(\mathbb{C})$ the variances of the a_j's are $\binom{n}{j}$ and the zeros are spread over $\mathbb{R}$ with local density $\sqrt{n}$, for $\nu(\mathbb{R})$ the action is on $\mathbb{P}^1(\mathbb{R})$ realized as $|z|=1$ with random trigonometric polynomials].

For $m=2$ and f nonsingular, $N(f)$ is equal to the number of ovals of f if n is even and is one more than the number of ovals if n is odd. It is a "global" invariant whose behavior is much more difficult to compute. That one can do anything at all is due to Nazarov and Sodin [N-S] who introduced new techniques to study the nodal lines of random spherical harmonics. Let $H(n)$ be the $2n+1$ dimensional linear subspaces of $W(2,n)$ consisting of $p(x_1,x_2,x_3)$ satisfying $\Delta p = 0$. They show that as $n\to\infty$ there is $c_1>0$ such that

$$E_{\mu(\mathbb{R}),H(n)}[N(f)] \sim c_1 n^2 . \qquad (10)$$

(They also show that $N(f)$ concentrates about its mean as $n\to\infty$ so that the random spherical harmonic has this many ovals)

Wigman and myself use their technique to show that there is $c_2>0$ such that as $n\to\infty$

$$E_{\nu(\mathbb{R}),\widetilde{W}(2,n)}[N(f)] \sim c_2 n^2 \qquad (11)$$

(and again there is concentration).

Maria Nastasescu [N] has developed an adaptive code to generate random elements f in $W(2,n)$ w.r.t. $V(\mathbb{R})$ and to count $N(f)$. Her experiments with n up to 100 yield

$$\left.\begin{aligned} c_1 &\simeq 0.0301\ldots \\ c_2 &\simeq 0.0195\ldots \end{aligned}\right\} \quad (12)$$

A quite striking suggestion by the physicists Bogomolny and Schmit [B-S] is that the behavior of the nodal lines of a random element in $H(n)$ can be modeled by a critical bond percolation model for the square lattice. Using some theoretical calculations for exactly solvable aspects of this model They arrive at a predicted value for c_1; $c_1 = \frac{3\sqrt{3}-5}{2\pi} = 0.0312\ldots$. It agrees with the Monte-Carlo value to two decimal places! I don't know any such explicit prediction for c_2.
(11) and (12) show that for the real Fubini Study ensemble the random plane curve is 4% Harnack in that Nastasescu's constant c_2 is about 4% of the maximal number of ovals as allowed by Harnack's inequality.

To contrast this behaviour for $\nu(\mathbb{C})$ Wigman and I have shown that

$$\overline{\lim_{n\to\infty}} \frac{E_{\nu(\mathbb{C}), \widetilde{W}(2,n)}[N(f)]}{n} \leq \frac{2}{\pi} \qquad (13)$$

In fact almost curves in this complex Fubini-Study ensemble have $O(n)$ ovals.

Ovals for hyperelliptic curves reduce to 1-dimension and so can be studied using (9). Consider the space $E(n)$ of affine plane curves

$$f(x,y) = y^2 - g(x) \qquad (14)$$

with g a real polynomial of degree n. Using the 1-variable Gaussian ensembles for g, we get the corresponding Gaussian ensembles $(E(n), \mu(S))$, $(E(n), \mu(\mathbb{C}))$ and $(E(n), \mu(\mathbb{R}))$. It is elementary that for g with distinct real roots

$$N(f) = \begin{cases} \frac{N(g)+1}{2} & \text{if } n \text{ is odd} \\ N(g)/2 & \text{if } n \text{ is even.} \end{cases} \qquad (15)$$

Hence as $n \to \infty$

$$E_{S, E(n)}[N(f)] \sim \log n$$

$$E_{\mu(\mathbb{C}), E(n)}[N(f)] \sim \frac{n}{2}$$

$$E_{\mu(\mathbb{R}), E(n)}[N(f)] \sim \frac{n}{2\sqrt{3}} \qquad (16)$$

For the cubic case $(n=3)$ of elliptic curves in the form (14) we can compute explicitly that

$$\mathrm{Prob}_{\mu(\mathbb{R}), E(3)}(N(f)=2) = 1 - \mathrm{Prob}_{\mu(\mathbb{R}), E(3)}(N(f)=1) = \frac{\sqrt{5}-1}{2} > \frac{1}{2},$$

while

$$\mathrm{Prob}_{\mu(\mathbb{C}), E(3)}(N(f)=2) = 1 - \mathrm{Prob}_{\mu(\mathbb{C}), E(3)}(N(f)=1) = \frac{\sqrt{3}-1}{2} < \frac{1}{2}. \qquad (17)$$

So if you have wondered when drawing a plane elliptic curve in $E(3)$, whether to draw two or one component it would depend on whether you have in mind the real or complex Fubini-Study ensemble.

For $m \geq 3$ one can prove the analogue of (11) for $(\widetilde{W}(m,n), \nu(\mathbb{R}))$ but I know of little else.

Best regards

Peter Sarnak

References:

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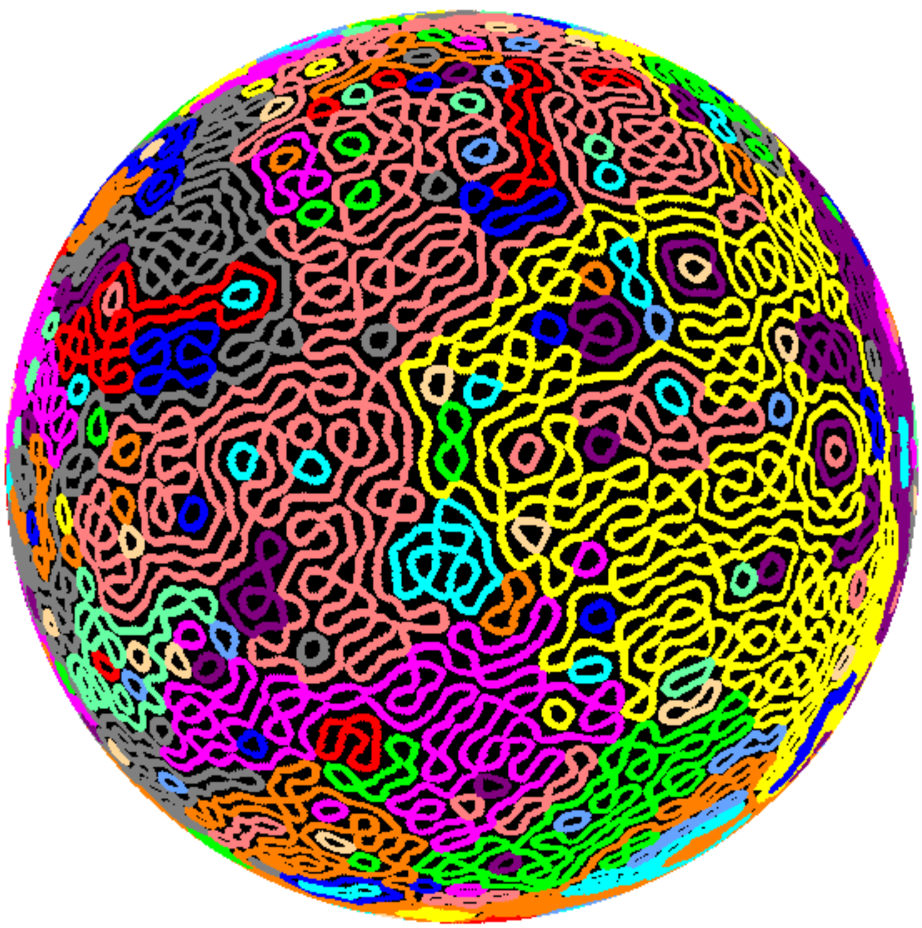

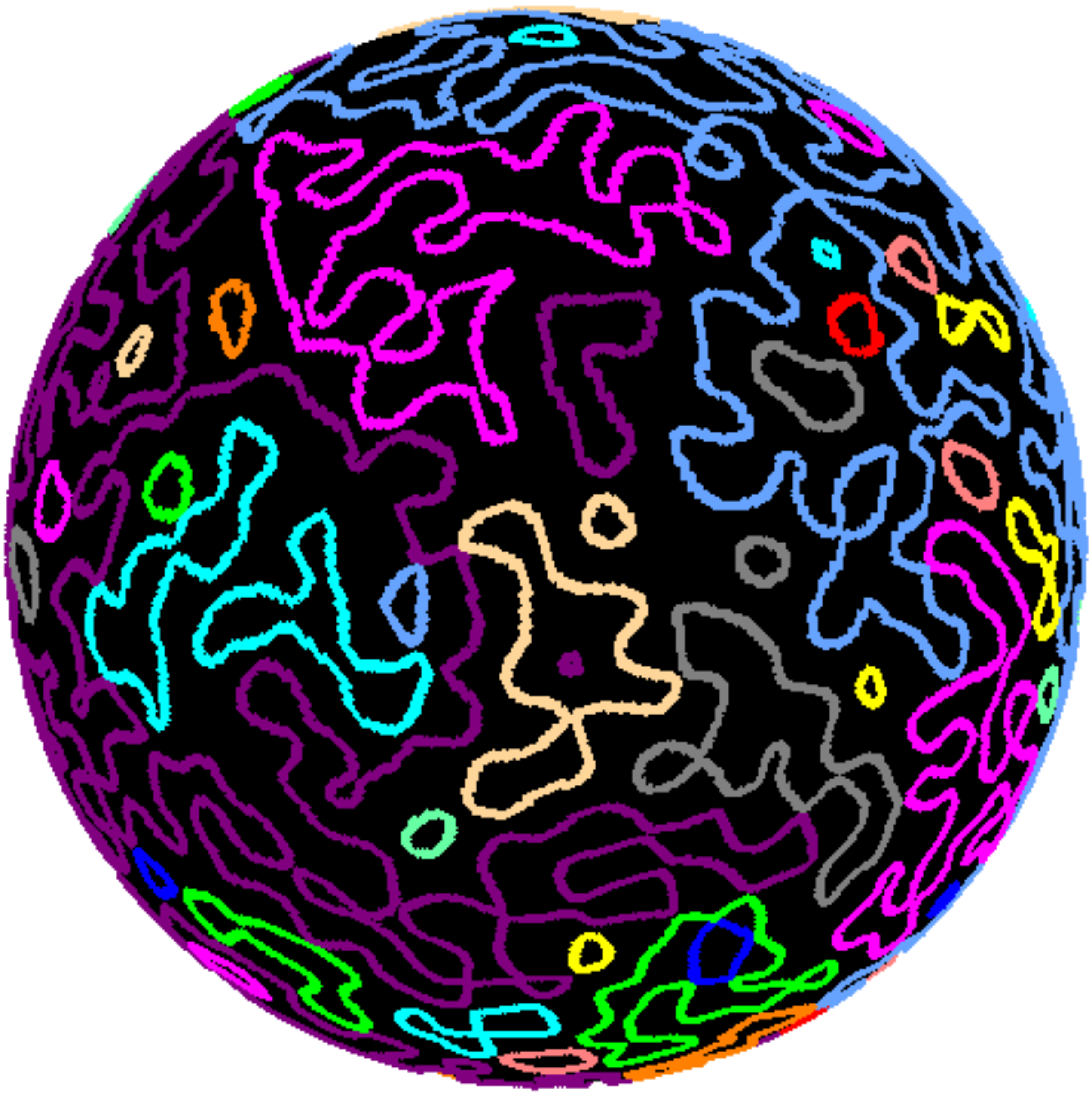

Notes added July 2011:

(1) The pictures on pages 12 and 13 are by Nastasescu. The first is of the ovals of a random harmonic curve of degree 100. The second is of a random real Fubini-Study curve of degree 50.

(2). The norm (5) used to define the complex Fubini-Study Ensemble was introduced (at least) in a paper of Beauzamy, Bombieri, Enflo and Montgomery [Jnl of number theory 36, 219-245 (1990)]. It is refered to as Bombieri norm in a number of papers.

(3). In a recent paper "Exponential rarefaction of real curves with many components" IHES Pub Math 113 (2011) 69-96, Gayet and Welschinger give upper estimates (exponentially small) for curves in the complex Fubini Study ensemble to be close to a Harnack curve.