Dear Professor Weil,

While trying to formulate clearly the question I was asking you before Christmas I was led to two more general questions. Your opinion of these questions would be appreciated. I have not had a chance to think over these questions seriously and I would not ask them except as the continuation of a casual conversation. I hope you will treat them with the tolerance they require at this stage. After I have asked them I will comment briefly on their genesis.

It will take a little discussion but I want to define some Euler products which I will call Artin-Hecke L-series because the Artin L-series, the L-series with Grössencharakter, and the series introduced by Hecke into the theory of automorphic forms are all special cases of these series. The first question will be of course whether or not these series define meromorphic functions with functional equations. I will say a few words about the functional equation later. The ~~next~~ other question I will formulate later. It is a generalization of the question of whether or not abelian L-series are L-series with Grössencharakter. Since I want to formulate the question for automorphic forms on any reductive group I have to assume that certain results in the reduction theory can be pushed a little further than they have been so far.

Unfortunately I must be rather pedantic. Let k be the rational field or a completion of it. Let G be a product of simple groups, perhaps abelian, split over k. Suppose the non-abelian factors are simply connected. The case that the product is empty and $\tilde{G} = \{1\}$ is not without interest. Fix a split Cartan subgroup $\tilde{T}$ and let $\tilde{L}$ be the lattice of weights of $\tilde{T}$. $\tilde{L}$ contains the roots. I want to define "the" conjugate group to $\tilde{G}$ and "the" conjugate lattice to $\tilde{L}$. It is enough to do this for a simple group for we can then take direct products and direct sums. If G is abelian and simple let $\tilde{L}'$ be any sublattice of $\tilde{L}$ and ${}^c\tilde{L}$, the conjugate lattice, be the dual of $\tilde{L}'$ (i.e. $\mathrm{Hom}(\tilde{L}', \mathbb{Z})$). It contains ${}^c\tilde{L}'$, the dual of $\tilde{L}$. Let ${}^c\tilde{G}$ be a one-dimensional split ~~group~~ torus whose lattice of weights is identified with ${}^c\tilde{L}$. If $\tilde{G}$ is simple and non-abelian let $\tilde{L}'$ be the lattice generated by the roots and let ${}^c\tilde{L}$ be the dual of $\tilde{L}'$. ${}^c\tilde{L}$ contains ${}^c\tilde{L}'$ the dual of $\tilde{L}$. Choose for each root α an element H_α in the Cartan subalgebra corresponding to $\tilde{T}$ in the usual way so that $\alpha(H_\alpha)$. The linear functions ${}^c\alpha(\lambda) = \lambda(H_\alpha)$ generate $\tilde{L}'$. There is a unique simply connected group ${}^c\tilde{G}$ whose lattice of weights is isomorphic to ${}^c\tilde{L}$ in such a way that the roots of ${}^c\tilde{G}$ correspond to the elements ${}^c\alpha$. Fix simple roots $\alpha_1, \dots, \alpha_\ell$ of $\tilde{G}$, then ${}^c\alpha_1, \dots, {}^c\alpha_\ell$ can be taken as the simple roots of ${}^c\tilde{G}$. Now return to the general case.

If L is a lattice lying between $\tilde{L}'$ and $\tilde{L}$ we can associate to L in a natural way a group G covered by $\tilde{G}$. The dual lattice ${}^cL$ of L lies between ${}^c\tilde{L}'$ and ${}^c\tilde{L}$. It determines a group ${}^cG$, covered by ${}^c\tilde{G}$, which I call the conjugate of G. Let $\underline{h}$ be the Lie algebra of $\tilde{T}$ and choose for each root α a root vector X_α so that the conditions of Chevalley are satisfied. Also let ${}^c\underline{h}$ be a split Cartan subalgebra of ${}^c\underline{g}$ for for each root ${}^c\alpha$ choose the root vector ${}^cX_\alpha$ so that the conditions of Chevalley are satisfied. Let Ω be the group of automorphisms of $\underline{g}$ which take $\underline{h}$ to itself, permute $\{X_\alpha \mid \alpha \text{ simple}\}$, and take $\tilde{L}, L, \tilde{L}'$ to themselves. Define ${}^c\Omega$ in a similar fashion. ${}^c\Omega$ is the contragredient of Ω so that Ω and ${}^c\Omega$ are canonically isomorphic. ~~If K is a finite extension~~ Ω thus acts as a group of automorphisms of G and of ${}^cG$. If K is a finite Galois extension of k and δ is a homomorphism of $\mathfrak{G} = \mathfrak{G}(K/k)$ into Ω with image Ω^δ let G^δ and ${}^cG^\delta$ be the associated forms of G and ${}^cG$.

In order to define the local factors of the L-series I have to recall some facts about the Hecke algebra of G^δ_k when K is an unramified extension of the p-adic field k. If we choose a maximal compact subgroup of G^δ_k in a suitable manner then, according to Bruhat and Satake, the Hecke algebra is isomorphic to the set of elements in the group algebra of ${}^cL^\delta$, the set of elements in ${}^cL$ fixed by ${}^c\Omega^\delta$, which are invariant under the restricted Weyl group ${}^cW^\delta$ of ${}^cG^\delta$. (Actually we have to stretch their results a little.) Thus any homomorphism χ of the Hecke algebra into the complex numbers can be extended to a homomorphism χ' of the group algebra of ${}^cL$ into the complex numbers. There is at least one element g of ${}^cT$ so that if $f = \sum_{\lambda \in L} a_\lambda \xi_\lambda$ (ξ_λ is λ written multiplicatively) then $\chi'(f) = \sum a_\lambda \xi_\lambda(g)$.

The semi-direct product $\mathfrak{G} \times_\delta {}^cG$ is a complex group. Let π be a complex representation of it. If σ is the Frobenius then

(x an indeterminate)

$$\frac{1}{\det(1 - x\pi(\sigma \times g))}$$

is the local zeta function corresponding to χ and π. I have to verify that it depends only on χ and not on g. If λ is any weight let n_λ be the lowest power of σ which fixes λ and if $n_\lambda | n$ and π acts on V let $t_\lambda(n)$ be the trace of σ^n on

$$\{v \in V \mid \pi(h)v = \xi_\lambda(h)v \text{ for all } h \in {}^cT\}.$$

Then

$$\log \frac{1}{\det(1 - x\pi(\sigma \times g))} = \sum_{n=0}^{\infty} \frac{x^n}{n} \sum_{\lambda \in L} \sum_{n_\lambda | n} t_\lambda(n)\, \xi_\lambda(g^{\sigma^{n-1}} g^{\sigma^{n-2}} \cdots g)$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n} \sum_{\lambda \in L} \sum_{n_\lambda | n} t_\lambda(n)\, \xi_{\frac{n}{n_\lambda}\left(\sum_{k=0}^{n_\lambda - 1} \lambda^{\sigma^k}\right)}(g)$$

Moreover if ω is an element of ${}^cW^\delta$ we can always choose a representative w of it which commutes with σ. Then the local zeta function does not change if g is replaced by $w^{-1}gw$ so it equals

$$\frac{1}{[{}^cW^\delta : 1]} \sum_{n=0}^{\infty} \frac{x^n}{n} \sum_{\lambda} \sum_{n_\lambda | n} t_\lambda(n) \sum_{\omega \in {}^cW^\delta : 1} \xi_{\frac{n}{n_\lambda}\left(\sum_{k=0}^{n_\lambda - 1} \lambda^{\sigma^k}\right)^\omega}(g)$$

Since

$$\sum_{\omega} \xi_{\frac{n}{n_\lambda}\left(\sum_{k=0}^{n_\lambda - 1} \lambda^{\sigma^k}\right)^\omega}$$

belong to the image of the Hecke algebra the assertion is verified.

I don't know if it is legitimate but let us assume that the characters of the complex representation separate the semi-simple conjugacy classes in $G_{\mathfrak{X}} \times_\sigma {}^cG$. Thus by the above I can associate to each homomorphism χ of the Hecke algebra into the complex numbers the conjugacy class of the semi-simple element $\sigma \times g$. Conversely given a semi-simple conjugacy class in $\mathfrak{G} \times_\sigma {}^cG$ it contains, by Borel-Mostow, an element in the normalizer of ${}^cT$. Then it even contains an element which takes positive roots into positive roots. Thus if the projection of the conjugacy class on $\mathfrak{G}$ (an abelian group) is σ the conjugacy class contains an element of the form $\sigma \times g$, $g \in {}^cT$. As above g determines a homomorphism of the Hecke algebra into the complex numbers. If this homomorphism χ is completely determined by the local zeta functions attached to it then it is completely determined by the conjugacy class of $\sigma \times g$ and we have a one-to-one correspondence between homomorphisms of the Hecke algebra into the complex numbers and semi-simple conjugacy classes in $\mathfrak{G} \times_\sigma {}^cG$ whose projection on $\mathfrak{G}$ is σ. It is enough to check that the value of χ on an element of the form $\sum_{\omega \in {}^cW^\sigma} \xi_{(\sum_{k=1}^{n_\lambda} \lambda^{\sigma^k})^\omega}$, where

$\sum_{k=1}^{n_\lambda} \lambda^{\sigma^k}$ belongs to the positive Weyl chamber, is determined by the local zeta functions. This can be done by the usual sort of induction for $\sum \lambda^{\sigma^k}$ is invariant under $\mathfrak{G}$ and thus the highest weight of a representation of $\mathfrak{G} \times_\sigma {}^cG$ whose restriction to ${}^cG$ is irreducible.

Now I am going to try to define the Artin-Hecke L series. To do this let us fix for each p an imbedding of $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$, in $\overline{\mathbb{Q}}_p$. We will have to come back later and check that the series are independent of these choices. The choice will be implicit in the next paragraph.

Suppose we have a twisted form $\overline{G}$ of G over the rationals. The twisting can be accomplished in two steps. First for a suitable Galois extension K of $\mathbb{Q}$ take a homomorphism σ of $\mathfrak{G} = \mathfrak{G}(K/\mathbb{Q})$ into $\mathfrak{A}$ to obtain G^σ. Then take an inner twisting of G^σ by means of the cocycle $\{a_\tau \mid \tau \in \mathfrak{G}\}$. Let me assume the truth of the following:

(i) Suppose G is a linear group acting on V. Let L be a Chevalley lattice in $V_{\mathbb{Q}}$. Then the intersection $G^\delta_{\mathbb{Z}_p}$ of $G^\delta_{\mathbb{Q}_p}$ with the stabilizer of $L \otimes_{\mathbb{Z}} \bar{\mathbb{Z}}_p$ ($\bar{\mathbb{Z}}_p$ is the ring of integers in $\bar{\mathbb{Q}}_p$) is, for almost all p, one of the maximal compact subgroups referred to above.

(ii) For almost all p the restriction of $\{a_\tau\}$ to $\mathfrak{G}(K\mathbb{Q}_p/\mathbb{Q}_p) = \mathfrak{G}(K_p/\mathbb{Q}_p) = \mathfrak{G}_p$ splits. Moreover for such p there is a b in the intersection of G_{K_p} with the stabilizer of $L \otimes_{\mathbb{Z}} \bar{\mathbb{Z}}_p$ so that $a_\sigma = b^{+\sigma} b^{-1}$, $\sigma \in \mathfrak{G}_p$.

Now take a p satisfying (i) and (ii) which does not ramify in K. Since $\bar{G}_{\mathbb{Q}_p} = \{g \in G^\delta_{K_p} \mid g^{\sigma a_\sigma} = g, \sigma \in \mathfrak{G}_p\}$ the map $g \to g^b$ is an isomorphism of $\bar{G}_{\mathbb{Q}_p}$ with $G^\delta_{\mathbb{Q}_p}$. Moreover we can take $\bar{G}_{\mathbb{Z}_p}$ to be the intersection of $\bar{G}_{\mathbb{Q}_p}$ with the stabilizer of $L \otimes_{\mathbb{Z}} \bar{\mathbb{Z}}_p$ so the map takes $\bar{G}_{\mathbb{Z}_p}$ to $G_{\mathbb{Z}_p}$. The induced isomorphism of the Hecke algebras is independent of the choice of b. Now $\bar{G}_A = \prod_p \bar{G}_{\mathbb{Q}_p}$. Suppose we have an automorphic form ϕ on $\bar{G}_{\mathbb{Q}} \backslash \bar{G}_A$ which is an eigenfunction of the Hecke algebras for almost all p. Then, for almost all p, we have a homomorphism of the Hecke algebra into the complex numbers and thus a semi-simple conjugacy class α_p in $\mathfrak{G}_p \times_\delta {}^cG \subseteq \mathfrak{G} \times_\delta {}^cG$. If π is a complex representation of $\mathfrak{G} \times_\delta {}^cG$ I define the Artin-Hecke L series as

$$L(s, \pi, \phi) = \prod_p \frac{1}{\det\left(1 - \frac{\pi(\alpha_p)}{p^s}\right)}$$

(Product is taken over almost all p)

I have to check that these series are independent of the imbeddings of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_p$. For the moment fix p. We have used the original imbedding to identify $\overline{\mathbb{Q}}$ with a subfield of $\overline{\mathbb{Q}}_p$. Let us preserve this identification. Any other imbedding is obtained by sending $x \to x^\tau$ with $\tau \in \mathfrak{G}(\overline{\mathbb{Q}}/\mathbb{Q})$. If we use the original imbedding to identify $\mathfrak{G}_p$ with a subgroup of $\mathfrak{G}$ then the map of $\mathfrak{G}_p$ into $\mathfrak{G}$ given by the new imbedding is $\sigma \to \tau\sigma\tau^{-1}$ (I identify τ with its image in $\mathfrak{G}$) The restriction of δ to $\mathfrak{G}_p$ is replaced by δ' with $\delta'(\sigma) = \delta(\tau\sigma\tau^{-1})$. Thus $G^{\delta}_{\mathbb{Q}_p}$ is replaced by $G^{\delta'}_{\mathbb{Q}_p}$. The map $g \to g^{\delta(\tau)}$ is an isomorphism of $G^{\delta'}_{\mathbb{Q}_p}$ with $G^{\delta}_{\mathbb{Q}_p}$. If $g \in G^{\delta}_{\mathbb{Q}} \subset G_{\mathbb{Q}_p}$ then g is the image of g^τ so this image commutes with the imbedding of $G^{\delta}_{\mathbb{Q}}$ in the two groups. The new cocycle $\{a'_\sigma\}$ is the image of $a_{\tau\sigma\tau^{-1}} = a_\tau^{\sigma\tau^{-1}} a_\sigma^{\tau^{-1}} a_{\tau^{-1}} = a_\tau^{\sigma\tau^{-1}} a_\sigma^{\tau^{-1}} a_\tau^{-\tau^{-1}}$ since $a_{\sigma\tau} = a_\sigma^\tau a_\tau$ for all σ and τ. The image is $\delta(\tau) a_\tau^\sigma a_\sigma a_\tau^{-1} \delta(\tau^{-1})$

Thus

$$\overline{G}'_{\mathbb{Q}_p} = \{ g \in G_{K_1} \mid g = g^{\delta(\tau)\,\sigma\delta(\sigma) a_\tau^\sigma a_\sigma a_\tau^{-1}\delta(\tau^{-1})} = g^{\delta(\tau) a_\tau\, \sigma\delta(\sigma) a_\sigma a_\tau^{-1}\delta(\tau^{-1})} \text{ for } \sigma \in \mathfrak{G}_p \}$$

and the map $g \to g^{\delta(\tau)a_\tau}$ is an isomorphism of $\overline{G}'_{\mathbb{Q}_p}$ with $\overline{G}_{\mathbb{Q}_p}$. It commutes with the imbeddings of $\overline{G}_{\mathbb{Q}}$ in these two groups since $\overline{G}_{\mathbb{Q}} = \{ g \in G_K \mid g^{\rho\delta(\rho)a_\rho} = g \text{ for all } \rho \in \mathfrak{G}\}$. Moreover for almost all p it takes $\overline{G}'_{\mathbb{Z}_p}$ to $\overline{G}_{\mathbb{Z}_p}$. If then we choose for each p a new imbedding we get a new adele group $\overline{G}'_A$. The above maps define an isomorphism of $\bar{G}'_A$ with $\bar{G}_A$ which takes $\bar{G}_Q$ to itself. Thus we have a map of $\bar{G}_Q\backslash\bar{G}'_A$ to $\bar{G}_Q\backslash\bar{G}_A$ and the automorphic form ϕ introduced above defines an automorphic form ϕ' on $\bar{G}_Q\backslash\bar{G}'_A$ with the same properties. We have to check that $L(s,\phi',\pi) = L(s,\phi,\pi)$.

Fix p again. Then if $a_\sigma = b^\sigma b^{-1}$,

$$a'_\sigma = \delta(z)\, a_z^\sigma\, b^\sigma\, \delta(z^{-1})\, \delta(z)\, b^{-1} a_z^{-1}\, \delta(z^{-1})$$

$$= [\delta'(\sigma^{-1})\sigma^{-1}\, \delta(z)\, a_z b\, \delta(z^{-1})\, \sigma\delta'(\sigma)]\, [\delta(z) a_z b\, \delta(z^{-1})]^{-1}$$

so a'_σ is split by $b' = \delta(z) a_z b\, \delta(z^{-1})$. For almost all p b' lies in the stabilizer of $L \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Thus we have the following commutative diagram

$$\begin{array}{ccc} \bar{G}'_{Q_p} & \xrightarrow{g \to g^{b'}} & G^{\delta'}_{O_p} \\ \Big\downarrow {\scriptstyle g \to g^{\delta(z) a_z}} & & \Big\downarrow {\scriptstyle g \to g^{\delta(z)}} \\ \bar{G}_{Q_p} & \xrightarrow{g \to g^{b}} & G^{\delta}_{Q_p} \end{array}$$

This means that if α'_p is the conjugacy class in $z\,U_p z^{-1} \times_\delta {}^cG$ associated to ϕ' then $\alpha'_p = z\alpha_p z^{-1}$. This shows that α'_p and α_p are conjugate for almost all p and this shows that $L(s,\phi',\pi)$ and $L(s,\phi,\pi)$ differ by a finite number of factors.

The first question is whether or not these products define functions meromorphic in the entire complex plane with poles of the usual type and whether or not for each ϕ there is an automorphic form ψ so that $\dfrac{L(s,\phi,\pi)}{L(s,\psi,\tilde{\pi})}$ is an elementary function for all π.

π is the representation contragredient to φ.

Before I go into the second question let me just say that I have been making some experiments with Eisenstein series and, although the work is far from completed, it looks as though I will get some series of the above type which because of their relation to the Eisenstein series will be meromorphic in the whole plane. It might even be possible to get a functional equation in a smaller number of cases from the functional equations of the Eisenstein series. The definitions above are the result of trying to find some class of Euler products which will contain the ones coming from the Eisenstein series but which is not restricted in any artificial fashion.

Now if $G = GL(n)$ and action of $\mathfrak{G}$ is trivial, and π is the representation $g \to g$ one can perhaps use the ideas of Tamagawa to handle the above series. This leads to the second question.

Suppose we have K, G, and δ as above, and also K', G', and δ'. If $K \subset K'$ we have a homomorphism $\mathfrak{G}' \to \mathfrak{G}$. Suppose moreover that ω is a homomorphism of $\mathfrak{G}' \times_{\delta'} {}^cG'$ into $\mathfrak{G} \times_{\delta} {}^cG$ so that the following diagram is commutative

$$\begin{array}{ccc} \mathfrak{G}' \times_{\delta'} {}^cG' & \longrightarrow & \mathfrak{G}' \\ \omega \downarrow & & \downarrow \\ \mathfrak{G} \times_{\delta} {}^cG & \longrightarrow & \mathfrak{G} \end{array}$$

If ϕ' is an automorphic form for some inner form of G' δ' satisfying the condition we had above then for almost all p ϕ' defines a conjugacy class α_p' in $\mathfrak{G}' \times_{\delta'} {}^cG'$.

Let α_p be the image of α_p' in $\mathfrak{g} \times_\delta {}^cG$. The second question is the following. Is there an automorphic form ϕ associated to some inner form of G^δ such that for almost all p the conjugacy class associated to it is α_p.

Let me give some idea of what an affirmative answer to the question entails

(i) Take $\mathfrak{g}' = \mathfrak{g}$ and let G' be a split torus of rank ℓ equal to the rational rank of ${}^cG^\delta$ on which $\mathfrak{g}'$ acts trivially. Let A be a maximal split torus of ${}^cG^\delta$. Since $\mathfrak{g}$ acts trivially on A $\mathfrak{g}' \times_{\delta'} {}^cG' \simeq \mathfrak{g} \times A \subseteq \mathfrak{g} \times_\delta {}^cG$. Since there are ℓ parameter families of automorphic forms on $G'_Q \backslash G'_A$ an affirmative answer implies the same is true for some inner form of G^δ. But this we know from the theory of Eisenstein series.

(ii) Let $\mathfrak{g}' = \mathfrak{g}$ and let $G' = \{1\}$. Map $\mathfrak{g}' \times_{\delta'} {}^cG'$ to $\mathfrak{g} \times \{1\} \subseteq \mathfrak{g} \times_\delta {}^cG$. In this case it should be possible to give an affirmative answer to the question by taking ϕ to be the automorphic form obtained by setting the parameter equal to zero in a suitable Eisenstein series. ϕ' is of course a constant.

(iii) Now let me say a few words about the relation of the question to the Artin reciprocity law. For the rational field take $\mathfrak{g}'$ abelian, $G' = \{1\}$ and let χ be a character of $\mathfrak{g}'$. Let $\mathfrak{g} = \mathfrak{g}'$ and let G be a one-dimensional split torus on which $\mathfrak{g}$ acts trivially. Let us take $z \times 1$ to $z \times \chi(z)$. Then an affirmative answer is just the Artin reciprocity law for cyclic extensions of the rationals.

Now suppose we have the following situation

K

|

K_1'

|

K_1

|

$\mathbb{Q}$.

$K/\mathbb{Q}$ is Galois and K_1'/K_1 is abelian. Let $\mathfrak{g} = \mathfrak{g}(K/\mathbb{Q})$, let $\mathfrak{g}_1$ be the elements of $\mathfrak{g}$ which fix K_1 and let $\mathfrak{g}_1'$ be the elements of $\mathfrak{g}$ which fix K_1'. Finally suppose ~~such~~ χ is a character of $\mathfrak{g}(K_1'/K_1) = \mathfrak{g}_1/\mathfrak{g}_1'$ and thus of $\mathfrak{g}_1$. I will take $\mathfrak{g}' = \mathfrak{g}$ and $G' = \{1\}$. Let $\ell = [\mathfrak{g}_1 : \mathfrak{g}]$ and let $\mathfrak{g} = \cup_{i=1}^{\ell} \mathfrak{g}_1 \tau_i$ with $\tau_1 \in \mathfrak{g}_1$. ~~Suppose~~ Let i_σ be such that $\tau_i \sigma \in \mathfrak{g}_1 \tau_{i_\sigma}$.

Let G be the direct product $T_1 \times \cdots \times T_\ell$ of ℓ one-dimensional split tori. Define δ by $(t_1 \times \cdots \times t_\ell)^{\delta(\sigma)} = t_{1_{\sigma^{-1}}} \times \cdots \times t_{\ell_{\sigma^{-1}}}$. It is easy to check that $\delta(\sigma)\delta(\tau) = \delta(\sigma\tau)$. Moreover G^r is just the multiplicative group of K_1. Also ${}^{c}G = G$. Define $\rho_i(\sigma)$ by $\tau_i \sigma^{-1} = \rho_i^{-1}(\sigma)\tau_{i_{\sigma^{-1}}}$. Then $\tau_i \tau^{-1} \sigma^{-1} = \rho_i^{-1}(\tau) \tau_{i_{\tau^{-1}}} \sigma^{-1} = \rho_i^{-1}(\tau) \rho_{i_{\tau^{-1}}}(\sigma) \tau_{i_{\tau^{-1}\sigma^{-1}}}$ so $\rho_i(\sigma\tau) = \rho_{i_{\tau^{-1}}}(\sigma)\rho_i(\tau)$. Define ω by

$$\omega(\sigma \times 1) = \sigma \times (\chi(\rho_1(\sigma)) \times \cdots \times \chi(\rho_\ell(\sigma)))$$

Then

$$\omega(\sigma \times 1)\omega(\tau \times 1) = \sigma\tau \times \prod_{i=1}^{\ell} \chi(\rho_{i_{\tau^{-1}}}(\sigma))\chi(\rho_i(\tau))$$

$$= \omega(\sigma\tau \times 1).$$

By the way if the Z_i, $1 \le i \le \ell$ are replaced by $Z_i' = \mu_i Z_i$ with $\mu_i \in \mathfrak{G}_1$ 12
then $\rho_i'(\sigma) = \mu_{i_{\sigma^{-1}}}^{-1} \rho_i(\sigma) \mu_i$ and

$$\omega'(\sigma \times 1) = \left(\chi(\mu_1) \times \cdots \times \chi(\mu_\ell)\right)^{-1} \omega(\sigma \times 1) \left(\chi(\mu_1) \times \cdots \times \chi(\mu_\ell)\right)$$

so the map does not depend in an essential way on the choice of coset representatives.

I will take ϕ' to be a constant. By the Artin reciprocity law there is associated to χ a character of $K_1^\times \backslash I_{K_1}$, that is, an automorphic form ϕ on $G_Q^\delta \backslash G_A^\delta$. I claim that ϕ is the automorphic form which provides an affirmative answer to the question

To show this we make use of the freedom we have in the choice of coset representatives. Let p be a prime which does not ramify in K. Fix an imbedding of K in $\bar{\mathbb{Q}}_p$. We identify K with its image. Let $\mathfrak{g}_1, \dots, \mathfrak{g}_r$ be the prime divisors of p in K_1. Choose $\mu_1, \dots, \mu_r$ in $\mathcal{G}$ so that the map $x \to x^{\mu_j}$ of K_1 into $\bar{\mathbb{Q}}_p$ extends to a continuous map of the completion of K_1 with respect to $\mathfrak{g}_j$ into $\bar{\mathbb{Q}}_p$. Let $L_j = [K_1^{\mu_j}\bar{\mathbb{Q}}_p : \mathbb{Q}_p]$ and let $n_j = [L_j : \mathbb{Q}_p]$. If σ_p is the Frobenius the automorphisms $\mu_j \sigma_p^k$, $1 \le j \le r$, $0 \le k \le n_j$ form a set of representatives for the cosets of $\mathcal{G}_1$. If $\tau_i = \mu_j \sigma_p^k$ then $\rho_i(\sigma_p) = 1$ unless $k = 0$ when $\rho_i(\sigma_p) = \mu_j \sigma_p^{n_j} \mu_j^{-1}$. Thus $w(\alpha_p^i) = \alpha_p$ is the conjugacy class of

$$\sigma_p \times \prod_{j=1}^{\ell} (\chi(\mu_j \sigma_p^{n_j} \mu_j^{-1}) \times 1 \times \dots \times 1)$$

$\mu_j \sigma_p^{n_j} \mu_j^{-1}$ belongs to the Frobenius conjugacy class in $\mathcal{G}_1$ corresponding to $\mathfrak{g}_j$.

On the other hand $G^{\delta}_{\mathbb{Q}_p} \subseteq G^{\delta}_A$ is the set of elements of the form $\prod_{j=1}^{r} \prod_{k=0}^{n_j-1} x_j^{\sigma^k}$ with x_j a non-zero element in L_j. The restriction of ϕ to such an element is, by its very definition,

$$\prod_{j=1}^{\ell} \chi(\mu_j \sigma_p^{n_j} \mu_j^{-1})^{o(x_j)}$$

if $|x_j| = p^{-o(x_j)}$. Since ${}^{c}G^{\delta} = G^{\delta}$ the associated conjugacy is the one determined by any element

$$\sigma_p \times \prod_{j=1}^{r} \prod_{k=0}^{n_j-1} \alpha_{jk}$$

such that

$$\prod_{j=1}^{r} \prod_{k=0}^{n_j-1} \alpha_{jk}^{o(x_j)} = \prod_{j=1}^{\ell} \chi(\mu_j \sigma_p^{n_j} \mu_j^{-1})^{o(x_j)}.$$

Looking above we see that $\omega(a_p')$ is such an element.

(iv) Finally I want to comment on the implications an affirmative answer to the second question ~~would~~ might have for the problem of finding a splitting law for non-abelian extensions. I had planned to discuss arbitrary ground fields but I realize now that I have to take the ground field to be $\mathbb{Q}$. However one could presumably go back and reformulate the two questions in the context of groups over a number field. The first question is not sensitive to the choice of ground field but the second is. I did not appreciate this until now; since little would be gained by rewriting the letter I content myself with taking the ground field to be $\mathbb{Q}$. ~~First~~

Let K be a Galois extension of $\mathbb{Q}$ and let $\mathfrak{G} = \mathfrak{G}(K/\mathbb{Q})$. We want a method of finding for almost all p the Frobenius conjugacy class $\{\sigma_p\}$ in $\mathfrak{G}$. Thus we have to find trace $\pi(\sigma_p)$ or the conjugacy class of $\pi(\sigma_p)$ in $GL(m, \mathbb{C})$, if $\pi: \mathfrak{G} \to GL(m, \mathbb{C})$, for all representations π of $\mathfrak{G}$. Let us fix π. As before I will take $\mathfrak{G}' = \mathfrak{G}$, $G' = \{1\}$, and ϕ' to be a constant function. I will take $G = GL(m)$. Let me check that ${}^cG$ is also $GL(m)$.

Take $\tilde{G} = {}^c\tilde{G} = A \times SL(m)$ where A is a ~~one~~ one dimensional split torus. Then

$${}^c\tilde{L} = \tilde{L} = \{(z, z_1, \dots, z_m) \mid z,\ z_i - z_j \in \mathbb{Z},\ \textstyle\sum_{i=1}^m z_i = 0\}$$

$${}^cL = L = \{(z, z_1, \dots, z_m) \mid z_i + \tfrac{z}{m} \in \mathbb{Z},\ \textstyle\sum_{i=1}^m z_i = 0\}$$

$${}^c\tilde{L}' = \tilde{L}' = \{(mz, z_1, z_2 - z_1, \dots, z_{n-1} - z_{n-2}, -z_{n-1}) \mid z, z_i \in \mathbb{Z}\}$$

The pairing is given by

$$\langle (z; z_1, \dots, z_m), (y; y_1, \dots, y_m) \rangle = \frac{zy}{m} + \sum_{i=1}^m z_i y_i$$

In any case $G = {}^cG = GL(n)$. Define ω by

$$\omega(\sigma \times 1) = \sigma \times \pi(\sigma)$$

The action of $\mathcal{O}$ on G is to be trivial. Since $\omega(\alpha_p') = \alpha_j$ is the conjugacy class of $\sigma_p \times \pi(\sigma_p)$ which of course determines the conjugacy class of $\pi(\sigma_p)$ all we need is a method of finding α_p.

Suppose there is an automorphic form ϕ on some inner form of $GL(n)$ which provides an affirmative answer to the above question. To find α_p all we need do is calculate the eigenvalues of a finite number of elements of the Hecke algebra H_p corresponding to the eigenfunction ϕ. Choose a finite set S of primes containing the infinite prime so that if $\bar{G}_S = \prod_{q \in S} \bar{G}_{Q_q}$ and $\bar{G}_{S'} = \prod_{q \notin S} \bar{G}_{Z_q}$ then $\bar{G}_A = \bar{G}_Q \bar{G}_S \bar{G}_{S'}$ and ϕ is a function on $\bar{G}_Q \backslash \bar{G}_A / \bar{G}_{S'}$.

Suppose $p \notin S$ and f is the characteristic function of $\bar{G}_{Z_p} a \bar{G}_{Z_p}$ which is the disjoint union $\cup_{i=1}^n a_i \bar{G}_{Z_p}$. If $g \in G_S$

$$\lambda(f)\phi(g) = \int_{\bar{G}_{Z_p}} \phi(gh) f(h)\, dh$$

$$= \Sigma_{i=1}^n \phi(g a_i) = \Sigma_{i=1}^n \phi(a_i g)$$

since $a_i \in \bar{G}_{Q_p}$. Choose $\bar{a}_1, \ldots, \bar{a}_n$ in $\bar{G}_Q$ so that $\bar{a}_i^{-1} a_i \in \bar{G}_{S'}$ and let b_i be the projection of $\bar{a}_i$ on $\bar{G}_S$. If $\chi(f)$ is the ~~characteristic function~~ eigenvalue of f

$$\chi(f)\phi(g) = \Sigma\ \phi(\bar{a}_i^{-1} a_i g)$$

$$= \Sigma\ \phi(b_i^{-1} g)$$

Now roughly speaking the elements $\bar{a}_1, \dots, \bar{a}_n$ are obtained by solving some diophantine equations involving p as a parameter. Then $\phi(b_i^{-1} g)$ depends upon the congruential properties of $\bar{a}_i$ modulo powers of the finite primes in S and the projection of $\bar{a}_i$ on $\bar{G}_{Q_\infty} = \bar{G}_{\mathbb{R}}$. If, for each g in $\bar{G}_S$, $\phi(hg)$ as a function of h in the connected component of $\bar{G}_{\mathbb{R}}$ were rational we would get a good splitting law. It would be rather complicated but in principle no worse than the splitting laws of Dedekind-Hasse for the splitting field of a cubic equation. However because of the strong approximation $\phi(hg)$ will probably not be rational unless $n = 1$ or 2. Thus we could only get a transcendental splitting law.

Nonetheless if we took G to be the symplectic group in $2n$ variables and ${}^cG$ to be the orthogonal group in $2n+1$ variables then strong approximation is no obstacle because G has inner forms for which $\bar{G}_{\mathbb{R}}$ is compact and we might hope to obtain laws about such things as the order of σ_p be considering imbeddings of $\mathfrak{G}$ in ${}^cG$.

Yours truly,

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Postscript: Let me add

(v) Let K be a quadratic extension of Q. Let $\mathfrak{G}' = \mathfrak{G} = \mathfrak{G}(K/Q)$. Let $G' = {}^cG' = A_1 \times A_2$ where A_1 and A_2 are one dimensional split tori. If σ is the non-trivial element of $\mathfrak{G}$ let $(t_1 \times t_2)^{\sigma'(\sigma)} = t_2 \times t_1$. Let $G = GL(2)$ and let $\mathfrak{G}$ act trivially on G. Define ω by

$$\omega(1 \times (t_1, t_2)) = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$$

$$\omega(\sigma \times (t_1, t_2)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} = \begin{pmatrix} 0 & t_2 \\ t_1 & 0 \end{pmatrix}$$

$G_A'^{\sigma'}$ is just the idele group of K. Take ϕ' to be a Grössencharakter. It is not inconceivable that the work of Hecke and Maass on the relation between

L series with Grössencharakter from a quadratic field and automorphic
forms will provide an affirmative answer to the second question in this case.