Hermitian differential geometry, Chern classes, and positive vector bundles

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0. Introduction, statement of results, and open questions

(a) *Statement of results.* Our purpose is to discuss the notion of *positivity* for holomorphic vector bundles. This paper is partly expository, and in so doing we hope to clarify, simplify, and unify, some of the existing material on the subject. There are several new results, mostly relating the analytic notion of positivity to the topological properties of the bundle. We also correct two errors in our previous paper [10].

We now give the main results to be proved in this paper.

Let V be a compact, complex manifold and $\mathbf{E} \to V$ a holomorphic vector bundle; we denote by $\Gamma(\mathbf{E})$ the space of holomorphic cross-sections of $\mathbf{E} \to V$. The relevant definitions are the following:

(0.1) $\mathbf{E} \to V$ is *positive* if there exists an hermitian metric in $\mathbf{E}$ whose *curvature tensor* $\Theta = \{\Theta^{\rho}_{\sigma i j}\}$ has the property that the hermitian quadratic form

$$\Theta(\xi, \eta) = \sum_{\rho,\sigma,i,j} \Theta^{\rho}_{\sigma i j} \xi^{\sigma} \bar{\xi}^{\rho} \eta^{i} \bar{\eta}^{j}$$

is positive definite in the two variables ξ, η;

(0.2) $\mathbf{E} \to V$ is *ample* if

(a) the *global sections generate each fibre*, so that we have $0 \to \mathbf{F}_z \to \Gamma(\mathbf{E}) \to \mathbf{E}_z \to 0$ (for all $z \in V$), and

(b) the natural mapping $\mathbf{F}_z \to \mathbf{E}_z \otimes \mathbf{T}^*_z$ is onto ($\mathbf{F}_z$ = sections of $\mathbf{E}$ vanishing at z);

(0.3) $\mathbf{E} \to V$ is *cohomologically positive* if, given any coherent sheaf $S \to V$, there is a $\mu_0 = \mu_0(S)$ such that $H^q(V, \mathcal{O}(E^{(\mu)}) \otimes S) = 0$ for $q > 0$, $\mu \geqq \mu_0$; and

(0.4) $\mathbf{E} \to V$ is *numerically positive* if, for any complex analytic subvariety $W \subset V$ and any quotient bundle $\mathbf{Q}$ of $\mathbf{E} \mid W$, we

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have $\int_W P(c_1, \cdots, c_s) > 0$ where $P(c_1, \cdots, c_s)$ is a *positive polynomial* in the *Chern classes* $c_1, \cdots, c_s$ of $\mathbf{Q} \to W$ (cf. § 5 (b)).

Our main general results are:

THEOREM A: *ample* $\Rightarrow$ *positive;*

THEOREM B: *positive* $\Rightarrow$ *cohomologically positive;*

THEOREM C: *positive* $\Rightarrow$ $\mathbf{E}^{(\mu)}$ *ample for all* $\mu \geqq \mu_0$;

THEOREM D: *ample* $\Rightarrow$ *numerically positive;*

THEOREM E: *numerically positive* $\Rightarrow$ *cohomologically positive.*

THEOREM F: *cohomologically positive* $\Rightarrow$ $\mathbf{E}^{(\mu)}$ *ample for* $\mu \geqq \mu_0$.

In summary, if we take the sequence of bundles $\mathcal{E} = (\mathbf{E}, \mathbf{E}^{(2)}, \cdots, \mathbf{E}^{(\mu)}, \cdots)$, then, modulo finitely many bundles, positive = ample = cohomologically positive.

Our principal specific results generalize the *Kodaira vanishing theorem* and the *Lefschetz theorem*. For example, we prove

THEOREM G. *If* $\mathbf{E} \to V$ *is generated by its sections, and if* $\mathbf{F} \to V$ *is a line bundle such that* $\mathbf{F}^* \otimes \mathbf{K} \otimes \det \mathbf{E} < 0$ *where* $\mathbf{K} \to V$ *is the canonical bundle, then:*

$$H^q(V, \mathcal{O}(\mathbf{E}^{(\mu)} \otimes \mathbf{F})) = 0 \qquad \textit{for } q > 0,\ \mu \geqq 0\,.$$

The Kodaira theorem is the case $\mathbf{E}$ = trivial line bundle in Theorem G. In § 3 (b) (cf. (3.25)) we shall give a precise vanishing theorem which has Theorem G as a consequence.

As a generalization of the Lefschetz theorem, we assume that $\mathbf{E} \to V$ is a positive bundle with fibre $\mathbf{C}^r$ and where $\dim V = n$. Let $\xi \in H^0(V, \mathcal{O}(\mathbf{E}))$ be a holomorphic section whose zero locus $S \subset V$ is a smooth subvariety of codimension r.

THEOREM H. *We have* $H_q(S, \mathbf{Z}) \to H_q(V, \mathbf{Z}) \to 0$ *for* $q \leqq n - r$ *and* $0 \to H_q(S, \mathbf{Z}) \to H_q(V, \mathbf{Z}) \to 0$ *for* $q \leqq n - r - 1$.

The Lefschetz theorem is the case $r = 1$.

As an application of Theorem H, we prove in § 3 (d) (cf. (3.51)) that the cup product

$$H^{p,q}(V) \xrightarrow{\omega} H^{p+r,q+r}(V),\ p + q = n - r\,, \tag{0.5}$$

is an isomorphism where $\omega \in H^{r,r}(V)$ is the r^{th} *Chern class* of $\mathbf{E} \to V$.

This result is the analogue of a well-known fact in Kähler geometry [26].

We now give another generalization of the *(coarse) Kodaira vanishing theorem*. Let $\mathbf{E} \to V$ be a positive vector bundle with a non-singular section $\xi \in H^0(V, \mathcal{O}(\mathbf{E}))$; denote by I the *ideal sheaf* of the zero-locus S of ξ. We introduce the curvature form

$$(0.1)' \quad \Theta^{\sharp}_{\mathbf{E}}(\xi, \eta) = (r+1) \sum_{\left\{\substack{\rho,\sigma \\ i,j}\right.} \Theta^{\rho}_{\sigma i} \xi_j{}^{\sigma} \bar{\xi}^{\rho} \eta^i \bar{\eta}^j - \sum_{\left\{\substack{\rho,\sigma \\ i,j}\right.} \Theta^{\rho}_{\rho i j} \xi^{\sigma} \bar{\xi}^{\sigma} \eta^i \bar{\eta}^j .$$

For the significance of this form, we refer to (3.25), Theorem G′ where it is proved that, if $\Theta^{\sharp}_{\mathbf{E}} > 0$, then $H^q(V, \mathcal{O}(\mathbf{E}^*)) = 0$ for $q \leqq n-1$. The Kodaira theorem is the case $r = 1$. Note that, if $\mathbf{L} \to V$ is a positive line bundle, then $\Theta^{\sharp}_{\mathbf{E}\otimes\mathbf{L}^{\mu}} > 0$ for μ sufficiently large; this is because of the $r+1$ factor in front of the first term.

THEOREM I. *Let* $\mathbf{F} \to V$ *be a holomorphic vector bundle. Then there exists a constant* $e = c(\mathbf{F}, V)$ *such that, if*

$$\Theta^{\sharp}_{\mathbf{E}}(\xi, \eta) > c\,|\,\xi\,|^2\,|\,\eta\,|^2 ,$$

then

$$H^q(V, I \otimes \mathcal{O}(\mathbf{F})) = 0 \qquad \textit{for } q \leqq n - r .$$

Another analogon of Kodaira's (coarse) vanishing theorem is

THEOREM J. *With the same assumption as in Theorem* I, *we have*

$$H^q(V, I^{\mu} \otimes \mathcal{O}(\mathbf{F})) = 0 \qquad \textit{for } \mu \geqq \mu_0(\mathbf{F}),\ q \leqq n - r .$$

As an unsolved problem, we would like to mention the following possible generalization of the *precise Kodaira vanishing theorem.*

(0.6) *Conjecture.* If $\mathbf{E} \to V$ is positive, then

$$H^q(V, \mathcal{O}(\mathbf{E}^*)) = 0 \qquad \textit{for } q \leqq n - r .$$

Remark. For $\mathbf{E} \to V$ a line bundle, (0.6) is just the Kodaira theorem. Taking $V = \mathbf{P}_2$ and $\mathbf{E} = \mathbf{T}(\mathbf{P}_2)$ the (positive) tangent bundle, $\mathcal{O}(\mathbf{E}^*) = \Omega^1$ and $H^1(\mathbf{P}_2, \Omega^1) \neq 0$, so that (0.6) is the best possible. We will prove the conjecture (cf. § 5 (e)) when $r = 2$ and $\mathbf{E} \to V$ has a non-singular section.

Another problem is

(0.7) *Conjecture.* If $\mathbf{E} \to V$ is positive, then $\mathbf{E} \to V$ is numerically positive.

Remark. We will prove (0.7) in case V is a surface ($n = 2$) and $\mathbf{E}$ has fibre dimension 2. This proof, given in the Appendix to

§ 5 (b), involves a Schwarz inequality for differential forms. In the context of algebraic geometry (characteristic zero), the assertion "$\mathbf{E} \to V$ cohomologically positive $\Rightarrow \mathbf{E} \to V$ numerically positive" has been proved for $r = 1$ (Nakai [21]), $n = 1$ and $r = 2$ (Hartshorne [14]), and $n = 2$ (Kleiman [17]). The first step in proving (0.7) would be to show that the *Chern classes* $c_q(\mathbf{E})$ of a positive bundle $\mathbf{E} \to V$ are positive. Still another problem we mention is

(0.8) *Problem.* Find a better definition of the cone of positive polynomials and prove that *cohomologically positive* $\Leftrightarrow$ *numerically positive.*

Remark. For $r = 1$, we have the theorem of Nakai [21] (cf. § 5 (c) below).

Another question is

(0.9) *Problem.* If $\mathbf{E} \to V$ is cohomologically positive, is $\mathbf{E} \to V$ positive?

If true, this would give the nicest general result, as we would have

(0.10) Positive (differential-geometric) $\Leftrightarrow$ cohomologically positive (algebro-geometric) $\Leftrightarrow$ numerically positive (topological) $\Leftrightarrow$ $\mathbf{E}^{\mu}$ ample for $\mu \geqq \mu_0$.

If $\mathbf{E} \to V$ is cohomologically positive, then there is a *non-linear positive metric* in $\mathbf{E}$ as follows: By sending ξ into $\underbrace{\xi \otimes \cdots \otimes \xi}_{\mu}$ (diagonal mapping), we have an embedding $\mathbf{E} \subset \mathbf{E}^{\mu}$, and $\mathbf{E}^{\mu}$ is ample for $\mu \geqq \mu_0$. Using this metric, a tubular neighborhood of the zero cross-section of $\mathbf{E}^* \to V$ is *strongly pseudo-convex* (cf. § 3 (a) below).

The reason that we use the differential-geometric notion of positivity (0.1) rather than the function-theoretic definition (cf. Grauert [9]) just mentioned is that the curvature is relevant for *precise vanishing theorems* and for the Lefschetz theorem, whereas pseudo-convexity will yield only coarse results. These precise vanishing theorems will have several uses in geometric problems; for example, Theorem G has been used by W. Schmid [23] to give a generalization of the *Borel-Weil theorem* to real, semi-simple Lie groups.

(b) *Complements and corrections to* [10]. The difficulty in [10] is that there were several definitions of positivity and ampleness given and these did not leave a clear picture of what positive and ample bundles should be. If we are not worried about the bundle $\mathbf{E} \to V$ itself but are content to take symmetric powers, then the various notions coincide, as indicated below Theorem F above. This is the approach taken by Hartshorne [14], whose definition of an ample bundle coincides with our cohomologically positive (cf. [10, Prop. (3.3)]).

It now seems that (0.1) is the best differential-geometric generalization of positivity for line bundles; other definitions are discussed in [10, § 3] and still another definition is given in [22]. Our positive here is the same as weakly positive in [10], and is a condition which turns up most naturally in geometric situations.

Our definition of ampleness (0.2) expresses the geometric assumption that $\mathbf{E} \to V$ should have "sufficiently many sections". In case $\mathbf{E}$ is a line bundle, "sufficiently many sections" means that the mapping into projective space is an immersion. However, in general the universal bundle over the grassmannian is *not* positive or ample (in any sense), and "sufficiently many sections" means that the immersion of V in a grassmannian is twisted.

The notion of sufficiently ample in [10] is not a good definition, nor is the definition of negative given above Proposition 4.1 in [10].

The definitions (0.3) and (0.4) of cohomologically positive and numerically positive are seemingly good notions.

The main error in [10] is Proposition 7.2. An application of this, Proposition 10.2 of [10], is incorrect, as the following example shows: Let $X = \mathbf{P}_2$ be projective 2-space and $\mathbf{T} \to X$ the tangent bundle. Then $\mathbf{T}$ is generated by its sections, as is $\Lambda^2\mathbf{T} = K^*$. Thus we have $0 \to \mathbf{F}^* \to \mathbf{E}^* \to \mathbf{K}^* \to 0$ where $\mathbf{E}$ is a trivial bundle; this dualizes to $0 \to \mathbf{K} \to \mathbf{E} \to \mathbf{F} \to 0$. Now $\mathbf{E}$ and $\mathbf{F}$ are generated by their sections and $\mathbf{K}$ is negative; if Proposition 10.2 were true, then $H^1(X, \mathcal{O}(\mathbf{F})) = 0 = H^2(X, \mathcal{O}(\mathbf{E}))$ and so $H^2(X, \mathcal{O}(\mathbf{K})) = H^2(X, \Omega^2) = 0$, which is absurd.

The mistake in the proof of Proposition 7.2 arises in equation (7.7), in which the row and column indices for the curvature matrix are interchanged. This error can be traced to just below equation (1.3), where the connexion form should be $\omega = \partial h \cdot h^{-1}$. With the correct formula $\Theta^{\rho}_{\sigma i \bar{j}} = -\sum A^{\alpha}_{\sigma i}\bar{A}^{\alpha}_{\rho j}$ (cf. equation (2.24) below), the curvature operator $\Theta(\xi, \xi)$, given by (4.2) in [10], on an E-valued (0.1) form $\xi = \{\xi^{\rho}_{j}\omega^{j}\}$ becomes

$$
\begin{aligned}
&-\sum A^\alpha_{\sigma i}\bar{A}^\alpha_{\rho i}\xi^\sigma_{\bar{j}}\xi^\rho_{\bar{j}} + \sum A^\alpha_{\sigma i}\bar{A}^\alpha_{\rho j}\xi^\sigma_{\bar{i}}\bar{\xi}^\rho_{\bar{j}} \\
&\qquad = \sum\nolimits_\alpha \{\text{Trace}\,(A^\alpha(\xi)^t\overline{A^\alpha(\xi)}) - |\,\text{Trace}\,A^\alpha(\xi)\,|^2\}
\end{aligned}
$$

where $A^\alpha(\xi)_{i\bar{j}} = \sum_\sigma A^\alpha_{\sigma i}\xi^\sigma_{\bar{j}}$, which is neither positive nor negative. With the incorrect formula, $\Theta(\xi, \xi) \leqq 0$ and this is the mistake.

The other error in [10] is Lemma 9.2, which is corrected in formula (2.38) below.

The corrected version of Proposition 10.2 in [10] is Theorem G above; and the corrected form of Proposition 7.2 is Theorem A above. The remaining results in [10] are presumably correct.

Referring again to Hartshorne's paper [14], he works on the associated projective bundle $\mathbf{P}(\mathbf{E}^*)$, as was done in of [10, § 9] and is done here. Many of the general results of [10] on positive bundles, such as the fact that a quotient of a positive (ample) bundle is positive (ample), also appear in [14]. The Theorems D and E on numerical positivity, which are proved for ample bundles in [11], are given in [14, Prop. 7.5] for the case of V a curve and $\mathbf{E}$ a bundle with fibre $\mathbf{C}^2$. Interestingly, the proofs are quite similar; both use the numerical criterion of Nakai [21] on the associated projective bundle. Finally, as mentioned above, Kleiman has proved (0.7) in case $\dim V = 2$ [17].

1. Discussion of methods and calculations

Suppose that $\mathbf{E} \to V$ is ample and let $\Gamma(\mathbf{E})$ be the trivial bundle $V \times \Gamma(\mathbf{E})$. Then there are exact sequences

$$
0 \longrightarrow \mathbf{F} \longrightarrow \Gamma(\mathbf{E}) \longrightarrow \mathbf{E} \longrightarrow 0\,, \tag{1.1}
$$

and

$$
\mathbf{F} \longrightarrow \mathbf{E} \otimes \mathbf{T}^* \longrightarrow 0\,, \tag{1.2}
$$

where the fibre $\mathbf{F}_z = \{s \in \Gamma(\mathbf{E}) : s(z) = 0\}$. The flat metric in the trivial bundle $\Gamma(\mathbf{E})$ induces a metric in $\mathbf{F}$ and, by orthogonality, a metric in $\mathbf{E}$. A computation shows that the curvature matrix Θ of this metric in $\mathbf{E}$ has the local form

$$
\Theta^\rho_\sigma = \sum\nolimits_\alpha A^\alpha_\sigma \wedge \bar{A}^\alpha_\sigma \tag{1.3}
$$

where $A^\alpha_\sigma = \sum_i A^\alpha_{\sigma i}dz^i$ is essentially the differential of the mapping of V into a grassmannian. The quadratic form $\Theta(\xi, \eta)$ in (0.1) is then $\sum_\alpha |\,A^\alpha(\xi, \eta)\,|^2$ where $A^\alpha(\xi, \eta) = \sum_{\rho, i} A^\alpha_{\rho i}\bar{\xi}^\rho\eta^i$, and from (1.2) it will follow that $\Theta(\xi, \eta)$ is positive definite, which proves Theorem A.

The formula (1.3) for the curvature is a special case of studying the hermitian geometry of an exact bundle sequence

$$0 \longrightarrow \mathbf{S} \longrightarrow \mathbf{E} \longrightarrow \mathbf{Q} \longrightarrow 0 . \tag{1.4}$$

A metric in $\mathbf{E}$ induces metrics in $\mathbf{S}$ and $\mathbf{Q}$, and the deviation of the induced connexion on $\mathbf{S}$ from the connexion of the induced metric is measured by an important invariant, the *second fundamental form of* $\mathbf{S}$ *in* $\mathbf{E}$, introduced in [11, § VI. 3], and discussed in some detail below. This tensor has both geometric and cohomological significance.

To prove Theorem D, we use the representation of Chern classes by differential forms [5] and [12], together with the form (1.3) of the curvature. Then, as was outlined in [11, § IV], it will follow that a positive polynomial

$$P(c_1, \cdots, c_s) \geqq \left(\frac{i}{2}\right)^q (-1)^{q(q-1)/2} \{\textstyle\sum_\lambda B_\lambda \wedge \bar{B}_\lambda\} \tag{1.5}$$

where B_λ is a $(q, 0)$ form ($q = \dim W$). Using (1.2) we find

$$\left(\frac{i}{2}\right)^q (-1)^{q(q-1)/2} \int_W \{\textstyle\sum_\lambda B_\lambda \wedge \bar{B}_\lambda\} > 0 ,$$

which proves Theorem D.

It is an open question whether or not a positive bundle, or perhaps a cohomologically positive bundle, is numerically positive.

The more difficult assertions are Theorems B, G, and, later on, E. A proof of Theorem B directly, by differential geometric methods, involves (for $q = 1$) the quadratic form (cf. [10, Prop. 5.2 and [22])

$$\Theta(\varphi, \varphi) = \textstyle\sum_{\rho,\sigma,i,j} \Theta^\rho_{\sigma i \bar{j}} \varphi^\sigma_i \bar{\varphi}^\rho_j , \tag{1.6}$$

and we need to show that $\Theta(\varphi, \varphi) \geqq 0$. Suppose that $\mathbf{E}$ has a lot of sections so that Θ^ρ_σ has the form (1.3). Using the identity

$$\begin{aligned} &A^\alpha_{\sigma i}\varphi^\rho_i \overline{A^\alpha_{\rho j}\varphi^\sigma_j} + A^\alpha_{\rho i}\varphi^\sigma_i \overline{A^\alpha_{\sigma j}\varphi^\rho_j} \\ &\quad = A^\alpha_{\rho i}\varphi^\sigma_i \overline{A^\alpha_{\rho j}\varphi^\sigma_j} + A^\alpha_{\sigma i}\varphi^\rho_i \overline{A^\alpha_{\sigma j}\varphi_j} \\ &\qquad - (A^\alpha_{\rho i}\varphi^\sigma_i - A^\alpha_{\sigma i}\varphi^\sigma_i)(\overline{A^\alpha_{\rho j}\varphi^\sigma_j - A^\alpha_{\sigma j}\varphi^\rho_j}) \end{aligned}$$

(no summation), we get

$$\Theta(\varphi, \varphi) = \textstyle\sum_{\alpha,\rho,\sigma} | A(\varphi)^\alpha_{\rho\sigma} |^2 - \frac{1}{2} \textstyle\sum_{\sigma,\rho,\alpha} | \hat{A}(\varphi)^\alpha_{\rho\sigma} |^2 , \tag{1.7}$$

where $A(\varphi)^{\alpha}_{\rho\sigma} = \sum_i A^{\alpha}_{\sigma i}\varphi^{\rho}_i$ and $\hat{A}(\varphi)^{\alpha}_{\rho\sigma} = \sum_i (A^{\alpha}_{\rho i}\varphi^{\sigma}_i - A^{\alpha}_{\sigma i}\varphi^{\rho}_i)$. If $\mathbf{E}$ is a line bundle, $\hat{A}(\varphi) = 0$ and, by (1.7), $\Theta(\varphi, \varphi) \geqq 0$. In general, however, the quadratic form (1.7) does not have a sign.

The conclusion is that a geometric assumption (ampleness) gives an inequality on the quadratic form $\Theta(\varphi, \varphi)$ where $\varphi = \xi \otimes \eta$ is a *decomposable tensor*, whereas to prove directly a vanishing theorem we need information on $\Theta(\varphi, \varphi)$ for *all* tensors.

A means around this trouble is suggested in [10, § 9]. Let $\mathbf{E}^* \to V$ be the dual bundle, $\mathbf{P}(\mathbf{E}^*) = \mathbf{P}$ the associated projective bundle, and $\mathbf{L} \to \mathbf{P}$ the standard line bundle whose restriction to each fibre of $\mathbf{P}(\mathbf{E}^*) \to V$ is the positive hyperplane bundle. There are two basic facts

(1.8) $\mathbf{E} \to V$ positive $\Rightarrow \mathbf{L} \to$ positive, (cf. [10, Prop. (9.1)] and § 3. (b) below); and

$$H^q(V, \mathcal{O}(\mathbf{E}^{(\mu)}) \otimes S) \cong H^q(\mathbf{P}, \mathcal{O}(\mathbf{L}^{\mu}) \otimes \pi^* S) \tag{1.9}$$

for any coherent sheaf S over V. These two facts, together with Theorem B for line bundles give the assertion for general vector bundles.

The proof of Theorem G follows from the usual Kodaira vanishing theorem on $\mathbf{P}(\mathbf{E}^*)$, coupled with a precise curvature computation. Passing from $\mathbf{E} \to V$ to $\mathbf{L} \to \mathbf{P}(\mathbf{E}^*)$ has the analytic effect of splitting all tensors, which in turn leads to the desired inequalities.

The proof of Theorem E, as outlined in [11, § A. 1], uses the result for line bundles (cf. Nakai [21]) and (1.9)), so that it will suffice to prove that $\mathbf{L} \to \mathbf{P}$ is numerically positive if $\mathbf{E} \to V$ is. This involves relating the algebraic homology ring of $\mathbf{P}$ with that of V and use of integration over the fibre in $\mathbf{P} \to V$.

The proof of Theorem H is done by suitably generalizing Bott's Morse-theoretic argument [2] for line bundles to the case of vector bundles. This result substantiates the definition of positivity. Along with the proof of Theorem H we show that $V - S$ is *r-convex* (i.e., there is an exhaustion function for $V - S$ whose *E. E. Levi form* has $n - r + 1$ positive eigenvalues).

The proof of Theorem I uses Theorem G′ and a standard locally free resolution of the ideal sheaf I of $S \subset V$. The more interesting Theorem J is proved by first blowing up V along S to obtain a codimension one situation of $\tilde{S} \subset \tilde{V}$ where $\tilde{S}$ is given by the zeros of a holomorphic section of a line bundle $\mathbf{L} \to \tilde{V}$. The metric on $\mathbf{E} \to V$ induces a metric on $\mathbf{L} \to \tilde{V}$ whose curvature we compute using the second fundamental forms. It follows that the curvature $\Theta_{\mathbf{L}}$ has everywhere $n - r + 1$ positive eigenvalues, and then a suitable vanishing theorem gives Theorem J.

In § 4 we discuss Chern classes. A direct geometric definition, involving an algebraic-geometric obstruction theory, and the definition using differential forms are given. Using the theorem of Weil, proved in § 4 (b) below, these definitions are proved to be the same. By putting a little more effort into this argument, we give in § 4 (c) another proof of the theorem of Bott-Chern [3].

In § 5 we discuss positive cohomology classes and give the proof of Theorem E. Remarks on the problem (0.7) are also given (cf. below the proof of Theorem D in § 5 (a)).

2. Hermitian differential geometry

(a) *The frame bundle.* Let V be a complex manifold and $\mathbf{E} \to V$ a *holomorphic vector bundle* with fibre $\mathbf{C}^r$. We think of $\mathbf{C}^r$ as column vectors

$$\xi = \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^r \end{pmatrix}$$

and, for a matrix $g = (g^\rho_\sigma) \in G = GL(r, \mathbf{C})$, we set

$$g\xi = \begin{pmatrix} \vdots \\ \sum_\sigma g^\rho_\sigma \xi^\sigma \\ \vdots \end{pmatrix}.$$

Let $\mathbf{P} \xrightarrow{\pi} V$ be the principal bundle, with group G, of all *holomorphic frames* $f = (e_1, \cdots, e_r)$ for $\mathbf{E} \to V$. Then G acts on $\mathbf{P}$ by $fg = (\cdots, \sum_\rho g^\rho_\sigma e_\rho, \cdots)$, and a section ξ of $\mathbf{E} \to V$ is given on $\mathbf{P}$ by $\xi = \sum_{\rho=1}^r \xi^\rho(f) e_\rho$ with

$$\xi^\rho(fg) = \sum_\sigma (g^{-1})^\rho_\sigma \xi^\sigma(f) .$$

Similarly, a differential form on V with values in $\mathbf{E}$ is given on $\mathbf{P}$ as $\varphi = \sum_\rho \varphi^\rho e_\rho$ where φ^ρ is a horizontal form on $\mathbf{P}$ satisfying equivariance conditions.

As an example, consider the *Grassmann manifold* $\mathbf{G} = \mathbf{G}(r, m)$ of r-planes in $\mathbf{C}^m$. We let $\mathbf{P}$ be the r-frames $f = (e_1, \cdots, e_r)$ in $\mathbf{C}^m$, where $e_\rho = (\xi^\alpha_\rho)$ is a column vector and $e_1 \wedge \cdots \wedge e_r \neq 0$. Then $\pi\colon \mathbf{P} \to \mathbf{G}$ is given by $\pi(f) =$ subspace spanned by $e_1, \cdots, e_r$. Observe that

$$fg = \left(\sum_\rho \xi^\alpha_\rho g^\rho_\sigma\right) = \left(\cdots, \sum_\rho g^\rho_\sigma e_\rho, \cdots\right)$$

so that our notation is consistent. The vector bundle $\mathbf{E} \to \mathbf{G}$ is the *universal bundle* whose fibre $\mathbf{E}_S$ at a subspace $S \in \mathbf{G}$ is the vector space S itself.

(b) *Metrics, connexions, and curvatures.* A *hermitian metric* in $\mathbf{E} \to V$ gives a matrix function h on $\mathbf{P}$ by the rule $h(f)_{\rho\sigma} = (e_\sigma, e_\rho)$. Then

$$\left(\sum_\rho \xi^\rho e_\rho, \sum_\sigma \eta^\sigma e_\sigma\right) = \sum_{\rho,\sigma} \bar{\eta}^\sigma h_{\sigma\rho} \xi^\rho = {}^t\bar{\eta} h \xi \ .$$

We have $h = {}^t\bar{h}$, $h > 0$, and $h(fg) = {}^t\bar{g} h(f) g$. From this last equation, we see that the (1.0) form θ on $\mathbf{P}$ defined by $\theta = h^{-1}\partial h$; i.e., $\theta^\rho_\sigma = \sum_\tau (h^{-1})_{\rho\tau} \partial h_{\tau\sigma}$, satisfies $\theta(fg) = g^{-1}\theta(f) g$ and gives a *connexion* in $\mathbf{P} \to V$. For a section $\xi = \sum_\rho \xi^\rho e_\rho$ of $\mathbf{E} \to V$, we define the *covariant differential* $D\xi = \sum_\rho d\xi^\rho e_\rho + \sum \xi^\rho D e_\rho$ where $De_\rho = \sum_\sigma \theta^\sigma_\rho e_\sigma$; i.e., $(D\xi)^\rho = d\xi^\rho + \sum_\sigma \theta^\rho_\sigma \xi^\sigma$. Then $D\xi$ is an $\mathbf{E}$-valued 1-form and $d(\xi, \eta) = (D\xi, \eta) + (\xi, D\eta)$ for sections ξ, η of $\mathbf{E}$. Writing $D = D' + D''$ where D' is type (1, 0) and D'' is of type (0, 1), we have that $D' = \partial + \theta$ and $D'' = \bar{\partial}$.

More generally, for an $\mathbf{E}$-valued q-form φ on V, we set $D\varphi = \sum_\rho (d\varphi^\rho e_\rho + (-1)^q \varphi^\rho \wedge De_\rho)$; $D\varphi$ is an $\mathbf{E}$-valued $q + 1$-form on V.

The *curvature form* $\Theta = (\Theta^\rho_\sigma)$ is given by $\Theta = d\theta + \theta \wedge \theta$; i.e., $\Theta^\rho_\sigma = d\theta^\rho_\sigma + \sum \theta^\rho_\tau \wedge \theta^\tau_\sigma$. Since $\partial(h^{-1}\partial h) + h^{-1}\partial h \wedge h^{-1}\partial h = 0$, it follows that

$$\Theta = \bar{\partial}\theta = -h^{-1}\partial\bar{\partial} h - h^{-1}\bar{\partial} h \wedge h^{-1}\partial h \ , \tag{2.1}$$

and Θ is of type (1, 1).

For an $\mathbf{E}$-valued form φ, we have the *Bianchi identity*:

$$D^2\varphi = \Theta \wedge \varphi = \sum_{\sigma,\rho} \Theta^\rho_\sigma \wedge \varphi^\sigma e_\rho \ . \tag{2.2}$$

Returning to our example of the principal bundle $\mathbf{P} \to \mathbf{G}$ over the grassmannian, we define a metric in the universal bundle by

$$h(f) = {}^t\bar{f} f \ , \tag{2.3}$$

where $f = (\xi^\alpha_\rho)$ is the $m \times r$ matrix whose columns give the frame f. Then $h(fg) = {}^t\bar{g}\,{}^t\bar{f} f g = {}^t\bar{g} h(f) g$ and the curvature:

$$\Theta = h^{-1}\,{}^t\overline{df} \wedge df - h^{-1}({}^t\overline{df} f) h^{-1}({}^t\bar{f} df) \ . \tag{2.4}$$

In case $r = 1$, $\mathbf{P} = \mathbf{C}^m - \{0\}$ and, by (2.4),

$$\Theta = \frac{(df, f)(f, df) - (f, f)(df, df)}{(f, f)^2} \ . \tag{2.5}$$

Thus Θ is the negative of the standard Kähler form on $\mathbf{P}_{m-1}$, which checks our signs since $\mathbf{E}$ is the dual of the positive line bundle given by the divisor $\mathbf{P}_{m-1} \subset \mathbf{P}_m$.

(c) *Calculations in local coordinates.* Let $z = (z^1, \cdots, z^n)$ be local holomorphic coordinates in V and $f(z) = (e_1(z), \cdots, e_r(z))$ a local holomorphic section of $\mathbf{P} \to V$. Then $h_{\rho\sigma}(z) = (e_\sigma(z), e_\rho(z))$ is a function of z and

$$\theta^\rho_\sigma(z) = \sum\nolimits_{\tau, j} (h^{-1}(z))_{\rho\tau} \frac{\partial h_{\tau\sigma}(z)}{dz^j} dz^j \ .$$

The curvature $\Theta(z)$ is given by (2.4), where $h = h(z)$ is a function of z.

If $g(z) = (g^\rho_\sigma(z))$ is a holomorphic matrix, then $f(z)g(z)$ is another holomorphic frame. Taking $g(z)$ to be a suitable constant matrix, we may assume that $h_{\rho\sigma}(0) = \delta^\rho_\sigma$; i.e., $h(0) = I$. Let $A(z) = \left(\sum_j A^\rho_{\sigma j} z^j\right)$ be a linear matrix with

$$A^\rho_{\sigma j} = -\frac{\partial h_{\rho\sigma}(0)}{\partial z^j} \ .$$

Then $dA(0) = -\partial h(0)$ and so $d({}^t\overline{g(z)}h(z)g(z))_{z=0} = 0$ with $g(z) = I + A(z)$. In summary, we may choose our frame $f(z) = (e_1(z), \cdots, e_r(z))$ such that, at $z = 0, h = I$ and $dh = 0$. By (2.1), at the origin,

$$\Theta = -\partial\bar{\partial} h \ . \tag{2.6}$$

For example, letting $Z = (\xi^\lambda_\rho)$ $(1 \leqq \lambda \leqq m - r, 1 \leqq \rho \leqq r)$ be an $(m - r) \times r$ matrix, the mapping $f(Z) = \begin{pmatrix} I_r \\ Z \end{pmatrix}$ gives a local cross-section of $\mathbf{P} \to \mathbf{G}$, the bundle over the grassmannian considered above. The metric $h(Z) = I + {}^t\bar{Z}Z$ and so $h(0) = I, dh(0) = 0$. The curvature is given, by (2.6), as

$$\Theta = {}^t\overline{dZ} \wedge dZ \ , \tag{2.7}$$

or

$$\Theta^\rho_\sigma = -\sum\nolimits_\lambda d\xi^\lambda_\sigma \wedge \overline{d\xi^\lambda_\rho} \ . \tag{2.8}$$

Observe that the general linear group $GL(m, \mathrm{C})$ acts transitively on $\mathbf{P}$ by $Af = (A^\alpha_\beta \xi^\beta_\rho)$; this action preserves the fibering $\mathbf{P} \to \mathbf{G}$. The unitary group $U(m)$ acts on $\mathbf{P}$ and preserves the metric, since $h(Af) = {}^t\overline{Af}Af = {}^t\bar{f}\,{}^t\bar{A}Af = {}^t\bar{f}f = h(f)$. The action of $U(m)$ on $\mathbf{G}$

is transitive, and so the curvature in $\mathbf{E} \to V$ is determined by (2.8), which is Θ at a point.

Let $e_1(z), \cdots, e_r(z)$ be a local frame for $\mathbf{E}$ and $e_1^*(z), \cdots, e_r^*(z)$ the dual frame for the *dual bundle* $\mathbf{E}^* \to X$. Then there is defined a connexion D^* in $\mathbf{E}^*$ by the requirement

$$0 = d\langle e_\rho, e_\sigma^* \rangle = \langle De_\rho, e_\sigma^* \rangle + \langle e_\rho, D^* e_\sigma^* \rangle .$$

This gives $\theta_\rho^\sigma + \theta^{*\rho}_\sigma = 0$ or $\theta^{*\rho}_\sigma = -\theta_\rho^\sigma$. The curvature

$$\Theta^{*\rho}_\sigma = -\Theta_\rho^\sigma . \tag{2.9}$$

It is easy to verify that θ^* is the metric connexion of the induced metric in $\mathbf{E}^*$. In fact, θ^* is obviously of type (1, 0) and preserves the metric in $\mathbf{E}^*$; by uniqueness, θ^* is the metric connexion.

If $\mathbf{E}$ and $\mathbf{F}$ are bundles with frames $e_1, \cdots, e_r; f_1, \cdots, f_s$, then $e_\rho \otimes f_\alpha$ is a frame for $\mathbf{E} \otimes \mathbf{F}$ and we may define a connexion in $\mathbf{E} \otimes \mathbf{F}$ by

$$\begin{aligned} D(e_\rho \otimes f_\alpha) &= De_\rho \otimes f_\alpha + e_\rho \otimes Df_\alpha \\ &= \textstyle\sum_\sigma \theta_\rho^\sigma e_\sigma \otimes f_\alpha + \sum_\beta \theta_\alpha^\beta e_\rho \otimes f_\beta ; \end{aligned}$$

i.e., the connexion $\theta_{\mathbf{E}\otimes\mathbf{F}} = \theta_{\mathbf{E}} \otimes 1 + 1 \otimes \theta_{\mathbf{F}}$ and the curvature

$$\Theta_{\mathbf{E}\otimes\mathbf{F}} = \Theta_{\mathbf{E}} \otimes I_{\mathbf{F}} + I_{\mathbf{E}} \otimes \Theta_{\mathbf{F}} . \tag{2.10}$$

We may consider the bundle $\mathbf{B} \subset \mathbf{P}$ of orthonormal frames. On $\mathbf{B}$, $h_{\rho\sigma} = \delta_\sigma^\rho$ and the connexion form θ satisfies $\theta_\sigma^\rho + \bar\theta_\rho^\sigma = 0$. For the curvature then we have the symmetry

$$\Theta_\sigma^\rho + \bar\Theta_\rho^\sigma = 0 , \tag{2.11}$$

or

$$\Theta_{\sigma i \bar j}^\rho + \bar\Theta_{\rho j \bar i}^\sigma = 0 . \tag{2.12}$$

The holomorphic frames $f(z)$ constructed above pass through a point $f(0) \in \mathbf{B}$ and are tangent to $\mathbf{B}$ at $f(0)$.

(d) *The second fundamental form of a sub-bundle.* Suppose that we have an exact sequence

$$0 \longrightarrow \mathbf{S} \longrightarrow \mathbf{E} \longrightarrow \mathbf{Q} \longrightarrow 0 , \tag{2.13}$$

of holomorphic vector bundles over V. In this case, we let $\mathbf{P}$ be the bundle of all frames $f = (e_1, \cdots, e_r)$ for $\mathbf{E}$ where $e_1, \cdots, e_s$ is a frame for $\mathbf{S}$. The group G of $\mathbf{P}$ is now the group $GL(s, r-s)$ of all $r \times r$ matrices $g = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ with A an $s \times s$ matrix. We agree on the range of indices $1 \leqq \rho, \sigma \leqq r$; $1 \leqq \lambda, \mu \leqq s$; and $s + 1 \leqq \alpha$,

$\beta \leqq r$.

Suppose now that we have in $\mathbf{E}$ a hermitian metric. Then there is a connexion $D_{\mathbf{E}}: A^0(\mathbf{E}) \to A^1(\mathbf{E})$, where $A^q(\mathbf{E})$ are the C^∞ q-forms with values in $\mathbf{E}$. This gives in particular $D_{\mathbf{E}}: A^0(\mathbf{S}) \to A^1(\mathbf{E})$.

Since $\mathbf{S} \subset \mathbf{E}$ is a sub-bundle, there is an induced hermitian metric which has its own metric connexion $D_{\mathbf{S}}: A^0(\mathbf{S}) \to A^1(\mathbf{S})$. The difference $D_{\mathbf{E}} - D_{\mathbf{S}}: A^0(\mathbf{S}) \to A^1(\mathbf{E})$ is then linear over the C^∞ functions and so is given by a Hom $(\mathbf{S}, \mathbf{E})$-valued 1-form $b \in A^1(\mathrm{Hom}(\mathbf{S}, \mathbf{E}))$. What we claim is that $b \in A^{1,0}(\mathrm{Hom}\,\mathbf{S}, \mathbf{Q}))$; i.e., b is a $(1, 0)$ form satisfying $(b(\mathbf{S}), \mathbf{S}) = 0$.

To see this, we let $h = \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix}$ be the metric function on $\mathbf{P}$ where h_1 is the induced metric on $\mathbf{S}$. We first choose a holomorphic frame

$$f(z) = (e_1(z), \cdots, e_s(z); \hat{e}_{s+1}(z), \cdots, \hat{e}_r(z))$$

such that $h_1(0) = I_s$, $dh_1(0) = 0$. This is done by varying the A part of $g = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$. Letting $e'_\alpha(z) = \hat{e}_\alpha(z) - \sum_\lambda h_{\alpha\lambda}(0) e_\lambda(z)$, we have a new holomorphic frame in $\mathbf{P}$ for which $h(0) = \begin{pmatrix} I & 0 \\ 0 & h_4(0) \end{pmatrix}$, $dh_1(0) = 0$. By using the C part of g, we may assume that $h(0) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$, $dh_1(0) = 0$, $dh_4(0) = 0$.

We let φ^ρ_σ be the connexion for $\mathbf{E}$ and θ^λ_μ the connexion for $\mathbf{S}$; $\varphi = h^{-1}\partial h$ and $\theta = h_1^{-1}\partial h_1$. Then

$$b(e_\lambda) = (D_{\mathbf{E}} - D_{\mathbf{S}})e_\lambda = \textstyle\sum_\rho \varphi^\rho_\lambda e_\rho - \sum_\mu \theta^\mu_\lambda e_\mu \, .$$

Since, at $z = 0$, $\varphi^\mu_\lambda(0) = 0$, $\theta^\mu_\lambda(0) = 0$, we have that $b(e_\lambda) = \sum_\alpha \varphi^\alpha_\lambda e_\alpha$, $\varphi^\alpha_\lambda = \partial h_{\alpha\lambda}$. This proves that b is of type $(1, 0)$ and $(b(e_\lambda), \mathbf{S}) = 0$ as required.

By definition, $b \in A^{1,0}(\mathrm{Hom}(\mathbf{S}, \mathbf{Q}))$ is the *second fundamental form* of $\mathbf{S}$ in $\mathbf{Q}$. This b has been used in [11, § VI. 3], where the terminology is justified.

We now compute the curvatures at $z = 0$ using (2.1). This gives $(\Theta_{\mathbf{S}})^\lambda_\mu = -\partial\bar{\partial} h_{\lambda\mu}$ and

$$\begin{aligned} (\Theta_{\mathbf{E}})^\lambda_\mu &= -\partial\bar{\partial} h_{\lambda\mu} - \textstyle\sum_\alpha \bar{\partial} h_{\lambda\alpha} \wedge \partial h_{\alpha\mu} \\ &= -\partial\bar{\partial} h_{\lambda\mu} - \textstyle\sum_\alpha \overline{\partial h_{\alpha\lambda}} \wedge \partial h_{\alpha\mu} \, . \end{aligned}$$

Combining, we have

$$(\Theta_{\mathbf{S}})^\lambda_\mu = (\Theta_{\mathbf{E}})^\lambda_\mu + \textstyle\sum_\alpha \overline{\partial h_{\alpha\lambda}} \wedge \partial h_{\alpha\mu} \, . \tag{2.14}$$

In invariant terms, this gives:

$$\Theta_S = \Theta_E \mid S + {}^t\bar{b} \wedge b \,, \tag{2.15}$$

where b is the 2^{nd} fundamental form.

Let us check our signs by computing an example. If $G = G(r, m)$ is the Grassmann variety and $S \to G$ is the universal bundle, $E = G \times C^m$ the trivial bundle, then we have $0 \to S \to E \to Q \to 0$. The metric in S is induced from the flat euclidean metric in $E = G \times C^m$. As above, we choose the frame $f(Z) = \begin{pmatrix} I \\ Z \end{pmatrix}$ for S over an open set in G. Writing $f(Z) = (e_1, \cdots, e_r)$ we complete this to the frame

$$(e_1, \cdots, e_r; e_{r+1}, \cdots, e_m) = \begin{pmatrix} I & 0 \\ Z & I \end{pmatrix} \qquad \text{for } E \,.$$

The metric function for E is

$$h = \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix} = \begin{pmatrix} I & {}^t\bar{Z} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ Z & I \end{pmatrix} = \begin{pmatrix} I + {}^t\bar{Z}Z & {}^t\bar{Z} \\ Z & I \end{pmatrix} .$$

Then $h_1(0) = I$, $dh_1(0) = 0$, $h_4(0) = I$, $dh_4(0) = 0$. Furthermore, $b = dZ$ and, by (2.15), we have $\Theta_S = {}^t\overline{dZ} \wedge dZ$, which agrees with (2.7).

(e) *Properties of the second fundamental form.* We want to discuss further the second fundamental form $b \in A^{1,0}(\text{Hom}(S, Q))$ given above. The basic facts are

$$D'b = 0 \,; \tag{2.16}$$

$$\bar{\partial} b = 0 \qquad \text{if } \Theta_E = 0 \,. \tag{2.17}$$

To check these, we use orthonormal frames. Thus let $B \subset P$ be all unitary frames $f = (e_1, \cdots, e_r)$ for E where $e_1, \cdots, e_s$ is a frame for S. Given $f \in B$, we can find a *holomorphic* frame $f(z)$ for E with $f(0) = f$, but, in general, this cross-section will *not* be tangent to B at f. The obstruction is essentially the second fundamental form b, as was seen above.

On B we write $D_E e_\rho = \sum_\sigma \varphi_\rho^\sigma e_\sigma$ and $D_S e_\lambda = \sum_\mu \theta_\lambda^\mu e_\mu$. Then $\varphi_\rho^\sigma + \bar{\varphi}_\sigma^\rho = 0$, $\theta_\mu^\lambda + \bar{\theta}_\lambda^\mu = 0$. We claim that $\varphi_\lambda^\mu = \theta_\lambda^\mu$ and that $b = \sum_{\alpha,\lambda} \varphi_\lambda^\alpha e_\alpha \otimes e_\lambda^*$.

Let $De_\lambda = \sum_\mu \varphi_\lambda^\mu e_\mu$; this gives a connexion in S which preserves the metric, and to show that $D = D_S$, we need to prove that $D'' = \bar{\partial}$. Choose a C^∞ frame $f(z) = (e_1(z), \cdots, e_r(z))$ for B. We may then find *holomorphic* sections $\xi_\mu(z) = \sum_\lambda \xi_\mu^\lambda(z) e_\lambda(z)$ of S with $\xi_\mu(0) = e_\mu(0)$. Then

$$0 = D_E'' \xi_\mu(0) = \sum_\lambda \bar{\partial} \xi_\mu^\lambda(0) e_\lambda(0) + \sum_{\lambda,\rho} \xi_\mu^\lambda(0) \varphi_\lambda^\rho(0)'' e_\rho(0)$$

which gives $\varphi_\lambda^{\alpha\,\prime\prime} = 0$, $(D_{\mathrm{E}} - D)'' = 0$, and $D'' = D''_{\mathrm{E}} = \bar{\partial}$. Furthermore $b(e_\lambda) = (D_{\mathrm{E}} - D_{\mathrm{S}})e_\lambda = \sum_\alpha \varphi_\lambda^\alpha e_\alpha$ where φ_λ^α is of type $(1, 0)$.

We now compute

$$\begin{aligned} Db &= \textstyle\sum_{\alpha,\lambda} d\varphi_\lambda^\alpha e_\alpha \otimes e_\lambda^* - \sum_{\alpha,\beta,\lambda} \varphi_\lambda^\alpha \wedge \varphi_\alpha^\beta e_\beta \otimes e_\lambda^* \\ &\quad + \textstyle\sum \varphi_\lambda^\alpha \wedge \varphi_\mu^\lambda e_\alpha \otimes e_\mu^* \\ &= \textstyle\sum_{\alpha,\lambda} (d\varphi_\lambda^\alpha + \sum_\beta \varphi_\beta^\alpha \wedge \varphi_\lambda^\beta + \sum_\mu \varphi_\mu^\alpha \wedge \varphi_\lambda^\mu) e_\alpha \otimes e_\lambda^* \ , \end{aligned}$$

which gives:

$$Db = \textstyle\sum_{\alpha,\lambda} \Theta_\lambda^\alpha e_\alpha \otimes e_\lambda^* \ . \tag{2.18}$$

Since b is of type $(1, 0)$ and Θ_{E} of type $(1, 1)$, we get $D'b = 0$, and clearly $\bar{\partial} b = 0$ if $\Theta_{\mathrm{E}} = 0$.

There are two applications of this. If we let $c = \sum_{\alpha,\lambda} \varphi_\alpha^\lambda e_\lambda \otimes e_\alpha^*$, then, since $\varphi_\lambda^{\alpha\,\prime\prime} = 0$ and $\varphi_\lambda^\alpha + \bar{\varphi}_\alpha^\lambda = 0$, $\varphi_\alpha^{\lambda\,\prime} = 0$ and $c \in A^{0,1}(\mathrm{Hom}(\mathbf{Q}, \mathbf{S}))$. From (2.16) we get $\bar{\partial} c = 0$, and it is proved in [11, § VI. 3] that the Dolbeault cohomology class of $c \in H^1(V, \mathrm{Hom}\,(\mathbf{Q}, \mathbf{S}))$ is the obstruction to splitting the sequence $0 \to \mathbf{S} \to \mathbf{E} \to \mathbf{Q} \to 0$ holomorphically.

Secondly, if we take V to be the grassmannian $\mathbf{G} = \mathbf{G}(r, m)$ of r-planes in $\mathbf{C}^m$ and $\mathbf{S} \to \mathbf{G}$ the universal bundle, then we have $0 \to \mathbf{S} \to \mathbf{E} \to \mathbf{Q} \to 0$ where $\mathbf{E} = \mathbf{G} \times \mathbf{C}^m$ is a trivial bundle. Taking the flat metric in $\mathbf{E}$, $\Theta_{\mathrm{E}} = 0$ and $\bar{\partial} b = 0$. But then b is a holomorphic section of $\mathrm{Hom}\,(\mathbf{T}, \mathrm{Hom}\,(\mathbf{S}, \mathbf{Q}))$ and from the local coordinate description of b above, we see that the second fundamental form b gives an isomorphism:

$$b\colon \mathbf{T} \longrightarrow \mathrm{Hom}\,(\mathbf{S}, \mathbf{Q}) \ . \tag{2.19}$$

(f) *Curvature of ample bundles and proof of Theorem* A. As an application of the second fundamental form, we let $\mathbf{E} \to V$ be a holomorphic vector bundle such that the global sections $\Gamma(\mathbf{E}^*)$ generate $\mathbf{E}^*$. Then we have $\Gamma(\mathbf{E}^*) \to \mathbf{E}^* \to 0$ and, by duality,

$$0 \longrightarrow \mathbf{E} \longrightarrow \mathbf{\Gamma}(\mathbf{E}^*)^* \longrightarrow \mathbf{F} \longrightarrow 0 \ , \tag{2.19}$$

where $\mathbf{\Gamma}(\mathbf{E}^*)^* = V \times \Gamma(\mathbf{E}^*)^*$.

Let $\mathbf{G}$ be the Grassmann manifold of r-planes in $\Gamma(\mathbf{E}^*)^*$; for each $z \in V$, $\mathbf{E}_z$ gives a subspace of $\Gamma(\mathbf{E}^*)^*$ and $\varphi\colon V \to \mathbf{G}$ given by $\varphi(z) = \mathbf{E}_z \subset \Gamma(\mathbf{E}^*)^*$ has the property that $\varphi^*(\mathbf{S}) = \mathbf{E}$ where $\mathbf{S} \to \mathbf{G}$ is the universal bundle. In fact, φ^* lifts the exact sequence

$$0 \longrightarrow \mathbf{S} \longrightarrow \mathbf{G} \times \Gamma(\mathbf{E}^*)^* \longrightarrow \mathbf{Q} \longrightarrow 0 \tag{2.20}$$

back to the exact sequence (2.19).

To find the induced metric in $\mathbf{E} \to V$, we choose a unitary basis $s^1, \cdots, s^m$ for $\Gamma(\mathbf{E}^*)$ and set

$$(2.21) \qquad (e_\rho, e_\sigma) = \sum_{j=1}^m \langle s,^j e_\rho \rangle \overline{\langle s^j, e_\sigma \rangle} ,$$

where $f = (e_1, \cdots, e_r)$ is a frame for $\mathbf{E}$. Thus

$$(2.22) \qquad h_{\sigma\rho} = \sum_{j=1}^m A_\rho^j \bar{A}_\sigma^j = ({}^t\bar{A}(f)A(f))_{\sigma\rho}$$

where $A_\rho^j = \langle s^j, e_\rho \rangle$.

If we let $\mathbf{P} \to V$ be the frame bundle for $\mathbf{E} \to V$ and $\mathbf{B} \to \mathbf{G}$ the frame bundle for $\mathbf{S} \to \mathbf{G}$, then the mapping $\varphi: \mathbf{P} \to \mathbf{B}$ given by $\varphi(f) = A(f)$ satisfies $\varphi(fg) = \varphi(f)g$ for $g \in G = GL(r, \mathbf{C})$; the induced mapping $\varphi: V \to \mathbf{G}$ $(V = \mathbf{P}/G, \mathbf{G} = \mathbf{B}/G)$ is the same as φ above.

We now give $\varphi: V \to \mathbf{G}$ locally. Let $z^1, \cdots, z^n$ be local coordinates on V and assume that $s^1, \cdots, s^r \in \Gamma(\mathbf{E}^*)$ are linearly independent near $z = 0$. We let $f(z) = (e_1(z), \cdots, e_r(z))$ be the frame for $\mathbf{E}$ with $\langle s^\rho, e_\sigma \rangle = \delta_\sigma^\rho$ $(1 \leqq \rho, \sigma \leqq r)$. Our range of indices is to be $1 \leqq \rho, \sigma \leqq r; r + 1 \leqq \alpha, \beta \leqq m$. Then $\varphi(z) = \begin{pmatrix} I \\ B(z) \end{pmatrix}$ where $B(z) = (b_\rho^\alpha(z)$ is an $(m - r) \times r$ matrix with $b_\rho^\alpha(z) = \langle s^\alpha(z), e_\rho(z) \rangle$. By (2.22),

$$h(z) = (I\,{}^t\overline{B(z)}) \begin{pmatrix} I \\ B(z) \end{pmatrix} = I + {}^t\overline{B(z)}B(z) .$$

Making a unitary change of $s^1, \cdots, s^m$, we may assume that $B(0) = 0$. Then $h(0) = I$, $dh(0) = 0$, and, by (2.6), the curvature in $\mathbf{E}$ at $z = 0$ is

$$(2.23) \qquad \Theta_{\mathbf{E}} = -\partial\bar{\partial}h = {}^t\overline{dB} \wedge dB ;$$

that is

$$(2.24) \qquad \Theta_\sigma^\rho = -\sum_{\alpha=r+1}^m db_\sigma^\alpha \wedge \overline{db_\rho^\alpha} .$$

We want to relate these formulas to the second fundamental forms. If $b_{\mathbf{E}} \in \operatorname{Hom}(\mathbf{T}(V), \operatorname{Hom}(\mathbf{E}, \mathbf{F}))$ is the 2nd fundamental form of $\mathbf{E}$ in $\Gamma(\mathbf{E}^*)^*$, then at $z = 0$, $b_{\mathbf{E}}$ is just db_σ^α. The formula (2.23) for the curvature is then the same as (2.15). Furthermore, the following diagram commutes.

$$(2.25) \qquad \begin{array}{ccc} \mathbf{T}(V) & \xrightarrow{\varphi_*} & \mathbf{T}(\mathbf{G}) \\ \downarrow{\scriptstyle b_{\mathbf{E}}} & & \downarrow{\scriptstyle b_{\mathbf{S}}} \\ \operatorname{Hom}(\mathbf{E}, \mathbf{F}) & \xrightarrow{\varphi} & \operatorname{Hom}(\mathbf{S}, \mathbf{Q}) . \end{array}$$

Now, as discussed in (e) above, b_S is an isomorphism (cf. (2.19)) and φ is an algebraic isomorphism. With these identifications, we conclude that $\varphi_*: \mathbf{T}(V) \to \mathbf{T}(\mathbf{G})$ is the same as

$$b_{\mathbf{E}}: \mathbf{T}(V) \longrightarrow \operatorname{Hom}(\mathbf{E}, \mathbf{F}) . \tag{2.26}$$

In terms of the local coordinates above $\varphi(z) = B(z)$ and (2.26) is clear.

PROOF OF THEOREM A. It will suffice to show that if $\mathbf{E}^* \to V$ is ample, then $\mathbf{E} \to V$ is negative in the sense that the quadratic form

$$\sum_{\rho,\sigma,i,j} (\Theta_{\mathbf{E}})^{\rho}_{\sigma i \bar{j}} \xi^{\sigma} \bar{\xi}^{\rho} \eta^{i} \bar{\eta}^{j} = \Theta_{\mathbf{E}}(\xi, \eta) \tag{2.27}$$

is negative definite. Indeed, $(\Theta_{\mathbf{E}^*})^{\rho}_{\sigma} = -(\Theta_{\mathbf{E}})^{\sigma}_{\rho}$ by (2.4) and so $-\Theta_{\mathbf{E}}(\xi, \eta) = \Theta_{\mathbf{E}^*}(\bar{\xi}, \eta)$.

At $z = 0$, write $db^{\alpha}_{\rho} = \sum_{i=1}^{n} b^{\alpha}_{\rho i} dz^{i}$. Then, by (2.24), $\Theta^{\rho}_{\sigma i \bar{j}} = -\sum_{\alpha} b^{\alpha}_{\sigma i} \bar{b}^{\alpha}_{\rho j}$ and so

$$\Theta_{\mathbf{E}}(\xi, \eta) = \sum_{\alpha} | \sum_{\sigma,i} b^{\alpha}_{\sigma i} \xi^{\sigma} \eta^{i} |^2 . \tag{2.28}$$

Since $\mathbf{E}^*$ is ample, we have

$$\begin{cases} 0 \longrightarrow \mathbf{F}^* \longrightarrow \Gamma(\mathbf{E}^*) \longrightarrow \mathbf{E}^* \longrightarrow 0 \\ \mathbf{F}^* \longrightarrow \mathbf{E}^* \otimes \mathbf{T}^* \longrightarrow 0 . \end{cases} \tag{2.29}$$

Taking the second exact sequence at $z = 0$ and dualizing, we get

$$0 \longrightarrow \mathbf{E}_z \otimes \mathbf{T}_z \xrightarrow{\psi} \mathbf{F}_z \longrightarrow 0 . \tag{2.30}$$

In terms of the frames $(e_1, \cdots, e_r)$ for $\mathbf{E}$, $(f_1, \cdots, f_{m-r})$ for $\mathbf{F}$, and coordinates above, we have that

$$\psi\left(e_{\sigma} \otimes \frac{\partial}{\partial z^{i}}\right) = \sum_{\alpha} b^{\alpha}_{\sigma i} f_{\alpha} . \tag{2.31}$$

Combining (2.31) and (2.28), we have that

$$\Theta_{\mathbf{E}}(\xi, \eta) = -| \psi(\xi \otimes \eta) |^2 , \tag{2.32}$$

which is negative by (2.30). This proves Theorem A.

Remark. If $r > 1$, the universal bundle $\mathbf{S} \to \mathbf{G}$ is *not* ample, and is in fact *not* positive, as follows easily from (2.8).

(g) *Curvatures in the associated projective bundle.* We now compute an example discussed in [10, Lemma 9.1]. Let $\mathbf{E} \to V$ be a holomorphic vector bundle in which we have a hermitian metric. On $\mathbf{E} - V$ we define a positive real function h by $h(z, \xi) = (\xi, \xi)_z =$

${}^t\bar{\xi}h(z)\xi$, where $\xi \in \mathbf{E}_z$, the fibre of $\mathbf{E}$ at $z \in V$. For $\lambda \in \mathbf{C}^*$, $h(z, \lambda\xi) = |\lambda|^2 h(z, \xi)$.

The quotient space $\mathbf{E} - V/\mathbf{C}^* = P(\mathbf{E})$ is a bundle $P(\mathbf{E}) \to V$, whose fibre $P(\mathbf{E})_z$ is the projective space $P(\mathbf{E}_z)$ of lines in $\mathbf{E}_z$. Clearly $\mathbf{E} - V \to P(\mathbf{E})$ is a principal bundle and $h(z, \xi)$ is a metric in the associated line bundle $\mathbf{L}(\mathbf{E})$. By (2.1), the curvature is given by $\Theta_{\mathbf{L}(\mathbf{E})} = -\partial\bar{\partial} \log h(z, \xi)$.

To calculate $\Theta_{\mathbf{L}(\mathbf{E})}$, we choose local coordinates $z^1, \cdots, z^n$ on V and a frame $f(z) = (e_1(z), \cdots, e_r(z))$ for $\mathbf{E} \to V$ such that $h(0) = I$, $dh(0) = 0$. Evaluated at $z = 0$, we have (cf. the proof of Lemma 9.1 in [10]).

$$\Theta_{\mathbf{L}(\mathbf{E})} = -\frac{{}^t\bar{\xi}\partial\bar{\partial}h\xi}{|\xi|^2} - \left\{\frac{(d\xi, d\xi)(\xi, \xi) - (d\xi, \xi)(\xi, d\xi)}{|\xi|^2}\right\}. \tag{2.33}$$

To interpret this, we consider the dual projective bundle $P(\mathbf{E}^*) \to V$ and the line bundle $\mathbf{L} \to P(\mathbf{E}^*)$ which, on each fiber of $P(\mathbf{E}^*) \to V$, is positive; thus $\mathbf{L} = \mathbf{L}(\mathbf{E}^*)^*$. By (2.33), the curvature of $\mathbf{L}$ at $(\xi, 0)$ is

$$\Theta_{\mathbf{L}} = -\frac{{}^t\bar{\xi}\partial\bar{\partial}h\xi}{|\xi|^2} + \left\{\frac{(d\xi, d\xi)(\xi, \xi) - (d\xi, \xi)(\xi, d\xi)}{|\xi|^2}\right\}. \tag{2.34}$$

If we set

$$\Theta_{\mathbf{E}}(\xi) = \sum\nolimits_{\rho,\sigma,i,j} \Theta^{\rho}_{\sigma i\bar{j}}\xi^{\sigma}\bar{\xi}^{\rho}dz^i \wedge d\bar{z}^j , \tag{2.35}$$

and

$$\omega(\xi) = \frac{(d\xi, d\xi)(\xi, \xi) - (d\xi, \xi)(\xi, d\xi)}{|\xi|^2} ,$$

then we have:

$$\Theta_{\mathbf{L}} = \frac{\Theta_{\mathbf{E}}(\xi)}{|\xi|^2} + \omega(\xi) . \tag{2.36}$$

Comparing (2.35) with the definition (0.1) of the introduction, we find (cf. [10, Prop. 9.1]):

(2.37) $\mathbf{L} \to P(\mathbf{E}^*)$ is *positive* if $\mathbf{E} \to V$ is.

We now want to compute the *canonical bundle* $\mathbf{K}_{\mathbf{P}}$ of $\mathbf{P}(\mathbf{E}^*)$. The formula to be verified is

$$\mathbf{K}_{\mathbf{P}} = \mathbf{L}^{-r} \otimes \det(\mathbf{E}) \otimes \mathbf{K}_V . \tag{2.38}$$

In (2.38), $\det(\mathbf{E}) \otimes \mathbf{K}_V$ is a line bundle on V which has been lifted in the fibering $\mathbf{P}(\mathbf{E}^*) \to V$.

In order to keep our signs straight, we need a few preliminary remarks. Let E be a vector space, E^* the dual space, and $\mathbf{P}(E^*)$ the projective space. If $\mathbf{L}^* \to \mathbf{P}(E^*)$ is the tautological line bundle whose fibre $\mathbf{L}^*_\lambda$ over a line $\lambda \subset E^*$ is just the 1-dimensional vector space λ, then E is the space $H^0(\mathbf{P}(E^*), \mathcal{O}(\mathbf{L}))$ of holomorphic cross-sections of the dual bundle $\mathbf{L} \to \mathbf{P}(E^*)$. In particular, each vector $e \in E$ gives a holomorphic function on $\mathbf{L}^*$; note that $\mathbf{L}^* - \mathbf{P}(E^*) = E^* - \{0\}$ and then it is clear how to think of e as a function on $\mathbf{L}^*$.

Now let $e_1, \cdots, e_r$ be a basis for E and let ξ_ρ be the function on $E^* - \{0\}$ defined by e_ρ. Then

$$\eta = \sum^r (-1)^{\sigma-1}\xi_\sigma d\xi_1 \wedge \cdots \wedge d\hat{\xi}_\sigma \wedge \cdots \wedge d\xi_r$$

is an $(r-1)$-form on $\mathbf{P}(E^*)$ with values in $\mathbf{L}^{*r} = \mathbf{L}^{-r}$. If we have a linear substitution $\hat{e}_\rho = \sum_\sigma g^\sigma_\rho e_\sigma$, then $\hat{\xi}_\rho = \sum g^\sigma_\rho \hat{\xi}_\sigma$ and $d\hat{\xi}_\rho = \sum_\sigma g^\sigma_\rho d\xi_\sigma$. Clearly then $\hat{\eta} = \det(g^\sigma_\rho)\eta$.

If now $z^1, \cdots, z^n$ are local coordinates on V and $e_1(z), \cdots, e_r(z)$ is a frame for $\mathbf{E}$, we let $\xi_\rho = \xi_\rho(z)$ be the corresponding functions on $\mathbf{E}^* - \{0\}$ and set

$$\varphi(z, \xi) = dz^1 \wedge \cdots \wedge dz^n \{\textstyle\sum_\sigma (-1)^{\sigma-1}\xi_\sigma d\xi_1 \wedge \cdots \wedge d\hat{\xi}_\sigma \wedge \cdots \wedge d\xi_r\} .$$

Then $\varphi(z, \xi)$ is an $(n + r - 1)$-form on $\mathbf{P}(\mathbf{E}^*)$ with values in $\mathbf{L}^{-r}$. If $\hat{e}_1(z), \cdots, \hat{e}_r(z)$ is a new frame for $\mathbf{E}$, then $\hat{e}_\rho(z) = \sum_\sigma g^\sigma_\rho(z)e_\sigma(z)$ and $d\hat{\xi}_\rho(z) = \sum_\sigma g^\sigma_\rho(z)d\xi_\sigma(z)$ modulo $dz^1, \cdots, dz^n$. Thus $\hat{\varphi} = (\det g)\varphi$. From this it follows that $\mathbf{K}_P = \mathbf{L}^{-r} \otimes \det(\mathbf{E}) \otimes \mathbf{K}_V$ as desired.

To close this section, let us prove (1.9) for S a locally free sheaf (this is the case to be used below). Thus, given $\mathbf{F} \to V$, we must show

$$H^q(V, \mathcal{O}(\mathbf{E}^{(\mu)} \otimes \mathbf{F})) \cong H^q(P, \mathcal{O}(\mathbf{L}^\mu \otimes \pi^*\mathbf{F})) . \tag{2.40}$$

For this, we use the *Leray spectral sequence* [8, p. 201]. The p^{th} derived sheaf $R^p_\pi(\mathbf{L}^\mu \otimes \pi^*\mathbf{F})$ for $\mathcal{O}(\mathbf{L}^\mu \otimes \pi^*\mathbf{F})$; $\mathbf{P}(\mathbf{E}^*) \xrightarrow{\pi} V$ is the sheaf arising from the pre-sheaf

$$U \longrightarrow H^p(\pi^{-1}(U), \mathcal{O}(\mathbf{L}^\mu \otimes \pi^*\mathbf{F}) \mid \pi^{-1}(U))$$

for $U \subset V$ an open set. Taking U for which $\mathbf{F} \mid U$ is trivial, we see that $R^p_\pi(\mathbf{L}^\mu \otimes \pi^*\mathbf{F}) \cong R^p_\pi(\mathbf{L}^\mu) \otimes \mathcal{O}_V(\mathbf{F})$. Taking U for which $\mathbf{E}^* \mid U$ is trivial, $\pi^{-1}(U) \cong U \times \mathbf{P}_{r-1}$ and $\mathbf{L}^\mu \mid \pi^{-1}(U) \cong \mathcal{O}_U \otimes \mathcal{O}_{\mathbf{P}_{r-1}}(\mathbf{H}^\mu)$ where $\mathbf{H} \to \mathbf{P}_{r-1}$ is the hyperplane bundle. If $\mathbf{P}_{r-1} = \mathbf{P}(E^*)$ for a vector space E, then $H^p(\mathbf{P}_{r-1}, \mathcal{O}(\mathbf{H}^\mu)) = 0$ for $p > 0, \mu \geqq 0$ and $H^0(\mathbf{P}_{r-1}, \mathcal{O}(\mathbf{H}^\mu)) \cong E^{(\mu)}$. It follows that $R^p_\pi(\mathbf{L}^\mu) = 0$ for $p > 0$ and $R^0_\pi(\mathbf{L}^\mu) = \mathcal{O}(\mathbf{E}^\mu)$. The assertion (2.40) now follows from the spectral sequence.

(h) PROOF OF THEOREM H. We let $\mathbf{E} \to V$ be a positive bundle with fibre $\mathbf{C}^r$, and we consider a section $\xi \in H^0(V, \mathcal{O}(\mathbf{E}))$ whose divisor $S = \{z \in V : \xi(z) = 0\}$ is a non-singular subvariety of dimension $n - r$, where $n = \dim V$. We define a non-negative function φ on V by $\varphi(z) = |\xi(z)|^2 = {}^t\xi(z)h(z)\xi(z)$, where $h(z)$ is the metric in $\mathbf{E} \to V$. We want to check first that S is a *non-degenerate critical manifold* of φ, which means that we must show:

$$(2.40) \qquad \begin{cases} d\varphi = 0 \text{ along } S\,; \\ \text{if } z \in S, \text{ the null-space of the } \textit{hessian } H(\varphi) \\ \text{at } z \text{ is the tangent space } \mathbf{T}_z(S)\,. \end{cases}$$

We may choose local coordinates $z^1, \cdots, z^n$ on V and a frame $e_1(z), \cdots, e_r(z)$ for $\mathbf{E}$ such that

$$\xi(z) = \begin{pmatrix} z^1 \\ \vdots \\ z_r \end{pmatrix}.$$

We may also assume that $h(0) = I$. Then $\varphi(z) = \sum_{\rho,\sigma} h_{\rho\sigma}(z)\bar{z}^\rho z^\sigma$ and the hessian $H(\varphi)$ of φ at the origin is

$$H(\varphi) = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

From this, (2.40) is clear.

Let now z_0 be a *critical point* of φ on $V - S$; i.e., $d\varphi(z_0) = 0$. In local coordinates $z^1, \cdots, z^n$, we may assume that z_0 is the origin, and we may compute the hessian $H(\varphi)$ of φ at z_0. The *index* of φ at z_0 is the dimension of the subspace of $\mathbf{T}_{z_0}(V)$ on which $H(\varphi)$ is negative definite. We want to show

(2.41) The index of φ at z_0 is no less than $n - r + 1$.

We assume that $h(0) = I$ and $dh(0) = 0$. Then $\varphi(z) = {}^t\overline{\xi(z)}h(z)\xi(z)$ and, at $z = 0$, we have ${}^t\overline{d\xi}\xi + {}^t\bar{\xi}d\xi = 0$ since z_0 is a critical point, and $\partial\bar{\partial}\varphi = {}^t\bar{\xi}\partial\bar{\partial}h\xi - {}^t\overline{d\xi} \wedge d\xi$. Thus, by (2.6),

$$(2.42) \qquad H(\varphi)_{i\bar{j}} = -\sum_{\rho,\sigma} \Theta^\rho_{\sigma i\bar{j}}\xi^\sigma\bar{\xi}^\rho + \sum_\rho \frac{\overline{\partial\xi^\rho}}{\partial z^j}\frac{\partial\xi^\rho}{\partial z^i}\,.$$

Let A be the $r \times n$ matrix $A^\rho_i = \partial\xi^\rho/\partial z^i$. For

$$\eta = \begin{pmatrix} \eta^1 \\ \vdots \\ \eta^n \end{pmatrix},$$

we have

$$(2.43) \qquad \sum_{i,j} H(\varphi)_{i\bar{j}}\eta^i\bar{\eta}^j = -\Theta(\xi, \eta) + ({}^t\overline{A\eta})(A\eta) ,$$

where $-\Theta(\xi, \eta)$ is negative definite since $\mathbf{E} \to V$ is positive. If we set $W = \{\eta \in \mathbf{C}^n : A\eta = 0\}$, from (2.43), to prove (2.41) it will suffice to show that $\dim W \geqq n - r + 1$.

Now $\dim W = n - \operatorname{rank}(A)$ where $\operatorname{rank}(A)$ is the number of independent row vectors $A^\rho = (A_1^\rho, \cdots, A_n^\rho)$ in A. From ${}^t\bar{\xi}d\xi = 0$ we get

$$\sum_\rho \bar{\xi}^\rho \frac{\partial \xi^\rho}{\partial z^i} = 0 \qquad (i = 1, \cdots, n) ,$$

so that $\sum_\rho \bar{\xi}^\rho A^\rho = 0$. Thus $\operatorname{rank}(A) \leqq r - 1$ and $\dim W \geqq n - r + 1$ as required.

The same argument as used in [2, Prop. 4.1 and Th. II] shows that

$$V = S \cup e_1 \cup \cdots \cup e_s \qquad \text{where } \dim e_k \geqq n - r + 1 .$$

This notation means that, up to homotopy type, V is obtained by attaching cells $e_1, \cdots, e_s$ to S. From this it follows that $H_q(S, \mathbf{Z})$ maps onto $H_q(V, \mathbf{Z})$ for $q \leqq n - r$ and $H_q(S, Z)$ maps into $H_q(V, Z)$ for $q \leqq n - r - 1$, which proves Theorem H.

Remark. We let $W = V - S$ be the open manifold obtained by removing S from V. Then $\psi(z) = -\log \varphi(z)$ is an *exhaustion function* on W, and we let $L(\psi) = \partial\bar{\partial}\psi$ be the *E. E. Levi form* of ψ.

(2.44) $L(\psi)$ has everywhere at least $n - r$ positive eigenvalues

PROOF. $L(\psi) = -\partial\bar{\partial} \log \varphi = -\partial\left(\dfrac{\bar{\partial}\varphi}{\varphi}\right) = \dfrac{-\partial\bar{\partial}\varphi}{\varphi} + \dfrac{\partial\varphi\bar{\partial}\varphi}{\varphi^2}$.

Now $\partial\varphi = \partial(\xi, \xi) = (D'\xi, \xi)$ (since $\bar{\partial}\xi = 0$); $\bar{\partial}\varphi = \bar{\partial}(\xi, \xi) = (\xi, D'\xi)$; and $\partial\bar{\partial}\varphi = \partial(\xi, D'\xi) = (D'\xi, D'\xi) + (\xi, \Theta\xi)$ (since $\bar{\partial}D'\xi = (\bar{\partial}D' + D'\bar{\partial})\xi = D^2\xi = \Theta\xi$). This gives that

$$(2.44)' \quad L(\psi) = \frac{-(D'\xi, D'\xi)(\xi, \xi) + (D'\xi, \xi)(\xi, D'\xi)}{\varphi^2} + \frac{(\Theta\xi, \xi)}{(\xi, \xi)} .$$

If $\eta = \{\eta^i\}$ is a vector and $\xi = \sum_{\rho=1}^r \xi^\rho e_\rho$ our section of $\mathbf{E}$, then

$$(\Theta\xi, \xi)\eta \wedge \bar{\eta} = \sum_{\rho,\sigma,i,j} \Theta^\rho_{\sigma i\bar{j}}\xi^\sigma\bar{\xi}^\rho\eta^i\bar{\eta}^j = \Theta(\xi, \eta) > 0$$

so that $L(\psi)$ is positive on the space of vectors η such that

$\langle D'\xi, \eta\rangle = 0$. But the dimension of this space is less than or equal r (since $D'\xi = \sum_{\rho=1}^{r} D'\xi^\rho e_\rho$), so that $L(\psi)$ has everywhere at least $n - r$ positive eigenvalues.

We observe that the calculation just used shows that $d\varphi = (D'\xi, \xi)$ and $H(\varphi) = (D'\xi, D'\xi) - (\Theta\xi, \xi)$ where $\varphi = (\xi, \xi)$ is the function used in the proof of Theorem H.

3. Positive, ample, and cohomologically positive bundles

(a) *General properties.* We want to give some functorial properties of positive, ample, and cohomologically positive bundles. The first are

$$(3.1) \quad \begin{cases} \text{If } \mathbf{E} \to V \text{ is positive, ample, or cohomologically} \\ \text{positive, then so is } \mathbf{L} \to \mathbf{P}(\mathbf{E}^*)\,; \end{cases}$$

$$(3.2) \quad \begin{cases} \text{If } \mathbf{L} \to \mathbf{P}(\mathbf{E}^*) \text{ is ample or cohomologically posi-} \\ \text{tive, then } \mathbf{E} \to V \text{ is also.} \end{cases}$$

We have already proved (cf. (2.37)) that

$$\mathbf{E} \text{ positive} \Longrightarrow \mathbf{L} \text{ positive}\,.$$

We now shall show that $\mathbf{E}$ ample $\Leftrightarrow$ $\mathbf{L}$ ample. Let $z_0 \in V$ and $\mathbf{E}_{z_0}$ be the fibre of $\mathbf{E}$ at z_0. If $(z_0, \xi) \in \mathbf{P}(\mathbf{E}^*)$ is a point lying over z_0, then ξ is a line in $\mathbf{E}_{z_0}^*$ and we let

$$\mathbf{F}_{(z_0,\xi)} = \{e \in \mathbf{E}_{z_0} : \langle e, \xi\rangle = 0\}\,.$$

As in §2 (g), we see that $\mathbf{L}_{(z_0,\xi)} = \mathbf{E}_{z_0}/\mathbf{F}_{(z_0,\xi)}$ and, in the exact sequence $0 \to \mathbf{F} \to \pi^*(\mathbf{E}) \to \mathbf{L} \to 0$ over $\mathbf{P} = \mathbf{P}(\mathbf{E}^*)$, $\mathbf{F}_{(z_0,\xi)}$ is the fibre of $\mathbf{F}$ at (z_0, ξ). Using the isomorphisms

$$H^0(V, \mathcal{O}(\mathbf{E})) \cong H^0(\mathbf{P}, \mathcal{O}(\pi^*\mathbf{E})) \cong H^0(\mathbf{P}, \mathcal{O}(\mathbf{L}))\,,$$

we see that $\mathbf{E}$ is generated by its sections if, and only if, $\mathbf{L}$ is. This checks the first condition in the definition (0.2) of ampleness. To verify the second condition, we choose local coordinates $z^1, \cdots, z^n$ on V such that z_0 is the origin and a frame for $\mathbf{E}$ so that $\mathbf{E}_{z_0} = \mathbf{C}^r$ is column vectors

$$\zeta = \begin{pmatrix} \zeta^1 \\ \vdots \\ \zeta^r \end{pmatrix}$$

and $\mathbf{F}_{(z_0,\xi)}$ is given by $\zeta^1 = 0$. Suppose that $\mathbf{F}_{z_0} = \{s \in \Gamma(\mathbf{E}) : s(z_0) = 0\}$ and that $\mathbf{F}_{z_0} \to \mathbf{E}_{z_0} \otimes \mathbf{T}_{z_0}^* \to 0$. Then we may find $s^j \in \mathbf{F}_{z_0}$ with

$$s^j(z) = \begin{pmatrix} z^j \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \text{(higher powers)} ,$$

and $\mathbf{s}^\rho \in \Gamma(\mathbf{E})$ with

$$\mathbf{s}^\rho(z) = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \text{(higher powers)} .$$

Then $s^1, \cdots, s^n; \mathbf{s}^2, \cdots, \mathbf{s}^r$ lie in $\mathbf{F}_{(z_0,\xi)}$ and the differentials $ds^1, \cdots, ds^n; d\mathbf{s}^2, \cdots, d\mathbf{s}^r$ span $\mathbf{L}_{(z_0,\xi)} \otimes \mathbf{T}_{(z_0,\xi)}(\mathbf{P})^*$, so that $\mathbf{L}$ is ample. By reversing this argument, we see that $\mathbf{L}$ ample $\Rightarrow$ $\mathbf{E}$ ample, which proves our assertion.

To prove cohomological positivity (cf. definition (0.3)), it will suffice to take S to be locally free sheaf $\mathcal{O}(\mathbf{F})$ where $\mathbf{F}$ is a holomorphic vector bundle (cf. § 3 (b), the proof of Theorem B). From (2.40) it follows that $\mathbf{L}$ cohomologically positive $\Rightarrow$ $\mathbf{E}$ cohomologically positive. To prove the converse, by examining the proof of (2.40) above, it will suffice to show that

(a) there is $\nu = \nu(\mathbf{F})$ such that $R^p_\pi(\mathbf{F} \otimes \mathbf{L}^\nu) = 0$ for $p > 0$, and

(b) $R^0_\pi(\mathbf{F} \otimes \mathbf{L}^\nu \otimes \mathbf{L}^\mu) \cong R^0_\pi(\mathbf{F} \otimes \mathbf{L}^\nu) \otimes R^0_\pi(\mathbf{L}^\mu)$.

In fact, given (a) and (b),

$$\begin{aligned} H^q(\mathbf{P}(\mathbf{E}^*), \mathcal{O}(\mathbf{F} \otimes \mathbf{L}^{\nu+\mu})) &\cong H^q(V, R^0_\pi(\mathbf{F} \otimes \mathbf{L}^{\nu+\mu})) \\ &\cong H^p(V, R^0_\pi(\mathbf{F} \otimes \mathbf{L}^\nu) \otimes \mathcal{O}(\mathbf{E}^{(\mu)})) = 0 \\ &\qquad \text{for } \mu > \mu_0(\mathbf{F}),\ q > 0 , \end{aligned}$$

since $R^0_\pi(\mathbf{F} \otimes \mathbf{L}^\nu)$ is locally free. Now both (a) and (b) are well-known.

We cannot prove that $\mathbf{L} \to \mathbf{P}(\mathbf{E}^*)$ positive $\Rightarrow$ $\mathbf{E} \to V$ positive. However, for the notion of *weakly positive*, due to Grauert [9], it is true that

(3.3) $\mathbf{E}$ weakly positive $\Leftrightarrow$ $\mathbf{L}$ weakly positive.

In fact, Grauert says that $\mathbf{E} \to V$ is weakly positive if, and only if, a tubular neighborhood of the zero section in $\mathbf{E}^* \to V$ is *strongly pseudo-convex* (cf. [13]). Since $\mathbf{E}^* - V = \mathbf{L}^* - \mathbf{P}(\mathbf{E}^*)$ (cf. 2(g)), it is clear that (3.3) is verified.

On the other hand, it is true that

(3.4) E positive $\Rightarrow$ E weakly positive.

The proof goes as follows. On $\mathbf{E} \to V$, we define a positive function φ by $\varphi(z, \xi) = |\xi|_z^2 = {}^t\bar{\xi}h(z)\xi$. The *tubular neighborhood* T of V in E is given by $\varphi < \varepsilon$. We must calculate the *E. E. Levi form* $L(\varphi) = \partial\bar{\partial}\varphi$ evaluated on the tangent space to the boundary ∂T_ε of T_ε. Choose coordinates $z^1, \cdots, z^n$ and a frame for E such that $h(0) = I, dh(0) = 0$. Then, at $(0, \xi)$, $d\varphi = \sum_\rho (d\xi^\rho\bar{\xi}^\rho + \xi^\rho d\bar{\xi}^\rho)$ and the tangent space to the boundary is all vectors

$$\sum\nolimits_j \varphi^j \frac{\partial}{\partial z^j} + \sum \eta^\rho \frac{\partial}{\partial \xi^\rho}$$

with $\sum_\rho \bar{\xi}^\rho\eta^\rho = 0$. The Levi form is given by

$$L(\varphi) = -\sum\nolimits_{\rho,\sigma,i,j} \Theta^\rho_{\sigma i j}\xi^\sigma\bar{\xi}^\rho dz^i d\bar{z}^j = \sum\nolimits_\gamma d\xi^\gamma d\bar{\xi}^\gamma\ , \tag{3.5}$$

since $\partial\bar{\partial}h = -\Theta_{\mathbf{E}}$ at $z = 0$ (cf. 2.6)). From this, (3.4) follows.

Grauert's weakly positive is a better function-theoretic notion than our positive. However, the differential-geometric methods lead to Theorems D, G, and H, and we know of no function-theoretic argument which gives these results.

Suppose now that $0 \to \mathbf{S} \to \mathbf{E} \to \mathbf{Q} \to 0$ is an exact sequence of holomorphic bundles. Then

(3.6) $\begin{cases}\text{If E is positive, ample, cohomologically positive,} \\ \text{or numerically positive, then so is } \mathbf{Q}\ . \end{cases}$

Assume that E is positive and let $f = (e_1, \cdots, e_r)$ be an orthonormal frame for E such that $e_1, \cdots, e_s$ is a frame for S (cf. 2(e); as done there, we let $1 \leqq \mu, \lambda \leqq s;\ s + 1 \leqq \alpha, \beta \leqq r;\ 1 \leqq \rho, \sigma \leqq r$). From (2.15) we have that

$$(\Theta_{\mathbf{Q}})^\alpha_\beta = (\Theta_{\mathbf{E}})^\alpha_\beta + \sum\nolimits_\lambda b^\alpha_\lambda \wedge \bar{b}^\beta_\lambda \tag{3.7}$$

where $b = (b^\alpha_\lambda) \in A^{1,0}(\mathrm{Hom}\,(\mathbf{S}, \mathbf{Q}))$ is the 2^{nd} fundamental form of S in E. Thus

$$\begin{aligned}\sum\nolimits_{\alpha,\beta,i,j} (\Theta_{\mathbf{Q}})^\alpha_{\beta i \bar{j}}\xi^\beta\bar{\xi}^\alpha\eta^i\bar{\eta}^j &= \sum\nolimits_{\alpha,\beta,i,j} (\Theta_{\mathbf{E}})^\alpha_{\beta i \bar{j}}\,\xi^\beta\bar{\xi}^\alpha\eta^i\bar{\eta}^j \\ &\quad + \sum\nolimits_{\alpha,\beta,\lambda,i,j} b^\alpha_{\lambda i}\bar{\xi}^\alpha\eta^i\overline{b^\beta_{\lambda j}\bar{\xi}^\beta\eta^j} \\ &\geqq \sum (\Theta_{\mathbf{E}})^\alpha_{\beta i j}\xi^\beta\bar{\xi}^\alpha\eta^i\bar{\eta} > 0\end{aligned}$$

since E is positive. This proves that E positive $\Rightarrow$ Q positive.

It is clear that: E ample $\Rightarrow$ Q ample.

Equally trivially, we note that: E weakly positive $\Rightarrow$ Q weakly positive.

Suppose now that $\mathbf{E}$ is cohomologically positive; we want to show that, for $\mathbf{F} \to V$ a bundle, there is $\mu = \mu_0(\mathbf{F})$ such that $H^q(V, \mathcal{O}(\mathbf{Q})^{(\mu)} \otimes \mathbf{F})) = 0$ for $\mu > \mu_0$, $q > 0$. An algebraic proof begins by observing the exact sequences

$$0 \longrightarrow \mathbf{\Sigma}_\mu \longrightarrow \mathbf{E}^{(\mu)} \longrightarrow \mathbf{Q}^{(\mu)} \longrightarrow 0\ , \tag{3.8}$$

$$0 \longrightarrow \Lambda^2\mathbf{S} \otimes \mathbf{E}^{(\mu-2)} \longrightarrow \mathbf{S} \otimes \mathbf{E}^{(\mu-1)} \longrightarrow \mathbf{\Sigma}_\mu \longrightarrow 0\ . \tag{3.9}$$

From (3.8), it will suffice to check that $\mathbf{\Sigma}_\mu$ is cohomologically positive; this follows from (3.9) and the fact that $\mathbf{E}$ is cohomologically positive. This proves that $\mathbf{E}$ cohomologically positive $\Rightarrow$ $\mathbf{Q}$ cohomologically positive.

A more instructive geometric proof can be given using the fact that $\mathbf{L}(\mathbf{E}) \to \mathbf{P}(\mathbf{E}^*)$ is cohomologically positive (cf. (3.1) above). For $z \in V$, let $\mathbf{P}(\mathbf{Q}^*)_z \subset \mathbf{P}(\mathbf{E}^*)_z = \mathbf{P}(\mathbf{E}_z^*)$ be those lines ξ in $\mathbf{P}(\mathbf{E}_z^*)$ with $\langle \xi, \mathbf{S}_z \rangle = 0$. Then $\bigcup_{z \in V} \mathbf{P}(\mathbf{Q}^*)_z = \mathbf{P}(\mathbf{Q}^*)$ is a sub-bundle of $\mathbf{P}(\mathbf{E}^*) \to V$. As the notation suggests, $\mathbf{P}(\mathbf{Q}^*)$ is the projective bundle associated to $\mathbf{Q}^* \to V$. Clearly $\mathbf{L}(\mathbf{E}) \mid \mathbf{P}(\mathbf{Q}^*) = \mathbf{L}(\mathbf{Q})$, and so we have the exact sheaf sequences

$$\begin{aligned} 0 \longrightarrow I \otimes \mathcal{O}(\mathbf{L}(\mathbf{E})^\mu \otimes \pi^*\mathbf{F}) &\longrightarrow \mathcal{O}(\mathbf{L}(\mathbf{E})^\mu \otimes \pi^*\mathbf{F}) \\ &\longrightarrow \mathcal{O}(\mathbf{L}(\mathbf{Q})^\mu \otimes \pi^*\mathbf{F}) \longrightarrow 0\ , \end{aligned} \tag{3.10}$$

where $I \subset \mathcal{O}(\mathbf{P}(\mathbf{E}^*))$ is the ideal sheaf of $\mathbf{P}(\mathbf{Q}^*)$. Using the cohomological positivity of $\mathbf{L}(\mathbf{E}) \to \mathbf{P}(\mathbf{E}^*)$, we find that

$$H^q(\mathbf{P}(\mathbf{Q}^*), \mathcal{O}(\mathbf{L}(\mathbf{Q})^\mu \otimes \pi^*\mathbf{F})) = 0 \qquad \text{for } \mu \geqq 0,\ q > 0\ .$$

From (2.40) it follows that $\mathbf{Q}$ is cohomologically positive.

The final general property is:

(3.11) $\begin{cases} \text{If } \mathbf{E} \text{ and } \mathbf{F} \text{ are positive, ample, or cohomo-} \\ \text{logically positive, then so is } \mathbf{E} \otimes \mathbf{F}\ . \end{cases}$

Letting $(e_1, \cdots, e_r) = (\cdots, e_\rho, \cdots)$ be a frame for $\mathbf{E}$ and $(f_1, \cdots, f_s) = (\cdots, f_\alpha, \cdots)$ a frame for $\mathbf{F}$, by (2.10) we have

$$(\Theta_{\mathbf{E}\otimes\mathbf{F}})^{\rho\alpha}_{\sigma\beta i\bar{j}} = \delta^\alpha_\beta (\Theta_\mathbf{E})^\rho_{\sigma i\bar{j}} + \delta^\rho_\sigma (\Theta_\mathbf{F})^\alpha_{\beta i\bar{j}}\ . \tag{3.12}$$

Then

$$\begin{aligned} \sum (\Theta_{\mathbf{E}\otimes\mathbf{F}})^{\sigma\alpha}_{\sigma\beta ij} \xi^{\sigma\beta} \bar{\xi}^{\rho\alpha} \eta^i \bar{\eta}^j &= \sum (\Theta_\mathbf{E})^\rho_{\sigma ij} \xi^{\sigma\alpha} \bar{\xi}^{\rho\alpha} \eta^i \eta^j \\ &\quad + \sum (\Theta_\mathbf{F})^\alpha_{\beta ij} \xi^{\rho\beta} \bar{\xi}^{\rho\alpha} \eta^i \bar{\eta}^j > 0\ . \end{aligned}$$

Note in fact that $\mathbf{E} \otimes \mathbf{F} > 0$ if $\mathbf{E} > 0$, $\mathbf{F} \geqq 0$.

It is trivial that, if E and F are ample, then $\mathbf{E} \otimes \mathbf{F}$ is ample. In fact, if E is ample and F is generated by its sections, then $\mathbf{E} \otimes \mathbf{F}$ is ample.

For the proof of E, F cohomologically positive $\Rightarrow \mathbf{E} \otimes \mathbf{F}$ cohomologically positive, we refer to Hartshorne's paper [14]. This paper contains a through account of the relationship between cohomological positivity and the algebraic operations $\oplus$, $\otimes$, etc. on vector bundles.

(b) *The Nakano inequalities and proofs of Theorems* B *and* G. The vanishing theorems we shall use are based on representing the cohomology groups $H^q(V, \mathcal{O}(\mathbf{E}))$ by *harmonic forms* [19], [22], and [15]. Suppose that we have a hermitian metric in $\mathbf{E} \to V$ and *Kähler metric* on V. We choose locally an orthonormal frame $f = (e_1, \cdots, e_r)$ for E and an orthonormal co-frame $\omega^1, \cdots, \omega^n$ for V; the Kähler form is then

$$\gamma = \frac{i}{2} \sum_{j=1}^{n} \omega^j \wedge \bar{\omega}^j .$$

An E-valued (p, q)-form φ is written

$$\varphi = \frac{1}{p!q!} \sum_{\rho, I, J} \varphi^{\rho}_{I\bar{J}} e_\rho \otimes \omega^I \otimes \bar{\omega}^J$$

where $I = (i_1, \cdots, i_p)$ and $J = (j_1, \cdots, j_q)$, $\omega^I = \omega^{i_1} \wedge \cdots \wedge \omega^{i_q}$, etc. The point inner product is

$$\langle \varphi, \psi \rangle = \frac{1}{p!q!} \left(\sum_{\rho, I, J} \varphi^{\rho}_{I\bar{J}} \bar{\psi}^{\rho}_{I\bar{J}} \right) .$$

Setting

$$(\varphi, \psi) = \int_V \langle \varphi, \psi \rangle \left\{ \left(\frac{i}{2} \right)^n \omega^1 \wedge \bar{\omega}^1 \wedge \cdots \wedge \omega^n \wedge \bar{\omega}^n \right\} ,$$

we obtain a global inner product and we let $\bar{\partial}^*$ be the adjoint of $\bar{\partial}$; thus $(\bar{\partial}^* \varphi, \psi) = (\varphi, \bar{\partial} \psi)$ for all ψ, and this equation defines $\bar{\partial}^* \varphi$. The *laplacian* $\square$ is defined by $\square = \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*$, and the space of *harmonic forms* is

(3.13) $$\mathbf{H}^{p,q}(\mathbf{E}) = \{\varphi : \square \varphi = 0\} = \{\varphi : \bar{\partial} \varphi = 0 = \bar{\partial}^* \varphi\} .$$

For φ an E-valued (p, q) form, we now write $\varphi = \sum_\rho \varphi^\rho \otimes e_\rho$ where φ^ρ is a (p, q) form and we set

(3.14) $$\Theta \wedge \varphi = \sum_{\rho, \sigma} \Theta^{\rho}_{\sigma} \wedge \varphi^\sigma \otimes e_\rho ,$$

$$L\varphi = \textstyle\sum_\rho \gamma \wedge \varphi^\rho \otimes e_\rho \ , \tag{3.15}$$

and we define the adjoint L^* of L by

$$\langle L^*\varphi, \psi\rangle = \langle \varphi, L\psi\rangle \ , \tag{3.16}$$

for all ψ. We remark that usually L^* is denoted by Λ and $\bar\partial^*$ by $\mathfrak{D}$. Comparing (3.14) with (2.2), we see that

$$\Theta \wedge \varphi = D^2\varphi = (D'\bar\partial + \bar\partial D')\varphi \ .$$

The following basic inequalities are due to S. Nakano [22]. For $\varphi \in H^{p,q}(\mathbf{E})$,

$$\frac{i}{2}(L^*\Theta \wedge \varphi, \varphi) \geqq 0 \ , \tag{3.17}$$

$$\frac{i}{2}(\Theta \wedge L^*\varphi, \varphi) \leqq 0 \ . \tag{3.18}$$

For the proofs, which use only one basic Kähler identity and no tensor calculations, we refer to [22], [10], or [4]. As hinted above the operator $\Theta \wedge \varphi$ in (3.14) arises from $\Theta \wedge \varphi = D^2\varphi$.

We shall use the Nakano inequalities primarily in case $\mathbf{E} \to V$ is a *negative line bundle*. Then we take as the Kähler metric $\gamma = (-i/2)\Theta$ and subtract (3.18) from (3.17) to get: For $\varphi \in \mathbf{H}^{p,q}(\mathbf{E})$,

$$\big(\{L^*L - LL^*\}\varphi, \varphi\big) \leqq 0 \ . \tag{3.19}$$

Combining this with the elementary identity $(L^*L - LL^*)\varphi = (n - p - q)\varphi$ [26, p. 21] gives

$$(n - p - q)(\varphi, \varphi) \leqq 0 \qquad \text{for } \varphi \in \mathbf{H}^{p,q}(\mathbf{E}) \ . \tag{3.20}$$

From (3.20) we obtain our vanishing theorem

$$\begin{cases} H^q(V, \Omega^p(\mathbf{E})) = 0 \text{ for } p + q < n \text{ and } \mathbf{E} \to V \text{ a} \\ \text{negative line bundle} \ . \end{cases} \tag{3.21}$$

For $p = 0$, (3.21) is the original Kodaira theorem [18], which may be dualized to read

$$H^q(V, \mathcal{O}(\mathbf{E})) = 0 \qquad \text{for } q > 0 \text{ if } \mathbf{E} \otimes \mathbf{K}^* \text{ is positive} \ . \tag{3.22}$$

Here $\mathbf{K} \to V$ is the *canonical bundle* and we have used the *duality theorem*: $H^q(V, \mathcal{O}(\mathbf{E})) \cong H^{n-q}(V, \mathcal{O}(\mathbf{K} \otimes \mathbf{E}^*))^*$ (cf. [15]).

PROOF OF THEOREM G. Let now $\mathbf{E} \to V$ be a general holomorphic vector bundle with fibre $\mathbf{C}^r$, $\mathbf{P}(\mathbf{E}^*) \xrightarrow{\pi} V$ and $\mathbf{L} \to \mathbf{P}(\mathbf{E}^*)$ the tautological bundles discussed in §2 (g), and $\mathbf{F} \to V$ a holomorphic vector bundle. We shall use the isomorphism (1.9) (cf. (2.40)).

$$(3.23) \qquad H^q(V, \mathcal{O}(\mathbf{E}^{(\mu)} \otimes \mathbf{F})) \cong H^q(\mathbf{P}, \mathcal{O}(\mathbf{L}^\mu \otimes \pi^*\mathbf{F})) ,$$

and the curvature calculations in §2 (g) to prove a vanishing theorem for the groups $H^q(V, \mathcal{O}(\mathbf{E}^\mu \otimes \mathbf{F}))$, $\mathbf{F} \to V$ being a line bundle. To state this result, suppose that we have hermitian metrics in $\mathbf{E} \to V$, $\mathbf{F} \to V$, and a Kähler metric on V. We denote the curvature forms for $\mathbf{E}$, $\mathbf{F}$, $\mathbf{K}_V$ by $\sum_{i,j} \Theta^\rho_{\sigma i\bar{j}} dz^i \wedge d\bar{z}^j$, $\sum_{i,j} \varphi_{i\bar{j}} dz^i \wedge d\bar{z}^j$, and $\sum_{i,j} k_{i\bar{j}} dz^i \wedge d\bar{z}^j$ respectively. We introduce the quadratic form:

$$(3.24) \qquad \begin{aligned} Q_\mu(\xi, \eta) = (\mu + r) \textstyle\sum_{\rho,\sigma,i,j} \Theta^\rho_{\sigma i\bar{j}} \xi^\sigma \bar{\xi}^\rho \eta^i \bar{\eta}^j \\ - \textstyle\sum_{i,j} \{k_{i\bar{j}} - \varphi_{i\bar{j}} + \sum_\rho \Theta^\rho_{\rho i\bar{j}}\} \eta^i \bar{\eta}^j \, | \xi |^2 . \end{aligned}$$

(3.25) THEOREM G′. *If $Q_\mu(\xi, \eta)$ is positive definite, then*

$$H^q(V, \mathcal{O}(\mathbf{E}^{(\mu)} \otimes \mathbf{F})) = 0 \qquad \text{for } q > 0 .$$

To prove Theorem G′, by (3.23) it will suffice to show that $H^q(\mathbf{P}, \mathcal{O}(\mathbf{L}^\mu \otimes \mathbf{F})) = 0$ for $q > 0$. This will be done using (3.22) on $\mathbf{P}$; thus, in (3.22), replace V by $\mathbf{P}$, $\mathbf{E}$ by $\mathbf{L}^\mu \otimes \mathbf{F}$, and $\mathbf{K}$ by $\mathbf{K}_P$. Using the formula (2.38) for $\mathbf{K}_P$, we must show

(3.26) $\mathbf{L}^{\mu+r} \otimes \mathbf{K}_V^* \otimes \det(\mathbf{E})^* \otimes \mathbf{F}$ is positive if Q_μ in (3.24) is.

But (3.24) follows immediately from (2.36), and this proves Theorem G′.

To prove Theorem G, we need to show that $Q_\mu(\xi, \eta)$ in (3.24) is positive definite for all $\mu \geqq 0$ if $\mathbf{F}^* \otimes \mathbf{K}_V \otimes \det(\mathbf{E}) < 0$, provided that $\mathbf{E}$ is generated by its sections. By (2.24), at a given point, $\Theta^\rho_{\sigma i\bar{j}} = \sum_\alpha A^\alpha_{\rho i} \bar{A}^\alpha_{\sigma j}$ and so

$$\textstyle\sum_{\rho,\sigma,i,j} \Theta^\rho_{\sigma i\bar{j}} \xi^\sigma \bar{\xi}^\rho \eta^i \bar{\eta}^j = \sum_{\alpha,\rho,\sigma i,j} A^\alpha_{\rho i} \bar{\xi}^\rho \eta^i \overline{A^\alpha_{\sigma j} \bar{\xi}^\sigma \eta^j} \geqq 0 .$$

Then $\Theta_\mu(\xi, \eta) \geqq$ (curvature form for $\mathbf{K}_V^* \otimes \mathbf{F} \otimes \det(\mathbf{E})^* \, | \xi |^2) > 0$. This completes the proof of Theorem G.

PROOF OF THEOREM B. Suppose we can prove that, for any *vector bundle* $\mathbf{F} \to V$,

$$(3.27) \qquad H^q(V, \mathcal{O}(\mathbf{E}^{(\mu)} \otimes \mathbf{F})) = 0 \qquad \text{for } q > 0,\ \mu > \mu_0(\mathbf{F}) .$$

Then, taking as *definition* of a coherent sheaf $S \to V$ a sheaf having a global resolution

$$(3.28) \qquad 0 \longrightarrow F_k \longrightarrow F_{k-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow S \longrightarrow 0$$

by locally free sheaves, using exact cohomology sequences we will find that $H^q(V, \mathcal{O}(\mathbf{E}^{(\mu)} \otimes S)) = 0$ for $q > 0$, $\mu \geqq \mu_0(S)$.

Now that usual definition of a coherent sheaf is that there are *locally* resolutions by free sheaves; then as a conseqence of cohomological positivity for some particular line bundle, we can find a global resolution (3.28). Taking a projective embedding $V \subset \mathbf{P}_N$, one of Serre's basic theorems [24] and [25] is the cohomological positivity of the standard positive line bundle. The conclusion then is that, to prove Theorem B, it will suffice to show (3.27).

We shall prove (3.27) by showing that (3.27) holds in case $\mathbf{E} \to V$ is a positive line bundle. Then, in the general case, $\mathbf{L} \to \mathbf{P}(\mathbf{E}^*)$ is a positive line bundle (cf. (2.37)) and so $H^q(\mathbf{P}, \mathcal{O}(\mathbf{L}^\mu \otimes \mathbf{F})) = 0$ for $q > 0$, $\mu \geqq \mu_0(\mathbf{F})$. Using the isomorphism (3.23), we get (3.27).

To prove (3.27) for $\mathbf{E} \to V$ a positive line bundle, it will suffice to show that

$$H^q(V, \mathcal{O}(\mathbf{E}^{-\mu} \otimes \mathbf{F})) = 0 \qquad \text{for } q < n,\ \mu \geqq \mu_0(\mathbf{F}) . \tag{3.29}$$

Indeed, one may pass from (3.27) to (3.29) and back by using the duality theorem. We shall prove (3.29) by using the first Nakano inequality (3.17). As our Kähler metric, we take $\gamma = (i/2)\Theta_{\mathbf{E}}$ where $\Theta_{\mathbf{E}}$ is the curvature in $\mathbf{E}$. By (2.10), we have

$$\Theta_{\mathbf{E}^{-\mu}\otimes\mathbf{F}} = -\mu\Theta_{\mathbf{E}} \otimes 1 + 1 \otimes \Theta_{\mathbf{F}} .$$

For φ a $(0, q)$ form with values in $\mathbf{E}^{-\mu} \otimes F$,

$$\begin{aligned} \frac{i}{2} L^*\Theta_{\mathbf{E}^{-\mu}\otimes\mathbf{F}} \wedge \varphi &= -\mu L^* L\varphi + \frac{i}{2} L^*\Theta_{\mathbf{F}} \wedge \varphi \\ &= -\mu(n - q)\varphi + \frac{i}{2} L^*\Theta_{\mathbf{F}} \wedge \varphi . \end{aligned}$$

From (3.17) we get:

$$\frac{i}{2}(L^*\Theta_{\mathbf{F}} \wedge \varphi, \varphi) \geqq \mu(n - q)(\varphi, \varphi) . \tag{3.30}$$

Taking μ large in (3.30), we get $\varphi = 0$, which implies that

$$H^q(V, \mathcal{O}(\mathbf{E}^{-\mu} \otimes \mathbf{F})) = 0 \qquad \text{for } \mu \geqq \mu_0(\mathbf{F}),\ q < n .$$

(c) PROOF OF THEOREMS C AND F. We shall prove Theorem C using Theorem G′ above (cf. (3.25)) and using *quadratic transformations* [20]. Let $z_0 \in V$ be a fixed point and $W \xrightarrow{\pi} V$ the quadratic transform of V at z_0. This may be described as follows. Let

$z^1, \cdots, z^n$ be coordinates in an open set $U \subset V$ with $z_0 = (0, \cdots, 0)$. We consider the principal bundle $\mathbf{C}^n - \{0\} \to \mathbf{P}_{n-1} = \mathbf{P}$ and let $\mathbf{H}^* \to \mathbf{P}$ be the corresponding line bundle. Then $\mathbf{H}^* - \mathbf{P} = \mathbf{C}^n - \{0\}$ and so U corresponds to a tubular neighborhood $T(U) \subset \mathbf{H}^*$ of the zero section, and W is obtained by replacing U with $T(U)$. Geometrically, we have replaced z_0 with the lines through z_0.

Letting $\mathbf{P} = \pi^{-1}(z_0)$, $\mathbf{P}$ is a $\mathbf{P}_{n-1}$ embedded in W and $W - \mathbf{P} \cong V - \{z_0\}$. If $\mathbf{L} \to W$ is a line bundle determined by the divisor $\mathbf{P} \subset W$, then $\mathbf{L} \mid P = \mathbf{H}^*$. To find a metric in $\mathbf{L} \to W$, we choose a concentric open set $U_1 \subset U$ with $\bar{U}_1 \subset U$, $z_0 \in U_1$ and let ρ be a function which is one on U_1 and zero on $W - U$. Then there is defined a metric in $\mathbf{L}$ whose curvature is [20]

$$\Theta_{\mathrm{L}} = \partial\bar{\partial}\left(\rho \log \left\{\textstyle\sum_{i=1}^{n} | z^i |^2\right\}\right) . \tag{3.31}$$

Thus $\Theta_{\mathrm{L}} = 0$ on $W - U$ and, on U_1,

$$\Theta_{\mathrm{L}} = -\left\{\frac{(dz, dz)(z, z) - (dz, z)(z, dz)}{(z, z)^2}\right\} \tag{3.32}$$

(cf. (2.5)). In particular, $\Theta_{\mathrm{L}} < 0$ on U_1.

Over W we consider the exact bundle sequences

$$\begin{cases} 0 \to \pi^*\mathbf{E}^{(\mu)} \otimes \mathbf{L}^* \to \pi^*\mathbf{E}^{(\mu)} \to \pi^*\mathbf{E}^{(\mu)} \mid \mathbf{P} \to 0 \ , \\ 0 \to \pi^*\mathbf{E}^{(\mu)} \otimes \mathbf{L}^{*2} \to \pi^*\mathbf{E}^{(\mu)} \otimes \mathbf{L}^* \to \pi^*\mathbf{E}^{(\mu)} \otimes \mathbf{L}^* \mid \mathbf{P} \to 0 \ . \end{cases} \tag{3.33}$$

We observe that

$$\begin{aligned} H^0(W, \mathcal{O}(\pi^*\mathbf{E}^{(\mu)})) &\cong H^0(V, \mathcal{O}(\mathbf{E}^{(\mu)}) \ , \\ H^0(\mathbf{P}, \mathcal{O}(\pi^*\mathbf{E}^{(\mu)})) &\cong \mathbf{E}_{z_0}^{(\mu)} \ , \\ H^0(\mathbf{P}, \mathcal{O}(\pi^*\mathbf{E}^{(\mu)} \otimes \mathbf{H})) &\cong \mathbf{E}_{z_0}^{(\mu)} \otimes \mathbf{T}_{z_0}^*(V) \end{aligned}$$

(since $H^0(\mathbf{P}, \mathcal{O}(\mathbf{H}))$ has basis $z^1, \cdots, z^n$). Thus, if

$$H^1(W, \mathcal{O}(\pi^*\mathbf{E}^{(\mu)} \otimes \mathbf{L}^*)) = 0 = H^1(W, \mathcal{O}(\pi^*\mathbf{E}^{(\mu)} \otimes \mathbf{L}^{*2})) \ ,$$

we get from (3.33) that

$$\begin{cases} 0 \longrightarrow \mathbf{F}_{z_0}^{(\mu)} \longrightarrow \Gamma(\mathbf{E}^{(\mu)}) \longrightarrow \mathbf{E}_{z_0}^{(\mu)} \longrightarrow 0 \\ \mathbf{F}_{z_0}^{(\mu)} \longrightarrow \mathbf{F}_{z_0} \otimes \mathbf{T}_{z_0}^* \longrightarrow 0 \ , \end{cases} \tag{3.34}$$

where $\mathbf{F}_{z_0}^{(\mu)} = \{s \in \Gamma(\mathbf{E}^{(\mu)}) \colon s(z_0) = 0\} \cong H^0(W, \mathcal{O}(\pi^*\mathbf{E}^{(\mu)} \otimes \mathbf{L}^*))$.

Thus, to prove Theorem C, we need to show that

$$\begin{cases} \mathbf{E} \text{ positive} \Longrightarrow H^1(W, \mathcal{O}(\pi^*\mathbf{E}^{(\mu)} \otimes \mathbf{L}^*)) = 0 = \\ H^1(W, \mathcal{O}(\pi^*\mathbf{E}^{(\mu)} \otimes \mathbf{L}^{*2})) \text{ for all } \mu > \mu_0, \text{ and } all\ z_0 \in V \ . \end{cases} \tag{3.35}$$

We want to look at the quadratic form (3.24) on W for $\pi^*\mathbf{E}$ and where $\mathbf{F} = \mathbf{L}^*$ or $\mathbf{L}^{*2}$. Since $\mathbf{E} \to V$ is positive, the quadratic form $\Theta_{\pi^*\mathbf{E}}(\xi, \eta)$ on W is positive definite outside of P. From (3.32), the curvature $\Theta_{\mathbf{L}^*}$ is positive definite in U_1 and zero outside U. Finally, the canonical bundle [20]

$$\mathbf{K}_W = \pi^*\mathbf{K}_V \otimes \mathbf{L}^{n-1} . \tag{3.36}$$

Thus $\Theta_{\mathbf{K}_W} = \Theta_{\pi^*\mathbf{K}_V} + (n-1)\Theta_{\mathbf{L}^*}$. From the explicit form of (3.24), we see that $Q_\mu(\xi, \eta)$ is positive definite on W for $\mu \geqq \mu_0$. In fact, $\Theta_\mu(\xi, \eta) = (\mu + r)\Theta_{\pi^*\mathbf{E}}(\xi, \eta) + \{n\Theta_{\mathbf{L}^*}(\eta) - \Theta_{\pi^*\det\mathbf{E}}(\eta) - \Theta_{\pi^*\mathbf{K}_V}(\eta)\} \,|\, \xi \,|^2$, from which our assertion is clear. Furthermore, by using obvious estimates from continuity, we see that the quadratic forms will be positive definite for $\mu \geqq \mu_0$ and *all* $z_0 \in V$. Then (3.35) follows from Theorem G′ ((3.25)).

This completes the proof of Theorem C.

PROOF OF THEOREM F. Let $z_0 \in V$ and $I_{z_0} \subset \mathcal{O}$ be the ideal sheaf of z_0. We can choose $\mu(z_0)$ such that

$$H^q(V, I_{z_0} \otimes \mathcal{O}(\mathbf{E}^\mu)) = 0 = H^q(V, I_{z_0}^2 \otimes \mathcal{O}(\mathbf{E}^{(\mu)})) \qquad \text{for } \mu \geqq \mu(z_0),\ q > 0 .$$

From the cohomology sequences of $0 \to I_{z_0} \otimes \mathcal{O}(\mathbf{E}^{(\mu)}) \to \mathcal{O}(\mathbf{E}^{(\mu)}) \to \mathbf{E}_{z_0}^{(\mu)} \to 0$ and $0 \to I_{z_0} \otimes \mathcal{O}(\mathbf{E}^{(\mu)}) \to I_{z_0} \otimes \mathcal{O}(\mathbf{E}^{(\mu)}) \to \mathbf{E}_{z_0}^{(\mu)} \otimes \mathbf{T}_{z_0}^* \to 0$, we get

$$\begin{cases} 0 \longrightarrow \mathbf{F}_{z_0}^{(\mu)} \longrightarrow \Gamma(\mathbf{E}^{(\mu)}) \longrightarrow \mathbf{E}_{z_0}^{(\mu)} \longrightarrow 0 , \\ \mathbf{F}_{z_0}^{(\mu)} \longrightarrow \mathbf{E}_{z_0}^{(\mu)} \otimes \mathbf{T}_{z_0}^* \longrightarrow 0 \end{cases}$$

for $\mu \geqq \mu(z_0)$. In particular then, there is a neighborhood $U(z_0)$ such that, for $z \in U(z_0)$, we have

$$\begin{cases} 0 \longrightarrow \mathbf{F}_z^{(\mu(z_0))} \longrightarrow \Gamma(\mathbf{E}^{(\mu(z_0))}) \longrightarrow \mathbf{E}_z^{(\mu(z_0))} \longrightarrow 0 , \\ \mathbf{F}_z^{(\mu(z_0))} \longrightarrow \mathbf{E}_z^{(\mu(z_0))} \otimes \mathbf{T}_z^* \longrightarrow 0 . \end{cases} \tag{3.37}$$

Observe that (3.37) holds for any $\mu = \mu_1\mu(z_0)$.

Given now $z_1 \in V$, we may find $\mu(z_1)$ such that

$$\begin{cases} 0 \longrightarrow \mathbf{F}_{z_1}^{(\nu_1)} \longrightarrow \Gamma(\mathbf{E}^{(\nu_1)}) \longrightarrow \mathbf{E}_{z_1}^{(\nu_1)} \longrightarrow 0 \\ \mathbf{F}_{z_1}^{(\nu_1)} \longrightarrow \mathbf{E}_{z_1}^{(\nu_1)} \otimes \mathbf{T}_{z_1}^* \longrightarrow 0 \end{cases} \tag{3.38}$$

where $\nu_1 = \nu(z_1)\mu(z_0)$. Also, (3.38) holds in a neighborhood $U(z_1)$, as well as in $U(z_0)$. Continuing, and using the compactness of V, we may find a μ such that $\mathbf{E}^{(\mu)}$ is ample. That is

(3.39) E cohomomologically positive $\Rightarrow \mathbf{E}^{(\mu)}$ ample for some μ.

This is weak version of Theorem F, which asserts that $\mathbf{E}^{(\mu)}$ is ample for *all* $\mu > \mu_0$. To prove this, it will suffice to show

$$(3.40)\quad \begin{cases} H^q(V, I_{z_0} \otimes \mathcal{O}(\mathbf{E}^{(\mu)})) = 0 = H^q(V, I^2_{z_0} \otimes \mathcal{O}(\mathbf{E}^{(\mu)})) \\ \text{for } q > 0,\ \mu \geqq \mu_0, \text{ and } all\ z_0 \in V\ . \end{cases}$$

Consider now $\mathbf{L} \to \mathbf{P}(\mathbf{E}^*)$. Since $\mathbf{E}$ is cohomologically positive, $\mathbf{L}$ is colomologically positive and, by (3.39) applied to $\mathbf{L}$, $\mathbf{L}^\mu \to \mathbf{P}$ is ample for some μ. Thus $\mathbf{L}^\mu$ is positive; i.e., there is a metric in $\mathbf{L}^\mu$ whose curvature $\theta_{\mathbf{L}^\mu} > 0$. Since $\Theta_{\mathbf{L}} = \dfrac{1}{\mu}(\Theta_{\mathbf{L}^\mu})$, we have that $\Theta_{\mathbf{L}} > 0$.

In the fibering $\mathbf{P} \xrightarrow{\pi} V$, let $P(z_0) = \pi^{-1}(z_0)$ and $I(z_0) \subset \mathcal{O}_{\mathbf{P}}$ be the ideal sheaf of $P(z_0)$. Then, by (1.9),

$$H^q(V, I^k_{z_0} \otimes \mathcal{O}(\mathbf{E}^{(\mu)})) \cong H^q(\mathbf{P}, I(z_0)^k \otimes \mathcal{O}(\mathbf{L}^\mu))$$

and, using (3.40), to prove Theorem F it will suffice to show:

$$(3.41)\qquad \begin{matrix} H^q(\mathbf{P}, I(z_0)^k \otimes \mathcal{O}(\mathbf{L}^\mu)) = 0 \\ k = 1, 2\ . \end{matrix} \qquad \text{for } q > 0;\ \mu \geqq \mu_0\ ;$$

We shall prove (3.41) by a method similar to the proof of Theorem C above.

Let $Q \xrightarrow{\varpi} \mathbf{P}$ be the quadratic transform of $\mathbf{P}$ along $P(z_0)$. For the equations giving Q, see [11, §V]. We set $S = \tilde{\omega}^{-1}(P(z_0))$ so that $Q - S = \mathbf{P} - P(z_0)$, and we let $\mathbf{J} \to Q$ be the line bundle determined by the divisor $S \subset Q$. We recall that $R^p_\varpi(\mathbf{J}^{*l}) = 0$ for $p > 0$ and $R^0_\varpi(\mathbf{J}^{*l}) = I(z_0)^l$. This, plus Leray's theorem, gives

$$(3.42)\qquad H^q(Q, \mathcal{O}(\mathbf{J}^{*l} \otimes \tilde{\omega}^{-1}\mathbf{L}^\mu)) = H^q(\mathbf{P}, I(z_0)^l \otimes \mathcal{O}(\mathbf{L}^\mu))\ .$$

We used special cases of (3.42) just below (3.33) above.

To prove (3.41), using (3.42) we need to show

$$(3.43)\qquad \begin{cases} H^q(Q, \mathcal{O}(\mathbf{J}^{*k} \otimes \tilde{\omega}^{-1}\mathbf{L}^\mu)) = 0 \text{ for } q > 0;\ \mu \geqq \mu_0\ ; \\ k = 1, 2; \text{ and for all } z_0 \in V\ . \end{cases}$$

As in the proof of Theorem C above, we shall prove (3.43) for fixed z_0 by a curvature estimate and then, using continuity of these estimates relative to z_0, we will get (3.43). Thus we need to compute the curvature $\Theta_{\mathbf{J}}$ on Q.

Let U be a polycylindrical neighborhood of z_0 such that $\mathbf{E} \mid U \cong U \times \mathbf{C}^r$. Then $\pi^{-1}(U) \cong U \times \mathbf{P}_{r-1}$ and $\pi^{-1}(U) \subset \mathbf{P}(\mathbf{E}^*)$ is a tubular neighborhood $N(P(z_0))$ of $P(z_0) = \{0\} \times \mathbf{P}_{r-1}$, where $z_0 = (0, \cdots, 0)$. Letting $\tilde{U}$ be the ordinary quadratic transform of U at $\{0\}$, $\tilde{\omega}^{-1}(N(P(z_0))) =$

$T(S)$ is a tubular neighborhood of S and $T(S) \cong \widetilde{U} \times \mathbf{P}_{r-1}$. If $z^1, \cdots, z^n$ are coordinates on U and $\xi^0, \cdots, \xi^r$ homogeneous coordinates on $\mathbf{P}_{r-1}$, we have that $\Theta_{\mathbf{L}} = \Theta_{\mathbf{L}}(dz^i, d\xi^\rho)$ is positive definite on $N(P(z_0))$. Furthermore, there is a metric in $\mathbf{J} \mid T(S)$ such that the curvature $\Theta_{\mathbf{J}} = \partial\bar{\partial} \log \left(\sum |z^i|^2\right)$ (cf. (3.32)). Thus $\Theta_{\varpi^{-1}(\mathbf{L})} + \Theta_{\mathbf{J}^*}$ is positive on $T(S)$. Since $\Theta_{\varpi^{-1}(\mathbf{L})} \mid Q - T(S)$ is positive, we may fit the above metric on $\mathbf{J} \mid T(S)$ is to get a metric on $\mathbf{J} \rightarrow Q$ such that $\nu\Theta_{\varpi^{-1}(\mathbf{L})} + \Theta_{\mathbf{J}^*} > 0$ for $\nu \geqq \nu_0$. By proceeding now just as in the proof of Theorem C, we will get (3.43) as desired.

(d) *Positive bundles and topological properties of algebraic varieties.* Let $\mathbf{E} \rightarrow V$ be a positive vector bundle and $\xi \in H^0(V, \mathcal{O}(\mathbf{E}))$ a holomorphic section whose divisor $S \subset V$ is a nice subvariety as discussed in § 2(h). Then Theorem H on the topology of S in V yields the following vanishing theorem for sheaf cohomology

$$H^q(V, I_S) = 0 \qquad \text{for } 0 \leqq q \leqq n - r\,, \tag{3.44}$$

where I_S is the ideal sheaf of S in V. For the proof of (3.44), we consider the exact sheaf sequence $0 \rightarrow I_S \rightarrow \mathcal{O}_V \rightarrow \mathcal{O}_S \rightarrow 0$. Since S and V are Kähler manifolds, we have a diagram

$$\begin{array}{ccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H^q(I_S) & \longrightarrow & H^q(\mathcal{O}_V) & \longrightarrow & H^q(\mathcal{O}_S) & \longrightarrow & \cdots \\ & & & & \downarrow & & \downarrow & & \\ & & & & (H^q(V,\mathbf{C}) & \longrightarrow & H^q(S,\mathbf{C})\,. & & \end{array} \tag{3.45}$$

Using that $0 \rightarrow H^q(V,\mathbf{C}) \rightarrow H^q(S,\mathbf{C})$ for $q \leqq n-r$ and $0 \rightarrow H^q(V,\mathbf{C}) \rightarrow H^q(S,\mathbf{C}) \rightarrow 0$ for $q \leqq n - r - 1$, we find $H^q(I_S) = 0$ for $0 \leqq q \leqq n - r$ as required.

In case S is a hypersurface ($\mathbf{E}$ is a line bundle), we have $I_S \cong \mathcal{O}_V(\mathbf{E}^*)$ and (3.44) becomes

$$H^q(V, \mathcal{O}(\mathbf{E}^*)) = 0 \qquad \text{for } 0 \leqq q \leqq n - 1\,, \tag{3.46}$$

which is the original Kodaira theorem [18].

Still considering the case when $S \subset V$ is a hypersurface, we shall show that Theorem H gives

$$H^q(V, \Omega_V^p(\mathbf{E}^*)) = 0 \qquad \text{for } p + q < n\,, \tag{3.47}$$

which generalizes (3.46). For the proof, consider the pair of exact sheaf sequences [15, page 127]

$$(3.48)\qquad \begin{cases} 0 \longrightarrow \Omega_V^p(\mathbf{E}^*) \xrightarrow{\ \xi\ } \Omega_V^p \longrightarrow \Omega_{V|S}^p \longrightarrow 0 \\ 0 \longrightarrow \Omega_S^{p-1}(\mathbf{E}^*) \xrightarrow{\ d\xi\ } \Omega_{V|S}^p \longrightarrow \Omega_S^p \longrightarrow 0 \,. \end{cases}$$

In cohomology, we have:

$$(3.49)\qquad \begin{cases} \to H^q(V, \Omega_V^p(\mathbf{E}^*)) \to H^q(V, \Omega_V^p) \to H^q(S, \Omega_{V|S}^p) \to \\ \hphantom{\to H^q(V, \Omega_V^p(\mathbf{E}^*)) \to H^q(V, \Omega_V^p) \to\ } \| \\ \qquad \cdots \to H^q(S, \Omega_S^{p-1}(\mathbf{E}^*)) \to H^q(S, \Omega_{V|S}^p) \to H^q(S, \Omega_S^p) \,. \end{cases}$$

Using that $0 \to H^q(V, \Omega_V^p) \to H^q(S, \Omega_S^p)$ for $p + q < n$, and that this mapping is onto for $p + q < n - 1$, we get (3.47).

Conversely, from (3.47), we obtain

$$(3.50)\qquad \begin{cases} 0 \longrightarrow H^q(V, \Omega_V^p) \longrightarrow H^q(S, \Omega_S^p) \longrightarrow 0 \,, & p + q < n - 1 \\ 0 \longrightarrow H^q(V, \Omega_V^p) \longrightarrow H^q(S, \Omega_S^p) \,, & p + q = n - 1 \,. \end{cases}$$

This in turn gives Theorem H over $\mathbf{Q}$.

In conclusion, for a positive line bundle $\mathbf{E} \to V$ which has a non-singular section, we see that Theorem H (over $\mathbf{Q}$) and the vanishing theorem (3.21) are equivalent.

As another application of topology to sheaf cohomology, we let $\mathbf{E} \to V$ be a positive bundle having a non-singular holomorphic section ξ with divisor S. Let $\omega \in H^{2r}(V, \mathbf{C})$ be the r^{th} *Chern class* [15] of $\mathbf{E} \to V$; then $\omega \in H^r(V, \Omega_V^r)$ is dual to the homology class $[\mathrm{S}] \in H_{2n-2r}(V, \mathbf{C})$ defined by S. We shall prove

$$(3.51)\qquad \begin{cases} \text{The cup product } H^q(V, \Omega_V^p) \xrightarrow{\ \omega\ } H^{q+r}(V, \Omega_V^{p+r}) \text{ is} \\ \text{an isomorphism for } p + q = n - r \,. \end{cases}$$

For $r = 1$, this is a well-known result in Kähler varieties [26]. We have to show that $H^{n-r}(V, \mathbf{C}) \xrightarrow{\ \omega\ } H^{n+r}(V, C)$ is an isomorphism. First we dualize $H_{n-r+1}(V, S) \to H_{n-r}(S) \to H_{n-r}(V) \to 0$ to get $H^{n+r-1}(V - S) \to H^{n-r}(S) \to H^{n+r}(V) \to 0$. Combining this with $0 \to H^{n-r}(V) \to H^{n-r}(S) \to H^{n-r-1}(V, S)$, we find a diagram

$$(3.52)\qquad \begin{tikzcd} & H^{n+r-1}(V - S) \arrow[d, "\partial^*"] \arrow[dr, dashed, "\psi^*"] & \\ 0 \arrow[r] & H^{n-r}(V) \arrow[r] \arrow[d, phantom] & H^{n-r}(S) \arrow[r] \arrow[d] & H^{n-r+1}(V, S) \\ & & H^{n+r}(V) \arrow[d] & \\ & & 0 \,. & \end{tikzcd}$$

From (3.52), to prove (3.51) it will suffice to show that kernel $\partial^* =$ kernel ψ^*. Dually, we must show that, in

$$\begin{tikzcd} H_{n-r+1}(V, S) \arrow[d, "\partial"] \arrow[dr, "\psi"] & \\ H_{n-r}(S) \arrow[r, "\tau"] & H_{n+r-1}(V-S) . \end{tikzcd}$$

ψ and ∂ have the same kernel; i.e., if $\partial\sigma \neq 0$, then $\psi(\sigma) \neq 0$.

At each point $z \in S$, the normal sphere S_z^{2r-1} to S in V at z is a $(2r-1)$-sphere. If $\gamma \in H_{n-r}(S)$ is a cycle, then $\tau(\gamma) \in H_{n+r-1}(V-S)$ is the cycle traced out by the spheres S_z^{2r-1} for $z \in \gamma$ (this follows from the Thom isomorphism [15]). Let $\sigma \in H_{n-r+1}(V, S)$ and $\partial\sigma = \delta \in H_{n-r}(S)$. Now find a cycle $\hat{\delta} \in H_{n-r}(S)$ whose intersection number with δ is ± 1. Then it is essentially clear that the intersection number $\sigma \cdot \tau(\hat{\delta}) = \pm 1$, and so $\psi(\sigma) \neq 0$. This proves (3.51).

We remark that (3.51) is *false* for the universal bundle over the grassmannian (cf. [11, § IV. 4, page 397]), of course, this bundle is *not* positive.

To close this section, let us give one of the most noteworthy "vanishing theorems" derived from topological considerations; this is the famous *regularity of the adjoint* theorem of Picard [27]. Let V be an algebraic surface, $\mathbf{E} \to V$ a positive line bundle, and $\xi \in H^0(V, \mathcal{O}(\mathbf{E}))$ a non-singular section with divisor $S \subset V$. We set $\mathbf{N} = \mathbf{E} \mid S$, $\mathbf{K}_V =$ canonical bundle of V, $\mathbf{K}_S =$ cannonical bundle of S; the *adjunction formula* gives (cf. [15]):

$$\mathbf{K}_S = \mathbf{L} \otimes \mathbf{K}_V \mid S . \tag{3.53}$$

Using (3.53), we get the exact sheaf sequences

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_V(\mathbf{K}_V \otimes \mathbf{E}^{r-1}) \xrightarrow{\ \xi\ } \mathcal{O}_V(\mathbf{K}_V \otimes \mathbf{E}^{r}) & \\ \longrightarrow \mathcal{O}_S(\mathbf{K}_S \otimes \mathbf{N}^{r-1}) \longrightarrow 0 & . \end{aligned} \tag{3.54}$$

The regularity of the adjoint theorem is

$$\begin{aligned} H^0(\mathcal{O}_V(\mathbf{K}_V \otimes \mathbf{E}^r)) \longrightarrow H^0(\mathcal{O}_S(\mathbf{K}_S \otimes \mathbf{N}^{r-1})) \longrightarrow 0 \ , & \\ r \geqq 2 \ . & \end{aligned} \tag{3.55}$$

In classical language, if $V \subset P_3$ is a surface of degree n with ordinary singularities given by an affine equation $f(x, y, z) = 0$, then the adjoint polynomials $P(x, y, z)$ of degree at least $n - 3 + r$ $(r \geqq 1)$ cut out on a generic plane section $x =$ constant of V a complete linear series.

It will suffice to prove that $H^1(\mathbf{K}_V \otimes \mathbf{E}^{r-1})) = 0$ for $r \geqq 2$. We shall do this for $r = 2$ by giving Picard's original argument. For $r = 1$, (3.54) becomes

$$0 \longrightarrow \Omega_V^2 \xrightarrow{\xi} \Omega_V^2(\mathbf{E}) \longrightarrow \Omega_S^1 \longrightarrow 0 . \tag{3.56}$$

In cohomology we get a diagram:

$$\begin{tikzcd} H^0(\Omega_S^1) \arrow[r] & H^1(\Omega_V^2) \arrow[r] & H^1(\Omega_V^2(\mathbf{E})) \arrow[r] & H^1(\Omega_S^1) \arrow[r, "\delta"] & H^2(\Omega_V^2) \\ H^0(\Omega_V^1) \arrow[u] \arrow[ur, "\omega"'] & & & & \\ 0 \arrow[u] & & & & \end{tikzcd} \tag{3.57}$$

As discussed above (cf. (3.51) and (3.52)), it is a purely topological fact that ω is an isomorphism, and obviously δ is an injection. Thus $H^1(\Omega_V^2(\mathbf{E})) = 0 = H^1(\mathcal{O}_V(\mathbf{K}_V \otimes \mathbf{E}))$. Using this in the exact cohomology sequences of (3.54) when $r = 2$, we get

$$H^0(\mathcal{O}_V(\mathbf{K}_V \otimes \mathbf{E}^2)) \longrightarrow H^0(\mathcal{O}_S(\mathbf{K}_S \otimes \mathbf{N})) \longrightarrow 0$$

as required.

PROOF OF THEOREMS I AND J. Let $\mathbf{E} \to V$ be a positive vector bundle and $\xi \in H^0(V, \mathcal{O}(\mathbf{E}))$ a holomorphic section such that $S = \{\xi(z) = 0\}$ is non-singular. Letting $I = I_S$ be the ideal sheaf of S and $\mathbf{F} \to V$ and arbitrary holomorphic bundle, we want to show that there is a constant $c = c(\mathbf{F}, V)$ such that, if the curvature form (0.1)′ has the property $\Theta_{\mathbf{E}}^{*}(\xi, \eta) \geqq c\,|\,\xi\,|^2\,|\,\eta\,|^2$, then we have

$$H^q(V, I \otimes \mathcal{O}(\mathbf{F})) = 0 \qquad \text{for } q \leqq n - r . \tag{3.58}$$

We remark that, for $\mathbf{F} = 1$, (3.58) follows from (3.44).

Now we use the *Koszul complex*

$$\begin{aligned} 0 \longrightarrow \mathcal{O}(\Lambda^r\mathbf{E}^*) \xrightarrow{\xi} \cdots \longrightarrow \mathcal{O}(\Lambda^2\mathbf{E}^*) \\ \xrightarrow{\xi} \mathcal{O}(\mathbf{E}^*) \xrightarrow{\xi} I \longrightarrow 0 , \end{aligned} \tag{3.59}$$

where $\Lambda^q\mathbf{E}^* \to \Lambda^{q-1}\mathbf{E}^*$ is contraction with $\xi \in \Gamma(\mathbf{E})$. For example, if $r = 2$ then (3.59) becomes

$$0 \longrightarrow \mathcal{O}(\Lambda^2\mathbf{E}^*) \longrightarrow \mathcal{O}(\mathbf{E}^*) \longrightarrow I \longrightarrow 0 . \tag{3.60}$$

The sequence (3.59) remains exact after tensoring with $\mathcal{O}(\mathbf{F})$. From (3.25) we see that, if for $1 \leqq q \leqq r$ we have

$$\Theta^{\sharp}_{\Lambda^q \mathbf{E}}(\xi, \eta) \geqq c(q) \mid \xi \mid^2 \mid \eta \mid^2 \tag{3.61}$$

where $c(q)$ becomes large as c becomes large, then we will have

$$H^q(V, \mathcal{O}(\Lambda^q \mathbf{E}^* \otimes \mathbf{F})) = 0 \tag{3.62}$$

for $0 \leqq p \leqq n - 1$. But, by using (3.59), (3.62) implies that

$$H^p(V, \mathcal{O}(\mathbf{F}) \otimes I) = 0 \qquad \text{for } q \leqq n - r$$

as required.

Suppose now that $\Theta^{\sharp}_{\mathbf{E}}(\xi, \eta) \geqq c \mid \xi \mid^2 \mid \eta \mid^2$ and that we have $\mathbf{E} = \mathbf{S} \oplus \mathbf{Q}$ as a direct sum of holomorphic bundles. Then, for the curvature of the induced metric in $\mathbf{S}$, we will have $\Theta^{\sharp}_{\mathbf{S}}(\xi, \eta) \geqq c \mid \xi \mid^2 \mid \eta \mid^2$. Since $\Lambda^q \mathbf{E}$ is such a unitary direct summand of $\bigotimes^q \mathbf{E}$, it will suffice to show that

$$\Theta^{\sharp}_{\mathbf{E}}(\xi, \eta) \geqq c \mid \xi \mid^2 \mid \eta \mid^2 \Longrightarrow \Theta^{\sharp}_{\otimes^q \mathbf{E}}(\xi, \eta) \geqq c_q \mid \xi \mid^2 \mid \eta \mid^2 , \tag{3.63}$$

where c_q becomes large as c does. This is easy to verify using the tensor product rule for curvatures.

Proof of (0.6) *for* $r = 2$. Here we simply use (3.60),

$$H^q(V, \mathcal{O}(\Lambda^2 \mathbf{E}^*)) = 0 \qquad \text{for } q \leqq n - 1$$

(Kodaira vanishing theorem), and $H^q(V, I) = 0$ for $q \leqq n - 2$ (by (3.58)) to conclude that $H^q(V, \mathcal{O}(\mathbf{E}^*)) = 0$ for $q \leqq n - 2$.

PROOF OF THEOREM J. Let $S \subset V$ be the zero locus of $\xi \in H^0(V, \mathcal{O}(\mathbf{E}))$ and $\widetilde{S} \subset \widetilde{V}$ the result of blowing V up along S. Thus, if $\mathbf{N} \to S$ is the normal bundle, $\widetilde{S} = \mathbf{P}(\mathbf{N})$ and so $z \in S$ is blown up by sending z into $\mathbf{P}(\mathbf{N}_z)$. Let $\mathbf{L} \to \widetilde{V}$ be the line bundle determined by $\widetilde{S}$; then $\mathbf{L} \mid \widetilde{S}$ is the normal bundle and $\mathbf{L} \mid \mathbf{P}(\mathbf{N}_z)$ is the negative of the hyperplane bundle. If our blowing up diagram is

$$\begin{array}{ccc} \widetilde{S} & \subset & \widetilde{V} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ S & \subset & V , \end{array}$$

we want to show that

(3.64) There is an exact bundle sequence $0 \to \mathbf{L} \to \pi^*(\mathbf{E}) \to \mathbf{Q} \to 0$.

(*Proof*. $\widetilde{S}$ determines a section $\sigma \in H^0(\widetilde{V}, \mathcal{O}(\mathbf{L}))$ and the locus $\sigma = 0$ is the same as $\pi^* \xi = 0$, where $\pi^* \xi \in H^0(\widetilde{V}, \mathcal{O}(\pi^* \mathbf{E}))$. It wiil suffice to show that $\pi^* \xi / \sigma$ is a non-vanishing holomorphic section of $\mathbf{L}^* \otimes \pi^* \mathbf{E}$. Locally on V we may choose coordinates $z^1, \cdots, z^n$ such that

$$\xi(z) = \begin{pmatrix} z^1 \\ \vdots \\ z^r \end{pmatrix}.$$

If $\lambda = [\lambda_1, \cdots, \lambda_r]$ are homogeneous coordinates on $\mathbf{P}_{r-1}$, locally $\widetilde{V}$ is the set of (z, λ) satisfying the quadratic relations $z^\alpha\lambda_\beta - z^\beta\lambda_\alpha = 0$ $(1 \leqq \alpha, \beta \leqq r)$. In the open set $\lambda_r \neq 0$, $\widetilde{S}$ is given by $z^r = 0$, since $z^r = 0$ and $z^\alpha\lambda_r - z^r\lambda_\alpha = 0$ taken together give $z^\alpha = 0$ for $1 \leqq \alpha \leqq r$. Thus σ is locally a unit times z^r, and it will suffice to have $\pi^*\xi/z^r$ a non-vanishing holomorphic vector. But $z^\alpha = z^r\lambda_\alpha$ and so

$$\frac{\pi^*\xi}{z^r} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{r-1} \\ 1 \end{pmatrix}$$

which is as required.)

We now recall that, setting $\widetilde{I} = \mathcal{O}(\mathbf{L}^*)$, the direct image sheaves $R^q_\pi(\widetilde{I}^\mu)$ are given by

$$R^q_\pi(\widetilde{I}^\mu) = 0 \text{ for } q > 0,\ \mu \geqq 0 \text{ and } R^0_\pi(\widetilde{I}^\mu) = I^\mu\,. \tag{3.65}$$

Using the Leray spectral sequence, it follows that

$$H^q(V, I^\mu \otimes \mathcal{O}(\mathbf{F})) \cong H^q(\widetilde{V}, \widetilde{I} \otimes \mathcal{O}(\pi^*\mathbf{F})) \tag{3.66}$$

for any holomorphic bundle $\mathbf{F} \rightarrow V$. Now by assumption $\mathbf{E} \rightarrow V$ has a metric such that the curvature form $\Theta_\mathbf{E}$ given by (0.1) is positive. Since $\pi^*\Theta_\mathbf{E} = \Theta_{\pi^*\mathbf{E}}$, there is induced a metric in $\pi^*\mathbf{E}$ whose curvature is positive semi-definite. In fact, $\Theta_{\pi^*\mathbf{E}}$ is zero on the $r-1$ dimensional tangent spaces to the fibering $\widetilde{S} \rightarrow S$.

By (3.64), there is induced a metric in $\mathbf{L}$ and, from § 2. (c), we have that

$$\Theta_\mathbf{L} = \Theta_{\pi^*\mathbf{E}} \,|\, \mathbf{L} - b \wedge {}^t\bar{b} \tag{3.67}$$

where $b \in A^{1,0}(\mathrm{Hom}\,(\mathbf{Q}, \mathbf{L}))$ is the *second fundamental form* of $\mathbf{L}$ in $\pi^*\mathbf{E}$. Choosing unitary frames $e_1, \cdots, e_r$ for $\pi^*\mathbf{E}$ such that e_1 is a frame for $\mathbf{L} \rightarrow \widetilde{V}$, (3.67) gives

$$\begin{aligned} \Theta_\mathbf{L} &= \Theta^1_1 - \textstyle\sum_{\alpha=2}^r b^\alpha_1 \wedge \bar{b}^\alpha_1\,, \\ b &= \textstyle\sum_{\alpha=2}^r b^\alpha_1 e_1 \otimes e^*_\alpha\,. \end{aligned} \tag{3.68}$$

We want to show that

(3.69) $\Theta_{\rm L}$ has everywhere at least $n-r+1$ positive eigenvalues.

From (3.68), this is clearly the case on $\widetilde{V} - \widetilde{S}$ where Θ_1^1 is positive definite. Let $\widetilde{z} \in \widetilde{S}$ and $z = \pi(\widetilde{z})$ so that $\widetilde{z} \in \mathbf{P}(\mathbf{N}_z)$. Since $b \wedge {}^t\bar{b}$ has rank less than or equal $r-1$, if we show that $b \wedge {}^t\bar{b}$ is non-singular on the tangent space to the projective space $\mathbf{P}(\mathbf{N}_z)$ passing thru $\widetilde{z}$, then $\Theta_1^1 - b \wedge {}^t\bar{b}$ will be positive definite on the $n-r+1$ dimensional null-space of b. This will prove (3.69). But $\Theta_{\rm L} \mid \mathbf{P}(\mathbf{N}_z) = -b \wedge {}^t\bar{b} \mid \mathbf{P}(\mathbf{N}_z)$ is the curvature of the negative of the hyperplane bundle over $\mathbf{P}_{r-1}$, and so $-b \wedge {}^t\bar{b}$ is negative definite on the tangent space to $\mathbf{P}(\mathbf{N}_z)$.

Now we use the theorem of Andreotti-Grauert [1] to conclude

$$H^q(\widetilde{V}, \mathcal{O}(\mathbf{L}^{-\mu} \otimes \pi^*\mathbf{F})) = 0 \qquad \text{for } \mu \geqq \mu_0,\ q \leqq n-r\ . \tag{3.70}$$

Combining (3.70) with (3.66) and using $\widetilde{I}^\mu \cong \mathcal{O}(\mathbf{L}^{-\mu})$ gives Theorem J.

4. Chern classes and numerically positive bundles

(a) *Chern homology classes of holomorphic bundles.* Let V be an algebraic manifold and $\mathbf{E} \to V$ an *ample* holomorphic vector bundle (cf. (0.2)). Then we have

$$0 \longrightarrow \mathbf{F} \longrightarrow \mathbf{\Gamma} \longrightarrow \mathbf{E} \longrightarrow 0 \tag{4.1}$$

where $\mathbf{\Gamma} = V \times \Gamma(\mathbf{E})$ is a trivial bundle. Dualizing (4.1) gives

$$0 \longrightarrow \mathbf{E}^* \longrightarrow \mathbf{\Gamma}^* \longrightarrow \mathbf{F}^* \longrightarrow 0\ . \tag{4.2}$$

We assume that $\mathbf{E}^*$ has fibre $\mathbf{C}^r$ and $\mathbf{F}^*$ has fibre $\mathbf{C}^{m+r}$. Choose a sequence of linear subspaces $\Gamma_1 \subset \Gamma_2 \subset \cdots \subset \Gamma_{m+r-1} \subset \mathbf{\Gamma}^*$ where $\dim \Gamma_\alpha = \alpha$. For each r-tuple of integers $\rho = (\rho_1, \cdots, \rho_r)$ with $\rho_1 \leqq \rho_2 \leqq \cdots \leqq \rho_r \leqq m$, we define $S_\rho \subset V$ as follows

$$S_\rho = \{z \in V \text{ such that } \dim(\mathbf{E}_z^* \cap \Gamma_{j+\rho_j}) \geqq j \text{ for } j = 1, \cdots, r\}\ . \tag{4.3}$$

Referring to Hodge-Pedoe [16, Ch. XIV], we see that S_ρ is the intersection of V with the *Schubert cycle* of symbol ρ on the grassmannian $G(r, m)$. Taking the Γ_α to be generic, we find

$$\begin{cases} S_\rho \text{ is an irreducible variety of dimension} \\ n + (\rho_1 + \cdots + \rho_r) - mr\ . \end{cases} \tag{4.4}$$

For later use, we record the trivial formula:

$$\dim(\mathbf{E}_z^* \cap \Gamma_\alpha) = r + \alpha - \dim(\mathbf{E}_z^* + \Gamma_\alpha)\ . \tag{4.5}$$

The dimension formula (4.4) above will be checked in the case of all S_ρ we shall use.

Let now ρ_q be the Schubert symbol

$$(4.6) \qquad (\underbrace{m-1, \cdots, m-1}_{q}, \underbrace{m, \cdots, m}_{r-q}) \,,$$

and $S_q = S_{\rho_q}$. By (4.4) $\dim S_q = n + q(m-1+m(r-q) - mr = n - q$ so that S_q defines $\sigma_q \in H_{2n-2q}(V, \mathbf{Z})$.

(4.7) *Definition.* σ_q is the q^{th} *Chern (homology) class* of $\mathbf{E} \to V$.

Remarks. (i) There are r irreducible algebraic subvarieties $S_1, \cdots, S_r$ which carry the Chern classes $\sigma_1, \cdots, \sigma_r$ of $\mathbf{E} \to V$. Since $S_\rho \subset S_\tau$ if $\rho \leqq \tau$ (i.e., $\rho_j \leqq \tau_j$ for $j = 1, \cdots, r$), we have $S_1 \supset \cdots \supset S_r$.

(ii) Let $\xi \in \Gamma(\mathbf{E})$ be a section of $\mathbf{E}$ and (ξ) the zero locus of ξ. Then (ξ) carries the homology class σ_r. In words, the r^{th} Chern class is the divisor of a (generic) section of $\mathbf{E}$.

PROOF. We consider ξ as a linear function on Γ^* and choose the Γ_α such that Γ_{m+r-1} is the set of vectors in Γ^* annilated by ξ. From $\dim(\mathbf{E}_z^* \cap \Gamma_{r+m-1}) = 2r + m - 1 - \dim(\mathbf{E}_z^* + \Gamma_{r+m-1})$, we see that: $\dim(\mathbf{E}_z^* \cap \Gamma_{r+m-1}) \geqq r \Leftrightarrow \dim(\mathbf{E}_z^* + \Gamma_{r+m-1}) \leqq r + m - 1 \Leftrightarrow \mathbf{E}_z^* \subset \Gamma_{r+m-1} \Leftrightarrow \xi(z) = 0$; i.e.,

$$(4.8) \qquad \dim(\mathbf{E}_z^* \cap \Gamma_{r+m-1}) \geqq r \Longleftrightarrow \xi(z) = 0 \,.$$

On the other hand, if $\dim(\mathbf{E}_z^* \cap \Gamma_{r+m-1}) \geqq r$, then $\mathbf{E}_z^* \subset \Gamma_{r+m-1}$ and $\dim(\mathbf{E}_z^* + \Gamma_{j+m-1}) \leqq r + m - 1$ for $j = 1, \cdots, r$. Using this in $\dim(\mathbf{E}_z^* \cap \Gamma_{j+m-1}) = r + j + m - 1 - \dim(\mathbf{E}_z^* + \Gamma_{j+m-1})$, we find:

$$(4.9) \qquad \dim(\mathbf{E}_z^* \cap \Gamma_{r+m-1}) \geqq r \Longleftrightarrow \dim(\mathbf{E}_z^* \cap \Gamma_{j+m-1}) \geqq j \qquad \text{for } j = 1, \cdots, r \,.$$

Combining (4.8) and (4.9) gives our assertion.

(iii) The general rule is: Let $\xi_1, \cdots, \xi_r$ be r generic sections of $\mathbf{E}$. Then $\xi_1 \cdots \xi_{r-q+1}$ is a section of $\Lambda^{r-q+1}\mathbf{E}$ and $S_q \subset V$ is the zero locus of this section.

Proof. We may let Γ_{j+m-1} be given by $\xi_1 = 0, \cdots, \xi_{r-j+1} = 0$ where the ξ's are elements of the dual space of Γ^*. The Schubert symbol of S_q is $(\underbrace{m-1, \cdots, m-1}_{q}, \underbrace{m, \cdots, m}_{r-q})$. Since $\dim(\mathbf{E}_z^* \cap \Gamma_{j+m}) \geqq m$, the only non-trivial conditions are

$$\dim(\mathbf{E}_z^* \cap \Gamma_{j+m-1}) \geqq j \qquad \text{for } j = 1, \cdots, q \,.$$

This condition is equivalent to

$$(4.10) \qquad \dim(\mathbf{E}_z^* + \Gamma_{j+m-1}) \leqq m + r - 1 \qquad \text{for } j = 1, \cdots, q \,.$$

Since all $\Gamma_{j+m-1} \subset \Gamma_{q+m-1}$ $(j = 1, \cdots, q)$, we see that $(4.10)_q \Rightarrow (4.10)_j$ for $j = 1, \cdots, q$. Thus

$$(4.11) \qquad z \in S_q \Longleftrightarrow \dim(\mathbf{E}_z^* \cap \Gamma_{q+m-1}) \geqq q \ .$$

Now $(\xi_1 \wedge \cdots \wedge \xi_{r-q+1})(z) = 0$ in $\Lambda^{r-q+1}\mathbf{E}_z \Leftrightarrow$ there exist $f_1, \cdots, f_q \in \mathbf{E}_z^*$ with $f_1 \wedge \cdots \wedge f_q \neq 0$ and $\langle \xi_j(z), f_\alpha \rangle = 0$ for $j = 1, \cdots, r - q + 1$ and $\alpha = 1, \cdots, q$. Combining this with (4.11), we have

$$(4.12) \qquad z \in S_q \Longleftrightarrow \xi_1(z) \wedge \cdots \wedge \xi_{r-q+1}(z) = 0 \ ,$$

as required.

(iv) A few comments about (4.12) are relevant. The section $\xi_1 \wedge \cdots \wedge \xi_{r-q+1}$ of $\Lambda^{r-q+1}\mathbf{E}$ is *not* generic unless $q = r$ or $q = 1$. The reason is, of course, that $\xi_1 \wedge \cdots \wedge \xi_{r-q+1}(z)$ is the vector of an $r - q + 1$-plane in $\mathbf{E}_z$, and so satisfies the Cayley-Grassman relations.

A better way to think of S_q is to let $\mathbf{P}(r - q + 1) \to V$ be the bundle whose fibre $\mathbf{P}(r - q + 1)_z$ is the complex *Stiefel manifold* of $(r - q + 1)$-frames in $\mathbf{E}_z$. For $q = r$, we have $\mathbf{P}(1) = \mathbf{E} - \{0\}$ is the bundle of non-zero vectors. Now $\xi_1 \wedge \cdots \wedge \xi_{r-q+1}$ gives a rational cross-section of $\mathbf{P}(r - q + 1) \to V$ and $S_q \subset V$ are precisely the points where this cross-section is not defined.

(4.13) PROPOSITION. *Let $T \subset V$ be an irreducible subvariety of dimension q and $\tau \in H_{2q}(V, \mathbf{Z})$ the homology class of T. Then the intersection number $\tau \cdot \sigma_q > 0$.*

(4.14) *Remark.* In words, we may say that the Chern classes of an ample bundle are *numerically positive.* For the universal bundle over the grassmannian, the Chern classes are non-negative, but not positive except in the projective space case.

PROOF. The proof is based on the following general fact: Let $\{S_\lambda\}_{\lambda \in P_N}$ be a rational system of $(n - q)$-dimensional subvarieties $S_\lambda \subset V$ such that the generic S_λ is irreducible. Suppose that, given $z_0 \in V$ and an $(n - q)$-plane $\prod \subset \mathbf{T}_{z_0}(V)$, there is an S_λ passing through z_0 and with $\mathbf{T}_{z_0}(S_\lambda) = \prod$. Then the homology class carried by a generic S_λ is numerically positive.

Let now z_0 be the origin in a coordinate system $z^1, \cdots, z^n$ on V. We may locally trivialize $\mathbf{E}$ so that sections of $\mathbf{E} \to V$ in a neighborhood of z_0 are $\mathbf{C}^r$-valued holomorphic functions of $z^1, \cdots, z^n$. Since $\mathbf{E} \to V$ is ample (cf. (2.29)), we may find $\xi_1, \cdots, \xi_{r-q+1} \in \Gamma(\mathbf{E})$ such that

$$\xi_1 \equiv \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \xi_{r-q} \equiv \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \xi_{r-q+1} \equiv \begin{pmatrix} 1 \\ 0 \\ \vdots \\ z^1 \\ \vdots \\ 0 \end{pmatrix},$$

where the symbol "$\equiv$" means "modulo quadratic terms in $z^1, \cdots, z^n$". Then $(\xi_1 \wedge \cdots \wedge \xi_{r-q+1})(z_0) = 0$ and the tangent space to $\xi_1 \wedge \cdots \wedge \xi_{r-q+1} = 0$ at $z_0 = 0$ is given by $z^1 = 0, \cdots, z^q = 0$. Using the general intersection number principle above, we complete the argument for Proposition (4.13).

It remains to define the Chern (homology) classes of a general holomorphic bundle $\mathbf{E} \to V$. Let $\mathbf{L} \to V$ be an ample line bundle such that $\mathbf{E} \otimes \mathbf{L}$ is ample. Then the Chern classes $d_1, \cdots, d_r$ of $\mathbf{E} \otimes \mathbf{L}$ are defined, as is the Chern class e of $\mathbf{L}$, by using (4.7). To determine the Chern classes $c_1, \cdots, c_r$ of $\mathbf{E}$, we use the formulas in [15, page 66]. Thus writing $1 + c_1 t + \cdots + c_r t^r = \prod_{i=1}^r (1 + \psi_i t)$, $1 + d_1 t + \cdots + d_r t^r = \prod_{i=1}^r (1 + \gamma_i t)$, $\gamma_i = \psi_i + e$, we can determine $c_1, \cdots, c_r$ from $d_1, \cdots, d_r$ and e. The multiplication is in the sense of intersection of homology classes. To show that this is consistent, we must prove.

(4.15) PROPOSITION. *Let* $\mathbf{E}_1, \mathbf{E}_2$ *be two ample bundles over* V *such that there is a line bundle* $\mathbf{L}$ *with* $\mathbf{E}_2 = \mathbf{E}_1 \otimes \mathbf{L}$. *Then the Chern (homology) classes of* $\mathbf{E}_1, \mathbf{E}_2, \mathbf{L}$, *as defined by* (4.7), *are consistent with the* $\otimes$ *rule given above.*

This proposition will be proved in § 4. (c) below. In particular it follows that the Chern (homology) classes are algebraic cycles [6].

(b) *Chern classes as differential forms.* We now give the definition of the Chern classes using differential forms [5] and a brief discussion of the Weil homomorphism [12] and [3].

Let $\mathbf{E} \to V$ be a vector bundle with fibre $\mathbf{C}^r$; for this discussion, we only need a C^∞ bundle. Let $\mathbf{P} \to V$ be the principle bundle of C^∞ frames $f = (e_1, \cdots, e_r)$ for $\mathbf{E} \to V$ and choose a connexion θ for $\mathbf{P}$. Thus θ is a matrix-valued 1-form on $\mathbf{P}$ satisfying $\theta(hg) = g^{-1}\theta(h)g$ and where θ is the Maurer-Cartan form on each fibre of $\mathbf{P} \to V$. If we choose a trivialization $\mathbf{E} \mid U \cong U \times \mathbf{C}^r$ for an open set $U \subset V$, then each

$$e_\rho = \begin{pmatrix} \xi^1_\rho \\ \vdots \\ \xi^r_\rho \end{pmatrix}$$

is a column vector and $f = (e_1, \cdots, e_r)$ is a non-singular $r \times r$ matrix. Letting $z = (z^1, \cdots, z^n)$ be a coordinate system in U, we easily have that $\theta(z, f) = f^{-1}\theta(z)f + f^{-1}df$ where $\theta(z)$ is a matrix-valued 1-form on U.

The curvature Θ of θ is given by

$$\Theta = d\theta + \theta \wedge \theta \,. \tag{4.16}$$

In terms of the local expressions above,

$$\Theta(z, f) = f^{-1}\Theta(z)f \tag{4.17}$$

where $\Theta(z) = d\theta(z) + \theta(z) \wedge \theta(z)$.

Consider now a multilinear form $P(A_1, \cdots, A_q)$ where the A_j are $r \times r$ matrices. By linearity, if $A_j = (A^\rho_{j\sigma})$, then,

$$P(A_1, \cdots, A_q) = \sum_{\substack{\rho=(\rho_1,\cdots,\rho_q) \\ \sigma=(\sigma_1,\cdots,\sigma_q)}} c_{\rho\sigma} A^{\rho_1}_{1\sigma_1} \cdots A^{\rho_q}_{\sigma_q} \,. \tag{4.18}$$

We will call P *symmetric* if $P(\cdots A_i, \cdots, A_j \cdots) = P(\cdots A_j, \cdots, A_i \cdots)$ and *invariant* if $P(g^{-1}A_1g, \cdots, g^{-1}A_qg) = P(A_1, \cdots, A_q)$ for all $g \in GL(r, \mathbf{C})$. The condition that P be invariant is, in infinitesimal form

$$\sum_{j=1}^q P(\cdots, [B, A_j], \cdots) = 0 \,, \tag{4.19}$$

where B is an arbitrary $r \times r$ matrix. These symmetric invariant q-linear forms will be called *invariant polynomials* and form a vector space $\mathbf{I}_q$. By an obvious multiplication, $\mathbf{I}_p \cdot \mathbf{I}_q \subset \mathbf{I}_{q+p}$ and we let $\mathbf{I} = \sum_{q \geqq 0} \mathbf{I}_q$ be the graded ring of invariant polynomials.

To justify the terminology, we remark that a symmetric q-linear invariant form P gives a polynomial $P(A) = P(\underbrace{A, \cdots, A}_{q})$ in the matrix entries A^ρ_σ of A satisfying $P(g^{-1}Ag)$. Conversely, given such an invariant polynomial in the A^ρ_σ, we may recover the corresponding q-linear invariant form. For example, when $q = 2$,

$$P(A_1, A_2) = 1/2\{P(A_1 + A_2) - P(A_1) - P(A_2)\} \,.$$

We now define the *Weil homomorphism*

$$W \colon \mathbf{I} \longrightarrow H^*(V, \mathbf{C}) \tag{4.20}$$

by letting, for $P \in \mathbf{I}$,

$$W(P) = P(\Theta)\ . \tag{4.21}$$

By definition, $P(\Theta) = P(\Theta, \cdots, \Theta)$ is obtained by plugging in the curvature matrix Θ for A. This makes sense, since Θ is a matrix-valued form of degree 2. From (4.17), $P(\Theta) = P(\Theta(z, f)) = P(f^{-1}\Theta(z)f) = P(\Theta)$ so that $P(\Theta)$ is a C^∞ form on V. Also, $dP(\Theta) = \sum P(\cdots, d\Theta, \cdots) = \sum P(\cdots, [\Theta, \theta], \cdots) = 0$ (by (4.19)) since $d\Theta = \Theta \wedge \theta - \theta \wedge \Theta$ by differentiating (4.16).

The main fact is:

(4.22) $W: \mathbf{I} \to H^*(V, \mathbf{C})$ is an algebra homomorphism which is independent of the connexion θ.

Weil's proof of (4.22) is short enough to be given here. Let θ, θ_1 be connexions in $\mathbf{P} \to V$ so that $\theta_1 - \theta = \eta$ is a Hom $(\mathbf{E}, \mathbf{E})$-valued one-form. Setting $\theta_t = \theta + t\eta$, we get connexions θ_t with curvatures $\Theta_t = d\theta + td\eta + (\theta + t\eta) \wedge (\theta + t\eta)$. Then $\dot{\Theta}_t = d\eta + [\theta + t\eta, \eta] = D_t\eta = D_t\dot{\theta}_t$ so that

$$\begin{aligned} \dot{P}(\Theta_t, \cdots, \Theta_t) &= \sum P(\Theta_t, \cdots, \dot{\Theta}_t, \cdots, \Theta_t) \\ &= \sum P(\Theta_t, \cdots, D_t\dot{\theta}_t, \cdots, \Theta_t) \\ &= d\{\sum P(\Theta_t, \cdots, \dot{\theta}_t, \cdots, \Theta_t)\}\ . \end{aligned}$$

Thus

$$P(\Theta_1) - P(\Theta) = d\Big\{\sum \int_0^1 P(\Theta_t, \cdots, \dot{\theta}_t, \cdots, \Theta_t)dt\Big\}$$

so that $P(\Theta_1) = P(\Theta)$ in $H^*(V, \mathbf{C})$. This proves (4.22), since it is trivial that W is an algebra homomorphism.

We also observe that, on $\mathbf{P}$,

$$P(\Theta, \cdots, \Theta) = dP(\theta, \Theta, \cdots, \Theta)\ . \tag{4.23}$$

To define Chern classes, we define invariant polynomials $P_q(A)$ by setting

$$\det\Big(\lambda I_r + \frac{1}{2\pi i}A\Big) = \sum\nolimits_{q=0}^r (-1)^q P_q(A)\lambda^{r-q}\ . \tag{4.24}$$

(4.25) *Definition.* The q^{th} *Chern class* (in cohomology) $c_q \in H^{2q}(V, \mathbf{C})$ is given by $c_q = P_q(\Theta) = W(P_q)$.

The *total Chern class* $c(\mathbf{E})$ is defined by

$$c(\mathbf{E}) = \sum\nolimits_{q=0}^r c_q t^q\ . \tag{4.26}$$

Remarks. (1) Suppose that $\mathbf{E} \to V$ is a holomorphic vector bundle in which we have a hermitian metric. We let $\mathbf{B} \subset \mathbf{P}$ be the bundle of unitary frames $f = (e_1, \cdots, e_r)$. Then the curvature form Θ of the metric connexion is of type (1, 1) (cf. (2.1)) and on $\mathbf{B}$, $\Theta^\rho_\sigma + \bar{\Theta}^\sigma_\rho = 0$. Thus

$$ {}^t\overline{\left(\frac{\Theta}{2\pi i}\right)} = \frac{\Theta}{2\pi i} $$

and so

$$ \det \overline{\left(\lambda I + \frac{\Theta}{2\pi i}\right)} = \det\left(\bar{\lambda} I + \frac{{}^t\Theta}{2\pi i}\right) = \det\left(\lambda I + \frac{\Theta}{2\pi i}\right) $$

(if λ is real), so that we have

(4.27) The Chern classes (4.25) of a holomorphic vector bundle are real and of type (q, q) .

(2) Suppose that $\mathbf{E}_1 \to V$, $\mathbf{E}_2 \to V$ are (C^∞) vector bundles. Then we have the *duality theorem*

$$ c(\mathbf{E}_1 \oplus \mathbf{E}_2) = c(\mathbf{E}_1)c(\mathbf{E}_2) \ . \tag{4.28} $$

PROOF. If θ_1, θ_2 are connexions for $\mathbf{E}_1$, $\mathbf{E}_2$ and with curvatures Θ_1, Θ_2, then

$$ \theta = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix} $$

is a connexion for $\mathbf{E}_1 \oplus \mathbf{E}_2$ with curvature

$$ \Theta = \begin{pmatrix} \Theta_1 & 0 \\ 0 & \Theta_2 \end{pmatrix} . $$

Letting $\lambda = 1/t$, we have by (4.24) and (4.26) that

$$ \begin{aligned} c(\mathbf{E}) &= t^r \det\left(\lambda I_r + \frac{i}{2\pi}\Theta\right) \\ &= t^{r_1} \det\left(\lambda I_{r_1} + \frac{i}{2\pi}\Theta_1\right) t^{r_2} \det\left(\lambda I_{r_2} + \frac{i}{2\pi}\Theta_2\right) \\ &= c(\mathbf{E}_1)c(\mathbf{E}_2) \ , \end{aligned} $$

where r_1, r_2 are the fibre dimensions of $\mathbf{E}_1$, $\mathbf{E}_2$.

(3) Let $\mathbf{E} \to V$, $\mathbf{F} \to V$ be (C^∞) bundles and write $c(\mathbf{E}) = \prod_{\rho=1}^r (1 + \gamma_\rho t)$, $c(\mathbf{F}) = \prod_{\alpha=1}^s (1 + \gamma_\alpha t)$ where r, s are the fibre dimensions of $\mathbf{E}$, $\mathbf{F}$. Then we have

$$c(\mathbf{E} \otimes \mathbf{F}) = \prod_{\rho,\alpha} (1 + (\gamma_\rho + \gamma_\alpha)t) \,. \tag{4.29}$$

PROOF. Choosing connexions $\theta_{\mathbf{E}}, \theta_{\mathbf{F}}$ in $\mathbf{E}, \mathbf{F}$, then $\theta_{\mathbf{E}\otimes\mathbf{F}} = \theta_{\mathbf{E}} \otimes 1 + 1 \otimes \theta_{\mathbf{F}}$ gives a connexion in $\mathbf{E} \otimes \mathbf{F}$ with curvature $\Theta_{\mathbf{E}\otimes\mathbf{F}} = \Theta_{\mathbf{E}} \otimes 1 + 1 \otimes \Theta_{\mathbf{F}}$ (cf. (2.10)). Using this, (4.29) will follow from the algebraic facts,

(i) $\det(A \otimes B) = (\det A)^s (\det B)^r$ if A is $r \times r$ and B is $s \times s$; and

(ii) if A, B are general $u \times u$ matrices satisfying $AB - BA = 0$, if $\det(I + tA) = \sum_{q=1}^{u} P_q(A)t^q = \prod_{j=1}^{u}(1 + \gamma_j(A)t)$ and $\det(I + tB) = \prod_{j=1}^{u}(1 + \gamma_j(B)t)$, then

$$\det(I + t(A + B)) = \prod_{j=1}^{u} \{1 + (\gamma_j(A) + \gamma_j(B))t\} \,.$$

Of course (i) is standard, and (ii) follows by simultaneously diagonalizing A and B. We apply (ii) letting

$$A = \frac{i}{2\pi}\Theta_{\mathbf{E}} \otimes 1,\ B = 1 \otimes \frac{i}{2\pi}\Theta_{\mathbf{F}}$$

(then $[A, B] = 0$) and $u = rs$. Then

$$\begin{aligned} c(\mathbf{E} \otimes \mathbf{F}) &= \det\Big(I + t\Big(\frac{i}{2\pi}\Theta_{\mathbf{E}} \otimes 1 + 1 \otimes \frac{i}{2\pi}\Theta_{\mathbf{F}}\Big)\Big) \\ &= \prod_{j=1}^{rs} \{1 + t(\gamma_j(A) + \gamma_j(B))\} \\ &= \prod_{\rho\ \alpha=1}^{r,s} \Big\{1 + t\Big(\gamma_\rho\Big(\frac{i}{2\pi}\Theta_{\mathbf{E}}\Big) + \gamma_\alpha\Big(\frac{i}{2\pi}\Theta_{\mathbf{F}}\Big)\Big)\Big\} \,, \end{aligned}$$

which proves (4.29).

(c) *Proof of the equivalence of definitions*. Let now V be an algebraic manifold and $\mathbf{E} \to V$ a holomorphic vector bundle. Then we have defined the Chern (homology) classes $\sigma_q \in H_{2n-2q}(V, \mathbf{Z})$ (cf. § 4 (a)) and the Chern (cohomology) classes $c_q \in H^{2q}(V, \mathbf{R})$.

(4.30) THEOREM. *σ_q is the Poincaré dual of c_q.*

Before giving the proof, we make some preliminary remarks. If we can prove (4.30) in case $\mathbf{E} \to V$ is an *ample* bundle, then (4.30) will be true for all bundles. This follows from the definition of σ_q for a general holomorphic bundle $\mathbf{E} \to V$, together with (4.29). At the same time, if Theorem (4.30) is true for ample bundles, then by (4.29) it follows that the definition of σ_q for general $\mathbf{E} \to V$ is consistent, which proves Proposition (4.15). So it will suffice to prove Theorem (4.30) for ample bundles, and we now give the argument in this case.

What has to be shown is this: Let $S_q \subset V$ be a general $2n - 2q$ dimensional subvariety which carries the homology class $\sigma_q \in H_{2n-2q}(V, \mathbf{Z})$.

We let Γ be a $2q$ cycle which meets S_q simply at a finite number of points. Then we need to have

$$\sigma_q \cdot \Gamma = \langle c_q, \Gamma \rangle \,, \tag{4.31}$$

where the left hand side of (4.31) is the intersection number.

PROOF. We assume that Γ is an algebraic submanifold, the general case is based on the same ideas. Let $\xi_1, \cdots, \xi_{r-q+1}$ be general holomorphic sections of $\mathbf{E} \to V$. Then S_q is given by $\xi_1 \wedge \cdots \wedge \xi_{r-q+1} = 0$ and we may assume that $\xi_1 \wedge \cdots \wedge \xi_{r-q} \neq 0$ on $S_q \cdot \Gamma$ (since $\xi_1 \wedge \cdots \wedge \xi_{r-q}$ defines a $2n - 2q - 2$ dimensional subvariety of V). Then $\xi_1 \wedge \cdots \wedge \xi_{r-q} \neq 0$ on Γ and so, over Γ, we have

$$0 \longrightarrow \mathbf{S} \longrightarrow \mathbf{E} \longrightarrow \mathbf{Q} \longrightarrow 0 \,, \tag{4.32}$$

where $\mathbf{S}$ is the trivial sub-bundle generated by $\xi_1, \cdots, \xi_{r-q}$. From $c(\mathbf{S})c(\mathbf{Q}) = c(\mathbf{E})$ and $c(\mathbf{S}) = 1$, it follows that $\langle c_q, \Gamma \rangle = \langle \omega, \Gamma \rangle$ where ω is the q^{th} Chern class of $\mathbf{Q} \to \Gamma$. Now ξ_{r-q+1} gives a section ξ of $\mathbf{Q} \to \Gamma$ such that $\sigma_q \cdot \Gamma$ is just the zero locus of ξ. In conclusion:

(4.33) To prove (4.31), it will suffice to take $\mathbf{E} \to V$ a holomorphic bundle with fibre $\mathbf{C}^n$ ($n = \dim V$), $q = n$, S_q the divisor of a general section ξ of $\mathbf{E}$, and ω the n^{th} Chern class of $\mathbf{E} \to V$.

There are two steps in the proof of (4.33). One is the geometric notion of *transgression*, due to Chern [7], and the other consists of applying certain formulas in the unitary geometry of $\mathbf{E} \to V$. These formulas have an independent interest.

Suppose then that ξ vanishes at z_0 (the case of several zeroes is the same) and let $S(\mathbf{E}) \to V$ be the bundle of unit vectors in $\mathbf{E} \to V$. Then $\pi^*(\mathbf{E}) \to S(\mathbf{E})$ has a non-vanishing section and so, by (4.28), $c_n(\pi^*(\mathbf{E})) = 0$. Thus $\pi^*\omega = d\psi$ where ψ is a $2n - 1$ form on $S(\mathbf{E})$. Letting $S_z \subset S(\mathbf{E})$ be the unit sphere in $\mathbf{E}_z$, we claim that $\int_{S_z} \psi$ is independent of z. (*Proof.* If γ is a curve joining z_1 and z_2 and $T = \bigcup_{z \in \gamma} S_z$, then $\int_{S_{z_1}} \psi - \int_{S_{z_2}} \psi = \int_{\partial T} \psi = \int_T \pi^*\omega = 0$ since ω is "horizontal".)

Let $\alpha = -\int_{S_z} \psi$; we claim that

(4.34) $\int_V \omega = \alpha \cdot (\text{number of zeroes of } \xi)$.

For the proof we let Δ be a spherical neighborhood of z_0 such that $\mathbf{E} \mid \Delta \cong \Delta \times \mathbf{C}^r$. Then $\xi(z) = (z, \zeta(z))$ for $z \in \Delta$, where ζ has an isolated zero at z_0. Using $\sim$ for "approximately equal to", we have:

$$\int_V \omega \sim \int_{V-\Delta} \omega = \int_{V-\Delta} \xi^* \pi^* \omega = \int_{V-\Delta} d\xi^* \psi = -\int_{\Delta-\Delta} \xi^* \psi = -\int_{\zeta(\partial\Delta)} \psi ,$$

where this last integral is taken over the unit sphere S_{z_0} in $\mathbf{E}_{z_0}$ and ζ is considered as a mapping of $\partial\Delta$ into S_{z_0}. Thus

$$\int_V \omega \sim \left(-\int_{S_{z_0}} \psi\right) (\deg \zeta)$$

and, letting Δ shrink to z_0, we get (4.34).

To prove (4.33), it will thus suffice to show that:

(4.35) $-\psi \mid S_z$ is the normalized volume element.

We shall prove (4.35) from the structure equations in the hermitian geometry of $\mathbf{E} \to V$.

Let then $\mathbf{E} \to V$ be an hermitian vector bundle and suppose that we have an exact sequence

$$0 \longrightarrow \mathbf{S} \longrightarrow \mathbf{E} \longrightarrow \mathbf{Q} \longrightarrow 0 \tag{4.36}$$

of holomorphic bundles. Then $\mathbf{S}$ and $\mathbf{Q}$ have each induced metrics with their respective hermitian geometries; the relevant formulas are discussed in § 1 (d). We let Θ be the curvature in $\mathbf{E}$ and $\hat{\Theta}$ the curvature in $\hat{\mathbf{E}} = \mathbf{S} \oplus \mathbf{Q}$. As C^∞ bundles, $\mathbf{E} \cong \hat{\mathbf{E}}$ so that Θ and $\hat{\Theta}$ are both curvatures in the *same* C^∞ complex vector bundle. By Weil's theorem (4.22), if $P \in \mathbf{I}_q$ is any invariant polynomial (cf. §4 (b)), we have

$$P(\Theta) - P(\hat{\Theta}) = d\varphi , \tag{4.37}$$

where φ is given in the proof of (4.22). We shall prove Bott and Chern's refinement [3] of Weil's theorem by showing that

$$P(\Theta) - P(\hat{\Theta}) = \partial\bar{\partial}\psi , \tag{4.38}$$

where ψ is a form of type $(q-1, q-1)$ which may be given explicitly. Our proof is a continuation of Weil's proof of (4.37) and is somewhat different from the proof in [3].

Let then $\mathbf{B} \to V$ be the principal bundle of unitary frames $f = (e_1, \cdots, e_r)$ for $\mathbf{E} \to V$ such that $e_1, \cdots, e_s$ is a frame for $\mathbf{S} \to V$ (cf. § 1 (e)). We write the connexion form for $\mathbf{E}$ as

$$\theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$$

where θ_{11} is an $s \times s$ matrix giving the induced connexion in $\mathbf{S}$, θ_{22} gives the induced connexion in $\mathbf{Q}$, $\theta_{21} \in A^{1,0}(\mathrm{Hom}(\mathbf{S}, \mathbf{Q}))$ gives the second fundamental form of $\mathbf{S}$ in $\mathbf{E}$, and $\theta_{12} = -{}^t\bar{\theta}_{21} \in A^{0,1}(\mathrm{Hom}(\mathbf{Q},\mathbf{S}))$. Then

$$\Theta = d\theta + \theta \wedge \theta \; , \tag{4.39}$$

and

$$\hat{\Theta} = d\hat{\theta} + \hat{\theta} \wedge \hat{\theta} \tag{4.40}$$

where

$$\hat{\theta} = \begin{pmatrix} \theta_{11} & 0 \\ 0 & \theta_{22} \end{pmatrix}$$

is the metric connexion in $\mathbf{S} \oplus \mathbf{Q}$.

Write $\hat{\theta} = \theta + \varphi$ where

$$\varphi = \begin{pmatrix} 0 & -\theta_{12} \\ -\theta_{21} & 0 \end{pmatrix}$$

and let

$$\varphi' = \begin{pmatrix} 0 & 0 \\ -\theta_{21} & 0 \end{pmatrix}$$

be the $(1, 0)$ part of φ,

$$\varphi'' = \begin{pmatrix} 0 & -\theta_{12} \\ 0 & 0 \end{pmatrix}$$

the $(0, 1)$ part. Then $\varphi = \varphi' + \varphi''$ and

$$\begin{aligned} D\varphi &= d\varphi + [\theta, \varphi] \\ &= \begin{pmatrix} 0 & -d\theta_{12} \\ -d\theta_{21} & 0 \end{pmatrix} + \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} 0 & -\theta_{12} \\ -\theta_{21} & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & -\theta_{12} \\ -\theta_{21} & 0 \end{pmatrix} \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\Theta_{12} \\ -\Theta_{21} & 0 \end{pmatrix} + 2\begin{pmatrix} -\theta_{12}\theta_{21} & 0 \\ 0 & -\theta_{21}\theta_{12} \end{pmatrix} . \end{aligned}$$

We collect this equation and the ones resulting from it by using decomposition into type as follows:

$$D\varphi = \begin{pmatrix} 0 & -\Theta_{12} \\ -\Theta_{21} & 0 \end{pmatrix} + 2\begin{pmatrix} -\theta_{12}\theta_{21} & 0 \\ 0 & -\theta_{21}\theta_{12} \end{pmatrix}, \tag{4.41}$$

$$\begin{cases} D'\varphi = D'\varphi'' = \begin{pmatrix} 0 & -\Theta_{12} \\ 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 \\ 0 & -\theta_{21}\theta_{12} \end{pmatrix} \\ D''\varphi = D''\varphi' = \begin{pmatrix} 0 & 0 \\ -\Theta_{21} & 0 \end{pmatrix} + 2\begin{pmatrix} -\theta_{12}\theta_{21} & 0 \\ 0 & 0 \end{pmatrix} \\ D'\varphi' = D''\varphi'' = 0\,. \end{cases} \tag{4.42}$$

Let $\theta_t = \theta + t\varphi$ be the linear 1-parameter family of connexions with $\theta_0 = \theta$, $\theta_1 = \hat{\theta}$. Then we have

$$\begin{cases} \Theta_t = \Theta + tD\varphi + \dfrac{t^2}{2}[\varphi, \varphi] \\ \dot{\Theta}_t = D\varphi + t[\varphi, \varphi]\,. \end{cases} \tag{4.43}$$

Here, as opposed to the proof of (4.22), D is always taken with respect to the connexion θ. We claim that (cf. the proof of (4.22)

$$\dot{P}(\Theta_t, \cdots, \Theta_t) = d\{\textstyle\sum P(\Theta_t, \cdot, \varphi, \cdot, \Theta_t)\}\,, \tag{4.44}$$

where $\dot{P} = dP/dt$. In fact, $\dot{P}(\Theta_t, \cdot, \Theta_t) = \sum P(\Theta_t, \cdot, \dot{\Theta}_t, \cdots, \Theta_t) = \sum P(\Theta_t, \cdot, D\varphi + t[\varphi, \varphi], \cdot, \Theta_t)$; while

$$\begin{aligned} &P(\Theta_t, \cdot, D\varphi + t[\varphi, \varphi], \cdot, \Theta_t) \\ &\quad = -\textstyle\sum P(\cdot, t[\varphi, \Theta_t], \cdot, \varphi, \cdot) + P(\Theta_t, \cdot, D\varphi, \cdot, \Theta_t) \\ &\quad = \textstyle\sum P(\cdot, D\Theta_t, \varphi, \cdot) + P(\Theta_t, D\varphi, \cdot, \Theta_t) \\ &\quad = dP(\Theta_t, \cdot, \varphi, \cdot, \Theta_t) \end{aligned}$$

since $D\Theta_t = d\Theta_t + [\theta, \Theta_t] = d\Theta_t + [\theta_t, \Theta_t] - [\theta_t - \theta, \Theta_t] = -t[\varphi, \Theta_t]$ because $D_t\Theta_t = 0$.

Taking types in (4.44), we obtain

$$\begin{aligned} \dot{P}(\Theta_t) = \partial\{\textstyle\sum P(\Theta_t, \cdot, \varphi'', \cdot, \Theta_t)\} \\ + \bar{\partial}\{\textstyle\sum P(\Theta_t, \varphi', \cdot, \Theta_t)\}\,. \end{aligned} \tag{4.45}$$

Let $\xi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ where 1 is an $s \times s$ unit matrix (ξ is the orthogonal projection on S). We set

$$Q_t = \frac{2}{(1-t)}\{\textstyle\sum P(\Theta_t, \cdot, \xi, \cdot, \Theta_t) - \sum P(\hat{\Theta}, \cdot, \xi, \cdot, \hat{\Theta})\} \tag{4.46}$$

and assert that

$$(4.47)\qquad \begin{cases} Q_t \text{ is a smooth family of } C^\infty \text{ forms on } V \text{ and} \\ \bar\partial Q_t = 2\{\sum P(\Theta_t, \cdot, \varphi'', \cdot, \Theta_t)\} \\ \partial Q_t = -2\{\sum P(\Theta_t, \cdot, \varphi', \cdot, \Theta_t)\}\,. \end{cases}$$

Combining (4.47) and (4.45) gives $\dot P(\Theta_t) = \partial\bar\partial Q_t$ or

$$(4.48)\qquad P(\hat\Theta) - P(\Theta) = \partial\bar\partial\left\{\int_0^1 Q_t\,dt\right\}.$$

Proof of (4.47). We have

$$D\xi = [\theta, \xi] = \begin{pmatrix}\theta_{11} & \theta_{12}\\ \theta_{21} & \theta_{22}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} - \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\theta_{11} & \theta_{12}\\ \theta_{21} & \theta_{22}\end{pmatrix} = \begin{pmatrix}0 & -\theta_{12}\\ \theta_{21} & 0\end{pmatrix},$$

which gives

$$\begin{cases} D'\xi = -\varphi' \\ D''\xi = \varphi''\,. \end{cases}$$

Now

$$\begin{aligned}
&\bar\partial P(\Theta_t, \cdot, \xi, \cdot, \Theta_t)\\
&\quad = \sum P(\cdot, D''\Theta_t, \cdot, \xi, \cdot) + P(\Theta_t, \cdot, D''\xi, \cdot, \Theta_t)\\
&\quad = -\sum P\big(\cdot, t[\varphi'', \Theta_t], \cdot, \xi, \cdot\big) + P(\Theta_t, \cdot, \varphi'', \cdot, \Theta_t)\\
&\quad = P\big(\Theta_t, \cdot, t[\varphi'', \xi], \cdot, \Theta_t\big) + P(\Theta_t, \cdot, \varphi'', \cdot, \Theta_t)\\
&\quad = (1-t)P(\Theta_t, \cdot, \varphi'', \cdot, \Theta_t)
\end{aligned}$$

since $[\varphi'', \xi] = -\varphi''$. This gives

$$(4.49)\qquad \bar\partial\frac{1}{1-t}\{P(\Theta_t, \cdot, \xi, \cdot, \Theta_t)\} = P(\Theta_t, \cdot, \varphi'', \cdot, \Theta_t)\,.$$

Since $P(\Theta_t, \cdot, \xi, \cdot, \Theta_t)$ is a polynomial in t whose coefficients are C^∞ forms on V, and since $P(\Theta_1, \cdot, \xi, \cdot, \Theta_1) = P(\hat\Theta, \cdot, \xi, \cdot, \hat\Theta)$ to prove (4.47) (for $\bar\partial$) from (4.49) we must show that $\bar\partial P(\hat\Theta, \cdot, \xi, \hat\Theta) = 0$. Letting $\hat D$ be covariant differentiation with respect to $\hat\theta$, we have $\hat D'' = \bar\partial$, $\hat D''\hat\Theta = 0$, $\hat D''\xi = 0$. Thus

$$\begin{aligned}
&\bar\partial P(\hat\Theta, \cdot, \xi, \cdot, \hat\Theta)\\
&\quad = \sum P(\cdot, \hat D''\hat\Theta, \cdot, \xi, \cdot) + P(\hat\Theta, \cdot, \hat D''\xi, \cdot, \hat\Theta)\\
&\quad = 0\,.
\end{aligned}$$

This proves (4.47) for $\bar\partial$; the proof for ∂ is similar.

PROOF OF (4.35.) What we have to show is this.

(4.50) Let $\mathbf{E} \to V$ be an hermitian vector bundle with fiber $\mathbf{C}^n$

and with a non-vanishing section σ. Let ω be the form of type (n, n) on V giving $c_n(\mathbf{E})$ computed from the curvature in $\mathbf{E}$. Then $\omega = d\psi$, where ψ gives the negative volume element on the unit spheres $S_z \subset \mathbf{E}_z$.

To prove (4.35) from (4.50), we look at $\pi^*\mathbf{E} \to \hat{\mathbf{E}}$ where $\hat{\mathbf{E}} = \mathbf{E} - \{0\}$ is the bundle of non-zero vectors in $\mathbf{E}$. There is a canonical section σ of $\pi^*\mathbf{E} \to \hat{\mathbf{E}}$, and (4.50) applied to this situation gives (4.35).

PROOF OF (4.50). Let $\mathbf{S} \subset \mathbf{E}$ be the bundle generated by σ so that we have an exact sequence (4.36) with curvatures $\Theta_{\mathbf{E}}$, $\Theta_{\mathbf{S}}$, $\Theta_{\mathbf{Q}}$. For a $q \times q$ matrix Φ, we set $c(\Phi) = [(1/2\pi i)]^q \det \Phi$; then, by (4.38),

$$c(\Theta_{\mathbf{E}}) = c(\Theta_{\mathbf{S}})c(\Theta_{\mathbf{Q}}) + \partial\bar{\partial}\gamma \ , \tag{4.51}$$

and

$$c(\Theta_{\mathbf{S}}) = d\eta \tag{4.52}$$

since $\mathbf{S}$ is trivial. Combining (4.51) and (4.52) gives

$$c(\Theta_{\mathbf{E}}) = d\{\eta c(\Theta_{\mathbf{Q}}) + \bar{\partial}\gamma\} = d\psi \tag{4.53}$$

where $\psi = \eta c(\Theta_{\mathbf{Q}}) + \bar{\partial}\gamma$. We want to compute $\psi \mid S_z$, and, in so doing, we may ignore all terms $\Theta_{\rho\sigma}$, which are horizontal forms. Using the frames in the proof of (4.38), and using "$\equiv$" to mean modulo terms $\Theta_{\rho\sigma}$", we need to show that

$$\psi \equiv -\left(\frac{1}{2\pi i}\right)^n \frac{1}{(n-1)!}\{\theta_{11}\theta_{21}\theta_{12} \cdots \theta_{n1}\theta_{1n}\} \ . \tag{4.54}$$

Now, by (4.44)

$$\bar{\partial}\gamma = \int_0^1 \{\textstyle\sum P(\Theta_t, \cdot, \varphi, \cdot, \Theta_t)\} dt \ , \tag{4.54}$$

where P is the invariant polynomial corresponding to $c(\Theta)$. Using (4.41) and (4.43),

$$\Theta_t \equiv tD\varphi + \frac{t^2}{2}[\varphi, \varphi] \equiv \left(-t + \frac{t^2}{2}\right)[\varphi, \varphi] \ .$$

Thus

$$\begin{aligned} \textstyle\sum P(\Theta_t, \cdot, \varphi, \cdot, \Theta_t) &\equiv \left(-t + \frac{t^2}{2}\right)^{n-1} \textstyle\sum P(\cdot, [\varphi, \varphi], \cdot, \varphi, \cdot) \\ &\equiv \left(t - \frac{t^2}{2}\right)^{n-1} P(\varphi, \cdot, [\varphi, [\varphi, \varphi]], \cdot, \varphi) \\ &= 0 \ . \end{aligned}$$

since $[\varphi, [\varphi, \varphi]] = 0$. Thus we obtain the equation:

$$\psi \equiv \eta c(\Theta_Q) . \tag{4.55}$$

Now, by (4.43) at $t = 1$,

$$\Theta_1 = \begin{pmatrix} \Theta_S & 0 \\ 0 & \Theta_Q \end{pmatrix} \equiv \begin{pmatrix} \theta_{12}\theta_{21} & 0 \\ 0 & \theta_{21}\theta_{12} \end{pmatrix} .$$

Also, θ_{11} is defined on V (using σ) and $d\theta_{11} \equiv -\theta_{12}\theta_{21}$ so that $\eta \equiv -(1/2\pi i)\theta_{11}$ and

$$\eta c(\Theta_Q) \equiv -\left(\frac{1}{2\pi i}\right)^n \theta_{11} \det(\theta_{21}\theta_{12}) . \tag{4.56}$$

But $\det(\theta_{21}\theta_{12}) = (n-1)!\ \theta_{21}\theta_{12}\theta_{31}\theta_{13} \cdots \theta_{n1}\theta_{1n}$ which, using (4.56) and (4.55), gives (4.54).

Remarks. (a) Let "$\sim$" denote "congruent modulo commutators" (so that, e.g., $AB \sim BA$). Referring to (4.43) we have

$$\dot{\Theta}_t \sim D\varphi , \tag{4.57}$$

$$\dot{\Theta}_t \sim D'D''\left(\frac{\dot{\xi}}{2}\right) . \tag{4.58}$$

(*Proof.* $D\varphi = (D' + D'')\varphi = (D' + D'')(D'' - D')\xi = (D'D'' - D''D')\xi$. Now $D^2\xi = (D'D'' + D''D')\xi = [\Theta, \xi] \sim 0$ so that $-D''D'\xi \sim D'D''\xi$.) Equation (4.57) which holds for general vector bundles, is the basis for Weil's theorem (4.37); it says that, modulo commutators (which essentially give zero in an invariant polynomial), the variation in Θ_t is an exact form. Similarly, (4.58), which holds for hermitian vector bundles, is the basis for (4.38).

(b) Referring to (4.38), suppose that $P(\Theta)$ is an invariant polynomial of degree q. Then we claim that, in (4.38),

$$\begin{aligned} \psi = q \textstyle\sum_{s+t>0} \left(\frac{1}{2}\right)^{s-1}\left(\frac{1}{2s+t}\right)\binom{q-1}{s+2}\binom{s+t}{s} \\ \times P(\hat{\Theta}; \underbrace{\tilde{\Theta}}_{t}; \underbrace{[\varphi, \varphi]}_{s}; \xi) \end{aligned} \tag{4.59}$$

where

$$P(\hat{\Theta}; \underbrace{\tilde{\Theta}}_{t}; \underbrace{[\varphi, \varphi]}_{s}; \xi) = P(\hat{\Theta}, \cdots, \hat{\Theta}, \underbrace{\tilde{\Theta}, \cdots, \tilde{\Theta}}_{t}, \underbrace{[\varphi, \varphi], \cdots, [\varphi, \varphi]}_{s}, \xi)$$

and

$$\widetilde{\Theta} = \begin{pmatrix} 0 & \Theta_{12} \\ \Theta_{21} & 0 \end{pmatrix}.$$

Proof. By (4.46), $Q_t = 2q/(1-t)\{P(\Theta_t, \cdots, \Theta_t, \xi) - P(\widehat{\Theta}, \cdots, \widehat{\Theta}, \xi)\}$. Now $\Theta_t = \widehat{\Theta} + \Phi$ where $\Phi = -(1-t)D\varphi - \{(1-t^2)/2\}[\varphi, \varphi] = (1-t)\{\widetilde{\Theta} + (1-t/2)[\varphi, \varphi]\}$, by (4.41) since $D\varphi = -\widetilde{\Theta} - [\varphi, \varphi]$. Thus

$$\begin{aligned} Q_t &= \frac{2q}{1-t}\{P(\widehat{\Theta} + \Phi, \cdots, \widehat{\Theta} + \Phi, \xi) - P(\widehat{\Theta}, \cdots, \widehat{\Theta}, \xi)\} \\ &= \frac{2q}{1-t}\left\{\sum\nolimits_{l>0}\binom{q-1}{l}P(\widehat{\Theta}, \cdot, \widehat{\Theta}, \Phi, \xi)\right\} \\ &= 2q\left\{\sum\nolimits_{l>0}\left(\frac{1}{2}\right)^m\binom{q-1}{l}\binom{l}{m}(1-t)^{l+m-1}P(\underbrace{\widehat{\Theta}; \widetilde{\Theta}}_{l-m}; \underbrace{[\varphi, \varphi]}_{m}; \xi)\right\} \end{aligned}$$

and

$$\begin{aligned} \int_0^1 Q_t &= 2q\left\{\sum\nolimits_{l>0}\left(\frac{1}{2}\right)^m\left(\frac{1}{l+m}\right)\binom{q-1}{l}\binom{l}{m}P(\underbrace{\widehat{\Theta}; \widetilde{\Theta}}_{l-m}; \underbrace{[\varphi, \varphi]}_{m}; \xi)\right\} \\ &= (4.59)\,. \end{aligned}$$

When $q = 1$, $P(\Theta)$ is a multiple of Trace Θ and $\operatorname{Tr}\Theta = \operatorname{Tr}\widehat{\Theta}$ since $\operatorname{Tr}[\varphi, \varphi] = 0 = \operatorname{Tr}\widetilde{\Theta}$. When $q = 2$, there are only two terms ($t = 0, s = 1$ and $t = 1, s = 0$) in (4.59) and $\psi = P([\varphi, \varphi], \xi) + 4P(\widetilde{\Theta}, \xi)$. If $r = 2$ and $P(\Theta) = \det\Theta$, then $\psi = P([\varphi, \varphi], \xi) = \theta_{21}\theta_{12}$, which means that

$$\text{(4.60)} \quad \det\begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix} - \det\begin{pmatrix} \Theta_{11} - \theta_{21}\theta_{12} & 0 \\ 0 & \Theta_{22} + \theta_{21}\theta_{12} \end{pmatrix} = \partial\bar{\partial}(\theta_{21}\theta_{12})\,,$$

an equation which may be verified directly.

(c) Suppose now that, in the exact sequence (4.36), $\mathbf{S}$ has fibre dimension one, $\mathbf{E}$ has fibre $\mathbf{C}^r$, and $P(\Theta) = \det\Theta$. Then ψ given by (4.59) may be written as

$$\psi = \psi_0 + \cdots + \psi_{r-1}$$

where ψ_q is homogeneous of degree q in terms $\theta_{\alpha 1}\theta_{1\beta}$. Recall that, using natural frames,

$$\varphi = -\begin{pmatrix} 0 & \theta_{12} \cdots \theta_{1r} \\ \theta_{21} & 0 \cdots 0 \\ \vdots & \vdots \quad\quad \vdots \\ \theta_{r1} & 0 \cdots 0 \end{pmatrix}$$

and

$$[\varphi, \varphi] = 2\begin{pmatrix} \sum_{\alpha=2}^{r} \theta_{1\alpha}\theta_{\alpha 1} & 0 & \cdots & 0 \\ 0 & \theta_{21}\theta_{12} & \cdots & \theta_{21}\theta_{1r} \\ \vdots & \vdots & & \vdots \\ 0 & \theta_{r1}\theta_{12} & \cdots & \theta_{r1}\theta_{1r} \end{pmatrix}.$$

Now $P(A_1, \cdots, A_r)$ is the polarized determinant function and, since

$$\widetilde{\Theta} = \begin{pmatrix} 0 & \Theta_{12} & \cdots & \Theta_{1r} \\ \Theta_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \Theta_{r1} & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad \xi = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & & \vdots \\ 0 & & \cdots & 0 \end{pmatrix}$$

we see that $P(\widetilde{\Theta}; \hat{\Theta}; [\varphi, \varphi]; \xi) = 0$ if $t > 0$. Thus

$$\psi = \sum_{s>0} \binom{r}{s}\left(\frac{1}{2}\right)^s \binom{r-1}{s} P(\hat{\Theta}; \widetilde{\Theta}; \underbrace{[\varphi, \varphi]}_{s}; \xi) .$$

Let

$$\gamma = \begin{pmatrix} \theta_{21}\theta_{12} & \cdots & \theta_{21}\theta_{1r} \\ \vdots & & \vdots \\ \theta_{r1}\theta_{12} & \cdots & \theta_{r1}\theta_{1r} \end{pmatrix} \quad \text{and} \quad \eta = \begin{pmatrix} \Theta_{22} & \cdots & \Theta_{2r} \\ & & \vdots \\ \Theta_{r2} & & \Theta_{rr} \end{pmatrix}$$

$(= \Theta_{\mathrm{Q}})$. Then $P(\hat{\Theta}; \underbrace{[\varphi, \varphi]}_{s}; \xi) = 2^s R(\underbrace{\eta + \gamma}_{r-s-1}, \underbrace{\gamma}_{s})$, where R is the polynomial obtained by polarizing the determinant function on $(r-1) \times (r-1)$ matrices. It is then clear that $\psi = \psi_0 + \cdots + \psi_{r-1}$ where $\psi_q = \lambda_q R(\eta, \underbrace{\gamma}_{q})$, the λ_q being suitable positive constants. This gives:

(4.61) In case S has fibre dimension one, we have

$$c_r(\Theta_{\mathrm{E}}) - c_1(\Theta_{\mathrm{S}})c_{r-1}(\Theta_{\mathrm{Q}}) = \partial\bar{\partial}(\psi_1 + \cdots + \psi_{r-1})$$

where

$$\psi_q = \frac{c_q}{(2\pi i)^r} \sum \operatorname{sgn} \pi \theta_{\alpha_1 1}\theta_{1\pi(\alpha_1)} \cdots \theta_{\alpha_q 1}\theta_{1\pi(\alpha_q)} \times \Theta_{\beta_1\pi(\beta_1)} \cdots \Theta_{\beta_{r-q-1}\pi(\beta_{r-q-1})} \tag{4.62}$$

and where the summation in (4.62) is over permutations π of $(2 \cdots r)$ into disjoint sets $(\alpha_1 \cdots \alpha_q)$, $(\beta_1 \cdots \beta_{r-q-1})$ of increasing indices. In particular,

$$\psi_{r-1} = \frac{c_r}{(2\pi i)^r}\theta_{21}\theta_{12} \cdots \theta_{r1}\theta_{1r} . \tag{4.63}$$

5. Numerically and arithmetically positive bundles

(a) *Positive forms and cohomology classes.* Let V be a complex manifold and ω a differential form of type (q, q) in an open set $U \subset V$. We say that ω is *positive*, written $\omega > 0$, if $\omega \neq 0$ and if there exist $(q, 0)$ forms φ_α such that

$$\omega = (-1)^{\frac{q(q-1)}{2}}\left(\frac{\sqrt{-1}}{2}\right)^q \{\textstyle\sum_\alpha \varphi_\alpha \wedge \bar{\varphi}_\alpha\} . \tag{5.1}$$

The signs are such that, in $\mathbf{C}^q$,

$$(-1)^{\frac{q(q-1)}{2}}\left(\frac{\sqrt{-1}}{2}\right)^q (dz^1 \wedge \cdots \wedge dz^q \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^q)$$
$$= dx^1 \wedge dy^1 \wedge \cdots \wedge dx^q \wedge dy^q .$$

The symbol $\omega \geqq 0$ has the obvious meaning. If $\omega \geqq 0$, $\varphi \geqq 0$, then $\omega \wedge \varphi \geqq 0$ and $\omega + \varphi \geqq 0$ if $\deg \omega = \varphi$.

We let $\mathbf{A}^{q,q}$ be the vector space of C^∞ (q, q) forms on V and $\mathbf{A}^* = \sum \mathbf{A}^{q,q}$. A form $\omega \in \mathbf{A}^{q,q}$ is positive if locally $\omega > 0$. The space $\mathbf{P}^q \subset \mathbf{A}^{q,q}$ of positive (q, q) forms is a convex cone.

If $\omega \geqq 0$ in an open set $U \subset V$ and $S \subset U$ is a q-dimensional analytic set, then $\int_S \omega \geqq 0$. If $Z \subset U$ is a subvariety, then $\omega \,|\, Z \geqq 0$.

Consider now $H^{q,q}(V)$ and let ω be a cohomology class which is real; $\omega = \bar{\omega}$. Then we write $\omega > 0$ if $\langle \omega, \sigma \rangle > 0$ for all $\sigma \in H_{2q}(V, \mathbf{Z})$ where σ is the cycle carried by an irreducible subvariety of dimension q lying in V. Obviously we have:

(5.2) Let $\omega \in H^{q,q}(V)$ and suppose $\omega \in \mathbf{A}^{q,q}$ is a closed (q, q) form representing the cohomology class ω. Then $\omega > 0$ if $\omega > 0$ and $\omega \geqq 0$ if $\omega \geqq 0$.

(b) *The cone of positive polynomials and proof of Theorem* D. Let V be a compact, complex manifold and $\mathbf{E} \to V$ a holomorphic vector bundle with Chern classes $c_1, \cdots, c_r$. For a q-tuple $I = (i_1, \cdots, i_q)$, we set $|I| = i_1 + \cdots + i_q$ and $c_I = c_{i_1} \cdots c_{i_1} \in H^{2|I|}(V, \mathbf{Z})$. We let $\mathbf{R} = \bigoplus_{q \geqq 0} \mathbf{R}_q$ be the graded ring of polynomials $P = \sum p_I c_I$ in $c_1, \cdots, c_r$ with rational coefficients; clearly $\mathbf{R}_q \cdot \mathbf{R}_s \subset \mathbf{R}_{q+s}$. We want to define the *cone of positive polynomials* $\mathbf{\Pi} = \bigoplus_{q \geqq 0} \mathbf{\Pi}_q$; $\mathbf{\Pi}$ will be a convex graded cone (over $\mathbf{Q}$) with $\mathbf{\Pi}_q \mathbf{\Pi}_s \subset \mathbf{\Pi}_{q+s}$. Then we will prove

$$\begin{cases} \text{If } \mathbf{E} \to V \text{ is ample and } P \in \mathbf{\Pi}_q\ (q \leqq \dim V), \text{ then} \\ \mathbf{P}(c_1, \cdots, c_r) > 0 \text{ in } H^{2q}(V, \mathbf{Q}) . \end{cases} \tag{5.3}$$

This is Theorem D. It should remain true when $\mathbf{E} \to V$ is positive, but we can only prove that certain $P \in \mathbf{\Pi}$ give positive cohomology classes (cf. the appendix to 5(b) below).

To describe $\mathbf{\Pi}$, we follow Hirzebruch [15] and write formally $1 + c_1t + \cdots + c_rt^r = (1 + \gamma_1 t) \cdots (1 + \gamma_r t)$.

$$(5.4) \quad \begin{cases} \text{Then } \mathbf{R} \cong \mathbf{R}^* \text{ where } \mathbf{R}^* \text{ is the ring of polynomials} \\ \text{in } \gamma_1, \cdots, \gamma_r \text{ which are invariant under the permu-} \\ \text{tation group .} \end{cases}$$

We remark that $\gamma_I = \gamma_{i_1} \cdots \gamma_{i_q}$ now has weight q, so that $\mathbf{R}_q^*$ consists of all invariant polynomials $p(\gamma) = \sum_{I=(i_1,\cdots,i_q)} p_I \gamma_I$.

We now let $B = (B_{\rho\sigma})$ be a variable $r \times r$ matrix $(1 \leqq \rho, \sigma \leqq r)$ and $\gamma_\rho = B_{\rho\rho}$. Then

$$(5.5) \quad \begin{cases} \text{The ring } \mathbf{R}^* \text{ of polynomials in } \gamma_1, \cdots, \gamma_r, \text{ invariant} \\ \text{under the permutation group, is isomorphic to the} \\ \text{ring } \mathbf{I} \text{ of polynomials in } B_{\rho\sigma} \text{ invariant under } B \to \\ MBM^{-1}(M \in GL(r)) \text{ .} \end{cases}$$

Proof. $\mathbf{I}$ is the ring of polynomials $P(B)$ satisfying $P(MBM^{-1}) = P(B)(M \in GL(r))$. The mapping $\mathbf{I} \to \mathbf{R}^*$ is given by $P(B) \to P(\gamma)$ where

$$\gamma = \begin{pmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_r \end{pmatrix}$$

(i.e., $\gamma_\rho = B_{\rho\rho}, B_{\rho\sigma} = 0$ for $\rho \neq \sigma$). To see that this makes sense, we let $\mathbf{h}$ be the vector space of diagonal matrices and $N = \{M \in GL(r): M\mathbf{h}\mathbf{M}^{-1} \subset \mathbf{h}\}$. Then $N \supset H$ where H is the group of non-singular diagonal matrices; H acts trivially on $\mathbf{h}$ (i.e., $M\gamma M^{-1} = \gamma$ for $M \in H$) and $N/H = W$ is the permutation group acting on $\mathbf{h}$. Consequently, if $P \in \mathbf{I}$, then $P(\gamma)$ is a polynomial in $\gamma_1, \cdots, \gamma_r$ invariant under $\gamma_\rho \to \gamma_{\pi(\rho)}$ where $\pi \in S(r)$, $S(r)$ being the permutation group on r symbols. Thus the ring homomorphism $\mathbf{I} \to \mathbf{R}^*$ is well-defined.

If $P(B) \in \mathbf{I}$ and $P(\gamma) \equiv 0$, then $P(B) = 0$ for any matrix B which can be diagonalized. This implies that $P = 0$ in $\mathbf{I}$, and so $\mathbf{I} \to \mathbf{R}^*$ is injective.

Similarly, if $P(\gamma) \in \mathbf{R}^*$ and B is diagonalizable, then we may set $P(B) = P(MBM^{-1})$ where $MBM^{-1} \in \mathbf{h}$. This is well-defined since

$P(\gamma)$ is invariant under N. By continuity, we may then define $P(B)$ for all B and so $\mathbf{I} \to \mathbf{R}^*$ is onto, which proves (5.5).

The gist of (5.4) and (5.5) is that the graded ring $\mathbf{I} = \bigoplus_{q \geq 0} \mathbf{I}_q$ of invariant polynomials (cf. 4 (b)) gives isomorphically the ring $\mathbf{R}$ of polynomials in the Chern classes. A direct mapping $W: \mathbf{I} \to \mathbf{R}$ is the *Weil homomorphism* (4.20).

We now describe those polynomials $P(B) \in \mathbf{I}$ which will be positive in $\mathbf{R}$. As motivation for this, we first observe that any $P(B) \in \mathbf{I}_q$ can be written as

$$(5.6) \qquad P(B) = \sum_{\left\{\substack{\rho=(\rho_1,\cdots\rho_q) \\ \pi,\tau \in S(q)}\right.} p_{\rho,\pi,\tau} B_{\rho_{\pi(1)}\rho_{\tau(1)}} \cdots B_{\rho_{\pi(q)}\rho_{\tau(q)}} .$$

PROOF. If we let $I_0(B) = 1$ and, for $q \geqq 1$,

$$(5.7) \qquad I_q(B) = \sum_{\substack{\rho_1<\cdots<\rho_q \\ \pi \in S(q)}} \operatorname{sgn} \pi B_{\rho_1\rho_{\pi(1)}} \cdots B_{\rho_q\rho_{\pi(q)}} ,$$

then $\mathbf{I}$ is just the ring of polynomials in $I_0(B), \cdots, I_r(B)$. This is because of (5.5) and the fact that $I_q(\gamma) = \sum_{\rho_1<\cdots<\rho_q} \gamma_{\rho_1} \cdots \gamma_{\rho_q}$ is the q^{th} elementary symmetric function of $\gamma_1, \cdots, \gamma_r$. Note that $I_q(B) \in \mathbf{I}_q$ because

$$(5.8) \qquad \det(B + tI) = \sum_{q=0}^{r} I_q(B) t^{r-q} .$$

Since

$$I_q(B) = \left(\frac{1}{q!}\right)^2 \sum_{\substack{\rho=(\rho_1,\cdots,\rho_q) \\ \pi,\tau \in S(q)}} \operatorname{sgn} \pi \operatorname{sgn} \tau B_{\rho_{\pi(1)}\rho_{\tau(1)}} \cdots B_{\rho_{\pi(q)}\rho_{\tau(q)}} ,$$

it is clear that any $P(B) \in \mathbf{I}_q$ is of the form (5.6). To define $\mathbf{\Pi}_q$, we only need to say which polynomials (5.6) are to be positive.

5.9 *Definition.* $P(B) \in \mathbf{I}_q$ is positive if

$$(5.10) \qquad P(B) = \sum_{\left\{\substack{\rho=(\rho_1,\cdots,\rho_q, \\ \pi,\tau \in S(q) \\ j}\right.} \lambda_{\rho,j} q_{\rho,j,\pi} \bar{q}_{\rho,j,\tau} B_{\rho_{\pi(1)}\rho_{\tau(1)}} \cdots B_{\rho_{\pi(q)}\rho_{\tau(q)}}$$

where $\lambda_{\rho,j} \geqq 0$.

Remarks. Using (5.6), $P(B) > 0$ if

$$(5.11) \qquad p_{\rho,\pi,\tau} = \sum_j \lambda_{\rho,j} q_{\rho,j,\pi} \bar{q}_{\rho,j,\tau} .$$

Observe that if $P(B) \in \mathbf{\Pi}_q$, then

$$\begin{aligned} P(\gamma) &= \sum_{\left\{\substack{\rho=(\rho_1,\cdots,\rho_q) \\ \pi \in S(q) \\ j}\right.} \lambda_{\rho,j} \, | q_{\rho,j,\pi} |^2 \gamma_{\rho_{\pi(1)}} \cdots \gamma_{\rho_{\pi(q)}} \\ &= q! \sum_\rho \left\{ \sum_{j,\pi} \lambda_{\rho,j} \, | q_{\rho,j,\pi} |^2 \right\} \gamma_{\rho 1} \cdots \gamma_{\rho q} \end{aligned}$$

so that, numerically, $P(\gamma) > 0$ if γ is real and positive (i.e., $\gamma_\rho > 0$).

More generally, we have

(5.12) If $B = {}^t\bar{B}$ and $B \geqq 0$, then $P(B) \geqq 0$ if $P \in \mathbf{\Pi}_q$.

PROOF. We may write $B = A{}^t\bar{A}$ for some matrix $A = (A_\rho^\alpha)$. Then $B_{\rho\sigma} = \sum_\alpha A_\rho^\alpha \bar{A}_\sigma^\alpha$ and

$$
\begin{aligned}
P(B) &= P(A{}^t\bar{A}) \\
&= \sum_{\begin{cases}\rho, j \\ \pi, \tau \in S(q) \\ \alpha = (\alpha_1, \cdots, \alpha_q)\end{cases}} \lambda_{\rho,j} q_{\rho,\pi,j} \bar{q}_{\rho,\tau,j} A_{\rho_{\pi(1)}}^{\alpha_1} \bar{A}_{\rho_{\tau(1)}}^{\alpha_1} \cdots A_{\rho_{\pi(q)}}^{\alpha_q} \bar{A}_{\rho_{\tau(q)}}^{\alpha_q} \\
&= \sum_{\begin{cases}\rho, j, \alpha \\ \pi, \tau\end{cases}} \lambda_{\rho,j} q_{\rho,\pi^{-1},j} \bar{q}_{\rho,\tau^{-1}j} A_{\rho_1}^{\alpha_{\pi(1)}} \cdots A_{\rho_q}^{\alpha_{\pi(q)}} \bar{A}_{\rho_1}^{\alpha_{\tau(1)}} \cdots \bar{A}_{\rho_q}^{\alpha_{\tau(q)}} \\
&= \sum_{\rho,j} \lambda_{\rho,j} \mid Q_{\rho,\alpha,j}(A) \mid^2
\end{aligned}
$$

where

$$
Q_{\rho,\alpha,j} = \sum_\pi q_{\rho,\pi^{-1},j} A_{\rho_1}^{\alpha_{\pi(1)}} \cdots A_{\rho_q}^{\alpha_{\pi(q)}} . \tag{5.13}
$$

From the definition it is clear that

(5.14) $\mathbf{\Pi}_q$ is a convex cone and $\mathbf{\Pi}_q \mathbf{\Pi}_s \subset \mathbf{\Pi}_{q+s}$.

Roughly speaking, $P(B) \in \mathbf{I}_q$ is positive if, upon writing $B = A{}^t\bar{A}$,

$$
P(B) = \sum_\lambda \mid Q_\lambda(A) \mid^2
$$

where $Q_\lambda(A)$ is a polynomial of degree q in A.

Examples of Positive Polynomials. (i) The q^{th} Chern class $c_q = \sum_{\rho_1 < \cdots < \rho_q} \gamma_{\rho_1} \cdots \gamma_{\rho_q}$ corresponds to the polynomial

$$
I_q(B) = \left(\frac{1}{q!}\right)^2 \sum_{\rho;\pi,\tau} \operatorname{sgn} \pi \operatorname{sgn} \tau B_{\rho_{\pi(1)}\rho_{\tau(1)}} \cdots B_{\rho_{\pi(q)}\rho_{\tau(q)}} .
$$

In (5.10) we then take $\lambda_{\rho,j} = (1/q!)^2$ and $q_{\rho,j,\pi} = \operatorname{sgn} \pi$. Thus $c_q > 0$ and, in fact, $c_I = c_{i_1} \cdots c_{i_q} > 0$. This gives

(5.15) Any polynomial $\sum_I p_I c_I$ with $p_I \geqq 0$ is positive .

(ii) If $\mathbf{E} \to V$ is a *line bundle* with Chern class ω, then $\mathbf{\Pi}_q$ are the classes $\lambda\omega^q$ with $\lambda > 0$.

(iii) Consider the polynomial $P_q(\gamma) = \sum_{\alpha_1 + \cdots + \alpha_r = q} \gamma_1^{\alpha_1} \cdots \gamma_r^{\alpha_r}$. Obviously $P_q(\gamma)$ is invariant under $S(q)$ and so $P_q(\gamma) \in \mathbf{R}_q^*$. We claim

$$
P_q(\gamma) = \sum_{\alpha_1 + \cdots + \alpha_r = q} \gamma_1^{\alpha_1} \cdots \gamma_r^{\alpha_r} \tag{5.16}
$$

is positive.

PROOF. Let $\boldsymbol{\rho} = (\rho_1, \cdots, \rho_q)$ be a q-tuple where the ρ_i need *not* be distinct. We let $\xi(\boldsymbol{\rho})$ be the number of permutations $\pi \in S(q)$

which leave ρ invariant; i.e. which satisfy $\rho_1 = \rho_{\pi(1)}, \cdots, \rho_q = p_{\pi(q)}$. If the ρ_i are distinct, then $\xi(\rho) = 1$; if the ρ_i are all equal, then $\xi(\rho) = q!$. It is clear that $q!/\xi(\rho)$ is the number of *distinct* q-tuples $\sigma = (\sigma_1, \cdots, \sigma_q)$ which are rearrangements of ρ. Thus we find

$$\sum\nolimits_{\alpha_1+\cdots+\alpha_r=q} \gamma_1^{\alpha_1} \cdots \gamma_r^{\alpha_q} = \frac{1}{q!} \sum\nolimits_{\rho=(\rho_1,\ldots,\rho_q)} \xi(\rho)\gamma_{\rho_1} \cdots \gamma_{\rho_q},$$

which gives us that

$$P_q(\gamma) = \left(\frac{1}{q!}\right)^2 \sum\nolimits_{\substack{\rho=(\rho_1,\ldots\rho_q) \\ \pi\in S(q)}} \xi(\rho)\gamma_{\rho_{\pi(1)}} \cdots \gamma_{\rho_{\pi(q)}}. \tag{5.17}$$

Consider now

$$P_q(B) = \left(\frac{1}{q!}\right)^2 \sum\nolimits_{\rho,\pi,\tau} B_{\rho_{\pi(1)}\rho_{\tau(1)}} \cdots B_{\rho_{\pi(q)}\rho_{\tau(q)}}. \tag{5.18}$$

If

$$B = \begin{pmatrix} B_{11} & & 0 \\ & \ddots & \\ 0 & & B_{rr} \end{pmatrix}$$

is diagonal, then

$$\begin{aligned} P_q(B) &= \left(\frac{1}{q!}\right)^2 \sum\nolimits_{\rho,\pi} \left\{ \sum\nolimits_{\substack{\tau(1)=\pi(1) \\ \vdots \\ \tau(q)=\pi(q)}} B_{\rho_{\pi(1)}\rho_{\tau(1)}} \cdots B_{\rho_{\pi(q)}\rho_{\tau(q)}} \right\} \\ &= \left(\frac{1}{q!}\right)^2 \sum\nolimits_{\rho,\pi} \xi(\rho) B_{\rho_{\pi(1)}\rho_{\tau(1)}} \cdots B_{\rho_{\pi(q)}\rho_{\tau(q)}} = P_q(\gamma), \end{aligned}$$

where $\gamma_\rho = B_{\rho\rho}$ and $P_q(\gamma)$ is given by (5.17) above. Thus we need to show that $P_q(B)$ is invariant and positive. For the latter assertion, we simply take $\lambda_\rho = (1/q!)^2$ and $q_{\rho,\pi} = 1$ in (5.10).

To see that $P_q(B)$ is invariant, we set $B = ACA^{-1}$ so that

$$\begin{aligned} &P_q(ACA^{-1}) \\ &= \left(\frac{1}{q!}\right)^2 \sum\nolimits_{\alpha,\gamma,\rho,\pi,\tau} A_{\rho_{\pi(1)}\alpha_1} C_{\alpha_1\gamma_1} (A^{-1})_{\gamma_1\rho_{\tau(1)}} \cdots A_{\rho_{\pi(q)}\alpha_q} C_{\alpha_q\gamma_q} (A^{-1})_{\gamma_q\rho_{\tau(q)}} \\ &= \left(\frac{1}{q!}\right)^2 \sum\nolimits_{\alpha,\gamma,\pi,\tau} \{ \sum\nolimits_{\rho} (A^{-1})_{\gamma_{\tau^{-1}(1)}\rho_1} A_{\rho_1\alpha_{\pi^{-1}(1)}} \cdots \\ &\qquad\qquad (A^{-1})_{\gamma_{\tau^{-1}(q)}\rho_q} \cdot A_{\rho_q\alpha_{\pi^{-1}(q)}} C_{\alpha_1\gamma_1} \cdots C_{\alpha_q\gamma_q} \} \\ &= \left(\frac{1}{q!}\right) \sum\nolimits_{\alpha,\gamma,\pi,\tau} \delta^{\gamma_{\tau^{-1}(1)}}_{\alpha_{\pi^{-1}(1)}} \cdots \delta^{\gamma_{\tau^{-1}(q)}}_{\alpha_{\pi^{-1}(q)}} C_{\alpha_1\gamma_1} \cdots C_{\alpha_q\gamma_q} \\ &= \left(\frac{1}{q!}\right)^2 \sum\nolimits_{\alpha,\pi,\tau} C_{\alpha_1\alpha_{\pi^{-1}\tau(1)}} \cdots C_{\alpha_q\alpha_{\pi^{-1}\tau(q)}} \\ &= P_q(C). \end{aligned}$$

This proves that $P_q(B)$ is invariant and completes the proof of (5.16).

We list here the first few polynomials $P_q(\gamma)$.

$$(5.19)\qquad \begin{cases} P_1(\gamma) = c_1 \\ P_2(\gamma) = c_1^2 - c_2 \\ P_3(\gamma) = c_1^3 - 2c_1c_2 + c_3 \end{cases}$$

Remark. For later interpretation, we record here a fact proved in [11, Lem. A. 1, p. 405]. Let $\mathbf{E} \to V$ be a holomorphic bundle with fibre $\mathbf{C}^r$, $P(\mathbf{E}^*) \xrightarrow{\pi} V$ the associated projective bundle, $\mathbf{L} \to P(\mathbf{E}^*)$ the standard line bundle, and $\omega \in H^2(P(\mathbf{E}^*), \mathbf{Z})$ the characteristic class of $\mathbf{L}$. Recall that there is defined the *integration over the fibre* ([3]):

$$(5.20)\qquad \pi_*\colon H^{2(q+r-1)}(P(\mathbf{E}^*)) \longrightarrow H^{2q}(V)\ ,$$

which satisfies

$$(5.21)\qquad \pi_*(\xi \cup \pi^*\eta) = (\pi_*\xi) \cup \eta \qquad (\xi \in H^*(P(\mathbf{E}^*)), \eta \in H^*(V))\ .$$

(5.22) PROPOSITION. $\pi_*(\omega^{q+r-1}) = P_q(\gamma) \in H^{2q}(V, \mathbf{Z})$.

(iv) Let $P(B) = \sum_{\rho,\pi,\tau} q_\pi \bar{q}_\tau B_{\rho_{\pi(1)}\rho_{\tau(1)}} \cdots B_{\rho_{\pi(q)}\rho_{\tau(q)}}$ where q_π is independent of ρ. Then

$$\begin{aligned} P(ACA^{-1}) &= \textstyle\sum_{\alpha,\gamma,\rho,\pi,\tau} q_\pi \bar{q}_\tau A_{\rho_{\pi(1)}\alpha_1} \cdots A_{\rho_{\pi(q)}\alpha_q} (A^{-1})_{\gamma_1\rho_{\tau(1)}} \cdots \\ &\qquad\qquad (A^{-1})_{\gamma_q\rho_{\tau(q)}} \cdot C_{\alpha_1\gamma_1} \cdots C_{\alpha_q\gamma_q} \\ &= \textstyle\sum_{\alpha,\gamma,\rho,\pi,\tau} q_{\pi^{-1}} \bar{q}_{\tau^{-1}} (A^{-1})_{\gamma_{\tau(1)}\rho_1} A_{\rho_1\alpha_{\pi(1)}} \cdots \\ &\qquad\qquad (A^{-1})_{\gamma_{\tau(q)}\rho_q} A_{\rho_q\alpha_{\pi(q)}} \cdot C_{\alpha_1\gamma_1} \cdots C_{\alpha_q\gamma_q} \\ &= \textstyle\sum_{\alpha,\gamma,\pi,\tau} q_{\pi^{-1}} \bar{q}_{\tau^{-1}} \delta^{\gamma_{\tau(1)}}_{\alpha_{\pi(1)}} \delta^{\gamma_{\tau(q)}}_{\alpha_{\pi(q)}} C_{\alpha_1\gamma_1} \cdots C_{\alpha_q\gamma_q} \\ &= \textstyle\sum_{\alpha,\pi,\tau} q_\pi \bar{q}_\tau C_{\alpha_1\alpha_{\pi^{-1}\tau(1)}} \cdots C_{\alpha_q\alpha_{\pi^{-1}\tau(q)}} \\ &= \textstyle\sum_{\rho,\pi,\tau} q_\pi \bar{q}_\tau C_{\alpha_{\pi(1)}\alpha_{\pi^{-1}\tau\pi(1)}} \cdots C_{\alpha_{\pi(q)}\alpha_{\pi^{-1}\tau\pi(q)}}\ . \end{aligned}$$

Thus we will have $P(ACA^{-1}) = P(C)$ if $q_\tau = q_{\pi^{-1}\tau\pi}$ for $\pi, \tau \in S(q)$. In other words:

$$(5.22)\quad \begin{cases} P(B) = \sum_{\rho,\pi,\tau} q_\pi \bar{q}_\tau B_{\rho_{\pi(1)}\rho_{\tau(1)}} \cdots B_{\rho_{\pi(q)}\rho_{\tau(q)}} \text{ is an in-} \\ \text{variant polynomial} \Leftrightarrow q_\tau = q_{\pi\tau\pi^{-1}} \text{ for all } \tau, \pi \in S(q)\ . \end{cases}$$

This allows us essentially to determine the positive polynomials of low degree. For example, when $q = 2 = r$, $S(q)$ has two elements e, f (e is $(1, 2) \to (1, 2)$ and f is $(1, 2) \to (2, 1)$). We let $q_e = \alpha$ and $q_f = \beta$ and, supposing that α, β are real (and rational), we have $P(B)=(\alpha+\beta)^2\{B_{11}B_{11}+B_{22}B_{22}\}+2(\alpha^2+\beta^2)B_{11}B_{22}+4\alpha\beta B_{12}B_{21}$. Thus $P(\gamma) = (\alpha+\beta)^2\{\gamma_1^2+2\gamma_1\gamma_2+\gamma_2^2\} + 2\{\alpha^2+\beta^2-(\alpha+\beta)^2\}\gamma_1\gamma_2 = (\alpha+\beta)^2c_1^2 - 4\alpha\beta c_2$; i.e.,

$$P(\gamma) = (\alpha+\beta)^2c_1^2 - 4\alpha\beta c_2 \tag{5.23}$$

is the general positive polynomial when $q = 2 = r$.

If $\alpha+\beta=0$ (this is essentially the case $q_\pi = \operatorname{sgn}\pi$ in example (i) above), then $P(\gamma) = \mu c_2$ where $\mu > 0$. If $\alpha = \beta$, then $P(\gamma) = \mu(c_1^2 - c_2)$ where $\mu > 0$, and this is essentially (iii) above.

Assume now that $\alpha+\beta \neq 0$. Then $P(\gamma) = c_1^2 - \{4\alpha\beta/(\alpha+\beta)^2\}c_2$. Now $\{4\alpha\beta/(\alpha+\beta)^2\} \leqq 1$ so that $P(\gamma) = (c_1^2 - c_2) + \mu c_2$ where $\mu \geqq 0$. Thus $\mathbf{\Pi}_2$ is generated by c_2 and $c_1^2 - c_2$.

PROOF OF THEOREM D. Suppose that $\mathbf{E} \to V$ is ample and $P(B) \in \mathbf{\Pi}_q$. It will suffice to find a metric in $\mathbf{E} \to V$ with curvature Θ such that

$$\left(\frac{1}{2\pi i}\right)^q P(\Theta) > 0$$

in the sense of 5(a). Of course we take in $\mathbf{E}$ the metric given by the global sections, which is the same as the metric induced in $\mathbf{E}$ from the universal bundle over the grassmannian (cf. § 1(f)).

Given $z_0 \in V$, we can find a local holomorphic frame $f(z) = (e_1(z), \cdots, e_r(z))$ for $\mathbf{E} \to V$ and a matrix $A = (A_\rho^\alpha(z))$ of (1, 0) forms such that $f(z_0)$ is unitary and

$$\Theta_{\rho\sigma}(z_0) = \sum\nolimits_\alpha A_\rho^\alpha(z_0)\bar{A}_\sigma^\alpha(z_0) \; . \tag{5.24}$$

This follows from (2.24).

Let $P(B)$ be given by (5.10). Then

$$\begin{aligned} P(\Theta) &= \sum\nolimits_{\left\{\substack{\rho,\pi,\tau,j \\ \alpha_1,\cdots,\alpha_q}\right.} \lambda_{\rho,j} q_{\pi,\rho,j} \bar{q}_{\tau,\rho,j} A^{\alpha_1}_{\rho_{\pi(1)}} \bar{A}^{\alpha_1}_{\rho_{\tau(1)}} \cdots A^{\alpha_q}_{\rho_{\pi(q)}} \bar{A}^{\alpha_q}_{\rho_{\tau(q)}} \\ &= (-1)^{[q(q-1)/2]} \sum \lambda_{\rho,j} q_{\pi^{-1},\rho,j} \bar{q}_{\tau^{-1},\rho,j} A^{\alpha_{\pi(1)}}_{\rho_1} \cdots \\ &\qquad\qquad A^{\alpha_{\pi(q)}}_{\rho_q} \bar{A}^{\alpha_{\tau(1)}}_{\rho_1} \cdots \bar{A}^{\alpha_{\tau(q)}}_{\rho_q} \\ &= (-1)^{[q(q-1)/2]} \sum\nolimits_{\alpha,\rho,j} \lambda_{\rho,j} \theta_{\rho,j,\alpha} \wedge \bar{\theta}_{\rho,j,\alpha} \end{aligned}$$

where $\theta_{\rho,j,\alpha} = \sum_\pi q_{\pi^{-1},\rho,j} A^{\alpha_{\pi(1)}}_{\rho_1} \cdots A^{\alpha_{\pi(q)}}_{\rho_q}$ is a form of type $(q, 0)$. This proves that $(1/2\pi i)^q P(\Theta) \geqq 0$; we need only show that some $\theta_{\rho,j,\alpha}(z_0) \neq 0$.

Changing notation slightly, let $\theta_{\rho,\alpha} = \sum_\pi q_{\pi,\rho} A^{\alpha_{\pi(1)}}_{\rho_1} \cdots A^{\alpha_{\pi(q)}}_{\rho_q}$. If all $\theta_{\rho,\alpha} = 0$, then we have

$$\sum\nolimits_\pi q_{\pi,\rho} A^{\alpha_{\pi(1)}}_{\rho_1} \cdots A^{\alpha_{\pi(q)}}_{\rho_q j_1} = 0 \tag{5.25}$$

for all ρ, α, and $j_1 < \cdots < j_q$ where $A^\alpha_\rho = \sum_j A^\alpha_{\rho j} dz^j$.

We need now to interpret the matrices A^α_ρ. This is given in Section 2 (f) where it is shown that: In terms of the holomorphic frame $f(z)$ above (with $f(z_0)$ unitary), we may choose a basis $s^1(z), \cdots, s^m(z) = (\cdots s^\alpha(z) \cdots)$ for the sections of $\mathbf{E} \to V$ which vanish at z_0 such that

$$s^\alpha(z) = \sum_{\rho, j} A^\alpha_{\rho j} z^j e_\rho + (\text{terms of order } 2)\ . \tag{5.26}$$

Since $\mathbf{E} \to V$ is ample, the forms $\sum_{\rho, j} A^\alpha_{\rho j} e_\rho \otimes dz^j$ span $\mathbf{E}_{z_0} \otimes \mathbf{T}^*_{z_0}$. We may choose the sections s^α such that the matrix $(A^\alpha_{\rho j})_{1 \leq \alpha \leq rn}$ is non-singular and $A^\alpha_{\rho j} = 0$ for $\alpha > rn$. Relabeling, we write $s^{\sigma k}(z) = \sum_{\rho, j} A^{\sigma k}_{\rho j} z^j e_\rho + (\cdots)$ where $(A^{\sigma k}_{\rho j})$ is non-singular and $ds^\alpha(z_0) = 0$ for $\alpha > rn$. Then (5.25) becomes:

$$\sum_\pi q_{\pi, \rho} A^{\sigma_{\pi(1)} k_{\pi(1)}}_{\rho_1 j_1} \cdots A^{\sigma_{\pi(q)} k_{\pi(q)}}_{\rho_q j_1} = 0\ . \tag{5.27}$$

Multiplying (5.27) on the right by $(A^{-1})^{\tau_1 i_1}_{\sigma_1 k_1} \cdots (A^{-1})^{\tau_q i_q}_{\sigma_q k_q}$ and summing on $\sigma_1, \cdots, \sigma_q$; $k_1, \cdots, k_q$, we get

$$\sum_\pi q_{\pi, \rho} \delta^{\tau_{\pi(1)}}_{\rho_1} \cdots \delta^{\tau_{\pi(q)}}_{\rho_q} \delta^{i_{\pi(1)}}_{j_1} \cdots \delta^{i_{\pi(q)}}_{j_q} = 0 \tag{5.28}$$

for all ρ, τ, $\mathbf{i} = (i_1, \cdots, i_q)$, and $j_1 < \cdots < j_q$. From (5.28) we get $q_{\pi, \rho} = 0$, a contradiction which completes the proof of Theorem D.

Remarks. As mentioned below (5.3), it should be the case that $P(c_1, \cdots, c_r) > 0$ if $\mathbf{E} \to V$ is *positive* and $P \in \mathbf{\Pi}_q$. In particular, we should be able to prove:

$$\begin{cases} c_q > 0 \\ P_q(c_1, \cdots, c_r) > 0\ , \qquad \text{where } P_q \text{ is given by (5.16)}\ . \end{cases} \tag{5.29}$$

Now in the Appendix to 5(b), we show that $c_2 > 0$ by proving that $(1/2\pi i)^2 I_2(\Theta) > 0$ if $\mathbf{E} \to V$ is positive. It is probably true that $(1/2\pi i)^q I_q(\Theta) > 0$, but this will require a better understanding of the algebraic properties of the curvature form Θ.

If $\mathbf{E} \to V$ is spanned by its sections, then $c_q \geqq 0$. Using (3.51), let us prove:

(5.30) $\begin{cases} \text{If } \mathbf{E} \to V \text{ is spanned by its sections, if } \mathbf{E} \to V \\ \text{is positive, and if } Z_q \subset V \text{ is an algebraic } \textit{sub-} \\ \textit{manifold}, \text{ then } \langle c_q, Z \rangle > 0\ . \end{cases}$

PROOF. Using a standard result, we have over Z an exact sequence

$$0 \longrightarrow \mathbf{S} \longrightarrow \mathbf{E} \mid Z \longrightarrow \mathbf{Q} \longrightarrow 0$$

where S is a trivial bundle with fibre $\mathbf{C}^{r-q}$. Now $\mathbf{E} \mid Z$ is positive and spanned by its sections; by (3.6) the same is true for $\mathbf{Q}$. Since $c_q(\mathbf{E} \mid Z) = c_q(\mathbf{Q})$ and $c_q\colon H^0(Z) \to H^{2q}(Z)$ is an isomorphism (by (3.51)), it follows that $\langle c_q, Z\rangle > 0$ as required.

A possible alternative approach to (5.29) is to split $\mathbf{E}$ into a sum of line bundles (certainly, if Θ is diagonal, then (5.29) holds). We shall show that this method gives $P_q(c_1, \cdots, c_r) > 0$, but not $c_q > 0$.

For simplicity, suppose that $\mathbf{E} \to V$ has fibre $\mathbf{C}^2$ and consider the exact sequence

$$(5.31) \qquad 0 \longrightarrow \mathbf{F} \longrightarrow \pi^*(\mathbf{E}) \longrightarrow \mathbf{L} \longrightarrow 0$$

over $P(\mathbf{E}^*)$. Here $\mathbf{F}_{(z,\xi)} = \{\lambda \in \mathbf{E}_z \colon \langle \xi, \lambda\rangle = 0\}$ where $z \in V, \xi \in P(\mathbf{E}_z^*)$. We choose unitary frames $f = (e_1, e_2)$ for $\pi^*(\mathbf{E})$ such that $e_1 \in \mathbf{F}$ and $e_2 \in \mathbf{L}$ (cf. 2(e)). Then, if

$$\Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix}$$

is the curvature for $\mathbf{E}$ (and also $\pi^*(\mathbf{E})$), the curvature in $\mathbf{F}$ is $\Theta_{\mathbf{F}} = \Theta_{11} - \theta \wedge \bar{\theta}$ and the curvature in $\mathbf{L}$ is $\Theta_{\mathbf{L}} = \Theta_{22} + \theta \wedge \bar{\theta}$. Here $\theta \in A^{1,0}(\mathrm{Hom}(\mathbf{F}, \mathbf{L}))$ is the 2^{nd} fundamental form of $\mathbf{F}$ in $\pi^*(\mathbf{E})$ (cf. 2(d)).

It is easy to check that $\theta \mid P(\mathbf{E}_z^*)$ is non-zero, so that $\Theta_{22} + \theta \wedge \bar{\theta} > 0$ on $P(\mathbf{E}^*)$ and $\mathbf{L} \to P(\mathbf{E}^*)$ is positive (cf. 2(g)).

We now prove that $P_q(\gamma) > 0$ in $H^{2q}(V, \mathbf{Z})$, where $P_q(\gamma)$ is given by (5.16). Let $Z \subset V$ be a q-dimensional algebraic subvariety and $Z_\pi = \pi^{-1}(Z) \subset P(\mathbf{E}^*)$. Then, by (5.22),

$$\langle P_q(c_1, \cdots, c_r), Z\rangle = \langle \pi_* \omega^{q+s-1}, Z\rangle = \langle \omega^{r+q-1}, Z_\pi\rangle > 0$$

since $\omega > 0$ on $P(\mathbf{E}^*)$.

If we try to use this argument to show that, e.g., $c_2 > 0$, we have $\langle c_2, Z\rangle = \langle \pi^* c_2 \omega, Z_\pi\rangle$ so that we need $\pi^* c_2 \omega > 0$. By (5.31), $\pi^* c_2 = c_1(\mathbf{F}) c_1(\mathbf{L})$, so we want

$$\Theta_{\mathbf{F}} \Theta_{\mathbf{L}} \Theta_{\mathbf{L}} = (\Theta_{22} - \theta\bar{\theta})(\Theta_{11} + \theta\bar{\theta})(\Theta_{11} + \theta\bar{\theta})$$

to be positive. The relevant term in this product is $(2\Theta_{11}\Theta_{22} - \Theta_{11}^2)\theta\bar{\theta}$, which however need *not* be positive.

Here we use $\pi^* c_2 \omega$ because $\pi_*(\pi^* c_2 \omega) = c_2$ (by (5.21)). Any $\eta \in H^6(P(\mathbf{E}^*))$ with $\pi_* \eta = c_2$ would work equally well (e.g., $\pi_* \omega^3 = P_2(\gamma) = \pi_*(\pi^*(c_1^2 - c_2)\omega)$); to prove $c_2 > 0$ by this method we need to choose the class η correctly and then make a judicious choice of a differential form representing η. This we are so far unable to do.

Appendix to § 5. (b). We shall prove.

$$(5.32)\quad \begin{cases} \text{Let } \mathbf{E} \to V \text{ be a positive holomorphic bundle} \\ \text{with fibre } \mathbf{C}^2. \text{ Then } c_2(\mathbf{E}) > 0 \, . \end{cases}$$

PROOF. It will suffice to assume that $\dim V = 2$ because, if $\mathbf{E} \to V$ is a positive bundle according to (0.1) and if $Z \subset V$ is an algebraic subvariety, then $\mathbf{E} \mid Z$ is positive.

By assumption, there is a metric in $\mathbf{E} \to V$ with curvature Θ such that the form

$$\Theta(\xi, \eta) = \sum\nolimits_{\rho,\sigma,i,j} \Theta^{\rho}_{\sigma i j}\, \xi^{\sigma}\bar{\xi}^{\rho}\eta^{i}\bar{\eta}^{j}$$

$(1 \leqq \rho, \sigma, i, j \leqq 2)$ is positive. Let $\Phi = (1/2\pi i)^2 \det \Theta$ be the 2^{nd} Chern class of $\mathbf{E} \to V$ according to (4.26). We shall prove that $\Phi > 0$.

Write

$$\Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix}.$$

Then, using at a point a unitary frame for $\mathbf{E} \to V$, we have $\Theta + {}^t\bar{\Theta} = 0$. This gives $\Theta_{11} + \bar{\Theta}_{11} = 0$, $\Theta_{22} + \bar{\Theta}_{22} = 0$, and $\Theta_{12} + \bar{\Theta}_{21} = 0$. Since $\Theta_{11} > 0$, we may choose a co-frame ω^1, ω^2 for V such that $\Theta_{11} = \omega^1 \wedge \bar{\omega}^1 + \omega^2 \wedge \bar{\omega}^2$. Since $\Theta_{22} > 0$, by a unitary change of ω^1, ω^2 we may assume that $\Theta_{22} = \alpha\omega^1\bar{\omega}^1 + \beta\omega^2\bar{\omega}^2$ where $\alpha, \beta > 0$ (we omit the "$\wedge$" symbol). Thus

$$\Theta = \begin{pmatrix} \omega^1\bar{\omega}^1 + \omega^2\bar{\omega}^2 & \theta \\ -\bar{\theta} & \alpha\omega^1\bar{\omega}^1 + \beta\omega^2\bar{\omega}^2 \end{pmatrix}$$

where $\theta = \sum_{i,j} h_{ij}\omega^i\bar{\omega}^j$. Letting $\omega = \omega^1\bar{\omega}^1\omega^2\bar{\omega}^2$,

$$\det \Theta = (\alpha + \beta - h_{11}\bar{h}_{22} - h_{22}\bar{h}_{11} + h_{12}\bar{h}_{12} + h_{21}\bar{h}_{21})\omega \, .$$

Now, by the Schwarz inequality, $h_{11}\bar{h}_{22} + h_{22}\bar{h}_{11} \leqq (| h_{11} |^2 + | h_{22}{}^2)$, so that:

$$(5.33)\qquad \det \Theta \geqq (\alpha - | h_{11} |^2 + \beta - | h_{22} |^2 + | h_{12} |^2 + | h_{21} |^2)\omega \, .$$

From (5.33), it will suffice to prove $\alpha > | h_{11} |^2$, $\beta > | h_{22} |^2$.

For $\xi = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}$, we let $\Theta(\xi) = \sum_{\rho,\sigma} \Theta^{\rho}_{\sigma ij}\xi^{\sigma}\bar{\xi}^{\rho}$. We have used that $\Theta\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \Theta_{11} > 0$, $\Theta\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \Theta_{22} > 0$. Now

$$0 < \Theta\begin{pmatrix}\lambda \\ 1\end{pmatrix} = \Theta_1^1 |\lambda|^2 + \Theta_2^1\lambda + \Theta_1\bar{\lambda} + \Theta_2^2$$
$$= \begin{pmatrix} |\lambda|^2 + \lambda h_{11} + \bar{\lambda}\bar{h}_{11} + \alpha & h_{12}\lambda + \bar{h}_{21}\bar{\lambda} \\ h_{21}\lambda + \bar{h}_{12}\bar{\lambda} & |\lambda|^2 + \lambda h_{22} + \bar{\lambda}\bar{h}_{22} + \beta \end{pmatrix}.$$

Write $\lambda = x + iy$ and $f(x, y) = |\lambda|^2 + \lambda h_{11} + \bar{\lambda}\bar{h}_{11} + \alpha$; then $f(x, y) > 0$ for all x, y. Seeking a minimum for $f(x,y)$, we set $\partial f/\partial x(x,y) = 0$, $\partial f/\partial y(x,y) = 0$ and find $x = -(h_{11} - \bar{h}_{11}/2)$, $y = i\{(-h_{11} + \bar{h}_{11})/2\}$, and $\lambda = -\bar{h}_{11}$. At this point, $0 < f(\lambda) = \alpha - |h_{11}|^2$ so that $|h_{11}|^2 < \alpha$, $|h_{22}|^2 < \beta$ as desired.

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