Hermitian Differential Geometry and the Theory of Positive and Ample Holomorphic Vector Bundles

PHILLIP A. GRIFFITHS

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In his well-known paper [10], Kodaira introduced the concept of a *positive line bundle* over a compact Kähler manifold. He then proved a "vanishing theorem" for the cohomology groups of the associated coherent analytic sheaf and, in [11] Kodaira used this vanishing theorem to show that, if $\mathbf{E}$ is a positive line bundle, then some power $\mathbf{E}^{\mu} = \mathbf{E} \otimes \cdots \otimes \mathbf{E}$ (μ times) has "sufficiently many sections." The converse is also true: if $\mathbf{E}$ is *ample*, then $\mathbf{E}$ is positive.

Now, although in [14] and [13] there was some mention of a definition of positivity for general holomorphic vector bundles, it seems that there has never been any explicit attempt to extend Kodaira's results just mentioned. As a possible exception to this, we mention Grauert's paper [6] in which he introduced the notion of a weakly positive holomorphic vector bundle.

On the other hand, there has recently been some attention directed to a problem of Nirenberg and Spencer ([15] and [16]), the so-called *rigidity problem for holomorphic embeddings*. Let $X \subset W$ be a germ of a holomorphic embedding with normal bundle $\mathbf{N}$. Then Nirenberg and Spencer showed that this germ of embedding is formally rigid whenever $\mathbf{N}$ is either positive or negative; in the positive case, uniqueness holds. In [6], Grauert established the actual (*i.e.* convergent) rigidity theorem when $\mathbf{N}$ is weakly negative by his definition. However, Hironaka has given a counter-example to the actual rigidity in case $\mathbf{N}$ is positive in some sense which has not yet been determined. It then seemed that, in order to further discuss the rigidity and related problems, it was necessary to have clearly in mind what is meant by a positive and/or ample holomorphic vector bundle. The motivation for this paper was to clarify these points for utilization in a companion manuscript on the extension and rigidity problems for positive embeddings.

We now briefly discuss the main points which we hope to make. Recall that a holomorphic line bundle $\mathbf{E}$ is said to be positive if a certain Hermitian form $\Theta(\xi, \xi)$ is positive definite, where Θ involves the curvature of $\mathbf{E}$ and ξ is an

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element of $\mathbf{E} \otimes \mathbf{T}^*$. Since $\mathbf{E}$ is a line bundle, $\xi \in \mathbf{E} \otimes \mathbf{T}^*$ may always be assumed to be *decomposable*; that is, of the form $\xi = \eta \otimes \psi$ where $\eta \in \mathbf{E}$ and $\psi \in \mathbf{T}^*$. For a general vector bundle we wish to have a Hermitian form $\Theta(\xi, \xi)$, $\xi \in \mathbf{E} \otimes \mathbf{T}^*$, involving the curvature in $\mathbf{E}$ which defines positivity and is such that the following requirements are met:

(i) For line bundles, we obtain Kodaira's original definition;
(ii) A suitable vanishing theorem holds;
(iii) If $\mathbf{E}$ is very positive, it has sufficiently many sections in some reasonable sense;
(iv) If $\mathbf{E}$ has sufficiently many sections, it is positive;
(v) If $\mathbf{E}$ is sufficiently positive, then a certain elliptic inequality (§6) holds; and
(vi) Certain standard bundles, such as $\mathbf{T}(\mathbf{P}_n)$, should be positive.

One may easily see by examples that none of the existing definitions of positivity will do, and, in fact, the above six conditions are more or less inconsistent. What we shall do then is to introduce two Hermitian forms Θ and $\boldsymbol{\Theta}$; we say that $\mathbf{E}$ is *weakly positive* if $\Theta(\xi, \xi)$ is positive for decomposable tensors $\xi = \eta \otimes \psi \in \mathbf{E} \otimes \mathbf{T}^*$, and we define $\mathbf{E}$ to be positive if $\boldsymbol{\Theta}(\xi, \xi)$ is positive definite on all tensors. For line bundles, weakly positive = positive. The condition of weak positivity is geometric and easy to check in examples, but (ii) and (v) do not hold. On the other hand, it turns out that the condition of positivity is algebraic and not easy to verify, but positive bundles have the desirable properties (ii), (iii), and (v).

The situation is reminiscent of Riemannian geometry where, for Riemannian manifolds of dimension $2n > 2$, one tries to draw conclusions about polynomials in the curvature (all tensors) from information on the sectional curvatures (decomposable tensors). This problem is in an unsatisfactory state. It is then perhaps of some interest that we are able to prove: If $\mathbf{E}$ is weakly positive, then, for μ sufficiently large, there exists a metric in $\mathbf{E}^\mu$ such that the associated curvature form $\boldsymbol{\Theta}^\mu(\xi, \xi)$ is positive definite. (If $\mathbf{E}$ is a line bundle and $\mathbf{E}^\mu$ is positive, then $\mathbf{E}$ is positive.) The way we do this is by means of the auxiliarly notions of *ample* and *sufficiently ample*. For line bundles, ample = sufficiently ample. Here again, the condition of ampleness is geometric and easy to check; furthermore, an ample bundle is weakly positive. On the other hand, sufficiently ample bundles are positive, but the condition is clumsy to verify. In §4 we prove that, if $\mathbf{E}$ is weakly positive, then for μ large, $\mathbf{E}^\mu$ is sufficiently ample.

We now list our notations. A holomorphic vector bundle $\mathbf{E}$ over a complex manifold X will be written as $\mathbf{E} \to X$; $\mathbf{E}^*$ is the dual bundle of $\mathbf{E}$. The holomorphic tangent bundle of X is $\mathbf{T}$, or $\mathbf{T}(X)$ if ambiguity is possible. The sheaf of germs of holomorphic sections of $\mathbf{E} \to X$ is generally denoted by $\Omega(\mathbf{E})$; the following are the exceptions; $\Theta = \Omega(\mathbf{T})$: $\Omega^q(\mathbf{E}) = \Omega(\mathbf{E} \otimes \wedge^q \mathbf{T}^*)$; and $\mathcal{O}$ is the structure sheaf. We denote by $\mathfrak{A}^{p,q}(\mathbf{E})$ the sheaf of germs of C^∞ $\mathbf{E}$-valued forms of type (p, q); $\mathbf{A}^{p,q}(\mathbf{E}) = H^0(X, \mathfrak{A}^{p,q}(\mathbf{E}))$. All other notations are the standard ones found in [9] or [17].

1. Hermitian differential geometry. Let X be a complex manifold and $\mathbf{E} \xrightarrow{\pi} X$ a holomorphic vector bundle with fibre $\mathbf{C}^r$. Associated to $\mathbf{E}$ there is a unique holomorphic principal bundle $\mathbf{P} \xrightarrow{\tilde{\omega}} X$, with structure group $GL(r, \mathbf{C})$ acting on $\mathbf{P}$ on the right, such that

$$\mathbf{E} = \mathbf{P} \times_{GL(r, \mathbf{C})} \mathbf{C}^r.$$

A point p in a fibre $\mathbf{P}_x$ is a frame $p = (e_1, \cdots, e_r)$, which is by definition a basis for the complex vector space $\mathbf{E}_x$.

If we let $H(r)$ be the manifold of $r \times r$ Hermitian positive definite matrices, then an Hermitian metric in $\mathbf{E}$ is given by C^∞ function $h : \mathbf{P} \to H(r)$ such that $h(pg) = {}^t\bar{g}h(p)g$ ($p \in P$, $g \in GL(r, \mathbf{C})$). Indeed, h is defined by

$$h(p)_{\rho\sigma} = (e_\rho, e_\sigma) \qquad p = (e_1, \cdots, e_r); \tag{1.1}$$

h is positive definite and $h = {}^t\bar{h}$.

There is uniquely defined a matrix $\omega = \{\omega^\rho{}_\sigma\}$ of differential forms of type $(1, 0)$ on $\mathbf{P}$ which satisfies

$$\omega(pg) = g\omega(p)g^{-1}, \tag{1.2}$$

and

$$dh = \omega h + h{}^t\bar{\omega}. \tag{1.3}$$

In fact, if we set $\omega = h^{-1}\,\partial h$, then ω satisfies (1.2) and (1.3), and defines the unique complex connexion in $\mathbf{P}$ such that the covariant differential $Dh = 0$.

If we let $\mathbf{B} \subset \mathbf{P}$ be the sub-bundle of unitary frames, then we write $\omega^\rho{}_\sigma|\mathbf{B} = \omega_{\rho\sigma}$ and then $\omega_{\rho\sigma} + \bar{\omega}_{\sigma\rho} = 0$ (from (1.3), since $dh = 0$). The curvature form $\Theta = \{\Theta_{\rho\sigma}\}$ on $\mathbf{B}$ is given, according to the Cartan structure equation, by

$$\Theta_{\rho\sigma} = d\omega_{\rho\sigma} + \sum_\tau \omega_{\rho\tau} \wedge \bar{\omega}_{\sigma\tau}. \tag{1.4}$$

We also have

$$\Theta_{\rho\sigma} + \bar{\Theta}_{\sigma\rho} = 0. \tag{1.5}$$

Let $U \subset X$ be an open set with holomorphic coordinates $z = (z_1, \cdots, z_n)$ and suppose that we have an analytic isomorphism $\tilde{\omega}^{-1}(U) \cong U \times GL(r, \mathbf{C})$, and subsequently $\pi^{-1}(U) \cong U \times \mathbf{C}^r$. The Hermitian metric is given by a C^∞ function $h : U \to H(r)$. From (1.2) it follows that $\omega(z, g) = g^{-1}\,\partial g + g^{-1}\theta(z)g$, where $\theta = (\theta^\rho{}_\sigma)$ is a matrix of $(1, 0)$ forms in U. From (1.3) it follows that $\theta = h^{-1}\,\partial h$, and from (1.4) it is immediate that $\Theta(z, g) = g^{-1}\Theta(z)g$ where $\Theta(z) = (\bar{\partial}\theta)(z)$.

For a C^∞ section $\xi : U \to \mathbf{E}|U$, the covariant differential $D\xi$ is uniquely defined by $D\xi = D'\xi + \bar{\partial}\xi$ and

$$D'\xi = \partial\xi + \theta \wedge \xi. \tag{1.6}$$

This formula is equally valid if ξ is a differential form with values in $\mathbf{E}$; in this case we write $\theta \wedge \xi = e(\theta)\xi$. It then follows immediately that, for a germ ξ in $\mathfrak{A}^{p,q}(\mathbf{E})$,

$$D^2\xi = (D'\bar{\partial} + \bar{\partial}D')\xi = e(\Theta)\xi, \tag{1.7}$$

where $e(\Theta)\xi = \Theta \wedge \xi$.

If we have in $\mathbf{E}$ a metric h with connexion ω and curvature Θ, then there is induced in the dual bundle $\mathbf{E}^*$ a metric $h^* = {}^t h^{-1}$ with connexion $\omega^* = -{}^t\omega$ and curvature

$$\Theta^* = -{}^t\Theta. \tag{1.8}$$

If, in another holomorphic vector bundle $\mathbf{E}'$, we have a metric h', then in $\mathbf{E} \otimes \mathbf{E}'$ there is a metric $h \otimes h'$ with connexion $\omega \otimes \mathbf{1}' + \mathbf{1} \otimes \omega'$ and curvature

$$\Theta \otimes \mathbf{1} + \mathbf{1} \otimes \Theta', \tag{1.9}$$

where $\mathbf{1}$ and $\mathbf{1}'$ are the identity maps in $\mathbf{E}$ and $\mathbf{E}'$ respectively.

Both of these statements are easy to check:

$$h^{*-1}\partial h^* = {}^t h^t(\partial h^{-1}) = -{}^t(h^{-1}\partial h), \quad \text{and} \quad (h \otimes h')^{-1}\partial(h \otimes h')$$
$$= (h^{-1} \otimes h'^{-1})(\partial h \otimes h' + h \otimes \partial h') = \theta \otimes \mathbf{1}' + \mathbf{1} \otimes \theta'.$$

2. Hermitian metric in the base space. Suppose now that on the tangent bundle $\mathbf{T}$ of X there is given an Hermitian metric. Denote by $\mathbf{E}^{p,q}$ the bundle $\mathbf{E} \otimes \wedge^p \mathbf{T}^* \otimes \wedge^q \bar{\mathbf{T}}^*$ of $\mathbf{E}$-valued (p, q) forms. There is defined the usual star operators $*\colon \mathbf{E}^{p,q} \to \mathbf{E}^{n-q,n-p}$ and in $\mathbf{E}_x^{p,q}$ an inner product by

$$\langle \xi, \eta \rangle \, dX = {}^t\xi \wedge \bar{h} * \bar{\eta}, \tag{2.1}$$

where $\xi, \eta \in \mathbf{E}_x^{p,q}$, h is the Hermitian metric in $\mathbf{E}_x$, and dX is the volume element.

For compact X, there is defined on $\mathbf{A}^{p,q}(\mathbf{E})$ an inner product by

$$(\varphi, \psi) = \int_X \langle \varphi, \psi \rangle \, dX.$$

In particular, the adjoint $\mathfrak{D}$ of $\bar{\partial}$ is defined; $\mathfrak{D} : \mathfrak{A}^{p,q}(\mathbf{E}) \to \mathfrak{A}^{p,q-1}(\mathbf{E})$ and $\mathfrak{D}^2 = 0$. Furthermore, we have the local expression

$$\mathfrak{D} = -*D'* = -*\partial* - *e(\theta)*. \tag{2.2}$$

Denoting by $\Box = \mathfrak{D}\bar{\partial} + \bar{\partial}\mathfrak{D}$ the Laplacian, $\Box\colon \mathbf{A}^{p,q}(\mathbf{E}) \to \mathbf{A}^{p,q}(\mathbf{E})$; and the null space $\mathbf{H}^{p,q}(\mathbf{E})$ is the finite dimensional vector space of harmonic forms in $\mathbf{A}^{p,q}(\mathbf{E})$; *i.e.* the forms $\varphi \in \mathbf{A}^{p,q}(\mathbf{E})$ which satisfy $\bar{\partial}\varphi = 0 = \mathfrak{D}\varphi$.

We recall the isomorphism

$$\mathbf{H}^{p,q}(\mathbf{E}) \cong H^q(X, \Omega^p(\mathbf{E})), \tag{2.3}$$

where $\Omega^q(\mathbf{E}) = \Omega(\mathbf{E}^{q,0})$. The mapping $\# : \mathbf{E}^{p,q} \to \mathbf{E}^{*n-p,n-q}$ defined locally by

$$\#\xi = *\bar{h}\bar{\xi} \tag{2.3}$$

is complex antilinear and $\square \# = \# \square$. Thus $\mathbf{H}^{p,q}(\mathbf{E}) \cong \mathbf{H}^{n-p,n-q}(\mathbf{E}^*)$ and from (2.3) we get the *duality theorem*

$$H^p(X, \Omega^q(\mathbf{E})) \cong H^{n-p}(X, \Omega^{n-q}(\mathbf{E}^*)). \tag{2.4}$$

In particular, when $q = 0$, we have

$$H^p(X, \Omega(\mathbf{E})) \cong H^{n-p}(X, \Omega(\mathbf{K} \otimes \mathbf{E}^*)), \tag{2.5}$$

where $\mathbf{K} = \wedge^n \mathbf{T}^*$ is the canonical bundle.

3. Remarks on the definitions of a positive bundle. There are, to the writer's knowledge, three extant definitions of positivity for holomorphic vector bundles; we shall now compare these. Let $\mathbf{E} \xrightarrow{\pi} X$ be a holomorphic vector bundle, of fibre dimension r, in which we have an Hermitian structure and associated connexion and curvature. Let $U \subset X$ be an open set in which we have holomorphic coordinates $z = (z^1, \cdots, z^n)$ and an isomorphism $\pi^{-1}(U) \cong U \times \mathbf{C}^r$. We agree on the ranges of indices $1 \leqq \rho, \sigma, \tau \leqq r$ and $1 \leqq i, j, k \leqq n$, and write the curvature tensor as $\Theta = \{\Theta^\rho_{\sigma i \bar{j}}\}$. Since, by (1.5), $\langle \Theta\eta, \xi\rangle + \langle \eta, \Theta\xi\rangle = 0$ for all η and ξ, we get $h\Theta + {}^t\bar{\Theta}h = 0$, h being the metric, which in turn gives

$$\sum_\tau h_{\rho\tau}\Theta^\tau_{\sigma i \bar{j}} = \sum_\tau \bar{h}_{\sigma\tau}\bar{\Theta}^\tau_{\rho j i} . \tag{3.1}$$

Setting

$$H(\Theta)_{\sigma i, \rho j} = \sum h_{\rho\tau}\Theta^\tau_{\sigma i \bar{j}} , \qquad H(\Theta)_{\sigma i, \rho j} = \overline{H(\Theta)_{\rho j, \sigma i}} ,$$

and consequently the quadratic form Θ defined on $\mathbf{E}^{1,0} = \mathbf{E} \otimes \mathbf{T}^*$ by

$$\Theta(\xi, \xi) = \sum_{\sigma, i, \rho, j} H(\Theta)_{\sigma i, \rho j}\xi^{\sigma i}\bar{\xi}^{\rho j} \tag{3.2}$$

is Hermitian. The first definition of positivity, which has been used by Spencer [14] and Nakano [13], is that $\mathbf{E}$ is positive if the Hermitian form Θ is positive definite.

The second definition, due to Nirenberg and Spencer [16], is as follows: For each non-zero vector $\lambda = (\lambda^1, \cdots, \lambda^n)$, define $H_\lambda(\Theta)_{\rho\sigma} = \sum h_{\rho\tau}\Theta^\tau_{\sigma i \bar{j}}\lambda^i\bar{\lambda}^j$. Then $\mathbf{E}$ is positive if the quadratic forms Θ_λ defined on $\mathbf{E}$ by

$$\Theta_\lambda(\eta, \eta) = \sum H_\lambda(\Theta)_{\rho\sigma}\, \bar{\eta}^\rho \eta^\sigma \tag{3.3}$$

are positive definite.

The third definition states that, if for each non-zero $\xi = (\xi^1, \cdots, \xi^r)$ we define

$$H_\xi(\Theta)_{i\bar{j}} = \sum_{\rho, \tau, \sigma} h_{\rho\tau}\Theta^\tau_{\sigma i \bar{j}}\xi^\sigma\bar{\xi}^\rho,$$

then the Hermitian form Θ_ξ defined on $\mathbf{T}^*$ by

$$\Theta_\xi(\varphi, \varphi) = \sum H_\xi(\Theta)_{i\bar{j}}\varphi_i\bar{\varphi}_j \tag{3.4}$$

should be positive definite in order that **E** be positive. This definition has been used in [7]. Also, Andreotti and Grauert [1] have shown that this third definition implies that there exists a strongly pseudo-convex tubular neighborhood of the zero cross-section in the dual bundle $\mathbf{E}^*$.

It is clear that the last two definitions essentially amount to saying that the Hermitian form Θ in (3.2) should be positive definite on the set of *decomposable* tensors of the form $\xi = \eta \otimes \varphi$ in $\mathbf{E}^{1,0}$. In particular, if **E** is a line bundle, then all the above definitions coincide and agree with the original one given by Kodaira [10].

If now **L** is a positive line bundle with curvature $\Xi = \{\Xi_{i\bar{j}}\}$, then the curvature $\Phi(\mu) = \{\Phi(\mu)^{\rho}_{\sigma i\bar{j}}\}$ in $\mathbf{E} \otimes \mathbf{L}^{\mu}$ is, by (1.9), given as

$$\Phi(\mu)^{\rho}_{\sigma i\bar{j}} = \Theta^{\rho}_{\sigma i\bar{j}} + \mu\delta^{\rho}_{\sigma}\Xi_{i\bar{j}} \,. \tag{3.5}$$

Thus $\mathbf{E} \otimes \mathbf{L}^{\mu}$ is positive in the sense of (3.2) for μ sufficiently large. Unfortunately, this is the only way known to the author of constructing bundles which are positive in this sense.

Definition. *We shall say that* **E** *is weakly positive if the quadratic form* Θ *given in* (3.2) *is positive on decomposable tensors* $\eta \otimes \varphi$ *in* $\mathbf{E}^{1,0}$.

Thus **E** is weakly positive if, and only if, the quadratic forms (3.3) and (3.4) are positive.

4. Positive and negative holomorphic vector bundles. We shall define what it means for **E** to be negative; **E** will then be positive if $\mathbf{E}^*$ is negative. This is justified by (1.8).

Suppose that we have on X a Kähler metric $ds^2 = \sum_{i=1}^{n} \omega^i\bar{\omega}^i$ where $(\omega^1, \cdots, \omega^n)$ is a local unitary co-frame in $\mathbf{T}^*$. We also assume fixed a local orthonormal frame $(e_1, \cdots, e_r)$ in **E**. We then write an element $\xi \in \mathbf{E}^{0,q}$ as $\xi = \sum_{\rho} (\xi^{\rho} e_{\rho})$ with

$$\xi^{\rho} = \frac{1}{q!} \sum_{j_1 \cdots j_q} \xi^{\rho}_{\bar{j}_1 \cdots \bar{j}_q} \bar{\omega}^{j_1} \wedge \cdots \wedge \bar{\omega}^{j_q} = \frac{1}{q!} \sum_{J} \xi^{\rho}_{\bar{J}} \bar{\omega}^{J},$$

where $J = (j_1, \cdots, j_q)$ runs over all q-tuples and where $\xi^{\rho}_{\bar{j}_1 \cdots \bar{j}_q}$ is anti-symmetric in $j_1, \cdots, j_q$.

Define now

$$\Theta(\xi, \xi) = \frac{1}{q!} \Big\{ \sum_{\rho,\sigma,i,K} \Theta_{\rho\sigma i i}\xi^{\sigma}_{K}\bar{\xi}^{\rho}_{K} - q \sum_{\rho,\sigma,i,j,K'} \Theta_{\rho\sigma i\bar{j}}\xi^{\sigma}_{iK'}\bar{\xi}^{\rho}_{jK'} \Big\}. \tag{4.1}$$

In order to write (4.1) more concisely, we set

$$\Theta^{\#}(\xi, \xi) = \frac{1}{q!} \sum_{\rho,\sigma,i,K} \Theta_{\rho\sigma i i}\xi^{\sigma}_{K}\bar{\xi}^{\rho}_{K} \,, \tag{4.2}$$

and

$$\Theta(\xi, \xi) = \frac{q}{q!} \sum_{\rho,\sigma,i,j,K'} \Theta_{\rho\sigma i\bar{j}}\xi^{\sigma}_{iK'}\bar{\xi}^{\rho}_{jK'} \,, \tag{4.3}$$

so that

$$\mathbf{\Theta}(\xi, \xi) = \Theta^{\#}(\xi, \xi) - \Theta(\xi, \xi). \tag{4.4}$$

Observe that, for $q = 1$, (4.3) coincides with (3.2).

Definition. $\mathbf{E}$ *is negative if* $\mathbf{\Theta}$ *is negative definite for* $0 \leqq q \leqq n - 1$.

In order to justify this, we first show

Proposition 4.1. *Negative in our sense agrees with the definition given by Kodaira when* $\mathbf{E}$ *is a line bundle.*

Proof. When $\mathbf{E}$ is a line bundle, $\Theta = \{\Theta_{i\bar{j}}\}$ is an ordinary differential form. Let $\xi^{(i)} = (\xi_j^{(i)})(1 \leqq i \leqq n)$ be the n eigenvectors of the quadratic form $\Theta_{i\bar{j}}$ relative to the Hermitian metric in $\mathbf{T}^*$, and denote by $\lambda^1, \cdots, \lambda^n$ the corresponding eigenvalues. Set $\xi^{(i_1 \cdots i_q)} = \bar{\xi}^{(i_1)} \wedge \cdots \wedge \bar{\xi}^{(i_q)}$. Then it is immediate that

$$\mathbf{\Theta}(\xi^{(i_1 \cdots i_q)}, \xi^{(i_1 \cdots i_q)}) = \left\{\operatorname{Trace} \Theta - \left(\sum_{\alpha=1}^{q} \lambda^{i_\alpha}\right)\right\} |\xi^{(i_1 \cdots i_q)}|^2. \tag{4.5}$$

Taking then $q = n - 1$, it follows from (4.5) that, if $\mathbf{E}$ is negative in our sense, then each $\lambda^i < 0$ so that $\sum \Theta_{i\bar{j}} \varphi^i \bar{\varphi}^j$ is negative definite and $\mathbf{E}$ is negative in Kodaira's sense.

If, conversely, $\mathbf{E}$ is negative in Kodaira's sense, then all $\lambda^i < 0$ and, by (4.5), the Hermitian form $\mathbf{\Theta}$ is negative definite. In fact, if $\sum \Theta_{i\bar{j}} \varphi^i \bar{\varphi}^j$ is negative definite, we may take the Kähler metric to be $-\sum \Theta_{i\bar{j}} \omega^i \bar{\omega}^j$ and then $\mathbf{\Theta}(\xi, \xi) = -(n - q)\, |\xi|^2$ for all $\xi \in \mathbf{E}^{0,q}$. *Q.E.D.*

Proposition 4.2. *If* $\mathbf{E}$ *is negative, then* $\mathbf{E}$ *is weakly negative.*

Proof. Fixing $\xi = (\xi^1, \cdots, \xi^r)$, then quadratic forms on $\wedge^q \mathbf{T}^*$ defined by

$$\begin{aligned} \mathbf{\Theta}_\xi(\eta, \eta) &= \frac{1}{q!} \{ \sum \Theta_{\rho\sigma i\bar{i}} \xi^\sigma \bar{\xi}^\rho \eta_J \bar{\eta}_J - q \sum \Theta_{\rho\sigma i\bar{j}} \xi^\sigma \bar{\xi}^\rho \eta_{iK} \bar{\eta}_{jK} \} \\ &= \left\{ \operatorname{Trace} \Theta_\xi \, |\eta|^2 - \frac{q}{q!} \sum (\Theta_\xi)_{i\bar{j}} \eta_{iK} \bar{\eta}_{jK} \right\} \end{aligned}$$

are negative definite for $0 \leqq q \leqq n - 1$. But then, just as in Proposition 4.1, we conclude that Θ_ξ in (3.4) is negative definite. Similarly, (3.5) is negative definite. *Q.E.D.*

Proposition 4.3. *Let* $\mathbf{L}$ *be a negative line bundle. Then the vector bundles* $\mathbf{E} \otimes \mathbf{L}^\mu$ *are negative for* $\mu \geqq \mu_0(\mathbf{E})$.

Proof. This follows easily from (3.5) and (4.5).

Let now $\mathbf{E}$ be a negative vector bundle. Then there exists $c' > 0$ such that $\mathbf{\Theta}(\xi, \xi) \leqq -c' \, |\xi|^2$ for all $\xi \in \mathbf{E}^{0,q}$, $0 \leqq q \leqq n - 1$. By (1.9) the curvature acts as a derivation when passing to symmetric powers of $\mathbf{E}$. Thus, letting Θ^μ be the curvature in $\mathbf{E}^\mu$, we find then a constant $c > 0$ such that

(4.6) $\Theta''(\xi, \xi) \leqq -\mu c\, |\xi|^2$, for all $\xi \in (\mathrm{E}'')^{0,q}$, $0 \leqq q \leqq n-1$ and $\mu \geqq 1$.

We shall close this section with a certain algebraic lemma, to be used later. We introduce the following notation: For a tensor $\varphi = \{\varphi_I\}$ where $I = (i_1, \cdots, i_q)$, we denote by $\varphi_{\langle I \rangle}$ the corresponding skew-symmetrized tensor. Suppose now that we are given a family $\{A_\alpha\}$ of $r \times n$ matrices $A_\alpha = (A_\alpha{}^\rho{}_i)$ $(1 \leqq \alpha \leqq m)$. Define a linear mapping $A : \mathbf{C}^r \otimes \wedge^q \mathbf{C}^n \to \mathbf{C}^m \otimes \wedge^{q+1} \mathbf{C}^n$ by

$$A(\xi)^\alpha_I = \hat{A}(\xi)^\alpha_{\langle i_1 \cdots i_{q+1} \rangle}, \qquad I = (i_1, \cdots, i_{q+1}), \tag{4.7}$$

where $\xi = (\xi^\rho_J) \in \mathbf{C}^r \otimes \wedge^q \mathbf{C}^n$ and

$$\hat{A}(\xi)^\alpha_{i_1 \cdots i_{q+1}} = \sum_\rho A^\rho_{\alpha i_1} \xi^\rho_{i_2 \cdots i_{q+1}} . \tag{4.8}$$

Assume that we have a relation

$$\Theta_{\rho\sigma i \bar{j}} = -\sum_{\alpha=1}^{m} A^\rho_{\alpha i} \bar{A}^\sigma_{\alpha j} . \tag{4.9}$$

Proposition 4.4. $\Theta(\xi, \xi) \doteq -\sum_{\alpha, K} |A(\xi)^\alpha_K|^2$.

Proof.

$$\begin{aligned} q!\,\Theta(\xi, \bar{\xi}) &= \sum_{\rho,\sigma,\alpha,i,K} A^\rho_{\alpha i} \bar{A}^\sigma_{\alpha i} \xi^\rho_K \bar{\xi}^\sigma_K - q \sum_{\rho,\sigma,\alpha,i,j,K} A^\rho_{\alpha i} \bar{A}^\sigma_{\alpha j} \xi^\rho_{\langle jK' \rangle} \bar{\xi}^\sigma_{\langle iK' \rangle} \\ &= \sum_{\alpha,i,j,K'} \Big(\sum_\rho A^\rho_{\alpha i} \xi^\rho_{\langle jK' \rangle}\Big)\Big(\sum_\sigma \bar{A}^\sigma_{\alpha i} \bar{\xi}^\sigma_{\langle jK' \rangle} - q \bar{A}^\sigma_{\alpha j} \bar{\xi}^\sigma_{\langle iK' \rangle}\Big) \\ &= \sum_{\alpha,i,j,K'} \hat{A}(\xi)^\alpha_{ijK'} \overline{\hat{A}(\xi)^\alpha_{\langle ijK' \rangle}} \\ &= \sum_{\alpha,i,j,K'} |\hat{A}(\xi)^\alpha_{\langle ijK' \rangle}|^2 = \sum_{\alpha,K} |A(\xi)^\alpha_K|^2. \qquad Q.E.D. \end{aligned}$$

Corollary. *If A is injective for $0 \leqq q \leqq n-1$, then the Hermitian form Θ given by* (4.1) *is negative definite for $0 \leqq q \leqq n-1$.*

5. The vanishing theorems. Let now X be a compact Kähler manifold and $\mathbf{E} \to X$ a holomorphic vector bundle with an Hermitian metric. We shall adhere to the notations previously established. A germ in $\mathfrak{A}^{p,q}(\mathbf{E})$ is then locally written as $\xi = (\xi^\rho)$ where

$$\xi^\rho = \frac{1}{p!\,q!} \sum_{I,J} \xi^\rho_{I\bar{J}} \omega^I \bar{\omega}^J ,$$

and $I = (i_1, \cdots, i_p)$ runs over all p-tuples, $J = (j_1, \cdots, j_q)$ runs over q-tuples, and $\xi^\rho_{I\bar{J}}$ is antisymmetric in the i_α and j_β separately. The inner product between germs ξ and η is

$$\langle \xi, \eta \rangle = \frac{1}{p!\,q!} \sum_{I,J,\rho,\sigma} \xi^\rho_{I\bar{J}} h_{\rho\sigma} \bar{\eta}^\sigma_{I\bar{J}} .$$

We shall write a lower semi-colon for covariant differentiation relative to the Kähler metric on X. Thus $\nabla_i(\xi^\rho_{IJ}) = \xi^\rho_{IJ;i}$. The connexion form of the metric connexion in **E** is given locally by

$$\theta^\rho_{\sigma i} = \sum_\tau h^{\rho\tau} h_{\tau\sigma;i} \,. \tag{5.1}$$

For the curvature, we have

$$\Theta^\rho_{\sigma i\bar{j}} = \theta^\rho_{\sigma i;\bar{j}} \,. \tag{5.2}$$

We shall write a lower bar for covariant differentiation in **E**; so that, for example,

$$\xi^\rho_{IJk} = \xi^\rho_{IJ;k} + \sum_\sigma \theta^\rho_{\sigma k} \xi^\sigma_{IJ} \tag{5.3}$$

(*c.f.* (1.6)).

With these notations, we now give the complex *Weitzenböck formula* ([4], [10], and [14]), which will be of double importance for us. In order to write this formula concisely, we first introduce the following operators:

$$\Theta(\xi)^\rho_{IJ} = \sum_{\sigma, j, s} \Theta^\rho_{\sigma i\bar{j}_s} \xi^\sigma_{I\bar{j}_1 \cdots (\bar{j})_s \cdots \bar{j}_q} \,, \tag{5.4}$$

and

$$\Theta(\xi, \xi) = \langle \Theta(\xi), \xi \rangle. \tag{5.5}$$

Observe that (5.5) is consistent with (4.3).

$$R(\xi)^\rho_{IJ} = \sum_{i, j s, t} R_{i i_s j_t \bar{j}} \xi^\rho_{i_1 \cdots (i)_s \cdots i_p \bar{j}_1 \cdots (\bar{j})_t \cdots \bar{j}_q} \,, \tag{5.6}$$

and

$$\hat{R}(\xi)^\rho_{IJ} = \sum_{j, j_s} R_{j_s \bar{j}} \xi^\rho_{I\bar{j}_1 \cdots (\bar{j})_s \cdots \bar{j}_q} \,, \tag{5.7}$$

where $R_{i\bar{j}k\bar{l}}$ is the *Riemann tensor* and

$$R_{i\bar{j}} = \sum_k R_{\bar{k}ki\bar{j}} \tag{5.8}$$

is the *Ricci tensor*

The Weitzenböck formula is

$$(\square\xi)^\rho_{IJ} = -\sum_k \xi^\rho_{IJ|\bar{k}|k} + \tfrac{1}{2}R(\xi)^\rho_{IJ} - \tfrac{1}{2}\hat{R}(\xi)^\rho_{IJ} + \tfrac{1}{2}\Theta(\xi)^\rho_{IJ} \,, \tag{5.9}$$

which may also be written symbolically as

$$\square\xi = -\sum_k \xi_{|\bar{k}|k} + \tfrac{1}{2}R(\xi) - \tfrac{1}{2}\hat{R}(\xi) + \tfrac{1}{2}\Theta(\xi). \tag{5.10}$$

The formula (5.9) is proven, using (5.1), (5.2), and (5.3), just as the corresponding equation in [4]. The factor of 2 is present to insure the relation $\square = \frac{1}{2}\Delta$ on scalar forms.

Lemma 5.1.

$$\frac{1}{p!\,q!}\int_X \{\sum \xi^{\rho}_{IJ|\bar{k}|k} h_{\rho\bar{\sigma}} \bar{\xi}^{\sigma}_{IJ}\}\, dX \leqq 0.$$

Proof. Set

$$\varphi_{\bar{k}} = \frac{1}{p!\,q!}\sum \xi^{\rho}_{IJ;\bar{k}} h_{\rho\bar{\sigma}} \bar{\xi}^{\sigma}_{IJ} \quad \text{and} \quad \varphi = \sum \varphi_{\bar{k}} \bar{\omega}^k.$$

Letting $-\delta\varphi = \sum_k \varphi_{\bar{k};k}$ be the co-differential (adjoint of d), it is immediate from (5.1) and (5.3) that

$$-\delta\varphi = \frac{1}{p!\,q!}\sum \xi^{\rho}_{IJ|\bar{k}|k} h_{\rho\bar{\sigma}} \bar{\xi}^{\sigma}_{IJ} + |\varphi|^2. \tag{5.11}$$

By Green's theorem,

$$\int_X \delta\varphi\, dX = 0 \quad \text{where} \quad dX = (\sqrt{-1})^n \omega^{\#} \wedge \bar{\omega}^{\#}, \qquad \# = (1, \cdots, n).$$

By (5.11),

$$\frac{1}{p!\,q!}\int_X \{\sum \xi^{\rho}_{IJ|\bar{k}|k} h_{\rho\bar{\sigma}} \bar{\xi}^{\sigma}_{IJ}\}\, dX = -\int_X |\varphi|^2\, dX \leqq 0. \quad Q.E.D.$$

If now $\Box\xi = 0$ so that ξ is harmonic, then (5.9), (5.5), and Lemma 5.1 give

$$\int_X \langle R(\xi) - \hat{R}(\xi), \xi\rangle\, dX + \int_X \Theta(\xi, \xi)\, dX \leqq 0. \tag{5.12}$$

In particular, if $p = n$, then by (5.8) $\langle R(\xi), \xi\rangle = \langle \hat{R}(\xi), \xi\rangle$ and we get

$$\int_X \Theta(\xi, \xi)\, dX \leqq 0, \tag{5.13}$$

where

$$\Theta(\xi, \xi) = \frac{1}{(q-1)!}\sum \Theta^{\rho}_{\sigma i\bar{j}} \xi^{\sigma}_{Ii\bar{j}_2\cdots\bar{j}_q} \bar{\xi}_{I\bar{i}\bar{j}_2\cdots\bar{j}_q}$$

is given by (4.3). From (2.3) we get

Proposition 5.1. *If the quadratic form* (4.3) *is sufficiently positive, then* $H^p(X, \Omega^q(\mathbf{E})) = 0$ *for* $1 \leqq p \leqq n$.

Proposition 5.2. *If the quadratic form* (4.3) *is positive on* $\mathbf{E}^{0,q}$, *then* $H^q(X, \Omega^n(\mathbf{E})) = 0$.

Proposition 5.3. ([13]). *If the quadratic form* (3.2) *is positive for* $\mathbf{K}^{-1} \otimes \mathbf{E}$, *then* $H^1(X, \Omega(\mathbf{E})) = 0$.

We observe also the original Kodaira theorem [10]:

Proposition 5.4. *If* $\mathbf{E}$ *is a negative line bundle, then* $H^q(X, \Omega(\mathbf{E})) = 0 = H^{n-q}(X, \Omega(\mathbf{K} \otimes \mathbf{E}^*))$ *for* $0 \leqq q \leqq n - 1$.

Actually, we may obtain a little more information from (5.13). Suppose then that $\mathbf{E}$ is a line bundle whose curvature form is positive semi-definite with at least α positive eigenvalues at each point. Then the quadratic form (4.3) is positive definite if $q > n - \alpha$, and so $H^q(X, \Omega(\mathbf{K} \otimes \mathbf{E})) = 0$ for $q > n - \alpha$. This result was announced in [7].

Finally, we shall prove:

Proposition 5.5. *Suppose that* $\mathbf{E}$ *is a line bundle and that the curvature form* $\Theta = \{\Theta_{i\bar{j}}\}$ *has* α *positive and* β *negative eigenvalues at each point. Then, for* $\mu \geqq \mu_0$,

$$H^q(X, \Omega(\mathbf{E}^\mu)) = 0 \quad \text{for} \quad 0 \leqq q < \beta \quad \text{and} \quad n - \alpha < q \leqq n. \tag{5.14}$$

Proof. At each point $x \in X$, write $\mathbf{T}_x = \mathbf{T}_x^+ \oplus \mathbf{T}_x^- \oplus \mathbf{T}_x^\#$ where T_x^+ is the positive eigenspace of Θ, $\mathbf{T}^-$ is the negative eigenspace, and $\mathbf{T}^\#$ is the complement to $\mathbf{T}^+ \oplus \mathbf{T}^-$. Then we may write

$$ds^2 = (ds^2)^+ + (ds^2)^- + (ds^2)^\#.$$

But, for positive constants λ and τ, $\lambda(ds^2)^+ + \tau(ds^2)^- + (ds^2)^\#$ is a metric such that, by choosing λ and τ suitably and passing to a power $\mathbf{E}^\mu$, the Hermitian form (4.3) will be positive for $\mathbf{E}^\mu \otimes \mathbf{K}^{-1}$ and $q > n - \alpha$ and for $(\mathbf{E}^*)^\mu$ and $q > n - \beta$. Now this metric will not in general be Kähler, and so one must generalize (5.10) and (5.13). The formula (5.10) is now replaced by

$$\Box\xi = -\sum_k \xi_{|\bar{k}|k} + \tfrac{1}{2}\Theta(\xi) + \sum T_{\bar{k}}\xi_{|\bar{k}} + \sum T_k\xi_{|k} + S(\xi), \tag{5.15}$$

where $S(\xi) = \sum S_\sigma^\rho \xi^\sigma$ and $T = \sum T_{\bar{k}}\bar{\omega}^k + T_k\omega^k$ and S are suitable tensors.

The same proof of Lemma 5.1 now gives more generally an inequality

$$\int_X \Theta(\xi, \xi) + \int_X (P\xi, \xi) \leqq 0, \tag{5.16}$$

where $P\xi \in \mathfrak{A}^q(\mathbf{E})$ and P is a tensor independent of $\mathbf{E}$ and which vanishes if the metric is Kähler. Thus, for μ sufficiently large, we will have $\Theta^\mu(\xi, \xi) + (P\xi, \xi) > |\xi|^2$ and from (5.16) it follows that $H^q(X, \Omega(\mathbf{E}^\mu)) = 0$. *Q.E.D.*

Remark. See [1], Proposition 28.

6. More vanishing theorems and an elliptic inequality. The discussion in §5 centered primarily around the quadratic form $\Theta(\xi, \xi)$ given in (3.2) and (4.3). We shall now give two interpretations of the Hermitian form $\boldsymbol{\Theta}(\xi, \xi)$ introduced in (4.1).

Lemma 6.1. *For* $\xi \in \mathfrak{A}^{0,q}(\mathbf{E})$, $\sqrt{-1}\,\langle \Lambda e(\Theta)\xi, \xi\rangle = \boldsymbol{\Theta}(\xi, \xi)$, *where* Λ *is the Hodge-Weil operator* [17].

Proof. We easily see that

$$q!\,(\sqrt{-1}\Lambda e(\Theta)\xi)^{\rho}_{\bar J} = \sum_{i \notin J} \Theta^{\rho}_{\sigma i \bar i}\xi^{\sigma}_{\bar J} - q \sum_{i \notin J = (j_1, \cdots, j_q)} \Theta^{\rho}_{\sigma i \bar j_1}\xi^{\sigma}_{\bar i \bar j_2 \cdots \bar j_q}$$

$$= \sum_{i,\sigma} \Theta^{\rho}_{\sigma i \bar i}\xi^{\sigma}_{\bar J} - \{ \sum_{\sigma, i \notin J} \Theta^{\rho}_{\sigma i \bar i}\xi^{\sigma}_{\bar J} + q \sum_{\sigma, i \notin J} \Theta^{\rho}_{\sigma i \bar j_1}\xi^{\sigma}_{\bar i \bar j_2 \cdots \bar j_q}\}$$

$$= \sum_{i,\sigma} \Theta^{\rho}_{\sigma i \bar i}\xi^{\sigma}_{\bar J} - q \sum_{i,\sigma} \Theta^{\rho}_{\sigma i \bar j_1}\xi^{\sigma}_{\bar i \bar j_2 \cdots \bar j_q}$$

(using the skew-symmetry of $\xi^{\rho}_{\bar J}$). The Lemma now follows from (4.2), (4.3), and (4.4). *Q.E.D.*

We now recall from [13] the inequality

$$\sqrt{-1}(\Lambda e(\Theta)\xi, \xi) \geqq 0, \tag{6.1}$$

for $\xi \in \mathbf{H}^{0,q}(\mathbf{E})$.

Proof. Letting $\mathfrak{D}' =$ adjoint of D',

$$\begin{aligned}\sqrt{-1}\langle \Lambda e(\Theta)\xi, \xi\rangle &= \sqrt{-1}\langle \Lambda D^2\xi, \xi\rangle \text{ (by (1.7))}\\ &= \sqrt{-1}\langle \Lambda\bar\partial D'\xi, \xi\rangle \text{ (by (1.7) and since } \bar\partial\xi = 0)\\ &= \sqrt{-1}\langle(\bar\partial\Lambda - \sqrt{-1}\mathfrak{D}')D'\xi, \xi\rangle \text{(since } \Lambda\bar\partial - \bar\partial\Lambda = \sqrt{-1}\mathfrak{D}')\\ &= \langle \mathfrak{D}'D'\xi, \xi\rangle \text{(since } \mathfrak{D}\xi = 0)\\ &= \langle D'\xi, D'\xi\rangle \geqq 0.\end{aligned}$$

Combining (6.1) and Lemma 6.1, we get then

Proposition 6.1. *If* $\mathbf{E}$ *is negative, then* $H^q(X, \Omega(\mathbf{E})) = 0$ *for* $0 \leqq q \leqq n - 1$.

This, together with the duality (2.5), gives

Proposition 6.2. *If* $\mathbf{E} \otimes \mathbf{K}^{-1}$ *is positive, then* $H^q(X, \Omega(\mathbf{E})) = 0$ *for* $1 \leqq q \leqq n$.

Definition. *We shall say that* $\mathbf{E}$ *is sufficiently negative if the Hermitian form*

$$\Theta(\xi, \xi) + \langle \hat{R}(\xi, \xi\rangle \tag{6.2}$$

is negative definite on $\mathbf{E}^{0,q}$ *for* $0 \leqq q \leqq n - 1$.

For an element $\xi \in \mathbf{A}^{0,q}(\mathbf{E})$, we set $||\xi|| = \sup_{x \in X} |\xi(x)|$.

Theorem 6.1. *If* $\mathbf{E}$ *is sufficiently negative, then there exists* $c > 0$ *such that*

$$||\xi|| \leqq c\, ||\Box\xi||, \tag{6.3}$$

for all $\xi \in \mathbf{A}^{0,q}(\mathbf{E})$, $0 \leqq q \leqq n - 1$.

In order to prove this theorem, we shall use Bochner's methods [2] together with the following

Lemma 6.2. *For a germ* $\xi \in \mathfrak{A}^{0,q}(\mathrm{E})$,

$$\square\, |\xi|^2 = 2\,\mathrm{Re}\,\langle \xi, \square\xi\rangle - |D'\xi|^2 - |D''\xi|^2 + \Theta(\xi,\xi) + \langle \hat{R}(\xi), \xi\rangle, \tag{6.4}$$

where D'' is a linear first order differential operator.

Proof. We use the fact that, for functions, $\square = -\sqrt{-1}\,\Lambda\partial\bar\partial$ ([17]). From this, a straightforward computation using (5.1) and (5.2) gives

$$\begin{aligned} q!\,\square\,|\xi|^2 = {}& \sqrt{-1}\Lambda\left(\sum \{\xi^\rho_{\bar J} h_{\rho\sigma} \overline{\Theta}^\sigma_{\tau i \bar j} \bar\xi^\tau_{J}\}\omega^i\bar\omega^j\right) - \sum (\xi^\rho_{J;\bar k} h_{\rho\tau} \bar\xi^\tau_{J;\bar k}) \\ & - \sum \{(\xi^\rho_{J;k} + \theta^\rho_{\sigma l}\xi^\sigma_J) h_{\rho\tau} (\xi^\tau_{J;k} + \theta^\tau_k \xi_J)\} - \sum (\xi^\rho_{J;\bar k;l} + \theta^\rho_{\sigma l}\xi^\sigma_{J;\bar k}) h_{\rho\tau}\bar\xi^\tau_J \\ & - \sum \xi^\rho_J h_{\rho\sigma}(\xi^\sigma_{J;\bar k;l} + \theta^\sigma_{\tau l}\xi^\tau_{J;\bar k}). \end{aligned} \tag{6.5}$$

Using (5.3) and (4.2), (6.5) may be simplified to

$$\square\,|\xi|^2 = \Theta^\#(\xi,\xi) - |D''\xi|^2 - |D'\xi|^2 - 2\,\mathrm{Re}\left\{\frac{1}{q!}\sum \xi^\rho_J h_{\rho\sigma}\bar\xi^\sigma_{J|\bar k|k}\right\}, \tag{6.6}$$

where $(D''\xi)^\rho_{J\bar k} = \xi^\rho_{J|\bar k}$.

From the Weitzenböck formula (5.9), we get

$$-\sum_k \xi^\sigma_{J|\bar k|k} = (\square\xi)^\rho_J + \tfrac{1}{2}\hat R(\xi)^\rho_J - \tfrac{1}{2}\Theta(\xi)^\rho_J\,.$$

Substituting this in (6.6), we get

$$\square\,|\xi|^2 = -|D'\xi|^2 - |D''\xi|^2 + \Theta^\#(\xi,\xi) - \Theta(\xi,\xi) + \langle\hat R(\xi),\xi\rangle + 2\,\mathrm{Re}\,\langle\square\xi,\xi\rangle,$$

and Lemma 6.2 follows. *Q.E.D.*

The proof of Theorem 6.1 is now easy. Namely, let x_0 be a point where $|\xi|^2_x$ has a local maximum. Then

$$\square\,|\xi|^2(x_0) = -\sum g^{i\bar j}\frac{\partial^2 |\xi|^2}{\partial z^i\,\partial \bar z^j}(x_0)$$

is non-negative. Now, since **E** is strongly negative, there exists $c' > 0$ and independent of ξ such that $|\xi|^2 \leqq -c'\{\Theta(\xi,\xi) + \langle\hat R(\xi),\xi\rangle\}$. Thus

$$\begin{aligned} \frac{1}{c'}|\xi|^2_{x_0} &\leqq -\{\Theta(\xi,\xi)_{x_0} + \langle\hat R(\xi),\xi\rangle_{x_0}\} \\ &= 2\,\mathrm{Re}\,\langle\xi,\square\xi\rangle_{x_0} - |D'\xi|^2_{x_0} - |D''\xi|^2_x - \square\,|\xi|^2_{x_0} \leqq 2\,\mathrm{Re}\,\langle\xi,\square\xi\rangle\rangle_{x_0}\,. \end{aligned}$$

Then $|\xi|^2_{x_0} \leqq c\,|\langle\xi,\square\xi\rangle|_{x_0} \leqq c\,|\xi|_{x_0}|\square\xi|_{x_0}$ or $|\xi|_{x_0} \leqq c|\square\xi|_{x_0}$. Thus $||\xi|| \leqq c\,||\square\,\xi||$ where $c = 2c'$ is independent of ξ.

Now it is obvious that, if **E** is negative, then E^μ is sufficiently negative for μ large enough. Combining this with (4.6) and Theorem 6.1, we get

Proposition 6.3. *Let* **E** *be a negative vector bundle. Then there exists a μ_0 and a constant $c > 0$ such that, for all $\mu \geqq \mu_0$ and all $\xi \in \mathrm{A}^{0,q}(\mathrm{E}^\mu)$ with $0 \leqq q \leqq n-1$, we have the inequality*

$$||\xi|| \leqq \mu c\,||\square\xi||. \tag{6.7}$$

7. On ample vector bundles. Let m,r be positive integers and $\mathbf{G}(m,r)$ the Grassmann variety of complex m-planes in $\mathbf{C}^{m+r}$. Denote by $\mathbf{F}(m, r)$ the holomorphic vector bundle over $\mathbf{G}(m, r)$ which assigns to each m-plane the corresponding m-dimensional subspace of $\mathbf{C}^{m+r}$. Letting $\mathfrak{C}^{m+r}$ be the trivial bundle $\mathbf{C}^{m+r} \times \mathbf{G}(m, r)$, there is an obvious exact sequence

$$0 \to \mathbf{F}(m, r) \to \mathfrak{C}^{m+r} \to \mathbf{E}(m, r) \to 0. \tag{7.1}$$

Consider the line bundle $\mathbf{L}(m, r) = \det \mathbf{F}(m, r)^* = \det \mathbf{E}(m, r)$. Let $e_1, \cdots, e_{m+r}$ give a basis for $\mathbf{C}^{m+r}$ and set

$$N = \binom{m+r}{m}.$$

Then $\wedge^m \mathbf{C}^{m+r}$ is an N-dimensional vector space with a basis consisting of the vectors $e_{i_1 \cdots i_m} = e_{i_1} \wedge \cdots \wedge \cdots \wedge e_{i_m}$, $(1 \leqq i_1 < \cdots < i_m \leqq m + r)$. Each point $\sigma \in \mathbf{G}(m, r)$ may be written as $\sigma = \sum \sigma^{i_1 \cdots i_m} e_{i_1 \cdots i_m}$ and the $\sigma^{i_1 \cdots i_m}$ give a basis for $\Gamma(\mathbf{L}(m, r))$. Furthermore, the mapping $\pi : \mathbf{G}(m, r) \to \mathbf{P}_{N-1}$ given in homogeneous coordinates by $\pi(\sigma) = [\cdots, \sigma^{i_1 \cdots i_m}, \cdots]$ is an embedding, the Plücker embedding.

Let now X be a compact, complex manifold and $\mathbf{E} \xrightarrow{\pi} X$ a holomorphic vector bundle of fibre dimension r. For each $x \in X$, denote by $\rho_x : \Gamma(\mathbf{E}) \to \mathbf{E}_x$ the restriction mapping. We shall say that $\mathbf{E}$ has *no base points* if ρ_x is everywhere onto. If this is the case, there is an exact bundle sequence

$$0 \to \mathbf{F} \to \mathbf{\Gamma}(\mathbf{E}) \xrightarrow{\rho} \mathbf{E} \to 0, \tag{7.2}$$

where $\mathbf{F}_x = \operatorname{Ker} \rho_x$ and where $\mathbf{\Gamma}(\mathbf{E}) = \Gamma(\mathbf{E}) \times X$ is a trivial bundle of fibre dimension $m + r$ for some $m \geqq 0$. The mapping $f_{\mathbf{E}} : X \to \mathbf{G}(m, r)$ given by $f_{\mathbf{E}}(x) = \mathbf{F}_x \subset \Gamma(\mathbf{E})$ is everywhere defined and $f_{\mathbf{E}}^{-1}((7.1)) = (7.2)$. Conversely, given $\mathbf{E}$, if we can find $f : X \to \mathbf{G}(m, r)$ such that $f^{-1}(\mathbf{E}(m, r)) = \mathbf{E}$, then $\mathbf{E}$ has no base points.

Definition. *We shall say that* $\mathbf{E}$ *is ample if, given any two distinct points* $x, x' \in X$ *and* $\xi \in \mathbf{E}_{x'}$, *there exists a* $\sigma \in \Gamma(\mathbf{E})$ *such that* $\sigma(x) = 0$ *and* $\sigma(x') = \xi$.

(We also require that the obvious infinitesimal condition should be satisfied).

Remark. If $\mathbf{E} \to X$ is ample, then the mapping $f_{\mathbf{E}} : X \to \mathbf{G}(m, r)$ is an embedding; however, $\mathbf{E}(m, r) \to \mathbf{G}(m, r)$ is *not* ample if $m > 1$ and $r > 1$.

Suppose now that $\mathbf{E}$ has no base points, and let $\mathbf{m}_x$ be the maximal ideal of $\mathcal{O}_x$. Then there is an obvious mapping

$$\tau_x : \mathbf{F}_x \to \mathbf{E}_x \otimes \mathbf{m}_x/\mathbf{m}_x^2 \; ; \tag{7.3}$$

and from the τ_x we get $\tau : \mathbf{F} \to \mathbf{E} \otimes \mathbf{T}^*$. We let $\tilde{\omega} : \mathbf{T}^* \otimes \wedge^q \mathbf{T}^* \to \wedge^{q+1} \mathbf{T}^*$ be the natural projection and define a mapping

$$\mathbf{A} : \mathbf{E}^* \otimes \wedge^q \mathbf{T}^* \to \operatorname{Hom}(\mathbf{F}, \wedge^{q+1} \mathbf{T}^*)$$

by

$$\mathbf{A}(e \otimes \varphi)(f) = \tilde{\omega}\{\langle \tau_x(f), e\rangle \otimes \varphi\}, \tag{7.4}$$

where $e \in \mathbf{E}_x^*$, $\varphi \in \wedge^q \mathbf{T}_x^*$, and $f \in \mathbf{F}_x$.

Definition. *We say that* $\mathbf{E}$ *is sufficiently ample if* $\mathbf{E}$ *is ample and if* $\mathbf{A}$ *is an injection for* $0 \leqq q \leqq n - 1$.

The following may be easily verified:

Proposition 7.1. *For* $\mathbf{E}$ *a line bundle, the following are equivalent:* (i) $\mathbf{E}$ *is sufficiently ample;* (ii) *the complete linear system* $|E|$ *gives a projective embedding;* and (iii) $\mathbf{E}$ *is ample and* τ_x *is surjective for all* $x \in X$.

Now we shall prove

Proposition 7.2. *If* $\mathbf{E}$ *is sufficiently ample, then* $\mathbf{E}$ *is positive.*

Proof. We shall prove that the dual bundle $\mathbf{E}^*$ is negative. Let $\sigma_1, \cdots, \sigma_{m+r}$ give a basis for $\Gamma(\mathbf{E})$, and define a metric in $\mathbf{E}^*$ by

$$\langle \xi, \eta\rangle_x = \sum_{\alpha=1}^{m+r} \langle \sigma_\alpha(x), \xi\rangle\overline{\langle \sigma_\alpha(x), \eta\rangle}, \tag{7.5}$$

for $\xi, \eta \in \mathbf{E}_x^*$. Since $\mathbf{E}$ has no base points, the inner product (7.5) is Hermitian positive definite.

Suppose now that $x_0 \in X$ and that $U \supset \{x_0\}$ is a holomorphic coordinate neighborhood such that we have an isomorphism $\pi^{-1}(U) \cong U \times \mathbf{C}^r$. Write explicitly $\sigma_\alpha = (\sigma_\alpha^1, \cdots, \sigma_\alpha^r)$ where the σ_α^ρ are holomorphic functions in U. The metric is given locally by an Hermitian matrix $h = (h_{\rho\tau})$ where

$$h_{\rho\tau} = \sum_\alpha \sigma_\alpha^\rho \bar{\sigma}_\alpha^\tau . \tag{7.6}$$

The curvature is given locally by $\Theta = \bar{\partial}(h^{-1}\,\partial h) = -h^{-1}\,\bar{\partial} h \wedge h^{-1}\,\partial h + h^{-1}\,\bar{\partial}\partial h$. Having fixed x_0, we may assume that our coordinatization is such that $h_{\rho\tau}(x_0) = \delta_\tau^\rho$ and $(\partial h)(x_0) = 0$. Then

$$\Theta_{\tau i \bar{j}}^\rho = -\partial^2 \frac{h_{\rho\tau}}{\partial z^i\, \partial \bar{z}^j},$$

which, by (7.6), is given by

$$\Theta_{\rho\tau i \bar{j}} = -\sum_\alpha A_{\alpha i}^\rho \bar{A}_{\alpha j}^\tau , \tag{7.7}$$

where

$$A_{\alpha i}^\rho = \left(\frac{\partial \sigma_\alpha^\rho}{\partial z^i}\right)(x_0). \tag{7.8}$$

Now the mapping $\mathbf{A}$ given by (7.4) is, when written in the local coordinates around x_0, the same as the mapping A defined in (4.7) and (4.8) where

$A_\alpha = (A^\rho_{\alpha i})$ is given by (7.8). Equations (4.9) and (7.7) then match and Proposition 7.2 follows from the Corollary to Proposition 4.4. *Q.E.D.*

The following Proposition is similar to Proposition 7.2:

Proposition 7.3. *If* $\mathbf{E}$ *is ample, then* $\mathbf{E}$ *is weakly positive.*

8. Positive and ample vector bundles.

Proposition 8.1. *If the bundle* $\mathbf{E} \to X$ *is sufficiently positive*, then it is ample.*

Proof. (See [11]) For $p \in X$, we denote by X_p the quadratic transform of X at p and we let $Q_p : X \to X_p$ be the corresponding birational transformation. Set $S = Q_p(p)$ and $\tilde{\mathbf{E}} = Q_p(\mathbf{E})$. Then S is a projective $(n - 1)$ space and $[S]|S = \mathbf{H}^{-1}$ where $\mathbf{H} = [h]$ is the line bundle determined by a hyperplane $h \subset \mathbf{P}_{n-1}$. Furthermore, $\tilde{\mathbf{E}}|S \cong \mathbf{E}_p \times S$ where $\mathbf{E}_p$ is the fibre of $\mathbf{E}$ at p.

Set now $\mathbf{F}_m = \tilde{\mathbf{E}} \otimes \{[S]^{-m}\}$. Then we have over X_p the exact sheaf sequence

$$0 \to \Omega(\mathbf{F}_{m+1}) \to \Omega(\mathbf{F}_m) \xrightarrow{r_m} \mathbf{E}_p \otimes \Omega_S(\mathbf{H}^m) \to 0. \tag{8.1}$$

Let $\mathbf{K}^\#$ be the canonical bundle over X_p. Then, as in [11], we may check the following: If $\mathbf{E} \to X$ is sufficiently positive, then the bundles $\tilde{\mathbf{E}} \otimes \{[S]^{-m}\} \otimes \{\mathbf{K}^{\#-1}\}$ are positive for $m = 1, 2$. In this case, $H^1(X_p, \Omega(\mathbf{F}_{m+1})) = 0$ for $m = 0, 1$. Thus we get

$$0 \to \Gamma(\mathbf{F}_{m+1}) \to \Gamma(\mathbf{F}_m) \xrightarrow{r_m} \mathbf{E}_p \otimes \Gamma_{P_{n-1}}(\mathbf{H}^m) \to 0, \tag{8.2}$$

for $m = 0, 1$. Since $\Gamma(\mathbf{F}_0) \cong \Gamma_X(\mathbf{E})$, $\mathbf{E}_p \otimes \Gamma_{P_{n-1}}(0) \cong \mathbf{E}_p$, and $\mathbf{E}_p \otimes \Gamma_{P_{n-1}}(\mathbf{H}) \cong \mathbf{E}_p \otimes \mathbf{m}_p/\mathbf{m}_p^2$, we get then

$$0 \to \Gamma(\mathbf{F}_1) \to \Gamma_X(\mathbf{E}) \xrightarrow{r_0} \mathbf{E}_p \to 0, \tag{8.3}$$

and

$$\Gamma(\mathbf{F}_1) \xrightarrow{r_1} \mathbf{E}_p \otimes \mathbf{m}_p/\mathbf{m}_p^2 \to 0. \tag{8.4}$$

But now it is immediate that $r_0 = \rho_p$, $\Gamma(\mathbf{F}_1) \cong \text{Kernel } \rho_p$, and $r_1 = \tau_p$ as defined by (7.3). Thus $\mathbf{E}$ has no base points and the condition for ampleness is satisfied locally. The global condition may then be checked just as in [11]. *Q.E.D.*

Remark. In Proposition (8.1), sufficiently positive then means the following: There exists on X a fixed line bundle $\mathbf{L}$ such that $\mathbf{E} \otimes \mathbf{L}^{-1}$ is positive.

Proposition 8.2. *If a suitable finite collection of cohomology groups*

$$\mathbf{H}^1(X, \Omega(\mathbf{E}) \otimes \mathcal{S}_\alpha) = 0,$$

where the $\mathcal{S}_\alpha$ *are coherent analytic sheaves, then* $\mathbf{E}$ *is sufficiently ample.*

Proof. Let $p \in X$ be a fixed point, $\Omega(\mathbf{F}_p) \subset \Omega(\mathbf{E})$ the germs of holomorphic sections of $\mathbf{E}$ which vanish at p, and denote by $\mathfrak{I}_p$ and $\mathcal{E}_p$ the "skyscraper sheaves" whose stalks at p are $\mathbf{T}_p$ and $\mathbf{E}_p$ respectively and which are zero elsewhere.

* See the remark below.

Define a sheaf homorphism

$$\varphi_p : \Omega(\mathbf{F}_p) \otimes \wedge^{q+1}\mathfrak{I}_p \to \mathcal{E}_p \otimes \wedge^q \mathfrak{I}_p \to 0 \tag{8.5}$$

as follows: For $f \in \Omega(\mathbf{F}_p)_x$ and $\gamma \in (\wedge^{q+1}\mathfrak{I}_p)_x$ with $x \neq p$, $\varphi_p(f \otimes \gamma) = 0$; and, for $f \in \Omega(\mathbf{F}_p)_p$ and $\gamma \in (\wedge^{q+1}\mathfrak{I}_p)_p$,

$$\varphi_p(f \otimes \gamma) = {}^t\mathbf{A}(f(p) \otimes \gamma), \tag{8.6}$$

where ${}^t\mathbf{A} : \mathbf{F}_p \otimes \wedge^{q+1}\mathbf{T}_p \to \mathbf{E}_p \otimes \wedge^q \mathbf{T}_p$ is the transpose of the mapping given by (7.4).

The point is that the kernel of φ_p is a sheaf of the form $\Omega(\mathbf{E}) \otimes_{\mathfrak{o}} \mathcal{S}_p$ where $\mathcal{S}_p$ is a coherent analytic sheaf on X, and thus we get an exact sequence

$$0 \to \Omega(\mathbf{E}) \otimes \mathcal{S}_p \to \Omega(\mathbf{F}_p) \otimes \wedge^{q+1}\mathfrak{I}_p \to \mathcal{E}_p \otimes \wedge^q \mathfrak{I}_p \to 0, \tag{8.7}$$

and, if $H^1(X, \Omega(\mathbf{E}) \otimes \mathcal{S}_p) = 0$, then the mapping $\mathbf{A}$ given by (7.4) is an injection at $p \in X$. Since then $\mathbf{A}$ is injective in a neighborhood, the Proposition follows.

Corollary 8.1. *If* $\mathbf{E}$ *is positive, then there exists a* μ_0 *such that* $\mathbf{E}^\mu$ *is ample for* $\mu \geqq \mu_0$.

Corollary 8.2. *If* $\mathbf{E}$ *is positive, then there exists a* μ_1 *such that* $\mathbf{E}^\mu$ *is sufficiently ample for* $\mu \geqq \mu_1$.

9. Weakly positive and ample vector bundles. We shall prove the following

Theorem 9.1. *Let* $\mathbf{E}$ *be a weakly positive vector bundle. Then, for* $\mu \geqq \mu_0$, $\mathbf{E}^\mu$ *is sufficiently ample.*

Proof. Associated to the holomorphic vector bundle $\mathbf{E} \xrightarrow{\pi} X$ there is a bundle of projective spaces $\mathbf{P}(\mathbf{E}) \xrightarrow{\tilde{\omega}} X$ where $\mathbf{P}(\mathbf{E})_x = \mathbf{P}(\mathbf{E}_x^*)$ is the projective space associated to $\mathbf{E}_x^*$. If we embed X in $\mathbf{E}^*$ as zero cross-section, then $\mathbf{E}^* - X$ is a principal $\mathbf{C}^*$ bundle over $\mathbf{P}(\mathbf{E})$, and we denote by $\mathbf{L}(\mathbf{E}) \to \mathbf{P}(\mathbf{E})$ the corresponding line bundle. Clearly $\mathbf{L}(\mathbf{E})|\mathbf{P}(\mathbf{E})_x$ is the hyperplane bundle.

We shall complete the proof in a sequence of three propositions, some of which may be of independent interest.

Proposition 9.1. *If* $\mathbf{E}$ *is weakly positive, then* $\mathbf{L}(\mathbf{E})$ *is positive in the sense of Kodaira.*

Proof. There is given on $\mathbf{E}$ an Hermitian structure and thus on $\mathbf{E}^* - X$ there is defined a positive function f by $f(x, \xi) = \langle \xi,\xi \rangle_x = {}^t\bar{\xi} h(x)\, \xi$, where $h(x)$ is the Hermitian metric on $\mathbf{E}$.

The reader may easily verify the following (see [11]):

Lemma 9.1. *The differential form* $\sqrt{-1}/2\pi\, \partial\bar{\partial} \log f$ *is closed differential form on* $\mathbf{P}(\mathbf{E})$ *which represents* $c_1(\mathbf{L}(\mathbf{E}))$.

To prove Proposition 9.1, we must then calculate $\partial\bar{\partial} \log \langle \xi, \xi \rangle_x$ as a function of x and ξ. In doing this, we shall adopt certain symbolic notations whose meaning should be obvious. First we get

$$\partial \log \langle \xi, \xi \rangle = \frac{\partial \langle \xi, \xi \rangle}{\langle \xi, \xi \rangle} = \frac{{}^t\bar{\xi}\partial h \xi + {}^t\bar{\xi} h \partial \xi + {}^t\partial \bar{\xi} h \xi}{\langle \xi, \xi \rangle}.$$

From this, it follows that

$$\bar{\partial}\partial \log \langle \xi, \xi \rangle = \frac{\langle \xi, \xi \rangle \{ {}^t\overline{\partial \xi} \partial h \xi + {}^t\bar{\xi} \bar{\partial}\partial h \xi + {}^t\overline{\partial \xi} h \partial \xi + {}^t\bar{\xi} \bar{\partial} h \partial \xi \}}{\langle \xi, \xi \rangle^2} - \frac{({}^t\bar{\xi}\partial h \xi + {}^t\bar{\xi} h \partial \xi)({}^t\overline{\partial \xi} h \xi + {}^t\bar{\xi}\bar{\partial} h \xi)}{\langle \xi, \xi \rangle^2}.$$

Expanding this, we get

$$\bar{\partial}\partial \log \langle \xi, \xi \rangle = \frac{\{ {}^t\overline{\partial \xi} h h^{-1} \partial h \xi + {}^t\bar{\xi} h h^{-1} \bar{\partial}\partial h \xi + {}^t\overline{\partial \xi} h \partial \xi + {}^t(\overline{h^{-1}\partial h \xi}) h \partial \xi \}}{\langle \xi, \xi \rangle}$$

$$= \frac{({}^t\bar{\xi} h h^{-1} \partial h \xi)({}^t\overline{\partial \xi} h \xi) - ({}^t\bar{\xi} h h^{-1} \partial h \xi)({}^t\{\overline{h^{-1}\partial h \xi}\} h \xi)}{\langle \xi, \xi \rangle^2}$$

$$- \frac{({}^t\bar{\xi} h \partial \xi)({}^t\overline{\partial \xi} h \xi) - ({}^t\xi h \partial \xi)({}^t\{\overline{h^{-1}\partial h \xi}\} h \xi)}{\langle \xi, \xi \rangle^2}.$$

We write now $\theta = h^{-1}\, \partial h$ and $\Theta = \bar{\partial}\theta = -(h^{-1}\, \bar{\partial} h)(h^{-1}\, \partial h) + h^{-1}(\bar{\partial}\partial h)$. Substituting these in the above expression and rearranging terms, we get

$$\bar{\partial}\partial \log \langle \xi, \xi \rangle_x = \Theta(x, \xi) + S(x, \xi) + T(x, \xi), \tag{9.1}$$

where

$$\Theta(x, \xi) = \frac{\langle \Theta \xi, \xi \rangle}{\langle \xi, \xi \rangle}; \tag{9.2}$$

$$S(x, \xi) = \frac{\langle \xi, \xi \rangle (\langle \partial \xi, \partial \xi \rangle - \langle \partial \xi, \xi \rangle \langle \xi, \partial \xi \rangle}{\langle \xi, \xi \rangle^2}; \tag{9.3}$$

and

$$T(x, \xi) = \left\{ \frac{\langle \theta \xi, \theta \xi \rangle + 2 \operatorname{Re} \langle \theta \xi, \partial \xi \rangle}{\langle \xi, \xi \rangle} \right\} - \left\{ 2 \operatorname{Re} \frac{\langle \partial \xi, \xi \rangle \langle \theta \xi, \xi \rangle}{\langle \xi, \xi \rangle^2} \right\}. \tag{9.4}$$

The restriction of $\bar{\partial}\partial \log \langle \xi, \xi \rangle$ to any fibre $\mathbf{P}(\mathbf{E})_{x_0}$ is $S(x_0, \xi)$ in (9.3), which we recognize as the fundamental form on the projective space $\mathbf{P}(\mathbf{E}_{x_0})$.

Fix now $(x_0; \xi_0) \in \mathbf{P}(\mathbf{E})$ and let $z = (z^1, \cdots, z^n)$ be a holomorphic coordinate system on X centered at x_0. Then, in these coordinates, the form $\Theta(x, \xi)$ in (9.2) is given locally as

$$\Theta(x, \xi) = \Theta_\xi(dz, dz), \tag{9.5}$$

where Θ_ξ is given by (3.4). Furthermore, as in the proof of Proposition 7.2, we may assume that $\theta(x_0) = 0$ so that $T(x_0, \xi_0) = 0$. By assumption, $\Theta_\xi(dz, dz)$ is a positive definite form in $dz^1, \cdots, dz^n$, and $S(x, \xi)$ is positive definite in

$[d\xi^1, \cdots, d\xi^r]$, the differentials of the homogeneous coordinates of the fibre $\mathbf{P}(\mathbf{E})_{x_0}$. From Lemma 9.1, we conclude that $\mathbf{L}(\mathbf{E}) \to \mathbf{P}(\mathbf{E})$ is positive.

Remark. As an application of the Proposition 9.1, we have the following result of Kodaira [11]:

Proposition. *Let* $\mathbf{B}$ *be a fibre bundle over a compact algebraic variety* X *with fibre* $\mathbf{P}_r$. *Then* $\mathbf{B}$ *is a compact algebraic variety.*

Denote by $\mathbf{K}^\#$ the canonical bundle on $\mathbf{P}(\mathbf{E})$. Combining Propositions 9.1 and 5.2, we get

Corollary. *If* $\mathbf{E}$ *is weakly positive, then* $H^p(\mathbf{P}(\mathbf{E}), \Omega(\mathbf{K}^\# \otimes \mathbf{L}(\mathbf{E}))) = 0$ *for* $p \geqq 1$.

In the fibering $\mathbf{P}(\mathbf{E}) \xrightarrow{\tilde{\omega}} X$, if we let $\mathbf{F}(\mathbf{E})$ be the bundle along the fibres, then there is an exact bundle sequence

$$0 \to \mathbf{F}(\mathbf{E}) \to \mathbf{T}(\mathbf{P}(\mathbf{E})) \to \tilde{\omega}^*(\mathbf{T}(X)) \to 0. \tag{9.6}$$

Furthermore, the reader may, with some pains, verify

Lemma 9.2. $\det \mathbf{F}(\mathbf{E}) = \mathbf{L}(\mathbf{E})^r$.

From (9.6) and Lemma 9.2, we get

$$\mathbf{K}^\# = \mathbf{L}(\mathbf{E})^{-r}\tilde{\omega}^*(\mathbf{K}). \tag{9.7}$$

Thus $\mathbf{K}^{\#-1} \otimes \mathbf{L}(\mathbf{E}) = \mathbf{L}(\mathbf{E})^{r+1} \otimes \tilde{\omega}^*(\mathbf{K}^{-1})$, and from Proposition 5.4 we get

Proposition 9.2. *If* μ_0 *is chosen so that* $\mathbf{E}^{\mu_0} \otimes \mathbf{K}^{-1}$ *is weakly positive, then* $\mathbf{L}(\mathbf{E}^\mu) \otimes \mathbf{K}^{\#-1}$ *is positive and*

$$H^q(\mathbf{P}(\mathbf{E}^\mu), \Omega(\mathbf{L}(\mathbf{E}^\mu))) = 0 \quad \textit{for} \quad \mu \geqq \mu_0, \qquad 1 \leqq q \leqq n. \tag{9.8}$$

Proposition 9.3. *There exist natural isomorphisms*

$$H^q(X, \Omega(\mathbf{E})) \cong H^q(\mathbf{P}(\mathbf{E}), \Omega(\mathbf{L}(\mathbf{E}))). \tag{9.9}$$

Proof. (This Proposition is implicitly contained in [9]; it is proven explicitly in a very general form in [3]. For completeness, we give here a proof). Denote the sheaf $\Omega(\mathbf{L}(\mathbf{E}))$ over $\mathbf{P}(\mathbf{E})$ by $\mathcal{S}$. We shall apply the Leray spectral sequence [5] to $\mathcal{S}$ and the proper holomorphic mapping $\tilde{\omega} : \mathbf{P}(\mathbf{E}) \to X$. Accordingly, there exists a spectral sequence $\{E_r\}$ such that $E_\infty \to H^*(\mathbf{P}(\mathbf{E}), \mathcal{S})$ and $E_2^{p,q} = H^p(X, \tilde{\omega}^q(\mathcal{S}))$, where $\tilde{\omega}^q(\mathcal{S})$ is the qth Leray sheaf. The Proposition will follow if we can show that $\tilde{\omega}^q(\mathcal{S}) = 0$ for $q > 0$ and $\tilde{\omega}^0(\mathcal{S}) \cong \Omega(\mathbf{E})$.

Now $\tilde{\omega}^q(\mathcal{S})$ is the sheaf associated to the presheaf which assigns to an open set $U \subset X$ the group $H^q(\tilde{\omega}^{-1}(U), \mathcal{S}|\tilde{\omega}^{-1}(U))$. Let $U \subset X$ be a polycylinder such that $\tilde{\omega}^{-1}(U) \cong U \times \mathbf{P}_{r-1}$ and $\pi^{-1}(U) \cong U \times \mathbf{C}^r$. The principal bundle of $\mathbf{L}(\mathbf{E})$, when restricted to $\tilde{\omega}^{-1}(U)$, is then given by $\mathbf{C}^* \to U \times (\mathbf{C}^r - \{0\}) \to U \times \mathbf{P}_{r-1}$. Thus, letting $\mathbf{H} \to \mathbf{P}_{r-1}$ be the hyperplane bundle, it follows that $\mathcal{S}|\tilde{\omega}^{-1}(U) \cong \mathcal{O}_U \hat{\otimes} \Omega_{\mathbf{P}_{r-1}}(\mathbf{H})$. Since $H^s(U, \mathcal{O}_U) = 0 = H^s(\mathbf{P}_{r-1}, \Omega_{\mathbf{P}_{r-1}}(\mathbf{H}))$ for $s > 1$ and since $H^0(\mathbf{P}_{r-1}, \Omega_{\mathbf{P}_{r-1}}(\mathbf{H})) \cong \mathbf{C}^r$, it follows immediately that

$$H^0(\tilde{\omega}^{-1}(U), \mathcal{S} \mid \tilde{\omega}^{-1}(U)) \cong \Gamma(U, \Omega(\mathbf{E}) \mid U), \qquad \tilde{\omega}^0(\mathcal{S}) = \Omega(\mathbf{E}),$$

and $\tilde{\omega}^q(\mathcal{S}) = 0$ for $q > 0$. *Q.E.D.*

Remark. The "obvious" Kunneth relations which we have used may be justified by Grothendieck's general machine [8] or perhaps more simply by applying the theory of harmonic integrals developed by J. Kohn [12].

Combining (1.9) and Propositions 9.1, 9.2, and 9.3, we find

Proposition 9.4. *Let* $\mathbf{E}$ *be a weakly positive vector bundle and* $\mathbf{F} \to X$ *any holomorphic vector bundle. Then there exists an integer* $\mu_0 = \mu_0(\mathbf{F})$ *such that*

$$H^q(X, \Omega(\mathbf{F} \otimes \mathbf{E}^\mu)) = 0 \quad \text{for} \quad q \geqq 1 \quad \text{and} \quad \mu \geqq \mu_0 . \tag{9.10}$$

It is of course obvious that (9.10) holds replacing $\mathbf{F}$ with any coherent sheaf on X. Theorem 9.1 now follows from Proposition 8.2.

Corollary 1. *Let* $\mathbf{E}$ *be weakly positive. Then for* μ *sufficiently large,* $\mathbf{E}^\mu$ *is positive.*

Corollary 2. *Let* $\mathbf{E}$ *be ample. Then, for* μ *sufficiently large,* $\mathbf{E}^\mu$ *is sufficiently ample and positive.*

Corollary 2 follows from Proposition 7.3.

Finally, as a consequence of Proposition 5.5, (9.1), and Proposition 9.3 we have

Proposition 9.5. *Suppose that for each* $\xi \neq 0$ *the Hermitian form* Θ_ξ *in* (3.4) *has* α *positive and* β *negative eigenvalues. Then, for* μ *sufficiently large,*

$$H^q(X, \Omega(\mathbf{E}^\mu)) = 0 \quad \text{for} \quad 0 \leqq q < \beta \quad \text{and} \quad n - \alpha < q \leqq n. \tag{9.11}$$

Remark. Compare with Proposition 2.8 in [1].

10. Applications and examples.

(i) Some of the differences between positivity for line bundles and general vector bundles may be accounted for by the following

Proposition 10.1. *The universal bundle* $\mathbf{E}(m, r) \to \mathbf{G}(m, r)$ *is not weakly positive if* $r > 1$.

Proof. Write $\mathbf{G}(m, r)$ as the coset space $\mathbf{U}(m + r)/\mathbf{U}(m) \times \mathbf{U}(r)$. Then $\mathbf{E}(m, r)$ is associated to the principal fibering

$$\mathbf{U}(m + r) \to \mathbf{U}(m + r)/\mathbf{U}(m) \times \mathbf{U}(r)$$

by means of the usual representation of $\mathbf{U}(r)$ on $\mathbf{C}^r$. The Euclidean metric on $\mathbf{C}^r$ induces an Hermitian structure in $\mathbf{E}(m, r)$ whose curvature we shall now compute.

Let $\omega = u^{-1}\, du$ be the left-invariant Maurer-Cartan form on $\mathbf{U}(m + r)$.

Since ${}^t\bar{u}u = 1$, $\omega + {}^t\bar{\omega} = 0$. Differentiating $u\omega = du$, we get the Maurer-Cartan equation

$$d\omega = -\omega \wedge \omega. \tag{10.1}$$

We now agree on the ranges of indices $1 \leqq \alpha, \beta, \gamma \leqq m; m + 1 \leqq i, j, k \leqq m + r$. Writing

$$\omega = \begin{bmatrix} \omega_{\alpha\beta} & \omega_{\alpha i} \\ \omega_{i\beta} & \omega_{ij} \end{bmatrix},$$

it follows easily that the forms ω_{ij} give the connexion form of the metric connexion in $\mathbf{E}(m, r)$.

By (1.4), the curvature $\Theta = \{\Theta_{ij}\}$ is given by

$$\Theta_{ij} = d\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj} .$$

By (10.1), we then get

$$\Theta_{ij} = \sum_{\alpha=1}^{m} \omega_{i\alpha} \wedge \bar{\omega}_{j\alpha} . \tag{10.2}$$

The complex dimension of $\mathbf{G}(m, r)$ is mr, and, letting $x_0 \in \mathbf{G}(m, r)$ be the identity coset, a tensor in $\mathbf{T}^*_{x_0}(\mathbf{G}(m, r))$ may be written as $\varphi = \{\varphi^{\alpha i}\}$. Furthermore, a typical element in $\mathbf{E}(m, r)_{x_0}$ is written as $\xi = \{\xi^j\}$. From (10.2) it then follows that the quadratic form (3.4) is given by

$$\Theta_\xi(\varphi, \varphi) = \sum_{\alpha, i, j} \xi^i \bar{\xi}^j \varphi^{i\alpha} \bar{\varphi}^{j\alpha}. \tag{10.3}$$

Clearly, if $r > 1$, Θ_ξ is only positive semi-definite.

Now $\mathbf{U}(m + r)$ acts transitively on $\mathbf{E}(m, r) \to \mathbf{G}(m, r)$, and so any metric in $\mathbf{E}(m, r)$ for which the Hermitian forms Θ_ξ are positive definite can be averaged to give an invariant such metric. But any invariant metric is unique up to a positive constant, and so $\mathbf{E}(m, r)$ is not weakly positive. *Q.E.D.*

(ii) Now we prove the following vanishing theorem:

Proposition 10.2. *Suppose that the canonical bundle* $\mathbf{K}$ *of* X *is negative. Then, if* $\mathbf{E}$ *is any holomorphic vector bundle without base points,* $H^q(X, \Omega(\mathbf{E})) = 0$ *for* $q \geqq 1$.

Proof. By the argument in Proposition 7.2, there exists in $\mathbf{E}^*$ an Hermitian structure such that the form (4.1) is negative semidefinite for $0 \leqq q \leqq n - 1$. Since $\mathbf{K}$ is negative, it follows from (1.9) that the Hermitian form (4.1) is negative definite for $\mathbf{K} \otimes \mathbf{E}^*$ and $0 \leqq q \leqq n - 1$. By Proposition 6.1, $H^q(X, \Omega(\mathbf{K} \otimes \mathbf{E}^*)) = 0$ for $0 \leqq q \leqq n - 1$. The result now follows from the duality formula (2.5). *Q.E.D.*

In order to apply this Proposition, we observe that $\mathbf{T}(\mathbf{G}(m, r)) = \operatorname{Hom}(\mathbf{F}(m, r), \mathbf{E}(m, r))$. Thus, the canonical bundle $\mathbf{K}$ for $\mathbf{G}(m, r)$ is equal to $\mathbf{L}(m, r)^{-2}$ and $\mathbf{K}$ is negative. Applying Proposition 10.2, we get

Proposition 10.3. *$H^q(\mathbf{G}(m, r), \Omega(\mathbf{E}(m, r))) = 0 = H^q(\mathbf{G}(m, r), \Theta)$ for $q > 0$.*

(iii) An example of an ample, and hence weakly positive, vector bundle is $\mathbf{T}(\mathbf{P}_n)$. The condition for ampleness follows immediately from the geometric statement that the projective group acts transitively on the tangent directions on $\mathbf{P}_n$. However, $\mathbf{T}(\mathbf{P}_n)$ is not sufficiently ample for $n > 1$. Indeed, if this were the case, then by Proposition 7.2 $\mathbf{T}(\mathbf{P}_n)^*$ would be negative and by Proposition 6.1, $H^1(\mathbf{P}_n, \Omega(\mathbf{T}(\mathbf{P}_n)^*)) = H^1(\mathbf{P}_n, \Omega^1) = 0$, which is absurd. This simple example illustrates the distinction which must be drawn between weakly positive and positive, and ample and sufficiently ample for holomorphic bundles of fibre dimensions greater than one.

(iv) We may easily verify the following

Proposition 10.4. *If we have an exact bundle sequence $\mathbf{E} \to \mathbf{F} \to 0$ and if $\mathbf{E}$ is ample, then $\mathbf{F}$ is ample.*

Remark. Proposition 10.4 is not true with sufficiently ample replacing ample. Indeed, letting $\mathbf{H} \to \mathbf{P}_n$ be the hyperplane bundle and $\mathbf{1}$ the trivial bundle, there is an exact sequence

$$0 \to \mathbf{1} \to \underbrace{\mathbf{H} \oplus \cdots \oplus \mathbf{H}}_{n+1} \to \mathbf{T}(\mathbf{P}_n) \to 0, \tag{10.4}$$

and $\underbrace{\mathbf{H} \oplus \cdots \oplus \mathbf{H}}_{n+1}$ is sufficiently ample but $\mathbf{T}(\mathbf{P}_n)$ is not.

Let now $X \subset \mathbf{P}_n$ be a non-singular algebraic variety; then $\mathbf{T}(\mathbf{P}_n)|X = \mathbf{T}$ is ample, and the normal bundle $\mathbf{N}$ to X in $\mathbf{P}_n$ is defined by $0 \to \mathbf{T}(X) \to \mathbf{T} \to \mathbf{N} \to 0$. Thus

Corollary. *The normal bundle to a projective algebraic variety is ample and hence weakly positive.*

(iv) For $\mathbf{E} \to X$ a holomorphic bundle let

$$c(\mathbf{E}) = c_0 + c_1 + \cdots + c_r, \qquad (c_i \in H^{2i}(X, \mathbf{Z}))$$

be the total Chern class. If we have in $\mathbf{E}$ an Hermitian structure, then there are defined (q, q) forms $\theta_q (1 \leqq q \leqq r)$ which represent c_q via the deRham isomorphism (θ_q is a polynomial of degree q in the entries of the curvature tensor Θ; in fact, we have $1 + \theta_1 t + \cdots + \theta_r t^r = \det\,(tI^r + \sqrt{-1/2\pi}\,\Theta)$). For a vector $I = (i_1, \cdots, i_r)$ of non-negative integers, set $\theta_I = \theta_1^{i_1} \cdots \theta_r^{i_r}$ and $|I| = i_1 + 2i_2 + \cdots + ri_r$. One can prove the following

Theorem. *Let $Y \subset X$ be any irreducible subvariety of dimension s. Then, if $\mathbf{E} \to X$ is ample,*

$$\int_Y \theta_I > 0, \quad \textit{for all} \quad I \quad \textit{with} \quad |I| = s. \tag{10.5}$$

If, conversely, (10.5) *is satisfied for every quotient bundle of* $\mathbf{E} \mid Y$ *and all* $Y \subset X$, *then some power of* $\mathbf{E}$ *is ample.*

Remark. (10.5) is not satisfied for $\mathbf{E}(m, r) \to G(m, r)$ if $r > 1$.

11. A synopsis. For the convenience of the reader, we now give a resume of the main implications which have been established in this paper.

The following is a list of the pertainent properties which may hold for a holomorphic vector bundle E over a compact Kähler manifold X:

(i) $\mathbf{E}$ is *weakly positive* (§3);
(ii) $\mathbf{E}$ is *positive* (§4);
(iii) the *vanishing* property; that is, $H^q(X, \Omega(\mathbf{E}^*)) = 0$ for $0 \leqq q \leqq n - 1$ (§§5 and 6);
(iv) *elliptic inequality*; that is, (6.3) holds;
(v) *stable vanishing* property; that is, for any coherent analytic sheaf $\mathcal{S}$, $H^q(X, \Omega(\mathbf{E}^\mu) \otimes \mathcal{S}) = 0$ for $\mu \geqq \mu_0(\mathcal{S})$, $1 \leqq q \leqq n$ (§8);
(vi) $\mathbf{E}$ is *ample* (§7);
(vii) $\mathbf{E}$ is *sufficiently ample* (§7); and
(viii) the line bundle $\mathbf{L}(\mathbf{E}) \to \mathbf{P}(\mathbf{E})$ is positive (§9).

With the obvious abbreviations, we list now the main inferences which have been drawn in this paper:

$$\text{(I)} \quad \left\{ \begin{array}{l} \mathbf{E} \text{ ample} \xrightarrow{\S 7} \mathbf{E} \text{ weakly positive} \xrightarrow{\S 9} \mathbf{L}(\mathbf{E}) \text{ positive} \\ \xrightarrow{\S\S 5 \text{ and } 9} \text{Stable vanishing} \xrightarrow{\S 8} \mathbf{E}^\mu \text{ sufficiently ample} \xrightarrow{\S 7} \\ \mathbf{E}^\mu \text{ positive} \end{array} \right.$$

$$\text{(II)} \quad \mathbf{E} \text{ positive} \begin{array}{l} \overset{\S 6}{\nearrow} \text{ stable vanishing} \\ \xrightarrow{\S 6} \text{vanishing} \\ \underset{\S 6}{\searrow} \text{ elliptic inequality for } \mathbf{E}^\mu \end{array}$$

$$\text{(III)} \quad \mathbf{E} \text{ sufficiently ample} \xrightarrow{\S 7} \mathbf{E} \text{ positive}$$

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University of California
Berkeley